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Optimal control of the thermistor problem in three spatial dimensions

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Abstract. This paper is concerned with the state-constrained optimal control of the three-dimensional thermistor problem, a fully quasilinear coupled system of a parabolic and elliptic PDE with mixed boundary conditions. This system models the heating of a conducting material by means of direct current. Local existence, uniqueness and continuity for the state system are derived by employing maximal parabolic regularity in the fundamental theorem of Prüss. Global solutions are addressed, which includes analysis of the linearized state system via maximal parabolic regularity, and existence of optimal controls is shown if the temperature gradient is under control. The adjoint system involving measures is investigated using a duality argument. These results allow to derive first-order necessary conditions for the optimal control problem in form of a qualified optimality system. The theoretical findings are illustrated by numerical results.

1. Introduction. In this paper, we consider the state-constrained optimal control of the three-dimensional thermistor problem. In detail the optimal control problem under consideration looks as follows:

$$\left.\begin{aligned} \min\quad & \frac{1}{2}\|\theta(T_1)-\theta_d\|^2_{L^2(E)} + \frac{\gamma}{s}\|\nabla\theta\|^s_{L^s(T_0,T_1;L^q(\Omega))} + \frac{\beta}{2}\int_{\Sigma_N}(\partial_t u)^2+|u|^p \,\mathrm{d}\omega\,\mathrm{d}t \\ \text{s.t.}\quad & (1.1)\text{–}(1.6) \\ \text{and}\quad & \qquad \theta(x,t) \le \theta_{\max}(x,t) \quad \text{a.e. in } \Omega\times(T_0,T_1), \\ & 0 \le u(x,t) \le u_{\max}(x,t) \quad \text{a.e. on } \Gamma_N\times(T_0,T_1) \end{aligned}\right\} \tag{P}$$

where (1.1)–(1.6) refer to the following coupled PDE system consisting of the instationary nonlinear heat equation and the quasi-static potential equation, which is also known as *thermistor problem*:

$$\partial_t\theta - \operatorname{div}(\eta(\theta)\kappa\nabla\theta) = (\sigma(\theta)\varepsilon\nabla\varphi)\cdot\nabla\varphi \qquad \text{in } Q := \Omega\times(T_0,T_1) \tag{1.1}$$

$$\nu\cdot\kappa\nabla\theta + \alpha\theta = \alpha\theta_l \qquad \text{on } \Sigma := \partial\Omega\times(T_0,T_1) \tag{1.2}$$

$$\theta(T_0) = \theta_0 \qquad \text{in } \Omega \tag{1.3}$$

$$-\operatorname{div}(\sigma(\theta)\varepsilon\nabla\varphi) = 0 \qquad \text{in } Q \tag{1.4}$$

$$\nu\cdot\sigma(\theta)\varepsilon\nabla\varphi = u \qquad \text{on } \Sigma_N := \Gamma_N\times(T_0,T_1) \tag{1.5}$$

$$\varphi = 0 \qquad \text{on } \Sigma_D := \Gamma_D\times(T_0,T_1). \tag{1.6}$$

Here θ is the temperature in a conducting material covered by the three dimensional domain Ω, while φ refers to the electric potential. The boundary of Ω is denoted by $\partial\Omega$ with the unit normal ν facing outward of Ω in almost every boundary point (w.r.t. the boundary measure ω). In addition, for the boundary we have $\Gamma_D \,\dot\cup\, \Gamma_N = \partial\Omega$, where Γ_D is closed within $\partial\Omega$. The functions $\eta(\cdot)\kappa$ and $\sigma(\cdot)\varepsilon$ represent heat- and electric conductivity. While κ and ε are given, prescribed functions, η and σ are allowed to depend on the temperature θ. Moreover, α is the heat transfer coefficient and θ_l and θ_0 are given boundary– and initial data, respectively. Finally, u stands for a current which is induced via the boundary part Γ_N and is to be controlled. The bounds in the optimization problem (P) as well as the desired temperature θ_d are given functions and β is the usual Tikhonov regularization parameter. The precise assumptions on the data in (P) and (1.1)–(1.6) will be specified in §2. In all what follows, the system (1.1)–(1.6) is frequently also called *state system*.

The PDE system (1.1)–(1.6) models the heating of a conducting material by means of a direct current, described by u, induced on the part Γ_N of the boundary, which is done for some time $T_1 - T_0$. At the grounding Γ_D, homogeneous Dirichlet boundary conditions are given, i.e., the potential is zero, inducing electron flow. Note that, usually, u will be zero on a subset Γ_{N_0} of Γ_N, which corresponds to having insulation at this part of the boundary. We emphasize that the different boundary conditions are essential for a realistic modeling of the process. The objective of (P) is to adjust the induced current u to minimize the L^2-distance between the desired and the resulting temperature at end time T_1 on the set $E \subseteq \Omega$, the latter representing the area of the material in which one is interested – realized in the objective functional by the first term. The other terms are in present to minimize thermal stresses (second term) and to ensure a certain smoothness of the controls (third term), whose influence to the objective functional, however, may be controlled by the weights γ and β. The actual form of these terms is motivated by functional-analytic considerations, see §4.1. Moreover, the optimization is subject to pointwise control and state constraints. The control constraints reflect a maximum heating power, while the state constraints limit the temperature evolution to prevent possible damage, e.g. by melting of the material. Similarly to the mixed boundary conditions, the inequality constraints in (P) are essential for a realistic model as demonstrated by the numerical example within this paper. Problem (P) is relevant in various applications, such as for instance the heat treatment of steel by means of an electric current. The example considered in the numerical part of this paper deals with an application of this type.

The state system (1.1)–(1.6) exhibits some non-standard features, in particular due to the quasilinear coupling of the parabolic and the elliptic PDE, the mixed boundary conditions in (1.5)–(1.6), and the inhomogeneity in the heat equation (1.1) as well as the temperature-dependent heat conduction coefficients. Besides the quasilinear state system, the pointwise state constraints on the temperature represent another challenging feature of the optimal control problem under consideration. The Lagrange multipliers associated with constraints of this kind only provide poor regularity in general, which especially complicates the analysis of the adjoint equation.

We briefly describe the genuine aspects of our work. First of all, the discussion of the quasilinear state system alone requires sophisticated up-to-date tools from maximal elliptic and parabolic regularity theory. This concerns already local-in-time existence for solutions of (1.5)–(1.6), let alone the characterization of global-in-time solutions. The corresponding maximal regularity results were established only recently, see e.g. [7, 33, 36] for the parabolic case and [43, Appendix], [19] for the elliptic one. Our key ingredient for the proof of local-in-time existence is a general result of Prüss on quasilinear parabolic equations [50]. To verify the assumptions required for the application of Prüss' result, we heavily rely on an isomorphism property of the elliptic differential operators in both equations of the state system. Assuming this isomorphism property only for the case of pure diffusion coefficients κ and ε in the differential operators, see Assumption 3.4 below, we show that the nonlinear differential operators involving $\eta(\theta)$ and $\sigma(\theta)$ then also enjoy it, by a technique developed in [43]. However, this analysis only guarantees the local-in-time existence, and the counterexample in [6] involving a blow-up criterion for a similar model of the thermistor system demonstrates that one can, in general, not expect global-in-time solutions. Nevertheless, based on recent results on non-autonomous parabolic equations [48], we prove that there are control functions that admit global-in-time solutions. Moreover, using the implicit function theorem, we show that these control functions form an open set, which is essential for the derivation of optimality conditions in qualified form that are useful for numerical computations. Concerning the existence of global minimizers for (P), we benefit from the pointwise state constraints and the second addend in the objective functional involving the gradient of the temperature. Both terms prevent a blow-up of the temperature and its gradient and allow to restrict the discussion of the optimization problem to control functions that admit a global-in-time solution of the state system. This approach is inspired by [4], where a similar technique was used to establish the existence of optimal controls.

Let us put our work into perspective. Up to the authors' best knowledge, there are only few contributions dealing with the optimal control of the thermistor problem. We refer to [45, 15, 41], where two-dimensional problems are discussed. In [45], a completely parabolic problem is discussed, while [41] considers the purely elliptic counterpart to (1.1)–(1.6). In [15, 5], the authors investigate a parabolic-elliptic system similar to (1.1)–(1.6), assuming a particular structure of the controls. In contrast to [45, 41], mixed boundary conditions are considered in [15]. However, all these contributions do not consider pointwise state constraints and non-smooth data. Thus, (P) differs significantly from the problems considered in the aforementioned papers. In a previous paper [39], two of the authors investigated the two-dimensional counterpart of (P). This contribution also accounts for mixed boundary conditions, non-smooth data, and pointwise state constraints. However, the analysis in [39] substantially differs from the three dimensional case considered here. First of all, in two spatial dimensions, the isomorphism-property of the elliptic operators mentioned above directly follows from the classical paper [30]. Moreover, the heat conduction coefficient in (1.1) is assumed not to depend on the temperature in [39]. Both features allow to derive a global existence result for a suitable class of control functions. Hence, main aspects of the present work do not appear in the two-dimensional setting. Let us finally take a broader look on state-constrained optimal control problems governed by PDEs. Compared to semilinear state-constrained optimal control problems, the literature concerning optimal control problems subject to quasilinear PDEs and pointwise state constraints is rather scarce. We exemplarily refer to [13, 12], where elliptic problems are studied. The vast majority of papers in this field deals with problems that possess a well defined control-to-state operator. By contrast, as indicated above, the state-system (1.1)–(1.6) in general just admits local-in-time solutions, which requires a sophisticated treatment of the optimal control problem under consideration.

The paper is organized as follows: We set the stage with notations and assumptions in §2 and discuss the state-system in §3. More precisely, §3.1 collects preliminary results, also interesting in their own sake, while §3.2 is devoted to the actual proof of existence and uniqueness of local-in-time solutions. We then proceed with the optimal control problem in §4. Before stating first order necessary conditions for (P) in §4.2, we give sufficient conditions for controls to produce global solutions and establish continuous differentiability of the control-to-state operator for global solutions in §4.1. The paper is wrapped-up with an illustrative numerical example in §5.

2. Notations and general assumptions. We introduce some notation and the relevant function spaces. All function spaces under our consideration are *real* ones. Let, for now, Ω be a domain in $\mathbb{R}^3$. We give precise geometric specifications for Ω in §2.1 below.

Let us fix some notations: The underlying time interval is called $J = (T_0, T_1)$ with $T_0 < T_1$. The boundary measure for the domain Ω is called ω. Generally, given an integrability order $q \in (1, \infty)$, we denote the conjugated of q by q', i.e., it always holds $1/q + 1/q' = 1$.

Definition 2.1. *For $q \in (1, \infty)$, let $W^{1,q}(\Omega)$ denote the usual Sobolev space on Ω. If $\Xi \subset \partial\Omega$ is a closed part of the boundary $\partial\Omega$, we set $W^{1,q}_\Xi(\Omega)$ to be the closure of the set $\{\psi|_\Omega : \psi \in C_0^\infty(\mathbb{R}^3),$ $\operatorname{supp} \psi \cap \Xi = \emptyset$ with respect to the $W^{1,q}$-norm.*

The dual space of $W^{1,q'}_\Xi(\Omega)$ is denoted by $W^{-1,q}_\Xi(\Omega)$; in particular, we write $W^{-1,q}_\emptyset(\Omega)$ for the dual of $W^{1,q'}(\Omega)$ (see Remark 2.3 below regarding consistency). The Hölder spaces of order δ on Ω or order ϱ on Q are denoted by $C^\delta(\Omega)$ and $C^\varrho(Q)$, respectively (note here that Hölder continuous functions on Ω or Q, respectively, possess an unique uniformly continuous extension to the closure of the domain, such that we will mostly use $C^\delta(\overline{\Omega})$ and $C^\varrho(\overline{Q})$ to emphasize on this).

We will usually abbreviate the function spaces on Ω by leaving out the Ω, e.g. we write $W^{1,q}_\Xi$ instead of $W^{1,q}_\Xi(\Omega)$ or L^p instead of $L^p(\Omega)$. Lebesgue spaces on subsets of $\partial\Omega$ are always to be considered with respect to the boundary measure ω, but we abbreviate $L^p(\partial\Omega, \omega)$ by $L^p(\partial\Omega)$ and do so analogously for any ω-measurable subset of the boundary. The norm in a Banach space X will be always indicated by $\|\cdot\|_X$. For two Banach spaces X and Y, we denote the space of linear, bounded operators from X into Y by $\mathcal{L}(X;Y)$. The symbol $\mathcal{LH}(X;Y)$ stands for the set of linear homeomorphisms between X and Y. If X, Y are Banach spaces which form an interpolation couple, then we denote by $(X,Y)_{\tau,r}$ the real interpolation space, see [54]. We use M_3 for the set of real, symmetric 3×3-matrices. In the sequel, a linear, continuous injection from X to Y is called an *embedding*, abbreviated by $X \hookrightarrow Y$. For Lipschitz continuous functions f, we denote the Lipschitz constants by L_f, while for bounded functions g we denote their bound by M_g (both over appropriate sets, if necessary). Finally, c denotes a generic positive constant.

2.1. Geometric setting for Ω and Γ_D. In all what follows, the symbol Ω stands for a bounded Lipschitz domain in $\mathbb{R}^3$ in the sense of [47, Ch. 1.1.9]; cf. [35] for the boundary measure ω on such a domain.

Remark 2.2. *The thus defined notion is different from* strong Lipschitz domain, *which is more restrictive and in fact identical with* uniform cone domain, *see again [47, Ch. 1.1.9]).*

A Lipschitz domain is formed e.g. by the topologically regularized union of two crossing beams (see [33, Ch. 7]), which is *not* a strong Lipschitz domain. Moreover, the interior of any three-dimensional connected polyhedron is a Lipschitz domain, if the polyhedron is, simultaneously, a 3-manifold with boundary, cf. [32, Thm. 3.10]. However, a ball minus half of the equatorial plate is *not* a Lipschitz domain, and a chisel, where the blade edge is bent onto the disc, is also not.

Remark 2.3. *The Lipschitz property of Ω implies the existence of a linear, continuous extension operator $\mathfrak{E} : W^{1,q}(\Omega) \to W^{1,q}(\mathbb{R}^3)$ (see [26, p.165]). This has the following consequences:*

- *Since any element from $W^{1,q}(\mathbb{R}^3)$ may be approximated by smooth functions in the $W^{1,q}$-norm, any element from $W^{1,q}(\Omega)$ may be approximated by restrictions of smooth functions in the $W^{1,q}(\Omega)$-norm. This tells us that the definitions of $W^{1,q}(\Omega)$ and $W^{1,q}_\Xi(\Omega)$ are consistent in case of $\Xi = \emptyset$, i.e., one has $W^{1,q}(\Omega) = W^{1,q}_\emptyset(\Omega)$. See also the detailed discussion in [29, Ch. 1.3.2].*
- *It is not hard to see that $\mathfrak{E}$ also provides a continuous extension operator $\mathfrak{E} : C^\delta(\Omega) \to C^\delta(\mathbb{R}^3)$ and $\mathfrak{E} : L^p(\Omega) \to L^p(\mathbb{R}^3)$, where $\delta \in (0,1), p \in [1, \infty]$.*
- *Finally, the existence of the extension operator $\mathfrak{E}$ provides the usual Sobolev embeddings $W^{1,q}(\Omega) \hookrightarrow L^p(\Omega)$. In particular, this yields, by duality, the embedding $L^{q/2}(\Omega) \hookrightarrow W^{-1,q}_\emptyset(\Omega)$ if*

q exceeds the space dimension three.

Next we define the geometric setting for the domains Ω and the Dirichlet boundary part. For this, we denote by K the open unit cube in $\mathbb{R}^n$, centered at $0 \in \mathbb{R}^n$, by K_- the lower half cube $K \cap \{\mathrm{x}\colon x_n < 0\}$, by $\Sigma_K = K \cap \{\mathrm{x}\colon x_n = 0\}$ the upper plate of K_- and by Σ_K^0 the left half of Σ, i.e. $\Sigma_K^0 = \Sigma_K \cap \{\mathrm{x}\colon x_{n-1} \le 0\}$.

DEFINITION 2.4. *Let $\Xi \subset \partial\Omega$ be closed within $\partial\Omega$.*

(i) *We say that $\Omega \cup \Xi$ is* regular *(in the sense of Gröger), if for any point $\mathrm{x} \in \partial\Omega$ there is an open neighborhood U_x of x, a number $a_\mathrm{x} > 0$ and a bi-Lipschitz mapping ϕ_x from U_x onto $a_\mathrm{x} K$ such that $\phi_\mathrm{x}(\mathrm{x}) = 0 \in \mathbb{R}^3$, and we have either $\phi_\mathrm{x}\big((\Omega \cup \Xi) \cap U_\mathrm{x}\big) = a_\mathrm{x} K_-$ or $a_\mathrm{x}(K_- \cup \Sigma_K)$ or $a_\mathrm{x}(K_- \cup \Sigma_K^0)$.*

(ii) *The regular set $\Omega \cup \Xi$ is said to satisfy the* volume-conservation condition*, if each mapping ϕ_x in Condition (i) is volume-preserving.*

Generally, Ξ is allowed to be empty in Definition 2.4. Then Definition 2.4 (i) merely describes a Lipschitz domain. Some further comments are in order:

REMARK 2.5.

(i) *Condition (i) exactly characterizes Gröger's* regular sets*, introduced in his pioneering paper [30]. Note that the volume-conservation condition also has been required in several contexts, cf. [27] and [31]. Clearly, the properties $\phi_\mathrm{x}(U_\mathrm{x}) = a_\mathrm{x} K$ and $\phi_\mathrm{x}\big(\Omega \cap U_\mathrm{x}\big) = a_\mathrm{x} K_-$ are already ensured by the Lipschitz property of Ω; the crucial point is the behavior of $\phi_\mathrm{x}(\Xi \cap U_\mathrm{x})$.*

(ii) *A simplifying topological characterization of Gröger's regular sets in the case of three space dimensions reads as follows (cf. [34, Ch. 5]):*

1. *Ξ is the closure of its interior within $\partial\Omega$,*
2. *the boundary $\partial\Xi$ within $\partial\Omega$ is locally bi-Lipschitz diffeomorphic to the open unit interval $(0, 1)$.*

(iii) *In particular, all domains with Lipschitz boundary (synonymous:* strong *Lipschitz domains) satisfy Definition 2.4: if, after a shift and an orthogonal transformation, the domain lies locally beyond a graph of a Lipschitz function ψ, then one can define $\phi(x_1, \ldots, x_d) = (x_1 - \psi(x_2, \ldots, x_d), x_2, \ldots, x_d)$. Obviously, the mapping ϕ is then bi-Lipschitz and the determinant of its Jacobian is identically 1.*

(iv) *It turns out that regularity together with the volume-conservation condition is not a too restrictive assumption on the mapping ϕ_x. In particular, there are such mappings—although not easy to construct—which map the ball onto the cylinder, the ball onto the cube and the ball onto the half ball, see [28, 23]. The general message is that this class has enough flexibility to map "non-smooth" objects onto smooth ones.*

(v) *The spaces $W^{1,q}_\Xi$ and $W^{-1,q}_\Xi$ still exhibit the usual interpolation properties, see [27] for details.*

(vi) *If Ξ is nonempty and $\Omega \cup \Xi$ is regular, then Ξ has interior points (with respect to the boundary topology in $\partial\Omega$), and, consequently, never has boundary measure 0.*

The following assumption is supposed to be valid for all the remaining considerations in the paper.

ASSUMPTION 2.6. *The set $\Omega \cup \Gamma_D$ is regular with $\Gamma_D \neq \emptyset$.*

For the moment, it is sufficient to impose only the regularity condition from Assumption 2.6 (i) on $\Omega \cup \Gamma_D$. The volume-conservation condition is not needed until Section 4, cf. Assumption 4.2 below. As explained in Remark 2.5, Assumption 2.6 in particular implies that $\omega(\Gamma_D) > 0$.

2.2. General assumptions on (P). Now we are in the position to state the main assumptions for the quantities in (P). Please note that in order to obtain sharp results we just give the assumptions on the quantities in (1.1)–(1.6) which are needed to obtain existence, uniqueness, and continuity of solutions to the state system. For further considerations in §4, in particular those which include Fréchet-differentiability of the associated solution operator, one has to require more restrictive conditions on the nonlinearities, which are formulated in Assumption 4.2, see §4.

We first address the assumptions regarding (local) existence and uniqueness for the state equation (1.1)–(1.6). This means in particular that we treat u as a fixed, given inhomogeneity in this context, whereas it is an unknown control function when considering the optimal control problem (P).

ASSUMPTION 2.7. *On the quantities in the state system (1.1)–(1.6) we generally impose:*

(i) *The functions $\sigma : \mathbb{R} \to (0, \infty)$ and $\eta\colon \mathbb{R} \to (0, \infty)$ are bounded and Lipschitzian on any bounded interval,*

(ii) *the function $\varepsilon \in L^\infty(\Omega; M_3)$ takes* symmetric *matrices as values, and satisfies the usual ellipticity condition, i.e.,*

$$\operatorname*{ess\,inf}_{x\in\Omega} \sum_{i,j=1}^{3} \varepsilon_{ij}(x)_{ij}\, \xi_i\, \xi_j \ge \underline{\varepsilon}\, \|\xi\|^2_{\mathbb{R}^3} \quad \forall\, \xi \in \mathbb{R}^3$$

with a constant $\underline{\varepsilon} > 0$,

(iii) *the function* $\kappa \in L^\infty(\Omega; M_3)$ *also takes* symmetric *matrices as values, and, additionally, satisfies an ellipticity condition, that is,*

$$\operatorname*{ess\,inf}_{x\in\Omega} \sum_{i,j=1}^{3} \kappa_{ij}(x)\,\xi_i\,\xi_j \geq \underline{\kappa}\,\|\xi\|^2_{\mathbb{R}^3} \quad \forall\, \xi \in \mathbb{R}^3$$

holds with a constant $\underline{\kappa} > 0$,

(iv) $\theta_l \in L^\infty(J; L^\infty(\partial\Omega))$,

(v) $\alpha \in L^\infty(\partial\Omega)$ *with* $\alpha(x) \geq 0$ *a.e. on* $\partial\Omega$ *and* $\int_{\partial\Omega} \alpha\, d\omega > 0$,

(vi) $u \in L^{2r}(J; W^{-1,q}_{\Gamma_D})$ *for some* $q > 3$ *to be specified in Assumption 3.4 below and* $r > \frac{2q}{q-3}$, *cf. Definition 3.10 and Theorem 3.13 below.*

Remark 2.8. *In assumption (vi), we implicitly made use of the embedding* $L^{\mathfrak{p}}(\Gamma_N) \hookrightarrow W^{-1,q}_{\Gamma_D}$ *for* $\mathfrak{p} > \frac{2}{3}q$, *realized by the adjoint operator of the continuous trace operator* $\tau_{\Gamma_N} : W^{1,q'}_{\Gamma_D} \to L^{\mathfrak{p}'}(\Gamma_N)$. *In this sense, a function* $u \in L^{2r}(J; L^{\mathfrak{p}}(\Gamma_N))$ *is considered as an element of* $L^{2r}(J; W^{-1,q}_{\Gamma_D})$. *In the same manner, we will treat the function* $\alpha\theta_l \in L^\infty(J; L^\infty(\partial\Omega))$ *as an element of* $L^\infty(J; W^{-1,q}_{\emptyset})$, *see [40, Lemma 2.7] for the required embeddings/trace operators.*

Next we turn to the assumptions concerning the optimal control problem (P). Now, u plays the role of the searched-for variable or function, whose regularity is implicitly determined by the objective functional in (P). As we will see in the sequel of § 4, our hypotheses on the objective functional stated below imply that it suffices to restrict to control functions in a function space $\mathbb{U}$, see (4.11), which continuously embeds in $L^{2r}(J; W^{-1,q}_{\Gamma_D})$ as required in Assumption 2.7 (vi), see Proposition 4.14 below.

Assumption 2.9. *The remaining quantities in* (P) *fulfill:*

(i) *The integrability exponents in the objective functional satisfy* $p > \frac{4}{3}q - 2$ *and* $s > \frac{2q}{q-3}(1 - \frac{3}{q} + \frac{3}{\varsigma})$, *where* q *and* ς *are specified in Assumption 3.4 and Definition 4.8 below.*

(ii) E *is an open (not necessarily proper) subset of* Ω.

(iii) $\theta_d \in L^2(E)$.

(iv) $\theta_{\max} \in C(\overline{Q})$ *with* $\max(\max_{\overline{\Omega}} \theta_0, \operatorname{ess\,sup}_\Sigma \theta_l) \leq \theta_{\max}(x,t)$ *for all* $(x,t) \in \overline{Q}$ *and* $\theta_0(x) < \theta_{\max}(T_0, x)$ *for all* $x \in \overline{\Omega}$.

(v) $u_{\max}$ *is a given function with* $u_{\max}(x,t) \geq 0$ *a.e. on* Σ_N.

(vi) $\beta > 0$.

Note that we do not impose any regularity assumptions on the function $u_{\max}$. In particular, it is allowed that $u_{\max} \equiv \infty$ so that no upper bound is present.

3. Rigorous formulation, existence and uniqueness of solutions for the thermistor problem. In this chapter we will present a precise analytical formulation for the thermistor-problem, see Definition 3.11 below. In order to do so, we first recall some background material. One of the most crucial points is the requirement of suitable mapping property for Poisson's operator, cf. Assumption 3.4. The reader should note that a similar condition was also posed in [6, Ch. 3] in order to get smoothness of the solution; compare also [24], where exactly this regularity for the solution of Poisson's equation is needed in order to show uniqueness for the semiconductor equations. We prove, in particular, some preliminary results which are needed later on and which may be also of independent interest. After having properly defined a solution of the thermistor problem, we establish some more preparatory results and afterwards show existence (locally in time) and uniqueness of the solution of the thermistor problem in Section 3.2. Finally, we show that our concept to treat the problem is not accidental, but—more or less—inevitable.

3.1. Prerequisites: Elliptic and parabolic regularity. We begin this subsection with the definition of the divergence operators. First of all, let us introduce the brackets $\langle\cdot,\cdot\rangle$ as the symbol for the dual pairing between $W^{-1,2}_{\Xi}$ and $W^{1,2}_{\Xi}$, extending the scalar product in L^2.

Definition 3.1. *Let* $\Xi \subset \partial\Omega$ *be closed. Assume that* μ *is any bounded, measurable,* M_3*-valued function on* Ω *and that* $\gamma \in L^\infty(\partial\Omega \setminus \Xi)$ *is nonnegative. We define the operators* $-\nabla\cdot\mu\nabla$ *and* $-\nabla\cdot\mu + \tilde{\gamma}$, *each mapping* $W^{1,2}_{\Xi}$ *into* $W^{-1,2}_{\Xi}$, *by*

$$\langle -\nabla\cdot\mu\nabla\psi, \xi\rangle := \int_\Omega \mu\nabla\psi\cdot\nabla\xi\,\mathrm{d}x \quad \text{for} \quad \psi,\xi \in W^{1,2}_{\Xi} \tag{3.1}$$

and

$$\langle(-\nabla\cdot\mu\nabla+\tilde{\gamma})\psi,\xi\rangle = \langle-\nabla\cdot\mu\nabla\psi,\xi\rangle + \int_{\partial\Omega\setminus\Xi}\gamma\,\psi\,\xi\,\mathrm{d}\omega \quad \textit{for} \quad \psi,\xi\in W^{1,2}_{\Xi}. \tag{3.2}$$

In all what follows, we maintain the same notation for the corresponding maximal restrictions to $W^{-1,q}_{\Xi}$*, where* $q>2$*.*

REMARK 3.2. *Let us denote the domain for the operator* $-\nabla\cdot\mu\nabla$*, when restricted to* $W^{-1,q}_{\Xi}$ $(q>2)$*, by* $\mathcal{D}_q$*, equipped with the graph norm. Then the estimate*

$$\|-\nabla\cdot\mu\nabla\psi\|_{W^{-1,q}_{\Xi}} = \sup_{\|\varphi\|_{W^{1,q'}_{\Xi}}=1}\left|\int_{\Omega}\mu\nabla\psi\cdot\nabla\varphi\,\mathrm{d}x\right| \le \|\mu\|_{L^\infty}\|\psi\|_{W^{1,q}_{\Xi}} \tag{3.3}$$

shows that $W^{1,q}_{\Xi}$ *is embedded in* $\mathcal{D}_q$ *for every bounded coefficient function* μ*. It is also known that* $\mathcal{D}_q\hookrightarrow C^\alpha(\overline{\Omega})$ *for some* $\alpha>0$ *whenever* $q>3$*, see [34, Thm. 3.3]. Additionally, (3.3) implies that the mapping*

$$L^\infty(\Omega;M_3)\ni\mu\mapsto\nabla\cdot\mu\nabla\in\mathcal{L}(W^{1,q}_{\Xi};W^{-1,q}_{\Xi})$$

is a linear and continuous contraction for every $q\in(1,\infty)$*.*

In the following, we consider the operators defined in Definition 3.1 mostly in two incarnations: firstly, the case $\Xi=\emptyset$ and $\mu=\kappa$; and secondly $\Xi=\Gamma_D$ with $\mu=\varepsilon$. We write $-\nabla\cdot\kappa\nabla$ and $-\nabla\cdot\kappa\nabla+\tilde{\alpha}$ in the first, and $-\nabla\cdot\varepsilon\nabla$ in the second case. We recall various properties of operators of the form $-\nabla\cdot\mu\nabla$.

PROPOSITION 3.3. *Let* $\Omega\cup\Xi$ *be regular in the sense of Definition 2.4 and suppose that the coefficient function* μ *in (3.2) is real, bounded and elliptic.*

(i) *Suppose that either* $\omega(\Xi)>0$ *or* $\Xi=\emptyset$ *and* $\int_{\partial\Omega}\gamma\,d\omega>0$*.*

1. *[35] The quadratic form corresponding to (3.2) is coercive.*
2. *[30] There is a number* $q_0>2$ *such that*

$$-\nabla\cdot\mu\nabla+\tilde{\gamma}\colon W^{1,q}_{\Xi}\to W^{-1,q}_{\Xi}$$

is a topological isomorphism for all $q\in[2,q_0]$*. The number* q_0 *may be chosen uniformly for all coefficient functions* μ *with the same ellipticity constant and the same* L^∞*-bound. Moreover, for each* $q\in[2,q_0]$*, the norm of the inverse of* $\nabla\cdot\mu\nabla+\tilde{\gamma}$ *as a mapping from* $W^{-1,q}_{\Xi}$ *to* $W^{1,q}_{\Xi}$ *may be estimated again uniformly for all coefficient functions with the same ellipticity constant and the same* L^∞*-bound.*

(ii) *Assume that* γ *is a nonnegative function from* $L^\infty(\partial\Omega\setminus\Xi)$ *and that the coefficient function* μ *takes symmetric matrices as values.*

1. *[36, Cor. 5.21] The operator* $-\nabla\cdot\mu\nabla+\tilde{\gamma}+1$ *is a* positive *one on any space* $W^{-1,q}_{\Xi}$*, if* $q\in[2,6]$*, i.e., one has the resolvent estimate*

$$\sup_{\lambda\in[0,\infty)}(\lambda+1)\|(-\nabla\cdot\mu\nabla+\tilde{\gamma}+1+\lambda)^{-1}\|_{\mathcal{L}(W^{-1,q}_{\Xi})}<\infty.$$

In particular, all fractional powers of $-\nabla\cdot\mu\nabla+\tilde{\gamma}+1$ *are well-defined and possess the usual properties, cf. [54, Ch. 1.14].*

2. *[36, Thm. 4.2] The square root satisfies* $(-\nabla\cdot\mu\nabla+\tilde{\gamma}+1)^{-1/2}\in\mathcal{L}(W^{-1,q}_{\Xi};L^q)$*, or in other words,* $\mathrm{dom}\big((-\nabla\cdot\mu\nabla+\tilde{\gamma}+1)^{1/2}\big)$ *embeds into* L^q*, if* $q\in[2,\infty)$*.*

See also [7] for recent results as in Proposition 3.3 (ii) in a broader context. Our next aim is to introduce the solution concept for the thermistor problem. To this end, we make the following assumption (cf. also Remark 3.25 below):

ASSUMPTION 3.4. *There is a* $q\in(3,4)$ *such that the mappings*

$$-\nabla\cdot\varepsilon\nabla : W^{1,q}_{\Gamma_D}\to W^{-1,q}_{\Gamma_D} \tag{3.4}$$

and

$$-\nabla\cdot\kappa\nabla+1 : W^{1,q}\to W^{-1,q}_{\emptyset} \tag{3.5}$$

each provide a topological isomorphism.

The papers [43, Appendix] and [19] provide a zoo of arrangements such that Assumption 3.4 is satisfied. Note that it is not presumptous to assume that both differential operators provide topological isomorphisms at the same time, since the latter property mainly depends on the behaviour of the discontinuous coefficient functions (versus the geometry of Γ_D), and these correspond to the material properties in the workpiece described by the domain Ω, i.e., the coefficient functions should exhibit similar properties with regard to jumps or discontinuities in general, the main obstacles to overcome for the isomorphism property. Since κ is not assumed to be continuous, Assumption (3.5) is not satisfied *a priori*, even though no mixed boundary conditions are present, see [21, Ch. 4] for a striking example. In this sense, mixed boundary conditions are not a stronger obstruction against higher regularity in the range $q \in (3, 4)$ than discontinuous coefficient functions are.

REMARK 3.5. *In case of mixed boundary conditions it does not make sense to demand Assumption 3.4—even if all data are smooth—for a $q \geq 4$, due to Shamir's famous counterexample [52]. Note further that the isomorphism properties in (3.4) and (3.5) remain valid for all other $\tilde{q} \in [2, q)$ due to interpolation, cf. Remark 2.5 (v).*

In order to treat the quasilinearity in (1.1), we need to ensure a certain uniformity of domains of the differential operator $-\nabla \cdot \eta(\theta)\kappa\nabla$ during the evolution. To this end, we first note that the isomorphism-property for $-\nabla \cdot \kappa\nabla + 1$ from Assumption 3.4 extends to a broader class of coefficient functions.

DEFINITION 3.6. *Let $\underline{C}(\overline{\Omega})$ denote the set of positive functions on Ω which are uniformly continuous and admit a positive lower bound.*

LEMMA 3.7. *Assume that Assumption 3.4 holds for some number $q \in [2, 4)$. If $\xi \in \underline{C}(\overline{\Omega})$, then (3.4) and (3.5) remain topological isomorphisms, if ε and κ are replaced by $\xi\varepsilon$ and $\xi\kappa$, respectively.*

A proof can be found in [19, Ch. 6].

COROLLARY 3.8. *Assume that (3.5) is a topological isomorphism for some $q \in [2, 4)$. Then, for every $\xi \in \underline{C}(\overline{\Omega})$, the domain of the operator $-\nabla \cdot \xi\kappa\nabla + \tilde{\alpha}$, considered in $W^{-1,q}_{\emptyset}$, is still $W^{1,q}$. In particular, for every function $\zeta \in C(\overline{\Omega})$, the operator $-\nabla \cdot \eta(\zeta)\kappa\nabla + \tilde{\alpha}$ has domain $W^{1,q}$.*

Proof. The first assertion follows from Lemma 3.7 and relative compactness of the boundary integral in $\tilde{\alpha}$ with respect to $-\nabla \cdot \xi\kappa\nabla$, compare [42, Ch. IV.1.3]. For the second assertion, note that η is assumed to be Lipschitzian on bounded intervals and bounded from below by 0 as in Assumption 2.7. Thus, $\eta(\zeta)$ is uniformly continuous and has a strictly positive lower bound. □

We are now in the position to define what is to be understood as a *solution* to the system (1.1)–(1.6).

DEFINITION 3.9. *We define*

$$\mathcal{A}(\zeta) := -\nabla \cdot \eta(\zeta)\kappa\nabla + \tilde{\alpha}$$

as a mapping $\mathcal{A}\colon C(\overline{\Omega}) \to \mathcal{L}(W^{1,q}; W^{-1,q}_{\emptyset})$.

DEFINITION 3.10. *The number $r^*(q) = \frac{2q}{q-3}$ is called the* critical exponent.

DEFINITION 3.11. *Let $q > 3$ and let r be from $(r^*(q), \infty)$. For given $J = (T_0, T_1)$, we call the pair (θ, φ) a* solution *of the thermistor-problem, if it satisfies the equations*

$$\theta'(t) + \mathcal{A}(\theta(t))\theta(t) = (\sigma(\theta(t))\varepsilon\nabla\varphi(t)) \cdot \nabla\varphi(t) + \alpha\theta_l(t) \qquad \text{in } W^{-1,q}_{\emptyset}, \tag{3.6}$$

$$-\nabla \cdot \sigma(\theta(t))\varepsilon\nabla\varphi(t) = u(t) \qquad \text{in } W^{-1,q}_{\Gamma_D} \tag{3.7}$$

with $\theta(T_0) = \theta_0$ for almost all $t \in (T_0, T_1)$, where

$$\varphi \in L^{2r}(J; W^{1,q}_{\Gamma_D}) \quad \text{and} \quad \theta \in W^{1,r}(J; W^{-1,q}_{\emptyset}) \cap L^r(J; W^{1,q}). \tag{3.8}$$

We call (θ, φ) a local solution, *if it satisfies (3.6) and (3.7) in the above sense, but only on $(T_0, T^\bullet) \subseteq (T_0, T_1)$.*

REMARK 3.12.

(i) *In the context of Definition 3.11, θ' always means the time derivative of θ in the sense of vector-valued distributions, see [1, Ch. III.1] or [25, Ch. IV].*

(ii) *Via (3.10) and Corollary 3.20 below, we will see that a solution θ in the above sense is in fact Hölder-continuous on $\overline{\Omega \times J}$. In particular, $\theta(t)$ is uniformly continuous on Ω for every $t \in J$, such that $\mathcal{A}(\theta(t))$ is well-defined according to Definition 3.9.*

(iii) *The reader will verify that the boundary conditions imposed on φ in (1.5) and (1.6) are incorporated in this definition in the spirit of [25, Ch. II.2] or [14, Ch. 1.2]. For an adequate interpretation of the boundary conditions for θ as in (1.2), see [46, Ch. 3.3.2] and the in-book references there.*

We are now going to formulate the main result of this part.

THEOREM 3.13. *Let $q \in (3,4)$ be a number for which Assumption 3.4 is satisfied, $r > r^*(q)$ and $u \in L^{2r}(J; W^{-1,q}_{\Gamma_D})$, where $r^*(q)$ is the critical exponent from Definition 3.10. If θ_0 is from $(W^{1,q}, W^{-1,q}_{\emptyset})_{\frac{1}{r},r}$, then there is a unique* local *solution of (3.6) and (3.7) in the sense of Definition 3.11.*

The proof of this theorem is given in the next subsection.

3.2. Local existence and uniqueness for the state system: the proof. Let us first briefly sketch the proof of Theorem 3.13 by giving an overview over the steps:

- The overall proof is based on a local existence result of Prüss for abstract quasilinear parabolic equations, whose principal part satisfies a certain maximal parabolic regularity property, see [50] and Proposition 3.17.
- For the application of this abstract result to our problem, we reduce the thermistor system to an equation in the temperature θ only by solving the elliptic equation for φ in dependence of θ. This gives rise to a nonlinear operator S appearing in the reduced equation for θ, see Definition 3.26 and Proposition 3.28.
- The key tool to verify the assumptions on S for the application of Prüss' result is Lemma 3.7, which is the basis for the proof of Lemma 3.27. The application of Lemma 3.7 requires to treat the temperature in a space which (compactly) embeds into $C(\overline{\Omega})$. This issue is addressed by Corollary 3.20.

Before we start with the proof itself, let us first recall the concept of maximal parabolic regularity, a crucial tool in the following considerations, and point out some basic facts on this:

DEFINITION 3.14. *Let X be a Banach space and A be a closed operator with dense domain* $\mathrm{dom}(A) \subset X$. *Suppose $\mathfrak{r} \in (1,\infty)$. Then we say that A has* maximal parabolic $L^{\mathfrak{r}}(J;X)$-regularity, *iff for every $f \in L^{\mathfrak{r}}(J;X)$ there is a unique function $w \in W^{1,\mathfrak{r}}(J;X) \cap L^{\mathfrak{r}}(J;\mathrm{dom}(A))$ which satisfies*

$$w'(t) + Aw(t) = f(t), \quad w(T_0) = 0 \tag{3.9}$$

in X for almost every $t \in J = (T_0, T_1)$.

REMARK 3.15.

(i) *As in Remark 3.12, w' in Definition 3.14 also always means the time derivative of w in the sense of vector-valued distributions.*

(ii) *We consider the concept of maximal parabolic regularity as adequate for the solution since it allows for discontinuous (in time) right hand sides—as are required in our context and in many other applications.*

REMARK 3.16. *The following results on maximal parabolic $L^{\mathfrak{r}}(J;X)$-regularity are well-known:*

(i) *If A satisfies maximal parabolic $L^{\mathfrak{r}}(J;X)$-regularity, then it does so for any other (bounded) time interval, see [20].*

(ii) *If A satisfies maximal parabolic $L^{\mathfrak{r}}(J;X)$-regularity for* some *$\mathfrak{r} \in (1,\infty)$, then it satisfies maximal parabolic $L^{\mathfrak{r}}(J;X)$-regularity for* all *$\mathfrak{r} \in (1,\infty)$, see [53] or [20].*

(iii) *Let Y be another Banach space, being dense in X with $Y \hookrightarrow X$. Then there is an embedding*

$$W^{1,\mathfrak{r}}(J;X) \cap L^{\mathfrak{r}}(J;Y) \hookrightarrow C^{\rho}(\overline{J};(Y,X)_{\zeta,1}) \tag{3.10}$$

where $0 < \rho \le \zeta - \frac{1}{\mathfrak{r}}$, see [3, Ch. 3, Thm. 3]. In the immediate context of maximal parabolic regularity, Y is taken as $\mathrm{dom}(A)$ equipped with the graph norm, of course.

According to (i) and (ii), we only say that A satisfies maximal parabolic regularity on X.

In the following, we establish some preliminary results for the proof of Theorem 3.13, which will heavily rest on the following fundamental theorem of Prüss, see [50]:

PROPOSITION 3.17. *Let Y, X be Banach spaces, Y dense in X, such that $Y \hookrightarrow X$ and set $J = (T_0, T_1)$ and $\mathfrak{r} \in (1,\infty)$. Suppose that A maps $(Y,X)_{\frac{1}{\mathfrak{r}},\mathfrak{r}}$ into $\mathcal{L}(Y;X)$ such that $A(w_0)$ satisfies maximal parabolic regularity on X with $\mathrm{dom}(A(w_0)) = Y$ for some $w_0 \in (Y,X)_{\frac{1}{\mathfrak{r}},\mathfrak{r}}$. Let, in addition, $S\colon J \times (Y,X)_{\frac{1}{\mathfrak{r}},\mathfrak{r}} \to X$ be a Carathéodory map and $S(\cdot,0)$ be from $L^{\mathfrak{r}}(J;X)$. Moreover, let the following two assumptions be satisfied:*

(A) *For every $M > 0$, there is a constant $L(M)$ such that for all $w, \bar{w} \in (Y,X)_{\frac{1}{\mathfrak{r}},\mathfrak{r}}$, where $\max(\|w\|_{(Y,X)_{\frac{1}{\mathfrak{r}},\mathfrak{r}}}, \|\bar{w}\|_{(Y,X)_{\frac{1}{\mathfrak{r}},\mathfrak{r}}}) \le M$, we have*

$$\|A(w) - A(\bar{w})\|_{\mathcal{L}(Y;X)} \le L(M)\|w - \bar{w}\|_{(Y,X)_{\frac{1}{\mathfrak{r}},\mathfrak{r}}}. \tag{3.11}$$

(S) *For every $M > 0$, assume that there is a function $h_M \in L^{\mathfrak{r}}(J)$ such that for all $w, \bar{w} \in (Y, X)_{\frac{1}{\mathfrak{r}},\mathfrak{r}}$, where* $\max(\|w\|_{(Y,X)_{\frac{1}{\mathfrak{r}},\mathfrak{r}}}, \|\bar{w}\|_{(Y,X)_{\frac{1}{\mathfrak{r}},\mathfrak{r}}}) \le M$, *it is true that*

$$\|S(t, w) - S(t, \bar{w})\|_X \le h_M(t)\|w - \bar{w}\|_{(Y,X)_{\frac{1}{\mathfrak{r}},\mathfrak{r}}} \tag{3.12}$$

for almost every $t \in J$.

Then, for each $w_0 \in (Y, X)_{\frac{1}{\mathfrak{r}},\mathfrak{r}}$, there exists $T_{\max} \in J$ such that the problem

$$\begin{cases} w'(t) + A(w(t))w(t) = S(t, w(t)) & \text{in } X, \\ w(T_0) = w_0 \end{cases} \tag{3.13}$$

admits a unique *solution $w \in W^{1,\mathfrak{r}}(T_0, T_\bullet; X) \cap L^{\mathfrak{r}}(T_0, T_\bullet^*; Y)$ on $(T_0, T_\bullet)$ for every $T_\bullet \in (T_0, T_{\max})$.*

Remark 3.18. *It is known that the solution of the thermistor problem possibly ceases to exist after finite time in general, cf. [6, Ch. 5] and the references therein. Thus, one has to expect here, in contrast to the two-dimensional case treated in [39], only a local-in-time solution. In this scope, Prüss' theorem will prove to be the adequate instrument.*

As indicated above, we will prove Theorem 3.13 by reducing the thermistor system to an equation in the temperature only and apply Proposition 3.17 to this equation. To be more precise, we first establish the assumptions **(A)** for $\mathfrak{r} = r > r^*(q)$ and $\mathcal{A}$ as defined in Definition 3.9. We then solve the elliptic equation (3.7) for φ (uniquely) for every time point t in dependence of a function ζ and $u(t)$, where ζ enters the equation inside the coefficient function $\sigma(\zeta)\varepsilon$. Then the right-hand side of the parabolic equation (3.6) may be written also as a function S solely of t and ζ. We then show that this function satisfies the suppositions **(S)** in Prüss' theorem.

To carry out this concept, we need several prerequisites: here our first central aim is to show that indeed the mapping $(W^{1,q}, W_\emptyset^{-1,q})_{\frac{1}{r},r} \ni \zeta \mapsto \mathcal{A}(\zeta)$ from Definition 3.9 satisfies the assumptions from Proposition 3.17 for $r > r^*(q)$, cf. Lemma 3.21 below. For doing so, we first investigate the spaces $(W^{1,q}, W_\emptyset^{-1,q})_{\zeta,1}$ in view of their embedding into Hölder spaces. For later use, the subsequent result is formulated slightly broader as presently needed.

Theorem 3.19. *Let $q \in (3, 4)$ and $\varsigma \in [2, q]$. For every $\tau \in (0, \frac{q-3}{2q}(1 - \frac{3}{q} + \frac{3}{\varsigma})^{-1})$, the interpolation space $(W^{1,q}, W_\emptyset^{-1,\varsigma})_{\tau,1}$ embeds into some Hölder space $C^\delta(\overline{\Omega})$ with $\delta > 0$.*

Proof. We apply the reiteration theorem [54, Ch. 1.10.2] to obtain

$$\begin{aligned}(W^{1,q}, W_\emptyset^{-1,\varsigma})_{\tau,1} &= (W^{1,q}, (W^{1,q}, W_\emptyset^{-1,\varsigma})_{\frac{1}{2},1})_{2\tau,1} \\ &\hookrightarrow (W^{1,q}, (W^{1,\varsigma}, W_\emptyset^{-1,\varsigma})_{\frac{1}{2},1})_{2\tau,1} \hookrightarrow (W^{1,q}, (W_\emptyset^{-1,\varsigma}, \mathcal{D}_\varsigma)_{\frac{1}{2},1})_{2\tau,1},\end{aligned} \tag{3.14}$$

where $\mathcal{D}_\varsigma$ denotes the domain of the Laplacian $-\Delta+1$ acting on the Banach space $W_\emptyset^{-1,\varsigma}$, cf. Remark 3.2. Denoting the domain of $(-\Delta+1)^{1/2}$, considered on the same space, by $\mathcal{D}_\varsigma^{\frac{1}{2}}$, one has $(W_\emptyset^{-1,\varsigma}, \mathcal{D}_\varsigma)_{\frac{1}{2},1} \hookrightarrow \mathcal{D}_\varsigma^{\frac{1}{2}}$, cf. [54, Ch. 1.15.2]. Due to Proposition 3.3 (ii), we already know the embedding $\mathcal{D}_\varsigma^{\frac{1}{2}} \hookrightarrow L^\varsigma$. Inserting in (3.14), this altogether yields $(W^{1,q}, W_\emptyset^{-1,\varsigma})_{\tau,1} \hookrightarrow (W^{1,q}, L^\varsigma)_{2\tau,1}$.

We define $p := \left(\frac{1-2\tau}{q} + \frac{2\tau}{\varsigma}\right)^{-1}$ and observe that $\delta := 1 - 2\tau - \frac{3}{p} \in (0, 1)$, due to our condition on τ. Denoting by $H^{t,p}$ the corresponding space of Bessel potentials (cf. [54, Ch. 4.2.1]) one has the embedding $H^{1-2\tau,p} \hookrightarrow C^\delta(\overline{\Omega})$, see [54, Thm. 4.6.1]. This, combined with the interpolation inequality for $H^{1-2\tau,p}$ ([27, Thm. 3.1]) gives for any $\psi \in W^{1,q}$ the estimate

$$\|\psi\|_{C^\delta(\overline{\Omega})} \le \|\psi\|_{H^{1-2\tau,p}} \le \|\psi\|_{W^{1,q}}^{1-2\tau}\|\psi\|_{L^\varsigma}^{2\tau}. \tag{3.15}$$

But it is well-known (cf. [54, Ch. 1.10.1] or [8, Ch. 5, Prop. 2.10]) that an inequality of type (3.15) is constitutive for the embedding $(W^{1,q}, L^\varsigma)_{2\tau,1} \hookrightarrow C^\delta(\overline{\Omega})$. □

Corollary 3.20.

(i) *Let $q > 3$ and $\varsigma \in [2, q]$. Then, for every $s > \frac{2q}{q-3}(1 - \frac{3}{q} + \frac{3}{\varsigma})$, the interpolation space $(W^{1,q}, W_\emptyset^{-1,\varsigma})_{\frac{1}{s},s}$ embeds into some Hölder space $C^\delta(\overline{\Omega})$, and thus even compactly into $C(\overline{\Omega})$.*

(ii) *Under the same supposition, there exists a $\varrho > 0$ such that*

$$W^{1,s}(J; W_\emptyset^{-1,\varsigma}) \cap L^s(J; W^{1,q}) \hookrightarrow C^\varrho(\overline{J}; C^\varrho(\overline{\Omega})).$$

(iii) *Let Assumption 3.4 hold true for some* $q \in (3,4)$. *Then the operator* $\mathcal{A}(\zeta)$ *satisfies maximal parabolic regularity on* $W^{-1,q}_\emptyset$ *with domain* $W^{1,q}$ *for every* $\zeta \in (W^{1,q}, W^{-1,q}_\emptyset)_{\frac{1}{r},r}$ *with* $r > r^*(q)$, *where* $r^*(q)$ *is the critical exponent from Definition 3.10.*

Proof. (i) We have $(W^{1,q}, W^{-1,\varsigma}_\emptyset)_{\frac{1}{s},s} \hookrightarrow (W^{1,q}, W^{-1,\varsigma}_\emptyset)_{\iota,1}$ for every $\iota \in (\frac{1}{s}, 1)$. The condition on s implies that the interval $\mathcal{I} := (\frac{1}{s}, \frac{q-3}{2q}(1 - \frac{3}{q} + \frac{3}{\varsigma})^{-1})$ is non-empty. Taking ι from $\mathcal{I}$, the assertion follows from Theorem 3.19. (ii) follows from Theorem 3.19 and Remark 3.16. (iii) The claim follows from uniform continuity of functions from $(W^{1,q}, W^{-1,q}_\emptyset)_{\frac{1}{r},r}$ by (i), Lemma 3.7 for $\xi := \eta(\zeta)$ and [36, Thm. 5.4/5.19 (ii)]. ∎

Setting $\varsigma = q$ in Corollary 3.20 (i) and (ii) gives the condition $r > r^*(q) = \frac{2q}{q-3}$ for the assertions to hold with $s = r$. We will use this special case frequently in the course of the remaining part of this section. Let us now turn to the operator $\mathcal{A}$.

Proposition 3.21. *Suppose that Assumption 3.4 holds true for some* $q \in (3,4)$ *and that* $\theta_0 \in (W^{1,q}, W^{-1,q}_\emptyset)_{\frac{1}{r},r}$ *where* $r > r^*(q)$. *With* $\mathcal{A}$ *as in Definition 3.9, the function* $(W^{1,q}, W^{-1,q}_\emptyset)_{\frac{1}{r},r} \ni \zeta \mapsto \mathcal{A}(\zeta)$ *then satisfies the assumptions from Proposition 3.17 for the spaces* $X = W^{-1,q}_\emptyset$ *and* $Y = W^{1,q}$.

Proof. With $\varsigma = q$, Corollary 3.20 shows that $(W^{1,q}, W^{-1,q}_\emptyset)_{\frac{1}{r},r} \hookrightarrow C(\overline{\Omega})$, such that the operator $\mathcal{A}$ indeed maps $(W^{1,q}, W^{-1,q}_\emptyset)_{\frac{1}{r},r}$ into $\mathcal{L}(W^{1,q}; W^{-1,q}_\emptyset)$ by Corollary 3.8. Using Lipschitz continuity of η on bounded sets and Remark 3.2, we also obtain **(A)**: Let $w, \bar{w} \in (W^{1,q}, W^{-1,q}_\emptyset)_{\frac{1}{r},r}$ with norms bounded by $M > 0$. Then we have

$$\begin{aligned}\|\mathcal{A}(w) - \mathcal{A}(\bar{w})\|_{\mathcal{L}(W^{1,q};W^{-1,q}_\emptyset)} &= \|\nabla \cdot (\eta(w) - \eta(\bar{w}))\, \kappa \nabla\|_{\mathcal{L}(W^{1,q},W^{-1,q}_\emptyset)} \\ &\le L_\eta \|\kappa\|_{L^\infty} \|w - \bar{w}\|_{C(\overline{\Omega})} \\ &\le C L_\eta \|\kappa\|_{L^\infty} \|w - \bar{w}\|_{(W^{1,q},W^{-1,q}_\emptyset)_{\frac{1}{r},r}}.\end{aligned}$$

Finally, the property of maximal parabolic regularity for $\mathcal{A}(\theta_0)$ follows immediately from Corollary 3.20. ∎

Next we will establish and investigate the right hand hand side of (3.13). For doing so, we now turn our attention to the elliptic equation (3.7).

Lemma 3.22. *For* $q \ge 2$ *and* $\zeta \in C(\overline{\Omega})$, $\mathfrak{a}_\zeta(\varphi_1, \varphi_2) := (\sigma(\zeta)\varepsilon\nabla\varphi_1) \cdot \nabla\varphi_2$ *defines a continuous bilinear form* $\mathfrak{a}_\zeta \colon W^{1,q}_{\Gamma_D} \times W^{1,q}_{\Gamma_D} \to L^{q/2}$. *Moreover,* $(\zeta, \varphi) \mapsto \mathfrak{a}_\zeta(\varphi, \varphi)$ *is Lipschitzian over bounded sets in* $C(\overline{\Omega}) \times W^{1,q}_{\Gamma_D}$.

Proof. Bilinearity and continuity of each $\mathfrak{a}_\zeta$ are clear. The second assertion follows from a straightforward calculation with the resulting estimate

$$\begin{aligned}\|\mathfrak{a}_{\zeta_1}(\varphi_1, \varphi_1) - \mathfrak{a}_{\zeta_2}(\varphi_2, \varphi_2)\|_{L^{q/2}} \le{}& \|\sigma(\zeta_1) - \sigma(\zeta_2)\|_{L^\infty} \|\varepsilon\|_{L^\infty} \|\varphi_1\|^2_{W^{1,q}_{\Gamma_D}} \\ &+ 2\|\sigma(\zeta_2)\|_{L^\infty} \|\varepsilon\|_{L^\infty} \|\varphi_1\|_{W^{1,q}_{\Gamma_D}} \|\varphi_1 - \varphi_2\|_{W^{1,q}_{\Gamma_D}},\end{aligned}$$

Lipschitz continuity of σ and boundedness of the underlying sets. ∎

Let us draw some further conclusions from Lemma 3.7. For this, we assume Assumption 3.4 for the rest of this chapter.

Theorem 3.23. *The mapping*

$$\underline{C}(\overline{\Omega}) \ni \phi \mapsto (-\nabla \cdot \phi\varepsilon\nabla)^{-1} \in \mathcal{LH}(W^{-1,q}_{\Gamma_D}; W^{1,q}_{\Gamma_D}) \tag{3.16}$$

is well-defined and even continuous.

Proof. The well-definedness assertion results from Lemma 3.7. The second assertion is implied by the first, Remark 3.2 and the continuity of the mapping $\mathcal{LH}(X;Y) \ni B \mapsto B^{-1} \in \mathcal{LH}(Y;X)$, see [51, Ch. III.8]. ∎

Corollary 3.24. *Let* $\mathfrak{C} \subset \underline{C}(\overline{\Omega})$ *be a compact set in* $C(\overline{\Omega})$ *which admits a common lower positive bound. Then the function*

$$\mathfrak{C} \ni \phi \mapsto \mathcal{J}(\phi) := (-\nabla \cdot \phi\varepsilon\nabla)^{-1} \in \mathcal{LH}(W^{-1,q}_{\Gamma_D}; W^{1,q}_{\Gamma_D})$$

is bounded and even Lipschitzian. The same holds for $\mathfrak{C} \times \mathfrak{B} \ni (\phi, v) \mapsto \mathcal{J}(\phi)v \in W^{1,q}_{\Gamma_D}$ *for every bounded set* $\mathfrak{B} \subset W^{-1,q}_{\Gamma_D}$.

Proof. Theorem 3.23 and the compactness of $\underline{\mathfrak{C}}$ in $C(\overline{\Omega})$ immediately imply boundedness of $\mathcal{J}$ on $\underline{\mathfrak{C}}$. In turn, Lipschitz continuity of $\mathcal{J}$ is obtained from boundedness and the resolvent-type equation

$$(-\nabla\cdot\phi_1\varepsilon\nabla)^{-1}-(-\nabla\cdot\phi_2\varepsilon\nabla)^{-1} = (-\nabla\cdot\phi_1\varepsilon\nabla)^{-1}(-\nabla\cdot(\phi_2-\phi_1)\varepsilon\nabla)(-\nabla\cdot\phi_2\varepsilon\nabla)^{-1} \quad (3.17)$$

(read: $A^{-1}-B^{-1}=A^{-1}(B-A)B^{-1}$) and Remark 3.2. Considering the assertion on the combined mapping, boundedness is obvious and further we have for $\phi_1,\phi_2\in\underline{\mathfrak{C}}$ and $v_1,v_2\in\mathfrak{B}$:

$$\|\mathcal{J}(\phi_1)v_1-\mathcal{J}(\phi_2)v_2\|_{W^{1,q}_{\Gamma_D}} \le \|\mathcal{J}(\phi_1)-\mathcal{J}(\phi_2)\|_{\mathcal{L}(W^{-1,q}_{\Gamma_D},W^{1,q}_{\Gamma_D})}\|v_1\|_{W^{-1,q}_{\Gamma_D}} + \|\mathcal{J}(\phi_2)\|_{\mathcal{L}(W^{-1,q}_{\Gamma_D},W^{1,q}_{\Gamma_D})}\|v_1-v_2\|_{W^{-1,q}_{\Gamma_D}}.$$

With Lipschitz continuity and boundedness of $\mathcal{J}$ over $\underline{\mathfrak{C}}$ and boundedness of $\mathfrak{B}$, this implies the claim. ∎

Remark 3.25. *At this point we are in the position to discuss the meaning of Assumption 3.4 in some detail. Under Assumption 2.6 (i) for a closed subset Ξ of $\partial\Omega$, it is known that, even for arbitrary measurable, bounded, elliptic coefficient functions μ, $(\mathcal{D}_q, W^{-1,q}_{\Xi})_{\tau,1}$ embeds into a Hölder space for suitable τ, cf. [34, Cor. 3.7] (for $\mathcal{D}_q$, see Remark 3.2). In particular, one does* not *need an assumption for the ismorphism property between $W^{1,q}_{\Xi}$ and $W^{-1,q}_{\Xi}$ for this result. The crucial point behind Assumption 3.4 is to achieve both independence of the domains for the operators $-\nabla\phi\mu\nabla$ within a suitable class of functions ϕ, as well as a well-behaved dependence on ϕ in the space $\mathcal{L}(\mathcal{D}_q;W^{-1,q}_{\Xi})$, cf. Lemma 3.7 and Corollaries 3.8 and 3.24.*

The next lemmata establish the right-hand side in (3.13) with the correct regularity and properties. Moreover, Lipschitz continuity with respect to the control u in the elliptic equation is shown along the way, which will become useful in later considerations. Recall that $\sigma\colon\mathbb{R}\to\mathbb{R}_+$ is Lipschitzian on any finite interval by Assumption 2.7.

Definition 3.26. *We assign to $\zeta\in C(\overline{\Omega})$ and $v\in W^{-1,q}_{\Gamma_D}$ the solution φ_v of $-\nabla\cdot\sigma(\zeta)\varepsilon\nabla\varphi_v=v$ via $\varphi_v=\mathcal{J}(\sigma(\zeta))v$ with $\mathcal{J}$ as in Corollary 3.24. Moreover, set*

$$\Psi_v(\zeta) := \mathfrak{a}_\zeta(\mathcal{J}(\sigma(\zeta))v,\mathcal{J}(\sigma(\zeta))v)$$

for $\zeta\in C(\overline{\Omega})$ with $\mathfrak{a}_\zeta$ as in Lemma 3.22.

Lemma 3.27. *Let $\mathfrak{C}$ be a compact subset of $C(\overline{\Omega})$ and $\mathfrak{B}$ a bounded set in $W^{-1,q}_{\Gamma_D}$. Then $(v,\zeta)\mapsto\Psi_v(\zeta)$ is Lipschitzian from $\mathfrak{C}\times\mathfrak{B}$ into $L^{q/2}$ and the Lipschitz constant of $\zeta\mapsto\Psi_v(\zeta)$ is bounded over $v\in\mathfrak{B}$.*

Proof. For every $\zeta\in\mathfrak{C}$, the function $\sigma(\zeta)$ belongs to $\underline{C}(\overline{\Omega})$, thus $\mathcal{J}(\sigma(\zeta))v$ is indeed from $W^{1,q}_{\Gamma_D}$ thanks to Lemma 3.7. Hence, $\Psi_v(\zeta)\in L^{q/2}$ is clear by Hölder's inequality. Let us show the Lipschitz property of Ψ: First, note that Nemytskii operators induced by Lipschitz functions preserve compactness in the space of continuous functions, and note further that the set of all $\sigma(\zeta)$ for $\zeta\in\mathfrak{C}$ admits a common positive lower bound by the Lipschitz property of σ. Hence, the set $\{\sigma(\zeta)\colon\zeta\in\mathfrak{C}\}$ satisfies the assumptions in Lemma 3.22 and Corollary 3.24. For $\zeta_1,\zeta_2\in\mathfrak{C}$ and $v_1,v_2\in W^{-1,q}_{\Gamma_D}$, we first obtain via Lemma 3.22

$$\|\Psi_{v_1}(\zeta_1)-\Psi_{v_2}(\zeta_2)\|_{L^{q/2}}\le L_{\mathfrak{a}}\left(\|\zeta_1-\zeta_2\|_{C(\overline{\Omega})}+\|\mathcal{J}(\sigma(\zeta_1))v_1-\mathcal{J}(\sigma(\zeta_2))v_2\|_{W^{1,q}_{\Gamma_D}}\right)$$

and further with Corollary 3.24

$$\|\mathcal{J}(\sigma(\zeta_1))v_1-\mathcal{J}(\sigma(\zeta_2))v_2\|_{W^{1,q}_{\Gamma_D}}\le L_{\mathcal{J}}\left(\|\sigma(\zeta_1)-\sigma(\zeta_2)\|_{C(\overline{\Omega})}+\|v_1-v_2\|_{W^{-1,q}_{\Gamma_D}}\right).$$

The assertion follows since σ was Lipschitz continuous. Uniformity of the Lipschitz constant of $\zeta\mapsto\Psi_v(\zeta)$ is immediate from the previous considerations. ∎

Following the strategy outlined above, we will specify the mapping S from Proposition 3.17 for our case and show that it satisfies the required conditions.

Proposition 3.28. *Let $q\in(3,4)$ be such that Assumption 3.4 is satisfied, $r>r^*(q)$, and $u\in L^{2r}(J;W^{-1,q}_{\Gamma_D})$. We set*

$$S(t,\zeta) := \Psi_{u(t)}(\zeta)+\alpha\theta_l(t).$$

Then S satisfies the conditions from Proposition 3.17 for the spaces $X = W^{-1,q}_\emptyset$ and $Y = W^{1,q}$.

Proof. We show that $S(\cdot,0) \in L^r(J;W^{-1,q}_\emptyset)$. The function $\alpha\theta_l$ is essentially bounded in time with values in $W^{-1,q}_{\Gamma_D}$ by virtue of Remark 2.8 and thus poses no problem here. For almost all $t \in J$, we further have

$$\big\|\Psi_{u(t)}(0)\big\|_{L^{q/2}} \le |\sigma(0)|\|\varepsilon\|_{L^\infty}\|\mathcal{J}(\sigma(0)\|^2_{\mathcal{L}(W^{-1,q}_{\Gamma_D};W^{1,q}_{\Gamma_D})}\|u(t)\|^2_{W^{-1,q}_{\Gamma_D}}.$$

Since u is $2r$-integrable in time, this means that $\Psi_{u(t)}(0) \in L^r(J;L^{q/2})$. Due to $q > 3$ and thus $L^{q/2} \hookrightarrow W^{-1,q}_\emptyset$ (cf. Remark 2.3), we hence have $S(\cdot,0) \in L^r(J;W^{-1,q}_\emptyset)$.
Let us now show the Lipschitz condition (3.12). If $\mathfrak{C} \subset (W^{1,q},W^{-1,q}_\emptyset)_{\frac{1}{r},r}$ is bounded, its closure $\overline{\mathfrak{C}}$ with respect to the sup-norm on $\overline{\Omega}$ forms a compact set in $C(\overline{\Omega})$ by Corollary 3.20. The desired Lipschitz estimate for $S(t,\cdot)$ now follows immediately from Lemma 3.27. ∎

Note that this is the point where the supposition on the time-integrability of u from Assumption 2.7 (vi) comes into play. Essentially, $\Psi_{u(t)}(\zeta)$ only admits half the time-integrability of u, but Propositions 3.21 and 3.28 both require $r > r^*(q)$ to make use of the (compact) embedding $(W^{1,q},W^{-1,q}_\emptyset)_{\frac{1}{r},r} \hookrightarrow C(\overline{\Omega})$. Hence, we need more than $2r^*(q)$-integrability for u in time.

Now we have established all ingredients to prove Theorem 3.13. For this purpose, let the assumptions of Theorem 3.13 hold. Combining Propositions 3.21 and 3.28 with Proposition 3.17, we obtain a local-in-time solution θ of the equation

$$\theta'(t) + \mathcal{A}(\theta(t))\theta(t) = S(t,\theta(t)), \quad \theta(T_0) = \theta_0$$

on (T_0,T_*) with $T_* \in (T_0,T_1]$, such that

$$\theta \in W^{1,r}(T_0,T_*;W^{-1,q}_\emptyset) \cap L^r(T_0,T_*;W^{1,q}) \hookrightarrow C([T_0,T_*];(W^{1,q},W^{-1,q}_\emptyset)_{\frac{1}{r},r}).$$

If $T_* < T_1$, we may apply Proposition 3.17 again on the interval (T_*,T_1) with initial value $\theta(T_*) \in (W^{1,q},W^{-1,q}_\emptyset)_{\frac{1}{r},r}$, thus obtaining another local solution on a subinterval of (T_*,T_1), "glue" the solutions together and start again (note that $\mathcal{A}(\theta(t))$ again satisfies maximal parabolic regularity for every $t \in [T_*,T_1)$ by Corollary 3.20). As we may let these intervals of local existence overlap, the uniqueness of local solutions by Proposition 3.17 implies that the "glued" solution satisfies the claimed regularity for the solutions as in (3.8). In this way, we either obtain a solution on the whole prescribed interval (T_0,T_1) or end up with a maximal interval of existence, denoted by $J_{\max} = (T_0,T_{\max})$, such that there exists a solution θ in the above sense on every interval $(T_0,T_\bullet)$ where $T_\bullet \in J_{\max}$ (or equivalently $(T_0,T_\bullet] \subset (T_0,T_{\max})$). The maximal time of existence $T_{\max}$ is characterized by the property that $\lim_{t\nearrow T_{\max}}\theta(t)$ does not exist in $(W^{1,q},W^{-1,q}_\emptyset)_{\frac{1}{r},r}$, see [50, Cor. 3.2].

Consider such $T_\bullet \in J_{\max}$. We now *define* the function $\varphi(t)$ for each $t \in (T_0,T_\bullet)$ as the solution of $-\nabla\cdot\sigma(\theta(t))\varepsilon\nabla\varphi = u(t)$, that is,

$$\varphi(t) := \mathcal{J}(\sigma(\theta(t)))u(t). \tag{3.18}$$

Then φ indeed belongs to $L^{2r}(T_0,T_\bullet;W^{1,q}_{\Gamma_D})$, since $\mathcal{J}(\sigma(\theta(t))$ is uniformly bounded in $\mathcal{L}(W^{-1,q}_{\Gamma_D};W^{1,q}_{\Gamma_D})$ over $[T_0,T_\bullet]$ due to the compactness of the set $\{\theta(t) \colon t \in [T_0,T_\bullet]\}$ in $C(\overline{\Omega})$ (cf. Corollary 3.20 and Corollary 3.24), and u was from $L^{2r}(J;W^{-1,q}_{\Gamma_D})$.

Obviously, (θ,φ) is then a solution of the thermistor-problem on $(T_0,T_\bullet)$ in the spirit of Definition 3.11 as claimed in Theorem 3.13.

We end this chapter with some explanations why the chosen setting in spaces of the kind $W^{-1,q}_\emptyset$ and $W^{-1,q}_{\Gamma_D}$ with $q > 3$ is adequate for the problem under consideration.

Let us inspect the requirements on the spaces in which the equations are formulated. Clearly, they need to contain Lebesgue spaces on Ω as well as on the boundary Γ (or on a subset of the boundary like Γ_N), in order to incorporate the nonhomogenous Neumann boundary data present in both equations. The boundary conditions should be reflected by the formulation of the equations in an adequate way, cf. Remark 3.12 (iii). These demands already strongly prejudice spaces of type $W^{-1,q_p}_\emptyset$ for the parabolic equation and $W^{-1,q_e}_{\Gamma_D}$ for the elliptic equation with probably different integrability orders q_p and q_e for each equation. Finally, in order to treat the nonlinear parabolic equation, we need maximal parabolic regularity for the second order divergence operators $\mathcal{A}(\zeta)$ over $W_\emptyset^{-1,q_p}$, which is generally available by Corollary 3.20 (iii) or [33, Thm. 5.16/Rem. 5.14] in a general context.

Further, aiming at continuous solutions θ, which are needed for having fulfillable Constraint Qualifications for (P) in the presence of state constraints, it is necessary that the domain $\mathcal{D}_{q_p}(\zeta)$ of the differential operators $\mathcal{A}(\zeta)$, cf. Remark 3.2, embeds into the space of continuous functions on $\overline{Q}$. But it is known that solutions y to equations $-\nabla\cdot\mu\nabla y = f$ for $\mu \in L^\infty(\Omega, M_n)$ elliptic with $f \in W_\emptyset^{-1,n}$, where n denotes the space dimension, may in general even be unbounded, see [44, Ch. 1.2]. On the other hand, $\mathcal{D}_{q_p}(\zeta)$ embeds into a Hölder space if $q_p > 3$, see Remark 3.2. These two facts make the requirement $q_p > n = 3$ expedient. Let us now assume that the elliptic equation admits solutions whose gradient is integrable up to some order q_g. Then the right hand side in the parabolic equation prescribes $q_g \geq \frac{6q_p}{q_p+3}$ in order to have the embedding $L^{q_g/2} \hookrightarrow W_\emptyset^{-1,q_p}$. From the requirement $q_p > 3$ then follows $q_g > 3$ as well, i.e., the elliptic equation must admit $W^{1,q_g}_{\Gamma_D}$-solutions with $q_g > 3$. With right-hand sides in $W^{-1,q_e}_{\Gamma_D}$, the best possible constellation is thus $q_e = q_g > 3$ again. Having q_e and q_p both in the same range, we simply choose $q = q_e = q_p > 3$.

Moreover, in order to actually have $W^{1,q}_{\Gamma_D}$-solutions to the elliptic equations for all right-hand sides from $W^{-1,q}_{\Gamma_D}$, the operator $-\nabla\cdot\sigma(\zeta)\varepsilon\nabla$ must be a topological isomorphism between $W^{1,q}_{\Gamma_D}$ and $W^{-1,q}_{\Gamma_D}$. It is also a well-established fact that solutions to elliptic equations with bounded and coercive, but discontinuous coefficient functions may admit almost arbitrarily poor integrability properties for gradients of their solutions, see [49] and [21, Ch. 4]. Under Assumption 3.4, we know that this is not the case for $-\nabla\cdot\varepsilon\nabla$ over $W^{-1,q}_{\Gamma_D}$, but it is clear that it is practically impossible to guarantee this also for the operators $-\nabla\cdot\sigma(\zeta)\varepsilon\nabla$ for all ζ, if $\sigma(\zeta)$ is discontinuous in general. However, from Lemma 3.7 we know that if $\sigma(\zeta)$ if uniformly continuous on Ω, then the isomorphism property carries over. This shows that continuous solutions for the parabolic equation are also needed purely from an analytical point of view, without the considerations coming from the optimal control problem, and also explains why Assumption 3.4 is, in a sense, a "minimal" assumption.

4. Global solutions and optimal control. The setting and results of § 3 are assumed as given from now on, i.e., we consider the assumptions of Theorem 3.13 to be fulfilled and fixed, that means, $q > 3$ and $r > r^*(q)$ are given from now on. In particular, for every $u \in L^{2r}(J; W^{-1,q}_{\Gamma_D})$, there exists a local solution θ_u such that $\theta_u \in W^{1,r}(T_0, T_\bullet; W^{1,q}) \cap L^r(T_0, T_\bullet; W_\emptyset^{-1,q})$ for every $T_\bullet \in J_{\max}(u)$, the maximal interval of existence for a given control u. We consider $\varphi \in L^{2r}(T_0, T_\bullet; W^{1,q}_{\Gamma_D})$ to be given in dependence of u and θ_u as in (3.18). Due to $q > 3$ and $r > r^*(q)$, each solution θ_u is Hölder-continuous on $[T_0, T_\bullet] \times \overline{\Omega}$, cf. Corollary 3.20 (ii).

Remark 4.1. *As noted above, if the solution θ_u for a given control u does* not *exist on the whole time interval J, there exists $T_{\max}(u) \leq T_1$, the maximal time of existence, such that $\lim_{t\nearrow T_{\max}(u)} \theta_u(t)$ does not exist in $(W^{1,q}, W_\emptyset^{-1,q})_{\frac{1}{r},r}$. For a proof and an equivalent formulation in the maximal regularity-norm, see [50, Cor. 3.2].*

Our aim in the following sections is to characterize the set of control functions which admit a solution on the *whole* time interval. These control functions will be called "global controls", see Definition 4.3. In view of the state constraints and the end time observation in the objective of (P), it is natural to restrict the optimal control problem to the set of global controls. Our characterization of this set will then allow to establish the existence of (globally) optimal controls. Let us give a brief roadmap of the upcoming analysis:

- We first show that the set of global controls is not empty, see Proposition 4.4, and that it is an open set, cf. Theorem 4.5. This property will be of major importance for the derivation of meaningful optimality conditions in Section 4.2.
- For global controls one can define a control-to-state operator in function spaces on the whole time interval, see Definition 4.3. The proof of Theorem 4.5 features an application of the implicit function theorem and thereby shows that the control-to-state operator is Fréchet-differentiable, which is also essential for the derivation of necessary optimality conditions.
- We then turn to the existence of optimal controls. The arguments follow the classical direct method of the calculus of variations, see Theorem 4.16. To this end, we need to establish a closedness result for the set of global controls in Theorem 4.9. This result requires a certain boundedness of the gradient of the temperatures which is ensured by the second addend in the

objective in (P). To pass to the limit in the thermistor system, we additionally need that the control space, induced by the third term in the objective functional, compactly embeds into $L^{2r}(J; W^{1,q}_{\Gamma_D})$. This issue is addressed in Proposition 4.14.

- Finally, Section 4.2 is devoted to the derivation of necessary optimality conditions. As the set of global controls is open, the standard generalized Karush-Kuhn-Tucker theory applies, see Theorem 4.22. We then introduce an adjoint system in Definition 4.23 and 4.24 and show by means of a classical duality argument that this system admits a unique solution, cf. Theorem 4.26. This allows to reformulate the necessary conditions in terms of a qualified optimality system, see Theorem 4.28.

4.1. Global solutions and existence of optimal controls. In [6], Antontsev and Chipot show that it is possible to give concrete conditions under which the solution to a thermistor-like problem does not exist globally. While the authors of [6] consider a slightly different setting (in particular no Robin boundary conditions for the parabolic equation), we devote a subsection to the question whether there is any relevant characterization of *global* controls u, i.e., controls such that the corresponding solution (θ_u, φ_u) does exist on the whole (prescribed) interval $J = (T_0, T_1)$.

We make the following assumption for the rest of this paper:

ASSUMPTION 4.2.

1. *The functions η and σ, each mapping $\mathbb{R} \to \mathbb{R}_+$, are continuously differentiable. The derivatives η' and σ' are each bounded and Lipschitz continuous on bounded sets.*
2. *In addition to Assumption 2.6, we from now on require that $\Omega \cup \Gamma_D$ satisfies the volume-conservation condition from Definition 2.4 (ii).*

DEFINITION 4.3. *We call a control $u \in L^{2r}(J; W^{-1,q}_{\Gamma_D})$, $r > r^*(q)$, a* global control *if the corresponding solution θ exists on the whole prescribed interval (T_0, T_1) and denote the set of global controls by $\mathcal{U}_g$. Moreover, we define the* control-to-state operator

$$\mathcal{S}\colon \mathcal{U}_g \ni u \mapsto \mathcal{S}(u) = \theta_u \in W^{1,r}(J; W^{-1,q}_{\emptyset}) \cap L^r(J; W^{1,q})$$

on $\mathcal{U}_g$.

Let us firstly show that the previous definition is in fact meaningful in the sense that $\mathcal{U}_g \neq \emptyset$. The natural candidate for a global control is $u \equiv 0$. One readily observes that the control $u \equiv 0$ leads to the solution $\varphi \equiv 0$ for the elliptic equation (3.7), hence the right-hand side in the parabolic equation reduces to $\alpha\theta_l(t)$ in this case. Indeed, we will show that there exists a global solution $\theta_{u=0}$ to the equation

$$\partial_t \theta + \mathcal{A}(\theta)\theta = \alpha\theta_l, \quad \theta(T_0) = \theta_0. \tag{4.1}$$

In order to obtain a global solution to (4.1), we need the volume-conservation condition. Under this additional assumption, the following result has been shown in [48, Thm. 5.3]. Note that the case of $\Omega \cup \Gamma_D$ regular is only a special case of the admissible geometries in [48].

PROPOSITION 4.4. *Assume that $\Omega \cup \Xi$ is regular and in addition satisfies the volume-conservation condition. Let μ be a coefficient function on Ω, measurable, bounded, elliptic. Assume that $\phi : \mathbb{R} \to [\underline{\phi}, \overline{\phi}]$, where $\underline{\phi} > 0$, is Lipschitz continuous on bounded sets. Suppose further that*

$$-\nabla \cdot \mu \nabla : W^{1,q}_{\Xi} \to W^{-1,q}_{\Xi}$$

is a topological isomorphism for some $q > 3$. Let w_0 be from $(W^{1,q}_{\Xi}, W^{-1,q}_{\Xi})_{\frac{1}{r},r}$ with $r > r^*(q) = \frac{2q}{q-3}$. Then, for every $f \in L^r(J; W^{-1,q}_{\Xi})$, there exists a unique global *solution w of the quasilinear equation*

$$w' - \nabla \cdot \phi(w)\mu\nabla w = f, \quad w(T_0) = w_0, \tag{4.2}$$

which belongs to $W^{1,r}(J; W^{-1,q}_{\Xi}) \cap L^r(J; W^{1,q}_{\Xi})$.

With $w_0 = \theta_0$, $\Xi = \emptyset$, $\phi = \eta$, $\mu = \kappa$ and $f = \alpha\theta_l$, we may use Proposition 4.4 to ensure the existence of a *global* solution of (4.1) in the sense of Definition 3.11 under Assumption 3.4 – in particular, $0 \in \mathcal{U}_g$ follows. In [48], Proposition 4.4 is proven for the case where the differential operator consists of the divergence-gradient operator only. However, it is clear that the result extends to the operators of the form $\mathcal{A}$ including the boundary form since the latter is relatively compact with respect to the main part, cf. Corollary 3.8 and the reference there, see also [33, Lem. 5.15].

The next theorem establishes continuous differentiability of the control-to-state operator $\mathcal{S}$. Given a control u, we use φ_u for the associated solution of the elliptic equation with u on the right-hand side, cf. (3.18).

THEOREM 4.5. *Let $r > r^*(q)$ be given. Then the set of global controls $\mathcal{U}_g$ forms an open set in $L^{2r}(J;W^{-1,q}_{\Gamma_D})$. Moreover, the control-to-state operator $\mathcal{S}$ is continuously differentiable. For every $h \in L^{2r}(J;W^{-1,q}_{\Gamma_D})$, its derivative $\zeta_h = \mathcal{S}'(u)h \in W^{1,r}(J;W^{-1,q}_{\emptyset}) \cap L^r(J;W^{1,q})$ is given by the unique solution of the equation*

$$\begin{aligned}\partial_t\zeta + \mathcal{A}(\theta_u)\zeta = (\sigma'(\theta_u)\zeta\varepsilon\nabla\varphi_u)\cdot\nabla\varphi_u + \nabla\cdot\eta'(\theta_u)\zeta\kappa\nabla\theta_u \\ - 2\,(\sigma(\theta_u)\varepsilon\nabla\varphi_u)\cdot\nabla\left[\mathcal{J}(\sigma(\theta_u))\left(-\nabla\cdot\sigma'(\theta_u)\zeta\varepsilon\nabla\varphi_u + h\right)\right], \quad (4.3)\end{aligned}$$

which has to hold for almost every $t \in J$ in the space $W^{-1,q}_{\emptyset}$, with $\zeta(T_0) = 0$.

Proof. Let $\bar u \in L^{2r}(J;W^{-1,q}_{\Gamma_D})$ be global, i.e., the associated solution $\theta_{\bar u} =: \bar\theta$ exists on the whole time horizon (T_0, T_1). We intend to apply the implicit function theorem. To this end, we show that the mapping

$$\mathcal{B}\colon \Big(W^{1,r}(J;W^{-1,q}_{\emptyset}) \cap L^r(J;W^{1,q})\Big) \times L^{2r}(J;W^{-1,q}_{\Gamma_D}) \\ \to L^r(J;W^{-1,q}_{\emptyset}) \times (W^{1,q}, W^{-1,q}_{\emptyset})_{\frac{1}{r},r},$$

where

$$\mathcal{B}(\theta,u) = \left(\partial_t\theta + \mathcal{A}(\theta)\theta - \Psi_u(\theta) - \alpha\theta_l, \theta(T_0) - \theta_0\right), \quad (4.4)$$

is continuously differentiable in $(\bar\theta, \bar u)$, and that the partial derivative $\partial_\theta\mathcal{B}(\bar\theta,\bar u)$ is continuously invertible. Note that $\mathcal{B}(\bar\theta,\bar u) = 0$. The term $\alpha\theta_l$ does not depend neither on u nor on θ and is thus neglected for the rest of this proof. Let us first consider the partial derivative with respect to u: For each $\theta \in C(\overline{Q})$, the mapping

$$L^{2r}(J;W^{-1,q}_{\Gamma_D})^2 \ni (u,v) \mapsto (\sigma(\theta)\varepsilon\nabla\varphi_u(\theta))\cdot\nabla\varphi_v(\theta) \in L^r(J;L^{q/2})$$

gives rise to a continuous symmetric bilinear form $b_\theta(u,v)$ (cf. also Lemma 3.22), since for fixed $\theta \in C(\overline{Q})$ we have

$$\|b_\theta(u,v)\|_{L^r(J;L^{q/2})} \le \|\sigma(\theta)\|_{C(\overline{Q})}\|\varepsilon\|_{L^\infty}\|\mathcal{J}(\sigma(\theta))\|^2_{C(\overline{J};\mathcal{L}(W^{-1,q}_{\Gamma_D},W^{1,q}_{\Gamma_D}))} \\ \cdot \|u\|_{L^{2r}(J;W^{-1,q}_{\Gamma_D})}\|v\|_{L^{2r}(J;W^{-1,q}_{\Gamma_D})}.$$

Accordingly, $u \mapsto \Psi_u(\bar\theta) = b_{\bar\theta}(u,u)$ is continuously differentiable, and its derivative in $\bar u$ is given by $h \mapsto 2b_{\bar\theta}(\bar u, h)$. The second component of $\mathcal{B}$ is independent of u. Next, we treat the derivative of $\mathcal{B}$ w.r.t. θ. First, note that, due to Assumption 4.2, the Nemytskii operator $\theta \mapsto \eta(\theta)$ is continuously differentiable from $C(\overline{Q})$ to $C(\overline{Q})$ and its derivative in $\bar\theta$ is given by $h \mapsto \eta'(\bar\theta)h$. With Remark 3.2, we thus find that the derivative of the function $\theta \mapsto \partial_t\theta + \mathcal{A}(\theta)\theta$ as a mapping from $W^{1,r}(J;W^{-1,q}_{\emptyset}) \cap L^r(J;W^{1,q})$ to $L^r(J;W^{-1,q}_{\emptyset})$ in the point $\bar\theta$ is given by

$$h \mapsto \partial_t h - \nabla\cdot\eta(\bar\theta)\kappa\nabla h + \tilde\alpha h - \nabla\cdot\eta'(\bar\theta)h\kappa\nabla\bar\theta = \partial_t h + \mathcal{A}(\bar\theta)h - \nabla\cdot\eta'(\bar\theta)h\kappa\nabla\bar\theta. \quad (4.5)$$

We turn to $\theta \mapsto \Psi_{\bar u}(\theta)$. As above, due to Assumption 4.2, $\theta \mapsto \sigma(\theta)$ is continuously differentiable as a mapping from $C(\overline{Q})$ to $C(\overline{Q})$ and with derivative $h \mapsto \sigma'(\bar\theta)h$ (in a point $\bar\theta$). Further, recall that the derivative of the (continuously differentiable) mapping $\mathcal{L}(X;Y) \ni A \mapsto A^{-1} \in \mathcal{L}(Y;X)$ in A is given by $H \mapsto -A^{-1}HA^{-1}$. The chain rule and Remark 3.2 thus yield continuous differentiability of $\theta \mapsto \mathcal{J}(\sigma(\theta))$ as a mapping from $C(\overline{J};C(\overline{\Omega}))$ to $C(\overline{J};\mathcal{L}(W^{1,q}_{\Gamma_D};W^{-1,q}_{\Gamma_D}))$ with the derivative

$$\left[(\mathcal{J}\circ\sigma)'(\bar\theta)\right]h = -\mathcal{J}(\sigma(\bar\theta))\left[-\nabla\cdot\sigma'(\bar\theta)h\varepsilon\nabla\right]\mathcal{J}(\sigma(\bar\theta)).$$

Hence, $\theta \mapsto \varphi_{\bar u}(\theta) = \mathcal{J}(\sigma(\theta))\bar u$ is also continuously differentiable, considered as a mapping from $C(\overline{J};C(\overline{\Omega}))$ to $L^{2r}(J;W^{1,q}_{\Gamma_D})$. Continuous differentiability of the function given by $C(\overline{J};C(\overline{\Omega})) \ni \theta \mapsto$

$\Psi_{\bar u}(\theta) \in L^r(J; L^{q/2}) \hookrightarrow L^r(J; W^{-1,q}_\emptyset)$ is now straightforward and its derivative in $\bar\theta$ is given by

$$\left[\partial_\theta \Psi_{\bar u}(\bar\theta)\right] h = -2\left(\sigma(\bar\theta)\varepsilon\nabla\left[\mathcal{J}(\sigma(\bar\theta))\bar u\right]\right)\cdot\nabla\left[\left(\left[(\mathcal{J}\circ\sigma)'(\bar\theta)\right]h\right)\bar u\right] \\ + \left(\sigma'(\bar\theta)h\varepsilon\nabla\left[\mathcal{J}(\sigma(\bar\theta))\bar u\right]\right)\cdot\nabla\left[\mathcal{J}(\sigma(\bar\theta))\bar u\right]. \quad (4.6)$$

The second component of $\mathcal{B}$, i.e., $\theta \mapsto \theta(T_0) - \theta_0$, is affine-linear and continuous from the maximal regularity space into $(W^{1,q}, W^{-1,q}_\emptyset)_{\frac{1}{r},r}$ and as such has the derivative $h \mapsto h(T_0)$. It remains to show the continuous invertibility of $\partial_\theta\mathcal{B}(\bar\theta,\bar u)$. For this, we identify for almost every $t \in (T_0,T_1)$ and $h \in C(\overline{J}; C(\overline{\Omega}))$ as follows:

$$B(t)h(t) = \left(\left[\partial_\theta\Psi_{\bar u}(\bar\theta)\right]h\right)(t) + \nabla\cdot\eta'(\bar\theta(t))h(t)\kappa\nabla\bar\theta(t),$$

such that $B(t)$ is from $\mathcal{L}(C(\overline{\Omega}); W^{-1,q}_\emptyset)$ and $t \mapsto B(t) \in L^r(J; \mathcal{L}(C(\overline{\Omega}); W^{-1,q}_\emptyset))$. Combining (4.5) and (4.6), in order to prove that $\mathcal{B}_\theta$ is continuously invertible we need to show that the equation

$$\partial_t\xi(t) + \mathcal{A}(\bar\theta(t))\xi(t) = B(t)\xi(t) + f(t), \qquad \xi(T_0) = \xi_0 \quad (4.7)$$

has a unique solution $\xi \in W^{1,r}(J; W^{-1,q}_\emptyset) \cap L^r(J; W^{1,q})$ for every $f \in L^r(J; W^{-1,q}_\emptyset)$ and $\xi_0 \in (W^{1,q}, W^{-1,q}_\emptyset)_{\frac{1}{r},r}$. This, however, is exactly what is obtained by [50, Cor. 3.4], hence we have

$$\partial_\theta\mathcal{B}(\bar\theta,\bar u) \in \mathcal{LH}\big(W^{1,r}(J; W^{-1,q}_\emptyset) \cap L^r(J; W^{1,q}); L^r(J; W^{-1,q}_\emptyset)\big) \times (W^{1,q}, W^{-1,q}_\emptyset)_{\frac{1}{r},r})).$$

Thus, all requirements for the implicit function theorem are satisfied, which yields neighbourhoods $\mathfrak{V}_{\bar u}$ of $\bar u$ in $L^{2r}(J; W^{-1,q}_{\Gamma_D})$ and $\mathfrak{V}_{\bar\theta}$ of $\bar\theta$ in the maximal regularity space, such that there exists a continuously differentiable mapping $\Phi\colon \mathfrak{V}_{\bar u} \to \mathfrak{V}_{\bar\theta}$ with $\mathcal{B}(\Phi(u),u) = \mathcal{B}(\bar\theta,\bar u) = 0$ for all $u \in \mathfrak{V}_{\bar u}$. This shows that the set of global controls is open. Moreover, Φ locally coincides with the control-to-state operator $u \mapsto \mathcal{S}(u)$, which implies continuous differentiability for the latter.
The stated expression for $\mathcal{S}'(u)h$ is obtained by differentiating the (constant) function $u \mapsto \mathcal{B}(\mathcal{S}(u),u)$. From the second component, we then find $(\mathcal{S}'(u)h)(T_0) = 0$ in $(W^{1,q}, W^{-1,q}_\emptyset)_{\frac{1}{r},r}$ for all h, and the chain rule yields

$$\mathcal{S}'(u)h = -[\partial_\theta\mathcal{B}(\mathcal{S}(u),u)]^{-1}\partial_u\mathcal{B}(\mathcal{S}(u),u)h,$$

meaning exactly that $\mathcal{S}'(u)h$ is the unique solution to the problem (4.7) with right-hand side $f = -\partial_u\mathcal{B}(\mathcal{S}(u),u)h$ and initial value 0. Inserting all formulas, we obtain the equation stated in the theorem. ∎

Remark 4.6. *One may split the equation solved by $\zeta_h = \mathcal{S}'(u)h$ in the previous Theorem 4.5 back into two equations: Introducing*

$$\Phi(\zeta) := \mathcal{J}(\sigma(\theta_u))\left(-\nabla\cdot\sigma'(\theta_u)\zeta\varepsilon\nabla\varphi_u + h\right) \in L^{2r}(J; W^{1,q}_{\Gamma_D}),$$

we find that, for every $h \in L^{2r}(J; W^{-1,q}_{\Gamma_D})$, the pair $(\zeta,\pi) := (\mathcal{S}'(u)h, \Phi(\mathcal{S}'(u)h))$ is the unique solution *of the system*

$$\partial_t\zeta + \mathcal{A}(\theta_u)\zeta = (\sigma'(\theta_u)\zeta\varepsilon\nabla\varphi_u)\cdot\nabla\varphi_u + \nabla\cdot\eta'(\theta_u)\zeta\kappa\nabla\theta_u + 2\left(\sigma(\theta_u)\varepsilon\nabla\varphi_u\right)\cdot\nabla\pi \\ -\nabla\cdot\sigma(\theta_u)\varepsilon\nabla\pi = -\nabla\cdot\sigma'(\theta_u)\zeta\varepsilon\nabla\varphi_u + h$$

with $\zeta(T_0) = 0$ (the first equation is supposed to hold in $W^{-1,q}_\emptyset$, the second one in $W^{-1,q}_{\Gamma_D}$, each for almost all $t \in J$). These equations are exactly the linearized state system *for* (3.6) *and* (3.7). *This also shows, expectedly, that from a functional-analytical point of view, it makes no difference working with θ only and considering φ as a function obtained by θ, instead of considering both functions at once.*

Combining Theorem 4.5 with Proposition 4.4 as explained above, we obtain the following

Corollary 4.7. *There is always a neighbourhood $\mathfrak{V}_0$ of 0 in $L^{2r}(J; W^{-1,q}_{\Gamma_D})$, containing only global controls, i.e., $\mathfrak{V}_0 \subseteq \mathcal{U}_g$.*

Now that we have established a certain richness of global controls, we turn to the question of existence of an optimal control of (P). Following the standard direct method of the calculus of variations, one soon encounters the problem of lacking uniform boundedness in a suitable space for solutions θ_u associated to a minimizing sequence of global controls u, which is a common obstacle to overcome when treating quasilinear equations. To circumvent this, we use Proposition 3.3 (i) to show that the solutions θ_u are uniformly bounded in $W^{1,s}(J; W_\emptyset^{-1,\varsigma})$, where $\varsigma \leq 3 < q$ (in general only $\varsigma \sim \frac{3}{2}$) and s is the exponent from the second addend in the objective function in (P). As this term in the objective gives an additional bound in $L^s(J; W^{1,q})$, we can employ Corollary 3.20 to "lift" this boundedness result to a Hölder space, which is suitable for passing to the limit with a minimizing sequence. However, in order to apply Corollary 3.20, the exponent s has to be sufficiently large. The precise bound for s is characterized by the following

Definition 4.8. *Let $\mathfrak{q} \in (2, \min\{q_0, 3\}]$ be given, where q_0 is the number from Proposition 3.3 (i), and set $\varsigma := \frac{3\mathfrak{q}}{6-\mathfrak{q}}$. Then we define the number $\bar{r}(q,\varsigma) > 0$ by*

$$\bar{r}(q,\varsigma) := \frac{2q}{q-3}\left(1 - \frac{3}{q} + \frac{3}{\varsigma}\right). \tag{4.8}$$

On account of $\varsigma \leq 3 < q$ it follows that $\bar{r}(q,\varsigma) > r^*(q) = \frac{2q}{q-3}$. Therefore, for a given number $s > \bar{r}(q,\varsigma)$, the previous results, in particular the assertions of Theorem 3.13, Theorem 4.5, and Corollary 4.13, hold with $r = s$. The next theorem precisely elaborates the argument depicted before Definition 4.8:

Theorem 4.9. *Let $\mathcal{U} \subseteq \mathcal{U}_g$ be bounded in $L^{2s}(J; W_{\Gamma_D}^{-1,q})$ with $s > \bar{r}(q,\varsigma)$ and suppose in addition that the associated set of solutions $\mathcal{K} = \{\theta_u : u \in \mathcal{U}\}$ is bounded in $L^s(J; W^{1,q})$. Then $\mathcal{K}$ is even compact in $C(\overline{Q})$ and the closure of $\mathcal{U}$ in $L^{2s}(J; W_{\Gamma_D}^{-1,q})$ is still contained in $\mathcal{U}_g$.*

As indicated above, the second addend in the objective functional together with the state constraints will guarantee the bound in $L^s(J; W^{1,q})$ for the minimizing sequence, see the proof of Theorem 4.16 below.

Proof of Theorem 4.9. We show that $\mathcal{K}$ is bounded in a suitable maximal-regularity-like space. To this end, we first investigate the right-hand side in the parabolic equation (3.6). Denote by (θ_u, φ_u) the solution for a given $u \in \mathcal{U}$. Thanks to Assumption 2.7 (i), Proposition 3.3 (i) shows that $-\nabla \cdot \sigma(\theta)\varepsilon\nabla$ is a topological isomorphism between $W_{\Gamma_D}^{1,\mathfrak{q}}$ and $W_{\Gamma_D}^{-1,\mathfrak{q}}$ with

$$\sup_{\theta \in \mathcal{K}} \| \left(-\nabla \cdot \sigma(\theta)\varepsilon\nabla\right)^{-1} \|_{\mathcal{L}(W_{\Gamma_D}^{-1,\mathfrak{q}}; W_{\Gamma_D}^{1,\mathfrak{q}})} < \infty. \tag{4.9}$$

Hence, for every $u \in \mathcal{U}$ there exists a unique $\psi = \psi_u \in L^{2s}(J; W_{\Gamma_D}^{1,\mathfrak{q}})$ such that

$$\psi_u(t) = \left(-\nabla \cdot \sigma(\theta_u(t))\varepsilon\nabla\right)^{-1} u(t) \quad \text{in } W_{\Gamma_D}^{1,\mathfrak{q}}$$

for almost every $t \in (T_0, T_1)$, and

$$\sup_{u \in \mathcal{U}} \|\psi_u\|_{L^{2s}(J; W_{\Gamma_D}^{1,\mathfrak{q}})} < \infty.$$

Since $W_{\Gamma_D}^{1,q} \hookrightarrow W_{\Gamma_D}^{1,\mathfrak{q}}$ and, by uniqueness of ψ_u, we in particular obtain $\varphi_u = \psi_u$, such that the family φ_u is bounded in $L^{2s}(J; W_{\Gamma_D}^{1,\mathfrak{q}})$ as well. Estimating as in Lemma 3.27, we find that also

$$\sup_{u \in \mathcal{U}} \|(\sigma(\theta_u)\varepsilon\nabla\varphi_u) \cdot \nabla\varphi_u\|_{L^s(J; L^{\mathfrak{q}/2})} < \infty.$$

Using the boundedness assumption on $\mathcal{K}$ in $L^s(J; W^{1,q})$, both the family of functionals $\tilde{\alpha}\theta_u$ and, here also employing boundedness of η, the divergence-operators $-\nabla \cdot \eta(\theta_u)\kappa\nabla\theta_u$ are uniformly bounded over $\mathcal{U}$, i.e.,

$$\sup_{u \in \mathcal{U}} \|\nabla \cdot \eta(\theta_u)\kappa\nabla\theta_u\|_{L^s(J; W_\emptyset^{-1,q})} + \|\tilde{\alpha}\theta_u\|_{L^s(J; W_\emptyset^{-1,q})} < \infty.$$

Sobolev embeddings give the embedding $L^{\mathfrak{q}/2} \hookrightarrow W_\emptyset^{-1,\varsigma}$ for $\varsigma = \frac{3\mathfrak{q}}{6-\mathfrak{q}}$, and certainly $W_\emptyset^{-1,q} \hookrightarrow W_\emptyset^{-1,\varsigma}$ due to $q > \varsigma$. Hence,

$$\partial_t\theta_u = \nabla \cdot \eta(\theta_u)\kappa\nabla\theta_u - \tilde{\alpha}\theta_u + (\sigma(\theta_u)\varepsilon\nabla\varphi_u) \cdot \nabla\varphi_u + \alpha\theta_l$$

is uniformly bounded over $\mathcal{U}$ in $L^{2s}(J; W_\emptyset^{-1,\varsigma})$. This shows that $\mathcal{K}$ is bounded in the space $W^{1,s}(J; W_\emptyset^{-1,\varsigma}) \cap L^s(J; W^{1,q})$. By Corollary 3.20, $\mathcal{K}$ is then also bounded in a Hölder space and thus a (relatively) compact set in $C(\overline{Q})$.

Next, let us show that the limit of a convergent sequence in $\mathcal{U}$ is still a global control. Denote by $(u_n) \subset \mathcal{U}$ such a sequence, converging in $L^{2s}(J; W_{\Gamma_D}^{-1,q})$ to the limit $\bar{u}$. We call the associated states $(\theta_n) := (\theta_{u_n})$. Compactness of $\mathcal{K}$ as shown above gives a subsequence of (θ_n), called (θ_{n_k}), which converges to some $\bar{\theta}$ in $C(\overline{Q})$. Lemma 3.27 shows that $\Psi_{u_{n_k}}(\theta_{n_k}) \to \Psi_{\bar{u}}(\bar{\theta})$ as $k \to \infty$. Note that $\bar{\theta} = \theta_{\bar{u}}$ is not clear yet, but of course we will show exactly this now. By [48, Lem. 5.5], the equations

$$\partial_t \zeta + \mathcal{A}(\theta_{n_k})\zeta = \Psi_{u_{n_k}}(\theta_{n_k}) + \alpha\theta_l, \quad \theta_{n_k}(T_0) = \theta_0$$

have solutions $\zeta_{n_k} \in W^{1,s}(J; W_\emptyset^{-1,q}) \cap L^s(J; W^{1,q})$, which, due to uniqueness of solutions, must coincide with θ_{n_k}. This means, on the one hand, that $\zeta_{n_k} = \theta_{n_k} \to \bar{\theta}$ in $C(\overline{Q})$ as $k \to \infty$. On the other hand, [48, Lem. 5.5] also shows that the sequence (ζ_{n_k}) has a limit $\bar{\zeta}$ in the maximal regularity space as k goes to infinity, where $\bar{\zeta}$ is the solution of the limiting problem

$$\partial_t \zeta + \mathcal{A}(\bar{\theta})\zeta = \Psi_{\bar{u}}(\bar{\theta}) + \alpha\theta_l, \quad \zeta(T_0) = \theta_0.$$

We do, however, already know that $\bar{\zeta} = \bar{\theta}$, such that $\bar{\theta}$ is the unique global solution to the nonlinear problem for the limiting control $\bar{u}$, i.e., $\bar{\zeta} = \bar{\theta} =: \theta_{\bar{u}}$. Hence, $\bar{u}$ is still a global control. ∎

REMARK 4.10. *Note that we used Proposition 3.3 (i) instead of Lemma 3.7 at the beginning of the proof of Theorem 4.9. This is indeed a crucial point, since Proposition 3.3 (i) implies the isomorphism property and a* uniform *bound of the inverse for all coefficient functions that share the same ellipticity constant and the same L^∞-bound. Thus, in our concrete situation, the norm of $(-\nabla\cdot\sigma(\theta)\varepsilon\nabla)^{-1}$ is completely determined by $\Omega\cup\Gamma_D$ and the data from Assumption 2.7 (i) and 2.7 (ii), which gives the estimate in* (4.9)*. By contrast, the application of Lemma 3.7 would require to control the norm of $\sigma(\theta)$ in $C(\overline{\Omega})$, see also Theorem 3.23. This however cannot be guaranteed a priori so that Proposition 3.3 (i) is indeed essential for the proof of Theorem 4.9. Since the integrability exponent from Proposition 3.3 (i) is in general less than 3 and therefore less than q, one needs an improved regularity in time to have the continuous embedding in the desired Hölder space, cf. Corollary 3.20. Therefore it is* not *sufficient to require $s > r^*(q)$ and the more restrictive condition $s > \bar{r}(q,\varsigma)$ is imposed instead.*

Next, we incorporate the control- and state constraints in (P) into the control problem. For this purpose, let us introduce the set

$$\mathcal{U}^{\mathrm{ad}} := \{u \in L^2(J; L^2(\Gamma_N)) \colon 0 \le u \le u_{\max} \text{ a.e. in } \Sigma_N\}. \tag{4.10}$$

DEFINITION 4.11. *We call a global control $u \in \mathcal{U}_g$* feasible*, if $u \in \mathcal{U}^{ad}$ and the associated state satisfies $\mathcal{S}(u)(x,t) \le \theta_{\max}(x,t)$ for all $(x,t) \in \overline{Q}$.*

While the state constraints give upper bounds on the values of feasible solutions, lower bounds are natural in the problem and implicitly contained in (1.1)–(1.6) in the sense that the temperature of the workpiece associated with Ω will not drop below the minima of the surrounding temperature (represented by θ_l) and the initial temperature distribution θ_0.

PROPOSITION 4.12. *For every solution (θ,φ) in the sense of Theorem 3.13 with maximal existence interval $J_{\max}$, we have $\theta(x,t) \ge m_{\inf} := \min(\operatorname{ess\,inf}_\Sigma \theta_l, \min_{\overline{\Omega}} \theta_0)$ for all $(x,t) \in \overline{\Omega}\times[T_0,T_\bullet]$, where $T_\bullet \in J_{\max}$.*

See Proposition A.1 in the Appendix for a proof. Analogously, we find that $u \equiv 0$ is a feasible control under Assumption 2.9 (iv), the latter demanding that the surrounding temperature and the initial temperature do not exceed the state bounds at any point.

COROLLARY 4.13. *The control $u \equiv 0$ is a feasible one.*

Proof. By Corollary 4.7, $u \equiv 0$ is a global control, it obviously satifies the control constraints, and using the same reasoning as in Proposition A.1 with Assumption 2.9 (iv), we obtain $\theta_{u\equiv 0} \le \theta_{\max}$. ∎

Let us next introduce a modified control space, fitting the norm in the objective functional in (P). So far, the controls originated from the space $L^{2s}(J; W_{\Gamma_D}^{-1,q})$ with $s > \bar{r}(q,\varsigma)$. For the optimization, we now switch to the more advanced control space

$$\mathbb{U} := W^{1,2}(J; L^2(\Gamma_N)) \cap L^p(J; L^p(\Gamma_N)) \tag{4.11}$$

with the standard norm $\|u\|_{\mathbb{U}} = \|u\|_{W^{1,2}(J;L^2(\Gamma_N))} + \|u\|_{L^p(J;L^p(\Gamma_N))}$. Since $p > \frac{4}{3}q - 2$ by Assumption 2.9, this space continuously embeds into $L^{2s}(J; W^{-1,q}_{\Gamma_D})$, which will give the boundedness required for Theorem 4.9. Moreover, this embedding is even *compact*, as the following result shows:

Proposition 4.14. *Let $p > 2$. The space $\mathbb{U}$ is embedded into a Hölder space $C^\varrho(\overline{J}; L^{\mathfrak{p}}(\Gamma_N))$ for some $\varrho > 0$ and $2 < \mathfrak{p} < \frac{p+2}{2}$. In particular, there exists a* compact *embedding $\mathcal{E} \colon \mathbb{U} \hookrightarrow L^s(J; W^{-1,q}_{\Gamma_D})$ for every $p > \frac{4}{3}q - 2$ and $s \in [1, \infty]$.*

Proof. From the construction of real interpolation spaces by means of the trace method it immediately follows that

$$\mathbb{U} \hookrightarrow C(\overline{J}; (L^p(\Gamma_N), L^2(\Gamma_N))_{\frac{2}{p+2}, \frac{p+2}{2}}) = C(\overline{J}; L^{\frac{p+2}{2}}(\Gamma_N)),$$

see [54, Ch. 1.8.1–1.8.3 and Ch. 1.18.4]. With similar reasoning as for (3.10), see also [39, Lem. 3.17] and its proof, we also may show $\mathbb{U} \hookrightarrow C^\varrho(\overline{J}; (L^p(\Gamma_N), L^2(\Gamma_N))_{\tau,1})$ for all $\tau \in (\frac{2}{2+p}, 1)$ and some $\varrho = \varrho(\tau) > 0$. Moreover,

$$\big(L^p(\Gamma_N), L^2(\Gamma_N)\big)_{\tau,1} \hookrightarrow \big[L^p(\Gamma_N), L^2(\Gamma_N)\big]_\tau = L^{\mathfrak{p}}(\Gamma_N)$$

with $\mathfrak{p} = \mathfrak{p}(\tau) = (\frac{1-\tau}{p} + \frac{\tau}{2})^{-1} \in (2, \frac{2+p}{2})$ for $\tau \in (\frac{2}{2+p}, 1)$, see [54, Ch. 1.10.1/3 and Ch. 1.18.4]. This means we have $\mathbb{U} \hookrightarrow C^\varrho(\overline{J}; L^{\mathfrak{p}}(\Gamma_N))$ for all $\mathfrak{p} \in (2, \frac{2+p}{2})$, with $\varrho > 0$ depending on $\mathfrak{p}$. If $\mathfrak{p} > \frac{2}{3}q$, then there is an embedding $L^{\mathfrak{p}}(\Gamma_N)) \hookrightarrow W^{-1,q}_{\Gamma_D}$, cf. Remark 2.8, and this is even compact in this case as we will show below. To make $\mathfrak{p} > \frac{3}{2}q$ possible, we need $\frac{p+2}{2} > \frac{2}{3}q$, which is equivalent to $p > \frac{4}{3}q - 2$. Now the vector-valued Arzelà-Ascoli Theorem yields the assertion.

It remains to show that $L^{\mathfrak{p}}(\Gamma_N) \hookrightarrow W^{-1,q}_{\Gamma_D}$ compactly for $\mathfrak{p} > \frac{2}{3}q$, or equivalently $W^{1,q'}_{\Gamma_D} \hookrightarrow L^{\mathfrak{p}'}(\Gamma_N)$ compactly. From [47, Ch. 1.4.7, Cor. 2] and [35, Lem. 3.2] we obtain

$$\|u\|_{L^{\mathfrak{p}'}(\partial\Omega)} \le C\|u\|^\tau_{W^{1,q'}} \|u\|^{1-\tau}_{L^{q'}} \quad \text{for all } u \in W^{1,q'}$$

for $\mathfrak{p}' \in (\frac{2}{3}q', \frac{2q'}{3-q'})$ and $\tau = \frac{3}{q'} - \frac{2}{\mathfrak{p}'}$. Note that $\tau \in (0,1)$ for the given range of $\mathfrak{p}'$. The preceding inequality implies $(L^{q'}, W^{1,q'})_{\tau,1} \hookrightarrow L^{\mathfrak{p}'}(\partial\Omega)$, cf. [54, Lem. 1.10.1] and hence, due to the compact embedding $W^{1,q'} \hookrightarrow L^{q'}$ as of [47, Ch. 1.4.6, Thm. 2], $W^{1,q'} \hookrightarrow L^{\mathfrak{p}'}(\partial\Omega)$ compactly for all $\mathfrak{p}' \in (0, \frac{2q'}{3-q'})$ by [54, Ch. 1.16.4]. With $W^{1,q'}_{\Gamma_D} \hookrightarrow W^{1,q'}$ and $L^{\mathfrak{p}'}(\partial\Omega) \hookrightarrow L^{\mathfrak{p}'}(\Gamma_N)$, this means $W^{1,q'}_{\Gamma_D} \hookrightarrow L^{\mathfrak{p}'}(\Gamma_N)$ compactly for $\mathfrak{p} > \frac{2}{3}q$. ∎

Definition 4.15. *Consider the embedding $\mathcal{E}$ from Proposition 4.14 with range in $L^{2s}(J; W^{-1,q}_{\Gamma_D})$, where $s > \bar{r}(q, \varsigma)$ is the integrability exponent from the objective functional. We set*

$$\mathbb{U}_g := \{u \in \mathbb{U} \colon \mathcal{E}(u) \in \mathcal{U}_g\}$$

and define the mapping

$$\mathcal{S}_{\mathcal{E}} := \mathcal{S} \circ \mathcal{E} \colon \mathbb{U}_g \to W^{1,s}(J; W^{1,q}) \cap L^s(J; W^{-1,q}_\emptyset).$$

Moreover, we define the reduced objective functional *j obtained by reducing the objective functional in* (P) *to u, i.e.,*

$$j(u) = \frac{1}{2}\int_E |\mathcal{S}_{\mathcal{E}}(u)(T_1) - \theta_d|^2 \,\mathrm{d}x + \frac{\gamma}{s}\|\nabla \mathcal{S}_{\mathcal{E}}(u)\|^s_{L^s(J;L^q)} + \frac{\beta}{2}\int_{\Sigma_N} (\partial_t u)^2 + |u|^p \,\mathrm{d}\omega\,\mathrm{d}t,$$

as a function on $\mathbb{U}_g$. Further, let $\mathbb{U}^{ad} := \mathbb{U} \cap \mathcal{U}^{ad}$ and $\mathbb{U}^{ad}_g := \mathbb{U}_g \cap \mathcal{U}^{ad}$, where $\mathcal{U}^{ad}$ is as defined in (4.10).

One readily observes that $\mathcal{S}_{\mathcal{E}}$ on $\mathbb{U}_g$ is still continuously differentiable with the derivative $h \mapsto \mathcal{S}'_{\mathcal{E}}(u)h = \mathcal{S}'(\mathcal{E}u)\mathcal{E}h$.

Theorem 4.16. *There exists an optimal solution $\bar{u} \in \mathbb{U}^{ad}_g$ to the problem*

$$\min_{u \in \mathbb{U}^{ad}_g} j(u) \quad \text{such that} \quad \mathcal{S}_{\mathcal{E}}(u)(x,t) \le \theta_{\max}(x,t) \quad \forall (x,t) \in \overline{Q}. \tag{P$_u$}$$

Proof. Thanks to the existence of the feasible control $u \equiv 0$, cf. Corollary 4.13, the objective functional is bounded from below by 0. Thus there exists a minimizing sequence of feasible controls (u_n) in $\mathbb{U}_g^{\mathrm{ad}}$ such that $j(u_n) \to \inf_{u \in \mathbb{U}_g^{\mathrm{ad}}} j(u)$ in $\mathbb{R}$. On account of

$$\int_{\Sigma_N} (\partial_t u)^2 + |u|^p \, \mathrm{d}\omega \, \mathrm{d}t \longrightarrow \infty \quad \text{when} \quad \|u\|_{\mathbb{U}} \longrightarrow \infty, \tag{4.12}$$

the objective functional is radially unbounded so that the minimizing sequence is bounded in $\mathbb{U}$ and, due to reflexivity of $\mathbb{U}$, has a weakly convergent subsequence (again (u_n)), converging weakly to some $\bar{u} \in \mathbb{U}$. As $\mathbb{U}^{\mathrm{ad}}$ is closed and convex, we have $\bar{u} \in \mathbb{U}^{\mathrm{ad}}$. By the compact embedding from Proposition 4.14, (u_n) converges strongly in $L^{2s}(J; W^{-1,q}_{\Gamma_D})$, also to $\bar{u} \in L^{2s}(J; W^{-1,q}_{\Gamma_D})$. The fact that state constraints are present and Proposition 4.12 imply that the family (θ_{u_n}) is uniformly bounded in time and space for every feasible control u. Together with the gradient term in the objective functional, Theorem 4.9 now shows $\bar{u} \in \mathbb{U}_g$, hence $\bar{u} \in \mathbb{U}_g^{\mathrm{ad}}$. Moreover, $\mathcal{S}_{\mathcal{E}}(u_n) \to \mathcal{S}_{\mathcal{E}}(\bar{u})$ in $W^{1,s}(J; W^{1,q}) \cap L^s(J; W^{-1,q}_{\emptyset})$, which immediately implies convergence of the first two terms in the objective functional (each as n goes to infinity). The third term, corresponding to $\mathbb{U}$, is clearly continuous and convex on $\mathbb{U}$ and as such weakly lower semicontinuous, hence we find

$$\inf_{u \in \mathbb{U}_g^{\mathrm{ad}}} j(u) = \lim_{n \to \infty} j(u_n) \geq j(\bar{u})$$

and thus $j(\bar{u}) = \inf_{u \in \mathbb{U}_g^{\mathrm{ad}}} j(u)$. ∎

REMARK 4.17. *In the proof of Theorem 4.16, boundedness of minimizing sequence (u_n) in the control space $\mathbb{U}$ was essential and followed from the radial unboundedness of the objective functional as seen in (4.12). Alternatively, one could also assume that the upper bound $u_{\max}$ in the control constraints satisfies $u_{\max} \in L^p(J; L^p(\Gamma_N))$ with $p > \frac{4}{3}q - 2$. In this case, an objective functional of the form*

$$\frac{1}{2}\|\theta(T_1) - \theta_d\|^2_{L^2(E)} + \frac{\gamma}{s}\|\nabla\theta\|^s_{L^s(T_0,T_1;L^q(\Omega))} + \frac{\beta}{2}\int_{\Sigma_N} (\partial_t u)^2 \, \mathrm{d}\omega \, \mathrm{d}t$$

is sufficient to establish the existence of a globally optimal control.

4.2. Necessary optimality conditions. This section is devoted to the derivation of necessary optimality conditions for (P) in the form (P_u). To this end, let us start with the definition of the Lagrangian function. It is well-known that the Lagrangian multipliers associated to the state constraints may, in general, only be regular Borel measures, see for instance [11]. Hence, we introduce the space $\mathcal{M}(\overline{Q})$ as the space of regular Borel measures on $\overline{Q}$ and, simultaneously, as the dual space of $C(\overline{Q})$.

DEFINITION 4.18. *The* Lagrangian function $\mathfrak{L} \colon \mathbb{U}_g \times \mathcal{M}(\overline{Q}) \to \mathbb{R}$ *associated with* (P_u) *is given by*

$$\mathfrak{L}(u, \mu) = j(u) + \langle \mu, \mathcal{S}_{\mathcal{E}}(u) - \theta_{\max} \rangle_{\mathcal{M}(\overline{Q}), C(\overline{Q})},$$

where j is the reduced objective functional.

DEFINITION 4.19. *We denote by* $\Delta_q \colon W^{1,q} \to W^{-1,q'}_{\emptyset}$ *the* (weak) q-Laplacian, *given by*

$$\langle \Delta_q \psi, \xi \rangle := \int_\Omega |\nabla\psi|^{q-2} \nabla\psi \cdot \nabla\xi \, \mathrm{d}x$$

for each $\psi, \xi \in W^{1,q}$.

The chain rule immediately yields the derivative of $\mathfrak{L}$ with respect to u:

LEMMA 4.20. *The Lagrangian function $\mathfrak{L}$ is continuously differentiable with respect to u. Abbreviating the states by $\theta_u := \mathcal{S}_{\mathcal{E}}(u)$ and $\theta_u' = \mathcal{S}_{\mathcal{E}}'(u)h$, the partial derivative in direction $h \in \mathbb{U}$ is given by*

$$\begin{aligned}\partial_u \mathfrak{L}(u, \mu)h = \int_E (\theta_u(T_1) - \theta_d)\theta_u'(T_1) \, \mathrm{d}x + \gamma \int_{T_0}^{T_1} \|\nabla\theta_u(t)\|^{s-q}_{L^q} \langle \Delta_q \theta_u(t), \theta_u'(t) \rangle \, \mathrm{d}t \\ + \beta \int_{\Sigma_N} \partial_t u \partial_t h + \frac{p}{2}|u|^{p-2} u h \, \mathrm{d}\omega \mathrm{d}t) + \langle \mu, \theta_u' \rangle_{\mathcal{M}(\overline{Q}), C(\overline{Q})}\end{aligned} \tag{4.13}$$

with Δ_q given as in Definition 4.19.

Using the Lagrangian function and its derivative, we characterize local optima of (P_u). We say that that a feasible control $\bar{u}$ is *locally optimal* if there exists an $\epsilon > 0$ such that $j(\bar{u}) \leq j(u)$ for all feasible $u \in \mathbb{U}_g^{\mathrm{ad}}$ with $\|u - \bar{u}\|_{\mathbb{U}} < \epsilon$. As we will see in the proof of Theorem 4.22, the restriction to global controls $u \in \mathbb{U}_g$ does not influence the derivation of optimality conditions, since $\mathbb{U}_g$ is an *open* set by Theorem 4.5.

Definition 4.21. *A measure $\bar{\mu} \in \mathcal{M}(\overline{Q})$ is called a* Lagrangian multiplier *associated with the state constraint in* (P_u)*, if for a locally optimal control $\bar{u}$ the* KKT conditions

$$\bar{\mu} \geq 0, \tag{4.14}$$

$$\langle \bar{\mu}, \mathcal{S}_{\mathcal{E}}(\bar{u}) - \theta_{\max} \rangle_{C(\overline{Q})} = 0, \tag{4.15}$$

$$\langle \partial_u \mathfrak{L}(\bar{u}, \bar{\mu}), u - \bar{u} \rangle_{\mathbb{U}} \geq 0 \quad \forall u \in \mathbb{U}^{ad} \tag{4.16}$$

hold true. Here, (4.14) *means that $\langle \bar{\mu}, f \rangle_{C(\overline{Q})} \geq 0$ for all $f \in C(\overline{Q})$ with $f(x,t) \geq 0$ for all $(x,t) \in Q$. Note that* (4.16) *has to be satisfied for* all *$u \in \mathbb{U}^{ad}$ instead of only in $\mathbb{U}_g^{ad}$, the latter being defined in Definition 4.15.*

It is well-known that, in general, a so-called regularity condition is needed in order to ensure the existence of a Lagrangian multiplier. In this case, we rely on the linearized Slater condition, which is a special form of Robinson's regularity condition.

Theorem 4.22. *Let $\bar{u}$ be a locally optimal control and let the following so-called* linearized Slater condition *be satisfied: There exists $\hat{u} \in \mathbb{U}_g^{ad}$ such that there is a $\delta > 0$ with the property*

$$\mathcal{S}_{\mathcal{E}}(\bar{u})(x,t) + \mathcal{S}_{\mathcal{E}}'(\bar{u})(\hat{u} - \bar{u})(x,t) \leq \theta_{\max}(x,t) - \delta \quad \textit{for all } (x,t) \in Q. \tag{4.17}$$

Then there exists a Lagrangian multiplier $\bar{\mu} \in \mathcal{M}(\overline{Q})$ associated with the state constraint in (P_u)*, i.e., such that* (4.14)-(4.16) *is satisfied.*

Proof. Since $\mathbb{U} \hookrightarrow L^{2s}(J; W_{\Gamma_D}^{-1,q})$ as seen in Proposition 4.14, Theorem 4.5 implies that there is an open ball $B_\delta(\bar{u}) \subset \mathbb{U}$ around $\bar{u}$ with radius $\delta > 0$ such that $B_\delta(\bar{u}) \cap \mathbb{U}^{\mathrm{ad}} \subset \mathbb{U}_g^{\mathrm{ad}}$. We consider the auxiliary problem

$$\left.\begin{array}{ll} \min & j(u) \\ \text{s.t.} & u \in B_\delta(\bar{u}) \cap \mathbb{U}^{\mathrm{ad}}, \quad \mathcal{S}_{\mathcal{E}}(u)(x,t) \leq \theta_{\max}(x,t) \quad \forall (x,t) \in \overline{Q}. \end{array}\right\} \tag{$\mathrm{P}_{\mathrm{aux}}$}$$

Clearly, $\bar{u}$ is also a local minimizer of this problem. Moreover, in contrast to $\mathbb{U}_g^{\mathrm{ad}}$ appearing in (P_u), the feasible set $B_\delta(\bar{u}) \cap \mathbb{U}^{\mathrm{ad}}$ is now *convex*. Therefore the standard Karush-Kuhn-Tucker (KKT) theory in function space can be applied to ($\mathrm{P}_{\mathrm{aux}}$), see e.g. [55, Thm. 3.1], [11, Thm. 5.2] or [9, Thm. 3.9]. Hence, on account of the linearized Slater condition in (4.17), there exists a Lagrange multiplier $\bar{\mu} \in \mathcal{M}(\overline{Q})$ so that (4.14), (4.15), and

$$\langle \partial_u \mathfrak{L}(\bar{u}, \bar{\mu}), v - \bar{u} \rangle_{\mathbb{U}} \geq 0 \quad \forall v \in B_\delta(\bar{u}) \cap \mathbb{U}^{\mathrm{ad}} \tag{4.18}$$

are fulfilled. Now, let $u \in \mathbb{U}^{ad}$ be arbitrary. Then, due to convexity, $\bar{u} + \tau(u - \bar{u}) \in B_\delta(\bar{u}) \cap \mathbb{U}^{\mathrm{ad}}$ for $\tau > 0$ sufficiently small such that this function can be chosen as test function in (4.18), giving in turn (4.16). ☐

Let us now transform (4.14)-(4.16) into an optimality system involving an adjoint state. To this end, we aim to reformulate the derivative expression for $\partial_u \mathfrak{L}(\bar{u}, \mu)$ from Lemma 4.20 in a designated locally optimal point $\bar{u}$. For brevity, we define

$$\mathbb{X} = W^{1,s}(J; W_\emptyset^{-1,q}) \cap L^s(J; W^{1,q}) \quad \text{and} \quad X_s = (W^{1,q}, W_\emptyset^{-1,q})_{\frac{1}{s},s}.$$

The plan is to use the adjoint of the derivative of the control-to-state operator. We will show that $\mathcal{S}_{\mathcal{E}}'(\bar{u})^*$ is associated to the solution operator (in an appropriate sense) to the *adjoint system*, which we formally introduce as follows:

Definition 4.23. *For given, fixed functions θ and φ, given terminal value ϑ_T and inhomogeneities f_1, f_2, g_1, g_2, we call the following system the* adjoint system*:*

$$
\left.\begin{aligned}
-\partial_t\vartheta - \operatorname{div}(\eta(\theta)\kappa\nabla\vartheta) &= (\sigma'(\theta)\vartheta\varepsilon\nabla\varphi)\cdot\nabla\varphi - (\sigma'(\theta)\varepsilon\nabla\varphi)\cdot\nabla\psi \\
&\quad - (\eta'(\theta)\kappa\nabla\vartheta)\cdot\nabla\theta + f_1 && \text{in } Q,\\
\nu\cdot\eta(\theta)\kappa\nabla\vartheta + \alpha\vartheta &= f_2 && \text{on } \Sigma,\\
\vartheta(T_1) &= \vartheta_T && \text{in } \Omega,\\[1ex]
-\operatorname{div}(\sigma(\theta)\varepsilon\nabla\psi) &= -2\operatorname{div}(\sigma(\theta)\vartheta\varepsilon\nabla\varphi) + g_1 && \text{in } Q,\\
\nu\cdot\nabla\sigma(\theta)\varepsilon\nabla\psi &= 2\nu\cdot\sigma(\theta)\vartheta\varepsilon\nabla\varphi + g_2 && \text{on } \Sigma_N,\\
\psi &= 0 && \text{on } \Sigma_D.
\end{aligned}\right\}\tag{4.19}
$$

More specified assumptions about the inhomogeneities f_1, f_2, g_1, g_2 and the terminal value ϑ_T will be given in the following. Note that (4.19) is only a *formal* representation of the adjoint of the linearized system of (1.1)-(1.6). We will work with the abstract version, referring to (3.6) and (3.7) and its linearizations, cf. (4.3) or Remark 4.6.

Definition 4.24. *Let $\theta \in \mathbb{X}$ be fixed and set $\varphi = \mathcal{J}(\sigma(\theta))u$. Further, let $f \in \mathbb{X}'$, $\vartheta_T \in X_r'$, and $g \in L^{(2s)'}(J; W^{-1,q'}_{\Gamma_D})$ be given with $(2s)' = \frac{2s}{2s-1}$. The* abstract adjoint system *is given by*

$$
\begin{aligned}
-\partial_t\vartheta + \partial_\theta\mathcal{A}(\theta)\vartheta = -(\eta'(\theta)\kappa\nabla\vartheta)\cdot\nabla\theta + (\sigma'(\theta)\vartheta\varepsilon\nabla\varphi)\cdot\nabla\varphi + \delta_{T_1}\otimes\vartheta_T - \delta_{T_0}\otimes\chi + f\\
- (\sigma'(\theta)\varepsilon\nabla\varphi)\cdot\nabla\left[\mathcal{J}(\sigma(\theta))^*(-2\nabla\cdot\sigma(\theta)\vartheta\varepsilon\nabla\varphi + g)\right].
\end{aligned}\tag{4.20}
$$

Here, δ_{T_0} and δ_{T_1} are Dirac measures in T_0 and T_1, obtained as the adjoints of the point evaluations in T_0 and T_1, respectively. The latter are continuous mappings from $C(\overline{J}; X_s)$ to X_s, such that $\delta_{T_0}\otimes\vartheta_T$ and $\delta_{T_1}\otimes\chi$ are seen as objects from $\mathcal{M}(J; X_s')$. We say that the functions $(\vartheta,\chi) \in L^{s'}(J; W^{1,q'}) \times X_s'$ are a weak solution *of* (4.20) *or* (4.19), *if*

$$
\begin{aligned}
\int_J \langle\partial_t\xi, \vartheta\rangle_{W^{1,q'}}\,\mathrm{d}t = &-\int_J\int_\Omega \langle(\eta(\theta)\kappa\nabla\vartheta)\nabla\xi\,\mathrm{d}x\mathrm{d}t - \int_J\int_\Gamma \alpha\vartheta\xi\,\mathrm{d}\omega\mathrm{d}t\\
&-\int_J\int_\Omega \left[(\eta'(\theta)\kappa\nabla\vartheta)\cdot\nabla\theta - (\sigma'(\theta)\vartheta\varepsilon\nabla\varphi)\cdot\nabla\varphi\right]\xi\,\mathrm{d}x\mathrm{d}t\\
&-\int_J\int_\Omega (\sigma'(\theta)\varepsilon\nabla\varphi)\cdot\nabla\left[\mathcal{J}(\sigma(\theta))^*(-2\nabla\cdot\sigma(\theta)\vartheta\varepsilon\nabla\varphi + g)\right]\xi\,\mathrm{d}x\mathrm{d}t\\
&+\langle\vartheta_T, \xi(T_1)\rangle_{X_r} - \langle\chi, \xi(T_0)\rangle_{X_r} + \langle f, \xi\rangle_{\mathbb{X}}
\end{aligned}\tag{4.21}
$$

is true for all $\xi \in \mathbb{X}$. Equivalently, (4.20) *holds true in $\mathbb{X}'$.*

Note that the functionals $\delta_{T_0}\times\chi$ and $\delta_{T_1}\otimes\vartheta_T$ are well-defined in $\mathbb{X}'$ due to $\mathbb{X} \hookrightarrow C(\overline{J}; X_s)$. Of course, the inhomogeneities f_1, f_2 and g_1, g_2 from (4.19) are represented by $f = f_1 + f_2$ and $g = g_1 + g_2$, respectively. Moreover, thanks to the symmetry of ε, one easily sees that $\mathcal{J}(\sigma(\theta))^*$ is formally selfadjoint, which is the basis of the following

Remark 4.25. *Similarly to Remark 4.6, we introduce*

$$
\psi(\vartheta) := \mathcal{J}(\sigma(\theta))^*(-2\nabla\cdot\sigma(\theta)\vartheta\varepsilon\nabla\varphi + g),
$$

which allows to split (4.20) *back into two equations, namely*

$$
\begin{aligned}
-\partial_t\vartheta + \partial_\theta\mathcal{A}(\theta)\vartheta &= (\sigma'(\theta)\vartheta\varepsilon\nabla\varphi)\cdot\nabla\varphi - (\eta'(\theta)\kappa\nabla\vartheta)\cdot\nabla\theta - (\sigma'(\theta)\varepsilon\nabla\varphi)\cdot\nabla\psi\\
&\quad + \delta_{T_1}\otimes\vartheta_T - \delta_{T_0}\otimes\chi + f,\\
-\nabla\cdot\sigma(\theta)\varepsilon\nabla\psi &= -2\nabla\cdot\sigma(\theta)\vartheta\varepsilon\nabla\varphi + g,
\end{aligned}
$$

to be understood as in (4.21). *This is exactly a very weak abstract formulation of the formal adjoint system* (4.19) *with inhomogeneities $f = f_1 + f_2$ and $g = g_1 + g_2$ and terminal value ϑ_T. Note that the first equation is supposed to hold in $\mathbb{X}'$, the second one in $L^{(2s)'}(J; W^{-1,q'}_{\Gamma_D})$.*

We next show that the abstract adjoint (4.20) always admits a unique weak solution for $f \in \mathbb{X}'$ and $g \in L^{(2s)'}(J; W^{-1,q'}_{\Gamma_D})$. This will follow directly from Theorem 4.5 using an adjoint-approach (see e.g. [2, Ch. 7]). Since the inhomogeneity f in (4.20) will in fact contain the Lagrange multiplier μ introduced in Definition 4.21, we will not investigate the adjoint system more specifically under additional regularity assumptions on f, since the Lagrange multipliers are in general only measures and thus limit said regularity in a crucial way anyhow. In particular, this lack of regularity is the very obstacle which permits time-derivatives for weak solutions to (4.20), cf. [2, Prop. 6.1]. Nevertheless, even in the absence of measure-valued Lagrange multipliers, the time regularity of the adjoint state is still limited by the differential operator itself, since $(\eta'(\theta)\kappa\nabla\vartheta)\cdot\nabla\theta$ is only integrable in time (as opposed to s'-integrable).

Theorem 4.26. *For every terminal value $\vartheta_T \in X_s' = (W^{1,q'}, W_\emptyset^{-1,q'})_{\frac{1}{s'},s'}$ and all imhomogeneities $f \in \mathbb{X}'$ and $g \in L^{(2s)'}(J; W_{\Gamma_D}^{-1,q'})$, there exists a unique weak solution $(\vartheta,\chi) \in L^{s'}(J;W^{1,q'}) \times X_s'$ of (4.20) in the sense of Definition 4.24.*

Proof. The equality $X_s' = (W^{1,q'}, W_\emptyset^{-1,q'})_{\frac{1}{s'},s'}$ follows from the usual duality properties of interpolation functors, see [54, Ch. 1.11.2 and 1.3.3]. Recall the operator

$$\mathcal{B}\colon \mathbb{X} \times L^{2s}(J; W_{\Gamma_D}^{-1,q}) \to L^s(J; W_\emptyset^{-1,q}) \times (W^{1,q}, W_\emptyset^{-1,q})_{\frac{1}{s},s},$$

from Theorem 4.5 with $r = s > \bar r(q,\varsigma) \geq r^*(q)$. The partial derivative w.r.t. θ of $\mathcal{B}$ was given by

$$\partial_\theta\mathcal{B}(\theta,u)\xi = (\partial_t\xi + \mathcal{A}(\theta)\xi - \nabla\cdot\eta'(\theta)\xi\kappa\nabla\theta - \partial_\theta\Psi_u(\theta)\xi, \xi(T_0))$$

with

$$\partial_\theta\Psi_u(\theta)\xi = -2(\sigma(\theta)\varepsilon\nabla\varphi)\cdot\nabla\left[\mathcal{J}(\sigma(\theta)(-\nabla\cdot\sigma'(\theta)\xi\varepsilon\nabla\varphi)\right] + (\sigma'(\theta)\xi\varepsilon\nabla\varphi)\cdot\nabla\varphi,$$

cf. (4.6), and $\varphi = \mathcal{J}(\sigma(\theta))u$. Now, let (ϑ,χ) be from $L^{s'}(J;W^{1,q'}) \times X_s'$. We easily find

$$\langle -\nabla\cdot\eta'(\theta)\xi\kappa\nabla\theta, \vartheta\rangle_{W^{1,q'}} = \int_\Omega (\eta'(\theta)\xi\kappa\nabla\vartheta)\cdot\nabla\theta \,\mathrm{d}x = \langle(\eta'(\theta)\kappa\nabla\vartheta)\cdot\nabla\theta, \xi\rangle_{W^{1,q}} \tag{4.22}$$

and

$$\langle(\sigma'(\theta)\xi\varepsilon\nabla\varphi)\cdot\nabla\varphi, \vartheta\rangle_{W^{1,q'}} = \langle(\sigma'(\theta)\vartheta\varepsilon\nabla\varphi)\cdot\nabla\varphi, \xi\rangle_{W^{1,q}}\,. \tag{4.23}$$

Let us turn to the complicated term in $\partial_\theta\Psi_u(\theta)$. Analogously to (4.22), we find

$$\begin{aligned}
&\langle 2(\sigma(\theta)\varepsilon\nabla\varphi)\cdot\nabla\left[\mathcal{J}(\sigma(\theta)(-\nabla\cdot\sigma'(\theta)\xi\varepsilon\nabla\varphi)\right], \vartheta\rangle_{W^{1,q'}} \\
&\qquad = \langle \mathcal{J}(\sigma(\theta))(-\nabla\cdot\sigma'(\theta)\xi\varepsilon\nabla\varphi), -2\nabla\cdot\sigma(\theta)\vartheta\varepsilon\nabla\varphi\rangle_{W_{\Gamma_D}^{-1,q'}} \\
&\qquad = \langle -\nabla\cdot\sigma'(\theta)\xi\varepsilon\nabla\varphi, \mathcal{J}(\sigma(\theta))^*\left(-2\nabla\cdot\sigma(\theta)\vartheta\varepsilon\nabla\varphi\right)\rangle_{W_{\Gamma_D}^{1,q'}} \\
&\qquad\qquad = \langle \xi, (\sigma'(\theta)\varepsilon\nabla\varphi)\nabla\left[\mathcal{J}(\sigma(\theta))^*\left(-2\nabla\cdot\sigma(\theta)\vartheta\varepsilon\nabla\varphi\right)\right]\rangle_{W_\emptyset^{-1,q'}}\,.
\end{aligned} \tag{4.24}$$

Symmetry of κ implies that $\mathcal{A}(\theta)$ is formally self-adjoint, i.e., $\mathcal{A}(\theta)^*$ maps $W^{1,q'}$ into $W_\emptyset^{-1,q'}$, but is still given as in Definitions 3.9 and 3.1, respectively. Using this and equations (4.22), (4.23) and (4.24), we obtain

$$\begin{aligned}
\langle\partial_\theta\mathcal{B}(\theta,u)^*(\vartheta,\chi),\xi\rangle_{\mathbb{X}} &= \langle(\vartheta,\chi),\partial_\theta\mathcal{B}(\theta,u)\xi\rangle_{L^s(J;W_\emptyset^{-1,q})\times X_s} \\
&= \int_J \langle\partial_t\xi,\vartheta\rangle_{W^{1,q'}}\,\mathrm{d}t + \int_J \langle\mathcal{A}^*(\theta)\vartheta,\xi\rangle_{W^{1,q}}\,\mathrm{d}t \\
&\quad + \int_J \langle(\eta'(\theta)\kappa\nabla\vartheta)\cdot\nabla\theta,\xi\rangle_{W^{1,q}}\,\mathrm{d}t \\
&\quad - \int_J \langle(\sigma'(\theta)\vartheta\varepsilon\nabla\varphi)\cdot\nabla\varphi,\xi\rangle_{W^{1,q}}\,\mathrm{d}t + \langle\chi,\xi(T_0)\rangle_{X_s} \\
&\quad + \int_J \langle(\sigma'(\theta)\varepsilon\nabla\varphi)\nabla\left[\mathcal{J}(\sigma(\theta))^*(-2\nabla\cdot\sigma(\theta)\vartheta\varepsilon\nabla\varphi)\right],\xi\rangle_{W^{1,q}}\,\mathrm{d}t
\end{aligned}$$

for all $\xi \in \mathbb{X}$. Moreover, in the proof of Theorem 4.5, $\partial_\theta\mathcal{B}(\theta,u)$ was found to be a topological isomorphism between the spaces $\mathbb{X}$ and $L^s(J;W_\emptyset^{-1,q}) \times X_s$ and consequently $\partial_\theta\mathcal{B}(\theta,u)^*$ is also a topological isomorphism between $L^{s'}(J;W^{1,q'}) \times X_s'$ and $\mathbb{X}'$. In particular, for every $\mathfrak{f} \in \mathbb{X}'$ there exists a unique $p = p_{\mathfrak{f}} \in L^{s'}(J;W^{1,q'}) \times X_s'$ such that $\partial_\theta \mathcal{B}(\theta,u)^* p = \mathfrak{f}$. Hence, setting

$$\bar{\mathfrak{f}} = f + \delta_{T_1} \otimes \vartheta_T - (\sigma'(\theta)\varepsilon\nabla\varphi)\,\nabla\left[\mathcal{J}(\sigma(\theta))^* g\right], \tag{4.25}$$

the pair $(\bar\vartheta, \bar\chi) := p_{\bar{\mathfrak{f}}}$ satisfies (4.21) by the above form of $\partial_\theta\mathcal{B}(\theta,u)^*$, and is exactly the searched-for unique solution as in Definition 4.24. □

As hinted above, we immediately obtain the following characterization of $\mathcal{S}'(u)^*$ for given $u \in \mathbb{U}_g$:

COROLLARY 4.27. *Let (ϑ,χ) be the solution of (4.21) in the sense of Definition 4.24 with inhomogeneites f and g and terminal value ϑ_T. The adjoint linearized solution operator $\mathcal{S}_\mathcal{E}'(u)^*$ then assigns to f, g and ϑ_T in the form $\mathfrak{f} \in \mathbb{X}'$ as in (4.25) the functional $\mathcal{E}^*\psi \in \mathbb{U}'$, where $\psi(\vartheta) \in L^{(2s)'}(J;W^{1,q'}_{\Gamma_D})$ is given by*

$$\psi(\vartheta) = \mathcal{J}(\sigma(\theta_u))^*(-\nabla\cdot\sigma(\theta_u)\vartheta\varepsilon\nabla\varphi_u),$$

similarly to Remark 4.25.

Proof. In Theorem 4.5, we found $\mathcal{S}'(u) = -[\partial_\theta\mathcal{B}(\mathcal{S}(u),u)]^{-1}\partial_u\mathcal{B}(\mathcal{S}'(u),u)$. Hence, with $\mathcal{S}_\mathcal{E}'(u) = \mathcal{S}'(u)\circ\mathcal{E}$, we obtain

$$\mathcal{S}_\mathcal{E}'(u)^*\mathfrak{f} = -\mathcal{E}^*\partial_u\mathcal{B}(\mathcal{S}_\mathcal{E}(u),u)^*\partial_\theta\mathcal{B}(\mathcal{S}_\mathcal{E}(u),u)^{-*}\mathfrak{f}.$$

In view of Theorem 4.26 and its proof, $\partial_\theta\mathcal{B}(\mathcal{S}_\mathcal{E}(u),u)^{-*}\mathfrak{f}$ is exactly the unique solution (ϑ,χ) of (4.21) in the sense of Definition 4.24 with inhomogeneites f, g and terminal value ϑ_T. Moreover, a repetition of the first lines of (4.24) shows that

$$-\partial_u\mathcal{B}(\mathcal{S}_\mathcal{E}(u),u)^*(\vartheta,\chi) = \mathcal{J}(\sigma(\theta_u))^*(-\nabla\cdot\sigma(\theta_u)\vartheta\varepsilon\nabla\varphi_u) = \psi(\vartheta).$$

An application of $\mathcal{E}^* : L^{(2s)'}(J;W^{1,q'}_{\Gamma_D}) \hookrightarrow \mathbb{U}'$ yields the claim. □

Having $\mathcal{S}_\mathcal{E}'(u)^*$ at hand, we now proceed to establish the actual necessary optimality conditions by manipulating the variational inequality in the KKT conditions (4.16).

For a concise "strong" formulation in the following theorem, we decompose measures $\mu \in \mathcal{M}(\overline{Q})$ by restriction into $\mu = \mu_{(T_0,T_1)} + \mu_{\{T_0\}\times\{T_1\}}$, with $\mu_{(T_0,T_1)} \in \mathcal{M}((T_0,T_1)\times\overline{\Omega})$ and $\mu_{\{T_0\}\times\{T_1\}} \in \mathcal{M}((\{T_0\}\times\{T_1\})\times\overline{\Omega})$. Both measures may in turn be further decomposed into $\mu_{(T_0,T_1)} = \mu_\Omega + \mu_\Gamma$, where $\mu_\Omega \in \mathcal{M}((T_0,T_1)\times\Omega)$ and $\mu_\Gamma \in \mathcal{M}((T_0,T_1)\times\Gamma)$, and $\mu_{\{T_0\}\times\{T_1\}} = \delta_{T_0}\otimes\mu_{T_0} + \delta_{T_1}\otimes\mu_T$ with $\mu_{T_0},\mu_T \in \mathcal{M}(\overline{\Omega})$.

THEOREM 4.28 (First Order Necessary Conditions). *Let $\bar u \in \mathbb{U}_g^{ad}$ be a locally optimal control such that the linearized Slater condition (4.17) is satisfied. Let $\theta_{\bar u} = \mathcal{S}_\mathcal{E}(\bar u)$ be the state associated with $\bar u$ and set $\varphi_{\bar u} := \varphi_{\bar u}(\theta_{\bar u})$. Then there exists a Lagrangian multiplier $\bar\mu \in \mathcal{M}(\overline{Q})$ in the sense of Definition 4.21 and adjoint states $\vartheta \in L^{s'}(J;W^{1,q'})$ and $\psi \in L^{(2s)'}(J;W^{1,q'}_{\Gamma_D})$, such that the formal system*

$$\begin{aligned}
-\partial_t\vartheta - \operatorname{div}(\eta(\theta_{\bar u})\kappa\nabla\vartheta) &= (\sigma'(\theta_{\bar u})\vartheta\varepsilon\nabla\varphi_{\bar u})\cdot\nabla\varphi_{\bar u} - (\sigma'(\theta_{\bar u})\varepsilon\nabla\varphi_{\bar u})\cdot\nabla\psi \\
&\quad - (\eta'(\theta)\kappa\nabla\vartheta)\cdot\nabla\theta + \|\nabla\theta_{\bar u}\|^{s-q}_{L^s(J;L^q)}\Delta_q\theta_{\bar u} + \bar\mu_\Omega && \text{in } Q,\\
\nu\cdot\eta(\theta_{\bar u})\kappa\nabla\vartheta + \alpha\vartheta &= \bar\mu_\Gamma && \text{on } \Sigma,\\
\vartheta(T_1) &= \chi_E(\theta_{\bar u}(T_1) - \theta_d) + \bar\mu_{T_1} && \text{in } \Omega,\\[1ex]
-\operatorname{div}(\sigma(\theta_{\bar u})\varepsilon\nabla\psi) &= -2\operatorname{div}(\sigma(\theta_{\bar u})\vartheta\varepsilon\nabla\varphi_{\bar u}) && \text{in } Q,\\
\nu\cdot\sigma(\theta_{\bar u})\varepsilon\nabla\psi &= 2\nu\cdot\sigma(\theta_{\bar u})\vartheta\varepsilon\nabla\varphi_{\bar u} && \text{on } \Sigma_N,\\
\psi &= 0 && \text{on } \Sigma_D.
\end{aligned}$$

is satisfied in the sense of Definition 4.24 and Remark 4.25. Moreover, $\bar u$ is the solution of the variational inequality

$$\begin{aligned}
\int_{\Sigma_N} \partial_t\bar u\,\partial_t(u-\bar u) + \frac{p}{2}|\bar u|^{p-2}(u-\bar u) + \frac{1}{\beta}(\tau_{\Gamma_N}\psi)(u-\bar u)\,\mathrm{d}\omega\,\mathrm{d}t \geq 0 \\
\text{for all } u \in \mathbb{U}^{ad} = \{u \in \mathbb{U}\colon 0 \leq u \leq u_{\max} \text{ a.e. in } \Sigma_N\}.
\end{aligned} \tag{4.26}$$

Note that the Lagrange multiplier $\bar{\mu}$ is not active on the set $\{T_0\} \times \overline{\Omega}$ due to Assumption 2.9 (iv) and the positivity and complementary conditions (4.14) and (4.15). Hence, $\bar{\mu}_{T_0}$ is zero and does not contribute to the system of equations in Theorem 4.28. Note moreover that the variational inequality in (4.26) is just a (semilinear) variational inequality of obstacle-type in time.

Proof. Let $\bar{u}$ be a locally optimal control such that the linearized Slater condition (4.17) is satisfied. Theorem 4.22 then yields the existence of a Lagrangian multiplier $\bar{\mu} \in \mathcal{M}(\overline{Q})$ such that (4.14)-(4.16) hold true. We show that these lead to the assertions.

First consider the linear continuous functional

$$\langle \chi_E(\theta_{\bar{u}}(T_1) - \theta_d), \Theta \rangle_{L^2(\Omega)} := \int_E (\theta_{\bar{u}}(T_1) - \theta_d)\Theta \,\mathrm{d}x.$$

Due to the choice of s, we have $X_s \hookrightarrow C(\overline{\Omega}) \hookrightarrow L^2(\Omega)$, such that the functional is also an element of X_s' and $\delta_{T_1} \otimes \chi_E(\theta_{\bar{u}}(T_1) - \theta_d) \in \mathbb{X}'$. Moreover, we set $\|\nabla \theta_{\bar{u}}\|_{L^q}^{s-q} \Delta_q \theta_{\bar{u}}$ as a functional on $\mathbb{X} \hookrightarrow L^s(J; W^{1,q})$ via

$$\left\langle \|\nabla \theta_{\bar{u}}\|_{L^q}^{s-q} \Delta_q \theta_{\bar{u}}, \xi \right\rangle_{\mathbb{X}} := \int_J \|\nabla \theta_{\bar{u}}(t)\|_{L^q}^{s-q} \langle \Delta_q \theta_{\bar{u}}(t), \xi(t) \rangle_{W^{1,q}} \,\mathrm{d}t.$$

The inclusion $\mathbb{X} \hookrightarrow C(\overline{Q})$ also implies $\bar{\mu} \in \mathcal{M}(\overline{Q}) \hookrightarrow \mathbb{X}'$. Hence, inserting $\theta_{\bar{u}}' = \mathcal{S}_{\mathcal{E}}'(u)h$ in (4.13), we immediately obtain

$$\begin{aligned}\partial_{\bar{u}} \mathfrak{L}(u, \mu)h = \langle \mathcal{S}_{\mathcal{E}}'(u)^* \left[\delta_{T_1}^* \chi_E(\theta_{\bar{u}}(T_1) - \theta_d) + \gamma \|\nabla \theta_{\bar{u}}\|_{L^q}^{s-q} \Delta_q \theta_{\bar{u}} + \mu\right], h \rangle_{\mathbb{U}} \\ + \beta \int_{\Sigma_N} \partial_t u \partial_t h + \frac{p}{2} |u|^{p-2} u h \,\mathrm{d}\omega \,\mathrm{d}t\end{aligned}$$

for $h \in \mathbb{U}$. Let us introduce (ϑ, χ) as the unique solution of (4.20) (cf. Theorem 4.26) with data $\vartheta_T = \chi_E(\theta_{\bar{u}}(T_1) - \theta_d) + \bar{\mu}_{T_1}$, $g = 0$ and $f = \gamma \|\nabla \theta_{\bar{u}}\|_{L^q}^{s-q} \Delta_q \theta_{\bar{u}} + \bar{\mu}_{(T_0, T_1)}$, which is then also the solution of the formal system (4.19) with the stated inhomogeneities f and g and terminal value ϑ_T. Here, ψ is obtained by $\psi(\vartheta) = \mathcal{J}(\sigma(\theta_{\bar{u}}))^*(-\nabla \cdot \sigma(\theta_{\bar{u}}) \vartheta \varepsilon \nabla \varphi_{\bar{u}})$, cf. Remark 4.25. Corollary 4.27 now shows that

$$\partial_{\bar{u}} \mathfrak{L}(\bar{u}, \bar{\mu})h = \langle \mathcal{E}^* \psi, h \rangle_{\mathbb{U}} + \beta \int_{\Sigma_N} \partial_t \bar{u} \partial_t h + \frac{p}{2} |\bar{u}|^{p-2} \bar{u} h \,\mathrm{d}\omega \,\mathrm{d}t \tag{4.27}$$

for $h \in \mathbb{U}$. It is convenient to write $\mathcal{E}$ as $\mathcal{E} = \tau_{\Gamma_N}^* \circ \mathfrak{E}$ with $\mathfrak{E} \colon \mathbb{U} \hookrightarrow L^{2s}(J; L^{\mathfrak{p}}(\Gamma_N))$ and $\tau_{\Gamma_N}^* \colon L^{2s}(J; L^{\mathfrak{p}}(\Gamma_N)) \to L^{2s}(J; W_{\Gamma_D}^{-1,q})$ with $\mathfrak{p} > \frac{2}{3}q$, see Proposition 4.14 and Remark 2.8. Then we have

$$\langle \mathcal{E}^* \psi, h \rangle_{\mathbb{U}} = \langle \tau_{\Gamma_N} \psi, \mathfrak{E} h \rangle_{L^{(2s)'}(J; L^{\mathfrak{p}'}(\Gamma_N)), L^{2s}(J; L^{\mathfrak{p}}(\Gamma_N))} = \int_{\Sigma_N} (\tau_{\Gamma_N} \psi) h \,\mathrm{d}\omega \,\mathrm{d}t, \tag{4.28}$$

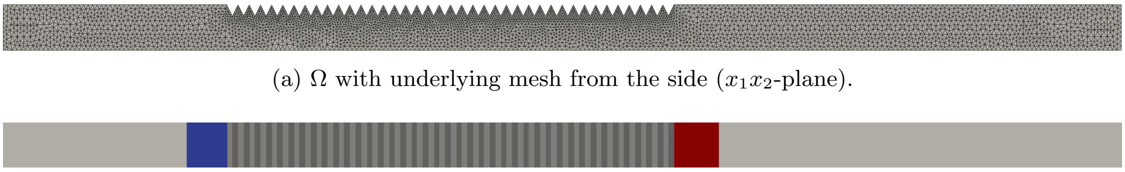
again $h \in \mathbb{U}$. Inserting (4.28) and (4.27) into (4.16), we obtain the stated variational inequality. ∎

Remark 4.29. *If the optimal control $\bar{u}$ in the previous theorem is an* interior *point of $\mathbb{U}^{ad}$, or if $\mathbb{U}^{ad}$ is not present at all, then one may transform the variational inequality (4.26) to the ordinary nonlinear differential equation of order two*

$$\partial_{tt} \bar{u} = \frac{1}{\beta} \tau_{\Gamma_N} \psi + \frac{p}{2} |\bar{u}|^{p-2} \bar{u}$$

in the space $L^{p'}(\Gamma_N)$ as a boundary value problem with $\partial_t \bar{u}(T_0) = \partial_t \bar{u}(T_1) = 0$. In particular, $\partial_{tt} \bar{u} \in L^{(2s)'}(J; L^{p'}(\Gamma_N))$ in this case.

5. Application and numerical example. As already outlined in [39] and the introduction, a typical example of an application for a problem in the form (P) is the optimal heating of a conducting material such as steel by means of an electric current. The aim of such procedures is to heat up a workpiece by electric current and to cool it down rapidly with water nozzles in order to harden it. In case of steel, this treatment indeed produces a hard martensitic outer layer, see for instance [10, Ch. 9.18] for a phase diagram and [10, Chapters 10.5/10.7 about Martensite], and is thus used for instance for rack-and-pinion actuators, to be found e.g. in steering mechanisms. The part of the workpiece to be heated up corresponds to the design area E in the objective functional in (P). In order to avoid thermal stresses in the material, it is crucial to produce a homogeneous temperature distribution in the design area, which is reflected by the first term of the objective functional. The gradient term in the objective functional further enforces minimal thermal stresses. Moreover, the temperatures necesssary for the hardening process as described above are rather close to the melting point of the material, thus the state constraints are used to prevent the temperature exceeding the melting temperature $\theta_{\max}$. The control constraints in (P) represent a maximum electrical current which can be induced in the workpiece.



(a) Ω with underlying mesh from the side (x_1x_2-plane).

(b) Ω from above (x_1x_3-plane) with Γ_N (left) and Γ_D (right) emphasized.

Fig. 5.1: The computational domain Ω used in the numerical example.

In the following we exhibit numerical examples for the optimal control of the three-dimensional thermistor problem in the form (P), underlining in particular the importance of the state-constraints. The considered computational domain Ω is a (simplified) three-dimensional gear-rack as seen in Figure 5.1, where the design area E consists of the sawteeth. The mesh consists of about 80000 nodes, inducing 400000 cells with cell diameteres ranging from $8.8 \cdot 10^{-4}$ to $7.6 \cdot 10^{-3}$.

The heat-equation we use in the computations is as follows:

$$\varrho C_p \partial_t \theta - \operatorname{div}(\eta(\theta)\kappa\nabla\theta) = (\sigma(\theta)\nabla\varphi)\cdot\nabla\varphi.$$

It deviates from (1.1) by the factor ϱC_p, the so-called the volumetric heat capacity, where ϱ is the density of the material and C_p is its specific heat capacity. However, since we assume ϱC_p to be constant, it certainly has no influence on the theory presented above. In [33, Remarks 6.13/15] and [37] it is laid out how to modify the analysis if one wants to incorporate a volumetric heat capacity depending on the temperature θ. For a realistic modeling of the process, we use the data gathered in [16], i.e., the workpiece Ω is supposedly made of non-ferromagnetic stainless steel (#1.4301). The constants used can be found in Table 5.1 and the conductivity functions are given by

$$\sigma(\theta) := \frac{1}{a_\sigma + b_\sigma\theta + c_\sigma\theta^2 + d_\sigma\theta^3} \quad \text{for} \quad \theta \in [0, 10000]\ \mathrm{K},$$

with the constants $a_\sigma = 4.9659 \cdot 10^{-7}$, $b_\sigma = 8.4121 \cdot 10^{-10}$, $c_\sigma = -3.7246 \cdot 10^{-13}$ and $d_\sigma = 6.1960 \cdot 10^{-17}$ for the electrical conductivity (in $\Omega^{-1}\mathrm{m}^{-1}$), and

$$\eta(\theta) := 100(a_\eta + b_\eta\theta) \quad \text{for} \quad \theta \in [0, 10000]\ \mathrm{K}$$

with $a_\eta = 0.11215$ and $b_\eta = 1.4087 \cdot 10^{-4}$ for the thermal conductivity (in $\mathrm{Wm}^{-1}\mathrm{K}^{-1}$). Both functions are extended outside of $[0, 10000]$ in a smooth and bounded way, such that Assumptions 2.7 and 4.2 are satisfied. Note that ε and κ are each chosen as the identity matrix, as we do not account for heterogeneous materials in this numerical example. To counter-act on the different scales inherent in the problem, cf. the value for $u_{\max}$ and θ_0 in Table 5.1, the model was nondimensionalized for the implementation.

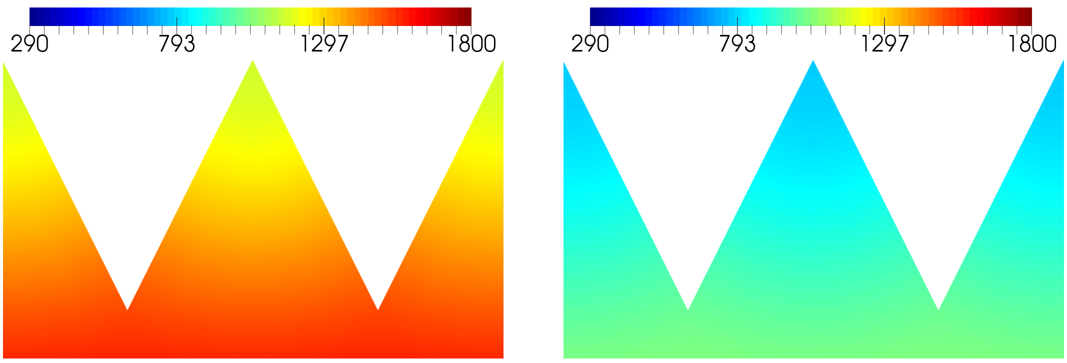
The optimization problem (P) is solved by means of a Nonlinear Conjugate-Gradients Method in the form as described in [17], modified to a projected method to account for the admissible set $\mathcal{U}_{\mathrm{ad}}$. The method needed up to 150 iterations to meet the stopping criterion, which required the relative change in the objective functional to be smaller than 10^{-5}. The state constraints in (P) are incorporated by a quadratic penalty approach—so-called Moreau-Yosida regularization—, cf. [38] and the references therein, where the penalty-parameter was increased up to a maximum of 10^{10}, stopping earlier if the violation of the state constraints was smaller than 10^{-2} K. This resulted in a violation of $9.54 \cdot 10^{-2}$ K,

ϱ	C_p	α	θ_0	θ_l	θ_d	$\theta_{\max}$	$u_{\max}$
7900 $\frac{\mathrm{kg}}{\mathrm{m}^3}$	455 $\frac{\mathrm{J}}{\mathrm{kg\,K}}$	20 $\frac{\mathrm{W}}{\mathrm{m}^3\,\mathrm{K}}$	290 K	290 K	1500 K	1700 K	$10 \cdot 10^7 \frac{\mathrm{A}}{\mathrm{m}^2}$

Table 5.1: Material parameters used in the numerical tests

which is about 0.0056% of the upper bound of 1700 K. In each step of the optimization algorithm, the nonlinear state equations (1.1)-(1.6) and the adjoint equations (4.19) have to be solved. We use an Implicit Euler Scheme for the time-discretization of these equations, whereas the spatial discretization is done via piecewise continuous linear finite elements. The nonlinear system of equations arising in each time-step is solved via Newton's method. Here, we do a semi-implicit pre-step to obtain a suitable initial guess for the discrete φ for Newton's method. For the control, we also choose piecewise continuous linear functions in space where the values in the first and last timestep were pre-set to 0. In the calculation of the gradient of the reduced objective functional j, the gradient representation with respect to the $L^2(J; L^2(\Gamma_N))$ scalar product of the derivative of $u \mapsto \frac{1}{2}(\partial_t u)^2$ is needed, which one formally computes as $\partial_{tt}^2 u$. We used the second order central difference quotient $\frac{u_{k+1}-2u_k-u_{k-1}}{\Delta t^2}$ to approximate $(\partial_{tt}^2 u)(t_k)$ at time step k with the appropriate modifications for the first and last time step, respectively. All computations were performed within the FEnICS framework [22].

For the experiment duration, we set $T_1 - T_0 = 2.0$ s with timesteps $\Delta t = 0.02$ s and $T_0 = 0.0$ s, while we use $\gamma = 10^{-8}$ and $\beta = 10^{-5}$ – this small value for β is only possible due to the nondimensionalization performed. In the following, we elaborate on two settings: one in which we enforce the state constraint $\theta \le \theta_{\max}$ and one in which we do not.



(a) Free optimization.

(b) State constrained optimization.

Fig. 5.2: Detail of the sawteeth in E at end time $t = 2.0$ s with distribution of the temperature θ in K.

Figure 5.2 shows the temperature distribution at end time $T_1 = 2.0$ s in E in both cases. The desired temperature distribution close to uniformly 1500 K has been nearly achieved in the free optimization, see Figure 5.2a, at the price of very high temperature values around Γ_D and Γ_N already early in the heating process. We come back to this below, cf. also Figure 5.6. For the state-constrained optimization, we achieve a much worse result (note the same scales in both Figure 5.2a and 5.2b), which again corresponds to the rapid evolution to high temperatures at the critical areas, since these crucially limit the maximal amount of energy induced into the workpiece if one wants to prevent the temperature rising higher than the given bounds $\theta_{\max}$. This can also be seen in the development of the optimal controls in both cases over time, see below.

The potential φ and its gradient $\nabla\varphi$ associated with the optimal control to the state-constrained optimization problem, at time $t = 1.0$ s are depicted in Figures 5.3 and 5.4. Here, $\nabla\varphi$ is to be understood as the projection of the potentially discontinuous gradient of φ to the space of continuous linear finite elements. The potential φ decreases from Γ_N to the grounding with prescribed value $\varphi \equiv 0$ at Γ_D, cf.

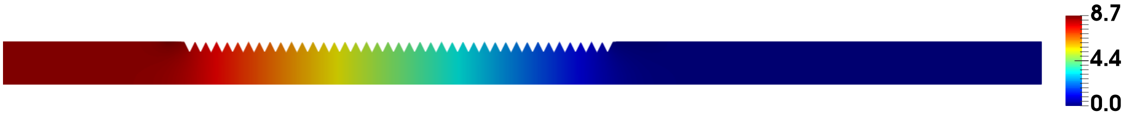


Fig. 5.3: The potential φ (in V) associated with the optimal solution at time $t = 1.0$ s, view from the side (x_1x_2-plane).

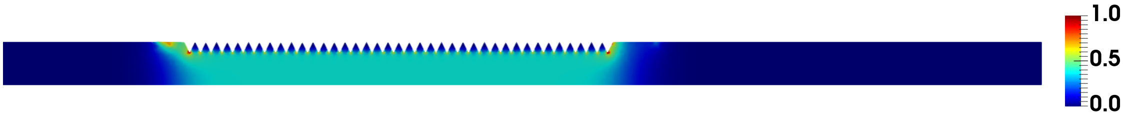
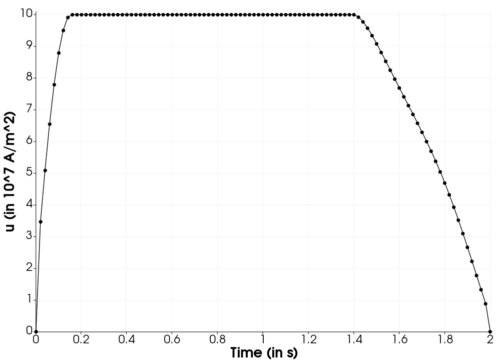
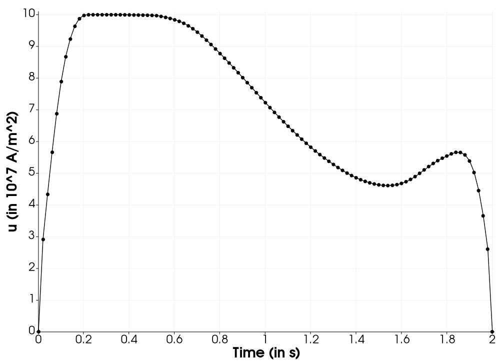


Fig. 5.4: Magnitude of the gradient $\nabla\varphi$ (in V/m) associated with the optimal solution at time $t = 1.0$ s, view from the side (x_1x_2-plane).

Figure 5.1b, thus inducing a current flow and acting as a heat source between Γ_D and Γ_N, since the corresponding term in the heat equation $\sigma(\theta)\varepsilon\nabla\varphi \cdot \nabla\varphi$ is proportional to $|\nabla\varphi|^2$ due to the coercivity and boundedness of ε and the bounds on σ. This is confirmed by the magnitude of $\nabla\varphi$ as seen in Figure 5.4. In particular one observes that $\nabla\varphi$ is very small or 0 in E, which means that the current flows only through the area between Γ_D and Γ_N and right *below* E, heating only this part of the workpiece.



(a) For the unconstrained problem.

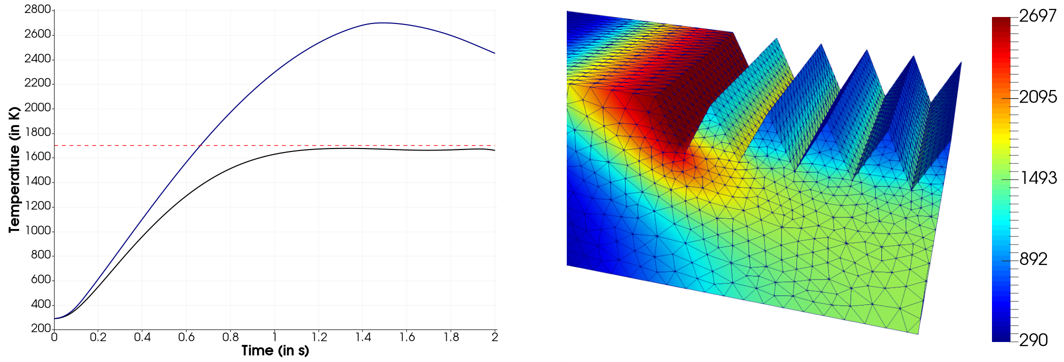


(b) For the state-constrained problem.

Fig. 5.5: Time plot of the optimal controls, taken at an arbitrary but fixed grid point in Γ_N.

The optimal controls are shown in Figure 5.5, taken at an arbitrary but fixed grid point in Γ_N. The high values in the control at the beginning of the process seem to be the result of the inability to heat up the tooth rack in the design-area E directly as explained above, which makes heating of the teeth reliant on diffusion. This in turn requires the needed total energy to be inserted into the system as fast as possible, resulting in high control values, which also agrees with the requirement to obtain a *uniform* temperature distribution in the tooth rack. These considerations also underline the necessity of control bounds in this example. In decreasing the control values after the inital period, the opimization procedure in the free optimization is avoiding to “over-shoot”, i.e., to produce a higher temperature than desired. In the case of state-constrained optimization, the presence of the state constraints forces an earlier decrease in control values in order to not violate the upper bound $\theta_{\max}$, which is then compensated by a slightly higher level of values towards the end of the simulation. This, however, is clearly not enough to make up for the earlier decrease as seen in Figure 5.2.

Figure 5.6 illustrates why state constraints are a necessary addition to an appropriate model of the industrial steel heating process. Figure 5.6a shows the temperature evolution in a point in one of the two critical regions, which are the points near Γ_D and Γ_N, see also Figure 5.6b and the magnitude of $\nabla\varphi$ at



(a) Time plot of the temperature in a point close to Γ_N.

(b) Temperature θ in K at the critical area near Γ_N at time $t = 1.4$ s.

Fig. 5.6: Influence and necessity of state constraints.

this region in Figure 5.4. In this case, the point lies in E close to Γ_N, but we emphasize that the state constraints hold in the whole Ω and are not limited to E. The upper line in Figure 5.6a corresponds to the temperature associated to the optimal solution of the unconstrained optimization, while the lower belongs to the state-constrained optimal solution, with the upper bound $\theta_{\max} = 1700$ K marked by the dashed line. In the free optimization case, the temperature exceeds the bounds already at about one third of the simulation time and continuous to rise to almost 1000 K above $\theta_{\max}$. On the other hand, the temperature obtained from the state-constrained case stays below the threshold, as required. Note here that the evaluated point is chosen as one of those where the temperature rises highest overall, compare the temperature distribution as seen in Figure 5.6b and the maximal temperature achieved in the free optimization case in Figure 5.6a.

Concluding from the results presented above, it becomes apparent that the prescribed time of 2.0 s is too short to heat up the workpiece in the given geometry enough to reach the required temperature for Austenite to form in the workpiece (cf. [10, Ch. 9.18]) in E, if melting is to be prevented.

Appendix A. A “minimum principle”.

Proposition A.1. *For every solution (θ, φ) in the sense of Theorem 3.13 with maximal existence interval $J_{\max}$, it is true that $\theta(x,t) \geq \min(\operatorname{ess\,inf}_\Sigma \theta_l, \min_{\overline{\Omega}} \theta_0)$ for all $(x,t) \in \overline{\Omega} \times [T_0, T_\bullet]$, where $T_\bullet \in J_{\max}$.*

Proof. We set $m_{\inf} := \min(\operatorname{ess\,inf}_\Sigma \theta_l, \min_{\overline{\Omega}} \theta_0)$ and $\zeta(t) = \theta(t) - m_{\inf}$ and decompose $\zeta(t)$ into its positive and negative part, that is, $\zeta(t) = \zeta^+(t) - \zeta^-(t)$ with both $\zeta^+(t)$ and $\zeta^-(t)$ being positive functions. By [18, Ch. IV, §7, Prop. 6/Rem. 12] we then have that $\zeta^-(t)$ is still an element of $W^{1,q}$ for almost every $t \in (T_0, T_\bullet)$. In particular, we may test (3.6) against $-\zeta^-(t)$, insert $\theta = \zeta + m_{\inf}$ and use that $m_{\inf}$ is constant:

$$-\int_\Omega \partial_t \zeta(t)\zeta^-(t)\,\mathrm{d}x - \int_\Omega (\eta(\theta(t))\kappa\nabla\zeta(t)) \cdot \nabla\zeta^-(t)\,\mathrm{d}x - \int_\Gamma \alpha\zeta(t)\zeta^-(t)\,\mathrm{d}x$$
$$= -\int_\Gamma \alpha(\theta_l(t) - m_{\inf})\zeta^-(t) - \int_\Omega \zeta^-(t)(\sigma(\theta(t))\varepsilon\nabla\varphi(t)) \cdot \nabla\varphi(t)\,\mathrm{d}x.$$

Observe that the support of products of $\zeta(t)$ and $\zeta^-(t)$ is exactly the support of $\zeta^-(t)$, and $\zeta(t) = -\zeta^-(t)$ there. We thus obtain

$$\frac{1}{2}\,\partial_t\|\zeta^-(t)\|_{L^2}^2 + \int_\Omega (\eta(\theta(t))\kappa\nabla\zeta^-(t)) \cdot \nabla\zeta^-(t)\,\mathrm{d}x + \int_\Gamma \alpha\zeta^-(t)^2\,\mathrm{d}x$$
$$= -\int_\Gamma \alpha(\theta_l(t) - m_{\inf})\zeta^-(t) - \int_\Omega \zeta^-(t)(\sigma(\theta(t))\varepsilon\nabla\varphi(t)) \cdot \nabla\varphi(t)\,\mathrm{d}x. \quad \text{(A.1)}$$

Let us show that $\partial_t \|\zeta^-(t)\|_{L^2}^2 \le 0$. By Assumption 2.7, $(\eta(\theta(t))\kappa\nabla\zeta^-(t)) \cdot \nabla\zeta^-(t) \ge \underline{\eta}\underline{\kappa}\|\nabla\zeta^-(t)\|_{\mathbb{R}^3}^2$ and $-(\sigma(\theta(t))\varepsilon\nabla\varphi(t)) \cdot \nabla\varphi(t) \le -\underline{\sigma}\underline{\varepsilon}\|\nabla\varphi(t)\|_{\mathbb{R}^3}^2$. This means that both integrals on the left-hand side in (A.1) are positive (since $\alpha \ge 0$), while the second term on the right-hand side is negative. The constant $m_{\inf}$ is constructed exactly such that $\theta_l(t) - m_{\inf}$ is greater or equal than zero almost everywhere, such that $-\alpha(\theta_l(t) - m_{\inf})\zeta^-(t) \le 0$. Hence, from (A.1) it follows that $\partial_t\|\zeta^-(t)\|_{L^2}^2 \le 0$. But, due to the construction of ζ, we have $\zeta(T_0) \ge 0$, which means that $\zeta^-(T_0) \equiv 0$ and thus $\zeta^-(t) \equiv 0$ for all $t \in (T_0, T_\bullet)$. ∎

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