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Imprint:

DEFINABLE ORTHOGONALITY CLASSES IN ACCESSIBLE CATEGORIES ARE SMALL

JOAN BAGARIA, CARLES CASACUBERTA,
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ABSTRACT. We lower substantially the strength of the assumptions needed for the validity of certain results in category theory and homotopy theory which were known to follow from Vopěnka's principle. We prove that the necessary large-cardinal hypotheses depend on the complexity of the formulas defining the given classes, in the sense of the Lévy hierarchy. For example, the statement that, for a class $\mathcal{S}$ of morphisms in an accessible category $\mathcal{C}$, the orthogonal class of objects $\mathcal{S}^{\perp}$ is a small-orthogonality class (hence reflective, if $\mathcal{C}$ is cocomplete) is provable in ZFC if $\mathcal{S}$ is $\mathbf{\Sigma_1}$, while it follows from the existence of a proper class of supercompact cardinals if $\mathcal{S}$ is $\mathbf{\Sigma_2}$, and from the existence of a proper class of what we call $C(n)$-extendible cardinals if $\mathcal{S}$ is $\mathbf{\Sigma_{n+2}}$ for $n \geq 1$. These cardinals form a new hierarchy, and we show that Vopěnka's principle is equivalent to the existence of $C(n)$-extendible cardinals for all n.

As a consequence, we prove that the existence of cohomological localizations of simplicial sets, a long-standing open problem in algebraic topology, follows from the existence of sufficiently large supercompact cardinals, since E^*-equivalences are $\mathbf{\Sigma_2}$-definable for every cohomology theory E^*. On the other hand, E_*-equivalences are $\mathbf{\Sigma_1}$-definable, from which it follows (as is well known) that the existence of homological localizations is provable in ZFC.

1. INTRODUCTION

The answers to certain questions in category theory turn out to depend on set theory. A typical example is whether every full limit-closed subcategory of a complete category $\mathcal{C}$ is reflective. On the one hand, there are counterexamples involving the category of topological spaces and continuous functions [37]. On the other hand, as explained in [2], an affirmative answer to this question for locally presentable categories is implied by a large-cardinal axiom called Vopěnka's principle (stating that, for every proper class of structures of the same type, there exists a nontrivial elementary embedding between two of them).

Large cardinals were used in a similar way in [15] to show that the existence of cohomological localizations, a famous unsolved problem, follows from Vopěnka's principle. Other relevant consequences of Vopěnka's principle in algebraic topology were found in [12], [13], [16], [35]. However, the precise consistency strength of many implications of this axiom in category theory or homotopy theory is not known, and in some cases the question of whether such statements are provable in ZFC remains unanswered. A relevant step in this direction was made in [34].

In another direction, it was pointed out in [7] that certain results about accessible categories that follow from Vopěnka's principle are still true under much weaker large-cardinal assumptions. This claim is based on the following finding, which is the subject of the present article: *the assumptions needed to infer reflectivity or smallness of orthogonality classes in accessible categories may depend on the complexity of the formulas in the language of set theory defining these classes.* Here "complexity" is meant in the sense of the Lévy hierarchy [25, Ch. 13]. Recall that Σ_n formulas and Π_n formulas are defined inductively as follows: Π_0 formulas are the same as Σ_0 formulas, namely formulas in which all quantifiers are bounded; Σ_{n+1} formulas are of the form $\exists x \, \varphi$, where φ is Π_n, and Π_{n+1} formulas are of the form $\forall x \, \varphi$, where φ is Σ_n.

For example, as we prove in this article, if $\mathcal{S}$ is a full limit-closed subcategory of a locally presentable category, and $\mathcal{S}$ can be defined by a Σ_2 formula (possibly with parameters), then the existence of a proper class of supercompact cardinals suffices to ensure reflectivity of $\mathcal{S}$. Remarkably, if $\mathcal{S}$ can be defined by a Σ_1 formula, then its reflectivity is provable in ZFC.

In case of a more complex definition of $\mathcal{S}$, its reflectivity follows from the existence of a proper class of what we call $C(n)$-*extendible cardinals*, for some n. These cardinals form a natural hierarchy ranging from extendible cardinals [25] when $n = 1$ to Vopěnka's principle. Indeed, as stated in Corollary 4.8 below, Vopěnka's principle is equivalent to the claim that there exists a $C(n)$-extendible cardinal for every $n < \omega$. We denote by $C(n)$ the proper class of cardinals α such that V_α is a Σ_n-elementary submodel of the set-theoretic universe V. Thus, a cardinal $\kappa \in C(n)$ is $C(n)$-extendible if, for all $\lambda > \kappa$ in $C(n)$, there is an elementary embedding $j \colon V_\lambda \to V_\mu$ for some $\mu \in C(n)$ with critical point κ, such that $j(\kappa) \in C(n)$ and $j(\kappa) > \lambda$.

By way of this approach, we prove that the existence of cohomological localizations of simplicial sets follows from the existence of a proper class of supercompact cardinals. This result uses the fact, proved in Theorem 7.3 below, that for each (Bousfield–Friedlander) spectrum E the class of E^*-acyclic simplicial sets (where E^* denotes the reduced cohomology theory represented by E) can be defined by means of a Σ_2 formula with E as a parameter. On the other hand, the class of E_*-acyclic simplicial sets (where E_* now denotes homology) can be defined with a Σ_1 formula. This is consistent with the fact that the existence of homological localizations can be proved in ZFC, as done indeed by Bousfield in [9]; see also [5].

The reason why classes of homology acyclics may have lower complexity than classes of cohomology acyclics is that, for a fibrant simplicial set Y, the statement "all maps $f \colon \mathbb{S}^n \to Y$ are nullhomotopic", where $\mathbb{S}^n$ is the

simplicial n-sphere, is absolute for transitive models of ZFC, since a simplicial map $\mathbb{S}^n \to Y$ is determined by a single n-simplex of Y satisfying certain conditions expressible in terms of Y with bounded quantifiers; cf. [32, 3.6]. However, if X and Y are arbitrary simplicial sets, then the statement "all maps $f: X \to Y$ are nullhomotopic" involves unbounded quantifiers, since it is formalized e.g. by stating

$$\forall f \, (f \text{ is a map from } X \text{ to } Y \to \exists h \, (h \text{ is a homotopy from } f \text{ to } y_0)).$$

Therefore, for a spectrum E, there might exist E^*-acyclic spaces in a model of ZFC containing E that fail to be E^*-acyclic in some larger model, while the class of E_*-acyclic spaces is absolute. See Section 7 for a detailed discussion of these facts.

Another consequence of this article is that the main theorem of [7] can now be proved for reflections, not necessarily epireflections. Thus, if there is a proper class of $C(n)$-extendible cardinals, then every reflection L on an accessible category is an $\mathcal{F}$-reflection for some set of morphisms $\mathcal{F}$, provided that the closure of the image of L under isomorphisms is $\boldsymbol{\Sigma_{n+1}}$ or the class of L-equivalences is $\boldsymbol{\Sigma_{n+2}}$; see Corollary 6.5 below. (Boldface types $\boldsymbol{\Sigma_n}$ or $\boldsymbol{\Pi_n}$ are used to denote the fact that the corresponding formulas may contain parameters.) Moreover, if the class of L-equivalences is $\boldsymbol{\Sigma_1}$, then no large-cardinal assumptions are necessary to infer the same result.

We also prove that the Freyd–Kelly orthogonal subcategory problem [19], asking if $\mathcal{S}^{\perp}$ is reflective for a class of morphisms $\mathcal{S}$ in a suitable category, has an affirmative answer in ZFC for $\boldsymbol{\Sigma_1}$ classes in locally presentable categories. It is also true for $\boldsymbol{\Sigma_2}$ classes if a proper class of supercompact cardinals exists, and for $\boldsymbol{\Sigma_{n+2}}$ classes if a proper class of $C(n)$-extendible cardinals exists for $n \geq 1$. We say that $\mathcal{S}$ is *definable with sufficiently low complexity* to encompass all these cases in a single phrase.

Essentially the same arguments hold in the homotopy category of simplicial sets, hence yielding a simpler and more accurate answer than in [15] (where Vopěnka's principle was used) to Farjoun's question in [17] of whether every homotopy reflection on simplicial sets is an f-localization for some map f. Localizations with respect to sets of maps were constructed in [10], [17], [22], and the extension to proper classes of maps was shown to exist under Vopěnka's principle [12], [15]. Here we prove that localizations with respect to proper classes of maps exist whenever the given classes are definable with sufficiently low complexity.

Lastly, a further corollary of our results is that, for a finitary operational signature Σ, every full subcategory of Σ-structures definable with sufficiently low complexity has only a set of implicit operations.

2. Preliminaries

2.1. **Categories of structures.** Most of the results in this article refer to categories of *structures* (possibly many-sorted, in a language of any cardinality). For the convenience of the reader, we recall terminology and background about structures in this subsection. Additional details can be found e.g. in [2, Ch. 5].

For a set S (called the set of *sorts*) and a regular infinite cardinal λ, a *λ-ary S-sorted signature* consists of a set of *operation symbols*, each of which has a certain *arity* $\prod_{i \in I} s_i \to s$, where s and all s_i are in S and $|I| < \lambda$, and another set of *relation symbols*, each of which has also a certain arity of the form $\prod_{j \in J} s_j$, where all s_j are in S and $|J| < \lambda$. We allow for an operation symbol to have arity $\emptyset \to s$ (with I empty), in which case it is called a *constant symbol*.

Given a signature Σ, a *Σ-structure* X consists of a nonempty set X_s for each $s \in S$ together with a function $\sigma_X \colon \prod_{i \in I} X_{s_i} \longrightarrow X_s$ for each operation symbol σ of arity $\prod_{i \in I} s_i \to s$ with $I \neq \emptyset$, a distinguished element of X_s for each constant symbol of arity $\emptyset \to s$, and a subset $\rho_X \subseteq \prod_{j \in J} X_{s_j}$ for each relation symbol ρ of arity $\prod_{j \in J} s_j$. For this, we use the notation $X = \langle \{X_s\}_{s \in S}, \{\sigma_X\}_\sigma, \{\rho_X\}_\rho \rangle$.

A *homomorphism* between two Σ-structures X and Y is a set of functions $\{f_s \colon X_s \to Y_s\}_{s \in S}$ that preserve operations and relations. If f is a homomorphism from X to Y, then f may be viewed as a single function from the disjoint union of $\{X_s\}_{s \in S}$ into the disjoint union of $\{Y_s\}_{s \in S}$ such that $f_s = f \restriction X_s$ takes values in Y_s for all $s \in S$, where $\restriction$ denotes restriction. For each signature Σ, the category of Σ-structures and their homomorphisms will be denoted by $\mathbf{Str}\,\Sigma$.

Given a regular infinite cardinal λ and a λ-ary S-sorted signature Σ, the *language* $\mathcal{L}_\lambda(\Sigma)$ consists of *variables*, *terms*, and *formulas*, together with a *satisfaction relation* $\models$. Thus, there is a set $W = \{W_s\}_{s \in S}$ of sets of cardinality λ, the elements of W_s being the *variables of sort s*. One defines *terms* by declaring that each variable is a term and, for each operation symbol σ of arity $\prod_{i \in I} s_i \to s$ and each collection of terms τ_i of sort s_i, the expression $\sigma(\tau_i)_{i \in I}$ is a term of sort s. *Atomic formulas* are expressions of the form $\tau_1 = \tau_2$ and $\rho(\tau_j)_{j \in J}$, where ρ is a relation symbol of arity $\prod_{j \in J} s_j$ and each τ_j is a term of sort s_j with $j \in J$. Formulas are built in finitely many steps from the atomic formulas by means of logical connectives and quantifiers, as follows. If φ and ψ are formulas, then so are the negation $\neg\varphi$, the implication $\varphi \to \psi$, conjunctions $\bigwedge_{j \in J} \varphi_j$ and disjunctions $\bigvee_{j \in J} \varphi_j$ indexed by sets J of cardinality smaller than λ, and quantification over sets X of variables of cardinality smaller than λ, namely $\forall X \, \varphi$ and $\exists X \, \varphi$. Satisfaction of a formula is defined inductively; see [2, 5.26].

A language $\mathcal{L}_\lambda(\Sigma)$ is called *finitary* if $\lambda = \omega$ (the first infinite cardinal); otherwise it is *infinitary*. An especially important finitary language is the *language of set theory*. This is the first-order finitary language corresponding to the signature with one sort, namely "sets", and one binary relation symbol ("membership"). Hence the atomic formulas are $x = y$ and $x \in y$, where x and y are sets.

Variables that appear unquantified in a formula are said to appear *free*. A formula without free variables is called a *sentence*. A set of sentences is called a *theory* (with signature Σ). A *model* of a theory T with signature Σ is a Σ-structure satisfying each sentence of T. For each theory T, we denote by $\mathbf{Mod}\,T$ the full subcategory of $\mathbf{Str}\,\Sigma$ consisting of all models of T.

Everything in this article is formulated in ZFC (Zermelo–Fraenkel set theory with the axiom of choice). Thus, a *class* $\mathcal{C}$ consists of all sets x for

which a certain formula $\varphi(x, x_1, \ldots, x_n)$ of the language of set theory with free variables $x, x_1, \ldots, x_n$ is satisfied; that is, $\mathcal{C} = \{x : \varphi(x, p_1, \ldots, p_n)\}$, where the sets $p_1, \ldots, p_n$, called *parameters*, are fixed values of the variables $x_1, \ldots, x_n$ under every variable assignment. Although, strictly speaking, formulas do not have parameters (but just variables), we say that $\mathcal{C}$ is defined by the formula $\varphi(x, p)$ with parameter set $p = \{p_1, \ldots, p_n\}$.

A class which is not a set is called a *proper class*.

2.2. **Absolute classes.** Define, recursively on the class of ordinals, $V_0 = \emptyset$, $V_{\alpha+1} = \mathcal{P}(V_\alpha)$ for all α, where $\mathcal{P}$ denotes the power-set operation, and $V_\lambda = \bigcup_{\alpha < \lambda} V_\alpha$ if λ is a limit ordinal. Then every set is an element of some V_α; see [24, Lemma 9.3] or [25, Lemma 6.3]. The *rank* of a set X is the least ordinal α such that $X \in V_{\alpha+1}$. Hence V_α is the set of all sets whose rank is less than α. The *set-theoretic universe* V of all sets is the union of V_α for all ordinals α.

A class M is *transitive* if every element of an element of M is an element of M. A *model* of set theory is a pair $\langle M, \in \rangle$ where M is a set or a proper class and $\in$ is the restriction of the membership relation to M, in which the formalized ZFC axioms are satisfied. Thus, if we neglect the fact that M can be a proper class, we may view $\langle M, \in \rangle$ as a Σ-structure where Σ has only one sort and one binary relation symbol. In what follows, we will always assume that models are transitive, but not necessarily inner (a model is called *inner* if it is transitive and contains all the ordinals).

We do not restrict absoluteness to inner models either. Thus, we say that a formula $\varphi(x_0, \ldots, x_n)$ of the language of set theory is *absolute* if for every transitive model M of set theory and all $a_0, \ldots, a_n \in M$,

$$\varphi(a_0, \ldots, a_n) \quad \text{if and only if} \quad M \models \varphi(a_0, \ldots, a_n),$$

that is, $\varphi(a_0, \ldots, a_n)$ holds in V if and only if it holds in M. A class $\mathcal{C}$ is *absolute* if it is definable by an absolute formula, possibly with parameters. A subcategory of the category of sets will be called *absolute*, as in [7], if its classes of objects and morphisms are absolute. See Section 5 for a more detailed discussion of this concept.

For each signature Σ, the category $\mathbf{Str}\,\Sigma$ embeds canonically into the category of sets as an absolute subcategory by assigning to each Σ-structure X the disjoint union of its constituent sets $\{X_s\}_{s \in S}$, its operation functions $\{\sigma_X\}_\sigma$, and its relation sets $\{\rho_X\}_\rho$. Homomorphisms $f \colon X \to Y$ correspond to functions $f_s \colon X_s \to Y_s$ for all $s \in S$, together with functions $f_\sigma \colon \sigma_X \to \sigma_Y$ for all operations σ and $f_\rho \colon \rho_X \to \rho_Y$ for all relations ρ, compatible with the collection $\{f_s\}_{s \in S}$.

For every theory T with signature Σ, the category $\mathbf{Mod}\,T$ is absolute if viewed as a subcategory of sets via the above embedding, since its objects and morphisms are definable by absolute formulas with the parameters Σ and T; cf. [7, Proposition 4.2]. To avoid confusion, note that, although the sentences of the theory T may belong to any (possibly infinitary) language $\mathcal{L}_\lambda(\Sigma)$, the class of models of T is defined by a formula of the language of set theory, namely a model of T is a Σ-structure X such that $\forall \varphi \in T\ (X \models \varphi)$.

2.3. **Accessible categories.** If $\mathcal{C}$ is any category, we denote by $\mathcal{C}(X, Y)$ the set of morphisms in $\mathcal{C}$ from X to Y. A category is *small* if its objects form a set, and *essentially small* if the isomorphism classes of its objects form a set. We normally do not make a distinction between a class of objects in a category and the full subcategory with those objects.

Let λ be a regular cardinal. A category $\mathcal{D}$ is called λ-*filtered* if, given any set of objects $\{A_i\}_{i \in I}$ where $|I| < \lambda$, there is an object A and a morphism $A_i \to A$ for each $i \in I$, and, moreover, given any set of parallel arrows between any two objects $\{f_j \colon B \to C\}_{j \in J}$ where $|J| < \lambda$, there is a morphism $g \colon C \to D$ such that $g \circ f_j$ is the same morphism for all $j \in J$.

If $\mathcal{C}$ is any category, a functor $F \colon \mathcal{D} \to \mathcal{C}$ where $\mathcal{D}$ is a λ-filtered small category is called a λ-*filtered diagram*, and, if F has a colimit L, then L is called a λ-*filtered colimit*. For example, every set is a λ-filtered colimit of its subsets of cardinality smaller than λ (partially ordered by inclusion).

For a category $\mathcal{C}$ and an object A of $\mathcal{C}$, we denote by $(\mathcal{C} \downarrow A)$ the *slice category* whose objects are morphisms $X \to A$ in $\mathcal{C}$, and whose morphisms are commutative triangles. Dually, the objects of the *coslice category* $(A \downarrow \mathcal{C})$ are morphisms $A \to X$. For each subcategory $\mathcal{D}$ of $\mathcal{C}$ and every object A in $\mathcal{C}$, the *canonical diagram* $(\mathcal{D} \downarrow A) \to \mathcal{C}$ sends each morphism $X \to A$ to X. Recall from [2, 1.23] that a full subcategory $\mathcal{D}$ of a category $\mathcal{C}$ is called *dense* if each object A of $\mathcal{C}$ is a colimit of the canonical diagram $(\mathcal{D} \downarrow A) \to \mathcal{C}$. A category $\mathcal{C}$ is *bounded* if it has a small dense full subcategory.

If $\mathcal{C}$ has a small dense full subcategory $\mathcal{D}$ such that the canonical diagram $(\mathcal{D} \downarrow A) \to \mathcal{C}$ is λ-filtered for every object A of $\mathcal{C}$, then we say that $\mathcal{C}$ is *weakly λ-preaccessible*. A category $\mathcal{C}$ will be called *weakly preaccessible* if it is weakly λ-preaccessible for some regular cardinal λ.

An object X of a category $\mathcal{C}$ is λ-*presentable* if the functor $\mathcal{C}(X, -)$ preserves λ-filtered colimits; that is, for each λ-filtered diagram $F \colon \mathcal{D} \to \mathcal{C}$ with a colimit L, each morphism $X \to L$ can be lifted to a morphism $X \to F(d)$ for some d in $\mathcal{D}$, and if two morphisms $X \to F(d)$ and $X \to F(d')$ compose to the same morphism $X \to L$, then there is some d'' and morphisms $d \to d''$ and $d' \to d''$ in $\mathcal{D}$ such that the two composites $X \to F(d'')$ are equal; see [20, §6.1] or [31, §2.1].

A category $\mathcal{C}$ is called λ-*preaccessible* if there is a set $\mathcal{A}$ of λ-presentable objects such that every object of $\mathcal{C}$ is a λ-filtered colimit of objects from $\mathcal{A}$. A category $\mathcal{C}$ is called *preaccessible* if it is λ-preaccessible for some λ. As shown in [3, p. 226], if $\mathcal{C}$ is λ-preaccessible, then the full subcategory $\mathcal{C}_\lambda$ of its λ-presentable objects is essentially small and dense in $\mathcal{C}$. Moreover, for each object X of $\mathcal{C}$, the slice category $(\mathcal{C}_\lambda \downarrow X)$ is λ-filtered and X is a colimit of the canonical diagram $(\mathcal{C}_\lambda \downarrow X) \to \mathcal{C}$. Thus, every preaccessible category is bounded (since we may replace $\mathcal{C}_\lambda$ by a set of representatives of all the isomorphism classes of its objects), and in fact weakly preaccessible.

Every bounded category $\mathcal{C}$ can be embedded into a category of relational structures (hence into the category **Set** of sets) as follows. Suppose that $\mathcal{A}$ is a small dense full subcategory of $\mathcal{C}$, and let $\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ denote the category of functors $\mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$, where $\mathcal{A}^{\mathrm{op}}$ denotes the opposite of $\mathcal{A}$. Then there are full embeddings

$$(2.1) \qquad\qquad \mathcal{C} \longrightarrow \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}} \longrightarrow \mathbf{Str}\,\Sigma,$$

defined as follows. The embedding of $\mathcal{C}$ into $\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ is of Yoneda type, sending each object X to the restriction of $\mathcal{C}(-, X)$ to $\mathcal{A}^{\mathrm{op}}$. The fact that it is full and faithful is proved in [2, Proposition 1.26]. Moreover, it preserves λ-filtered colimits if every object of $\mathcal{A}$ is λ-presentable. The signature Σ is chosen by picking the objects of $\mathcal{A}$ as sorts and the morphisms of $\mathcal{A}^{\mathrm{op}}$ as relation symbols. The full embedding of $\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ into $\mathbf{Str}\,\Sigma$ is described in [2, Example 1.41], where it is also shown that it preserves all filtered colimits. Therefore, if $\mathcal{C}$ is λ-preaccessible and $\mathcal{A}$ is a set of representatives of all isomorphism classes of objects in $\mathcal{C}_\lambda$, then (2.1) preserves λ-filtered colimits.

If $\mathcal{C}$ is λ-preaccessible and λ-filtered colimits exist in $\mathcal{C}$, then $\mathcal{C}$ is called λ-*accessible*. It is called *accessible* if it is λ-accessible for some λ, and *locally presentable* if it is accessible and cocomplete, i.e., if all colimits exist.

By [2, Theorem 5.35], for each accessible category $\mathcal{C}$, the image of the full embedding (2.1) is precisely the category $\mathbf{Mod}\,T$ of models of a basic theory T, hence absolute. Therefore, every accessible category can be viewed as an absolute subcategory of $\mathbf{Set}$ via the full embedding (2.1) followed by the canonical embedding of $\mathbf{Str}\,\Sigma$ into $\mathbf{Set}$. We will implicitly do so in this article, unless otherwise chosen in some cases for convenience.

2.4. **Elementary embeddings.** An *elementary embedding* of a structure A into another structure B with the same signature Σ is a function $j\colon A \to B$ that preserves and reflects truth. That is, for every formula $\varphi(x_1, \ldots, x_k)$ of the language of Σ and $a_1, \ldots, a_k$ in A, the sentence $\varphi(a_1, \ldots, a_k)$ is satisfied in A if and only if $\varphi(j(a_1), \ldots, j(a_k))$ is satisfied in B.

Elementary embeddings between models of set theory (which may be proper classes) are defined in the same way. If $j\colon V \to M$ is a nontrivial elementary embedding of the set-theoretic universe V into a transitive class M, then its critical point (i.e., the least ordinal moved by j) is a measurable cardinal. In fact, the existence of a nontrivial elementary embedding of the set-theoretic universe into a transitive class is equivalent to the existence of a measurable cardinal; see [25] or [27].

Given a subcategory $\mathcal{C}$ of $\mathbf{Set}$ and an elementary embedding $j\colon V \to M$, we say that j is *supported* by $\mathcal{C}$ if, for every object X in $\mathcal{C}$, the set $j(X)$ is also an object of $\mathcal{C}$ and the restriction function $j \restriction X : X \longrightarrow j(X)$ is a morphism in $\mathcal{C}$.

Lemma 2.1. *Let $j\colon V \to M$ be an elementary embedding with critical point κ and let Σ be an S-sorted signature such that S and Σ are in V_κ. If X is a Σ-structure, then $j(X)$ is also a Σ-structure and $j \restriction X : X \to j(X)$ is a homomorphism of Σ-structures.*

Proof. Apply j to the operations and relations of X, and use the fact that $j(S) = S$ and $j(\Sigma) = \Sigma$. $\square$

More generally, if $\mathcal{C}$ is an absolute full subcategory of $\mathbf{Str}\,\Sigma$ for some S-sorted signature Σ, then $\mathcal{C}$ supports elementary embeddings whose critical point κ is sufficiently large, namely such that S and Σ are in V_κ. This is a more precise restatement of [7, Proposition 4.4].

2.5. **The Lévy hierarchy.** Let us recall the following terminology due to Lévy; see [25, Ch. 13]. A formula of the language of set theory is said to be Σ_0 if all its quantifiers are bounded, that is, of the form $\exists x \in a$ or $\forall x \in a$. Then Σ_n *formulas* and Π_n *formulas* are defined inductively as follows: Π_0 formulas are the same as Σ_0 formulas; Σ_{n+1} formulas are of the form $\exists x\, \varphi$, where φ is Π_n; and Π_{n+1} formulas are of the form $\forall x\, \varphi$, where φ is Σ_n.

Mathematical concepts can be formalized in the language of set theory in many different ways. We say that a statement (property, relation, etc.) is Σ_n if it can be formalized with a Σ_n formula, and similarly with Π_n. A statement is Δ_n if it is both Σ_n and Π_n.

A class $\mathcal{C}$ is called $\boldsymbol{\Sigma_n}$ (with boldface letters) if there is a Σ_n formula $\varphi(x, p)$ such that $\mathcal{C} = \{x : \varphi(x, p)\}$, where p is a finite set of parameters. Similarly, a class is $\boldsymbol{\Pi_n}$ if it can be defined by some Π_n formula with parameters. If $\mathcal{C}$ is both $\boldsymbol{\Sigma_n}$ and $\boldsymbol{\Pi_n}$, then we say that $\mathcal{C}$ is $\boldsymbol{\Delta_n}$. If no parameters are involved, then we say that $\mathcal{C}$ is Σ_n, Π_n, or Δ_n, using lightface types.

If a class $\mathcal{C}$ is $\boldsymbol{\Sigma_1}$, then it is *upwards absolute* for transitive classes. That is, if $\mathcal{C}$ is defined by a Σ_1 formula $\varphi(x, p)$ and M is a transitive class containing the set p of parameters, then, for every $a \in M$, if $M \models \varphi(a, p)$ then $a \in \mathcal{C}$.

As we next show, the converse is also true, that is, if a class $\mathcal{C}$ is upwards absolute for transitive classes, then $\mathcal{C}$ is $\boldsymbol{\Sigma_1}$. In fact, in order to infer that $\mathcal{C}$ is $\boldsymbol{\Sigma_1}$, it suffices that $\mathcal{C}$ be upwards absolute for adequate transitive sets. (We say that M is *adequate* for $\mathcal{C}$, as in [25, p. 184], if a sufficiently large finite set of ZFC axioms hold in M, including all those involved in a proof of the fact that $\mathcal{C}$ is upwards absolute.) Thus, suppose that a class $\mathcal{C}$ is defined by a formula $\varphi(x, p)$ and that $\mathcal{C}$ is upwards absolute for adequate transitive sets. By the Reflection Principle [25, Theorem 12.14], every sentence which is true in the universe V holds in V_α for some α. Therefore, a set a is in $\mathcal{C}$ if and only if

(2.2) $\quad \exists M\, (M$ is transitive and adequate $\wedge\ \{a, p\} \in M\ \wedge\ M \models \varphi(a, p))$,

which is indeed a Σ_1 formula.

Similarly, if a class $\mathcal{C}$ is defined by a Π_1 formula $\varphi(x, p)$, then it is *downwards absolute* for transitive classes (that is, if $a \in \mathcal{C}$, then $M \models \varphi(a, p)$ for every transitive class M containing a and p), and if $\mathcal{C}$ is downwards absolute for adequate transitive sets then it is $\boldsymbol{\Pi_1}$, similarly as in (2.2). Thus, $\boldsymbol{\Delta_1}$ classes are absolute in the sense of Subsection 2.2 and, conversely, absolute classes are $\boldsymbol{\Delta_1}$.

For every theory T with signature Σ, the category $\mathbf{Mod}\, T$ is $\boldsymbol{\Delta_0}$ in the parameters Σ and T. Hence, every accessible category is equivalent to a $\boldsymbol{\Delta_0}$ category, according to [2, 5.35].

An example of a $\boldsymbol{\Sigma_2}$ class of structures is the class $\mathcal{C}$ of all groups of the form $\mathbb{Z}^\kappa$, where κ is a cardinal. To see this, recall first that "x is a cardinal" is a Π_1 statement [25, Lemma 13.13]. Then $G \in \mathcal{C}$ if and only if

(2.3) $\quad \exists x\, (x$ is a cardinal $\wedge\ \forall y\, (y \in G \leftrightarrow y$ is a function from x to $\mathbb{Z}))$,

which is a Σ_2 formula, since the expression written within the outer parentheses is Π_1. The following is a related example. In a model of ZFC with measurable cardinals, the following sentence is true:

$\exists \kappa\, \exists f\, (\kappa$ is a cardinal $\wedge\ f$ is a homomorphism $\mathbb{Z}^\kappa / \mathbb{Z}^{<\kappa} \to \mathbb{Z}\ \wedge\ f \neq 0)$,

while if this holds then the smallest κ with this property is measurable [14, Theorem 3]. Therefore, this statement is false in a model of ZFC without measurable cardinals and it is true in a model with measurable cardinals.

Let us also observe that the class of topological spaces is Π_1, since the union of every collection of open sets must be open. Indeed, a topology on a set X in some model may fail to be a topology on X in a larger model.

For sets A and B, although the statement $A \subseteq B$ is absolute, since it amounts to writing $\forall a \in A \, (a \in B)$, the statement $A = \mathcal{P}(B)$ —where $\mathcal{P}$ denotes power set— is formalized with the following Π_1 formula:

$$\forall a \in A \, (a \subseteq B) \wedge \forall x \, (x \subseteq B \rightarrow x \in A).$$

To see that, in fact, $A = \mathcal{P}(B)$ cannot be formalized by any absolute formula, pick a countable model M of ZFC. Then, if $A = \mathcal{P}(\mathbb{N})$ is true in M, the set A (which is countable) is distinct from the power set of $\mathbb{N}$ in the universe V.

On the other hand, if we denote by $\mathcal{P}_\omega(B)$ the set of all *finite* subsets of B, then the statement $A = \mathcal{P}_\omega(B)$ is absolute between adequate transitive classes, as the following argument shows. Let $M \subseteq N$ be transitive classes in which the pairing and union axioms hold, and let B be a set in M. Then every finite subset of B in M is also a finite subset of B in N, and, conversely, if X is a finite subset of B in N, then $X \in M$, as inferred by repeatedly using (finitely many times, as X is finite) the pairing and union axioms; so, X is a finite subset of B in M. Therefore, $A = \mathcal{P}_\omega(B)$ is a $\mathbf{\Delta_1}$ statement.

3. Vopěnka's principle and supercompact cardinals

We say that A and B are *structures of the same type* if they are both Σ-structures for some signature Σ. *Vopěnka's principle* is the following statement; cf. [2, Ch. 6], [25, (20.29)]:

VP: *For every proper class $\mathcal{C}$ of structures of the same type, there exist distinct A and B in $\mathcal{C}$ and an elementary embedding of A into B.*

This is a statement about classes. In the language of set theory, one can also formulate VP, but as an axiom schema, that is, an infinite set of axioms; in fact, one axiom for each formula $\varphi(x, y)$ of the language of set theory with two free variables, as follows:

$$\forall x \, [(\forall y \, \forall z \, (\varphi(x,y) \wedge \varphi(x,z) \rightarrow y \text{ and } z \text{ are structures of the same type}) \wedge$$
$$\forall \alpha \in \mathrm{Ord} \, \exists y \, (\mathrm{rank}(y) > \alpha \wedge \varphi(x,y))) \rightarrow$$
$$\exists y \, \exists z \, (\varphi(x,y) \wedge \varphi(x,z) \wedge y \neq z \wedge \exists e \, (e \colon y \rightarrow z \text{ is elementary}))].$$

In this article, VP will be understood as this axiom schema, and similarly with the variants of VP defined below.

In the statement of VP, the requirement that there is an elementary embedding between two distinct structures is sometimes replaced by the requirement that there is a nontrivial elementary embedding between two (possibly equal) structures. The two formulations are seen to be equivalent using rigid graphs; see [2, §6.A].

The theory ZFC+VP is very strong. It implies, for instance, that the class of extendible cardinals is stationary, i.e., every club proper class contains an extendible cardinal [30]. The consistency of ZFC + VP follows from that of ZFC plus the existence of an almost-huge cardinal; see [25] or [27].

For any two structures $M \subseteq N$ of the language of set theory and $n < \omega$, we write $M \preceq_n N$ and say that M is a Σ_n-*elementary substructure* of N if, for every Σ_n formula $\varphi(x_1, \ldots, x_k)$ and all $a_1, \ldots, a_k \in M$,

$$M \models \varphi(a_1, \ldots, a_k) \text{ if and only if } N \models \varphi(a_1, \ldots, a_k).$$

For a cardinal λ, we denote by $H(\lambda)$ the set of all sets whose transitive closure has cardinality less than λ. Thus $H(\lambda)$ is a transitive set contained in V_λ, and, if λ is strongly inaccessible, then $H(\lambda) = V_\lambda$; see [28, Lemma 6.2].

A class C of ordinals is *unbounded* if it contains arbitrarily large ordinals, and it is *closed* if, for every ordinal α, if $\bigcup(C \cap \alpha) = \alpha$ then $\alpha \in C$. The abbreviation *club* means closed and unbounded. As a consequence of the Reflection Principle, for every n there exists a club class of cardinals λ such that $H(\lambda) \preceq_n V$. In addition, if λ is uncountable, then $H(\lambda) \preceq_1 V$.

Recall that a cardinal κ is λ-*supercompact* if there is an elementary embedding $j: V \to M$ with M transitive and with critical point κ, such that $j(\kappa) > \lambda$ and M is closed under λ-sequences; i.e., every sequence $\langle X_\alpha \rangle_{\alpha < \lambda}$ of elements of M is an element of M. Note that it then follows that $H(\lambda) \in M$. Note also that, if an elementary embedding $j: V \to M$ has critical point κ, then $j(X) = X$ for every set $X \in V_\kappa$.

A cardinal κ is called *supercompact* if it is λ-supercompact for every $\lambda > \kappa$.

The following theorem is an upgraded version of [7, Theorem 4.5], where a similar result was proved for absolute classes.

Theorem 3.1. *Let C be a class of structures of the same type definable with a Σ_2 formula $\varphi(x, p)$. Suppose that there exists a supercompact cardinal κ bigger than the rank of the set p of parameters. Then for every $B \in C$ there exists $A \in C \cap V_\kappa$ and an elementary embedding of A into B.*

Proof. Suppose that κ is a supercompact cardinal with $p \in V_\kappa$. Fix $B \in C$, and let λ be a cardinal bigger than κ such that $B \in H(\lambda)$ and $H(\lambda) \preceq_2 V$. Let $j: V \to M$ be an elementary embedding with M transitive and critical point κ, such that $j(\kappa) > \lambda$ and M is closed under λ-sequences. Hence, B and the restriction $j \restriction B : B \to j(B)$ are in M, and $H(\lambda) \in M$ as well.

Since being a cardinal is definable by a Π_1 formula, hence downwards absolute, λ is a cardinal in M. This implies that $H(\lambda)$ in the sense of M is the same as $H(\lambda)$ in V. Hence $H(\lambda) \preceq_1 M$. Therefore, Σ_2 formulas are upwards absolute between $H(\lambda)$ and M.

Since $H(\lambda) \preceq_2 V$ and the class C is defined by a Σ_2 formula $\varphi(x, p)$, we have that $H(\lambda) \models \varphi(B, p)$, and hence $M \models \varphi(B, p)$.

Thus, in M it is true that there exists an object X with $M \models \text{rank}(X) < j(\kappa)$ and $M \models \varphi(X, p)$, and there is an elementary embedding $X \to j(B)$. (Indeed, B is such an object.) Moreover, $p = j(p)$, since $p \in V_\kappa$. Therefore, by elementarity of j, the same holds in V; that is, there exists an object X with $\text{rank}(X) < \kappa$ such that $\varphi(X, p)$ holds, and there exists an elementary embedding $X \to B$. Letting A be such an X, we are done. $\square$

Theorem 3.1 tells us that the existence of sufficiently large supercompact cardinals implies that VP holds for $\boldsymbol{\Sigma_2}$ classes. The following theorem yields a strong converse of this fact.

Theorem 3.2. *Suppose that, for every* $\mathbf{\Delta_2}$ *proper class* $\mathcal{C}$ *of structures in the language of set theory with one additional constant symbol, there exist distinct* A *and* B *in* $\mathcal{C}$ *and an elementary embedding of* A *into* B. *Then there exists a proper class of supercompact cardinals.*

Proof. Let ξ be any ordinal and suppose, towards a contradiction, that there are no supercompact cardinals bigger than ξ. Then the class function F given as follows is well defined on ordinals $\zeta > \xi$: $F(\zeta)$ equals the least cardinal $\lambda > \zeta$ such that no cardinal κ such that $\xi < \kappa \leq \zeta$ is λ-supercompact. Since the assertion "ζ is λ-supercompact" is Δ_2 in ZFC (see [27, 22]), F is Δ_2-definable with ξ as a parameter. Let

$$C_0 = \{\alpha : \alpha \text{ is a limit ordinal, } \xi < \alpha, \text{ and } \forall \zeta \, (\xi < \zeta < \alpha \to F(\zeta) < \alpha)\}.$$

Then C_0 is a club class Δ_2-definable with ξ as a parameter.

Fix a rigid binary relation (i.e., a rigid graph) R on $\xi + 1$ (see, e.g., [33]). For each ordinal α, let λ_α be the least element of C_0 greater than λ. The proper class $\mathcal{C} = \{\langle V_{\lambda_\alpha+2}, \in, \langle \alpha, R \rangle \rangle\}_{\alpha > \xi}$ is a $\mathbf{\Delta_2}$ class definable with R as a parameter. By our assumption, there exist $\alpha < \beta$ greater than ξ and an elementary embedding

$$j : \langle V_{\lambda_\alpha+2}, \in, \langle \alpha, R \rangle \rangle \longrightarrow \langle V_{\lambda_\beta+2}, \in, \langle \beta, R \rangle \rangle.$$

Since j must send α to β, it is not the identity. Hence, by Kunen's Theorem [25], we have $\lambda_\alpha < \lambda_\beta$. Let $\kappa \leq \alpha$ be the critical point of j. Then, as in [30, Lemma 2], it follows that κ is λ_α-supercompact. But this is impossible, since $F(\kappa) < \lambda_\alpha$ because $\lambda_\alpha \in C_0$. $\square$

In order to summarize what we have proved so far, we introduce some useful notation. Let Γ be one of Σ_n, Π_n, Δ_n, or $\mathbf{\Sigma_n}$, $\mathbf{\Pi_n}$, $\mathbf{\Delta_n}$, for any n. For an infinite cardinal κ and a signature Σ, we write:

$\mathrm{VP}^\Sigma(\Gamma)$: *For every* Γ *proper class* $\mathcal{C}$ *of* Σ-*structures, there exist distinct* A *and* B *in* $\mathcal{C}$ *and an elementary embedding of* A *into* B.

$\mathrm{SVP}^\Sigma_\kappa(\Gamma)$: *For every proper class* $\mathcal{C}$ *of* Σ-*structures admitting a* Γ *definition whose parameters, if any, are in* $H(\kappa)$, *and for every* $B \in \mathcal{C}$, *there exists* $A \in \mathcal{C} \cap H(\kappa)$ *and an elementary embedding of* A *into* B.

In both cases, if Σ is omitted from the notation, we mean that the corresponding statement holds for all signatures. Even though $\mathrm{SVP}^\Sigma_\kappa(\Gamma)$ is an apparently stronger statement than $\mathrm{VP}^\Sigma(\Gamma)$ —hence the notation SVP—, in the case of $\mathbf{\Sigma_2}$ classes of structures they turn out to be equivalent, as we next prove.

Corollary 3.3. *The following statements are equivalent:*
  (1) $\mathrm{SVP}_\kappa(\mathbf{\Sigma_2})$ *holds for a proper class of cardinals* κ.
  (2) $\mathrm{VP}(\mathbf{\Sigma_2})$ *holds.*
  (3) $\mathrm{VP}^\Sigma(\mathbf{\Delta_2})$ *holds if* Σ *is the signature of the language of set theory with one additional constant symbol.*
  (4) *There exists a proper class of supercompact cardinals.*

Proof. In order to check that $(1) \Rightarrow (2)$, suppose that (1) is true, and let Σ be any signature. Let $\mathcal{C}$ be any proper class of Σ-structures defined by a Σ_2 formula with parameters, and let κ be bigger than the ranks of the

parameters and such that $\mathrm{SVP}_\kappa^\Sigma(\mathbf{\Sigma_2})$ holds. Since $\mathcal{C}$ is a proper class, we may choose B of rank bigger than κ, so any $A \in \mathcal{C} \cap H(\kappa)$ will necessarily be distinct from B. Hence, there exist distinct A and B such that A is elementarily embeddable into B, so $\mathrm{VP}^\Sigma(\mathbf{\Sigma_2})$ holds, as needed. The implication $(2) \Rightarrow (3)$ is trivial, and Theorem 3.2 implies that $(3) \Rightarrow (4)$. Finally, to see that $(4) \Rightarrow (1)$, let ξ be any cardinal and pick a supercompact cardinal $\kappa > \xi$. Since $H(\kappa) = V_\kappa$, Theorem 3.1 tells us that $\mathrm{SVP}_\kappa(\mathbf{\Sigma_2})$ holds. $\qquad\square$

The following is a corresponding version without parameters, with the same (in fact, simpler) proof.

Corollary 3.4. *The following statements are equivalent:*
  (1) $\mathrm{SVP}_\kappa(\Sigma_2)$ *holds for some cardinal κ.*
  (2) $\mathrm{VP}(\Sigma_2)$ *holds.*
  (3) $\mathrm{VP}^\Sigma(\Delta_2)$ *holds if Σ is the signature of the language of set theory.*
  (4) *There exists a supercompact cardinal.*

We finish this section by observing that, remarkably, $\mathrm{SVP}_\kappa^\Sigma(\mathbf{\Sigma_1})$ *can be proved in ZFC* for every uncountable cardinal κ and every S-sorted signature Σ such that S and Σ are in $H(\kappa)$. In fact, this result is more general, since it holds for sets as well as proper classes.

Theorem 3.5. *Let κ be an uncountable cardinal and Σ an S-sorted signature such that S and Σ are in $H(\kappa)$. Let $\mathcal{C}$ be a class of Σ-structures definable with a Σ_1 formula with parameters in $H(\kappa)$. Then for every $B \in \mathcal{C}$ there exists $A \in \mathcal{C} \cap H(\kappa)$ and an elementary embedding of A into B.*

Proof. Suppose that a class $\mathcal{C}$ of Σ-structures is definable by a Σ_1 formula; say, $\mathcal{C} = \{A : \exists x\, \varphi(x, A, p)\}$, where φ is Σ_0 and p is a set of parameters. Let κ be an uncountable cardinal such that $\{p, S, \Sigma\} \in H(\kappa)$.

Given $B \in \mathcal{C}$, let λ be a regular cardinal with $B \in H(\lambda)$. By the Löwenheim–Skolem Theorem, we can find an elementary substructure N of $H(\lambda)$, of cardinality less than κ, with $B \in N$ and with the transitive closure of $\{p, S, \Sigma\}$ contained in N.

Let $\bar{N}$ and $\bar{B}$ be the transitive collapses of N and B, respectively, and let $j : \bar{N} \to N$ be the isomorphism given by the collapse. Note that $j(p) = p$, $j(S) = S$ and $j(\Sigma) = \Sigma$. Then $\bar{B} \in H(\kappa)$ and the restriction $j \upharpoonright \bar{B} : \bar{B} \to B$ is an elementary embedding. Finally, since $\bar{N} \models \exists x\, \varphi(x, \bar{B}, p)$, and Σ_1 formulas are upwards absolute for transitive models, we have that $\bar{B} \in \mathcal{C}$. Take $A = \bar{B}$. $\qquad\square$

4. Vopěnka's principle and extendible cardinals

For cardinals $\kappa < \lambda$, we say that κ is λ-*extendible* if there is an elementary embedding $j : V_\lambda \to V_\mu$ for some μ, with critical point κ and such that $j(\kappa) > \lambda$. A cardinal κ is called *extendible* if it is λ-extendible for all cardinals $\lambda > \kappa$. As shown in [25, Theorem 20.24], extendible cardinals are supercompact; see [25] or [27] for more information about extendible cardinals.

For each $n < \omega$, let $C(n)$ denote the club proper class of infinite cardinals κ that are Σ_n-*correct* in V, that is, $V_\kappa \preceq_n V$. Since the satisfaction relation $\models_n$ for Σ_n sentences is Σ_n-definable for $n \geq 1$ [27, §0.2], it follows that, for

$n \geq 1$, the class $C(n)$ is Π_n. To see this, note first that $C(0)$ is the class of all infinite cardinals, and therefore it is Π_1-definable. For κ an infinite cardinal, $\kappa \in C(1)$ if and only if κ is an uncountable cardinal and $V_\kappa = H(\kappa)$, which implies that $C(1)$ is Π_1-definable. In general, for $n \geq 1$ and for any infinite cardinal κ, we have $V_\kappa \preceq_{n+1} V$ if and only if

$$\kappa \in C(n) \wedge (\forall \varphi(x) \in \Sigma_{n+1}) (\forall a \in V_\kappa) (\models_{n+1} \varphi(a) \rightarrow V_\kappa \models \varphi(a)),$$

which is a Π_{n+1} formula showing that $C(n+1)$ is Π_{n+1}-definable.

We shall use the following new strong form of extendibility.

Definition 4.1. For C a club proper class of cardinals and $\kappa < \lambda$ in C, we say that κ is λ-C-*extendible* if there is an elementary embedding $j\colon V_\lambda \to V_\mu$ for some $\mu \in C$, with critical point κ, such that $j(\kappa) > \lambda$ and $j(\kappa) \in C$.

We say that $\kappa \in C$ is C-*extendible* if it is λ-C-extendible for all λ in C greater than κ.

Note that, for all n, if κ is $C(n)$-extendible, then κ is extendible. Therefore, a cardinal is $C(0)$-extendible if and only if it is extendible.

Proposition 4.2. *Every extendible cardinal is $C(1)$-extendible.*

Proof. Suppose that κ is extendible and $\lambda \in C(1)$ is greater than κ. Note that the existence of an extendible cardinal implies the existence of a proper class of inaccessible cardinals, as the image of κ under any elementary embedding $j\colon V_\lambda \to V_\mu$, with critical point κ and λ a cardinal, is always an inaccessible cardinal in V. So we can pick an inaccessible cardinal $\lambda' \geq \lambda$. Let $j'\colon V_{\lambda'} \to V_{\mu'}$ be an elementary embedding with critical point κ and such that $j'(\kappa) > \lambda'$. Since $V_{\lambda'} = H(\lambda')$, it follows by elementarity of j' that $V_{\mu'} = H(\mu')$. Hence, $\mu' \in C(1)$.

Let us see that $j = j' \restriction V_\lambda\colon V_\lambda \to V_{j'(\lambda)}$ witnesses the λ-$C(1)$-extendibility of κ. We only need to check that $\mu = j'(\lambda) \in C(1)$. But since $V_\lambda \preceq_1 V_{\lambda'}$, it follows by elementarity of j' that $V_\mu \preceq_1 V_{\mu'}$. Hence, since $\mu' \in C(1)$, also $\mu \in C(1)$. $\square$

Hence, a cardinal is $C(1)$-extendible if and only if it is extendible. Let us also observe that, if there exists a $C(n+2)$-extendible cardinal, then there exists a proper class of $C(n)$-extendible cardinals.

Lemma 4.3. *If κ is $C(n)$-extendible, then $\kappa \in C(n+2)$.*

Proof. By induction on n. For $n = 0$, since $\kappa \in C(1)$, we only need to show that if $\exists x\, \varphi(x)$ is a Σ_2 sentence, where φ is Π_1 and has parameters in V_κ, that holds in V, then it holds in V_κ. So suppose that a is such that $\varphi(a)$ holds in V. Let $\lambda \in C(n)$ be greater than κ and with $a \in V_\lambda$, and let $j\colon V_\lambda \to V_\mu$ be elementary, with critical point κ and with $j(\kappa) > \lambda$. Then $V_{j(\kappa)} \models \varphi(a)$, and so, by elementarity, $V_\kappa \models \exists x\, \varphi(x)$.

Now suppose that κ is $C(n)$-extendible and $\exists x\, \varphi(x)$ is a Σ_{n+2} sentence, where φ is Π_{n+1} and has parameters in V_κ. If $\exists x\, \varphi(x)$ holds in V_κ, then, since by the induction hypothesis $\kappa \in C(n+1)$, we have that $\exists x\, \varphi(x)$ holds in V. Now suppose that a is such that $\varphi(a)$ holds in V. Let $\lambda \in C(n)$ be greater than κ and such that $a \in V_\lambda$, and let $j\colon V_\lambda \to V_\mu$ be elementary with critical point κ and with $j(\kappa) > \lambda$. Then, since $j(\kappa) \in C(n)$, we have $V_{j(\kappa)} \models \varphi(a)$, and so, by elementarity, $V_\kappa \models \exists x\, \varphi(x)$. $\square$

Theorem 4.4. *For every $n \geq 1$, if κ is a $C(n)$-extendible cardinal, then $\mathrm{SVP}_\kappa(\mathbf{\Sigma_{n+2}})$ holds.*

Proof. Fix a Σ_{n+2} formula $\exists x\, \varphi(x, y, z)$, where φ is Π_{n+1}, such that, for some set $p \in V_\kappa$,
$$\mathcal{C} = \{B : \exists x\, \varphi(x, B, p)\}$$
is a proper class of structures of the same type.

Fix $B \in \mathcal{C}$ and let $\lambda \in C(n+2)$ be greater than κ and the ranks of p and B. Thus,
$$V_\lambda \models \exists x\, \varphi(x, B, p).$$

Let $j \colon V_\lambda \to V_\mu$ for some $\mu \in C(n)$ be an elementary embedding with critical point κ, with $j(\kappa) > \lambda$ and $j(\kappa) \in C(n)$. Note that both B and $j \upharpoonright B \colon B \to j(B)$ are in V_μ.

Since $\kappa, \lambda \in C(n+2)$ by Lemma 4.3, and $\kappa < \lambda$, we have $V_\kappa \preceq_{n+2} V_\lambda$. It follows that $V_{j(\kappa)} \preceq_{n+2} V_\mu$. Indeed, the following holds:
$$V_\lambda \models (\forall x \in V_\kappa)\,(\forall \theta \in \Sigma_{n+2})\,(V_\kappa \models \theta(x) \leftrightarrow {\models_{n+2}} \theta(x)).$$

Hence, by elementarity,
$$V_\mu \models (\forall x \in V_{j(\kappa)})\,(\forall \theta \in \Sigma_{n+2})\,(V_{j(\kappa)} \models \theta(x) \leftrightarrow {\models_{n+2}} \theta(x)),$$
which implies that $V_{j(\kappa)} \preceq_{n+2} V_\mu$.

Since $j(\kappa) \in C(n)$, we have $V_\lambda \preceq_{n+1} V_{j(\kappa)}$, and therefore $V_\lambda \preceq_{n+1} V_\mu$. It follows that $V_\mu \models \exists x\, \varphi(x, B, b)$.

Thus, in V_μ it is true that there exists $X \in V_{j(\kappa)}$ such that $X \in \mathcal{C}$, namely B, and there exists an elementary embedding $e \colon X \to j(B)$, namely $j \upharpoonright B$. Therefore, by elementarity of j, the same is true in V_λ, that is, there exists $X \in V_\kappa$ such that $X \in \mathcal{C}$, and there exists an elementary embedding $e \colon X \to B$. Let $A \in V_\kappa$ be such an X, and let $e \colon A \to B$ be an elementary embedding. Since $\lambda \in C(n+2)$, we have $A \in \mathcal{C}$ and we are done. $\qquad\square$

Corollary 4.5. *If κ is an extendible cardinal, then $\mathrm{SVP}_\kappa(\mathbf{\Sigma_3})$ holds.*

We say that a class $\mathcal{C}$ is $\mathbf{\Sigma_n \wedge \Pi_n}$ if it is definable, with parameters, by a formula that is a conjunction of a Σ_n formula and a Π_n formula. If no parameters are involved, we use lightface types as usual. The notation $\mathrm{VP}^\Sigma(\Gamma)$ and $\mathrm{SVP}_\kappa^\Sigma(\Gamma)$ is used as before, now including the cases $\Gamma = \Sigma_n \wedge \Pi_n$ and $\Gamma = \mathbf{\Sigma_n \wedge \Pi_n}$ as well.

The following theorem yields a converse to Theorem 4.4.

Theorem 4.6. *Let $n \geq 1$, and suppose that $\mathrm{VP}^\Sigma(\Sigma_{n+1} \wedge \Pi_{n+1})$ holds when Σ is the signature of the language of set theory with finitely many additional 1-ary relation symbols. Then there exists a $C(n)$-extendible cardinal.*

Proof. Suppose, to the contrary, that there is no $C(n)$-extendible cardinal. Then the class function F on ordinals given by defining $F(\zeta)$ to be the least $\lambda > \zeta$ such that $\lambda \in C(n)$ and ζ is not λ-$C(n)$-extendible is well defined.

For $\lambda \in C(n)$, the relation "ζ is λ-$C(n)$-extendible" is Σ_{n+1}, for it holds if and only if $\zeta \in C(n)$ and
$$\exists \mu\, \exists j \colon V_\lambda \to V_\mu\, (j \text{ is elementary} \wedge \mathrm{cp}(j) = \zeta \wedge j(\zeta) > \lambda \wedge \mu, j(\zeta) \in C(n)),$$
where $\mathrm{cp}(j)$ denotes the critical point of j. Hence F is $\Sigma_{n+1} \wedge \Pi_{n+1}$.

Let $C = \{\alpha : \alpha$ is a limit ordinal and $\forall \zeta < \alpha \ F(\zeta) < \alpha\}$. So, C is a $\Sigma_{n+1} \wedge \Pi_{n+1}$ closed unbounded proper class.

For each ordinal α, let λ_α be the first limit point of $D = C \cap C(n)$ above α. Let

$$\mathcal{C} = \{\langle V_{\lambda_\alpha}, \in, \{\alpha\}, C \cap \alpha, C(n) \cap \alpha \rangle\}_{\alpha \in D}.$$

So $\mathcal{C}$ is a $\Sigma_{n+1} \wedge \Pi_{n+1}$ proper class of structures of the same type in the language of set theory with three additional relation symbols. By our assumption, there are $\alpha < \beta$ in D and an elementary embedding

$$j \colon \langle V_{\lambda_\alpha}, \in, \{\alpha\}, C \cap \alpha, C(n) \cap \alpha \rangle \longrightarrow \langle V_{\lambda_\beta}, \in, \{\beta\}, C \cap \beta, C(n) \cap \beta \rangle.$$

Since j sends α to β, it is not the identity. Let κ be the critical point of j.

Since $\alpha \in C$, we have $\kappa < F(\kappa) < \alpha$. Thus,

$$j \restriction V_{F(\kappa)} \colon V_{F(\kappa)} \longrightarrow V_{j(F(\kappa))}$$

is elementary, with critical point κ.

We claim that $\kappa \in D$. Otherwise, $\gamma = \sup(D \cap \kappa) < \kappa$. Let δ be the least ordinal in D greater than γ with $\kappa < \delta < \lambda_\alpha$. Since δ is definable from γ in the structure $\langle V_{\lambda_\alpha}, \in, \{\alpha\}, C \cap \alpha, C(n) \cap \alpha \rangle$, and since $j(\gamma) = \gamma$, we must also have $j(\delta) = \delta$. But then $j \restriction V_{\delta+2} \colon V_{\delta+2} \to V_{\delta+2}$ is an elementary embedding, contradicting Kunen's Theorem [27].

By elementarity, $j(\kappa) \in C(n)$. Moreover, since $F(\kappa) \in C(n)$ and $\lambda_\beta \in C(n)$, we have $j(F(\kappa)) \in C(n)$. Since $\kappa \in C$, by elementarity we also have $j(\kappa) \in C$. Hence, $j(\kappa) > F(\kappa)$. This shows that $j \restriction V_{F(\kappa)}$ witnesses that κ is $F(\kappa)$-$C(n)$-extendible, and this contradicts the fact that κ belongs to C. $\quad\square$

The proof of Theorem 4.6 easily generalizes to the boldface case (see the proof of Theorem 3.2), namely if $\mathrm{VP}(\boldsymbol{\Sigma_{n+1}} \wedge \boldsymbol{\Pi_{n+1}})$ holds, then there is a proper class of $C(n)$-extendible cardinals. In fact it is sufficient to assume that $\mathrm{VP}^\Sigma(\boldsymbol{\Sigma_{n+1}} \wedge \boldsymbol{\Pi_{n+1}})$ holds when Σ is the signature of the language of set theory with a finite number of additional 1-ary relation symbols.

The following corollaries summarize our results in this section.

Corollary 4.7. *The following statements are equivalent for $n \geq 1$:*

(1) $\mathrm{SVP}_\kappa(\boldsymbol{\Sigma_{n+2}})$ *holds for some cardinal κ.*
(2) $\mathrm{VP}(\Sigma_{n+1} \wedge \Pi_{n+1})$ *holds.*
(3) $\mathrm{VP}^\Sigma(\Sigma_{n+1} \wedge \Pi_{n+1})$ *holds when Σ is the signature of the language of set theory with a finite number of additional 1-ary relation symbols.*
(4) *There exists a $C(n)$-extendible cardinal.*

Corollary 4.8. *The following statements are equivalent:*

(1) *For every n, $\mathrm{SVP}_\kappa(\boldsymbol{\Sigma_n})$ holds for a proper class of cardinals κ.*
(2) $\mathrm{VP}(\Sigma_n)$ *holds for all n.*
(3) $\mathrm{VP}^\Sigma(\Sigma_n)$ *holds for all n when Σ is the signature of the language of set theory with a finite number of additional 1-ary relation symbols.*
(4) *There exists a $C(n)$-extendible cardinal for every n.*
(5) *Vopěnka's principle holds.*

To prove the latter, note that Vopěnka's principle is indeed equivalent to the statement that $\mathrm{VP}(\boldsymbol{\Sigma_n})$ holds for all n.

5. Definable categories

If $\mathcal{C}$ is a category, in order to simplify formulas we denote by $X \in \mathcal{C}$ the statement that X is an object of $\mathcal{C}$ and by $f \in \mathcal{C}(A, B)$ the statement that A and B are objects of $\mathcal{C}$ and f is a morphism from A to B.

Definition 5.1. A category $\mathcal{C}$ is *definable* with a set of parameters p if there is a formula $\varphi_{\mathcal{C}}(x_1, \ldots, x_7, p)$ of the language of set theory such that the sentence

$$(5.1) \qquad \varphi_{\mathcal{C}}(A, B, C, f, g, h, i, p)$$

is true if and only if $f \in \mathcal{C}(A, B)$, $g \in \mathcal{C}(B, C)$, $h \in \mathcal{C}(A, C)$, $i \in \mathcal{C}(A, A)$, $h = g \circ f$, and $i = \mathrm{id}_A$.

Equivalently, a category $\mathcal{C}$ is definable with a set of parameters p if there are formulas

$$(5.2) \qquad \psi_{\mathrm{Ob}}(x, p), \quad \psi_{\mathrm{Mor}}(x, y, z, p), \quad \psi_{\circ}(x_1, \ldots, x_6, p), \quad \psi_{\mathrm{id}}(x, y, p)$$

such that:

(1) A is an object of $\mathcal{C}$ if and only if $\psi_{\mathrm{Ob}}(A, p)$ is true.
(2) The sentence $\psi_{\mathrm{Mor}}(A, B, f, p)$ is true if and only if $f \in \mathcal{C}(A, B)$.
(3) The sentence $\psi_{\circ}(A, B, C, f, g, h, p)$ is true if and only if $f \in \mathcal{C}(A, B)$, $g \in \mathcal{C}(B, C)$, $h \in \mathcal{C}(A, C)$, and h is the composite of f and g.
(4) The sentence $\psi_{\mathrm{id}}(A, i, p)$ is true if and only if A is an object of $\mathcal{C}$ and i is the identity of A.

This approach is clearer for some purposes, although it is redundant. For example, the formula $\exists i \, \psi_{\mathrm{id}}(x, i, p)$ also defines the class of objects of $\mathcal{C}$. From a single formula $\varphi_{\mathcal{C}}$ as in (5.1) we can obtain each of (5.2) using existential quantifiers. For instance, $\psi_{\mathrm{Mor}}(x, y, z, p)$ can be chosen to be

$$\exists i \, \varphi_{\mathcal{C}}(x, x, y, i, z, z, i, p).$$

We call $\mathcal{C}$ a $\mathbf{\Sigma_n}$ *category* (resp. $\mathbf{\Pi_n}$) if it is definable by a Σ_n formula (resp. Π_n) with parameters, in the sense of Definition 5.1. If $\mathcal{C}$ is given by Σ_n formulas (resp. Π_n formulas) in the sense of (5.2), then it is $\mathbf{\Sigma_n}$ (resp. $\mathbf{\Pi_n}$). Conversely, if $\mathcal{C}$ is $\mathbf{\Sigma_n}$, then there exist Σ_n formulas (5.2) defining it.

A category is said to be $\mathbf{\Delta_n}$ if it is both $\mathbf{\Sigma_n}$ and $\mathbf{\Pi_n}$.

Note that, if A and B are objects of $\mathcal{C}$ and $\mathcal{C}$ is definable with a set of parameters p, then the set $\mathcal{C}(A, B)$ is defined by the formula $\psi_{\mathrm{Mor}}(A, B, x, p)$, where A, B are now additional parameters. The assertion "f is a morphism of $\mathcal{C}$" can thus be formalized as

$$\exists A \, \exists B \, \psi_{\mathrm{Mor}}(A, B, f, p).$$

If $\mathcal{C}$ is $\mathbf{\Sigma_n}$, then "$f \in \mathcal{C}(A, B)$" is $\mathbf{\Sigma_n}$. However, "$X = \mathcal{C}(A, B)$" is formalized with the following $\Sigma_n \wedge \Pi_n$ formula:

$$(\forall f \in X) \, f \in \mathcal{C}(A, B) \, \wedge \, \forall g \, (g \in \mathcal{C}(A, B) \to g \in X).$$

The domain and codomain of a morphism f of $\mathcal{C}$ can be defined as follows, where f is treated as a parameter. The set $\{\mathrm{dom}(f)\}$ is defined by the formula $\exists B \, (f \in \mathcal{C}(x, B))$, and $\{\mathrm{codom}(f)\}$ is defined by $\exists A \, (f \in \mathcal{C}(A, x))$.

For a category $\mathcal{C}$ and an object A of $\mathcal{C}$, the slice category $(\mathcal{C} \downarrow A)$ and the coslice category $(A \downarrow \mathcal{C})$ are definable with the complexity of $\mathcal{C}$ with

an additional unbounded existential quantifier and with A as an additional parameter, since f is an object of $(\mathcal{C} \downarrow A)$ if and only if $\exists X \, (f \in \mathcal{C}(X, A))$, and α is a morphism from f to g in $(\mathcal{C} \downarrow A)$ if and only if

$$\alpha \in \mathcal{C}(\operatorname{dom} f, \operatorname{dom} g) \,\wedge\, g \circ \alpha = f.$$

If $\mathcal{C}$ is a subcategory of the category of sets, then composition and identities in $\mathcal{C}$ are prescribed by those of sets. Hence, suppose that there is a formula $\psi(x, y, z, p)$ such that $\psi(A, B, f, p)$ is true if and only if $f \in \mathcal{C}(A, B)$. Then $\mathcal{C}$ can be defined in the sense of Definition 5.1 by the following formula:

$$\psi(x_1, x_2, x_4, p) \,\wedge\, \psi(x_2, x_3, x_5, p) \,\wedge\, \psi(x_1, x_3, x_6, p)$$
$$\wedge \, x_6 = x_5 \circ x_4 \,\wedge\, x_7 = \operatorname{id}_{x_1}.$$

Therefore, the definition of an absolute subcategory of sets given in [7] is in agreement with our present definition of a subcategory of sets definable by means of a formula $\varphi_{\mathcal{C}}$ that is absolute (for inner models). In particular, all accessible categories are absolute, if viewed as subcategories of **Set** by means of the embedding (2.1).

Many important categories which cannot be embedded into **Set** are nevertheless definable in our sense. For example, the homotopy category of simplicial sets can be defined with a Σ_2 formula, since F is a morphism from X to Y if and only if there exists a simplicial map f from X to a fibrant replacement of Y (see Section 7 for details) such that F is equal to the equivalence class of f under the homotopy relation. The latter can be expressed with a $\Sigma_1 \wedge \Pi_1$ formula, stating that every element of F is a map from X to a fibrant replacement of Y that is homotopic to f and that every such map is an element of F.

Theorem 5.2. *For an uncountable cardinal κ and an S-sorted signature Σ, let $\mathcal{C}$ be a full subcategory of Σ-structures defined by a Σ_1 formula with a set of parameters p. Suppose that p, S and Σ are in $H(\kappa)$. Then every object $B \in \mathcal{C}$ has a subobject $A \in \mathcal{C} \cap H(\kappa)$.*

Proof. This follows from Theorem 3.5, since every elementary embedding of structures is injective and, in a subcategory of sets, every injective morphism is a monomorphism; see [1, Proposition 7.37]. □

The next two theorems hold for more general categories than subcategories of structures and extend Theorem 4.5 in [7]. They are proved in the same way as Theorem 3.1 and Theorem 4.4 above.

Theorem 5.3. *For every supercompact cardinal κ and every $\mathbf{\Sigma_2}$ subcategory $\mathcal{C}$ of sets defined with parameters of rank less than κ and supporting elementary embeddings with critical point κ, every object $B \in \mathcal{C}$ has a subobject $A \in \mathcal{C} \cap V_{\kappa}$.*

Theorem 5.4. *For every $C(n)$-extendible cardinal κ and every $\mathbf{\Sigma_{n+2}}$ subcategory $\mathcal{C}$ of sets defined with parameters of rank less than κ and supporting elementary embeddings with critical point κ, every object $B \in \mathcal{C}$ has a subobject $A \in \mathcal{C} \cap V_{\kappa}$.*

6. Small-orthogonality classes

Recall from Subsection 2.3 that, for a regular cardinal λ, a category $\mathcal{C}$ is called *weakly λ-preaccessible* if it has a small full subcategory $\mathcal{D}$ such that, for every object A of $\mathcal{C}$, the canonical diagram $(\mathcal{D} \downarrow A) \to \mathcal{C}$ is λ-filtered and A is a colimit of it, and we say that $\mathcal{C}$ is *weakly preaccessible* if it is weakly λ-preaccessible for some λ. Every weakly preaccessible category is bounded, and the converse is true under Vopěnka's principle, as shown in [3]. In fact, Vopěnka's principle is equivalent to the statement that each bounded category is preaccessible.

For the next result, we also need to recall that, if a category $\mathcal{C}$ is λ-accessible, then it is also κ-accessible for every regular cardinal κ that is *sharply bigger* than λ in the sense of [31, §2.3]. For any regular cardinal λ, if $\mu \geq \lambda$ is regular, then, as shown in [31, Proposition 2.3.5], $(2^{\mu})^{+}$ is sharply bigger than λ. Hence, for every regular cardinal λ there are arbitrarily large regular cardinals that are sharply bigger than λ. Moreover, if κ is strongly inaccessible and $\kappa > \lambda$, then κ is sharply bigger than λ; see [2, 2.13(4)].

Theorem 6.1. *Let $\mathcal{C}$ be an accessible category. Then:*
   (1) *Every $\mathbf{\Sigma_1}$ full subcategory of $\mathcal{C}$ is weakly preaccessible.*
   (2) *If there is a proper class of supercompact cardinals, then every $\mathbf{\Sigma_2}$ full subcategory of $\mathcal{C}$ is weakly preaccessible.*
   (3) *For $n \geq 1$, if there is a proper class of $C(n)$-extendible cardinals, then every $\mathbf{\Sigma_{n+2}}$ full subcategory of $\mathcal{C}$ is weakly preaccessible.*

Proof. Pick a regular cardinal λ such that $\mathcal{C}$ is λ-accessible. Let $\mathcal{C}_\lambda$ denote the full subcategory of λ-presentable objects in $\mathcal{C}$ and let $\mathcal{A}$ be a set of representatives of all isomorphism classes of objects in $\mathcal{C}_\lambda$. By [2, 2.8], $\mathcal{A}$ is dense in $\mathcal{C}$. Moreover, $(\mathcal{A} \downarrow X)$ is λ-filtered for every object X in $\mathcal{C}$, and the embedding $\mathcal{C} \to \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ described in (2.1) preserves λ-filtered colimits.

We assume, as we may, that $\mathcal{C} = \mathbf{Mod}\,T$ for some basic theory T with S-sorted signature Σ. Let $\mathcal{S}$ be a full subcategory of $\mathcal{C}$. Choose a formula $\varphi(x,p)$ defining $\mathcal{S}$, and suppose that this formula is Σ_1 in case (1), Σ_2 in case (2), and Σ_{n+2} with $n \geq 1$ in case (3). Choose a regular cardinal κ such that p, S, Σ, T and $\mathcal{A}$ are in $H(\kappa)$, and κ is sharply bigger than λ. Moreover, in case (2) choose κ supercompact, and in case (3) choose it $C(n)$-extendible.

Consider the small full subcategory $\mathcal{D} = \mathcal{S} \cap H(\kappa)$ of $\mathcal{S}$. We aim at proving that every object X of $\mathcal{S}$ is a colimit of the canonical diagram $(\mathcal{D} \downarrow X) \to \mathcal{S}$, and moreover that $(\mathcal{D} \downarrow X)$ is κ-filtered. For this, note first that, since the embedding $\mathcal{C} \to \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ preserves λ-filtered colimits (hence κ-filtered colimits as well), and colimits in $\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ are computed objectwise, every object X of $\mathcal{C}$ whose transitive closure has cardinality smaller than κ is κ-presentable in $\mathcal{C}$. Therefore, all objects in $\mathcal{D}$ are κ-presentable in $\mathcal{C}$ (yet possibly not in $\mathcal{S}$, since the inclusion of $\mathcal{S}$ into $\mathcal{C}$ need not preserve κ-filtered colimits).

Let $\mathcal{C}_\kappa$ denote the full subcategory of all κ-presentable objects of $\mathcal{C}$, and let X be any object of $\mathcal{S}$. Since κ is sharply bigger than λ and hence $\mathcal{C}$ is κ-accessible, we know that X is a colimit of the canonical diagram $(\mathcal{C}_\kappa \downarrow X) \to \mathcal{C}$, which is κ-filtered, by [2, 2.8]. Therefore, if we prove that $(\mathcal{D} \downarrow X)$ is *cofinal* in $(\mathcal{C}_\kappa \downarrow X)$, it will then follow that X is a colimit of the

canonical diagram $(\mathcal{D} \downarrow X) \to \mathcal{C}$, and that $(\mathcal{D} \downarrow X)$ is κ-filtered. Moreover, since X is in $\mathcal{S}$, we will be able to conclude that X is a colimit of the canonical diagram $(\mathcal{D} \downarrow X) \to \mathcal{S}$, as we wanted to show.

Thus, towards proving that $(\mathcal{D} \downarrow X)$ is cofinal in $(\mathcal{C}_\kappa \downarrow X)$, let A be any object of $\mathcal{C}_\kappa$ and let a morphism $f \colon A \to X$ be given. By [31, Proposition 2.3.11], A is a λ-filtered colimit of λ-presentable objects indexed by a category with less than κ morphisms. Since the set $\mathcal{A}$ is in $H(\kappa)$, it follows that $A \in H(\kappa)$ as well.

Now the coslice category $(A \downarrow \mathcal{S})$ can be defined by a formula with the same complexity as the formula $\varphi(x, p)$ chosen above defining $\mathcal{S}$, since we are only adding an unbounded existential quantifier, together with A as an additional parameter. Thus, Theorem 5.2 in case (1), Theorem 5.3 in case (2), or Theorem 5.4 in case (3) tell us that $f \colon A \to X$ has a subobject $f' \colon A \to A'$ where $A' \in H(\kappa)$ and hence $A' \in \mathcal{D}$. (Recall that $H(\kappa) = V_\kappa$ if κ is strongly inaccessible.) Let $i \colon A' \to X$ be the inclusion:

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & X \\
& \searrow{\scriptstyle f'} & \nearrow{\scriptstyle i} \\
& A' &
\end{array}
$$

Thus f' can also be viewed as a morphism from $f \colon A \to X$ to $i \colon A' \to X$ in $(\mathcal{C}_\kappa \downarrow X)$. Since $(\mathcal{C}_\kappa \downarrow X)$ is filtered, this is sufficient to infer that $(\mathcal{D} \downarrow X)$ is cofinal in $(\mathcal{C}_\kappa \downarrow X)$, as we wanted to show. $\qquad\square$

Corollary 6.2. *If there is a proper class of supercompact cardinals, then every accessible category is co-wellpowered.*

Proof. Our argument to prove this fact is similar to the one used in the proof of [2, Theorem 2.49]. Let $\mathcal{C}$ be accessible and let X be any object of $\mathcal{C}$. Let $\mathcal{E}_X$ be the full subcategory of $(X \downarrow \mathcal{C})$ whose objects are the epimorphisms, and let $\bar{\mathcal{E}}_X$ be a skeleton of $\mathcal{E}_X$, i.e., a full subcategory with a representative of each isomorphism class of objects in $\mathcal{E}_X$. Then $\bar{\mathcal{E}}_X$ is partially ordered, since between any two of its objects there is at most one morphism. Now observe that $(X \downarrow \mathcal{C})$ is $\mathbf{\Sigma_1}$ and $\mathcal{E}_X$ is $\mathbf{\Sigma_2}$, since $f \in \mathcal{E}_X$ if and only if

$$\exists Y \, [f \in \mathcal{C}(X, Y) \, \wedge \, \forall Z \, \forall g \, \forall h$$
$$((g \in \mathcal{C}(Y, Z) \, \wedge \, h \in \mathcal{C}(Y, Z) \, \wedge \, g \circ f = h \circ f) \, \to \, g = h)].$$

Therefore, part (2) of Theorem 6.1 implies that $\mathcal{E}_X$ is bounded. Hence $\bar{\mathcal{E}}_X$ is also bounded, and every bounded partially ordered category is small. $\quad\square$

On the other hand, as shown in [2, A.19], if each accessible category is co-wellpowered then there exists a proper class of measurable cardinals. Therefore, the statement that every accessible category is co-wellpowered is set-theoretical. Its precise consistency strength is not known; see [2, Open Problem 11]. By part (i) of Theorem 6.3.8 in [31], together with the fact that categories of epimorphisms can be sketched by a pushout sketch (as done in [2, p. 101]), the statement that every accessible category is co-wellpowered is implied by the existence of a proper class of compact cardinals, a large-cardinal assumption that is not known to be weaker, consistency-wise, than the existence of a proper class of supercompact cardinals.

In order to simplify the statements of several corollaries of Theorem 6.1, we use from now on the following terminology.

Definition 6.3. We say that a class $\mathcal{C}$ is *definable with sufficiently low complexity* if either of the following conditions is satisfied:

(1) $\mathcal{C}$ is $\mathbf{\Sigma_1}$.
(2) There is a proper class of supercompact cardinals and $\mathcal{C}$ is $\mathbf{\Sigma_2}$.
(3) There is a proper class of $C(n)$-extendible cardinals for some $n \geq 1$ and $\mathcal{C}$ is $\mathbf{\Sigma_{n+2}}$.

By Corollary 4.8, if Vopěnka's principle holds, then all classes are definable with sufficiently low complexity.

An object X and a morphism $f \colon A \to B$ in a category $\mathcal{C}$ are called *orthogonal* if the function

$$\mathcal{C}(f, X) \colon \mathcal{C}(B, X) \longrightarrow \mathcal{C}(A, X)$$

is bijective. That is, X and f are orthogonal if and only if for every morphism $g \colon A \to X$ there is a unique morphism $h \colon B \to X$ such that $h \circ f = g$.

For a class of objects $\mathcal{X}$, we denote by $^{\perp}\mathcal{X}$ the class of morphisms that are orthogonal to all the objects of $\mathcal{X}$. Similarly, for a class of morphisms $\mathcal{F}$, we denote by $\mathcal{F}^{\perp}$ the class of objects that are orthogonal to all the morphisms of $\mathcal{F}$. Classes of objects of the form $\mathcal{F}^{\perp}$ are called *orthogonality classes*, and, if $\mathcal{F}$ is a set (not a proper class), then $\mathcal{F}^{\perp}$ is called a *small-orthogonality class*. In what follows, we keep confusing a class of objects and the full subcategory with those objects.

Note that, if $\mathcal{D}$ is dense in $\mathcal{F}$, then $\mathcal{D}^{\perp} = \mathcal{F}^{\perp}$. To prove this claim, only the inclusion $\mathcal{D}^{\perp} \subseteq \mathcal{F}^{\perp}$ needs to be checked. Given an object $X \in \mathcal{D}^{\perp}$ and any morphism $f \in \mathcal{F}$, we may write $f = \operatorname{colim}_{i \in I} d_i$ for a small indexing category I and with every d_i in $\mathcal{D}$. Then $\mathcal{C}(f, X) = \lim_{i \in I} \mathcal{C}(d_i, X)$ is a bijection, so $X \in \mathcal{F}^{\perp}$, as needed.

If $\mathcal{S}$ is a full subcategory of a category $\mathcal{C}$, then the class of morphisms $^{\perp}\mathcal{S}$ can be defined as follows: $f \in {}^{\perp}\mathcal{S}$ if and only if

$$
(6.1) \quad
\begin{aligned}
&\forall X \, \forall g \, [(X \in \mathcal{S} \,\wedge\, g \in \mathcal{C}(\operatorname{dom}(f), X)) \to \\
&\exists h \in \mathcal{C}(\operatorname{codom}(f), X) \, (h \circ f = g \,\wedge\, \text{such an } h \text{ is unique})].
\end{aligned}
$$

Note that $g \in \mathcal{C}(\operatorname{dom}(f), X)$ can be restated as

$$(6.2) \quad \exists A \, \exists B \, (f \in \mathcal{C}(A, B) \,\wedge\, g \in \mathcal{C}(A, X)),$$

and recall that $P \to Q$ means $\neg(P \wedge \neg Q)$, or $\neg P \vee Q$. Therefore, (6.1) is at least Π_2, and it is Π_n if $\mathcal{S}$ is $\mathbf{\Sigma_n}$ with $n \geq 2$.

Theorem 6.4. *Assume the existence of a proper class of $C(n)$-extendible cardinals, where $n \geq 1$. Then each $\mathbf{\Sigma_{n+1}}$ orthogonality class in an accessible category $\mathcal{C}$ is a small-orthogonality class.*

Proof. We assume, as we may, that $\mathcal{C} = \mathbf{Mod}\, T$ for some basic theory T with S-sorted signature Σ. Let $\mathcal{S}$ be a full subcategory of $\mathcal{C}$ whose objects form a $\mathbf{\Sigma_{n+1}}$ orthogonality class. Thus $\mathcal{S} = \mathcal{F}^{\perp}$ for some $\mathcal{F}$, and this implies that

$$({}^{\perp}\mathcal{S})^{\perp} = ({}^{\perp}(\mathcal{F}^{\perp}))^{\perp} = \mathcal{F}^{\perp} = \mathcal{S}.$$

From (6.1) and (6.2) we infer that the class $^\perp\mathcal{S}$ is $\mathbf{\Pi_{n+1}}$ (in the parameters of some Σ_{n+1} definition of $\mathcal{S}$, plus S, Σ and T) if $n \geq 1$, and it is $\mathbf{\Pi_2}$ if $n = 0$. By part (3) of Theorem 6.1, $^\perp\mathcal{S}$ is bounded. Let $\mathcal{D}$ be a small dense full subcategory of $^\perp\mathcal{S}$. Then $\mathcal{D}^\perp = (^\perp\mathcal{S})^\perp = \mathcal{S}$, so $\mathcal{S}$ is a small-orthogonality class. $\square$

This result can be sharpened so as to yield the following improvement of [7, Corollary 4.6], where the assumption that L be an epireflection, made in [7], is no longer necessary. A *reflection* on a category is a left adjoint (when it exists) of the inclusion of a full subcategory [29], which is then called *reflective*. For example, in the category of groups, the abelianization functor is a reflection onto the reflective full subcategory of commutative groups. For every reflection L, the closure under isomorphisms of its image is an orthogonality class, and it is in fact orthogonal to the class of *L-equivalences*, i.e., morphisms f such that Lf is an isomorphism.

A reflection L is called an *$\mathcal{F}$-reflection*, where $\mathcal{F}$ is a set or a proper class of morphisms, if the closure under isomorphisms of the image of L is equal to $\mathcal{F}^\perp$. This notion is particularly relevant when $\mathcal{F}$ can be chosen to be a set (or even better a single morphism). In the previous example, abelianization is an f-reflection where f is the canonical projection of a free group on two generators onto a free abelian group on two generators, since the groups orthogonal to f are precisely the commutative groups.

Corollary 6.5. *Let L be a reflection on an accessible category $\mathcal{C}$. Then L is an $\mathcal{F}$-reflection for some set $\mathcal{F}$ of morphisms under either of the following assumptions:*

(1) *The class of L-equivalences is definable with sufficiently low complexity.*
(2) *The class of objects isomorphic to LX for some X is $\mathbf{\Sigma_{n+1}}$ for $n \geq 1$ and there is a proper class of $C(n)$-extendible cardinals.*

Proof. To prove case (1), let $\mathcal{S}$ be the full subcategory of L-equivalences in the category of arrows of $\mathcal{C}$. Since the category of arrows of $\mathcal{C}$ is also accessible, it follows from Theorem 6.1 that $\mathcal{S}$ is bounded. Choose a small full subcategory $\mathcal{F}$ which is dense in $\mathcal{S}$. Then $\mathcal{S}^\perp = \mathcal{F}^\perp$, as needed. Case (2) follows as a special case of Theorem 6.4. $\square$

As already shown in [15, Theorem 6.3], the assertion that every reflection on an accessible category is an $\mathcal{F}$-reflection for some set $\mathcal{F}$ of morphisms cannot be proved in ZFC. Specifically, if one assumes that measurable cardinals do not exist and considers reflection on the category of groups with respect to the class $\mathcal{Z}$ of homomorphisms of the form $\mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa} \to \{0\}$, where κ runs over all cardinals, then there is no set $\mathcal{F}$ of group homomorphisms such that $\mathcal{F}$-reflection coincides with $\mathcal{Z}$-reflection. This class $\mathcal{Z}$ is $\mathbf{\Sigma_2}$, according to (2.3). This example was also discussed in [7].

Corollary 6.6. *If $\mathcal{C}$ is a locally presentable category, then every full limit-closed subcategory $\mathcal{S}$ definable with sufficiently low complexity is reflective.*

Proof. By Theorem 6.1, for each X in $\mathcal{C}$ the category $(X \downarrow \mathcal{S})$, viewed as a full subcategory of the locally presentable category $(X \downarrow \mathcal{C})$, is bounded if $\mathcal{S}$ is definable with sufficiently low complexity. Thus there is a set $\mathcal{F}$ of

morphisms in $(X \downarrow \mathcal{S})$ such that each $f \colon X \to Y$ with Y in $\mathcal{S}$ can be written as $f = \mathrm{colim}_{i \in I} f_i$ for some small indexing category I, with $f_i \colon X \to Z_i$ in $\mathcal{F}$ for all i. This implies that f factors through f_i for each i. Hence the inclusion $\mathcal{S} \hookrightarrow \mathcal{C}$ satisfies the solution-set condition for every X in $\mathcal{C}$, as required in the Freyd Adjoint Functor Theorem [29, V.6], from which the existence of a reflection of $\mathcal{C}$ onto $\mathcal{S}$ follows. $\qquad \square$

The following result is a further improvement, since, if $\mathcal{S}$ is $\boldsymbol{\Sigma_1}$, then $\mathcal{S}^{\perp}$ is $\boldsymbol{\Pi_2}$; yet, as we next show, if $\mathcal{S}$ is $\boldsymbol{\Sigma_1}$, then the reflectivity of $\mathcal{S}^{\perp}$ is provable in ZFC. This yields, in particular, a solution of the Freyd–Kelly orthogonal subcategory problem [19] in ZFC for $\boldsymbol{\Sigma_1}$ classes.

Corollary 6.7. *Let $\mathcal{S}$ be any class of morphisms in a locally presentable category. If $\mathcal{S}$ is definable with sufficiently low complexity, then $\mathcal{S}^{\perp}$ is reflective.*

Proof. Theorem 6.1 ensures that $\mathcal{S}$ is bounded. Thus there is a set $\mathcal{F} \subseteq \mathcal{S}$ such that $\mathcal{F}^{\perp} = \mathcal{S}^{\perp}$, from which the reflectivity of $\mathcal{S}^{\perp}$ follows, since small-orthogonality classes are reflective in a locally presentable category. $\qquad \square$

If we weaken the assumption that $\mathcal{S}$ is closed under limits in Corollary 6.6, by imposing only that it is closed under products and retracts, then we may infer similarly that $\mathcal{S}$ is weakly reflective, under the hypotheses made in the statement. On the other hand, it is shown in [13] that, assuming the nonexistence of measurable cardinals, there is a $\boldsymbol{\Sigma_2}$ full subcategory $\mathcal{S}$ of the category of abelian groups which is closed under products and retracts but not weakly reflective. Specifically, $\mathcal{S}$ is the closure of the class of groups $\mathbb{Z}^{\kappa}/\mathbb{Z}^{<\kappa}$ under products and retracts, where κ runs over all cardinals. Hence, the statement that all $\boldsymbol{\Sigma_2}$ full subcategories closed under products and retracts in locally presentable categories are weakly reflective implies the existence of measurable cardinals, while it follows from the existence of supercompact cardinals.

Theorem 6.8. *Every full colimit-closed subcategory definable with sufficiently low complexity in a locally presentable category is coreflective.*

Proof. Argue as in [2, 6.28]. $\qquad \square$

7. Consequences in homotopy theory

Hovey conjectured in [23] that for every cohomology theory (defined on spectra) there is a homology theory with the same acyclics. This conjecture remains so far unsolved. In a different but closely related direction, the existence of cohomological localizations is also an open problem in ZFC, although it is known that it follows from Vopěnka's principle, both in unstable homotopy and in stable homotopy, by [15] and [12, Theorem 1.5].

Motivated by these problems, in this section we compare homological acyclic classes with cohomological acyclic classes from the point of view of complexity of their definitions. We consider homology theories and cohomology theories defined on simplicial sets and represented by spectra.

Spectra will be meant in the sense of Bousfield–Friedlander [11]. Thus, a spectrum E consists of a sequence of pointed simplicial sets (E_n, p_n) for

$0 \leq n < \omega$ together with pointed simplicial maps $\sigma_n \colon SE_n \to E_{n+1}$ for all n. Here S denotes *suspension*, i.e., $SX = \mathbb{S}^1 \wedge X$. For $m \geq 1$, we denote by $\mathbb{S}^m$ the simplicial m-sphere, namely $\mathbb{S}^m = \Delta[m]/\partial\Delta[m]$, where $\Delta[m]$ is the standard m-simplex and $\partial\Delta[m]$ is its boundary. For pointed simplicial sets X and Y, the *smash product* $X \wedge Y$ is the quotient of the product $X \times Y$ by the one-point union $X \vee Y$, and we denote by $\mathrm{map}_*(X, Y)$ the *pointed function complex* from X to Y, whose n-simplices are the pointed maps $X \wedge \Delta[n]_+ \to Y$, where the subscript $+$ denotes a disjoint basepoint.

A simplicial set is *fibrant* if it is a Kan complex [26]. For the purposes of this article, it will be convenient to use Kan's Ex^∞ construction as a fibrant replacement functor. Thus, there is a natural (injective) weak equivalence $j_Y \colon Y \hookrightarrow \mathrm{Ex}^\infty Y$ for all Y, where $\mathrm{Ex}^\infty Y$ is fibrant.

Let $[X, Y]$ denote the set of morphisms from X to Y in the pointed homotopy category of simplicial sets, which can be described as the set of pointed homotopy classes of maps $X \to \mathrm{Ex}^\infty Y$. If Y is fibrant, then this is in bijective correspondence, via j_Y, with the set of pointed homotopy classes of maps $X \to Y$.

A spectrum E is an Ω-*spectrum* if each E_n is fibrant and the adjoints $\tau_n \colon E_n \to \Omega E_{n+1}$ of the structure maps $\sigma_n \colon SE_n \to E_{n+1}$ are weak equivalences, where Ω denotes the *loop space* functor $\Omega X = \mathrm{map}_*(\mathbb{S}^1, X)$.

Each spectrum E defines a reduced homology theory E_* on simplicial sets by

$$(7.1) \qquad E_k(X) = \mathrm{colim}_n \, \pi_{n+k}(X \wedge E_n) = \mathrm{colim}_n \, [\mathbb{S}^{n+k}, X \wedge E_n]$$

for $k \in \mathbb{Z}$, and, if E is an Ω-spectrum, then E defines a reduced cohomology theory E^* on simplicial sets by

$$(7.2) \qquad E^k(X) = \mathrm{colim}_n \, \pi_{n-k}(\mathrm{map}_*(X, E_n)) = \mathrm{colim}_n \, [S^n X, E_{n+k}]$$

for $k \in \mathbb{Z}$. Note that, if $k \geq 0$, then simply $E^k(X) \cong [X, E_k]$.

Such homology or cohomology theories are called *representable*, and we will only consider these in this article. Although not every generalized homology or cohomology theory in the sense of Eilenberg–Steenrod is representable [36, Example II.3.17], homological localizations have only been constructed and studied assuming representability [5], [9]. According to Brown's representability theorem, every cohomology theory which is *additive* (i.e., sending coproducts to products) is represented by some Ω-spectrum. Similarly, homology theories that preserve filtered colimits are representable. See [4] or [36] for further details.

In most of what follows, we assume that E is an Ω-spectrum. A simplicial set X is called E_*-*acyclic* if $E_k(X) = 0$ for all $k \in \mathbb{Z}$, and, similarly, X is E^*-*acyclic* if $E^k(X) = 0$ for all $k \in \mathbb{Z}$. Observe that, by (7.2), the statement that X is E^*-acyclic is equivalent to the statement that the pointed function complex $\mathrm{map}_*(X, E_n)$ is weakly contractible (i.e., it is connected and its homotopy groups are zero) for all n.

A map $f \colon X \to Y$ is an E_*-*equivalence* if

$$E_k(f) \colon E_k(X) \longrightarrow E_k(Y)$$

is an isomorphism of abelian groups for all $k \in \mathbb{Z}$, and similarly for cohomology. Let Cf denote the *mapping cone* of f. Thus, Cf is obtained from

the disjoint union of Y and $X \times \Delta[1]$ by identifying $X \times \{0\}$ with $f(X) \subseteq Y$ using f, and collapsing $X \times \{1\}$ to a point. Then, by the Mayer–Vietoris axiom, f is an E_*-equivalence if and only if Cf is E_*-acyclic, and analogously for cohomology.

The category of simplicial sets is $\mathbf{\Delta_0}$, since it is locally presentable (see Section 2). In fact, it is easy to write down explicitly a formula without unbounded quantifiers and with the ordinal ω as a parameter which is equivalent to the statement that X and Y are simplicial sets and f is a simplicial map from X to Y. This amounts to formalizing the claim that a simplicial set X is a sequence of sets $\langle X_n \rangle_{n < \omega}$ (where the elements of X_n are called n-*simplices*), together with functions $d_i^n \colon X_n \to X_{n-1}$ (called *faces*) for $n \geq 1$ and $0 \leq i \leq n$, and functions $s_i^n \colon X_n \to X_{n+1}$ (called *degeneracies*) for $n \geq 0$ and $0 \leq i \leq n$, satisfying the simplicial identities; see [32, Definition 1.1]. A simplicial map $f \colon X \to Y$ is a sequence of functions $\langle f_n \colon X_n \to Y_n \rangle_{n < \omega}$ compatible with faces and degeneracies.

Similarly, the category of Bousfield–Friedlander spectra is $\mathbf{\Delta_0}$, since a spectrum E consists of a sequence of pointed simplicial sets $\langle (E_n, p_n) \rangle_{n < \omega}$, where $p_n \in (E_n)_0$, and a sequence of pointed maps $\langle \sigma_n \colon SE_n \to E_{n+1} \rangle_{n < \omega}$. A pointed map $SE_n \to E_{n+1}$ can be defined as a map $\Delta[1] \times E_n \to E_{n+1}$ sending $\partial\Delta[1] \times E_n$ and $\Delta[1] \times p_n$ to the basepoint p_{n+1}.

Proposition 7.1. *The following are* $\mathbf{\Delta_0}$ *classes:*

(1) *Fibrant simplicial sets.*
(2) *Weak equivalences of simplicial sets.*
(3) *Weakly contractible spectra.*
(4) Ω-*spectra.*

Proof. The assertion that a given simplicial set X is fibrant can be formalized by means of the Kan extension condition, as in [32, Definition 1.3]. Explicitly, a simplicial set X is fibrant if and only if for every $1 \leq n < \omega$ and every $k \leq n+1$, the following sentence holds: For all $x_0, x_1, \ldots, x_{n+1} \in X_n$ such that $d_i^n x_j = d_{j-1}^n x_i$ for $i < j$, $i \neq k$ and $j \neq k$, there exists $x \in X_{n+1}$ such that $d_i^{n+1} x = x_i$ for $i \neq k$. Hence, fibrant simplicial sets are models of a certain theory.

To make the latter claim precise, let us expand the language of set theory to a finitary ω-sorted language (see Subsection 2.1) by adding an operation symbol $\bar{d}_i^n$ of arity $n \to n-1$ for every $1 \leq n < \omega$ and $0 \leq i \leq n$, and an operation symbol $\bar{s}_i^n$ of arity $n \to n+1$ for every $0 \leq n < \omega$ and $0 \leq i \leq n$. Let $\varphi_{n,k}$ be the following formula of this expanded formal language:

$$\forall x_0, x_1, \ldots, x_{n+1} \left(\bigwedge_{i < j;\, i,j \neq k} \bar{d}_i^n x_j = \bar{d}_{j-1}^n x_i \ \to \ \exists x \bigwedge_{i \neq k} \bar{d}_i^{n+1} x = x_i \right),$$

where $x_0, x_1, \ldots, x_{n+1}$ are variables of sort n and x is a variable of sort $n+1$. Let T consist of the collection of sentences $\bigwedge_{k \leq n+1} \varphi_{n,k}$ for $1 \leq n < \omega$ together with the simplicial identities among the operation symbols $\bar{d}_i^n$ and $\bar{s}_i^n$. Then a fibrant simplicial set is an ω-sorted structure

$$\langle \langle X_n \rangle_{0 \leq n < \omega},\ \langle d_i^n \rangle_{i \leq n;\, 1 \leq n < \omega},\ \langle s_i^n \rangle_{i \leq n;\, 0 \leq n < \omega} \rangle$$

satisfying all the sentences of T. This proves that the class of fibrant sim-plicial sets is $\boldsymbol{\Delta_0}$ in the parameters ω and T.

Towards (2), recall that a map of simplicial sets $f\colon X \to Y$ is a weak equivalence if and only if it induces a bijection of connected components and isomorphisms of homotopy groups for every choice of a basepoint. Let us assume first that X and Y are fibrant. Then f induces a bijection of connected components if and only if, for all x_0 and x_1 of X_0, if there exists $v \in Y_1$ with $d_0^1 v = f(x_0)$ and $d_1^1 v = f(x_1)$, then there exists $u \in X_1$ with $d_0^1 u = x_0$ and $d_1^1 u = x_1$, and moreover for each $y \in Y_0$ there exist $x \in X_0$ and $v \in Y_1$ such that $d_0^1 v = y$ and $d_1^1 v = f(x)$. Hence, the statement that f induces a bijection of connected components is $\boldsymbol{\Delta_0}$.

Similarly, if a simplicial set X is fibrant, then the nth homotopy group $\pi_n(X, p)$ with basepoint $p \in X_0$ is the quotient of the set of all $x \in X_n$ such that $d_i^n x = sp$ for all i (where $s = s_{n-2}^{n-2} \circ \cdots \circ s_0^0$) by the homotopy relation, where $x \sim x'$ if $d_i^n x = d_i^n x'$ for all i and there exists $z \in X_{n+1}$ with $d_{n+1}^{n+1} z = x$, $d_n^{n+1} z = x'$, and $d_i^{n+1} z = s_{n-1} d_i^n x$ for $0 \le i < n$; cf. [32, Definition 3.1]. Therefore, if X and Y are fibrant, then f induces an isomorphism $\pi_n(X, p) \cong \pi_n(Y, q)$, where $p \in X_0$ and $q = f(p)$, if and only if the following sentence holds:

$$\forall y \in Y_n \left[\forall i \le n \left(d_i^n y = sq \right) \to \left[\exists x \in X_n \left(\forall i \le n \left(d_i^n x = sp \right) \wedge \right. \right. \right.$$

$$f_n(x) \sim y \wedge \forall x' \in X_n \left(\left(\forall i \le n \left(d_i^n x' = sp \right) \wedge f_n(x') \sim y \right) \to x \sim x' \right) \right) \right].$$

This shows that the statement that a map between *fibrant* simplicial sets is a weak equivalence is $\boldsymbol{\Delta_0}$.

Next we analyze the complexity of a fibrant replacement. For a simplicial set X, the map $j_X\colon X \hookrightarrow \mathrm{Ex}^\infty X$ can be defined as the inclusion of X into a simplicial set $\mathrm{Ex}^\infty X$ defined as follows. Let $\mathrm{Ex}^1 X$ be the simplicial set whose set of n-simplices is the set of all maps from the barycentric subdivision of $\Delta[n]$ into X. The barycentric subdivision $\mathrm{sd}\,\Delta[n]$ is the nerve of the poset of non-degenerate simplices of $\Delta[n]$ (see [21, Ch. III, §4]). The *last vertex map* $\mathrm{sd}\,\Delta[n] \to \Delta[n]$ yields an inclusion $X \hookrightarrow \mathrm{Ex}^1 X$. Then $\mathrm{Ex}^\infty X$ is the union of a sequence of inclusions $\mathrm{Ex}^k X \hookrightarrow \mathrm{Ex}^{k+1} X$ for $k \ge 1$, where Ex^k is the composite of Ex^1 with itself k times.

Let p be any vertex of X. Each element in $\pi_n(\mathrm{Ex}^\infty Y, f(p))$ is represented by a map $\mathbb{S}^n \to \mathrm{Ex}^k Y$ based at $f(p)$ for some $k < \omega$, that is, a map from $\Delta[n]$ to $\mathrm{Ex}^k Y$ sending the boundary of $\Delta[n]$ to $f(p)$. By adjointness, the maps $\Delta[n] \to \mathrm{Ex}^k Y$ correspond bijectively with the maps $\mathrm{sd}^k \Delta[n] \to Y$, where sd^k is an iterated barycentric subdivision. Let $a_{k,n}$ be the number of non-degenerate n-simplices of $\mathrm{sd}^k \Delta[n]$ and let $R_{k,n}$ be the set of all relations among their faces. For example, $a_{2,1} = 4$ and $R_{2,1}$ consists of the equalities

$$d_1^1 x_{(0 \to 001)} = d_1^1 x_{(01 \to 001)}, \qquad d_0^1 x_{(01 \to 001)} = d_0^1 x_{(01 \to 011)},$$

$$d_1^1 x_{(01 \to 011)} = d_1^1 x_{(1 \to 011)}.$$

Thus, each map $\Delta[n] \to \mathrm{Ex}^k Y$ is determined by a sequence of $a_{k,n}$ (not necessarily distinct) elements of Y_n satisfying a set $R_{k,n}$ of equalities among their faces. In what follows, when we write "a map $\beta\colon \mathbb{S}^n \to \mathrm{Ex}^k Y$" we implicitly formalize it as an ordered sequence of $a_{k,n}$ elements of Y_n satisfying

a set $S_{k,n}$ of sentences, including those of $R_{k,n}$ and those needed to express the fact that $\partial\Delta[n]$ is sent to the basepoint $f(p)$. Homotopies into $\mathrm{Ex}^k Y$ are formalized similarly.

The assertion that $f\colon X \to Y$ induces $\pi_n(\mathrm{Ex}^\infty X, p) \cong \pi_n(\mathrm{Ex}^\infty Y, f(p))$ for every $p \in X_0$ can therefore be expressed with a formula ψ_n (of an infinitary language, allowing countably many conjunctions and disjunctions) stating that for every $k < \omega$ and every map $\beta\colon \mathbb{S}^n \to \mathrm{Ex}^k Y$ based at $f(p)$ there exist $l < \omega$ and a map $\alpha\colon \mathbb{S}^n \to \mathrm{Ex}^l X$ based at p and a homotopy $H\colon \mathbb{S}^n \wedge \Delta[1]_+ \to \mathrm{Ex}^r Y$ from $(\mathrm{Ex}^r f) \circ \alpha$ to β, where $r \geq k$ and $r \geq l$, and, moreover, if $\alpha'\colon \mathbb{S}^n \to \mathrm{Ex}^m X$ is based at p and there is a homotopy from $(\mathrm{Ex}^r f) \circ \alpha'$ to β with $r \geq k$ and $r \geq m$, then there is a homotopy $H\colon \mathbb{S}^n \wedge \Delta[1]_+ \to \mathrm{Ex}^s X$ from α to α' with $s \geq l$ and $s \geq m$. Therefore, analogously as in the proof of part (1), it follows that weak equivalences between simplicial sets are $\mathbf{\Delta_0}$, where the sequence $\langle \psi_n \rangle_{n<\omega}$ is used as a parameter.

Having proved (1) and (2), we next address (3). A spectrum F is weakly contractible if and only if all its homotopy groups vanish, that is,

$$\mathrm{colim}_n \, [\mathbb{S}^{n+k}, F_n] = 0 \text{ for all } k \in \mathbb{Z}.$$

This is equivalent to imposing that, for all $k \in \mathbb{Z}$ and $n \geq 0$ such that $n + k \geq 0$, each pointed map $\beta\colon \mathbb{S}^{n+k} \to \mathrm{Ex}^\infty F_n$ becomes nullhomotopic after suspending it a finite number of times (say, m times) and composing with the structure maps $\sigma_n\colon SF_n \to F_{n+1}$. More precisely, on the one hand, we have:

$$(7.3) \qquad \mathbb{S}^{n+m+k} \xrightarrow{\; S^m\beta \;} S^m \mathrm{Ex}^\infty F_n \xrightarrow{\; j \;} \mathrm{Ex}^\infty S^m \mathrm{Ex}^\infty F_n,$$

and, on the other hand, there are maps

$$\mathrm{Ex}^\infty S^m \mathrm{Ex}^\infty F_n \xleftarrow{\; \mathrm{Ex}^\infty S^m j \;} \mathrm{Ex}^\infty S^m F_n \xrightarrow{\; \mathrm{Ex}^\infty \sigma \;} \mathrm{Ex}^\infty F_{n+m},$$

where σ is an abbreviation for $\sigma_{n+m-1} \circ S\sigma_{n+m-2} \circ \cdots \circ S^{m-2}\sigma_{n+1} \circ S^{m-1}\sigma_n$. The maps j and $\mathrm{Ex}^\infty S^m j$ are natural weak equivalences.

Hence, F is weakly contractible if and only if, for each $k \in \mathbb{Z}$ and each $(n + k)$-simplex $x \in \mathrm{Ex}^\infty F_n$ whose faces are the basepoint, there is an $(n + m + k)$-simplex $y \in \mathrm{Ex}^\infty S^m F_n$ whose faces are the basepoint and an $(n + m + k + 1)$-simplex $z \in \mathrm{Ex}^\infty F_{n+m}$ whose top face is y and all its other faces are equal to the basepoint, and $(\mathrm{Ex}^\infty S^m j)y \sim j(S^m x)$.

We finally prove (4). In order to formalize the fact that a spectrum E is an Ω-spectrum, we first need that each simplicial set E_n be fibrant. Then we need to define the adjoint maps $\tau_n\colon E_n \to \Omega E_{n+1}$ and we need to impose that each τ_n be a weak equivalence. To define τ_n, let x be a k-simplex of E_n. Its image in $\Omega E_{n+1} = \mathrm{map}_*(\mathbb{S}^1, E_{n+1})$ is a map $\mathbb{S}^1 \wedge \Delta[k]_+ \to E_{n+1}$ which is determined by imposing that

$$(\tau_n(x))(se_1, e_k) = \sigma_n(se_1, x),$$

where e_1 is the non-degenerate 1-simplex of $\mathbb{S}^1$ and e_k is the non-degenerate k-simplex of $\Delta[k]$, and s denotes a composition of degeneracies. $\qquad \square$

In what follows, let us denote by $\mathbf{sSet}_*$ the category of pointed simplicial sets and pointed maps.

Theorem 7.2. *The class of E_*-acyclic simplicial sets is $\mathbf{\Delta_1}$ for every spectrum E.*

Proof. If (X, p) and (Y, q) are pointed simplicial sets, then $W = X \vee Y$ is a pointed simplicial set contained in $X \times Y$ such that W_n contains all elements of the form (x, sq) with $x \in X_n$ and all those of the form (sp, y) with $y \in Y_n$, where s is a composition of degeneracies, with basepoint (p, q). The smash product $X \wedge Y$ is obtained from $X \times Y$ by collapsing $X \vee Y$ to a point. Hence, $(X \wedge Y)_n = (X_n \times Y_n) \setminus (W_n \setminus \{(sp, sq)\})$ for all n, and setting equal to (sp, sq) all faces of elements of $X_{n+1} \times Y_{n+1}$ and all degeneracies of elements of $X_{n-1} \times Y_{n-1}$ taking values in W_n.

If (X, p) is a pointed simplicial set and E is a spectrum with structure maps $\langle \sigma_n \rangle_{n < \omega}$, then $X \wedge E$ is a spectrum with $(X \wedge E)_n = X \wedge E_n$ and structure maps $(\mathrm{id} \wedge \sigma_n) \circ (\tau \wedge \mathrm{id})$ for all n, where $\tau \colon \mathbb{S}^1 \wedge X \to X \wedge \mathbb{S}^1$ is the twist map. By part (3) of Proposition 7.1, the statement that $X \wedge E$ is weakly contractible does not require any unbounded quantifiers. However, a formula expressing this fact has to contain a definition of $X \wedge E$, where E is a given spectrum treated as a parameter. This can be done in two equivalent ways, as follows:

$$
(7.4) \quad
\begin{aligned}
&X \in \mathbf{sSet}_* \wedge \exists F \, [F \text{ is a spectrum} \wedge (\forall n < \omega)((F_n = X \wedge E_n) \wedge \\
&\qquad \sigma_n^F = (\mathrm{id} \wedge \sigma_n^E) \circ (\tau \wedge \mathrm{id})) \wedge F \text{ is weakly contractible}];
\end{aligned}
$$

$$
(7.5) \quad
\begin{aligned}
&X \in \mathbf{sSet}_* \wedge \forall F \, [[F \text{ is a spectrum} \wedge (\forall n < \omega)((F_n = X \wedge E_n) \wedge \\
&\qquad \sigma_n^F = (\mathrm{id} \wedge \sigma_n^E) \circ (\tau \wedge \mathrm{id}))] \to F \text{ is weakly contractible}].
\end{aligned}
$$

Since (7.4) is Σ_1 and (7.5) is Π_1, the theorem is proved. $\qquad \square$

As explained in Subsection 2.5, the fact that homological acyclic classes are $\mathbf{\Delta_1}$ implies that they are absolute for transitive models. This means that, if E is a spectrum and M is a transitive model of ZFC such that $E \in M$ (in which case E is a spectrum in M as well, since the class of spectra is $\mathbf{\Delta_0}$), then a simplicial set $X \in M$ is E_*-acyclic in M if and only if it is E_*-acyclic in the universe V.

Note, however, that if E is not treated as a parameter but is defined by a formula $\varphi(x, p)$ of the language of set theory, then the corresponding class of E_*-acyclic simplicial sets needs no longer be absolute. For example, the complexity of the formula

$$
(7.6) \quad
\begin{aligned}
&X \in \mathbf{sSet}_* \wedge \exists E \, [E \text{ is a spectrum} \wedge \\
&\forall x (x \in E \leftrightarrow \varphi(x, p)) \wedge X \text{ is } E_*\text{-acyclic}]
\end{aligned}
$$

depends on the complexity of φ. The formula φ may define distinct spectra in different models, or even not define a spectrum at all in some model M, in which case (7.6) just defines the empty set in M.

We thank Federico Cantero for pertinent remarks about the argument given in the proof of the next result.

Theorem 7.3. *The class of E^*-acyclic simplicial sets is $\mathbf{\Delta_2}$ for every Ω-spectrum E.*

Proof. Let E be an Ω-spectrum, which will be used as a parameter. A simplicial set X is E^*-acyclic if and only if, for all $k \in \mathbb{Z}$ and $n \geq 0$ with $n + k \geq 0$,

every map $S^n X \to E_{n+k}$ becomes nullhomotopic after suspending it a finite number of times and composing with the structure maps of E as in (7.3). This claim leads to a Π_2 formula —note that a map $S^n X \to E_{n+k}$ is no longer determined by any finite set of simplices of E_{n+k}. Next we show that it is possible to restate it by means of a Σ_2 formula.

A pointed simplicial set (X, p) is E^*-acyclic if and only if for all $n < \omega$ the simplicial set $\mathrm{map}_*(X, E_n)$ is weakly contractible, assuming that E is an Ω-spectrum. Thus, X is E^*-acyclic if and only if the following formula holds, where we need to define $M = \mathrm{map}_*(X, E_n)$:

$$X \in \mathbf{sSet}_* \wedge (\forall n < \omega) \, \exists M \, [M \in \mathbf{sSet}_* \wedge$$

$$(\forall k < \omega) \, [(\forall f \in M_k) \, f \in \mathbf{sSet}_*(X \wedge \Delta[k]_+, E_n) \wedge$$

$$\forall g \, (g \in \mathbf{sSet}_*(X \wedge \Delta[k]_+, E_n) \to g \in M_k)] \wedge M \text{ is weakly contractible]}.$$

According to Proposition 7.1, this is a Σ_2 formula. $\qquad\square$

It seems plausible, although we have not proved it, that there exist cohomological acyclic classes that fail to be upwards absolute, i.e., that cannot be defined with any Σ_1 formula with parameters.

In order to state and prove the next results, we use the term *homotopy reflection* (also called *homotopy localization* in other articles) to designate a coaugmented functor on the category of pointed simplicial sets (i.e., a functor $L \colon \mathbf{sSet}_* \to \mathbf{sSet}_*$ equipped with a natural transformation $\eta \colon \mathrm{Id} \to L$) which preserves weak equivalences and becomes a reflection when passing to the homotopy category. Recall from [15] or [18] that, for a homotopy reflection L, an *L-equivalence* is a map $f \colon X \to Y$ such that $Lf \colon LX \to LY$ is an isomorphism in the homotopy category, and a simplicial set X is called *L-local* if it is fibrant and weakly equivalent to LX for some X.

We also recall that, for a map $f \colon A \to B$, a fibrant simplicial set X is *f-local* if the induced map of unpointed function complexes

$$\mathrm{map}(f, X) : \mathrm{map}(B, X) \longrightarrow \mathrm{map}(A, X)$$

is a weak equivalence. The same terminology is used for a set or a proper class of maps $\mathcal{F}$; that is, a simplicial set is $\mathcal{F}$-*local* if it is f-local for all $f \in \mathcal{F}$. An $\mathcal{F}$-*localization* is a homotopy reflection L such that the class of L-local spaces coincides with the class of $\mathcal{F}$-local spaces.

For the following results we need to observe that, given any class of maps $\mathcal{S}$ between simplicial sets, if there is a set $\mathcal{F} \subseteq \mathcal{S}$ such that each element of $\mathcal{S}$ is a filtered colimit of elements of $\mathcal{F}$, then every $\mathcal{F}$-local space is $\mathcal{S}$-local. This is inferred, as in [15, Lemma 5.2], from the fact that the natural map $\mathrm{hocolim}_{i \in I} X_i \to \mathrm{colim}_{i \in I} X_i$ is a weak equivalence for every filtered diagram of simplicial sets $X \colon I \to \mathbf{sSet}_*$.

Theorem 7.4. *Assume the existence of arbitrarily large supercompact cardinals. Then for every additive cohomology theory E^* defined on simplicial sets there is a homotopy reflection L such that the L-equivalences are precisely the E^*-equivalences.*

Proof. Let $\mathcal{S}$ be the class of E^*-equivalences for a given additive cohomology theory E^*, and view it as a full subcategory of the category of pointed maps between simplicial sets (which is locally presentable —in fact, locally finitely

presentable). Since the class of E^*-equivalences coincides with the class of maps whose mapping cone is E^*-acyclic, Theorem 7.3 tells us that $\mathcal{S}$ is $\mathbf{\Sigma_2}$. Hence, it follows from part (2) of Theorem 6.1 that $\mathcal{S}$ is weakly preaccessible; that is, there is a regular cardinal λ and a set $\mathcal{F}$ of E^*-equivalences such that every E^*-equivalence is a λ-filtered colimit of elements of $\mathcal{F}$.

To conclude the proof, let $f\colon A \to B$ be the coproduct of all the elements of $\mathcal{F}$, and let L be f-localization, as constructed in [10], [18] or [22]. Since all the elements of $\mathcal{F}$ are E^*-equivalences and E^* is additive, f is an E^*-equivalence. Let E be an Ω-spectrum representing E^*. Then f induces bijections $[B, E_n] \cong [A, E_n]$ for all n, and in fact weak equivalences $\mathrm{map}_*(B, E_n) \simeq \mathrm{map}_*(A, E_n)$ for all n. In other words, the basepoint component of E_n is f-local for all n. Since E_n is a loop space, all its connected components have the same homotopy type and therefore E_n itself is f-local for all n. It follows that every L-equivalence $g\colon X \to Y$ induces a weak equivalence $\mathrm{map}_*(Y, E_n) \simeq \mathrm{map}_*(X, E_n)$ for all n, and using (7.2) we conclude that all L-equivalences are E^*-equivalences.

Conversely, every E^*-equivalence is, as said above, a λ-filtered colimit of objects from $\mathcal{F}$, hence filtered. According to [15, Lemma 5.2], the class of L-equivalences is closed under filtered colimits. This implies that every E^*-equivalence is an L-equivalence and completes the argument. $\qquad\square$

What we have proved is that localization with respect to any additive cohomology theory exists on the homotopy category of simplicial sets if arbitrarily large supercompact cardinals exist. This is a substantial improvement of [15, Corollary 5.4], where it was proved that the existence of cohomological localizations follows from Vopěnka's principle.

We also emphasize that from Theorem 7.2 it follows, by a similar method as in the proof of Theorem 7.4 (or using Theorem 7.6 below), that the existence of *homological* localizations (for representable homology theories) is provable in ZFC. Bousfield did it indeed in [9]; see also the Epilogue of [5], where Adams' original approach is repaired.

The same line of argument provides an answer to Farjoun's question in [17] of whether all homotopy reflections are f-localizations for some map f. It was shown in [15] that the answer is affirmative under Vopěnka's principle, and Przeździecki proved in [34] that an affirmative answer is in fact equivalent to Vopěnka's principle. Here we prove an analogue of Corollary 6.5.

Theorem 7.5. *A homotopy reflection L on simplicial sets is an f-localization for some map f under either of the following assumptions:*

    (1) *The class of L-equivalences is definable with sufficiently low complexity.*

    (2) *The class of L-local simplicial sets is $\mathbf{\Sigma_{n+1}}$ for $n \geq 1$ and there is a proper class of $C(n)$-extendible cardinals.*

Proof. For (1), since the category of pointed maps between simplicial sets is locally finitely presentable, we may choose, by Theorem 6.1, a set $\mathcal{F}$ of L-equivalences such that every L-equivalence is a filtered colimit of elements of $\mathcal{F}$. Let f be the coproduct of all the elements of $\mathcal{F}$. Then f is an L-equivalence, since the class of L-equivalences is closed under coproducts. Therefore, every L-local simplicial set is f-local, by [15, Corollary 4.4].

Conversely, let X be an f-local simplicial set and pick any L-equivalence g. From the fact that g is a filtered colimit of elements of $\mathcal{F}$, it follows, by [15, Lemma 5.2], that X is g-local. Since this is true for every L-equivalence g, we conclude that X is L-local, as needed.

In order to prove (2), note that, if the class of L-local simplicial sets is $\mathbf{\Sigma_{n+1}}$, then the class of L-equivalences is $\mathbf{\Pi_{n+1}}$, since $f\colon A \to B$ is an L-equivalence if and only if the induced function $[B, X] \to [A, X]$ is a bijection for each L-local space X, which can be formalized as

$$\forall X \,\forall g \,[(X \text{ is an } L\text{-local simplicial set} \wedge g \in \mathbf{sSet}_*(A, X)) \to$$

$$(\exists h\,(h \in \mathbf{sSet}_*(B, X) \wedge h \circ f \simeq g) \wedge \text{ any two such maps are homotopic})].$$

(The statement "any two such maps are homotopic" can be formally written as a Π_2 formula.) Hence the same argument as in part (1) applies under the assumption that a proper class of $C(n)$-extendible cardinals exists, by part (3) of Theorem 6.1. $\qquad\square$

The corresponding analogue of Corollary 6.7 is the next result.

Theorem 7.6. *Let $\mathcal{S}$ be any (possibly proper) class of maps of simplicial sets. If $\mathcal{S}$ is definable with sufficiently low complexity, then an $\mathcal{S}$-localization exists.*

Proof. As above, Theorem 6.1 implies that there is a set $\mathcal{F} \subseteq \mathcal{S}$ such that every $f \in \mathcal{S}$ is a filtered colimit of elements of $\mathcal{F}$. Then $\mathcal{F}$-localization exists since $\mathcal{F}$ is a set, and every $\mathcal{F}$-local simplicial set is $\mathcal{S}$-local by [15, Lemma 5.2]. Since $\mathcal{F} \subseteq \mathcal{S}$, the converse implication is obvious. $\qquad\square$

8. Bergman's question

A finitary operational signature Σ consists of a set of finitary operation symbols. Then Σ-structures are universal algebras. If $\mathcal{S}$ is a full subcategory of $\mathbf{Str}\,\Sigma$ and n is a non-negative integer, an *n-ary implicit operation* f on $\mathcal{S}$ is a collection of maps $f_X\colon X^n \to X$ indexed by objects X of $\mathcal{S}$ such that the square

$$
\begin{array}{ccc}
X & \xrightarrow{\ h\ } & Y \\
\uparrow{\scriptstyle f_X} & & \uparrow{\scriptstyle f_Y} \\
X^n & \xrightarrow{\ h^n\ } & Y^n
\end{array}
$$

commutes for each homomorphism $h\colon X \to Y$. Such implicit operations are very useful in finite universal algebra; see [6]. If $\mathcal{S}$ is a proper class with no homomorphisms except identities, then each collection $\{f_X\}_{X \in \mathcal{S}}$ is an implicit operation. Thus, assuming the negation of Vopěnka's principle, there is a proper class of implicit operations on $\mathcal{S}$. In connection with [8], Bergman asked whether this can happen assuming Vopěnka's principle.

Theorem 8.1. *For a finitary operational signature Σ, Vopěnka's principle implies that there is only a set of implicit operations on each full subcategory of $\mathbf{Str}\,\Sigma$.*

Proof. Let $\mathcal{S}$ be a full subcategory of $\mathbf{Str}\,\Sigma$. By [3], Vopěnka's principle implies that there is a regular cardinal κ and a set $\mathcal{A}$ of objects in $\mathcal{S}$ such that each object of $\mathcal{S}$ is a κ-filtered colimit of objects of $\mathcal{A}$. Since the forgetful functor $\mathbf{Str}\,\Sigma \to \mathbf{Set}$ and the power functor $(-)^n \colon \mathbf{Set} \to \mathbf{Set}$ preserve filtered colimits, each implicit operation f_X with $X \in \mathcal{S}$ is uniquely determined by $\{f_A\}_{A\in\mathcal{A}}$. Hence there is only a set of implicit operations on $\mathcal{S}$. $\qquad\square$

We improve this result as follows.

Theorem 8.2. *For a finitary operational signature Σ, every full subcategory $\mathcal{S}$ of $\mathbf{Str}\,\Sigma$ definable with sufficiently low complexity has only a set of implicit operations.*

Proof. We have proved in Theorem 6.1 that, for each object X of $\mathcal{S}$, the slice category $(\mathcal{S} \cap H(\kappa) \downarrow X)$ is cofinal in $((\mathbf{Str}\,\Sigma)_\kappa \downarrow X)$ for some regular cardinal κ, where $(\mathbf{Str}\,\Sigma)_\kappa$ is the (essentially small) class of κ-presentable objects in $\mathbf{Str}\,\Sigma$. Thus each object of $\mathcal{S}$ is a κ-filtered colimit of objects from $\mathcal{S} \cap H(\kappa)$. The rest is the same as in Theorem 8.1. $\qquad\square$

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