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Dual representations for general multiple stopping problems

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Abstract

In this paper, we study the dual representation for generalized multiple stopping problems, hence the pricing problem of general multiple exercise options. We derive a dual representation which allows for cashflows which are subject to volume constraints modeled by integer valued adapted processes and refraction periods modeled by stopping times. As such, this extends the works by Schoenmakers [2010], Bender [2011a], Bender [2011b], Aleksandrov and Hambly [2010] and Meinshausen and Hambly [2004] on multiple exercise options, which either take into consideration a refraction period or volume constraints, but not both simultaneously. We also allow more flexible cashflow structures than the additive structure in the above references. For example some exponential utility problems are covered by our setting. We supplement the theoretical results with an explicit Monte Carlo algorithm for constructing confidence intervals for the price of multiple exercise options and exemplify it by a numerical study on the pricing of a swing option in an electricity market.

1 Introduction

The last decades have seen ground breaking developments of Monte Carlo methods for American options based on multi-dimensional underlying price processes. In the late nineties the regression based methods by Carriere [1996], Longstaff and Schwartz [2001], and Tsitsiklis and Van Roy [2001] may be considered as main breakthroughs. In general these methods provide lower bounds on the option price by constructing an approximation to the optimal exercise (stopping) time via regression on a set of basis functions. As such these approaches are termed “primal”. At the beginning of this century Rogers [2002] and independently Haugh and Kogan [2004] provided the next breakthrough by presenting a “dual” representation for the optimal stopping problem corresponding to the American pricing problem. In this representation the option price is expressed as infimum of an expectation over a set of martingales. (The key behind his dual representation can already be found in Davis and Karatzas [1994], in fact.) While in the primal methods the central problem is to find a “good” stopping time, in the dual problem one needs to find a “good” martingale, which leads to an upper bound for the price of an American option. As one of the standard numerical approaches to compute dual upper bounds for American options by Monte Carlo we refer to Andersen and Broadie [2004].

During the same time, in the emerging electricity markets products with a multiple of exercise opportunities, such as “swing options”, became popular. Naturally, pricing of such a product leads to a multiple stopping problem, and so numerical methods for solving multiple stopping problems were called for. In this respect, generalization of the existing primal regression methods for standard optimal stopping was just a matter of routine. Further, Bender and Schoenmakers [2006] developed a kind of policy iteration for multiple stopping. However, regarding the dual approach the situation was not so clear. Meinshausen and Hambly [2004] proposed a dual representation for the multiple stopping problem via expressing the excess value due to each additional exercise right by an infimum of an expectation over a set of martingales and a set of stopping times. This line of research was carried out further by Aleksandrov and Hambly [2010] and Bender [2011b] in the context of dual pricing of multi-exercise options under volume constraints. Recently, Schoenmakers [2010] introduced a dual representation for the price of a multiple exercise option in contrast to the dual representation for the excess value of an additional right. This new dual representation involves an infimum over martingales only and can thus be considered as a more natural extension of the dual for single exercise options. This approach was generalized by Bender [2011a] to a continuous time setting involving (constant) refraction periods.

In the mean time Kobylanski et al. [2011] introduced and studied multiple stopping problems in the primal sense in a far more general context, where the pay-off is considered to be some abstract functional of an (ordered) sequence of stopping times. The goal of the present paper is to find (pure) martingale dual representations for such generalized multiple stopping problems in a discrete time setting. As we will show, such representations can be constructed even in a most general setting. However, for practical implementation these general representations unfold their full strength only, if applied to some more specifically structured cashflows. In this respect we study a generic pay-off structure with both multiplicative and additive structure that incorporates (integer valued) volume constraints and refraction periods given by stopping times. We furthermore provide an explicit Monte Carlo based algorithm and give a detailed numerical study exemplifying the pricing of swing options. Comparing to existing works, the numerical experiments reveal that, by and large, the dual algorithms due to our new representations applied to the problem type considered in Aleksandrov and Hambly [2010] and Bender [2011b] produce tighter upper bounds on the option price, in particular when the number of exercise rights is large. We moreover present a numerical example which involves swing options subject to both volume constraints and refraction periods and give tight confidence intervals for the respective option prices. We underline that the latter example cannot be treated by the dual methods presented in the literature so far.

The structure of the paper is as follows: In Section 2 we derive a dual representation for general multiple stopping problems in terms of a family of martingales. As this family of martingales is typically too large for practical purposes in general, we specialize to a generic cash-flow with additive and multiplicative structure which incorporates volume constraints and refraction periods in Section 3. In this section we then prove two dual representations. One is in terms of the Doob decomposition of the Snell envelopes of an auxiliary family of stopping problems, the other one only requires approximations of these Snell envelopes. In Section 4 we explain how to build a Monte-Carlo algorithm for computing confidence intervals on the value of the multiple stopping problems based on the results of Section 3 and perform some numerical experiments in the context of swing option pricing.

2 General multiple stopping problem

In this section we consider a multiple stopping problem in discrete time $i = 0, \ldots, T$, where $T \in \mathbb{N}$ is a fixed an finite time horizon. We further introduce a "cemetery time" $\partial := T + 1$ where all rights will be exercised, which are not exercised up to time T. For a given filtration $(\mathcal{F}_i)_{0 \le i \le \partial}$ and a number L of exercise dates we next consider a cashflow X as a map $X : \{0, ..., T, \partial\}^L \times \Omega \to \mathbb{R}$ which satisfies for all $0 \le i_1 \le \cdots \le i_L \le \partial$,

$$X_{i_1,\ldots,i_L} \text{ is } \mathcal{F}_{i_L}\text{-measurable,}$$
$$\mathbb{E}\,|X_{i_1,\ldots,i_L}| < \infty.$$

Now consider the stopping problem (sup:=ess.sup, $\mathbb{E}_i := \mathbb{E}_{\mathcal{F}_i}$),

$$Y_i^{*,L} = \sup_{i \le \tau^{(1)} \le \cdots \le \tau^{(L)}} \mathbb{E}_i \, X_{\tau^{(1)},\ldots,\tau^{(L)}},$$

where the supremum runs over a family of ordered stopping times $\tau^{(k)}, 1 \le k \le L$.

Let us define for $k = 2, \ldots, L$ and $0 \leq j_1 \ldots \leq j_{k-1} \leq r \leq \partial$,

$$Y_r^{*L-k+1, j_1, \ldots, j_{k-1}} := \sup_{r \leq \tau^{(k)} \leq \cdots \leq \tau^{(L)}} \mathbb{E}_r X_{j_1, \ldots, j_{k-1}, \tau^{(k)}, \ldots, \tau^{(L)}}, \tag{2.1}$$

with the convention that for $k = 1$, we put $Y_r^{*L, \varnothing} := Y_r^{*L}$, and for $k = L+1$, we put $Y_r^{*0, j_1, \ldots, j_L} = X_{j_1, \ldots, j_L}$.

Proposition 1 *We have the following reduction principle*

$$Y_r^{*L-k+1, j_1, \ldots, j_{k-1}} = \sup_{\tau \geq r} \mathbb{E}_r Y_\tau^{*L-k, j_1, \ldots, j_{k-1}, \tau}, \quad r \geq j_{k-1}. \tag{2.2}$$

Proof. This principle can be straightforwardly proved in an inductive manner, but it can also be considered as a discrete time version of a related result in a continuous time setting from Kobylanski et al. [2011]. ■

In what follows the following remark turns out to be useful.

Remark 2 *We say that a martingale $(M_r)_{r \geq p}$ is a Doob martingale of $(Y_r)_{r \geq p}$, whenever there exists a predictable process $(A_r)_{r \geq p}$, such that $Y_r - M_r + A_r$ is $\mathcal{F}_p$-measurable for any $r \geq p$. In particular, for any two Doob martingales $(M_r)_{r \geq p}$ and $(\widetilde{M}_r)_{r \geq p}$ of $(Y_r)_{r \geq p}$, it holds*

$$M_r - M_{r'} = \widetilde{M}_r - \widetilde{M}_{r'} = \sum_{k=r'}^{r-1} \left(Y_{k+1} - \mathbb{E}_k Y_{k+1} \right)$$

for any $r \geq r' \geq p$.

We can now state and prove a dual representation for the general multiple stopping problem in terms of martingales.

Theorem 3 (Dual representation) *In the setting described above, we have that:*

(i) *For any $0 \leq i \leq \partial$ and any set of martingales $\left(M_r^{L-k+1, j_1, \ldots, j_{k-1}} \right)_{r \geq j_{k-1}}$, where $1 \leq k \leq L$, and $i =: j_0 \leq j_1 \leq \cdots \leq j_{k-1}$, it holds*

$$Y_i^{*,L} \leq \mathbb{E}_i \max_{i \leq j_1 \leq \cdots \leq j_L \leq \partial} \left(X_{j_1, \ldots, j_L} + \sum_{k=1}^{L} \left(M_{j_{k-1}}^{L-k+1, j_1, \ldots, j_{k-1}} - M_{j_k}^{L-k+1, j_1, \ldots, j_{k-1}} \right) \right). \tag{2.3}$$

(ii) *It holds for $i \geq 0$*

$$Y_i^{*,L} = \max_{i \leq j_1 \leq \cdots \leq j_L \leq \partial} \left(X_{j_1, \ldots, j_L} + \sum_{k=1}^{L} \left(M_{j_{k-1}}^{*L-k+1, j_1, \ldots, j_{k-1}} - M_{j_k}^{*L-k+1, j_1, \ldots, j_{k-1}} \right) \right)$$

where for $1 \leq k \leq L$, and $i =: j_0 \leq j_1 \leq \cdots \leq j_{k-1}$, $\left(M_r^{*L-k+1, j_1, \ldots, j_{k-1}} \right)_{r \geq j_{k-1}}$ is a Doob martingale of $\left(Y_r^{*L-k+1, j_1, \ldots, j_{k-1}} \right)_{r \geq j_{k-1}}$.

Proof. ***(i)*** For the martingale family as stated we have for any chain of stopping times $0 \leq \tau^{(1)} \leq \cdots \leq \tau^{(L)} \leq \partial$,

$$\mathbb{E}_i \sum_{k=1}^{L} \left(M_{\tau^{(k-1)}}^{L-k+1,\tau^{(1)},...,\tau^{(k-1)}} - M_{\tau^{(k)}}^{L-k+1,\tau^{(1)},...,\tau^{(k-1)}} \right)$$
$$= \sum_{k=1}^{L} \mathbb{E}_i \mathbb{E}_{\tau^{(k-1)}} \left(M_{\tau^{(k-1)}}^{L-k+1,\tau^{(1)},...,\tau^{(k-1)}} - M_{\tau^{(k)}}^{L-k+1,\tau^{(1)},...,\tau^{(k-1)}} \right) = 0,$$

hence,

$$Y_i^{*,L} = \sup_{i \leq \tau^{(1)} \leq \cdots \leq \tau^{(L)}} \mathbb{E}_i \left(X_{\tau^{(1)},...,\tau^{(L)}} + \sum_{k=1}^{L} \left(M_{\tau^{(k-1)}}^{L-k+1,\tau^{(1)},...,\tau^{(k-1)}} - M_{\tau^{(k)}}^{L-k+1,\tau^{(1)},...,\tau^{(k-1)}} \right) \right)$$

from which (i) follows directly.

(ii) For any chain $i \leq j_1 \leq \cdots \leq j_L \leq \partial$ we may write (recalling $j_0 := i$)

$$X_{j_1,...,j_L} + \sum_{k=1}^{L} \left(M_{j_{k-1}}^{*L-k+1,j_1,...,j_{k-1}} - M_{j_k}^{*L-k+1,j_1,...,j_{k-1}} \right)$$
$$= X_{j_1,...,j_L} + \sum_{k=1}^{L} \left(Y_{j_{k-1}}^{*L-k+1,j_1,...,j_{k-1}} - Y_{j_k}^{*L-k+1,j_1,...,j_{k-1}} \right)$$
$$+ \sum_{k=1}^{L} \sum_{l=j_{k-1}}^{j_k-1} \left(\mathbb{E}_l Y_{l+1}^{*L-k+1,j_1,...,j_{k-1}} - Y_l^{*L-k+1,j_1,...,j_{k-1}} \right)$$
$$= Y_i^{*L} + \sum_{k=1}^{L} \left(Y_{j_k}^{*L-k,j_1,...,j_k} - Y_{j_k}^{*L-k+1,j_1,...,j_{k-1}} \right)$$
$$+ \sum_{k=1}^{L} \sum_{l=j_{k-1}}^{j_k-1} \left(\mathbb{E}_l Y_{l+1}^{*L-k+1,j_1,...,j_{k-1}} - Y_l^{*L-k+1,j_1,...,j_{k-1}} \right). \tag{2.4}$$

By the reduction principle (2.2) it follows that $\left(Y_r^{*L-k+1,j_1,...,j_{k-1}} \right)_{r \geq j_{k-1}}$ is a super-martingale which dominates the (virtual) cash-flow $Y_r^{*L-k,j_1,...,j_{k-1},r}$ for $k = 1, ..., L$. Hence, expression (2.4) is less than or equal to Y_i^{*L}. It thus follows that

$$\max_{i \leq j_1 \leq \cdots \leq j_L \leq \partial} \left(X_{j_1,...,j_L} + \sum_{k=1}^{L} \left(M_{j_{k-1}}^{*L-k+1,j_1,...,j_{k-1}} - M_{j_k}^{*L-k+1,j_1,...,j_{k-1}} \right) \right) \leq Y_i^{*L},$$

and then, an application of (i) finishes the proof. ■

A straightforward consequence of Theorem 3 is the following dual representation in terms of approximate Snell envelopes.

Corollary 4 *For any set* $\left(Y_r^{L-k+1,j_1,\ldots,j_{k-1}}\right)_{r\geq j_{k-1}}$ *of approximations to the Snell envelopes* $\left(Y_r^{*L-k+1,j_1,\ldots,j_{k-1}}\right)_{r\geq j_{k-1}}$ *with* $Y_{j_L}^{0,j_1,\ldots,j_L} := X_{j_1,\ldots,j_L}$, *it holds for* $i \geq 0$

$$Y_i^{*,L} \leq Y_i^L + \mathbb{E}_i \max_{i\leq j_1\leq\cdots\leq j_L\leq\partial} \sum_{k=1}^{L} \left(Y_{j_k}^{L-k,j_1,\ldots,j_k} - Y_{j_k}^{L-k+1,j_1,\ldots,j_{k-1}} \right.$$
$$\left. + \sum_{l=j_{k-1}}^{j_k-1} \left(\mathbb{E}_l Y_{l+1}^{L-k+1,j_1,\ldots,j_{k-1}} - Y_l^{L-k+1,j_1,\ldots,j_{k-1}} \right) \right). \tag{2.5}$$

Equality holds when the Snell envelopes are plugged in.

Proof.
Given $\left(Y_r^{L-k+1,j_1,\ldots,j_{k-1}}\right)_{r\geq j_{k-1}}$, we denote a corresponding family of Doob martingales by $\left(M_r^{L-k+1,j_1,\ldots,j_{k-1}}\right)_{r\geq j_{k-1}}$. Following the same manipulations as in (2.4) and recalling that by definition $Y_{j_L}^{0,j_1,\ldots,j_L} := X_{j_1,\ldots,j_L}$, we get

$$\begin{aligned} & Y_i^L + \sum_{k=1}^{L} \left(\left(Y_{j_k}^{L-k,j_1,\ldots,j_k} - Y_{j_k}^{L-k+1,j_1,\ldots,j_{k-1}} \right) \right. \\ & \qquad \left. + \sum_{l=j_{k-1}}^{j_k-1} \left(\mathbb{E}_l Y_{l+1}^{L-k+1,j_1,\ldots,j_{k-1}} - Y_l^{L-k+1,j_1,\ldots,j_{k-1}} \right) \right) \\ = \quad & X_{j_1,\ldots,j_L} + \sum_{k=1}^{L} \left(M_{j_{k-1}}^{L-k+1,j_1,\ldots,j_{k-1}} - M_{j_k}^{L-k+1,j_1,\ldots,j_{k-1}} \right) \end{aligned}$$

Hence, the assertion is a mere reformulation of Theorem 3.

■

At this point, we stress that the dual representation from Theorem 3 relies on families of martingales $\left(M_r^{L-k+1,j_1,\ldots,j_{k-1}}\right)_{r\geq j_{k-1}}$ whose size is parametrized via the $(k-1)$-tuples $(j_1,\ldots,j_{k-1}), k = 1,\ldots,L.$ Hence, depending on the time horizon T and the number of exercise rights L a huge number of martingales $M^{L-k+1,j_1,\ldots,j_{k-1}}$, $k = 1,\ldots,L$, $0 \leq j_1 \leq \ldots \leq j_{k-1} \leq \partial = T+1$ is required in order to compute an upper price bound by the above dual formulation. It is thus of great importance to single out situations, in which a family of optimal martingales can be constructed from a much smaller family of auxiliary processes. This will be the topic of Section 3. A motivating example in this respect is the standard multiple stopping problem.

Example 5 (Standard multiple stopping) *Let* Z *be a non-negative adapted process with* $Z_j = 0$ *for* $j = \partial$, *i.e. no penalty is imposed for unexercised rights. The standard multiple stopping problem is to maximize* $\mathbb{E}[\sum_{k=1}^{L} Z_{\tau^{(k)}}]$ *over the set of ordered stopping times* $\tau^{(1)} \leq \cdots \leq \tau^{(L)}$ *such that* $\tau^{(k)} < \tau^{(k+1)}$ *or* $\tau^{(k)} = \tau^{(k+1)} = \partial$. *This means that at most one right can be exercised per day, but an arbitrary number of rights can be left unexercised, i.e. is exercised at time* ∂. *This problem can be put into our general setting by* *considering the cashflow*

$$X_{i_1,\dots,i_L} = \begin{cases} \sum_{k=1}^{L} Z_{i_k}, & \text{if } i_{j+1} = i_j \Rightarrow i_j = \partial\,, \\ -N, & \text{else }, \end{cases}$$

for $N \in \mathbb{N}$. Note that the Snell envelope $Y_i^{*,L}$ does not depend on the choice of N, because it is never optimal to exercise X in a way which gives a negative payment. Hence, letting N tend to $-\infty$, Theorem 3 yields,

$$Y_i^{*,L} = \max_{\substack{i \le j_1 \le \cdots \le j_L \\ j_k = j_{k+1} \Rightarrow j_k = \partial}} \left(\sum_{k=1}^{L} Z_{j_k} + \sum_{k=1}^{L} \left(M_{j_{k-1}}^{*L-k+1,j_1,\dots,j_{k-1}} - M_{j_k}^{*L-k+1,j_1,\dots,j_{k-1}} \right) \right)$$

where for $1 \le k \le L$, $i \le j_1 < \cdots < j_{k-1}$, $\left(M_r^{*L-k+1,j_1,\dots,j_{k-1}}\right)_{r \ge j_{k-1}}$ is a Doob martingale of

$$Y_r^{*L-k+1,j_1,\dots,j_{k-1}} = \sum_{p=1}^{k-1} Z_{j_p} + \sup_{\substack{r \le \tau^{(k)} \le \cdots \le \tau^{(L)} \text{ with} \\ \tau^{(p)} = \tau^{(p+1)} \Rightarrow \tau^{(p)} = \partial \text{ and } \tau^{(k)} = j_{k-1} \Rightarrow j_{k-1} = \partial}} \mathbb{E}_r \sum_{p=k}^{L} Z_{\tau^{(p)}}.$$

for $r \ge j_{k-1}$. Define, for $r \ge 0$,

$$Y_r^{*L-k+1} := \sup_{\substack{r \le \tau^{(k)} \cdots \le \tau^{(L)} \text{ with} \\ \tau^{(p)} = \tau^{(p+1)} \Rightarrow \tau^{(p)} = \partial}} \mathbb{E}_r \left(\sum_{p=k}^{L} Z_{\tau^{(p)}} \right)$$

and denote the Doob martingale of $(Y_r^{*L-k+1})_{r \ge 0}$ by $(M_r^{*L-k+1})_{r \ge 0}$. As

$$Y_r^{*L-k+1} - Y_r^{*L-k+1,j_1,\dots,j_{k-1}}$$

is $\mathcal{F}_{j_{k-1}}$-measurable for $r \ge j_{k-1}$, we can conclude by Remark 2 that $(M_r^{*L-k+1})_{r \ge j_{k-1}}$ is a Doob martingale of $\left(Y_r^{*L-k+1,j_1,\dots,j_{k-1}}\right)_{r \ge j_{k-1}}$. Hence we end up with the dual representation

$$Y_i^{*,L} = \max_{\substack{i \le j_1 \le \cdots \le j_L \\ j_k = j_{k+1} \Rightarrow j_k = \partial}} \left(\sum_{k=1}^{L} Z_{j_k} + \sum_{k=1}^{L} \left(M_{j_{k-1}}^{*L-k+1} - M_{j_k}^{*L-k+1} \right) \right)$$

of Schoenmakers [2010]. Here the potentially large family of optimal martingales $\left(M^{*L-k+1,j_1,\dots,j_{k-1}}\right)$, $k = 1,\dots,L$, $0 \le j_1, \dots \le j_{k-1} \le \partial$, collapses, in fact, to a family of L martingales, namely the Doob martingales of Y^{*k}.

3 Generic cashflow with additive and multiplicative structure

We now introduce a generic cashflow structure for which the dual representation simplifies in a similar way than for the standard multiple stopping problem in Example 5. To this end let us consider for each $k = 1, \dots, L$ and

$l = 1, ..., L-1$ two adapted processes $U^{(k)}$ and $V^{(l)}$. We define a “pre-cashflow”

$$\widetilde{X}_{j_1,\dots,j_L} = \sum_{k=1}^{L} U_{j_k}^{(k)} \prod_{l=1}^{k-1} V_{j_l}^{(l)},$$

which is assumed to satisfy $\widetilde{X}_{j_1,\dots,j_L} > -N$ for some (possibly large) $N \in \mathbb{N}$. Concerning the processes $U^{(k)}$ and $V^{(l)}$, we suppose that $U_i^{(k)}$ is integrable for every $k = 1, \dots, L$ and $i = 0, \dots, \partial$, and that $V_i^{(l)}$ is strictly positive and bounded from above for every $l = 1, \dots, L-1$ and $i = 0, \dots, \partial$. The multiple stopping problem which we have in mind is to optimally exercise this pre-cashflow under some constraints on the set of admissible stopping times, which we now formulate. We first define an adapted volume constraint process v with values in $\{1, ..., L\}$ such that $1 \le v_t \le L$ is the maximum number of rights one may exercise at t, and such that $v_\partial = L$. In order to formalize this constraint, we introduce for $p \ge 1$ the mapping $\mathcal{E}_p$ which acts on a non-decreasing p-tuple $(j_1, \dots, j_p)$ by

$$\mathcal{E}_p(j_1, ..., j_p) := \#\{r : 1 \le r \le p,\ j_r = j_p\}.$$

Hence, $\mathcal{E}_p$ denotes the number of rights exercised at j_p in the non-decreasing chain $0 \le j_1 \le \cdots \le j_p \le \partial$. Obviously, an ordered chain of stopping times $\tau^{(1)} \le \cdots \le \tau^{(L)}$ satisfies the volume constraint if and only if $\mathcal{E}_p(\tau^{(1)}, ..., \tau^{(p)}) \le v_{\tau^{(p)}}$ for every $p = 1, \dots, L$. The second constraint, which we want to impose, is a refraction period which specifies the minimal waiting time between to exercises at different times. We admit random refraction periods, i.e. at each time i, $0 \le i < \partial$, we fix a stopping time $\rho^{(i)}$ taking values in $\{i+1, ..., \partial\}$. If at least one right is exercised at time i, then the refraction period constraint imposes that the next right must either be exercised at the same time (if consistent with the volume constraint) or otherwise no earlier than $\rho^{(i)}$. A standard case is $\rho^{(i)} = (i + \delta) \wedge \partial$, where $1 \le \delta \le T$ is deterministic. Both constraints can be summarized by the binary $\mathcal{F}_{j_p}$-measurable random variable

$$\mathcal{C}_p(j_1, ..., j_p) := \begin{cases} 1, & \forall_{1 \le l \le p} :\ \mathcal{E}_l(j_1, ..., j_l) \le v_{j_l} \text{ and } \forall_{1 \le l \le p} :\ j_l > j_{l-1} \Longrightarrow j_l \ge \rho^{(j_{l-1})} \\ 0, & \text{else,} \end{cases}$$

which is equal to 1, if and only if the constraints are satisfied when exercising at the p times $j_1 \le \cdots \le j_p$.

The dynamic multiple stopping problem which we now study is

$$Y_i^{*L} = \sup_{\substack{i \le \tau^{(1)} \le \cdots \le \tau^{(L)} \le \partial \\ \mathcal{C}_L(\tau^{(1)}, ..., \tau^{(L)}) = 1}} \mathbb{E}_i \left[\sum_{k=1}^{L} U_{\tau^{(k)}}^{(k)} \prod_{l=1}^{k-1} V_{\tau^{(l)}}^{(l)}, \right] \tag{3.1}$$

i.e. the supremum is taken over all stopping times with values in $\{i, \dots, T, \partial\}$ which satisfy the volume constraint and the refraction period constraint. This problem fits in our general (unconstrained) setting by considering the cashflow

$$X_{j_1,\dots,j_L} = \begin{cases} \widetilde{X}_{j_1,\dots,j_L}, & \text{if } \mathcal{C}_L(j_1, , ..., j_L) = 1, \\ -N, & \text{else.} \end{cases} \tag{3.2}$$

To illustrate our motivation for studying the previous cashflow, let us have a look at the following examples.

Example 6 (Swing options) *We extend the situation in Example 5 by imposing volume constraints and refraction periods as decribed above. Hence, we have*

$$V_j^{(l)} := 1,\ l = 1, \ldots, L-1,\ j = 0, \ldots, \partial,$$
$$U_j^{(p)} := Z_j\ p = 1, \ldots, L,\ j = 0, \ldots, \partial,$$

where we recall that Z is a nonnegative adapted process with $Z_\partial = 0$. The multiple stopping problem then becomes

$$\sup_{\substack{\tau^{(1)} \leq \cdots \leq \tau^{(L)} \\ \mathcal{C}_L(\tau^{(1)}, \ldots, \tau^{(L)}) = 1}} \mathbb{E}\left[\sum_{k=1}^{L} Z_{\tau^{(k)}}\right],$$

leading to

$$X_{j_1,\ldots,j_L} = \begin{cases} \sum_{k=1}^{L} Z_{j_k}, & \text{if } \mathcal{C}_L(j_1, , \ldots, j_L) = 1, \\ -N, & \text{else.} \end{cases}$$

Here any $N \in \mathbb{N}$ can be chosen because Z is nonnegative.

A dual approach for this multiple stopping probelm was studied by Bender [2011b] and Aleksandrov and Hambly [2010] under volume constraints, but with unit refraction period, i.e. $\rho^{(i)} = i + 1$. The case with non-trivial constant refraction period is treated in Bender [2011a], but only under unit volume constraint, i.e. $v_i = 1$. A typical problem in the context of electricity markets which leads to this type of multiple stopping problem is the pricing of Swing option contracts, in which volume constraints and refraction periods are often imposed. This option pricing problem will be explained in more detail in our numerical study in Section 4.

Example 7 (Exponential utility) *Under the assumptions of the previous example we can also maximize the exponential utility of exercising the cashflow Z_i L-times while obeying the constraints. Given the risk aversion parameter $\alpha \in (0, \infty)$ the corresponding multiple stopping problem becomes*

$$\sup_{\substack{\tau^{(1)} \leq \cdots \leq \tau^{(L)} \\ \mathcal{C}_L(\tau^{(1)}, \ldots, \tau^{(L)}) = 1}} \mathbb{E}\left[-e^{-\alpha \sum_{k=1}^{L} Z_{\tau^{(k)}}}\right].$$

This problem fits in our setting by considering

$$X_{j_1,\ldots,j_L} = \begin{cases} \sum_{k=1}^{L} U_{j_k}^{(k)} \prod_{l=1}^{k-1} V_{j_l}^{(l)}, , & \text{if } \mathcal{C}_L(j_1, , \ldots, j_L) = 1, \\ -N, & \text{else} \end{cases}$$

with

$$V_j^{(l)} := e^{-\alpha Z_j} > 0 \quad \text{and} \quad U_j^{(k)} := \begin{cases} 0, & \text{if } k = 1, \ldots, L-1, \\ -e^{-\alpha Z_j}, & \text{if } k = L, \end{cases}$$

for $j = 0, 1, \ldots, \partial$ and $N \geq 2$.

Example 8 (Portfolio liquidation) *Suppose a (large) investor on a illiquid market wants to sell out (liquidate) L shares of a stock during the period $\{0, ..., T\}$. We assume that $\widetilde{S}_j > 0$, $j = 0, \ldots, T$, is the virtual stock price process reflecting the stock price evolution in the absence of the large investor's trading. In the spirit of Schied and Slynko [2011], Section 3.1, we model the price impact of the large investor by a resilience function G which we here apply to the log-price. Hence, the log-stock price $\ln S_{j_k}^{j_1,\ldots,j_{k-1}}$ at time j_k of the sale of the k-th share, where $k-1$ shares were already sold at dates $0 \le j_1 \le \cdots \le j_{k-1}$, is given by*

$$\ln S_{j_k}^{j_1,\ldots,j_{k-1}} = \ln \widetilde{S}_{j_k} - \sum_{l=1}^{k-1} G(j_k - j_l).$$

We here choose the capped linear resilience function $G(t) = b(1 - at)_+$ for constants $a, b > 0$. Assuming a short time horizon $T \le 1/a$, the investor is thus faced with a multiple stopping problem

$$\sup_{\tau^{(1)} \le \cdots \le \tau^{(L)}} \mathbb{E}\left[\sum_{k=1}^{L} S_{\tau^{(k)}}^{\tau^{(1)},\ldots,\tau^{(k-1)}}\right],$$

which fits in our framework by applying, for $0 \le j_1 \le \cdots \le j_L \le T$, the cashflow

$$\begin{aligned} X_{j_1,\ldots,j_L} &:= \sum_{k=1}^{L} S_{j_k}^{j_1,\ldots,j_{k-1}} = \sum_{k=1}^{L} \widetilde{S}_{j_k} \exp\left(-\sum_{l=1}^{k-1} b\left(1 - a\left(j_k - j_l\right)\right)\right) \\ &= \sum_{k=1}^{L} \widetilde{S}_{j_k} \exp\left[b\left(a\, j_k - 1\right)(k-1)\right] \prod_{l=1}^{k-1} \exp\left(-abj_l\right) \\ &= \sum_{k=1}^{L} U_{j_k}^{(k)} \prod_{l=1}^{k-1} V_{j_l}^{(k)} \end{aligned}$$

with

$$U_j^{(k)} := \widetilde{S}_j \exp\left[b\left(aj - 1\right)(k-1)\right] \quad \textit{and} \quad V_l^{(k)} := \exp\left(-abj\right).$$

(Note that the cemetery time ∂ is irrelevant in this setting and we can e.g. set $U_\partial^{(k)} = 0$, $V_\partial^{(l)} = 1$ to make sure that it is never optimal to exercise at this time).

Similarly to the situation in Example 5, we now introduce a family of auxiliary multiple stopping problems Y_r^{*L-k+1}, which are not parametrized by the times $j_1, \ldots, j_{k-1}$, at which the first rights were exercised. We will then show that a family of optimal martingales for the original multiple stopping problem (3.1) can be constructed via the Doob decomposition of the auxiliary problems. This then leads to a simplified dual representation for (3.1), which can be implemented in practice even when the maturity T and the number of rights L are large.

Define

$$Y_r^{*L-k+1} := \sup_{\substack{\tau^{(k)},\cdots,\tau^{(L)} \text{ with} \\ r \le \tau^{(k)} \le \cdots \le \tau^{(L)} \text{ and } \mathcal{C}_{L-k+1}(\tau^{(k)},\ldots,\tau^{(L)})}} \mathbb{E}_r\left(\sum_{p=k}^{L} U_{\tau^{(p)}}^{(p)} \prod_{l=k}^{p-1} V_{\tau^{(l)}}^{(l)}\right) \tag{3.3}$$

with the convention $Y_r^{*0} := 0$. The following proposition states the Bellman principle for this multiple stopping problem.

Proposition 9 (Dynamic program) *For $r \geq 0$ and $1 \leq k \leq L$ we have,*

$$Y_r^{*L-k+1} = \max \left(\mathbb{E}_r Y_{r+1}^{*L-k+1}, \max_{1 \leq n \leq v_r \wedge L-k+1,} \left(\sum_{p=k}^{k+n-1} U_r^{(p)} \prod_{l=k}^{p-1} V_r^{(l)} + \prod_{l=k}^{k+n-1} V_r^{(l)} \mathbb{E}_r Y_{\rho^{(r)}}^{*L-k-n+1} \right) \right).$$

Proof. From (3.3) we derive straightforwardly,

$$\begin{aligned}
& Y_r^{*L-k+1} \\
= \; & \max_{0 \leq n \leq v_r \wedge L-k+1} \sup_{\substack{r \leq \tau^{(k+n)} \leq \cdots \leq \tau^{(L)} \\ \mathcal{C}_{L-k+1}(r,\ldots,r,\tau^{(k+n)},\ldots,\tau^{(L)})=1}} \mathbb{E}_r \left(\sum_{p=k}^{k+n-1} U_r^{(p)} \prod_{l=k}^{p-1} V_r^{(l)} + \sum_{p=k+n}^{L} U_{\tau^{(p)}}^{(p)} \prod_{l=k}^{p-1} V_{\tau^{(l)}}^{(l)} \right) \\
= \; & \max \left(\mathbb{E}_r Y_{r+1}^{*L-k+1}, \max_{1 \leq n \leq v_r \wedge L-k+1} \left(\sum_{p=k}^{k+n-1} U_r^{(p)} \prod_{l=k}^{p-1} V_r^{(l)} \right. \right. \\
& \left. \left. + \prod_{l=k}^{k+n-1} V_r^{(l)} \sup_{\substack{\rho^{(r)} \leq \tau^{(k+n)} \leq \cdots \leq \tau^{(L)} \\ \mathcal{C}_{L-k-n+1}(\tau^{(k+n)},\ldots,\tau^{(L)})=1}} \mathbb{E}_r \Big(\sum_{p=k+n}^{L} U_{\tau^{(p)}}^{(p)} \prod_{l=k+n}^{p-1} V_{\tau^{(l)}}^{(l)} \Big) \right) \right).
\end{aligned}$$

A standard argument shows that

$$\begin{aligned}
& \sup_{\substack{\rho^{(r)} \leq \tau^{(k+n)} \leq \cdots \leq \tau^{(L)} \\ \mathcal{C}_{L-k-n+1}(\tau^{(k+n)},\ldots,\tau^{(L)})=1}} \mathbb{E}_r \Big(\sum_{p=k+n}^{L} U_{\tau^{(p)}}^{(p)} \prod_{l=k+n}^{p-1} V_{\tau^{(l)}}^{(l)} \Big) \\
= \; & \mathbb{E}_r \sup_{\substack{\rho^{(r)} \leq \tau^{(k+n)} \leq \cdots \leq \tau^{(L)} \\ \mathcal{C}_{L-k-n+1}(\tau^{(k+n)},\ldots,\tau^{(L)})=1}} \mathbb{E}_{\rho^{(r)}} \Big(\sum_{p=k+n}^{L} U_{\tau^{(p)}}^{(p)} \prod_{l=k+n}^{p-1} V_{\tau^{(l)}}^{(l)} \Big) = \mathbb{E}_r Y_{\rho^{(r)}}^{*L-k-n+1},
\end{aligned} \tag{3.4}$$

which concludes the proof. ∎

We now establish a crucial relationship between the Snell envelopes $Y^{*L-k+1,j_1,\ldots,j_{k-1}}$ and Y^{*L-k+1} defined in (3.3). The following Proposition shows that $Y^{*L-k+1,j_1,\ldots,j_{k-1}}$, parametrized by the j_k's can be represented in terms of Y^{*L-k+1} which avoids the j_k's. Notice that, for $k=1$, both Snell envelopes coincide by definition.

Proposition 10 *Suppose $1 < k \leq L+1$. Under the condition $\mathcal{C}_{k-1}(j_1,\ldots,j_{k-1}) = 1$, we have*

(i) for $r > j_{k-1}$ it holds

$$Y_r^{*L-k+1,j_1,\ldots,j_{k-1}} = \sum_{p=1}^{k-1} U_{j_p}^{(p)} \prod_{l=1}^{p-1} V_{j_l}^{(l)} + \mathbb{E}_r Y_{\max\{\rho^{(j_{k-1})},r\}}^{*L-k+1} \prod_{l=1}^{k-1} V_{j_l}^{(l)}. \tag{3.5}$$

(ii) Further it holds

$$\begin{aligned}
Y_{j_{k-1}}^{*L-k+1,j_1,\ldots,j_{k-1}} = \; & \sum_{p=1}^{k-1} U_{j_p}^{(p)} \prod_{l=1}^{p-1} V_{j_l}^{(l)} \\
+ \; & \prod_{l=1}^{k-1} V_{j_l}^{(l)} \max_{n \in N(j_1,\ldots,j_{k-1})} \left\{ \sum_{p=k}^{k-1+n} U_{j_{k-1}}^{(p)} \prod_{l=k}^{p-1} V_{j_{k-1}}^{(l)} + \mathbb{E}_{j_{k-1}} Y_{\rho^{(j_{k-1})}}^{*L-k+1-n} \prod_{l=k}^{k-1+n} V_{j_{k-1}}^{(l)} \right\}
\end{aligned} \tag{3.6}$$

where the maximum runs over the $\mathcal{F}_{j_{k-1}}$-measurable set

$$N(j_1,\dots,j_{k-1}) := \{n;\ 0 \le n \le (v_{j_{k-1}} - \mathcal{E}_{k-1}(j_1,...,j_{k-1})) \wedge (L-k+1)\}.$$

Proof. For $k = L+1$ both assertions are implied by the conventions $Y_r^{*0,j_1,\dots,j_L} = X_{j_1,\dots,j_L}$ and $Y_r^{*0} = 0$. Hence we assume for the remainder of the proof that $1 < k \le L$.

(i) Under $\mathcal{C}_{k-1}(j_1,...,j_{k-1}) = 1$, we have for $r > j_{k-1}$

$$\begin{aligned} & Y_r^{*L-k+1,j_1,\dots,j_{k-1}} \\ = \ & \sum_{p=1}^{k-1} U_{j_p}^{(p)} \prod_{l=1}^{p-1} V_{j_l}^{(l)} + \prod_{l=1}^{k-1} V_{j_l}^{(l)} \sup_{\substack{r\le\tau^{(k)}\le\dots\le\tau^{(L)} \\ \mathcal{C}_L(j_1,\dots,j_{k-1},\tau^{(k)},\dots,\tau^{(L)})=1}} \mathbb{E}_r\left(\sum_{p=k}^{L} U_{\tau^{(p)}}^{(p)} \prod_{l=k}^{p-1} V_{\tau^{(l)}}^{(l)}\right). \end{aligned} \tag{3.7}$$

As $r > j_{k-1}$, we obtain, thanks to (3.4),

$$\begin{aligned} & \sup_{\substack{r\le\tau^{(k)}\le\dots\le\tau^{(L)} \\ \mathcal{C}_L(j_1,\dots,j_{k-1},\tau^{(k)},\dots,\tau^{(L)})=1}} \mathbb{E}_r\left(\sum_{p=k}^{L} U_{\tau^{(p)}}^{(p)} \prod_{l=k}^{p-1} V_{\tau^{(l)}}^{(l)}\right) \\ = \ & \mathbb{1}_{r<\rho^{(j_{k-1})}} \sup_{\substack{\rho^{(j_{k-1})}\le\tau^{(k)}\le\dots\le\tau^{(L)} \\ \mathcal{C}_{L-k+1}(\tau^{(k)},\dots,\tau^{(L)})=1}} \mathbb{E}_r\left(\sum_{p=k}^{L} U_{\tau^{(p)}}^{(p)} \prod_{l=k}^{p-1} V_{\tau^{(l)}}^{(l)}\right) \\ & + \mathbb{1}_{r\ge\rho^{(j_{k-1})}} \sup_{\substack{r\le\tau^{(k)}\le\dots\le\tau^{(L)} \\ \mathcal{C}_{L-k+1}(\tau^{(k)},\dots,\tau^{(L)})=1}} \mathbb{E}_r\left(\sum_{p=k}^{L} U_{\tau^{(p)}}^{(p)} \prod_{l=k}^{p-1} V_{\tau^{(l)}}^{(l)}\right) \\ = \ & \mathbb{1}_{r<\rho^{(j_{k-1})}} \mathbb{E}_r Y_{\rho^{(j_{k-1})}}^{*L-k+1} + \mathbb{1}_{r\ge\rho^{(j_{k-1})}} Y_r^{*L-k+1}. \end{aligned} \tag{3.8}$$

Hence, by combining (3.7) and (3.8) we get (i).

(ii) Given that the first $(k-1)$ rights have been exercised at times $j_1 \le \dots \le j_{k-1}$ the number of the remaining $(L-k+1)$ rights which are also exercised at time j_{k-1} must be chosen from the $\mathcal{F}_{j_{k-1}}$-measurable set $N(j_1,\dots,j_{k-1}) = \{n;\ 0 \le n \le (v_{j_{k-1}} - \mathcal{E}_{k-1}(j_1,...,j_{k-1})) \wedge (L-k+1)\}$. These are the only choices which obey the volume constraint at time j_{k-1}. Hence,

$$Y_{j_{k-1}}^{*L-k+1,j_1,\dots,j_{k-1}} = \max_{n\in N(j_1,\dots,j_{k-1})} \sup_{\substack{\rho^{(j_{k-1})}\le\tau^{(n+k)}\le\dots\le\tau^{(L)} \\ \mathcal{C}_{L-k-n+1}(\tau^{(k+n)},\dots,\tau^{(L)})=1}} \mathbb{E}X_{j_1,\dots,j_{k-1},\cdots,j_{k-1},\tau^{(k+n)},\dots,\tau^{(L)}},$$

where the time index j_{k-1} appears $(n+1)$ times in the nth term. It then follows, for fixed $n \in N(j_1, \ldots, j_{k-1})$,

$$
\begin{aligned}
&\sup_{\substack{\rho^{(j_{k-1})} \leq \tau^{(n+k)} \leq \cdots \leq \tau^{(L)} \\ \mathcal{C}_{L-k-n+1}(\tau^{(k+n)}, \ldots, \tau^{(L)})=1}} \mathbb{E} X_{j_1, \ldots, j_{k-1}, \cdots, j_{k-1}, \tau^{(k+n)}, \ldots, \tau^{(L)}} \\
= &\sum_{p=1}^{k-1} U_{j_p}^{(p)} \prod_{l=1}^{p-1} V_{j_l}^{(l)} + \prod_{l=1}^{k-1} V_{j_l}^{(l)} \sum_{p=k}^{k-1+n} U_{j_{k-1}}^{(p)} \prod_{l=k}^{p-1} V_{j_{k-1}}^{(l)} \\
&+ \prod_{l=1}^{k-1} V_{j_l}^{(l)} \prod_{l=k}^{k-1+n} V_{j_{k-1}}^{(l)} \sup_{\substack{\rho^{(j_{k-1})} \leq \tau^{(n+k)} \leq \cdots \leq \tau^{(L)} \\ \mathcal{C}_{L-k-n+1}(\tau^{(k+n)}, \ldots, \tau^{(L)})=1}} \mathbb{E}_{j_{k-1}} \sum_{p=k+n}^{L} U_{\tau^{(p)}}^{(p)} \prod_{l=k+n}^{p-1} V_{\tau^{(p)}}^{(l)} \\
= &\sum_{p=1}^{k-1} U_{j_p}^{(p)} \prod_{l=1}^{p-1} V_{j_l}^{(l)} + \prod_{l=1}^{k-1} V_{j_l}^{(l)} \sum_{p=k}^{k-1+n} U_{j_{k-1}}^{(p)} \prod_{l=k}^{p-1} V_{j_{k-1}}^{(l)} + \prod_{l=1}^{k-1} V_{j_l}^{(l)} \prod_{l=k}^{k-1+n} V_{j_{k-1}}^{(l)} \mathbb{E}_{j_{k-1}} Y_{\rho^{(j_{k-1})}}^{*L-k-n+1},
\end{aligned}
$$

making again use of (3.4). This implies (3.6).

■

3.1 Dual representation based on Doob decompositions

The goal of this subsection is to prove and discuss the following simplified version of the dual representation from Theorem 3 for multiple stopping problems of the form (3.1).

Theorem 11 *Suppose $Y_i^{*,L}$ is given by (3.1). Then:*

(i) *For any set of martingales $\left(M_r^{(L-k+1)}\right)_{r \geq 0}$, $k = 1, \ldots, L$, and any set of integrable adapted processes $\left(A_r^{(L-k+1)}\right)_{r \geq 0}$, $k = 1, \ldots, L$, it holds for $i \geq 0$ and with $j_0 := i$*

$$
\begin{aligned}
Y_i^{*,L} \quad \leq \quad \mathbb{E}_i \max_{\substack{i \leq j_1 \leq \cdots \leq j_L \leq \partial \\ \mathcal{C}_L(j_1, \ldots, j_L)=1}} &\left(\sum_{k=1}^{L} U_{j_k}^{(k)} \prod_{l=1}^{k-1} V_{j_l}^{(l)} \right. \\
&\left. + \sum_{k=1}^{L} \prod_{l=1}^{k-1} V_{j_l}^{(l)} \left(M_{j_{k-1}}^{(L-k+1)} - M_{j_k}^{(L-k+1)} + \mathbb{E}_{j_k} A_{\rho^{(j_{k-1})}}^{(L-k+1)} - \mathbb{E}_{j_{k-1}} A_{\rho^{(j_{k-1})}}^{(L-k+1)} \right) \right).
\end{aligned}
$$

(ii) *For every $i \geq 0$ it holds with $j_0 := i$*

$$
\begin{aligned}
Y_i^{*,L} \quad = \quad \max_{\substack{i \leq j_1 \leq \cdots \leq j_L \leq \partial \\ \mathcal{C}_L(j_1, \ldots, j_L)=1}} &\left(\sum_{k=1}^{L} U_{j_k}^{(k)} \prod_{l=1}^{k-1} V_{j_l}^{(l)} \right. \\
&\left. + \sum_{k=1}^{L} \prod_{l=1}^{k-1} V_{j_l}^{(l)} \left(M_{j_{k-1}}^{*L-k+1} - M_{j_k}^{*L-k+1} + \mathbb{E}_{j_k} A_{\rho^{(j_{k-1})}}^{*L-k+1} - \mathbb{E}_{j_{k-1}} A_{\rho^{(j_{k-1})}}^{*L-k+1} \right) \right),
\end{aligned}
$$

where M^{*L-k+1}, A^{*L-k+1} are the martingale part and the predictable part of the Doob decomposition of the auxiliary Snell envelopes Y^{*L-k+1} in (3.3), respectively.

We here recall that the Doob decomposition of Y^{*L-k+1} is the unique decomposition of the form

$$Y_r^{*L-k+1} = Y_0^{*L-k+1} + M_r^{*L-k+1} - A_r^{*L-k+1},$$

where the martingale M_r^{*L-k+1} and the predictable process A_r^{*L-k+1} start in zero at time zero. In order to prove Theorem 11 we need the following auxiliary result.

Proposition 12 *Under the assumption of Theorem 11, a Doob martingale of* $\mathbb{E}_r Y^{*L-k+1}_{\rho^{(j_{k-1})}\vee r}$, *say* $\overline{M}_r^{*L-k+1}$, *is determined or* $r \geq j_{k-1}$ *by*

$$\overline{M}_r^{*L-k+1} - \overline{M}_{j_{k-1}}^{*L-k+1} = M_r^{*L-k+1} - M_{j_{k-1}}^{*L-k+1} + \mathbb{E}_{j_{k-1}} A^{*L-k+1}_{\rho^{(j_{k-1})}} - \mathbb{E}_r A^{*L-k+1}_{\rho^{(j_{k-1})}}.$$

Proof. Using the Doob decomposition we may write

$$\begin{aligned}
&\mathbb{E}_r Y^{*L-k+1}_{\rho^{(j_{k-1})}\vee r} \\
&= \mathbb{1}_{r<\rho^{(j_{k-1})}} \mathbb{E}_r Y^{*L-k+1}_{\rho^{(j_{k-1})}} + \mathbb{1}_{r\geq\rho^{(j_{k-1})}} Y_r^{*L-k+1} \\
&= \mathbb{1}_{j_{k-1}\leq r<\rho^{(j_{k-1})}} \left(Y^{*L-k+1}_{j_{k-1}} + M_r^{*L-k+1} - M^{*L-k+1}_{j_{k-1}} - \mathbb{E}_r A^{*L-k+1}_{\rho^{(j_{k-1})}} + A^{*L-k+1}_{j_{k-1}} \right) \\
&\quad + \mathbb{1}_{r\geq\rho^{(j_{k-1})}} \left(Y^{*L-k+1}_{j_{k-1}} + M_r^{*L-k+1} - M^{*L-k+1}_{j_{k-1}} - A_r^{*L-k+1} + A^{*L-k+1}_{j_{k-1}} \right) \\
&= Y^{*L-k+1}_{j_{k-1}} + M_r^{*L-k+1} - M^{*L-k+1}_{j_{k-1}} \\
&\quad - \mathbb{1}_{j_{k-1}\leq r<\rho^{(j_{k-1})}} \mathbb{E}_r A^{*L-k+1}_{\rho^{(j_{k-1})}} - \mathbb{1}_{r\geq\rho^{(j_{k-1})}} A_r^{*L-k+1} + A^{*L-k+1}_{j_{k-1}} \\
&= Y^{*L-k+1}_{j_{k-1}} + A^{*L-k+1}_{j_{k-1}} - \mathbb{1}_{r\geq\rho^{(j_{k-1})}} \left(A_r^{*L-k+1} - A^{*L-k+1}_{\rho^{(j_{k-1})}} \right) && (3.9) \\
&\quad + M_r^{*L-k+1} - M^{*L-k+1}_{j_{k-1}} - \mathbb{E}_r A^{*L-k+1}_{\rho^{(j_{k-1})}}. && (3.10)
\end{aligned}$$

Since line (3.9) is the sum of a $\mathcal{F}_{k-1}$-measurable random variable and a predictable process and line (3.10) is a martingale, the proposition follows.

■

We now can prove the dual representation.

Proof of Theorem 11. ***(i)*** Suppose that, for $k = 1, \ldots, L$, $M^{(L-k+1)}$ is a martingale and $A^{(L-k+1)}$ is an adapted and integrable process. Then, the process $M_r^{L-k+1, j_1, \ldots, j_{k-1}}$ defined, for $r \geq j_{k-1}$, via

$$M_r^{L-k+1, j_1, \ldots, j_{k-1}} := \prod_{l=1}^{k-1} V_{j_l}^{(l)} \left(M_r^{(L-k+1)} - M_{j_{k-1}}^{(L-k+1)} + \mathbb{E}_{j_{k-1}} A^{(L-k+1)}_{\rho^{(j_{k-1})}} - \mathbb{E}_r A^{(L-k+1)}_{\rho^{(j_{k-1})}} \right)$$

is a martingale due to the boundedness of the $V^{(l)}$'s. By Theorem 3-***(i)***, we have

$$\begin{aligned}
Y_i^{*L} &\leq \mathbb{E}_i \max_{i\leq j_1\leq\cdots\leq j_L\leq\partial} \left(X_{j_1,\ldots,j_L} + \sum_{k=1}^{L} \left(M_{j_{k-1}}^{L-k+1, j_1, \ldots, j_{k-1}} - M_{j_k}^{L-k+1, j_1, \ldots, j_{k-1}} \right) \right) \\
&= \mathbb{E}_i \max_{i\leq j_1\leq\cdots\leq j_L\leq\partial} \Bigg(X_{j_1,\ldots,j_L} \\
&\qquad + \sum_{k=1}^{L} \prod_{l=1}^{k-1} V_{j_l}^{(l)} \left(M_{j_{k-1}}^{(L-k+1)} - M_{j_k}^{(L-k+1)} + \mathbb{E}_{j_k} A^{(L-k+1)}_{\rho^{(j_{k-1})}} - \mathbb{E}_{j_{k-1}} A^{(L-k+1)}_{\rho^{(j_{k-1})}} \right) \Bigg)
\end{aligned}$$

with X as defined in (3.2) for sufficiently large $N \in \mathbb{N}$. Letting N tend to infinity, we observe that maximization only takes place over those $j_1 \leq \cdots \leq j_L$ which satisfy $\mathcal{C}_L(j_1, \ldots, j_L) = 1$. Plugging in the definition of X for those $j_1 \leq \cdots \leq j_L$ yields the assertion.

(ii) We now apply Theorem 3-***(ii)*** for X as defined in (3.2) with sufficiently large $N \in \mathbb{N}$. Letting N tend to infinity again and substituting the definition of X, we obtain,

$$Y_i^{*,L} = \max_{\substack{i \leq j_1 \leq \cdots \leq j_L \leq \partial \\ \mathcal{C}_L(j_1,\ldots,j_L)=1}} \left(\sum_{k=1}^{L} U_{j_k}^{(k)} \prod_{l=1}^{k-1} V_{j_l}^{(l)} + \sum_{k=1}^{L} \left(M_{j_{k-1}}^{*L-k+1,j_1,\ldots,j_{k-1}} - M_{j_k}^{*L-k+1,j_1,\ldots,j_{k-1}} \right) \right),$$

whenever the $\left(M_r^{*L-k+1,j_1,\ldots,j_{k-1}}\right)_{r \geq j_{k-1}}$ are Doob martingales of $\left(Y_r^{*L-k+1,j_1,\ldots,j_{k-1}}\right)_{r \geq j_{k-1}}$. By Proposition 10-***(i)*** and Proposition 12 we can take

$$\begin{aligned} & M_{j_{k-1}}^{*L-k+1,j_1,\ldots,j_{k-1}} - M_{j_k}^{*L-k+1,j_1,\ldots,j_{k-1}} \\ = & \prod_{l=1}^{k-1} V_{j_l}^{(l)} \left(M_{j_{k-1}}^{*L-k+1} - M_{j_k}^{*L-k+1} + \mathbb{E}_{j_k} A_{\rho^{(j_{k-1})}}^{*L-k+1} - \mathbb{E}_{j_{k-1}} A_{\rho^{(j_{k-1})}}^{*L-k+1} \right). \end{aligned}$$

■

Theorem 11 gives a straightforward generic way to calculate upper bounds for multiple stopping problems of the form (3.1) at time $i = 0$ via Monte Carlo by performing the following steps in a Markovian setting:

1 Solve the dynamic program in Proposition 9 for the auxiliary problems Y^{*L-k+1} approximately, and let $\hat{Y}^{L-k+1}$, $k = 1, \ldots, L$, denote the respective approximations.

2 Perform the Doob decomposition of $\hat{Y}^{L-k+1}$, $k = 1, \ldots, L$, numerically, e.g. by one layer of nested Monte Carlo as suggested by Andersen and Broadie [2004] in the context of options with a single early exercise right.

3 Plug the processes which stem from the numerical Doob decomposition into the formula of Theorem 11-***(i)*** and replace the outer expectation by the sample mean.

This program will be carried out in more detail in Section 4 in the context of Swing options.

Notice that for a large maturity and a large number of exercise rights, the pathwise maximum in the dual representation of Theorem 11 runs over a huge set. We will now show that, due to the special structure of the payoff in (3.1), this maximum can be computed efficiently by a recursion over the time steps and exercise levels.

Given any L-tuple of martingales $M = (M^{(1)}, \ldots, M^{(L)})$ and any L-tuple of adapted processes $A = (A^{(1)}, \ldots, A^{(L)})$, define, for $n = 0, \ldots, L$ and $i = 0, \ldots, \partial$,

$$\begin{aligned} \theta_i^{n,L}(M, A) := & \max_{\substack{j_0 = i \leq j_1 \leq \cdots \leq j_{L-n} \\ \mathcal{C}_{L-n}(j_1,\ldots,j_{L-n})=1}} \sum_{k=1}^{L-n} \left(\prod_{l=1}^{k-1} V_{j_l}^{(l+n)} \right) \left(U_{j_k}^{(n+k)} - \left(M_{j_k}^{(L-n-k+1)} - M_{j_{k-1}}^{(L-n-k+1)} \right) \right. \\ & \left. + \mathbf{1}_{\{k>1 \wedge j_k > j_{k-1}\}} \left(A_{\rho^{j_{k-1}}}^{(L-k-n+1)} - \mathbb{E}_{j_{k-1}} A_{\rho^{j_{k-1}}}^{(L-k-n+1)} \right) \right). \end{aligned}$$

By Theorem 11,

$$Y_0^{*,L} \leq \mathbb{E}[\theta_0^{0,L}(M,A)]$$

for any pair of L-tuples (M, A), and

$$Y_0^{*,L} = \theta_0^{0,L}(M^*, A^*)$$

for an optimal pair of L-tuples (M^*, A^*). Generalizing a related formula in Balder et al. [2011] in the context of flexible (or chooser) caps, the expression $\theta_0^{0,L}(M, A)$ can be recursively calculated by the following proposition.

Proposition 13 *For every L-tuple of martingales $M = (M^{(1)}, \ldots, M^{(L)})$ and L-tuple of adapted processes $A = (A^{(1)}, \ldots, A^{(L)})$ it holds for $i = 0, \ldots, T$ and $n = 0, \ldots, L$,*

$$\begin{aligned}
\theta_i^{n,L}(M,A) \;=\; & \max\Big\{\theta_{i+1}^{n,L}(M,A) - (M_{i+1}^{(L-n)} - M_i^{(L-n)}), \max_{\nu=1,\ldots,v_i\wedge(L-n)} \sum_{k=1}^{\nu}\left(\prod_{l=1}^{k-1} V_i^{(l+n)}\right) U_i^{(n+k)} \\
& + \left(\prod_{\lambda=1}^{\nu} V_i^{(\lambda+n)}\right)\Big(\theta_{\rho^i}^{n+\nu,L}(M,A) - (M_{\rho^i}^{(L-n-\nu)} - M_i^{(L-n-\nu)}) \\
& + A_{\rho_i}^{(L-n-\nu)} - \mathbb{E}_i A_{\rho_i}^{(L-n-\nu)}\Big)\Big\},
\end{aligned}$$

with

$$\theta_\partial^{n,L}(M,A) = \sum_{k=1}^{L-n}\left(\prod_{l=1}^{\nu} V_\partial^{(l+n)}\right) U_\partial^{(n+k)}.$$

Proof. The formula for $\theta_\partial^{n,L}(M,A)$ is obvious by definition. In order to prove the recursive formula, we denote

$$\begin{aligned}
F^{n,L}(j_0,\ldots,j_{L-n}) \;=\; & \sum_{k=1}^{L-n}\left(\prod_{l=1}^{k-1} V_{j_l}^{(l+n)}\right)\Big(U_{j_k}^{(n+k)} - (M_{j_k}^{(L-n-k+1)} - M_{j_{k-1}}^{(L-n-k+1)}) \\
& + \mathbf{1}_{\{k>1\wedge j_k>j_{k-1}\}}(A_{\rho^{j_{k-1}}}^{(L-k-n+1)} - \mathbb{E}_{j_{k-1}} A_{\rho^{j_{k-1}}}^{(L-k-n+1)})\Big).
\end{aligned}$$

Then,

$$\theta_i^{n,L}(M,A) = \max_{\nu=0,\ldots,v_i\wedge(L-n)}\Big\{\max_{\substack{j_0=\cdots=j_\nu=i<j_{\nu+1}\leq\cdots\leq j_{L-n}\\ \mathcal{C}_{L-n}(j_1,\ldots,j_{L-n})=1}} F^{n,L}(j_0,\ldots,j_{L-n})\Big\}. \tag{3.11}$$

For $\nu = 0$ we get

$$\begin{aligned}
& \max_{\substack{j_0=i<j_1\leq\cdots\leq j_{L-n}\\ \mathcal{C}_{L-n}(j_1,\ldots,j_{L-n})=1}} F^{n,L}(j_0,\ldots,j_{L-n}) \\
= \; & \max_{\substack{j_0=i+1\leq j_1\leq\cdots\leq j_{L-n}\\ \mathcal{C}_{L-n}(j_1,\ldots,j_{L-n})=1}} F^{n,L}(j_0,\ldots,j_{L-n}) - (M_{i+1}^{(L-n)} - M_i^{(L-n)}) \\
= \; & \theta_{i+1}^{n,L}(M,A) - (M_{i+1}^{(L-n)} - M_i^{(L-n)}).
\end{aligned} \tag{3.12}$$

For $\nu > 0$ we obtain

$$
\begin{aligned}
&\max_{\substack{j_0=\cdots=j_\nu=i<j_{\nu+1}\le\cdots\le j_{L-n}\\ \mathcal{C}_{L-n}(j_1,\ldots,j_{L-n})=1}} F^{n,L}(j_0,\ldots,j_{L-n})\\
=\;& \sum_{k=1}^{\nu}\left(\prod_{l=1}^{k-1} V_i^{(l+n)}\right) U_i^{(n+k)}\\
&+\left(\prod_{\lambda=1}^{\nu} V_i^{(\lambda+n)}\right) \max_{\substack{j_\nu=i,\,\rho^i\le j_{\nu+1}\le\cdots\le j_{L-n}\\ \mathcal{C}_{L-n-\nu}(j_{\nu+1},\ldots,j_{L-n})=1}} \sum_{k=\nu+1}^{L-n}\left(\prod_{l=\nu+1}^{k-1} V_{j_l}^{(l+n)}\right)\Big(U_{j_k}^{(n+k)}-M_{j_k}^{(L-n-k+1)}\\
&\qquad +M_{j_{k-1}}^{(L-n-k+1)}+\mathbf{1}_{\{j_k>j_{k-1}\}}(A_{\rho^{j_{k-1}}}^{(L-k-n+1)}-\mathbb{E}_{j_{k-1}}A_{\rho^{j_{k-1}}}^{(L-k-n+1)})\Big)\\
=\;& \sum_{k=1}^{\nu}\left(\prod_{l=1}^{k-1} V_i^{(l+n)}\right) U_i^{(n+k)}\\
&+\left(\prod_{\lambda=1}^{\nu} V_i^{(\lambda+n)}\right)\Big((A_{\rho^i}^{(L-\nu-n)}-\mathbb{E}_iA_{\rho^i}^{(L-\nu-n)})-(M_{\rho^i}^{(L-\nu-n)}-M_i^{(L-\nu-n)})\Big)\\
&+\left(\prod_{\lambda=1}^{\nu} V_i^{(\lambda+n)}\right) \max_{\substack{j_\nu=\rho^i\le j_{\nu+1}\le\cdots\le j_{L-n}\\ \mathcal{C}_{L-n-\nu}(j_{\nu+1},\ldots,j_{L-n})=1}} \sum_{k=\nu+1}^{L-n}\left(\prod_{l=\nu+1}^{k-1} V_{j_l}^{(l+n)}\right)\Big(U_{j_k}^{(n+k)}-M_{j_k}^{(L-n-k+1)}\\
&\qquad +M_{j_{k-1}}^{(L-n-k+1)}+\mathbf{1}_{\{(k>\nu+1)\wedge(j_k>j_{k-1})\}}(A_{\rho^{j_{k-1}}}^{(L-k-n+1)}-\mathbb{E}_{j_{k-1}}A_{\rho^{j_{k-1}}}^{(L-k-n+1)})\Big)\\
=\;& \sum_{k=1}^{\nu}\left(\prod_{l=1}^{k-1} V_i^{(l+n)}\right) U_i^{(n+k)}\\
&+\left(\prod_{\lambda=1}^{\nu} V_i^{(\lambda+n)}\right)\Big((A_{\rho^i}^{(L-\nu-n)}-\mathbb{E}_iA_{\rho^i}^{(L-\nu-n)})-(M_{\rho^i}^{(L-\nu-n)}-M_i^{(L-\nu-n)})\Big)\\
&+\left(\prod_{\lambda=1}^{\nu} V_i^{(\lambda+n)}\right) \max_{\substack{j_0=\rho^i\le j_1\le\cdots\le j_{L-n-\nu}\\ \mathcal{C}_{L-n-\nu}(j_1,\ldots,j_{L-n-\nu})=1}} \sum_{k=1}^{L-n-\nu}\left(\prod_{l=1}^{k-1} V_{j_l}^{(l+n+\nu)}\right)\Big(U_{j_k}^{(n+\nu+k)}-M_{j_k}^{(L-n-\nu-k+1)}\\
&\qquad +M_{j_{k-1}}^{(L-n-\nu-k+1)}+\mathbf{1}_{\{(k>1)\wedge(j_k>j_{k-1})\}}(A_{\rho^{j_{k-1}}}^{(L-k-n-\nu+1)}-\mathbb{E}_{j_{k-1}}A_{\rho^{j_{k-1}}}^{(L-k-n-\nu+1)})\Big)\\
=\;& \sum_{k=1}^{\nu}\left(\prod_{l=1}^{k-1} V_i^{(l+n)}\right) U_i^{(n+k)}\\
&+\left(\prod_{\lambda=1}^{\nu} V_i^{(\lambda+n)}\right)\Big((A_{\rho^i}^{(L-\nu-n)}-\mathbb{E}_iA_{\rho^i}^{(L-\nu-n)})-(M_{\rho^i}^{(L-\nu-n)}-M_i^{(L-\nu-n)})\Big)\\
&+\left(\prod_{\lambda=1}^{\nu} V_i^{(\lambda+n)}\right)\theta_{\rho^i}^{n+\nu,L}(M,A).
\end{aligned}
$$

Plugging this identity and (3.12) into (3.11) yields the assertion. ■

3.2 Dual representation based on Snell envelopes

In this subsection we present a simplified version of the dual representation in Corollary 4 in terms of approximate Snell envelopes for the multiple stopping problem of the form (3.1). It reads as follows.

Theorem 14 *Suppose Y_i^{*L} is given by (3.1) for some fixed $0 \le i \le \partial$. Let $(Y^{(k)})_{1\le k\le L}$ be any set of integrable approximations to $(Y^{*k})_{1\le k\le L}$ defined in* (3.3). *We then have, with the conventions $j_0 := -1$, $\rho^{(j_0)} := i$, and $Y^{(0)} = 0$,*

$$\begin{aligned}
&Y_i^{*,L} - Y_i^{(L)} \\
\le\ & \mathbb{E}_i \max_{\substack{i\le j_1\le\cdots\le j_L\le\partial,\\ \mathcal{C}_L(j_1,\ldots,j_L)=1}} \sum_{k=1}^{L}\Bigg\{\sum_{r=\rho^{(j_{k-1})}}^{j_k-1}\prod_{l=1}^{k-1}V_{j_l}^{(l)}\left(\mathbb{E}_r Y_{r+1}^{(L-k+1)} - Y_r^{(L-k+1)}\right)\\
&+\mathbb{1}_{j_k>j_{k-1}}\prod_{l=1}^{k-1}V_{j_l}^{(l)}\Bigg(\max_{1\le n\le v_{j_k}\wedge(L-k+1)}\Bigg\{\sum_{p=k}^{k+n-1}U_{j_k}^{(p)}\prod_{l=k}^{p-1}V_{j_k}^{(l)}\\
&\quad+\prod_{l=k}^{k+n-1}V_{j_k}^{(l)}\mathbb{E}_{j_k}Y_{\rho^{(j_k)}}^{(L-k-n+1)}\Bigg\} - Y_{j_k}^{(L-k+1)}\Bigg)\Bigg\}.
\end{aligned}$$

Moreover, the righthand side becomes zero if $Y^{(k)} = Y^{*k}$, for $k = 1, ..., L$.

Proof. Suppose $0 \le i \le \partial$ is fixed and assume that integrable and adapted processes $Y^{(L-k+1)}$, $k = 1,\ldots,L$, are given which we consider as approximations of the Snell envelopes of the auxiliary multiple stopping problems Y^{*L-k+1}. Following the relationships for the Snell envelopes Y^{*L-k+1} and $Y^{*L-k+1,j_1,\ldots,j_{k-1}}$ in Proposition 10, we define for $k > 1$ approximations to $Y^{*L-k+1,j_1,\ldots,j_{k-1}}$ via

$$Y_r^{L-k+1,j_1,\ldots,j_{k-1}} := \sum_{p=1}^{k-1}U_{j_p}^{(p)}\prod_{l=1}^{p-1}V_{j_l}^{(l)} + \mathbb{E}_r Y_{\rho^{(j_{k-1})}\vee r}^{(L-k+1)}\prod_{l=1}^{k-1}V_{j_l}^{(l)}, \quad r > j_{k-1}, \tag{3.13}$$

and (for $r = j_{k-1}$)

$$\begin{aligned}
&Y_{j_{k-1}}^{L-k+1,j_1,\ldots,j_{k-1}}\\
:=\ & \sum_{p=1}^{k-1}U_{j_p}^{(p)}\prod_{l=1}^{p-1}V_{j_l}^{(l)}\\
&+\prod_{l=1}^{k-1}V_{j_l}^{(l)}\max_{n\in N(j_1,\ldots,j_{k-1})}\left\{\sum_{p=k}^{k-1+n}U_{j_{k-1}}^{(p)}\prod_{l=k}^{p-1}V_{j_{k-1}}^{(l)} + \prod_{l=k}^{k-1+n}V_{j_{k-1}}^{(l)}\mathbb{E}_{j_{k-1}}Y_{\rho^{(j_{k-1})}}^{(L-k+1-n)}\right\}.
\end{aligned}\tag{3.14}$$

Clearly, we define, for $k = 1$, $Y^{L,\emptyset} = Y^{(L)}$.

Applying Corollary 4 for X as defined in (3.1) and the above approximations we obtain,

$$\begin{aligned}
Y_i^{*,L} \le\ & Y_i^L + \mathbb{E}_i \max_{\substack{i\le j_1\le\cdots\le j_L\le\partial,\\ \mathcal{C}_L(j_1,\ldots,j_L)=1}}\sum_{k=1}^{L}\Bigg(Y_{j_k}^{L-k,j_1,\ldots,j_k} - Y_{j_k}^{L-k+1,j_1,\ldots,j_{k-1}}\\
&+\sum_{l=j_{k-1}}^{j_k-1}\left(\mathbb{E}_l Y_{l+1}^{L-k+1,j_1,\ldots,j_{k-1}} - Y_l^{L-k+1,j_1,\ldots,j_{k-1}}\right)\Bigg),
\end{aligned}\tag{3.15}$$

where we again observe the the pathwise maximum is attained on the set $\mathcal{C}_L(j_1,...,j_L)=1$ by letting N (in the definition of X) tend to infinity.

In order to prove the upper bound, it is, in view of (3.15), sufficient to show that, for $i\le j_1\le\cdots\le j_L\le\partial$ with $\mathcal{C}_L(j_1,...,j_L)=1$ the following assertions are true:
(i) If $k=2,\dots,L$ and $j_k>j_{k-1}$ or if $k=1$, then

$$\begin{aligned}&Y_{j_k}^{L-k,j_1,\dots,j_k}-Y_{j_k}^{L-k+1,j_1,\dots,j_{k-1}}\\ =\;&\prod_{l=1}^{k-1}V_{j_l}^{(l)}\left(\max_{0\le n\le(v_{j_k}-1)\wedge(L-k)}\left\{\sum_{p=k}^{k+n}U_{j_k}^{(p)}\prod_{l=k}^{p-1}V_{j_k}^{(l)}+\prod_{l=k}^{k+n}V_{j_k}^{(l)}\mathbb{E}_{j_k}Y_{\rho^{(j_k)}}^{(L-k-n)}\right\}-Y_{j_k}^{(L-k+1)}\right).\end{aligned}$$

(ii) If $=2,\dots,L$ and $j_k=j_{k-1}$, then

$$Y_{j_k}^{L-k,j_1,\dots,j_k}-Y_{j_k}^{L-k+1,j_1,\dots,j_{k-1}}\le 0.$$

(iii) If $k=1$ and $i\le r\le j_1-1$, or if $k=2,\dots,L$ and $\rho^{(j_{k-1})}\le r\le j_k-1$, then

$$\mathbb{E}_rY_{r+1}^{L-k+1,j_1,\dots,j_{k-1}}-Y_r^{L-k+1,j_1,\dots,j_{k-1}}=\prod_{l=1}^{k-1}V_{j_l}^{(l)}\left(\mathbb{E}_rY_{r+1}^{(L-k+1)}-Y_r^{(L-k+1)}\right).$$

(iv) For $k=2,\dots,L$ and $j_{k-1}\le r<\rho^{(j_{k-1})}$

$$\mathbb{E}_rY_{r+1}^{L-k+1,j_1,\dots,j_{k-1}}-Y_r^{L-k+1,j_1,\dots,j_{k-1}}\le 0.$$

We first show (i). To this end suppose that $k\ge 2$ and $j_k>j_{k-1}$. Then, $\mathcal{E}_k(j_1,\dots,j_k)=1$, which implies $N(j_1,\dots,j_k)=\{n;\ 0\le n\le(v_k-1)\wedge(L-k)\}$. Hence, by (3.14),

$$\begin{aligned}&Y_{j_k}^{L-k,j_1,\dots,j_k}\\ =\;&\sum_{p=1}^{k-1}U_{j_p}^{(p)}\prod_{l=1}^{p-1}V_{j_l}^{(l)}+\prod_{l=1}^{k-1}V_{j_l}^{(l)}\max_{0\le n\le(v_{j_k}-1)\wedge(L-k)}\left\{\sum_{p=k}^{k+n}U_{j_k}^{(p)}\prod_{l=k}^{p-1}V_{j_k}^{(l)}+\prod_{l=k}^{k+n}V_{j_k}^{(l)}\mathbb{E}_{j_k}Y_{\rho^{(j_k)}}^{(L-k-n)}\right\}.\end{aligned}$$

Substracting the defining equation (3.13) for $Y_{j_k}^{L-k+1,j_1,\dots,j_{k-1}}$ from the above expression, we obtain (i), because $j_k\ge\rho^{(j_{k-1})}$. For $k=1$, we again get $\mathcal{E}_1(j_1)=1$ and (i) follows in the same way, taking the definition $Y_{j_1}^{L,\emptyset}=Y_{j_1}^{(L)}$ into account.

In order to derive (ii), we note that $\mathcal{E}_k(j_1,\ldots,j_k)=\mathcal{E}_{k-1}(j_1,\ldots,j_{k-1})+1$ for $j_k=j_{k+1}$. Thus,

$$\begin{aligned}
& Y_{j_k}^{L-k,j_1,\ldots,j_k} \\
= \;& \sum_{p=1}^{k-1} U_{j_p}^{(p)} \prod_{l=1}^{p-1} V_{j_l}^{(l)} \\
& + \prod_{l=1}^{k-1} V_{j_l}^{(l)} \max_{0\le n\le (v_{j_k}-\mathcal{E}_{k-1}(j_1,\ldots,j_{k-1})-1)\wedge(L-k)} \left\{ \sum_{p=k}^{k+n} U_{j_k}^{(p)} \prod_{l=k}^{p-1} V_{j_k}^{(l)} + \prod_{l=k}^{k+n} V_{j_k}^{(l)} \mathbb{E}_{j_k} Y_{\rho^{(j_k)}}^{(L-k-n)} \right\} \\
\le \;& \sum_{p=1}^{k-1} U_{j_p}^{(p)} \prod_{l=1}^{p-1} V_{j_l}^{(l)} \\
& + \prod_{l=1}^{k-1} V_{j_l}^{(l)} \max_{0\le n\le (v_{j_k}-\mathcal{E}_{k-1}(j_1,\ldots,j_{k-1}))\wedge(L-k+1)} \left\{ \sum_{p=k}^{k+n} U_{j_k}^{(p)} \prod_{l=k}^{p-1} V_{j_k}^{(l)} + \prod_{l=k}^{k+n} V_{j_k}^{(l)} \mathbb{E}_{j_k} Y_{\rho^{(j_k)}}^{(L-k-n)} \right\} \\
= \;& Y_{j_{k-1}}^{L-k+1,j_1,\ldots,j_{k-1}} = Y_{j_k}^{L-k+1,j_1,\ldots,j_{k-1}}
\end{aligned}$$

We next prove (iii). The case $k=1$ is trivial in view of the definition of $Y^{L,\emptyset}$. Hence, we assume that $k\ge 2$ and $\rho^{(j_{k-1})}\le r\le j_k-1$. Then, $r+1>r\ge\rho^{(j_{k-1})}>j_{k-1}$ and, thus, by (3.13)

$$\begin{aligned}
\mathbb{E}_r Y_{r+1}^{L-k+1,j_1,\ldots,j_{k-1}} &= \sum_{p=1}^{k-1} U_{j_p}^{(p)} \prod_{l=1}^{p-1} V_{j_l}^{(l)} + \mathbb{E}_r Y_{r+1}^{(L-k+1)} \prod_{l=1}^{k-1} V_{j_l}^{(l)}, \\
Y_r^{L-k+1,j_1,\ldots,j_{k-1}} &= \sum_{p=1}^{k-1} U_{j_p}^{(p)} \prod_{l=1}^{p-1} V_{j_l}^{(l)} + Y_r^{(L-k+1)} \prod_{l=1}^{k-1} V_{j_l}^{(l)}.
\end{aligned}$$

Taking the difference of both equations yields (iii).

It remains to show (iv). For $k\ge 2$ and $j_{k-1}<r<\rho^{(j_{k-1})}$, (3.13) implies

$$\mathbb{E}_r Y_{r+1}^{L-k+1,j_1,\ldots,j_{k-1}} = \sum_{p=1}^{k-1} U_{j_p}^{(p)} \prod_{l=1}^{p-1} V_{j_l}^{(l)} + \mathbb{E}_r Y_{\rho^{(j_{k-1})}}^{(L-k+1)} \prod_{l=1}^{k-1} V_{j_l}^{(l)} = Y_r^{L-k+1,j_1,\ldots,j_{k-1}}.$$

Finally, for $k\ge 2$ and $r=j_{k-1}$, by (3.13) and (3.14),

$$\begin{aligned}
& \mathbb{E}_{j_{k-1}} Y_{j_{k-1}+1}^{L-k+1,j_1,\ldots,j_{k-1}} = \sum_{p=1}^{k-1} U_{j_p}^{(p)} \prod_{l=1}^{p-1} V_{j_l}^{(l)} + \mathbb{E}_{j_{k-1}} Y_{\rho^{(j_{k-1})}}^{(L-k+1)} \prod_{l=1}^{k-1} V_{j_l}^{(l)} \\
\le \;& \sum_{p=1}^{k-1} U_{j_p}^{(p)} \prod_{l=1}^{p-1} V_{j_l}^{(l)} \\
& + \prod_{l=1}^{k-1} V_{j_l}^{(l)} \max_{n\in N(j_1,\ldots,j_{k-1})} \left\{ \sum_{p=k}^{k-1+n} U_{j_{k-1}}^{(p)} \prod_{l=k}^{p-1} V_{j_{k-1}}^{(l)} + \prod_{l=k}^{k-1+n} V_{j_{k-1}}^{(l)} \mathbb{E}_{j_{k-1}} Y_{\rho^{(j_{k-1})}}^{(L-k+1-n)} \right\} \\
= \;& Y_{j_{k-1}}^{L-k+1,j_1,\ldots,j_{k-1}}.
\end{aligned}$$

Hence, the asserted upper bound for $Y_i^{*L} - Y_i^{(L)}$ is shown. This upper bound is zero, if $Y^{(k)} = Y^{*k}$, for $k = 1, ..., L$, because, by Proposition 9, Y^{*L-k+1} is a supermartingale and $Y_{j_k}^{*L-k+1}$ dominates

$$\max_{1\leq n\leq v_{j_k}\wedge(L-k+1)}\Big\{\sum_{p=k}^{k+n-1} U_{j_k}^{(p)}\prod_{l=k}^{p-1}V_{j_k}^{(l)} + \prod_{l=k}^{k+n-1} V_{j_k}^{(l)}\mathbb{E}_{j_k}Y_{\rho(j_k)}^{*L-k-n+1}\Big\}.$$

■

As a spin-off result from Theorem 14, we may write the following upper bound for $Y_i^{*,L}$ which avoids the computation of the recursive maximum from Proposition 13 (cf. Schoenmakers [2010][Remark 3.3] for a related result in the context of the standard multiple stopping problem).

Corollary 15 *Suppose all assumptions and all conventions of Theorem 14 are in force. Then,*

$$\begin{aligned} & Y_i^{*,L} - Y_i^{(L)} \\ \leq\ & \mathbb{E}_i\left\{\sum_{r=i}^{T-1}\max_{0\leq k<L}\left(\mathcal{V}_{\max}^{(k)}\left(\mathbb{E}_r Y_{r+1}^{(L-k)} - Y_r^{(L-k)}\right)^+\right) + \sum_{k=1}^{L}\mathcal{V}_{\max}^{(k-1)}\right. \\ & \left.\times\max_{i\leq j\leq\partial}\left(\max_{1\leq n\leq v_{j_k}\wedge(L-k+1)}\left(\sum_{p=k}^{k+n-1}U_j^{(p)}\prod_{l=k}^{p-1}V_j^{(l)}\prod_{l=k}^{k+n-1}V_j^{(l)}\mathbb{E}_j Y_{\rho(j)}^{(L-k-n+1)}\right) - Y_j^{(L-k+1)}\right)^+\right\}, \end{aligned}$$

where

$$\mathcal{V}_{\max}^{(k)} := \prod_{l=1}^{k}\max_{j\geq i}V_j^{(l)}.$$

Moreover, the righthand side becomes zero if $Y^{(k)} = Y^{*k}$, for $k = 1, ..., L$.

Proof. It is straightforward to check that the upper bound in this corollary is actually an upper bound to the righthand side of the estimate in Theorem 14. That the bound is still tight, i.e. that the righthand side becomes zero, if $Y^{(k)} = Y^{*k}$, for $k = 1, ..., L$, follows from the same argument as at the end of the proof of Theorem 14. ■

4 A numerical example

We provide a numerical example for the dual representation of multiple stopping problems in the context of swing option pricing. Throughout this section, we assume $i = 0$, i.e. we provide confidence bounds for the swing option price at time 0. Precisely, we consider a stylized swing option, similar to those considered in Meinshausen and Hambly [2004] and Bender [2011a]. In our setting, the holder of a swing option has the right to buy a certain quantity of electricity in the period from $j = 0, \ldots, T$, for a fixed strike price $K > 0$, subject to the restriction that the option allows up to $L \geq 1$ exercise opportunities under the volume constraints v_j, and where a refraction period has to be taken into account. Here we choose $T = 50$ and recall that $\partial := T + 1$. The price of electricity, $(S_t)_{t=0,...,T}$, is modeled by the following discretized exponential Gaussian Ornstein-Uhlenbeck process

$$\log(S_j) = (1-k)\big(\log(S_{j-1}) - \mu\big) + \mu + \sigma\epsilon_j,\ S_0 = s_0 > 0, \tag{4.1}$$

where $(\epsilon_j)_{j=1,\ldots,T}$ is a family of independent standard normal random variables and the parameters are specified by

$$\sigma = 0.5,\ k = 0.9,\ \mu = 0,\ s_0 = 1.$$

We set $S_\partial = 0$, which means that no penalty is imposed, if the holder of the option does not exercise all rights. The payoff of the swing option is then given by X in (3.1) with

$$\begin{aligned} &V_j^{(l)} := 1,\ l = 1,\ldots,L-1,\ j = 0,\ldots,\partial, \\ &U_j^{(p)} := Z_j := Z(S_j) := (S_j - K)^+,\ j = 0,\ldots,\partial,\ p = 1,\ldots,L. \end{aligned}$$

In our numerical study we assume that the strike price is $K = 1$. As volume constraints we consider the situation of a unit volume constraint $v_i = 1$ for $i = 0,\ldots,T$ and the situation of an off-peak swing option with $v_i = 1$ on weekdays and $v_i = 2$ on Saturdays and Sundays. The refraction period which we impose is a constant refraction period, i.e. $\rho^{(i)} = (i+\delta) \wedge \partial$ for various choices of the constant $\delta \in \mathbb{N}$.

In this Markovian framework, we produce confidence intervals for the price of the swing option at time $i = 0$ by applying the following steps. This procedure can be easily generalized to the generic cash-flow structure of Section 3, provided the problem has a Markovian structure. (For notational convenience we only spell out the algorithm for the swing option case.)

4.1 Implementation

Step 1: Precompute an approximation of the continuation value. We employ least squares Monte Carlo regression to obtain an approximation to the continuation values

$$\begin{aligned} C_j^{*,1,l}(S_j) &:= \mathbb{E}\Big[Y_{j+1}^{*,l}\big|\mathcal{F}_j\Big] = \mathbb{E}\Big[Y_{j+1}^{*,l}\big|S_j\Big],\ C_T^{*,1,l}(S_T) := 0, \\ C_j^{*,\delta,l}(S_j) &:= \mathbb{E}\Big[Y_{j+\delta}^{*,l}\big|\mathcal{F}_j\Big] = \mathbb{E}\Big[Y_{j+\delta}^{*,l}\big|S_j\Big],\ C_T^{*,\delta,l}(S_T) := 0, \end{aligned}$$

with $l = 1,\ldots,L$. Recall that $(Y_j^{*,l})_{j=0,\ldots,T}$ is given by the dynamic program from Proposition 9. We simulate N_1 independent paths $(S_j^m)_{j=0,\ldots,T}^{m=1,\ldots,N_1}$. Choosing as basis functions

$$\psi_1(x) := x, \quad \psi_2(x) := (x-K)^+,$$

we use $(S_j^m)_{j=0,\ldots,T}^{m=1,\ldots,N_1}$ in a straightforward least squares regression procedure to solve the dynamic program approximately, replacing the conditional expectations by the least-squares Monte Carlo estimator. This yields approximations to $C_j^{*,1,l}(\cdot)$ and $C_j^{*,\delta,l}(\cdot)$, denoted by $C_j^{1,l}(\cdot)$ and $C_j^{\delta,l}(\cdot)$.

Step 2: Compute lower bounds. Given the functions $C_j^{1,l}(\cdot)$ and $C_j^{\delta,l}(\cdot)$, we define a (suboptimal) stopping rule $\big(\tau_j^{p,l}\big)_{1\le p\le l}^{1\le l\le L}$ for $0 \le j \le T$ along a given trajectory $(S_j)_{j=0\ldots,T}$ (which we suppress in the notation below) using the following iteration. Here $\tau_j^{p,l}$ is interpreted as the time at which the investor exercises the pth right, if l

rights are left at time j.

$$
\begin{aligned}
&\tau_j^{0,l} := j - \delta; \\
&p := k := 0; \\
&while\,(p < l) \quad do \\
&\qquad \tau_j^{p+1,l} := \inf\left\{(\tau_j^{p,l} + \delta) \wedge \partial \leq r \leq \partial : \max_{1 \leq n \leq v_r \wedge (l-p)} \left(nZ_r + C_r^{\delta,l-p-n}\right) \geq C_r^{1,l-p}\right\}, \\
&\qquad s := \tau_j^{p+1,l}, \\
&\qquad k := \operatorname{argmax}_{1 \leq n \leq v_s \wedge (l-p)} \left(nZ_s + C_s^{\delta,l-p-n}\right), \\
&\qquad \tau_j^{p+1,l} := \tau_j^{p+2,l} := \ldots := \tau_j^{p+k,l} := s, \\
&\qquad p := p + k, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.2) \\
&end
\end{aligned}
$$

When $C_j^{1,l}(\cdot)$ and $C_j^{\delta,l}(\cdot)$ are replaced by $C_j^{*,1,l}(\cdot)$ and $C_j^{*,\delta,l}(\cdot)$, then this family of stopping times is optimal. Hence, $\left(\tau_j^{p,l}\right)_{1 \leq p \leq l}^{1 \leq l \leq L}$ is a good family of stopping times, if the approximations of the continuation values in Step 1 are reasonably close to the true continuation values.

Remark 16 *(i) In the situation of unit volume constraint (i.e. $v \equiv 1$), the stopping rule* (4.2) *simplifies to* $\tau_j^{0,l} = j - \delta$ *and*

$$
\tau_j^{p,l} = \inf\left\{(\tau_j^{p-1,l} + \delta) \wedge \partial \leq r \leq \partial : Z_r + C_r^{\delta,l-p} \geq C_r^{1,l-p+1}\right\}, \quad 1 \leq l \leq L, \quad 1 \leq p \leq l,
$$

compare with Eq. (3.7) in Bender [2011a].

(ii) In the situation of a trivial refraction period (i.e. $\delta = 1$), the above construction of approximate stopping rules is also used in Aleksandrov and Hambly [2010].

Setting

$$
\underline{Y}_0^l := \mathbb{E}_0 \sum_{p=1}^{l} Z_{\tau_0^{p,l}}, \; \underline{Y}_1^l := \mathbb{E}_1 \sum_{p=1}^{l} Z_{\tau_1^{p,l}}, \; \underline{Y}_\delta^l := \mathbb{E}_\delta \sum_{p=1}^{l} Z_{\tau_\delta^{p,l}},
$$

we have that $\underline{Y}_0^l$ is a lower biased estimate for $Y_0^{*,l}$. By the tower property of the conditional expectation, we also have

$$
\mathbb{E}_0 \underline{Y}_1^l = \mathbb{E}_0 \sum_{p=1}^{l} Z_{\tau_1^{p,l}}, \quad \mathbb{E}_0 \underline{Y}_\delta^l = \mathbb{E}_0 \sum_{p=1}^{l} Z_{\tau_\delta^{p,l}}.
$$

As for simulations, we generate a new set of N_2 independent paths of the underlying price process, which we again denote, in abuse of notation, by $(S_j^m)_{j=0,\ldots,T}^{m=1,\ldots,N_2}$. Along theses N_2 trajectories we compute $\tau_0^{p,l}$ and apply the notation

$$
\tau_0^{p,l,m}, \quad 1 \leq p \leq l, \; 1 \leq m \leq N_2.
$$

Now the lower biased estimate $\widehat{\underline{Y}}_0^l$ for $Y_0^{*,l}$ is calculated by averaging over the N_2 realizations of $\sum_{p=1}^l Z_{\tau_0^{p,l}}$, i.e.

$$\widehat{\underline{Y}}_0^l = \frac{1}{N_2} \sum_{m=1}^{N_2} \sum_{p=1}^{l} Z(S^m_{\tau_0^{p,l,m}}), \quad 1 \le l \le L. \tag{4.3}$$

Similarly, we can also construct approximations

$$\hat{\mathbb{E}}_0 \underline{Y}_1^l = \frac{1}{N_2} \sum_{m=1}^{N_2} \sum_{p=1}^{l} Z(S_{\tau_1^{p,l,m}}), \quad \hat{\mathbb{E}}_0 \underline{Y}_\delta^l = \frac{1}{N_2} \sum_{m=1}^{N_2} \sum_{p=1}^{l} Z(S_{\tau_\delta^{p,l,m}})$$

of $\mathbb{E}_0 \underline{Y}_1^l$ and $\mathbb{E}_0 \underline{Y}_\delta^l$.

For constructing confidence intervals, we also save the empirical standard deviation $\mathrm{stddev}(\widehat{\underline{Y}}_0^l)$.

Step 3: Compute approximations to the Snell envelopes. Using the stopping rule (4.2), we consider a family of random variables

$$\underline{Y}_j^l := \mathbb{E}_j \sum_{p=1}^{l} Z_{\tau_j^{p,l}}, \; 1 \le l \le L, \; 0 \le j \le T, \tag{4.4}$$

which is an approximation to the Snell envelope $\left(Y_j^{*,l}\right)_{0 \le j \le T}^{1 \le l \le L}$. We apply the following procedure to simulate $\underline{Y}_j^l$:

We simulate a new set of N_3 paths of the underlying $(S_j^m)_{0 \le j \le T}^{1 \le m \le N_3}$ (abusing the notation, again). We refer to these paths as the outer paths. We now fix a pair (m, j) and compute approximations of $\underline{Y}_j^l$, $\mathbb{E}_j \underline{Y}_{j+1}^l$, and $\mathbb{E}_j \underline{Y}_{j+\delta}^l$ along the mth outer path which are denoted by $\widehat{\underline{Y}}_j^{l,m}$, $\widehat{\mathbb{E}}_j^m \underline{Y}_{j+1}^l$, and $\widehat{\mathbb{E}}_j^m \underline{Y}_{j+\delta}^l$, respectively. In these approximations the conditional expectations are replaced by the sample mean over a set of inner simulations. Hence, for the fixed path S^m and the fixed time point j, we generate N_4 independent sample paths of $(S_r)_{r=j,\dots,T}$ under the conditional law given that $S_j = S_j^m$. These inner paths are denoted by $(\bar{S}_r^\nu)_{r=j,\dots,T}^{\nu=1,\dots,N_4}$, suppressing here and in the following the dependence on (m, j). Along the inner paths $\bar{S}^\nu$ we compute the stopping times $\tau_i^{p,l}$ for $i = j, j+1, j+\delta$ in (4.2) and apply the notation

$$\tau_i^{p,l,\nu}, \; 1 \le p \le l, \; 1 \le l \le L, \; \nu = 1, \dots, N_4.$$

We now define

$$\widehat{\underline{Y}}_j^{l,m} := \widehat{\mathbb{E}}_j^m \sum_{p=1}^{l} Z_{\tau_j^{p,l}} := \frac{1}{N_4} \sum_{\nu=1}^{N_4} \sum_{p=1}^{l} Z\big(\bar{S}^\nu_{\tau_j^{p,l,\nu}}\big).$$

Similarly, we approximate $\mathbb{E}_j \underline{Y}_{j+1}^l$ for the fixed j along the fixed mth outer path by

$$\widehat{\mathbb{E}}_j^m \underline{Y}_{j+1}^l := \frac{1}{N_4} \sum_{\nu=1}^{N_4} \sum_{p=1}^{l} Z\big(\bar{S}^\nu_{\tau_{j+1}^{p,l,\nu}}\big),$$

taking the tower property of the conditional expectation into account. The approximation $\widehat{\mathbb{E}}_j^m \underline{Y}_{j+\delta}^l$ is obtained analogously.

Remark 17 *Note that, for $j = 0$ approximations $\widehat{\underline{Y}}_0^l$, $\widehat{\mathbb{E}}_0\underline{Y}_1^l$, and $\widehat{\mathbb{E}}_0\underline{Y}_\delta^l$ of $\underline{Y}_0^l$, $\mathbb{E}_0\underline{Y}_1^l$, and $\mathbb{E}_0\underline{Y}_\delta^l$ were alredy obtained based on the N_2-samples in Step 2. As typically $N_2 > N_4$ these approximations are more accurate. Hence, one can perform Step 3 for $j \geq 1$ only and set*

$$\widehat{\underline{Y}}_0^{l,m} := \widehat{\underline{Y}}_0^l, \quad \widehat{\mathbb{E}}_0^m\underline{Y}_1^l = \widehat{\mathbb{E}}_0\underline{Y}_1^l, \quad \widehat{\mathbb{E}}_0^m\underline{Y}_\delta^l = \widehat{\mathbb{E}}_0\underline{Y}_1^l, \quad m = 1, \ldots, M.$$

This trick of applying the more accurate non-nested Monte Carlo simulation of Step 2 at time 0 leads to a significant decrease of the variance in the simulation of the upper bound. This is in the same spirit as the computation of low variance upper bounds for the standard stopping problem from Andersen and Broadie [2004]).

Step 4: Compute the upper bounds. The Doob decomposition of $\underline{Y}_j^l$ yields the pair $(\underline{M}_j^l, \underline{A}_j^l)$. Note that due to

$$\underline{M}_{i+1}^l - \underline{M}_i^l = \underline{Y}_{i+1}^l - \mathbb{E}_i\underline{M}_{i+1}^l,$$

and

$$-(\underline{M}_{i+\delta}^l - \underline{M}_i^l) - \underline{A}_{i+\delta}^l - \mathbb{E}_i\underline{A}_{i+\delta}^l = \mathbb{E}_i\underline{Y}_{i+\delta}^l - \underline{Y}_{i+\delta}^l,$$

we can rewrite the recursion formula in Proposition 13 as

$$\theta_i^{n,L} = \max\left\{\theta_{i+1}^{n,L} + \mathbb{E}_i\underline{Y}_{i+1}^{L-n} - \underline{Y}_{i+1}^{L-n}, \max_{1\leq\nu\leq v_i\wedge(L-n)}\left(\nu Z_i + \theta_{i+\delta}^{n+\nu,L} + \mathbb{E}_i\underline{Y}_{i+\delta}^{L-n-\nu} - \underline{Y}_{i+\delta}^{L-n-\nu}\right)\right\}.$$

We now introduce approximations $\theta_i^{n,L,m}$ of $\theta_i^{n,L}$ along the mth outer path of Step 3 by replacing $\underline{Y}_j^l$, $\mathbb{E}_j\underline{Y}_{j+1}^l$, and $\mathbb{E}_j\underline{Y}_{j+\delta}^l$ with their simulated counterparts $\widehat{\underline{Y}}_j^{l,m}$, $\widehat{\mathbb{E}}_j^m\underline{Y}_{j+1}^l$, and $\widehat{\mathbb{E}}_j^m\underline{Y}_{j+\delta}^l$ constructed in Step 3.

As simulation based estimate for the upper bound, we use

$$Y_0^{up,L} := \frac{1}{N_3}\sum_{m=1}^{N_3}\theta_0^{0,L,m}.$$

Replacing the conditional expectations by the sample mean in $\theta_0^{0,L,m}$ introduces an additional bias up thanks to Jensen's inequality and the convexity of the maximum. Hence, the estimator $Y_0^{up,L}$ is biased up by Theorem 11 and Proposition 13.

Finally, a 95% confidence interval on the price of the swing option is given by

$$\left[\widehat{\underline{Y}}_0^L - 1.96 \times \mathsf{stddev}(\widehat{\underline{Y}}_0^L), Y_0^{up,L} + 1.96 \times \mathsf{stddev}(Y_0^{up,L})\right].$$

δ	L	$\widehat{\underline{Y}}_0^L$	$Y_0^{up,L}$	95% confidence interval	L	$\widehat{\underline{Y}}_0^L$	$Y_0^{up,L}$	95% confidence interval
1	2	3.3116	3.3211	[3.30738, 3.32229]	3	4.53627	4.54806	[4.53118, 4.54938]
2	2	3.27513	3.28469	[3.27094, 3.28587]	3	4.43753	4.45154	[4.43252, 4.45295]
3	2	3.2525	3.26286	[3.2483, 3.26414]	3	4.36706	4.38245	[4.36204, 4.38392]
4	2	3.2313	3.24083	[3.22716, 3.242]	3	4.29996	4.31656	[4.29502, 4.31813]
5	2	3.20906	3.22061	[3.20496, 3.22199]	3	4.29996	4.31656	[4.29502, 4.31813]
6	2	3.18613	3.19809	[3.18197, 3.19948]	3	4.15557	4.17514	[4.15063, 4.17697]
8	2	3.13625	3.14984	[3.13213, 3.15143]	3	3.99773	4.01954	[3.99289, 4.02158]
10	2	3.09022	3.10332	[3.08613, 3.1048]	3	3.83377	3.8528	[3.82898, 3.85464]
12	2	3.03874	3.05196	[3.03468, 3.05356]	3	3.65492	3.67658	[3.65023, 3.67868]
14	2	2.98727	3.00048	[2.98321, 3.00199]	3	3.47017	3.49061	[3.46558, 3.49258]
16	2	2.92751	2.94214	[2.9235, 2.9438]	3	3.27524	3.29482	[3.27077, 3.29674]
18	2	2.87368	2.8888	[2.86964, 2.89049]	3	3.09209	3.11002	[3.08775, 3.11186]
20	2	2.81521	2.83005	[2.81123, 2.83173]	3	2.91951	2.93649	[2.91536, 2.9383]

Table 4.1: *Unit volume constraints ($v_j \equiv 1$).* Numerical results based on the approximation $\widehat{\underline{Y}}_j^l$ to the Snell envelope via the stopping rule (4.2) for two and three exercise rights.

4.2 Numerical results: swing options with unit volume constraints

We now present some numerical results which the above algorithm produces for the swing option contract as specified at the beginning of this section. Let us first consider the situation of a unit volume constraint, i.e. $v_j := 1$ for $j = 0, \dots, T$. We recall that $\delta \in \mathbb{N}$ denotes a constant refraction period. In this setting the dual representation of Theorem 11 reduces to the one derived in Bender [2011a]. In the latter paper the same swing option example is treated numerically but for up to three exercise rights only. Thanks to the new recursion formula in Proposition 13 we can now efficiently treat the case of a large number of exercise rights (here up to $L = 10$). Moreover, the upper bound algorithm in Bender [2011a] differs slightly from the one we propose here. In Bender [2011a] the upper bound is calculated based on the numerical Doob decomposition of

$$Y_j^l = \max\{Z_j + C_j^{\delta,l-1}, C_j^{1,l}\}$$

while we here utilize the numerical Doob decomposition of $\underline{Y}_j^l := \mathbb{E}_j \sum_{p=1}^l Z_{\tau_j^{p,l}}$.

The choice of simulation parameters in our study is as follows: in Step 1, we choose $N_1 = 1000$ paths for the least squares Monte Carlo regression to approximate the continuation function. In Step 2, the lower bound is simulated using $N_2 = 300000$ paths and in Step 3, we employ $N_3 = 2000$ outer and $N_4 = 100$ inner paths for the computation of the upper bound. Moreover, we use the variance reduction method from Remark 17.

Table 4.1 depicts the numerical results for the case of two and three exercise rights for a refraction period ranging from 1 to 20. We observe that the relative length of the 95%-confidence intervals is less than 1% in all cases. A comparison with the numerical results in Bender [2011a] shows that the differences in the upper price estimator based on Y^l and $\underline{Y}^l$ are negligible, but the variance reduction method of Remark 17 shrinks the confidence interval significantly.

The numerical results for the case of a larger number of exercise rights ($L = 4, 6, 8, 10$) are presented in Table 4.2. Due to the time horizon of 50 days, it may happen that, for a large number of rights and a large refraction period, some exercise rights cannot be used by the investor. This explains why e.g. the price bounds for the swing option with refraction period $\delta = 14$ are the same for $L = 4$ and $L = 6$ rights. Concerning the accuracy of our numerical procedure we emphasize that the relative difference between lower and upper bound is still less than 1% even in the case of 10 exercise rights.

δ	L	$\underline{\widehat{Y}}_0^L$	$Y_0^{up,L}$	95% confidence interval	L	$\underline{\widehat{Y}}_0^L$	$Y_0^{up,L}$	95% confidence interval
1	4	5.60136	5.614	[5.59554, 5.61527]	6	7.38677	7.40107	[7.37977, 7.4023]
2	4	5.41347	5.43091	[5.4078, 5.43249]	6	6.94554	6.97364	[6.93882, 6.97562]
3	4	5.27248	5.29342	[5.2668, 5.29509]	6	6.58739	6.62054	[6.58071, 6.62272]
4	4	5.13119	5.15543	[5.12562, 5.15741]	6	6.2151	6.25165	[6.2086, 6.2541]
5	4	4.98353	5.01117	[4.97802, 5.0133]	6	5.81445	5.85591	[5.80811, 5.85862]
6	4	4.82479	4.85239	[4.81928, 4.85454]	6	5.4079	5.44248	[5.40174, 5.44491]
8	4	4.49057	4.51822	[4.48525, 4.52041]	6	4.68469	4.71574	[4.67908, 4.71808]
10	4	4.13658	4.16231	[4.13141, 4.16444]	6	4.1662	4.19164	[4.16098, 4.19373]
12	4	3.78981	3.81429	[3.78491, 3.81652]	6	3.78992	3.81448	[3.78502, 3.81671]
14	4	3.50023	3.52138	[3.49558, 3.52334]	6	3.50023	3.52138	[3.49558, 3.52334]
1	8	8.83286	8.84907	[8.82488, 8.85034]	10	10.0219	10.0391	[10.0131, 10.0404]
2	8	8.04508	8.08421	[8.03754, 8.08651]	10	8.80264	8.85093	[8.79443, 8.85353]
3	8	7.36943	7.41474	[7.36205, 7.41734]	10	7.73096	7.78117	[7.72312, 7.78394]
4	8	6.66726	6.71129	[6.66021, 6.71389]	10	6.74649	6.78596	[6.73929, 6.78847]
5	8	5.99864	6.0388	[5.99202, 6.0414]	10	6.0035	6.04305	[5.99687, 6.04558]
6	8	5.45188	5.48518	[5.44563, 5.48748]	10	5.45187	5.48518	[5.44563, 5.48748]

Table 4.2: *Unit volume constraints ($v_j \equiv 1$).* Numerical results based on the approximation $\underline{\widehat{Y}}_j^l$ to the Snell envelope via the stopping rule (4.2) for higher exercise rights.

4.3 Numerical results: off peak swing option

We now consider a swing option which allows for buying at most one package of electricity on weekdays and two packages on Saturdays and Sundays (off peak period). Hence, we have for $j = 0, \dots, 50$ the volume constraints

$$v_j := \begin{cases} 1, & \text{if } j \text{ is a week day,} \\ 2, & \text{if } j \text{ is a weekend day,} \end{cases} \tag{4.5}$$

where we start in $j = 0$ on a Monday.

We run the above algorithm with $N_1 = 10000$, $N_2 = 300000$, $N_3 = 2000$, and $N_4 = 100$ sample paths. The numerical results for this off-peak swing option are presented in Table 4.3 for various choices of the number L of exercise rights and the length δ of the refraction period. Notice that the dual representations yet available in the literature do not cover the case of a nontrivial refraction period ($\delta \neq 1$) in combination with nontrivial volume constraints ($v \neq 1$). Due to the feature of allowing for exercising twice on weekends, the swing option prices are now higher than in the example with unit volume constraint. Moreover, additional rights can now become beneficial in situations in which they could not be exercised under the unit volume constraint (e.g. the additional

δ	L	$\widehat{\underline{Y}}_0^L$	$Y_0^{up,L}$	95% confidence interval	L	$\widehat{\underline{Y}}_0^L$	$Y_0^{up,L}$	95% confidence interval
1	2	3.39804	3.40779	[3.39342, 3.409]	3	4.72241	4.73543	[4.71667, 4.73682]
2	2	3.36368	3.37544	[3.35908, 3.37676]	3	4.63568	4.65304	[4.62998, 4.65468]
3	2	3.34728	3.35873	[3.34265, 3.36017]	3	4.58105	4.59793	[4.57532, 4.59971]
4	2	3.32763	3.34072	[3.32302, 3.34223]	3	4.52367	4.54475	[4.51799, 4.5467]
5	2	3.30829	3.32101	[3.30366, 3.32246]	3	4.46437	4.48607	[4.45867, 4.48806]
6	2	3.28626	3.29915	[3.28162, 3.30067]	3	4.40272	4.42552	[4.39701, 4.42758]
8	2	3.24383	3.26104	[3.23921, 3.26284]	3	4.29259	4.31753	[4.2869, 4.31985]
10	2	3.20872	3.22244	[3.20409, 3.22404]	3	4.18401	4.2091	[4.1783, 4.21161]
12	2	3.16994	3.18626	[3.16529, 3.18807]	3	4.06657	4.09375	[4.06086, 4.09637]
14	2	3.12722	3.14467	[3.12253, 3.14666]	3	3.95351	3.97621	[3.94778, 3.97861]
16	2	3.08634	3.10272	[3.08169, 3.10464]	3	3.86402	3.88597	[3.85829, 3.88827]
18	2	3.05186	3.06793	[3.04713, 3.06991]	3	3.75608	3.77801	[3.75036, 3.78047]
20	2	3.01557	3.03044	[3.01088, 3.03233]	3	3.65138	3.67244	[3.64572, 3.67486]
1	4	5.89744	5.91312	[5.89078, 5.91464]	6	7.91351	7.93364	[7.90538, 7.93527]
2	4	5.73736	5.76003	[5.73078, 5.76192]	6	7.55394	7.58547	[7.54595, 7.58773]
3	4	5.62688	5.65097	[5.62027, 5.65305]	6	7.28486	7.32349	[7.27684, 7.32611]
4	4	5.51763	5.5468	[5.51105, 5.54915]	6	7.01995	7.06275	[7.01198, 7.06577]
5	4	5.40154	5.43295	[5.39496, 5.43546]	6	6.7418	6.78724	[6.73383,6.79042]
6	4	5.27976	5.30967	[5.27316, 5.31227]	6	6.45197	6.49774	[6.44401, 6.50102]
8	4	5.06733	5.10055	[5.06078, 5.10335]	6	5.9401	5.98546	[5.93238, 5.989]
10	4	4.85039	4.88637	[4.84386, 4.88953]	6	5.46672	5.50782	[5.45916, 5.51116]
12	4	4.63227	4.66583	[4.62575, 4.66884]	6	5.08473	5.11779	[5.07729, 5.12079]
14	4	4.43104	4.45997	[4.42453, 4.46279]	6	4.76734	4.79957	[4.76006, 4.80269]
16	4	4.25079	4.27955	[4.24441, 4.28237]	6	4.39537	4.42311	[4.38864, 4.42584]
18	4	4.07804	4.10338	[4.07164, 4.10605]	6	4.18139	4.20744	[4.17473, 4.21004]
20	4	3.94562	3.96789	[3.93923, 3.97034]	6	4.02465	4.04716	[4.01803, 4.04968]
1	8	9.60253	9.62348	[9.59318, 9.62507]	10	11.0436	11.0661	[11.0332, 11.0677]
2	8	8.97188	9.01806	[8.96279, 9.02078]	10	10.0822	10.1411	[10.0721, 10.1443]
3	8	8.48335	8.53629	[8.47425, 8.53952]	10	9.31393	9.3793	[9.30393, 9.38283]
4	8	8.00789	8.06203	[7.99887, 8.06551]	10	8.58082	8.63832	[8.57102, 8.64178]
5	8	7.5251	7.57926	[7.5161, 7.58278]	10	7.9058	7.961	[7.89611, 7.96454]
6	8	7.06562	7.11754	[7.05669, 7.12102]	10	7.33533	7.38481	[7.32577, 7.38835]
8	8	6.18418	6.23051	[6.17596, 6.23403]	10	6.20274	6.24781	[6.19445, 6.2513]
10	8	5.54885	5.58774	[5.54107, 5.59089]	10	5.54885	5.58774	[5.54107, 5.59089]

Table 4.3: *Off peak volume constraints.* Numerical results for the off peak swing option for various exercise rights and refraction periods. The simulations are based on the approximation $\widehat{\underline{Y}}_t^l$ to the Snell envelope via the stopping rule (4.2).

8th right when the refraction period is $\delta = 14$. As for accuracy, we again observe that the relative length of the 95% confidence interval is less than 1% in all cases, which demonstrates that the algorithm performs equally well in the presence of volume constraints.

δ	L	$Y_0^{up,L}$	upper bound using Bender [2011b]
1	1	1.86485 (0.0019)	1.8638 (0.0019)
1	2	3.40832 (0.003)	3.4078 (0.003)
1	3	4.73509 (0.0037)	4.7368 (0.0038)
1	4	5.90956 (0.0043)	5.9170 (0.0045)
1	5	6.96665 (0.00047)	6.98 (0.0052)
1	6	7.92669 (0.005)	7.9470 (0.0058)
1	7	8.80743 (0.0055)	8.8327 (0.0062)
1	8	9.61643 (0.0058)	9.6493 (0.0069)
1	9	10.3642 (0.0061)	10.4040 (0.0074)
1	10	11.0553 (0.00064)	11.1035 (0.0079)

Table 4.4: *Off peak volume constraints.* A comparison between our upper bounds and the upper bounds obtained via the algortihm from Bender [2011b] for the case of unit refraction period. Standard deviations are displayed in parentheses.

In the case of unit refraction period $\delta = 1$, upper price bounds for the off-peak swing option can also be computed by the dual representation of Bender [2011b] for the marginal price of a multiple exercise option. This approach generalizes the ideas of Meinshausen and Hambly [2004]: An upper biased estimate for the marginal price of having an additional lth right is computed in terms of one martingale and $(l-1)$ stopping times. By summing up these upper bounds for the marginal prices, one finally ends with an upper biased estimate for the option price. This approach is based on the fact that, roughly speaking, under the assumption of a trivial refraction period ($\delta = 1$) optimal exercise times for the problem with $(l-1)$ rights are also optimal for the problem with l rights, if one adds one additional exercise time in a clever way. This is clearly not possible in general in the presence of a nontrivial refraction period. So it seems that this alternative approach cannot be easily generalized to include refraction periods.

Table 4.4 compares the upper bounds obtained using our method and the method from Bender [2011b] for the unit refraction case $\delta = 1$. We mention that in Table 4.4, the variance reduction method from Remark 17 is not applied for both algorithms. As both methods are run with the same number of sample paths and the nested maximum in our method can be efficiently calculated by the recursion formula in Proposition 13, the computational effort is roughly the same for both algorithms. We observe that, as the number of exercise rights increases, our method of directly tackling the Snell envelope produces upper bounds that become lower than the algorithm tackling the marginal values from Bender [2011b]. Whereas the differences for $L = 1, \ldots, 4$ are numerically not significant yet, they however become noticeable starting from $L = 5$ and are striking for e.g. $L = 10$. We also note that the larger L, the better our method performs concerning the variance of the upper bounds. At large, we conclude that if one is mainly interested in the price (and not the marginal price) of the swing option, our new method performs better than the algorithm from Bender [2011b]. Moreover, it is applicable to a larger class of problems.

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