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An inverse electromagnetic scattering problem for a bi-periodic inhomogeneous layer on a perfectly conducting plate

Guanghui Hu[1], Jiaqing Yang [2], Bo Zhang[2]

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[1] Weierstrass Institute
for Applied Analysis and Stochastics
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: Guanghui.Hu@wias-berlin.de

[2] LSEC and Institute of Applied Mathematics
Academy of Mathematics and Systems Science
Chinese Academy of Sciences
100190 Beijing
China
E-Mail: jiaqingyang@amss.ac.cn
b.zhang@amt.ac.cn

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Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

This paper is concerned with uniqueness for reconstructing a periodic inhomogeneous medium covered on a perfectly conducting plate. We deal with the problem in the frame of time-harmonic Maxwell systems without TE or TM polarization. An orthogonal relation for two refractive indices is obtained, and then inspired by Kirsch's idea, the refractive index can be identified by utilizing the eigenvalues and eigenfunctions of a quasi-periodic Sturm-Liouville eigenvalue problem.

1 Introduction

Scattering theory in periodic structures has many applications in micro-optics, radar imaging and nondestructive testing. We refer to [22] for historical remarks and details of these applications. Consider a time-harmonic electromagnetic plane wave incident on a bi-periodic layer sitting on a perfectly conducting plate in $\mathbb{R}^3$. We assume that the medium inside the layer consists of some inhomogeneous isotropic conducting or dielectric material, whereas the medium above the layer consists of some homogeneous dielectric material. Suppose the magnetic permeability is a fixed positive constant throughout the whole space. The material properties of the media are then characterized completely by an index of refraction in the layer and a positive constant above the layer. The direct scattering problem is, given the incident field and the bi-periodic refractive index, to study the electromagnetic distributions, whereas the inverse scattering problem is to determine the refractive index from the knowledge of the incident waves and their corresponding measured scattered fields.

Adopting the Cartesian axis $ox_1x_2x_3$ with the x_3-axis vertically upwards, perpendicular to the plate. If the refractive index is invariant in the x_2 direction, the direct and inverse problems as indicated above can be dealt with in the TE polarization case where the electric field $E(x)$ is transversal to the (x_1, x_3)-plane by assuming $E = (0, u(x_1, x_3), 0)$, or in the TM polarization case where the magnetic field $H(x)$ is transversal to the (x_1, x_3)-plane by assuming $H = (0, u(x_1, x_3), 0)$. In the case of TE polarization, Kirsch [17] has studied the direct scattering problem via the variational method, and for the inverse problem, instead of constructing the complex geometrical optical solutions as in the Calderóns problem (see [19, 28]), he considered a class of eigenfunctions to a special kind of quasi-periodic Sturm-Liouville eigenvalue problem. Relying on the asymptotic behavior of those eigenvalues, the uniqueness result for the inverse problem can be proved once the orthogonal relation for two different refractive indexes has obtained. See also [25, 26] for the direct and inverse acoustic scattering by periodic, inhomogeneous, penetrable medium in the whole $\mathbb{R}^2$. Other uniqueness results for reconstructing the profile of a bi-periodic perfectly conducting grating can be seen in [2, 5, 6].

In this paper, we are mainly concerned with the uniqueness issue for reconstructing the refractive index in the framework of time-harmonic Maxwell equations without TE or TM polarization. The uniqueness result for the inverse problem in this paper is most closely related in term of result and method of argument to Kirsch on the determination of the refractive index in the TE polarization. Inspired by [27] and [15], we obtain an orthogonality relation for two different refractive indexes by using a D-to-N map on an artificial boundary on which the tangential electric fields are identical for an integral type of incident electric field. It should be remarked that the method for constructing geometry optical solutions in [19, 15, 27] for non-periodic inverse conductivity problems does not work since the solutions are required to be quasi-periodic in the periodic case. To reconstruct the refractive index, we follow Kirsch's idea [17] (see also [26]) by considering a kind of Sturm-Liouville eigenvalue problems. We shall prove the uniqueness result when the index depends only on one direction (x_1 or x_2). However, we expect the result to hold in a more general case by constructing special solutions with suitable asymptotic behaviors for the Maxwell equations.

Scattering by bi-periodic structures have been studied by many authors using both integral equation methods and variational methods (see, e.g. [1], [4], [12], [13], [14], [16], [20] and [24]). It is known that, for all but possibly a discrete set of frequencies, the direct scattering problem has a unique weak solution in the case of bi-periodic inhomogeneous medium in the whole $\mathbb{R}^3$, of which an absorbing medium always leads to a uniqueness result for any frequency. When the refractive index is non-absorbing, uniqueness can be guaranteed in the TE mode if the refractive index satisfies an increasing criterion in the x_3-direction ([25, 7]). See also [10] and [30] for the uniqueness results of more general rough surface scattering by an inhomogeneous medium in a half space in the TE or TM mode. In this paper, we assume that the medium inside the layer is absorbing so that the uniqueness result for the direct problem holds, implying that the D-to-N map T (at the end of Section 3), which depends on the refractive index, is well-defined.

The rest of the paper is organized as follows. In the next section we set up the precise mathematical framework and introduce some quasi-periodic function spaces needed. In Section 3, we consider a quasi-periodic boundary value problem (QPBVP) in a periodic cell via the variational approach which is used for the study of the inverse problem. Uniqueness and existence of solutions to the QPBVP are justified by the classic Hodge decomposition and the Fredholm alternative. This leads to the definition of a D-to-N map on an artificial boundary which is continuous and depends on the refractive index. In Section 4, based on the property of the transparent boundary condition defined on the artificial boundary, we give a solvability result of the direct scattering problem. In Section 5, we establish a uniqueness result for the inverse scattering problem.

2 Time-harmonic Maxwell equations and quasi-periodic function spaces

2.1 Time-harmonic Maxwell equations

Let $\mathbb{R}^3_+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, x_3 > 0\}$ and assume that $\mathbb{R}^3_+$ is filled with an inhomogeneous, isotropic, conducting or dielectric medium of electric permittivity $\epsilon > 0$, magnetic permeability $\mu > 0$ and electric conductivity $\sigma \geq 0$. Suppose the medium is non-magnetic, that is, the magnetic permeability μ is a fixed constant in $\mathbb{R}^3_+$ and the field is source free. Then the electromagnetic wave propagation is governed by the time-harmonic Maxwell equations (with the time variation of the form $e^{-i\omega t}$, $\omega > 0$)

$$\operatorname{curl} E - i\omega\mu H = 0, \qquad \operatorname{curl} H + i\omega(\epsilon + i\frac{\sigma}{\omega})E = 0, \tag{1}$$

where E and H are the electric field and magnetic field, respectively. Suppose the inhomogeneous medium is 2π-periodic with respect to x_1 and x_2 directions, that is, for all $n_1, n_2 \in \mathbb{Z}^2$,

$$\begin{aligned} \epsilon(x_1 + 2\pi n_1, x_2 + 2\pi n_2, x_3) &= \epsilon(x_1, x_2, x_3), \\ \sigma(x_1 + 2\pi n_1, x_2 + 2\pi n_2, x_3) &= \sigma(x_1, x_2, x_3). \end{aligned}$$

Further, assume that $\epsilon(x) = \epsilon_0$, $\sigma = 0$ for $x_3 > b$ (which means that the medium above the layer is lossless) and that the inhomogeneous medium has a perfectly conducting boundary $\Gamma_0 := \{x_3 = 0\}$. Consider a time-harmonic plane wave

$$E^i = pe^{ikx\cdot d}, \qquad H^i = qe^{ikx\cdot d},$$

incident on the periodic inhomogeneous layer from the top region $\Omega := \{x \in \mathbb{R}^3 \,|\, x_3 > b\}$, where $d = (\alpha_1, \alpha_2, -\beta) = (\cos\theta_1 \cos\theta_2, \cos\theta_1 \sin\theta_2, -\sin\theta_1)$ is the incident wave vector whose direction is specified by θ_1 and θ_2 with $0 < \theta_1 < \pi$, $0 < \theta_2 \leq 2\pi$ and the vectors p and q are polarization directions satisfying that $p = \sqrt{\mu/\varepsilon}(q \times d)$ and $q \perp d$. The problem of scattering of time-harmonic electromagnetic waves in this model leads to the following problem:

$$\operatorname{curl}\operatorname{curl} E - k^2 E = 0 \quad \text{in} \quad x_3 > b, \tag{2}$$

$$\operatorname{curl}\operatorname{curl} E - k^2 qE = 0 \quad \text{in} \quad \Omega_b, \tag{3}$$

$$\nu \times E = 0 \quad \text{on} \quad \Gamma_0, \tag{4}$$

$$E = E^i + E^s \quad \text{in} \quad \mathbb{R}^3_+, \tag{5}$$

where $k = \sqrt{\epsilon_0 \mu}\omega$ is the wave number, $q(x) = \frac{1}{\epsilon_0}(\epsilon(x) + i\frac{\sigma(x)}{\omega})$ is the refractive index and ν is the unit normal at the boundary.

Set $\alpha = (\alpha_1, \alpha_2, 0) \in \mathbb{R}^3$ and $n = (n_1, n_2) \in \mathbb{Z}^2$. The periodicity of the medium motivates us to look for α-quasi-periodic solutions in the sense that $E(x_1, x_2, x_3)e^{-i\alpha\cdot x}$ is 2π periodic with respect to x_1 and x_2, respectively. Since the domain is unbounded in the x_3-direction, a radiation condition must be imposed. It is required physically that the diffracted fields remain bounded as x_3 tends to $+\infty$, which leads to the so-called outgoing wave condition in the form of

$$E^s(x) = \sum_{n\in\mathbb{Z}^2} E_n e^{i(\alpha_n \cdot x + \beta_n x_3)}, \qquad x_3 > b, \tag{6}$$

where $\alpha_n = (\alpha_1 + n_1, \alpha_2 + n_2, 0) \in \mathbb{R}^3$, $E_n = (E_n^{(1)}, E_n^{(2)}, E_n^{(3)}) \in \mathbb{C}^3$ are constant vectors and

$$\beta_n = \begin{cases} (k^2 - |\alpha_n|^2)^{\frac{1}{2}} & \text{if } |\alpha_\mathrm{n}| < \mathrm{k}, \\ i(|\alpha_n|^2 - k^2)^{\frac{1}{2}} & \text{if } |\alpha_\mathrm{n}| > \mathrm{k}, \end{cases}$$

with $i^2 = -1$. Furthermore, we assume that $\beta_n \neq 0$ for all $n \in \mathbb{Z}^2$. The series expansion in (6) is considered as the Rayleigh series of the scattered field and the condition is called the Rayleigh expansion radiation condition. The coefficients E_n in (6) are also called the Rayleigh sequence. From the fact that $\operatorname{div} \mathrm{E^s(x)} = 0$ it is clear that

$$\alpha_n \cdot E_n + \beta_n E_n^{(3)} = 0.$$

The direct problem (DP) is to compute the total field E in $\mathbb{R}^3_+$, given the incident wave E^i, the refractive index $q(x)$ and the boundary condition on Γ_0. Since only a finite number of terms in (6) are upward propagating plane waves and the rest is evanescent modes that decay exponentially with distance away from the periodic medium, we use the near field data rather than the far field data to reconstruct the refractive index $q(x)$. Thus, our inverse problem (IP) is to determine the periodic medium $q(x)$ from a knowledge of the incident wave E^i and the total tangential electric field $\nu \times E$ on a plane $\Gamma_a = \{x \in \mathbb{R}^3 \,|\, x_3 = a\}(a > b)$ above the layer.

2.2 Quasi-periodic function spaces

In this section we introduce some function spaces needed for the scattering problem (2)-(5). These spaces will play a crucial role not only in the study of the direct problem but also in the inverse problem. In [4, 12, 24], the authors always seek the H^1-variational solution for the magnetic field H, based on the facts that the magnetic permeability $\mu > 0$ is a constant and that any vector field $H \in L^2(D)^3$ satisfying that $\nabla \times H \in L^2(D)^3$ and $\nabla \cdot H \in L^2(D)^3$ belongs to $H^1_{loc}(D)^3$ for any bounded domain $D \subset \mathbb{R}^3$. In this paper, based on the classic Hodge decomposition, we are interested in weak solutions in $H(\mathrm{curl}\,)$ of the problem (2)-(5), that is, both E and $\nabla \times E$ belong to $L^2_{loc}(\mathbb{R}^3_+)^3$. This allows us to solve the scattering problem in a general case when μ is a periodic variable function other than a constant.

The scattering problem can be reduced to a single periodic cell. To this end, we reformulate the following notations.

$$\Gamma_b = \{x_3 = b \,|\, 0 < x_1, x_2 < 2\pi\},\ \ \Omega_b = \{x \in \mathbb{R}^3_+ \,|\, x_3 < b,\ 0 < x_1, x_2 < 2\pi\}.$$

We also need the following scalar quasi-periodic Sobolev space:

$$H^1(\Omega_b) = \{u(x) = \sum_{n\in\mathbb{Z}^2} u_n(x_3)\exp(i\alpha_n \cdot x) \,|\, u \in L^2(\Omega_b), \nabla u \in (L^2(\Omega_b))^3, u_n \in \mathbb{C}\}.$$

Denote by $H^{\frac{1}{2}}(\Gamma_b)$ the trace space of $H^1(\Omega_b)$ on Γ_b with the norm

$$||f||^2_{H^{\frac{1}{2}}(\Gamma_b)} = \sum_{n\in\mathbb{Z}^2} |f_n|^2(1 + |\alpha_n|^2)^{\frac{1}{2}}, \qquad f \in H^{\frac{1}{2}}(\Gamma_b),$$

where $f_n = (f, \exp(i\alpha_n \cdot x))_{L^2(\Gamma_b)}$ and write $H^{-\frac{1}{2}}(\Gamma_b) = (H^{\frac{1}{2}}(\Gamma_b))'$, the dual space to $H^{\frac{1}{2}}(\Gamma_b)$.

We now introduce some vector spaces. Let

$$\begin{aligned} H(\mathrm{curl}\,, \Omega_b) \;&=\; \{E(x) = \sum_{n\in\mathbb{Z}^2} E_n(x_3)\exp(i\alpha_n \cdot x) \,|\, E_n \in \mathbb{C}^3, \\ &\qquad E \in (L^2(\Omega_b))^3,\ \mathrm{curl}\, E \in (L^2(\Omega_b))^3\} \end{aligned}$$

with the norm

$$||E||^2_{H(\mathrm{curl}\,,\Omega_b)} = ||E||^2_{L^2(\Omega_b)} + ||\mathrm{curl}\, E||^2_{L^2(\Omega_b)}.$$

For $x' = (x_1, x_2, b) \in \Gamma_b$, $s \in \mathbb{R}$ define

$$\begin{aligned}
H_t^s(\Gamma_b) &= \{E(x') = \sum_{n\in\mathbb{Z}^2} E_n \exp(i\alpha_n \cdot x') \,|\, E_n \in \mathbb{C}^3,\ e_3 \cdot E = 0, \\
&\qquad \|E\|^2_{H^s(\Gamma_b)} = \sum_{n\in\mathbb{Z}^2} (1 + |\alpha_n|^2)^s |E_n|^2 < +\infty\} \\
H_t^s(\mathrm{div}\,, \Gamma_b) &= \{E(x') = \sum_{n\in\mathbb{Z}^2} E_n \exp(i\alpha_n \cdot x') \,|\, E_n \in \mathbb{C}^3,\ e_3 \cdot E = 0, \\
&\qquad ||E||^2_{H^s(\mathrm{div}\,,\Gamma_b)} = \sum_{n\in\mathbb{Z}^2} (1 + |\alpha_n|^2)^s (|E_n|^2 + |E_n \cdot \alpha_n|^2) < +\infty\} \\
H_t^s(\mathrm{curl}\,, \Gamma_b) &= \{E(x') = \sum_{n\in\mathbb{Z}^2} E_n \exp(i\alpha_n \cdot x') \,|\, E_n \in \mathbb{C}^3,\ e_3 \cdot E = 0, \\
&\qquad ||E||^2_{H^s(\mathrm{curl}\,,\Gamma_b)} = \sum_{n\in\mathbb{Z}^2} (1 + |\alpha_n|^2)^s (|E_n|^2 + |E_n \times \alpha_n|^2) < +\infty\}
\end{aligned}$$

and write $L_t^2(\Gamma_b) = H_t^0(\Gamma_b)$. Recall that

$$H_t^{-1/2}(\mathrm{div}\,, \Gamma_b) = \{e_3 \times E|_{\Gamma_b} \,|\, E \in H(\mathrm{curl}\,, \Omega_b)\}$$

and that the trace mapping from $H(\mathrm{curl}\,, \Omega_b)$ to $H_t^{-1/2}(\mathrm{div}\,, \Gamma_b)$ is continuous and surjective (see [8] and the references there).

It is well-known (see [20]) that the free space α-quasi-periodic Green function for the Helmholtz equation $(\Delta + k^2)u = 0$ in $\mathbb{R}^3$ is given by

$$G(x, y) = \frac{1}{8\pi^2} \sum_{n\in\mathbb{Z}^2} \frac{1}{i\beta_n} \exp(i\alpha_n \cdot (x - y) + i\beta_n |x_3 - y_3|). \tag{7}$$

We assume throughout this paper that q satisfies the following conditions:

(**A1**) $q \in C^1(\overline{\Omega_b})$ and $q(x) = 1$ when $x_3 > b$;

(**A2**) $\mathrm{Im}\,[q(x)] \geq 0$ for all $x \in \overline{\Omega_b}$ and $\mathrm{Im}\,[q(x_0)] > 0$ for some $x_0 \in \overline{\Omega_b}$;

(**A3**) $\mathrm{Re}\,[q(x)] \geq \gamma$ for all $x \in \overline{\Omega_b}$ for some positive constant γ.

3 A quasi-periodic boundary value problem

Before studying the original problem (2)-(6), we first consider the following quasi-periodic boundary value problem in Ω_b:

$$\begin{aligned}
\mathrm{curl}\,\mathrm{curl}\, E - k^2 q(x) E &= 0 \qquad \text{in } \Omega_b, &(1)\\
\nu \times E &= 0 \qquad \text{on } \Gamma_0, &(2)\\
\nu \times E &= f \qquad \text{on } \Gamma_b, &(3)
\end{aligned}$$

where $f \in H_{\mathrm{div}}^{-1/2}(\Gamma_b)$.

Lemma 3.1 *If the conditions* $(A1)-(A3)$ *are satisfied, then there exists a unique solution* $E \in H(\mathrm{curl}\,, \Omega_b)$ *to the problem* $(1)-(3)$ *such that*

$$||E||_{H(\mathrm{curl}\,,\Omega_b)} \le C||f||_{H_{\mathrm{div}}^{-1/2}(\Gamma_b)},$$

where C *is a positive constant independent of* f*.*

Proof. We first prove the uniqueness part. Let $f=0$. Multiplying both sides of (1) by $\overline{E}$ it follows from Green's vector formula, the quasi-periodic property of E and the boundary conditions (2) and (3) that

$$\int_{\Omega_b} [|\mathrm{curl}\, E|^2 - k^2 q|E|^2] dx = 0. \tag{4}$$

Take the imaginary part of the above equation and use the assumption on $q(x)$ to find that

$$\int_{B_\epsilon(x_0)} |E(x)|^2 dx = 0,$$

where $B_\epsilon(x_0) \subset \Omega_b$ is a small ball centered at x_0 with radius ϵ. Thus $E(x) \equiv 0$ in $B_\epsilon(x_0)$. By [9, Theorem 6] we have $E \in (H^1(\Omega_b))^3$. Thus, by the unique continuation principle (see [21, Theorem 2.3]) we have $E \equiv 0$ in Ω_b.

We are now in a position to prove the existence of solutions. For any $V \in H(\mathrm{curl}\,, \Omega_b)$ such that $\nu \times E = 0$ on $\Gamma_0 \cup \Gamma_b$, multiplying both sides of (1) by $\overline{V}$ yields

$$\int_{\Omega_b} [\mathrm{curl}\, E \cdot \mathrm{curl}\, \overline{V} - k^2 q E \cdot \overline{V}] dx = 0. \tag{5}$$

There exists at least one element $W \in H(\mathrm{curl}\,, \Omega_b)$ satisfying that $\nu \times W = 0$ on Γ_0 and $\nu \times W = f$ on Γ_b. Then the equation (5) can be rewritten as

$$\int_{\Omega_b} [\mathrm{curl}\,(E-W) \cdot \mathrm{curl}\, \overline{V} - k^2 q (E-W) \cdot \overline{V}] dx = -\int_{\Omega_b} [\mathrm{curl}\, W \cdot \mathrm{curl}\, \overline{V} - k^2 q W \cdot \overline{V}] dx.$$

Let $X := \{U \in H(\mathrm{curl}\,, \Omega_b),\ \nu \times U = 0 \text{ on } \Gamma_0 \cup \Gamma_b\}$. Then $U := E - W \in X$. Thus the problem (1)-(3) is equivalent to the following variational problem: Find $U \in X$ such that for any $V \in X$,

$$\int_{\Omega_b} [\mathrm{curl}\, U \cdot \mathrm{curl}\, \overline{V} - k^2 q U \cdot \overline{V}] dx = F_W(V), \tag{6}$$

where $F_W(V) = -\displaystyle\int_{\Omega_b} [\mathrm{curl}\, W \cdot \mathrm{curl}\, \overline{V} - k^2 q W \cdot \overline{V}] dx$. The proof is broken down into the following steps.

Step 1. To establish the Hodge decomposition:

$$X = X_0 \oplus \nabla S, \tag{7}$$

where $S = \{p \in H^1(\Omega_b),\ p = 0 \text{ on } \Gamma_0 \cup \Gamma_b\}$ and $X_0 = \{\xi \in X \mid \displaystyle\int_{\Omega_b} q(x) \xi \cdot \nabla \overline{p} = 0,\ \forall p \in S\}$.

For $U, V \in X$ define

$$a(U,V) = \int_{\Omega_b} [\operatorname{curl} U \cdot \operatorname{curl} \overline{V} - k^2 q U \cdot \overline{V}] dx.$$

It follows from the assumptions (A1)-(A3) on $q(x)$ that

$$|a(\nabla p, \nabla p)| \geq k^2 \int_{\Omega_b} \operatorname{Re}[q(x)] |\nabla p|^2 dx \geq k^2 \gamma ||\nabla p||^2_{L^2(\Omega_b)} = k^2 \gamma ||\nabla p||^2_{H(\operatorname{curl}, \Omega_b)}.$$

Thus, for every $E \in X$ there exits a unique $p \in S$ such that $a(\nabla p, \nabla q) = a(E, \nabla q)$ for all $q \in S$. Let $\xi := E - \nabla p$. Then it is easy to show that $\xi \in X_0$ and $X_0 \cap S = \emptyset$, which implies the Hodge decomposition (7).

Step 2. To prove the existence of a unique solution $U \in X$ to the problem ((6).

By (7) we may assume that $U = \xi + \nabla p$, $V = \eta + \nabla q$ with $\xi, \eta \in X_0$ and $p, q \in S$. Then the problem (6) becomes the following one: Find $\xi \in X_0$ and $p \in S$ such that

$$a(\nabla p, \nabla q) + a(\xi, \eta) = F_W(\nabla q) + F_W(\eta).$$

Since $a(\cdot,\cdot)$ is coercive on ∇S, there exists a unique $p \in S$ such that

$$a(\nabla p, \nabla q) = F_W(\nabla q) \qquad \forall\, q \in S$$

with the estimate $||\nabla p||_{H(curl,\Omega_b)} \leq C||W||_{H(curl,\Omega_b)}$. It remains to find $\xi \in X_0$ such that $a(\xi, \eta) = F_W(\eta)$ for all $\eta \in X_0$. The bilinear form $a(\cdot,\cdot)$ can be decomposed into the sum of the following two forms:

$$\begin{aligned} a_1(\xi, \eta) &= \int_{\Omega_b} \operatorname{curl} \xi \cdot \operatorname{curl} \overline{\eta} + \xi \cdot \overline{\eta} dx, \\ a_2(\xi, \eta) &= -k^2 \int_{\Omega_b} (1+q) \xi \cdot \overline{\eta} dx. \end{aligned}$$

Obviously, $a_1(\cdot,\cdot)$ is coercive on X_0, and it follows from [3, Lemma 3.2] that X_0 is compactly imbedded into $(L^2(\Omega_b))^3$. Thus, by the standard Fredholm alternative theory there exists a unique $\xi \in X_0$ satisfying that $a(\xi, \eta) = F_W(\eta)$ for all $\eta \in X_0$. Furthermore, $||\xi||_{H(curl,\Omega_b)} \leq C||W||_{H(curl,\Omega_b)}$.

Step 3. To establish the estimate (4).

By Steps 1 and 2 we know that $E = \xi + \nabla p + W \in H(\operatorname{curl}, \Omega_b)$ is a solution to the problem (1)-(3) with the estimate

$$||E||_{H(\operatorname{curl},\Omega_b)} \leq ||\xi||_{H(\operatorname{curl},\Omega_b)} + ||\nabla p||_{H(\operatorname{curl},\Omega_b)} + ||W||_{H(\operatorname{curl},\Omega_b)} \leq C||W||_{H(\operatorname{curl},\Omega_b)}. \tag{8}$$

Recalling that

$$||f||_{H^{-1/2}_{\operatorname{div}}(\Gamma_b)} = \inf\{||W||_{H(\operatorname{curl},\Omega_b)} \,|\, \nu \times W = 0 \text{ on } \Gamma_0 \text{ and } \nu \times W = f \text{ on } \Gamma_b\},$$

it follows from (8) that $||E||_{H(\operatorname{curl},\Omega_b)} \leq C||f||_{H^{-1/2}_{\operatorname{div}}(\Gamma_b)}$. □

For $f \in H^{-1/2}_{\operatorname{div}}(\Gamma_b)$ define the operator T by

$$T(f) = \nu \times (\operatorname{curl} E \times \nu) \qquad \text{on } \Gamma_b,$$

where E solves the quasi-periodic boundary value problem (1)-(3). By Lamma 3.1, the operator T is well-defined. Note that $T(f)$ belongs to the dual space $(H_{\rm div}^{-1/2}(\Gamma_b))' = H_{\rm curl}^{-1/2}(\Gamma_b)$ of $H_{\rm div}^{-1/2}(\Gamma_b)$ with the duality defined by

$$< T(f), g >= \int_{\Omega_b} [\operatorname{curl} E \cdot \operatorname{curl} \overline{V} - k^2 qE \cdot \overline{V}]dx$$

for $g \in H_{\rm div}^{-1/2}(\Gamma_b)$, where $V \in H(\operatorname{curl}, \Omega_b)$ satisfies that $\nu \times V = g$ on Γ_b and $\nu \times V = 0$ on Γ_0. The operator T can be considered as a Dirichlet-to-Neumann map associated with the problem (1)-(3) and depending on the index $q(x)$. Under the assumptions (A1)-(A3), the above definition of $T(f)$ is independent of the choice of V and therefore $T : H_{\rm div}^{-1/2}(\Gamma_b) \to (H_{\rm div}^{-1/2}(\Gamma_b))' = H_{\rm curl}^{-1/2}(\Gamma_b)$ is well-defined. Moreover, it follows from the above equality and Lemma 3.1 that

$$||T(f)||_{H_{\rm curl}^{-1/2}(\Gamma_b)} \le C||E||_{H(\operatorname{curl}, \Omega_b)} \le C||f||_{H_{\rm div}^{-1/2}(\Gamma_b)}.$$

This implies that T is continuous from $H_{\rm div}^{-1/2}(\Gamma_b)$ to $H_{\rm curl}^{-1/2}(\Gamma_b)$.

4 Solvability of the scattering problem

In this section we will establish the solvability of the scattering problem (2)-(6), employing the variational method. To this end, we propose a variational formulation of the scattering problem in a truncated domain by introducing a transparent boundary condition on Γ_b. The existence and uniqueness of solutions to the problem will then be proved using the Hodge decomposition together with the Fredholm alternative.

4.1 Transparent boundary condition and variational formulation

Let $x' = (x_1, x_2, b) \in \Gamma_b$ for $b > 0$. For $\widetilde{E} \in H_t^{-\frac{1}{2}}(\operatorname{div}, \Gamma_b)$ with $\widetilde{E}(x') = \sum_{n\in\mathbb{Z}^2} \widetilde{E}_n \exp(i\alpha_n \cdot x')$, define $\mathcal{R} : H_t^{-\frac{1}{2}}(\operatorname{div}, \Gamma_b) \to H_t^{-\frac{1}{2}}(\operatorname{curl}, \Gamma_b)$ by

$$(\mathcal{R}\widetilde{E})(x') = (e_3 \times \operatorname{curl} E) \times e_3 \quad \text{on } \Gamma_b, \tag{1}$$

where E satisfying the Rayleigh expansion condition (6) is the unique quasi-periodic solution to the problem

$$\operatorname{curl}\operatorname{curl} E - k^2 E = 0 \quad \text{for } x_3 > b, \qquad \nu \times E = \widetilde{E}(x') \quad \text{on } \Gamma_b.$$

The map $\mathcal{R}$ is well-defined and can be used to replace the radiation condition (6) on Γ_b. Then the direct scattering problem (2)-(6) can be transformed into the following boundary value problem in the truncated domain Ω_b:

$$\begin{aligned} \operatorname{curl}\operatorname{curl} E - k^2 qE &= 0 \quad \text{in } \Omega_b, & (2)\\ \nu \times E &= 0 \quad \text{on } \Gamma_0, & (3)\\ (\operatorname{curl} E)_T - \mathcal{R}(e_3 \times E) &= (\operatorname{curl} E^i)_T - \mathcal{R}(e_3 \times E^i) \quad \text{on } \Gamma_b, & (4)\end{aligned}$$

where, for any vector function U, $U_T = (\nu \times U) \times \nu$ denotes its tangential component on a surface. The variational formulation for the problem (2)-(4) can be given as follows: find $E \in X := \{E \in H(\mathrm{curl}\,, \Omega_b) \mid \nu \times E = 0 \text{ on } \Gamma_0\}$ such that

$$\begin{aligned} B(E, \varphi) &:= \int_{\Omega_b} \left[\mathrm{curl}\, E \cdot \mathrm{curl}\, \overline{\varphi} - k^2 q E \cdot \overline{\varphi}\right] dx - \int_{\Gamma_b} \mathcal{R}(e_3 \times E) \cdot (e_3 \times \overline{\varphi}) ds \qquad (5) \\ &= \int_{\Gamma_b} \left[(\mathrm{curl}\, E^i)_T - \mathcal{R}(e_3 \times E^i)\right] \cdot (e_3 \times \overline{\varphi}) ds \end{aligned}$$

for all $\varphi \in X$. We have the following properties of $\mathcal{R}$:

1) $\mathcal{R} : H_t^{-\frac{1}{2}}(\mathrm{div}\,, \Gamma_b) \to H_t^{-\frac{1}{2}}(\mathrm{curl}\,, \Gamma_b)$ is continuous and can be explicitly represented as

$$(\mathcal{R}\widetilde{E})(x') = -\sum_{n \in \mathbb{Z}^2} \frac{1}{i\beta_n} \left[k^2 \widetilde{E}_n - (\alpha_n \cdot \widetilde{E}_n)\alpha_n\right] \exp(i\alpha_n \cdot x'). \qquad (6)$$

2) Let $P = \{n = (n_1, n_2) \in \mathbb{Z}^2 \mid \beta_n \text{ is a real number}\}$. Then

$$\mathrm{Re} < \mathcal{R}\widetilde{E}, \widetilde{E} > \; = \; 4\pi^2 \sum_{n \in \mathbb{Z}^2 \setminus P} \frac{1}{|\beta_n|} \left[k^2 |\widetilde{E}_n|^2 - |\alpha_n \cdot \widetilde{E}_n|^2\right], \qquad (7)$$

$$-\mathrm{Re} < \mathcal{R}\widetilde{E}, \widetilde{E} > \; \geq \; C_1 ||\mathrm{div}\, \widetilde{E}||^2_{H_t^{-1/2}(\Gamma_b)} - C_2 ||\widetilde{E}||^2_{H_t^{-1/2}(\Gamma_b)}, \qquad (8)$$

where C_1 and C_2 are positive constants and $< \cdot, \cdot >$ denotes the inner product of $L_t^2(\Gamma_b)$.

3)

$$\mathrm{Im} < \mathcal{R}\widetilde{E}, \widetilde{E} >= 4\pi^2 \sum_{n \in P} \frac{1}{\beta_n} \left[k^2 |\widetilde{E}_n|^2 - |\alpha_n \cdot \widetilde{E}_n|^2\right] \geq 0. \qquad (9)$$

The representation (6) of $\mathcal{R}$ can be computed directly from its definition (1) (see [1]) and the properties (7)-(9) can be easily obtained using this representation. Furthermore, there exists a $C > 0$ such that for every $\eta > 0$ and $E \in H(\mathrm{curl}\,, \Omega_b)$ we have (see [1])

$$||\nu \times E||_{H_t^{-1/2}(\Gamma_b)} \leq C \left[\eta ||\mathrm{curl}\, E||_{L^2(\Omega_b)} + (1 + 1/\eta) ||E||_{L^2(\Omega_b)}\right]. \qquad (10)$$

Let

$$\begin{aligned} S &= \{p \in H^1(\Omega_a) \mid p = 0 \text{ on } \Gamma_0\} \\ X_0 &= \{E \in X \mid B(E, \nabla p) = 0 \; \forall p \in S\} \end{aligned}$$

Then in a completely similar manner as in the proof of Lemma 4.2, we can establish the Hodge decomposition $X = X_0 \oplus \bigtriangledown S$.

4.2 Solvability of the direct scattering problem

Lemma 4.1 *The bilinear form $B(\cdot, \cdot)$ defined by* (5) *is strongly elliptic on X_0, that is, for all $w_0 \in X_0$,*

$$\mathrm{Re}\, B(w_0, w_0) \geq C ||w_0||_X - \rho(w_0, w_0)$$

for some constant $C > 0$ and a compact bilinear form $\rho(\cdot, \cdot)$.

Proof. Let M be a positive constant to be determined later and let

$$\begin{aligned} b_1(w_0,\varphi_0) &= \int_{\Omega_b}\left[\operatorname{curl} w_0\cdot\operatorname{curl}\overline{\varphi_0}+(M-k^2q)w_0\cdot\overline{\varphi_0}\right]dx-\int_{\Gamma_b}\mathcal{R}(e_3\times w_0)\cdot(e_3\times\overline{\varphi}_0)ds \\ b_2(w_0,\varphi_0) &= -M\int_{\Omega_b} w_0\cdot\overline{\varphi_0}dx. \end{aligned}$$

Then $B(w_0,\varphi_0)=b_1(w_0,\varphi_0)+b_2(w_0,\varphi_0)$ for $w_0,\ \varphi_0\in X_0$. By the properties of $\mathcal{R}$ it follows that

$$\begin{aligned} &-\mathrm{Re} < \mathcal{R}(e_3\times w_0), e_3\times\overline{w}_0 > \\ &\quad\geq C_1||\mathrm{div}\,(e_3\times w_0)||^2_{H_t^{-1/2}(\Gamma_b)}-C_2||e_3\times w_0||^2_{H_t^{-1/2}(\Gamma_b)} \\ &\quad\geq C_1||\mathrm{div}\,(e_3\times w_0)||^2_{H_t^{-1/2}(\Gamma_b)}-C_3\eta^2||\operatorname{curl} w_0||^2_{L^2(\Omega_b)}-C_3(1+\frac{1}{\eta})^2||w_0||^2_{L^2(\Omega_b)}, \end{aligned}$$

where C_1, C_2 and C_3 are three positive constants and $\eta>0$ is arbitrary. Thus we have

$$\begin{aligned} \mathrm{Re}\, b_1(w_0,w_0) &\geq \|\operatorname{curl} w_0\|^2_{L^2(\Omega_b)}+(M-k^2q_\infty)\|w_0\|^2_{L^2(\Omega_b)} \\ &\quad -C_3\eta^2||\operatorname{curl} w_0||^2_{L^2(\Omega_b)}-C_3(1+1/\eta)^2||w_0||^2_{L^2(\Omega_b)} \\ &= (1-C_3\eta^2)||\operatorname{curl} w_0||^2_{L^2(\Omega_b)}+[M-k^2q_\infty-C_3(1+1/\eta)^2]||w_0||^2_{L^2(\Omega_b)}, \end{aligned}$$

where $q_\infty=\max_{x\in\mathbb{R}^3_+}|q(x)|<\infty$. Choose η sufficiently small and M sufficiently large so that

$$\mathrm{Re}\, b_1(w_0,w_0)\geq C_0(||\operatorname{curl} w_0||^2_{L^2(\Omega_b)}+||w_0||^2_{L^2(\Omega_b)}) \tag{11}$$

for some constant $C_0>0$. This, together with the fact that X_0 is compactly imbedded in $(L^2(\Omega_b))^3$, yields the desired result. □

Theorem 4.2 *Assume that the conditions* $(A1)-(A3)$ *are satisfied. Then the problem* $(2)-(6)$ *has a unique solution* $E\in H_{loc}(\operatorname{curl},\mathbb{R}^3_+)$ *such that*

$$||E||_{H_{loc}(\operatorname{curl},\mathbb{R}^3_+)}:=\max_{a>b}||E||_{H(\operatorname{curl},\Omega_a)}\leq C||E^i||_{H(\operatorname{curl},\Omega_b)},$$

where C *is a positive constant depending on the domain and* q.

Proof. It follows from Lemma 4.1 and the proof of Lemma 3.1 that there exists a unique solution $E\in H(\operatorname{curl},\Omega_b)$ satisfying that $||E||_{H(\operatorname{curl},\Omega_b)}\leq C||E^i||_{H(\operatorname{curl},\Omega_b)}$. It remains to extend $E(x)$ to be a function in $H_{loc}(\operatorname{curl},\mathbb{R}^3_+)$. Suppose $e_3\times(E-E^i)|_{\Gamma_b}=\sum_{n\in\mathbb{N}\times\mathbb{N}}A_ne^{i\alpha_n\cdot x}\in H^{-1/2}(\mathrm{div}\,,\Gamma_b)$. Let

$$E^s(x)=\sum_{n\in\mathbb{N}\times\mathbb{N}}(A_n\times e_3+B_ne_3)e^{i\alpha_n\cdot x+i\beta_n(x_3-b)},\qquad x_3>b$$

and let E^s satisfy that $\mathrm{div}\,E^s(x)=0$ for $x_3>b$. Then we have $B_n=\frac{1}{\beta_n}(e_3\times A_n)\cdot\alpha_n$. Thus

$$E^s(x)=\sum_{n\in\mathbb{N}\times\mathbb{N}}\left[A_n\times e_3+\frac{1}{\beta_n}(e_3\times A_n)\cdot\alpha_ne_3\right]e^{i\alpha_n\cdot x+i\beta_n(x_3-b)},\qquad x_3>b.$$

Define $E(x)=E^i(x)+E^s(x)$ for $x_3>b$. Then it is easy to prove that $E\in H(\operatorname{curl},\Omega_a\backslash\Omega_b)$ with $||E||_{H(\operatorname{curl},\Omega_a\backslash\Omega_b)}\leq C||E^i||_{H(\operatorname{curl},\Omega_b)}$ for any $a>b$, so $E\in H(\operatorname{curl},\Omega_a)$ for any $a>b$, that is, $E\in H_{loc}(\operatorname{curl},\mathbb{R}^3_+)$ with the required estimate (12). The proof is thus completed. □

5 The inverse problem

Let $a > b$ and assume that there are two refractive index functions q_i $(i = 1, 2)$ satisfying the assumptions (A1)-(A3). For $g \in L_t^2(\Gamma_a)$ let the incident waves be of the form:

$$E^i(x, g) = \mathrm{curl}_x \mathrm{curl}_x \int_{\Gamma_a} G(x, y) g(y) ds(y), \qquad x < a. \tag{1}$$

Write the scattered electric field and the total electric field as $E_i^s(x, g)$ and $E_i(x, g)$, respectively, indicating their dependance on g and the refractive index function q_i $(i = 1, 2)$.

For the refractive index q_i denote by T_i the corresponding Dirichlet-to-Neumann map associated with the problem (1)-(3) with q replaced by q_i $(i = 1, 2)$, as defined at the end of Section 3.

Lemma 5.1 *If $T_1(f) = T_2(f)$ for all $f \in H_t^{-1/2}(\mathrm{div}\,, \Gamma_b)$, then*

$$\int_{\Omega_b} E_1(x) \cdot \overline{E}_2(x) \, [q_1(x) - q_2(x)] \, dx = 0,$$

where E_1, $E_2 \in H(\mathrm{curl}\,, \Omega_b)$ solve the problem $(1) - (3)$ with q replaced by q_1 and $\overline{q}_2$, respectively.

Proof. Let E_1 and $F_2 \in H(\mathrm{curl}\,, \Omega_b)$ be the solution of the problems

$$\mathrm{curl}\,\mathrm{curl}\, E_1 - k^2 q_1 E_1 = 0 \quad \text{in } \Omega_b, \qquad \nu \times E_1 = 0 \quad \text{on } \Gamma_0$$

and

$$\mathrm{curl}\,\mathrm{curl}\, F_2 - k^2 q_2 F_2 = 0 \quad \text{in } \Omega_b, \quad \nu \times F_2 = 0 \quad \text{on } \Gamma_0, \quad \nu \times F_2 = \nu \times E_1 \quad \text{on } \Gamma_b,$$

respectively. Let $E = F_2 - E_1$. Then it is easy to see that

$$\begin{aligned} \mathrm{curl}\,\mathrm{curl}\, E - k^2 q_2 E &= k^2 (q_2 - q_1) E_1 \quad \text{in } \Omega_b, \\ \nu \times E &= 0 \quad \text{on } \Gamma_0 \cup \Gamma_b, \\ \nu \times \mathrm{curl}\, E &= 0 \quad \text{on } \Gamma_b, \end{aligned}$$

where the last quality is obtained from the assumption $T_1 = T_2$. Thus, it follows from the Green vector formula that

$$\begin{aligned} \int_{\Omega_b} (q_2 - q_1) E_1 \cdot \overline{E}_2 dx &= \frac{1}{k^2} \int_{\Omega_b} (\mathrm{curl}\,\mathrm{curl}\, E - k^2 q_2 E) \cdot \overline{E}_2 dx \\ &= \frac{1}{k^2} \int_{\Omega_b} (\mathrm{curl}\, E \cdot \mathrm{curl}\, \overline{E}_2 - k^2 q_2 E \cdot \overline{E}_2) dx \\ &= \frac{1}{k^2} \int_{\Omega_b} (E \cdot \mathrm{curl}\,\mathrm{curl}\, \overline{E}_2 - k^2 q_2 E \cdot \overline{E}_2) dx \\ &= \frac{1}{k^2} \int_{\Omega_b} (E \cdot k^2 q_2 \overline{E}_2 - k^2 q_2 E \cdot \overline{E}_2) dx = 0. \end{aligned}$$

The proof is thus completed. □

For $g \in L_t^2(\Gamma_a)$ appearing in the incident waves (1), we define an operator $F : L_t^2(\Gamma_a) \to H_t^{-1/2}(\mathrm{div}\,,\Gamma_b)$ by

$$F(g) = e_3 \times E(x,g) \quad \text{on} \quad \Gamma_b,$$

where $E(x,g)$ solves the problem (2)-(5) with the incident wave $E^i(x,g)$. The operator F can be considered as an input-output operator mapping the sum of the electric dipoles to the tangential component of the corresponding total field on Γ_b. Moreover, for all $g \in L_t^2(\Gamma_a)$, the operator F has a dense range in $H_t^{-1/2}(\mathrm{div}\,,\Gamma_b)$, as stated in the following lemma.

Lemma 5.2 *The operator F has a dense range in $H_t^{-1/2}(\mathrm{div}\,,\Gamma_b)$.*

Proof. We only need to prove that $F^* : H_t^{-1/2}(\mathrm{curl}\,,\Gamma_b) \to L_t^2(\Gamma_a)$ is injective. First, we show that for any $f \in H_t^{-1/2}(\mathrm{curl}\,,\Gamma_b)$, $F^*(f)$ is given by

$$F^*(f) = \left[\mathrm{curl}\,_y \mathrm{curl}\,_y \int_{\Gamma_b} \overline{G(x,y)} \mathrm{curl}\,(\overline{V^+(x) - W(x)}) \times e_3 ds(x)\right]_T, \tag{2}$$

where the superscripts $+$ and $-$ indicate the limit obtained from $\mathbb{R}_3\backslash\Omega_b$ and Ω_b, respectively, and for any $a > b$ the function $V \in H(\mathrm{curl}\,,\Omega_b) \cap H(\mathrm{curl}\,,\Omega_a\backslash\Omega_b)$ solves the problem

$$\mathrm{curl}\,\mathrm{curl}\, V - k^2 V = 0 \qquad \text{for} \quad x_3 > b, \tag{3}$$

$$\mathrm{curl}\,\mathrm{curl}\, V - k^2 q V = 0 \qquad \text{in} \quad \Omega_b, \tag{4}$$

$$\nu \times V = 0 \qquad \text{on} \quad \Gamma_0, \tag{5}$$

$$\nu \times V^+ - \nu \times V^- = 0 \qquad \text{on} \quad \Gamma_b, \tag{6}$$

$$\left[\mathrm{curl}\, V^+ - \mathrm{curl}\, V^-\right]_T = \overline{f} \qquad \text{on} \quad \Gamma_b \tag{7}$$

and satisfies the Rayleigh expansion condition (6) with α replaced by $-\alpha$ for $x_3 > b$, that is,

$$V(x) = \sum_{n\in\mathbb{Z}^2} V_n e^{i(\alpha'_n \cdot x + \beta'_n x_3)}, \qquad x_3 \geq b \tag{8}$$

with $\alpha'_n = (-\alpha_1 + n_1, -\alpha_2 + n_2, 0) \in \mathbb{R}^3$, $V_n \in \mathbb{C}^3$ and

$$\beta'_n = \begin{cases} (k^2 - |\alpha'_n|^2)^{\frac{1}{2}} & \text{if } |\alpha'_{\mathrm{n}}| < \mathrm{k}, \\ i(|\alpha'_n|^2 - k^2)^{\frac{1}{2}} & \text{if } |\alpha'_{\mathrm{n}}| > \mathrm{k}. \end{cases}$$

In addition, the function W is given by

$$W(x) = \sum_{n\in\mathbb{Z}^2} V_n e^{i((\alpha'_n \cdot x + \beta'_n (2b - x_3))}, \qquad x_3 \leq b. \tag{9}$$

In fact, for any $f \in H_t^{-1/2}(\mathrm{curl}\,,\Gamma_b)$ and $g \in H_t^{-1/2}(\mathrm{div}\,,\Gamma_b)$ we have

$$\begin{aligned}
&< Fg, f >_{H_t^{-1/2}(\mathrm{div}\,,\Gamma_b)\times H_t^{-1/2}(\mathrm{curl}\,,\Gamma_b)} \\
&= \int_{\Gamma_b} \nu \times E(\cdot,g) \cdot \overline{f} ds \\
&= \int_{\Gamma_b} \nu \times E(\cdot,g) \cdot [\mathrm{curl}\, V^+ - \mathrm{curl}\, V^-] ds \\
&= \int_{\Gamma_b} [(\nu \times E \cdot \mathrm{curl}\, V^+ - \nu \times V^+ \cdot \mathrm{curl}\, E) - (\nu \times E \cdot \mathrm{curl}\, V^- - \nu \times V^- \cdot \mathrm{curl}\, E)] ds,
\end{aligned}$$

where the transmission conditions (6) and (7) have been used. It follows from the Maxwell equations (4) and (3) and the boundary conditions (5) and (4) that

$$\int_{\Gamma_b} [\nu \times E \cdot \mathrm{curl}\, V^- - \nu \times V^- \cdot \mathrm{curl}\, E] ds = 0. \tag{10}$$

On the other hand, from the Rayleigh expansion conditions (6) and (8) it is derived that

$$\begin{aligned} &\int_{\Gamma_b} [\nu \times E \cdot \mathrm{curl}\, V^+ - \nu \times V^+ \cdot \mathrm{curl}\, E] ds \\ &= \int_{\Gamma_b} [(\nu \times E^i \cdot \mathrm{curl}\, V^+ - \nu \times V^+ \cdot \mathrm{curl}\, E^i) + (\nu \times E^s \cdot \mathrm{curl}\, V^+ - \nu \times V^+ \cdot \mathrm{curl}\, E^s)] ds \\ &= \int_{\Gamma_b} [\nu \times E^i \cdot \mathrm{curl}\, V^+ - \nu \times V^+ \cdot \mathrm{curl}\, E^i] ds. \end{aligned} \tag{11}$$

Similarly, from the definition of E^i and the Rayleigh expansion condition (9) it follows that

$$\int_{\Gamma_b} [\nu \times E^i \cdot \mathrm{curl}\, W - \nu \times W \cdot \mathrm{curl}\, E^i] ds = 0. \tag{12}$$

The equations (10)-(12) together with the fact that $V = W$ on Γ_b yield

$$\begin{aligned} < Fg, f > &= \int_{\Gamma_b} [\nu \times E^i \cdot \mathrm{curl}\, V^+ - \nu \times V^+ \cdot \mathrm{curl}\, E^i] ds \\ &= \int_{\Gamma_b} [\nu \times E^i \cdot \mathrm{curl}\, V^+ - \nu \times W \cdot \mathrm{curl}\, E^i] ds \\ &= \int_{\Gamma_b} [\nu \times E^i \cdot \mathrm{curl}\, V^+ - \nu \times E^i \cdot \mathrm{curl}\, W] ds \\ &= \int_{\Gamma_b} \nu \times E^i \cdot (\mathrm{curl}\, V^+ - \mathrm{curl}\, W) ds. \end{aligned}$$

Substituting the expression (1) of E^i into the above equation and exchanging the order of integration we get

$$< Fg, f >= \int_{\Gamma_a} g(y) \cdot \mathrm{curl}_y \mathrm{curl}_y \left[\int_{\Gamma_b} G(x,y) \mathrm{curl}\, [V^+(x) - W(x)] \times e_3 ds(x) \right] ds(y),$$

which implies (2).

We now prove that F^* is injective. Suppose $F^*(f) = 0$ for some $f \in H_t^{-1/2}(\mathrm{curl}\,, \Gamma_b)$. Define U by

$$U(y) := \mathrm{curl}_y \mathrm{curl}_y \left[\int_{\Gamma_b} \overline{G(x,y)} h(x) ds(x) \right], \qquad y \in \mathbb{R}^3 \backslash \Gamma_b,$$

where $h = \mathrm{curl}\, (\overline{V^+ - W}) \times e_3$. Then $e_3 \times U(y) = 0$ on Γ_a. It is clear that $U(y)$ is a $-\alpha$-quasi-periodic function satisfying the Rayleigh expansion condition (6) when $y_3 > a$. By the uniqueness of solutions to the exterior Dirichlet problem (see [2]) we have $U(y) = 0$ when $y_3 > a$, which together with the unique continuation principle ([11]) implies that $U(y) = 0$ when $y_3 > b$. Now from the jump relation $e_3 \times U^+(y) - e_3 \times U^-(y) = 0$ on Γ_b and again the uniqueness of solutions for the exterior Dirichlet problem for $y_3 < b$ we get that $U(y) = 0$ when $y_3 < b$. Thus, $h(y) = e_3 \times \mathrm{curl}\,[U^+(y) - U^-(y)] = 0$ on Γ_b, which, together with (8) and (9), implies that

$$e_3 \times V^+ = e_3 \times W, \quad e_3 \times \mathrm{curl}\, V^+ = e_3 \times \mathrm{curl}\, W \quad \text{on } \Gamma_b. \tag{13}$$

Since V and W satisfy the Maxwell equation $\mathrm{curl}\,\mathrm{curl}\, E - k^2 E = 0$ in the regions $x_3 > b$ and $x_3 < b$, respectively, then it follows easily from the transmission condition (13) and the Rayleigh expansion conditions (8) and (9) that $V = 0$ for $x_3 > b$ and $W = 0$ for $x_3 < b$. Thus, by (6) we have $\nu \times V^- = 0$ on Γ_b, so $V \in H(\mathrm{curl}\,, \Omega_b)$ satisfies the problem (1)-(3) with $f = 0$. By Lemma 3.1 we have $V = 0$ in Ω_b. Thus, $f = [\mathrm{curl}\, \overline{V}^+ - \mathrm{curl}\, \overline{V}^-]_T = 0$, which completes the proof of Lemma 5.2. □

Combining Lemmas 5.1 and 5.2, we have the following orthogonality relation for two different functions q_i $(i = 1, 2)$.

Lemma 5.3 *Let the incident waves* $E^i(x, g)$ *be defined by* (1). *If*

$$e_3 \times E_1(x, g) = e_3 \times E_2(x, g) \qquad \text{on } \Gamma_a \tag{14}$$

for all $g \in L_t^2(\Gamma_a)$ *and some* $a > b$, *then the following orthogonality relation holds:*

$$\int_{\Omega_b} E_1(x) \cdot \overline{E}_2(x)(q_1(x) - q_2(x))dx = 0,$$

where E_1, $E_2 \in H(\mathrm{curl}\,, \Omega_b)$ *solve the problem* $(1) - (3)$ *with* q *replaced by* q_1 *and* $\overline{q}_2$, *respectively.*

Proof. From the equation (14), the uniqueness of solutions for the exterior Dirichlet problem and the unique continuation principle it follows that $E_1(x, g) = E_2(x, g)$ for all $x_3 > b$. This implies that

$$e_3 \times \mathrm{curl}\, E_1^+(x, g) = e_3 \times \mathrm{curl}\, E_2^+(x, g) \qquad \text{on } \Gamma_b.$$

Since $[e_3 \times \mathrm{curl}\, E_j^+(x; g)]|_{\Gamma_b} = 0$ for $j = 1, 2$, then we have

$$e_3 \times \mathrm{curl}\, E_1^-(x, g) = e_3 \times \mathrm{curl}\, E_2^-(x, g) \qquad \text{on } \Gamma_b.$$

By the above two equalities and the definition of T_i we have

$$T_1(e_3 \times E_1(x, g)) = T_2(e_3 \times E_2(x, g))$$

for all $g \in L_t^2(\Gamma_a)$. The continuity of T_j $(j = 1, 2)$ and Lemma 5.2 lead to

$$T_1(f) = T_2(f) \qquad \forall f \in H_t^{-1/2}(\mathrm{div}\,, \Gamma_b).$$

This together with Lemma 5.1 gives the desired result. □

We are now ready to prove our main result for the inverse scattering problem.

Theorem 5.4 *Let* q_j $(j = 1, 2)$ *satisfy the assumptions* $(A1) - (A3)$ *and let* q_j *depend on only one direction* x_1 *or* x_2 *with* $j = 1, 2$. *If*

$$e_3 \times E_1(x, g) = e_3 \times E_2(x, g) \quad \text{on } \Gamma_a$$

for all $g \in L_t^2(\Gamma_a)$ *with some* $a > b$, *where* $E_j(x, g)$ *solves the problem* $(2) - (5)$ *with* $q = q_j$ $(j = 1, 2)$ *corresponding to the incident wave* $E^i(x, g)$ *given by* (1), *then* $q_1 = q_2$.

Proof. By Lemma 5.3 we have the orthogonality relation:

$$\int_{\Omega_b} E_1(x)\cdot \overline{E}_2(x)\,[q_1(x)-q_2(x)]\,dx = 0, \tag{15}$$

where $E_1,\ E_2 \in H(\mathrm{curl}\,,\Omega_b)$ solve the problem (1)-(3) with q replaced by q_1 and $\overline{q}_2$, respectively.

We now look for solutions to the problem (1)-(3) in the following form:

$$E(x) = (0,0,E_3(x_1,x_2)) = (0,0,v(x_1)u(x_2))$$

with the scalar functions v and u satisfying the following quasi-periodic conditions:

$$v(x_1)e^{2i\alpha_1\pi} = v(x_1+2\pi), \qquad u(x_2)e^{2i\alpha_2\pi} = v(x_2+2\pi).$$

It is clear that such a function E is α-quasi-periodic and satisfies the boundary condition (2). Without loss of generality, we may assume that $q_j(x)=q_j(x_1)$, that is, q_j depends only the x_1-direction with $j=1,2$. Substituting such E into the Maxwell equation (1) and noting that $\mathrm{curl}\,\mathrm{curl} = -\triangle + \nabla(\nabla\cdot)$, we find that

$$v''(x_1)u(x_2) + v(x_1)u''(x_2) + k^2 q(x_1)v(x_1)u(x_2) = 0, \qquad x_1,\ x_2 \in (0,2\pi),$$

which implies that

$$\frac{v''(x_1)}{v(x_1)} + k^2 q(x_1)v(x_1) = \frac{u''(x_2)}{u(x_2)} = \lambda$$

for some constant λ, where $x_1, x_2 \in (0,2\pi)$. Following the idea of Kirsch [17], we construct a special kind of solutions v by considering the following quasi-periodic Sturm-Liouville eigenvalue problem:

$$(\mathrm{I}):\quad \begin{cases} v''(x_1) + k^2 q(x_1)v(x_1) = \lambda v(x_1), & x_1 \in (0,2\pi)\\ v(x_1)e^{2i\alpha_1\pi} = v(x_1+2\pi), & \\ v'(x_1)e^{2i\alpha_1\pi} = v'(x_1+2\pi). & \end{cases}$$

The eigenvalues λ_n and the corresponding eigenfunctions v_n, normalized to $v_n(0)=1$, have the following asymptotic behaviors as $n\to\infty$ (see [29]):

$$\begin{aligned} \lambda_n^{\pm} &= \left(n \pm \frac{\alpha_1}{2\pi}\right)^2 - \frac{k^2}{2\pi}\int_0^{2\pi} q(s)ds + \mathcal{O}\left(\frac{1}{n}\right), \\ v_n^{\pm}(x_1) &= \exp\left[i(\pm n + \frac{\alpha_1}{2\pi})x_1\right] + \mathcal{O}\left(\frac{1}{n}\right) \end{aligned}$$

which are uniform in $x_1 \in [0,2\pi]$. We also consider the following quasi-periodic boundary problem for u:

$$(\mathrm{II}):\quad \begin{cases} u''(x_2) - \lambda_n u(x_2) = 0, & x_2 \in (0,2\pi)\\ u(x_2)e^{2i\alpha_2\pi} = v(x_2+2\pi). & \end{cases}$$

The non-trivial solutions to the problem (II) can be written explicitly as

$$u_n(x_2) = c_{n,1}e^{\sqrt{\lambda_n}x_2} + c_{n,1}e^{-\sqrt{\lambda_n}x_2}, \quad \lambda_n \neq 0,$$

where $c_{n,1}$ and $c_{n,2}$ are constants satisfying

$$c_{n,1} = c_{n,2} \left(e^{-2\pi\sqrt{\lambda_n}} - e^{i2\pi\alpha_2}\right) \Big/ \left(e^{i2\pi\alpha_2} - e^{2\pi\sqrt{\lambda_n}}\right). \tag{16}$$

Now, let $E^{\pm}_{3,n} = v^{\pm}_n(x_1)u^{\pm}_n(x_2)$ be the third component of $E^{\pm}_n = (0, 0, E^{\pm}_{3,n})$ corresponding to $q_1(x_1)$ and let $E^{\pm}_{3,m} = v^{\pm}_m(x_1)u^{\pm}_m(x_2)$ be the third component of $E^{\pm}_n$ corresponding to $\overline{q_2}(x_1)$. It follows from (15) that

$$0 = \int_{\Omega_b} E_{3,n}(x_1, x_2) \cdot \overline{E}_{3,m}(x_1, x_2) \,[q_1(x_1) - q_2(x_1)]\, dx = bA^{n,m}_1 A^{n,m}_2, \tag{17}$$

where

$$\begin{aligned} A^{n,m}_1 : &= \int_0^{2\pi} [q_1(x_1) - q_2(x_1)] e^{i(n-m)x_1} dx_1 + \mathcal{O}\left(\frac{1}{n}\right) + \mathcal{O}\left(\frac{1}{m}\right), \\ A^{n,m}_2 : &= \int_0^{2\pi} \left(c_{n,1} e^{\sqrt{\lambda_n} x_2} + c_{n,2} e^{-\sqrt{\lambda_n} x_2}\right) \left(\overline{c_{m,1} e^{\sqrt{\lambda_m} x_2} + c_{m,2} e^{-\sqrt{\lambda_m} x_2}}\right) dx_2 \end{aligned}$$

and $c_{n,j},\ c_{m,j}$ satisfy (16) with $j = 1, 2$. For arbitrarily fixed $l \in \mathbb{N}$, letting $m = n - l$ gives

$$\begin{aligned} A^{m+l,m}_1 &= \int_0^{2\pi} [q_1(x_1) - q_2(x_1)] e^{ilx_1} d(x_1) + \mathcal{O}\left(\frac{1}{m}\right), \\ A^{m+l,m}_2 &= \int_0^{2\pi} \left(c_{m+l,1} e^{\sqrt{\lambda_{m+l}} x_2} + c_{m+l,2} e^{-\sqrt{\lambda_{m+l}} x_2}\right) \left(\overline{c_{m,1} e^{\sqrt{\lambda_m} x_2} + c_{m,2} e^{-\sqrt{\lambda_m} x_2}}\right) dx_2. \end{aligned}$$

We can always choose appropriate constants $c_{m,2}$ and $c_{m,1}$ satisfying (16) such that $A^{m+l,m}_2 \neq 0$ for sufficiently large m. In fact, we may assume that l is a positive number since otherwise we can take $n = m - l'$ for some positive l' instead of l. Now choose $c_{m,2} = e^{2\pi\sqrt{\lambda_m}}$. Then, by (16), $|c_{m,1}| \geq C_1$ for large m with some positive constant C_1 independent of m and $\left|\int_0^{2\pi} c_{m,2} e^{-2\pi\sqrt{\lambda_m} x_2} dx_2\right|$ tends to $+\infty$ as $m \to \infty$. This implies that $|A^{m+l,m}_2| \to +\infty$ as $m \to +\infty$. Letting $m \to +\infty$ we conclude from (17) and the above discussion that

$$\int_0^{2\pi} (q_1(x_1) - q_2(x_1)) e^{ilx_1} dx_1 = 0$$

for every $l \in \mathbb{N}$, which implies that $q_1 = q_2$. The proof is thus completed. □

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