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Lagrange multiplier and singular limit of double obstacle problems for Allen–Cahn equation with constraint

Dedicated to Professor Nobuyuki Kenmochi on the occasion of his 70th birthday

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Abstract

We consider an Allen–Cahn equation with a constraint of double obstacle-type. This constraint is a subdifferential of an indicator function on the closed interval, which is a multivalued function. In this paper we study the properties of the Lagrange multiplier to our equation. Also, we consider the singular limit of our system and clarify the limit of the solution and the Lagrange multiplier to our double obstacle problem. Moreover, we give some numerical experiments of our problem by using the Lagrange multiplier.

1 Introduction

In this paper, for each $\varepsilon \in (0,1]$ we consider the following constrained Allen–Cahn equation:

$$u_t^\varepsilon - \Delta u^\varepsilon + \frac{W'(u^\varepsilon)}{\varepsilon^2} + \frac{\partial I_{[\sigma_*,\sigma^*]}(u^\varepsilon)}{\varepsilon^2} \ni 0 \ \text{ in } Q := (0,T)\times\Omega, \tag{1.1}$$

$$\frac{\partial u^\varepsilon}{\partial \nu} = 0 \ \text{ on } \Sigma := (0,T)\times\Gamma, \tag{1.2}$$

$$u^\varepsilon(0,x) = u_0^\varepsilon(x), \ \ x\in\Omega, \tag{1.3}$$

where $0 < T < +\infty$, Ω is a bounded domain in $\mathbb{R}^N$ $(1 \le N < +\infty)$ with smooth boundary $\Gamma := \partial\Omega$, σ_*, σ^* are given constants with $-\infty < \sigma_* < \sigma^* < +\infty$, ν is an outward normal vector on Γ and u_0^ε is a given initial data. W is a potential so that W' is Lipschitz on $\mathbb{R}$. The typical examples of W are the following:

$$W(z) = \frac{(1-z^2)^2}{4}, \quad W(z) = \frac{1-z^2}{2}, \quad W(z) = 1+\cos z, \quad \cdots \text{etc.}$$

Furthermore, $\partial I_{[\sigma_*,\sigma^*]}(\cdot)$ is the subdifferential of the indicator function $I_{[\sigma_*,\sigma^*]}(\cdot)$ on the closed interval $[\sigma_*,\sigma^*]$ defined by

$$I_{[\sigma_*,\sigma^*]}(z) := \begin{cases} 0, & \text{if } z\in[\sigma_*,\sigma^*], \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.4}$$

More precisely, $\partial I_{[\sigma_*,\sigma^*]}(\cdot)$ is a set-valued mapping defined by

$$\partial I_{[\sigma_*,\sigma^*]}(z) := \begin{cases} \emptyset & \text{if } z<\sigma_* \text{ or } z>\sigma^*, \\ [0,\infty) & \text{if } z=\sigma^*, \\ \{0\} & \text{if } \sigma_*<z<\sigma^*, \\ (-\infty,0] & \text{if } z=\sigma_*. \end{cases} \tag{1.5}$$

The Allen–Cahn equation was proposed to describe the macroscopic motion of phase boundaries. In physical context, the function $u^\varepsilon = u^\varepsilon(t,x)$ in (P)$^\varepsilon$:={(1.1), (1.2), (1.3)} denotes the nonconserved order parameter that characterizes physical structure. For instance, let $v = v(t,x)$ be the local ratio of the volume of pure liquid relative to that of pure solid at time t and position $x \in \Omega$, defined by

$$v(t,x) := \lim_{r \downarrow 0} \frac{\text{the volume of pure liquid in } B_r(x) \text{ at time } t}{|B_r(x)|},$$

where $B_r(x)$ is the ball in $\mathbb{R}^N$ with center x and radius r and $|B_r(x)|$ denotes its volume. Put $u^\varepsilon(t,x) := (\sigma^* - \sigma_*)v(t,x) + \sigma_*$ for any $(t,x) \in Q$. Then, we easily see that $u^\varepsilon(t,x)$ is the nonconserved order parameter that characterizes the physical structure:

$$\begin{cases} u^\varepsilon(t,x) = \sigma^* & \text{on the pure liquid region,} \\ u^\varepsilon(t,x) = \sigma_* & \text{on the pure solid region,} \\ \sigma_* < u^\varepsilon(t,x) < \sigma^* & \text{on the mixture region.} \end{cases}$$

There are vast literatures of Allen–Cahn equation with or without the double obstacle constraint $\partial I_{[\sigma_*,\sigma^*]}(\cdot)$. For such works, we refer to [1, 3, 7, 8, 9, 10, 11, 15, 20, 21, 22, 24, 26], for instance. In particular, Bronsard and Kohn [7] studied the singular limit of (P)$^\varepsilon$ as $\varepsilon \longrightarrow 0$ with a bistable potential W with both wells of equal depth and without the constraint $\partial I_{[\sigma_*,\sigma^*]}(\cdot)$. Also, Chen and Elliott [9] considered the asymptotic behavior of the solution to (P)$^\varepsilon$ with $W'(z) = -z$ and with the constraint $\partial I_{[-1,1]}(\cdot)$ as $\varepsilon \longrightarrow 0$. But there was no information of an element of $\partial I_{[-1,1]}(u^\varepsilon)$ in [9] as $\varepsilon \longrightarrow 0$. Recently, the authors [12] gave the results of an element of $\partial I_{[-1,1]}(u^\varepsilon)$ in (P)$^\varepsilon$ as $\varepsilon \longrightarrow 0$ in the case of $W'(z) = -z$, $\sigma_* = -1$ and $\sigma^* = 1$.

On the other hand, elliptic and parabolic variational inequalities were considered in connection with Lagrange multipliers (cf. [3, 4, 13, 14, 17, 25]). In particular, there were some numerical experiments of PDE's by using the Lagrange multiplier (cf. [3, 25]). Note from the constraint that the notion of solution to (P)$^\varepsilon$ is given in variational sense (cf. Remark 2.1 below). Also, our constraint $\partial I_{[\sigma_*,\sigma^*]}(\cdot)$ is a set-valued mapping (cf. (1.5)). Therefore, it is very difficult to make numerical experiments to (P)$^\varepsilon$. Hence, it is worthy considering the Lagrange multiplier to (P)$^\varepsilon$ in order to analyze (P)$^\varepsilon$ numerically.

In this paper, for each $\varepsilon \in (0,1]$ we consider an element $\lambda^\varepsilon \in \partial I_{[\sigma_*,\sigma^*]}(u^\varepsilon)$, which is called the Lagrange multiplier to (P)$^\varepsilon$. Also, we investigate the singular limit of our system (P)$^\varepsilon$ and clarify the limit of the solution u^ε and the Lagrange multiplier λ^ε to (P)$^\varepsilon$ as $\varepsilon \to 0$. Moreover, we give numerical experiments to (P)$^\varepsilon$ in one dimension of space for sufficient small $\varepsilon \in (0,1]$. Namely, the main novelties found in this paper are the following:

(a) We give the characterization of the Lagrange multiplier λ^ε to (P)$^\varepsilon$.

(b) We show the convergence of the solution u^ε and the Lagrange multiplier λ^ε to (P)$^\varepsilon$ as $\varepsilon \longrightarrow 0$.

(c) We clarify the properties of the limit of u^ε and λ^ε as $\varepsilon \to 0$.

(d) We give numerical experiments to (P)$^\varepsilon$ in one dimension of space for sufficient small $\varepsilon \in (0,1]$.

The plan of this paper is as follows. In Section 2, we state the main result in this paper. In Section 3 we recall the decomposition result of the subdifferential of convex functions. Also, we prove the main result (Theorem 2.1) concerning the existence-uniqueness of solutions to (P)$^\varepsilon$ and properties of the Lagrange multiplier λ^ε. In Section 4, we prove Theorem 2.2 corresponding to the items (b) and (c) listed above. In Section 5, we give numerical experiments to (P)$^\varepsilon$ in one dimension of space for sufficient small $\varepsilon \in (0,1]$.

Notations and basic assumptions

Throughout this paper, for any reflexive Banach space B, we denote by $|\cdot|_B$ the norm of B, and denote by B^* the dual space of B.

In particular, we put $H := L^2(\Omega)$ with usual real Hilbert space structure, and denote by $(\cdot,\cdot)_H$ the inner product in H. Also, we put $V := H^1(\Omega)$ with the usual norm

$$|z|_V := \left\{ |z|_H^2 + |\nabla z|_H^2 \right\}^{\frac{1}{2}}, \quad z \in V,$$

and denote by $\langle \cdot, \cdot \rangle$ the duality pairing between V^* and V. By identifying H with its dual space, we have $V \subset H \subset V^*$ with compact and dense embeddings; then,

$$\langle u, v \rangle = (u, v)_H \text{ for } u \in H \text{ and } v \in V. \tag{1.6}$$

In the proof of Theorem 2.1, we use some techniques of proper (that is, not identically equal to infinity), l.s.c. (lower semi-continuous), convex functions and their subdifferentials, which are useful in the systematic study of variational inequalities. Therefore, let us outline some notations and definitions. For a proper, l.s.c. and convex function $\psi : H \to \mathbb{R} \cup \{+\infty\}$, the effective domain $D(\psi)$ is defined by

$$D(\psi) := \{z \in H;\ \psi(z) < \infty\}.$$

The subdifferential of ψ is a possibly multi-valued operator in H and is defined by $z^* \in \partial\psi(z)$ if and only if

$$z \in D(\psi) \quad \text{and} \quad (z^*, y - z)_H \le \psi(y) - \psi(z) \quad \text{for all } y \in H.$$

For various properties and related notions of the proper, l.s.c., convex function ψ and its subdifferential $\partial\psi$, we refer to a monograph by Brézis [5].

Next, we give assumptions on the data. Throughout this paper,

- (A1) σ_*, σ^* are constants with $-\infty < \sigma_* < \sigma^* < +\infty$.
- (A2) W is a C^1-function on $\mathbb{R}$ such that $W \ge 0$ on $\mathbb{R}$ and W' is Lipschitz continuous on $[\sigma_*, \sigma^*]$. Moreover, the equation $W + I_{[\sigma_*,\sigma^*]} = 0$ has at most k roots $\xi_1, \xi_2, \cdots, \xi_k$ in $[\sigma_*, \sigma^*]$ so that $\sigma_* \le \xi_1 < \xi_2 < \cdots < \xi_k \le \sigma^*$.
- (A3) $u_0^\varepsilon \in K := \{z \in V\ ;\ \sigma_* \le z \le \sigma^* \text{ a.e. in } \Omega\}$ for all $\varepsilon \in (0, 1]$.

Example 1.1. *If $W(z) = (1 - z^2)^2/4$, $\sigma_* = -1$ and $\sigma^* = 1$, then, the equation $W + I_{[\sigma_*,\sigma^*]} = 0$ has exactly two roots $\xi_1 = -1$ and $\xi_2 = 1$.*

Example 1.2. *If $W(z) = (1.5^2 - z^2)/2$, $\sigma_* = -1$ and $\sigma^* = 1.5$, then, the equation $W + I_{[\sigma_*,\sigma^*]} = 0$ has exactly one root $\xi_1 = 1.5$.*

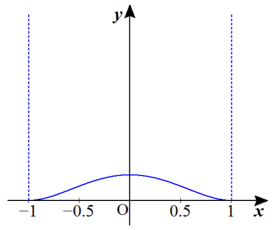

Figure 1: The graph of $\frac{(1-z^2)^2}{4} + I_{[-1,1]}(z)$.

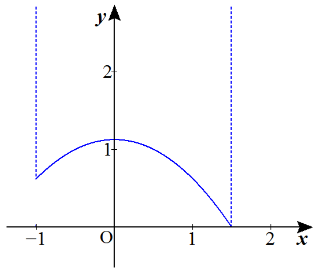

Figure 2: The graph of $\frac{(1.5^2-z^2)}{2} + I_{[-1,1.5]}(z)$.

Example 1.3 (cf. [20]). *If $W(z) = 1 + \cos z$, $\sigma_* = -3\pi$ and $\sigma^* = 3\pi$, then, the equation $W + I_{[\sigma_*,\sigma^*]} = 0$ has exactly four roots $-3\pi, -\pi, \pi, 3\pi$.*

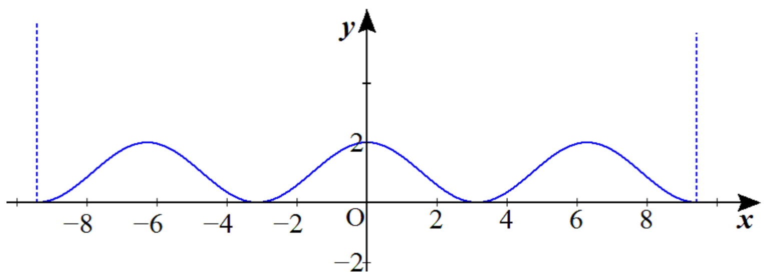

Figure 3: The graph of $1 + \cos z + I_{[-3\pi,3\pi]}(z)$.

Finally, throughout this paper, $C_i = C_i(\cdot), i = 1, 2, 3, \cdots$, denote positive (or nonnegative) constants depending only on its arguments.

2 Main results

We begin by giving the rigorous definition of solutions to our problem (P)$^\varepsilon$ ($\varepsilon \in (0, 1]$).

Definition 2.1. *A function $u^\varepsilon : [0, T] \to H$ is called a solution to* (P)$^\varepsilon$ *on $[0, T]$, if the following conditions are satisfied:*

(i) $u^\varepsilon \in W^{1,2}(0, T; H) \cap L^\infty(0, T; V)$.

(ii) *There is a function $\lambda^\varepsilon \in L^2(0,T;H)$ with $\lambda^\varepsilon \in \partial I_{[\sigma_*,\sigma^*]}(u^\varepsilon)$ a.e. in Q such that*

$$(u_t^\varepsilon(t), z)_H + (\nabla u^\varepsilon(t), \nabla z)_H + \frac{1}{\varepsilon^2}\left(W'(u^\varepsilon(t)), z\right)_H + \frac{1}{\varepsilon^2}\left(\lambda^\varepsilon(t), z\right)_H = 0$$
$$\text{for all } z \in V \text{ and a.e. } t \in (0,T).$$

(iii) $u^\varepsilon(0) = u_0^\varepsilon$ *in* H.

We call λ^ε in (ii) *a Lagrange multiplier to* $(\mathrm{P})^\varepsilon$ *on* $[0,T]$.

Remark 2.1. *It follows from the constraint $\partial I_{[\sigma_*,\sigma^*]}(\cdot)$ and* (ii) *of Definition 2.1 that the equation* (1.1) *is equivalent to the following variational inequality:*

$$\left(u_t^\varepsilon(t) + \frac{1}{\varepsilon^2}W'(u^\varepsilon(t)), u^\varepsilon(t) - z\right)_H + (\nabla u^\varepsilon(t), \nabla u^\varepsilon(t) - \nabla z)_H \leq 0$$
$$\text{for all } z \in K \text{ and a.e. } t \in (0,T).$$

Now, let us mention the first main result in this paper, which is concerned with the existence and basic property of the solution and the Lagrange multiplier to (P)$^\varepsilon$ on $[0,T]$.

Theorem 2.1. *Suppose that the assumptions* (A1)–(A3) *are satisfied and let $\varepsilon \in (0,1]$. Then, there exists a unique solution u^ε to* $(\mathrm{P})^\varepsilon$ *on $[0,T]$. Also, there exists a Lagrange multiplier λ^ε to (P)$^\varepsilon$ on $[0,T]$ in the sense of Definition 2.1 such that*

$$\lambda^\varepsilon(t,x) \begin{cases} \geq 0 & \text{on } \{(t,x) \in Q\ ;\ u^\varepsilon(t,x) = \sigma^*\}, \\ = 0 & \text{on } \{(t,x) \in Q\ ;\ \sigma_* < u^\varepsilon(t,x) < \sigma^*\}, \\ \leq 0 & \text{on } \{(t,x) \in Q\ ;\ u^\varepsilon(t,x) = \sigma_*\}. \end{cases} \tag{2.1}$$

In next Section 3, we give the proof of Theorem 2.1.

Next, we consider the convergence of (P)$^\varepsilon$ as $\varepsilon \to 0$. To this end, we use the following energy functional:

$$\mathscr{F}^\varepsilon(u) := \int_\Omega \left\{\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}I_{[\sigma_*,\sigma^*]}(u) + \frac{1}{\varepsilon}W(u)\right\} dx, \ \ u \in V. \tag{2.2}$$

Now we state the second main result in this paper, which is concerned with the singular limit of (P)$^\varepsilon$ as $\varepsilon \to 0$:

Theorem 2.2. *Assume* (A1)–(A3). *For each $\varepsilon \in (0,1]$, let u^ε be the unique solution to* $(\mathrm{P})^\varepsilon$ *on $[0,T]$. Also, let λ^ε be the Lagrange multiplier to* $(\mathrm{P})^\varepsilon$ *on $[0,T]$ in the sense of Definition 2.1. Assume that there are a function $u_0 \in L^1(\Omega)$ and a positive constant M, independent of $\varepsilon \in [0,1]$, satisfying $u_0(x)$ takes only a value which is the zero point of $W + I_{[\sigma_*,\sigma^*]}$, namely, $W(u_0(x)) + I_{[\sigma_*,\sigma^*]}(u_0(x)) = 0$ for a.e. $x \in \Omega$,*

$$\sup_{\varepsilon\in[0,1]} \mathscr{F}^\varepsilon(u_0^\varepsilon) < M \tag{2.3}$$

and

$$\lim_{\varepsilon\to 0}\int_\Omega |u_0^\varepsilon(x) - u_0(x)| dx = 0. \tag{2.4}$$

Then, there are a subsequence $\{\varepsilon_k\}$ of $\{\varepsilon\}$ with $\varepsilon_k \searrow 0$ as $k \to \infty$, functions $u \in L^2(0,T;H)$ and $\lambda \in L^2(0,T;V^*)$ and a positive number N_0 independent of $\varepsilon \in (0,1]$ such that $u(t,x)$ also takes only a value which is the zero point of $W + I_{[\sigma_*,\sigma^*]}$, namely, $W(u(t,x)) + I_{[\sigma_*,\sigma^*]}(u(t,x)) = 0$ for a.e. $(t,x) \in Q$,

$$\lim_{k\to\infty} u^{\varepsilon_k}(t,x) = u(t,x), \ \text{ a.e. } (t,x) \in Q, \tag{2.5}$$

$$\int_\Omega |u(t_1,x) - u(t_2,x)|\,dx \le N_0|t_1 - t_2|^{\frac{1}{2}}, \quad \forall t_1, t_2 \in [0,T], \tag{2.6}$$

$$\lim_{t\to 0} u(t,x) = u_0(x), \ \text{ a.e. } x \in \Omega, \tag{2.7}$$

$$\int_\Omega |\nabla u(t)| \le N_0, \quad \text{ a.e. } t \in (0,T) \tag{2.8}$$

and

$$\lambda^{\varepsilon_k} \longrightarrow \lambda \text{ weakly in } L^2(0,T;V^*) \ \text{ as } k \to \infty, \tag{2.9}$$

where $\int_\Omega |\nabla u(t)|$ is the total variation measure of $u(t)$. Moreover, $\lambda + W'(u) = 0$ in $L^2(0,T;V^*)$, hence,

$$\lambda(t,x) = -W'(u(t,x)), \quad \text{ a.e. } (t,x) \in Q. \tag{2.10}$$

In Section 4 we prove Theorem 2.2 by using a priori estimates of u^ε and λ^ε.

Example 2.1. *If $W(z) = (1 - z^2)^2/4$, $\sigma_* = -1$ and $\sigma^* = 1$, then, the equation $W + I_{[\sigma_*,\sigma^*]} = 0$ has exactly two roots $\xi_1 = -1$ and $\xi_2 = 1$ (cf. Example 1.1). Then, if the initial data $u_0(x)$ takes only a value -1 or 1 for a.e. $x \in \Omega$, we infer from Theorems 2.1–2.2 that the limit function $u(t,x)$ takes only a value -1 or 1 for a.e. $(t,x) \in Q$. Therefore, we observe that the limit function λ of Lagrange multiplier has the following property:*

$$\lambda(t,x) = -W'(u(t,x)) = -(u(t,x))^3 + u(t,x) = 0, \ \text{ a.e. } (t,x) \in Q.$$

Example 2.2. *If $W(z) = (1.5^2 - z^2)/2$, $\sigma_* = -1$ and $\sigma^* = 1.5$, then, the equation $W + I_{[\sigma_*,\sigma^*]} = 0$ has exactly one root $\xi_1 = 1.5$ (cf. Example 1.2). Then, if the initial data $u_0(x)$ takes only a value 1.5 for a.e. $x \in \Omega$, we infer from Theorems 2.1–2.2 that the limit function $u(t,x)$ also takes only a value 1.5 for a.e. $(t,x) \in Q$. Therefore, we observe that the limit function λ of Lagrange multiplier has the following property:*

$$\lambda(t,x) = -W'(u(t,x)) = u(t,x) = 1.5, \ \text{ a.e. } (t,x) \in Q.$$

Example 2.3. *If $W(z) = 1 + \cos x$, $\sigma_* = -3\pi$ and $\sigma^* = 3\pi$, then, the equation $W + I_{[\sigma_*,\sigma^*]} = 0$ has exactly four roots -3π, $-\pi$, π, 3π (cf. Example 1.3). Then, if the initial data $u_0(x)$ takes a value -3π, $-\pi$, π, or 3π for a.e. $x \in \Omega$, we infer from Theorems 2.1–2.2 that the limit function $u(t,x)$ also takes a value -3π, $-\pi$, π, or 3π for a.e. $(t,x) \in Q$. Therefore, we observe that the limit function λ of Lagrange multiplier has the following property:*

$$\lambda(t,x) = -W'(u(t,x)) = \sin(u(t,x)) = 0, \ \text{ a.e. } (t,x) \in Q.$$

3 Solvability of (P)$^\varepsilon$

In this section we consider (P)$^\varepsilon$ for each $\varepsilon \in (0,1]$. In fact, we study (P)$^\varepsilon$ by arguments similar to [12, 19, 24], namely by abstract evolution equations governed by subdifferentials.

Now, we define a functional φ_0 on H by

$$\varphi_0(z) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla z|^2 dx, & \text{if } z \in V, \\ \infty, & \text{otherwise.} \end{cases} \tag{3.1}$$

Clearly, φ_0 is proper, l.s.c. and convex on H.

Also, we define the proper, l.s.c. and convex functional $\mathcal{I}_{[\sigma_*,\sigma^*]}$ of H by

$$\mathcal{I}_{[\sigma_*,\sigma^*]}(z) := \int_\Omega I_{[\sigma_*,\sigma^*]}(z)dx \quad \text{for any } z \in H,$$

where $I_{[\sigma_*,\sigma^*]}$ is the indicator function defined in (1.4).

Next, we consider the functional φ defined by the form:

$$\varphi(z) = \varphi_0(z) + \frac{1}{\varepsilon^2}\mathcal{I}_{[\sigma_*,\sigma^*]}(z) \quad \text{for any } z \in H.$$

Clearly, φ is proper, l.s.c. and convex on H with the effective domain $D(\varphi) = K$, where K is the set defined in (A3).

Here, we recall the following decomposition result of the subdifferential $\partial\varphi$.

Proposition 3.1 (cf. [6, Section 3], [24, Theorem 3.1])**.** *The subdifferential $\partial\varphi$ of φ is decomposed into the following form:*

$$\partial\varphi(z) = \partial\varphi_0(z) + \frac{1}{\varepsilon^2}\partial\mathcal{I}_{[\sigma_*,\sigma^*]}(z) \text{ in } H \quad \text{for any } z \in H.$$

By arguments similar to [6, Section 3] and [24, Theorem 3.1], we can show Proposition 3.1, thus, omit its detailed proof.

Now, we prove Theorem 2.1 by using Proposition 3.1 and applying the abstract theory of evolution equation associated with subdifferential $\partial\varphi$.

Proof of Theorem 2.1. By the same argument as in [12, Section 3], we can show the existence-uniqueness of a solution u^ε to (P)$^\varepsilon$ on $[0,T]$ for each $\varepsilon \in (0,1]$. In fact, we easily prove the uniqueness of solutions to (P)$^\varepsilon$ on $[0,T]$ by the quite standard arguments: monotonicity and Gronwall's inequality.

Now, we show the existence of solutions to (P)$^\varepsilon$ on $[0,T]$. We easily observe that the problem (P)$^\varepsilon$ can be rewritten as in an abstract framework of the form:

$$\text{(CP)}^\varepsilon \begin{cases} \dfrac{d}{dt}u^\varepsilon(t) + \partial\varphi(u^\varepsilon(t)) + \dfrac{1}{\varepsilon^2}W'(u^\varepsilon(t)) \ni 0 \text{ in } H, \text{ for a.e. } t > 0, \\ u^\varepsilon(0) = u_0^\varepsilon \text{ in } H. \end{cases} \tag{3.2}$$

Therefore, applying the Lipschitz perturbation theory of abstract evolution equations (cf. [6, 16, 23]), we can show the existence of a solution u^ε to (P)$^\varepsilon$ on $[0,T]$ for each $\varepsilon \in (0,1]$ in the variational sense (cf. Remark 2.1).

Also, note from Proposition 3.1 that (CP)$^\varepsilon$ is equivalent to:

$$\widetilde{\text{(CP)}}^\varepsilon \begin{cases} \dfrac{d}{dt}u^\varepsilon(t) + \partial\varphi_0(u^\varepsilon(t)) + \dfrac{1}{\varepsilon^2}\partial\mathcal{I}_{[\sigma_*,\sigma^*]}(u^\varepsilon(t)) + \dfrac{1}{\varepsilon^2}W'(u^\varepsilon(t)) \ni 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } H, \ \ \text{for a.e. } t > 0, \\ u^\varepsilon(0) = u_0^\varepsilon \ \text{ in } H. \end{cases} \tag{3.3}$$

Namely, there are functions $v^\varepsilon \in L^2(0,T;H)$ and $\lambda^\varepsilon \in L^2(0,T;H)$ such that $v^\varepsilon(t) \in \partial\varphi_0(u^\varepsilon(t))$ a.e. $t \in (0,T)$, $\lambda^\varepsilon \in \partial I_{[\sigma_*,\sigma^*]}(u^\varepsilon)$ a.e. in Q and (3.3) holds in the following sense:

$$\frac{d}{dt}u^\varepsilon(t) + v^\varepsilon(t) + \frac{1}{\varepsilon^2}\lambda^\varepsilon(t) + \frac{1}{\varepsilon^2}W'(u^\varepsilon(t)) = 0 \ \text{ in } H, \ \ \text{for a.e. } t > 0.$$

Thus, from the characterization of $\partial\varphi_0$, we easily observe that u^ε is a solution to (P)$^\varepsilon$ on $[0,T]$ and λ^ε is the Lagrange multiplier to (P)$^\varepsilon$ on $[0,T]$ in the sense of Definition 2.1.

Furthermore, taking account of the definition (1.5) of $\partial I_{[\sigma_*,\sigma^*]}(\cdot)$, we conclude from $\lambda^\varepsilon \in \partial I_{[\sigma_*,\sigma^*]}(u^\varepsilon)$ a.e. in Q that the signature result (2.1) of the Lagrange multiplier λ^ε holds. Thus, the proof of Theorem 2.1 has been completed. $\square$

4 Singular limit of (P)$^\varepsilon$ as $\varepsilon \to 0$

In this section we consider the singular limit of (P)$^\varepsilon$ as $\varepsilon \to 0$, and clarify the limit of the solution u^ε and the Lagrange multiplier λ^ε.

We begin by giving the uniform estimates for u^ε and λ^ε with respect to $\varepsilon \in (0,1]$.

Lemma 4.1. *Suppose all the same conditions in Theorem 2.2. For each $\varepsilon \in (0,1]$, let u^ε be the unique solution to* (P)$^\varepsilon$ *on $[0,T]$. Also, let λ^ε be the Lagrange multiplier to* (P)$^\varepsilon$ *on $[0,T]$ in the sense of Definition 2.1. Moreover, assume that there is a positive constant M, independent of $\varepsilon \in [0,1]$, satisfying*

$$\sup_{\varepsilon\in[0,1]} \mathscr{F}^\varepsilon(u_0^\varepsilon) < M.$$

Then, there is a positive number $N_1 > 0$, dependent on M and independent of $\varepsilon \in (0,1]$, such that

$$\varepsilon\int_0^T |u_t^\varepsilon(\tau)|_H^2 d\tau + \sup_{\tau\in[0,T]} \mathscr{F}^\varepsilon(u^\varepsilon(\tau)) + \int_0^T |\lambda^\varepsilon(\tau)|_{V^*}^2 d\tau \le N_1. \tag{4.1}$$

Proof. Multiplying (1.1) by $\varepsilon u_t^\varepsilon$, we get

$$\varepsilon|u_t^\varepsilon(\tau)|_H^2 + \frac{d}{d\tau}\mathscr{F}^\varepsilon(u^\varepsilon(\tau)) = 0 \quad \text{for a.e. } \tau > 0, \tag{4.2}$$

where $\mathscr{F}^\varepsilon(\cdot)$ is the functional defined in (2.2). By integrating (4.2) (in τ) over $[0,t]$ $(\subset [0,T])$, we get

$$\varepsilon\int_0^t |u_t^\varepsilon(\tau)|_H^2 d\tau + \mathscr{F}^\varepsilon(u^\varepsilon(t)) = \mathscr{F}^\varepsilon(u_0^\varepsilon) < M \quad \text{for all } t \in [0,T]. \tag{4.3}$$

Also, taking account of the constraint $\partial I_{[\sigma_*,\sigma^*]}(\cdot)$ (cf. (1.5)), we easily see that

$$\sigma_* \le u^\varepsilon \le \sigma^*, \quad \text{a.e. in } Q. \tag{4.4}$$

Since W' is Lipschitz on $[\sigma_*, \sigma^*]$ (cf. (A2)), we easily infer from (4.4) that

$$|W'(u^\varepsilon(t,x))| \le C_1, \quad \text{a.e. } (t,x) \in Q \tag{4.5}$$

for some constant $C_1 > 0$. By (1.6), (4.5) and (ii) of Definition 2.1, we observe from the Hölder inequality that:

$$\begin{aligned}
&\left|\int_0^T \langle \lambda^\varepsilon(t), z(t)\rangle dt\right| = \left|\int_0^T (\lambda^\varepsilon(t), z(t))_H dt\right| \\
&\le \int_0^T \left|(\varepsilon^2 u_t^\varepsilon(t), z(t))_H\right| dt + \int_0^T \left|\varepsilon^2 \left(\nabla u^\varepsilon(t), \nabla z(t)\right)_H\right| dt \\
&\qquad + \int_0^T \left|(W'(u^\varepsilon(t)), z(t))_H\right| dt \\
&\le \left(\varepsilon^2 |u_t^\varepsilon|_{L^2(0,T;H)} + \varepsilon^2 \sqrt{T} \sup_{t\in[0,T]} |\nabla u^\varepsilon(t)|_H + C_1\sqrt{T|\Omega|}\right) |z|_{L^2(0,T;V)}
\end{aligned} \tag{4.6}$$

for any $z \in L^2(0,T;V)$, where $|\Omega|$ denotes the volume of Ω. Therefore, from $\varepsilon \in (0,1]$, (2.2), (4.3)–(4.6), we infer that:

$$|\lambda^\varepsilon|_{L^2(0,T;V^*)} \le \sqrt{M} + \sqrt{2MT} + C_1\sqrt{T|\Omega|} \quad \text{for all } \varepsilon \in (0,1]. \tag{4.7}$$

From (4.3) and (4.7), we infer that the uniform estimate (4.1) holds for some positive constant N_1. Thus, the proof of Lemma 4.1 has been completed. □

Corollary 4.1. *Suppose all the same conditions in Lemma 4.1. For each $\varepsilon \in (0,1]$, let u^ε be the unique solution to $(\mathrm{P})^\varepsilon$ on $[0,T]$. Also, let $N_1 > 0$ be the positive number obtained in Lemma 4.1. Put*

$$h(s) := \int_{\sigma_*}^s \sqrt{W(\sigma)}\, d\sigma \quad \text{for } s \in [\sigma_*, \sigma^*]. \tag{4.8}$$

Then, the following estimates hold:

$$\sup_{t\in[0,T]} \int_\Omega |\nabla h(u^\varepsilon(t,x))| dx \le N_1 \tag{4.9}$$

and

$$\int_{t_1}^{t_2} \int_\Omega |(h(u^\varepsilon(t,x)))_t| dx dt \le N_1 (t_2 - t_1)^{\frac{1}{2}} \quad \text{for all } t_1, t_2 \text{ with } 0 \le t_1 < t_2 \le T. \tag{4.10}$$

Proof. First, note from (4.8) that $h'(s) = \sqrt{W(s)}$ for $s \in [\sigma_*, \sigma^*]$.

Now, we show the estimate (4.9). By (4.1), (4.4) and the Schwarz inequality, we have:

$$\begin{aligned}
\int_\Omega |\nabla h(u^\varepsilon(t,x))| dx &= \int_\Omega |h'(u^\varepsilon(t,x))| |\nabla u^\varepsilon(t,x)| dx \\
&= \int_\Omega \sqrt{W(u^\varepsilon(t,x))} |\nabla u^\varepsilon(t,x)| dx \\
&\le \frac{1}{2\varepsilon} \int_\Omega W(u^\varepsilon(t,x)) dx + \frac{\varepsilon}{2} \int_\Omega |\nabla u^\varepsilon(t,x)|^2 dx \\
&\le \mathscr{F}^\varepsilon(u^\varepsilon(t)) \\
&\le N_1 \quad \text{for all } t \in [0,T].
\end{aligned}$$

Thus, (4.9) holds.

Next, we show (4.10). By (4.1), (4.4) and the Hölder inequality, we have:

$$\begin{aligned}\int_{t_1}^{t_2}\int_\Omega |(h(u^\varepsilon(t,x)))_t|dxdt &= \int_{t_1}^{t_2}\int_\Omega |h'(u^\varepsilon(t,x))||u_t^\varepsilon(t,x)|dxdt \\ &\le \left(\int_{t_1}^{t_2}\int_\Omega |h'(u^\varepsilon(t,x))|^2 dxdt\right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} |u_t^\varepsilon(t)|_H^2 dt\right)^{\frac{1}{2}} \\ &= \left(\int_{t_1}^{t_2}\int_\Omega W(u^\varepsilon(t,x)) dxdt\right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} |u_t^\varepsilon(t)|_H^2 dt\right)^{\frac{1}{2}} \\ &\le (t_2-t_1)^{\frac{1}{2}}\{\varepsilon \sup_{t\in[t_1,t_2]} \mathscr{F}^\varepsilon(u^\varepsilon(t))\}^{\frac{1}{2}} \frac{N_1^{\frac{1}{2}}}{\varepsilon^{\frac{1}{2}}} \\ &\le N_1(t_2-t_1)^{\frac{1}{2}}\end{aligned}$$

for all t_1, t_2 with $0 \le t_1 < t_2 \le T$.

Thus, the proof of Corollary 4.1 has been completed. □

Now, we prove the main Theorem 2.2, which is concerned with the singular limit of (P)$^\varepsilon$ as $\varepsilon \to 0$.

Proof of Theorem 2.2. Let $\{\xi_1, \xi_2, \cdots, \xi_k\}$ be the set of all solutions to the equation $W + I_{[\sigma_*,\sigma^*]} = 0$ so that $\sigma_* \le \xi_1 < \xi_2 < \cdots < \xi_k \le \sigma^*$. Then, we first show the existence of a subsequence $\{\varepsilon_k\}$ of $\{\varepsilon\}$ and a function $u \in L^2(0,T;H)$ such that $u(t,x)$ takes only a value, which is an element of $\{\xi_1, \xi_2, \cdots, \xi_k\}$, for a.e. $(t,x) \in Q$. Also, we show (2.5).

By the definition of h (cf. (4.8)), we easily see that the function h is bounded on $[\sigma_*, \sigma^*]$:

$$|h(s)| \le C_h \text{ for all } s \in [\sigma_*, \sigma^*] \tag{4.11}$$

for some positive constant $C_h > 0$. Therefore we infer from (4.4) and (4.11) that:

$$\int_0^T\int_\Omega |h(u^\varepsilon(t,x))|dxdt \le T|\Omega|C_h. \tag{4.12}$$

Taking account of (4.9), (4.10) and (4.12), we observe that $\{h(u^\varepsilon)\}$ is bounded in $BV((0,T)\times\Omega)$ uniformly in $\varepsilon \in (0,1]$, where $BV((0,T)\times\Omega)$ is the space of all functions of bounded variation on $(0,T)\times\Omega$. Since $BV((0,T)\times\Omega)$ is compactly embedded into $L^1((0,T)\times\Omega)$ (cf. [2, Corollary 3.49]), there are a subsequence $\{\varepsilon_k\} \subset \{\varepsilon\}$ and a function $\tilde{h} \in BV((0,T)\times\Omega)$ such that $\varepsilon_k \to 0$ and

$$h(u^{\varepsilon_k}) \longrightarrow \tilde{h} \text{ in } L^1((0,T)\times\Omega) \text{ as } k \to \infty. \tag{4.13}$$

Therefore, taking a subsequence if necessary, we obtain:

$$h(u^{\varepsilon_k}(t,x)) \longrightarrow \tilde{h}(t,x), \text{ a.e. } (t,x) \in (0,T)\times\Omega \text{ as } k \to \infty. \tag{4.14}$$

Since h is continuous and strictly increasing on $[\sigma_*, \sigma^*]$ (cf. (A2) and (4.8)), we can find a unique function $u(t,x)$ such that

$$\tilde{h}(t,x) = h(u(t,x)), \text{ a.e. } (t,x) \in (0,T)\times\Omega \tag{4.15}$$

and

$$u^{\varepsilon_k}(t,x) \longrightarrow u(t,x), \text{ a.e. } (t,x) \in (0,T)\times\Omega \text{ as } k \to \infty, \tag{4.16}$$

hence (2.5) holds. Clearly, it follows from (4.4) and (4.16) that:

$$\sigma_* \le u \le \sigma^*, \text{ a.e. in } (0,T)\times\Omega \tag{4.17}$$

We easily see from (A2) and (4.4) that

$$|W(u^{\varepsilon_k}(t,x))| \le C_2, \quad \text{a.e. } (t,x) \in (0,T)\times\Omega \tag{4.18}$$

for some constant $C_2 > 0$ independent of k. Therefore, from (A2), (4.1), (4.4), (4.16)–(4.18) and Lebesgue's dominated convergence theorem, we infer that:

$$\begin{aligned} 0 \le \int_0^T\int_\Omega W(u(t,x))dxdt &= \lim_{k\to\infty}\int_0^T\int_\Omega W(u^{\varepsilon_k}(t,x))dxdt \\ &\le N_1 T \lim_{k\to\infty}\varepsilon_k = 0. \end{aligned} \tag{4.19}$$

Thus, we obtain

$$W(u(t,x)) = 0, \text{ a.e. } (t,x) \in (0,T)\times\Omega,$$

which implies from (4.17) that the limit function u of u^{ε_k} takes only a value that is an element of $\{\xi_1, \xi_2, \cdots, \xi_k\}$.

Next, we show (2.6). Note that from (4.10) the following inequality follows:

$$\begin{aligned} \int_\Omega |h(u^{\varepsilon_k}(t_1,x)) - h(u^{\varepsilon_k}(t_2,x))|\,dx &\le \int_\Omega\int_{t_1}^{t_2} |(h(u^{\varepsilon_k}(t,x)))_t|dtdx \\ &\le N_1(t_2-t_1)^{\frac{1}{2}} \end{aligned} \tag{4.20}$$

$$\text{for all } t_1, t_2 \text{ with } 0 \le t_1 < t_2 \le T.$$

Taking the limit in (4.20) as $k \to \infty$, we infer from (4.4), (4.11), (4.13)–(4.15) and Lebesgue's dominated convergence theorem that

$$\begin{gathered} \int_\Omega |h(u(t_1,x)) - h(u(t_2,x))|\,dx \le N_1(t_2-t_1)^{\frac{1}{2}} \\ \text{for a.e. } t_1, t_2 \text{ with } 0 \le t_1 < t_2 \le T. \end{gathered} \tag{4.21}$$

Note that the limit function u of u^{ε_k} takes only a value that is an element of $\{\xi_1, \xi_2, \cdots, \xi_k\}$. Therefore, we easily see that:

$$\begin{gathered} |h(u(t_1,x)) - h(u(t_2,x))| \ge C_3|u(t_1,x) - u(t_2,x)|, \\ \text{a.e. } x \in \Omega \text{ and a.e. } t_1, t_2 \in (0,T), \end{gathered} \tag{4.22}$$

where $C_3 > 0$ is a positive constant defined by:

$$C_3 := \frac{\min\left\{ |h(\xi_i) - h(\xi_j)|; \begin{array}{c} i,j = 1,2,\cdots,k \\ \text{with } i \ne j \end{array} \right\}}{\sigma^* - \sigma_*}.$$

Therefore, we observe from (4.21) and (4.22) that:

$$\int_{\Omega} |u(t_1,x)-u(t_2,x)|\,dx \le \frac{N_1}{C_3}(t_2-t_1)^{\frac{1}{2}} \quad \text{for a.e. } t_1,t_2 \text{ with } 0\le t_1<t_2\le T. \tag{4.23}$$

By (4.23), we can redefine the function u in order that $u(t) \in L^1(\Omega)$ is continuous with respect to $t \in [0,T]$. Therefore, (4.23) holds for all $t_1, t_2 \in [0,T]$ with $t_2 > t_1$, thus, (2.6) holds by defining $N_0 := N_1/C_3$.

Next, we show (2.7). At first, we note from (2.4) that

$$u_0^{\varepsilon_k}(x) \longrightarrow u_0(x), \text{ a.e. } x\in\Omega \text{ as } k\to\infty \tag{4.24}$$

by taking a subsequence if necessary. Therefore, by (2.3), (4.11), (4.24) and the continuity of h, we easily obtain from Lebesgue's dominated convergence theorem that:

$$\lim_{k\to\infty}\int_{\Omega} |h(u_0^{\varepsilon_k}(x))-h(u_0(x))|\,dx = 0. \tag{4.25}$$

Now, taking $t_1 = 0$ in (4.20), we have:

$$\int_{\Omega} |h(u_0^{\varepsilon_k}(x))-h(u^{\varepsilon_k}(t_2,x))|\,dx \le N_1 t_2^{\frac{1}{2}} \text{ for all } t_2\in(0,T]. \tag{4.26}$$

Taking the limit in (4.26) as $k \to \infty$, we observe from (4.13)–(4.16), (4.24) and (4.26) that:

$$\int_{\Omega} |h(u_0(x))-h(u(t_2,x))|\,dx \le N_1 t_2^{\frac{1}{2}}, \quad \text{a.e. } t_2\in(0,T]. \tag{4.27}$$

Note that $u_0(x)$ takes only a value that is an element of $\{\xi_1, \xi_2, \cdots, \xi_k\}$ for a.e. $x \in \Omega$. Also, note that $u(t) \in L^1(\Omega)$ is continuous with respect to $t \in [0,T]$. Therefore, we easily see from (4.22) and (4.27) that:

$$\int_{\Omega} |u_0(x)-u(t_2,x)|\,dx \le \frac{N_1}{C_3} t_2^{\frac{1}{2}} \text{ for all } t_2\in(0,T], \tag{4.28}$$

Thus, by taking the limit $t_2 \to 0$ in (4.28), we observe that (2.7) holds.

Next, we show (2.8). By (4.9), (4.13) and the lower semicontinuity of the total variation under L^1-convergence (cf. [2, Proposition 3.6]), we observe that

$$\int_{\Omega} |\nabla \tilde{h}(t)| \le N_1, \quad \text{a.e. } t\in(0,T), \tag{4.29}$$

where $\int_\Omega |\nabla \tilde{h}(t)|$ is the total variation measure of $\tilde{h}(t)$. Since $u(t,x)$ takes only a value, that is an element of $\{\xi_1, \xi_2, \cdots, \xi_k\}$, for a.e. $(t,x) \in (0,T)\times\Omega$, we infer from (4.15) and (4.22) that

$$\int_{\Omega} |\nabla u(t)| \le \frac{N_1}{C_3}, \quad \text{a.e. } t\in(0,T). \tag{4.30}$$

Hence, (2.8) holds.

Finally, we show (2.9)–(2.10). By the uniform estimate (4.1), we easily see that there is a subsequence of $\{\varepsilon_k\}$ (which we denote ε_k for simplicity) and a function $\lambda \in L^2(0,T;V^*)$ satisfying (2.9).

From (A2), (4.4), (4.5), (4.16) and Lebesgue's dominated convergence theorem, we infer that:

$$W'(u^{\varepsilon_k}) \longrightarrow W'(u) \text{ in } L^2(0,T;H) \text{ as } k \to \infty. \tag{4.31}$$

By (1.6), (4.1) and (ii) of Definition 2.1, we have that:

$$\begin{aligned}
&\int_0^T \langle \lambda^{\varepsilon_k}(t) + W'(u^{\varepsilon_k}(t)), z(t) \rangle dt \\
&= \int_0^T (\lambda^{\varepsilon_k}(t) + W'(u^{\varepsilon_k}(t)), z(t))_H dt \\
&\le \int_0^T \left| (\varepsilon_k^2 u_t^{\varepsilon_k}(t), z(t))_H \right| dt + \int_0^T \left| \varepsilon_k^2 (\nabla u^{\varepsilon_k}(t), \nabla z(t))_H \right| dt \\
&\le \varepsilon_k^{\frac{3}{2}} \sqrt{N_1} |z|_{L^2(0,T;H)} + \varepsilon_k^{\frac{3}{2}} \sqrt{2TN_1} |z|_{L^2(0,T;V)} \\
&\qquad\qquad\qquad\qquad \text{for any } z \in L^2(0,T;V).
\end{aligned}$$

From (2.9), (4.31) and the above inequality, we see that

$$\int_0^T \langle \lambda(t) + W'(u(t)), z(t) \rangle dt = \lim_{k\to\infty} \int_0^T \langle \lambda^{\varepsilon_k}(t) + W'(u^{\varepsilon_k}(t)), z(t) \rangle dt \le 0. \tag{4.32}$$

Since $z \in L^2(0,T;V)$ is arbitrary, we infer from (4.32) that

$$\lambda + W'(u) = 0 \left(\in L^2(0,T;H) \right) \text{ in } L^2(0,T;V^*). \tag{4.33}$$

Hence, we conclude from (4.33) that (2.10) holds. Thus, the proof of Theorem 2.2 has been completed.

□

5 Numerical experiments in one dimension of space

In this section, for each $\varepsilon \in (0,1]$, we consider the following special case of (P)$^\varepsilon$, denoted by (1D)$^\varepsilon$:

$$\text{(1D)}^\varepsilon \begin{cases} u_t^\varepsilon - u_{xx}^\varepsilon - \dfrac{u^\varepsilon}{\varepsilon^2} + \dfrac{\partial I_{[-1,1]}(u^\varepsilon)}{\varepsilon^2} \ni 0 \quad \text{in } Q := (0,T) \times (0,1), \\ u_x^\varepsilon(t,0) = u_x^\varepsilon(t,1) = 0, \quad t \in (0,T), \\ u^\varepsilon(0,x) = u_0^\varepsilon(x), \quad x \in (0,1). \end{cases}$$

Remark 5.1 (cf. [12, Theorem 2.2]). *The problem* $(1\mathrm{D})^\varepsilon$ *is a special case of* $(\mathrm{P})^\varepsilon$ *when we use the following setting:* $\Omega := (0,1)$*,* $W(z) = (1-z^2)/2$*,* $\sigma_* = -1$ *and* $\sigma^* = 1$*. Therefore, we easily see that the equation* $W + I_{[\sigma_*,\sigma^*]} = 0$ *has exactly two roots* $\xi_1 = -1$ *and* $\xi_2 = 1$*. Hence, if the initial value* $u_0(x)$ *takes only a value* -1 *or* 1 *for a.e.* $x \in \Omega$*, we infer from Theorems 2.1–2.2 that the limit function* $u(t,x)$ *takes only a value* -1 *or* 1 *for a.e.* $(t,x) \in (0,T) \times \Omega$*. Also, we observe that the limit function* λ *of Lagrange multiplier has the following property:*

$$\lambda = \begin{cases} 1 & \text{on } \{(t,x) \in Q \,;\, u(t,x) = 1\}, \\ -1 & \text{on } \{(t,x) \in Q \,;\, u(t,x) = -1\}. \end{cases} \tag{5.1}$$

In this section, we consider the problem (1D)$^\varepsilon$ numerically for sufficient small $\varepsilon \in (0,1]$.

Note that the constraint $\partial I_{[-1,1]}(\cdot)$ is a set-valued mapping (cf. (1.5)). Therefore, it is very difficult to do numerical simulations (to (1D)$^\varepsilon$). To handle the set-valued constraint $\partial I_{[-1,1]}(\cdot)$, we use the Lagrange multiplier λ^ε (to (1D)$^\varepsilon$). Taking account of (5.1) (cf. Theorems 2.1 and 2.2), we propose the following numerical algorithm, denoted by (NA)$^\varepsilon$.

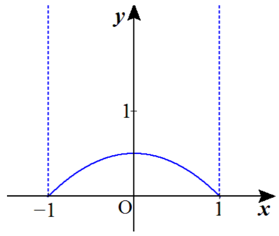

Figure 4: The graph of $\frac{(1-z^2)}{2} + I_{[-1,1]}(z)$.

Numerical Algorithm (NA)$^\varepsilon$ to (1D)$^\varepsilon$

(Step 0) Fix the small parameter $\varepsilon \in (0,1]$, and choose the initial data $u_0^\varepsilon \in K$. Put $u_n = u_0^\varepsilon$ and $u_{n-1} = u_0^\varepsilon$.

(Step 1) If $-1 < u_n^\varepsilon < 1$, then, go to (Step 3); Otherwise, compute a Lagrange multiplier λ_n^ε by:

$$\lambda_n^\varepsilon := \varepsilon^2 \frac{-u_n^\varepsilon + u_{n-1}^\varepsilon}{\triangle t} + \varepsilon^2 (u_n^\varepsilon)_{xx} + u_n^\varepsilon,$$

where $\triangle t$ is the time mesh size and $(u_n^\varepsilon)_{xx}$ is the second-order central difference approximation for the Laplacian of u_n^ε.

(Step 2) Test: if $\lambda_n^\varepsilon = 1$ (resp. -1), then, set $u_{n+1}^\varepsilon = 1$ (resp. -1); Otherwise, go to (Step 3).

(Step 3) To determine u_{n+1}^ε, solve (1D)$^\varepsilon$ without the constraint $\partial I_{[-1,1]}(\cdot)$.

(Step 4) Redefine u_{n+1}^ε so that $u_{n+1}^\varepsilon \in [-1,1]$.

(Step 5) Set $n = n+1$, and go to (Step 1).

Remark 5.2. *Blank et al. [3] had already proposed the numerical algorithm, called a primal-dual active set method, which is similar to* (NA)$^\varepsilon$. *However, there was no result on the value of a Lagnange multiplier in [3] (cf.* (5.1)*).*

To make numerical experiments, we use the following data.

Numerical Data

- $\Omega = (0,1)$ and $T = 0.01$.
- The mesh size of space $\triangle h = 0.005 = 1/200$.
- The mesh size of time $\triangle t = 0.2 * \triangle h^2 = 0.000005 = 1/200000$.

Now, we consider the following initial data $u_0^\varepsilon(x)$, which converges to $u_0(x)$ defined by

$$u_0(x) = \begin{cases} 1, & \text{if } x \in J := [0.4, 0.7], \\ -1, & \text{if } x \in \Omega \setminus J. \end{cases}$$

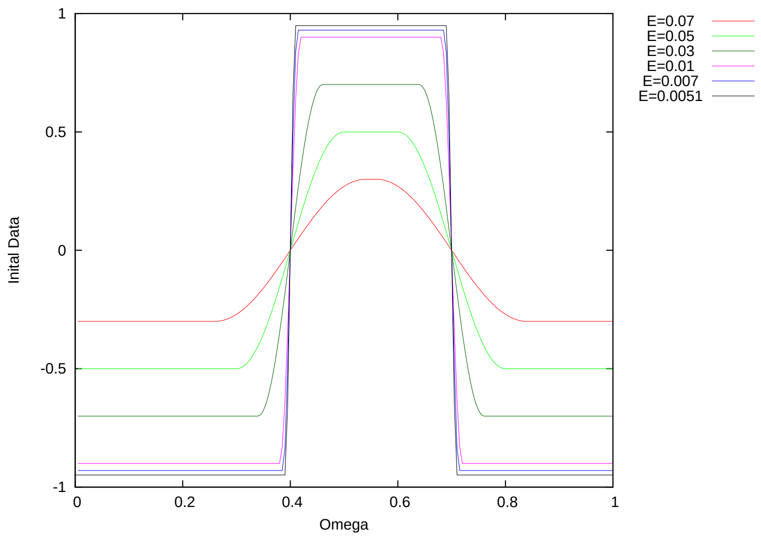

Figure 5: The graph of initial data $u_0^\varepsilon(x)$ in the case $\varepsilon = 0.07, 0.05, 0.03, 0.01, 0.007$ and 0.0051.

By using the numerical algorithm (NA)$^\varepsilon$ with numerical data as before, we have the following numerical experiments (Figures 6–11) to (1D)$^\varepsilon$. From these results, we easily see that the solution $u^\varepsilon(t,x)$ takes a value 1 or -1 in very short time if ε is sufficient small.

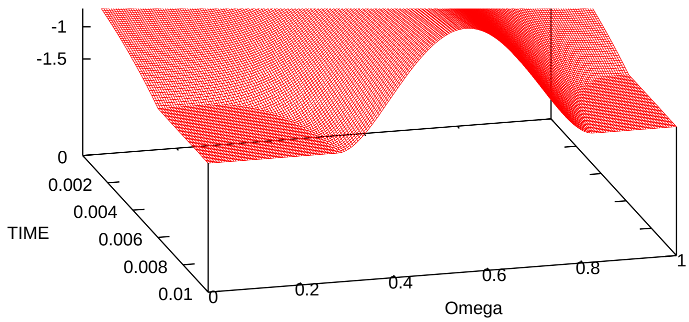

Figure 6: Behaviour of a solution $u^\varepsilon(t,x)$ with $\varepsilon = 0.07$.

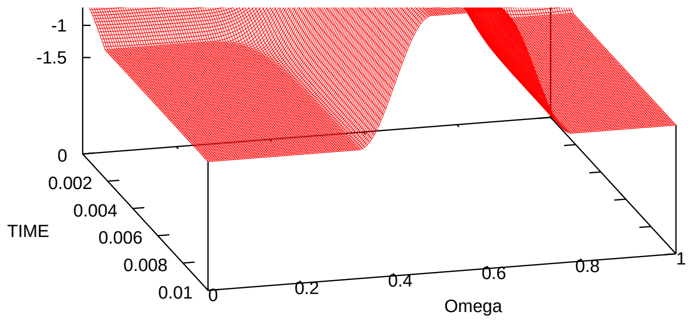

Figure 7: Behaviour of a solution $u^{\varepsilon}(t,x)$ with $\varepsilon = 0.05$.

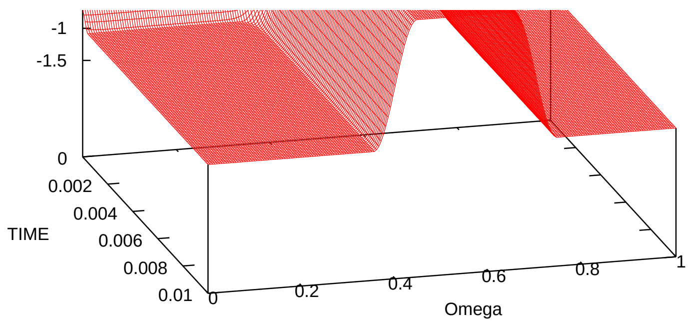

Figure 8: Behaviour of a solution $u^{\varepsilon}(t,x)$ with $\varepsilon = 0.03$.

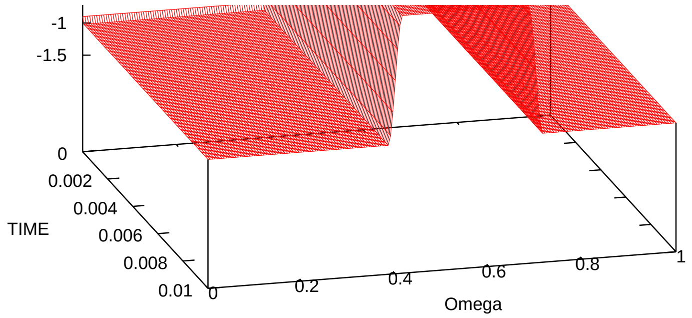

Figure 9: Behaviour of a solution $u^{\varepsilon}(t,x)$ with $\varepsilon = 0.01$.

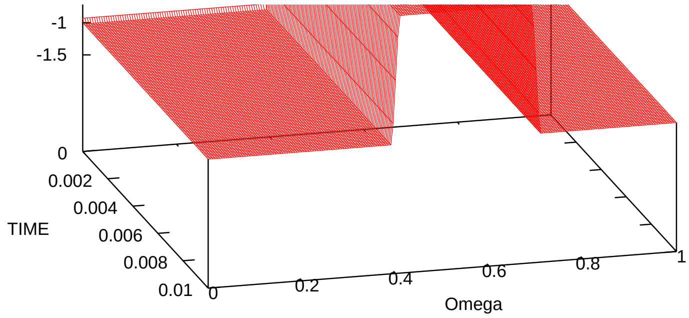

Figure 10: Behaviour of a solution $u^\varepsilon(t,x)$ with $\varepsilon = 0.007$.

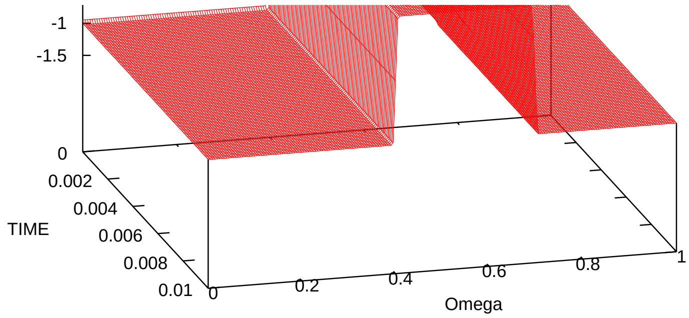

Figure 11: Behaviour of a solution $u^\varepsilon(t,x)$ with $\varepsilon = 0.0051$.

Remark 5.3. *To handle the set-valued constraint $\partial I_{[-1,1]}(\cdot)$, we can consider some approximating method. For instance, for $\delta > 0$, we use the following Yosida approximation $(\partial I_{[-1,1]})_\delta(\cdot)$ of $\partial I_{[-1,1]}(\cdot)$ defined by:*

$$(\partial I_{[-1,1]})_\delta(z) = \frac{[z-1]^+ - [-1-z]^+}{\delta} \quad \text{for all } z \in \mathbb{R},$$

where $[z]^+$ is the positive part of z. Then, we observe from [5, Chapter 2, Section 4] that $(\partial I_{[-1,1]})_\delta(\cdot) \to \partial I_{[-1,1]}(\cdot)$ as $\delta \to 0$. Now, for each $\delta > 0$, we consider the approximation problem of $(\mathrm{1D})^\varepsilon$, denoted by $(\mathrm{1D})^\varepsilon_\delta$:

$$(\mathrm{1D})^\varepsilon_\delta \begin{cases} u^\varepsilon_t - u^\varepsilon_{xx} - \dfrac{u^\varepsilon}{\varepsilon^2} + \dfrac{(\partial I_{[-1,1]})_\delta(u^\varepsilon)}{\varepsilon^2} = 0 \quad \text{in } Q := (0,T)\times(0,1), \\ u^\varepsilon_x(t,0) = u^\varepsilon_x(t,1) = 0, \quad t \in (0,T), \\ u^\varepsilon(0,x) = u^\varepsilon_0(x), \quad x \in (0,1). \end{cases}$$

Then, we have the following numerical result to $(\mathrm{1D})^\varepsilon_\delta$ by using the standard forward Euler method for $(\mathrm{1D})^\varepsilon_\delta$ with the same numerical data as before.

From Figure 12, we easily observe that in order to get stable numerical results, we have to choose the suitable constants ε, δ and the mesh size $\triangle t$, $\triangle x$. Therefore, if we make a numerical experiment of $(\mathrm{P})^\varepsilon$ for sufficient small ε, we had better consider the original problem by using a Lagrange multiplier method: a primal-dual active set method in [3], $(\mathrm{NA})^\varepsilon$ as above, and so on.

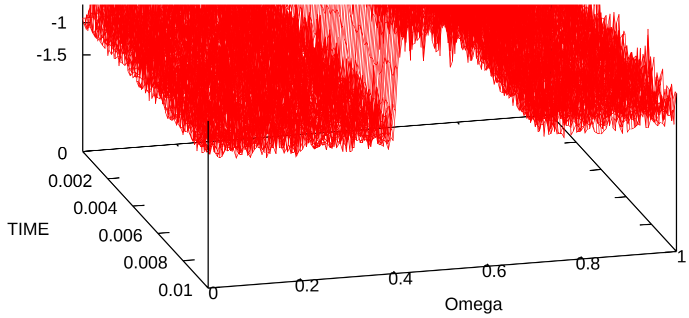

Figure 12: Behaviour of a solution $u^{\varepsilon}(t,x)$ with $\varepsilon = 0.007$ and $\delta = 0.01$.

References

[1] S. Allen and J. Cahn, A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening, Acta Metall., **27**(1979), 1084-1095.

[2] L. Ambrosio, N. Fusco and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.

[3] L. Blank, H. Garcke L. Sarbu and V. Styles, Primal-dual active set methods for Allen-Cahn variational inequalities with nonlocal constraints, Numer. Methods Partial Differential Equations, **29**(2013), 999–1030.

[4] V. Barbu and T. Precupanu, *Convexity and optimization in Banach spaces, Fourth edition,* Springer Monographs in Mathematics, Springer, Dordrecht, 2012.

[5] H. Brézis, *Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert*, North-Holland, Amsterdam, 1973.

[6] H. Brézis, M. G. Crandall and A. Pazy, Perturbations of nonlinear maximal monotone sets in Banach space, Comm. Pure Appl. Math., **23**(1970), 123–144.

[7] L. Bronsard and R.V. Kohn, Motion by mean curvature as the singular limit of Ginzburg-Landau dynamics, J. Differential Equations, **90**(1991), 211–237.

[8] X. Chen, Generation and propagation of interfaces for reaction-diffusion equations, J. Differential Equations, **96**(1992), 116–141.

[9] X. Chen and C. M. Elliott, Asymptotics for a parabolic double obstacle problem, Proc. Roy. Soc. London Ser. A, **444**(1994), 429–445.

[10] P. de Mottoni and M. Schatzman, Evolution géométrique d'interfaces, C. R. Acad. Sci. Paris Ser. I Math., **309**(1989), 453–458.

[11] P. C. Fife, Dynamics of internal layers and diffusive interfaces, CBMS-NSF Regional Conf. Ser. in Appl. Math., **53**, SIAM, Philadelphia, 1988.

[12] M. H. Farshbaf-Shaker, T. Fukao and N. Yamazaki, Singular limit of Allen-Cahn equation with constraints and its Lagrange multiplier, submitted.

[13] T. Fukao and N. Kenmochi, Lagrange multipliers in variational inequalities for nonlinear operators of momotone type, Adv. Math. Sci. Appl., **23**(2013), 545–574.

[14] E. Ginder, Construction of solutions to heat-type problems with time-dependent volume constraints, Adv. Math. Sci. Appl., **20**(2010), 467–482.

[15] A. Ito, Asymptotic stability of Allen-Cahn model for nonlinear Laplacian with constraints, Adv. Math. Sci. Appl., **9**(1999), 137–161.

[16] A. Ito, N. Yamazaki and N. Kenmochi, Attractors of nonlinear evolution systems generated by time-dependent subdifferentials in Hilbert spaces, *Dynamical systems and differential equations, Vol. I (Springfield, MO, 1996)*, Discrete Contin. Dynam. Systems 1998, Added Volume I, 327–349.

[17] K. Ito and K. Kunisch, *Lagrange multiplier approach to variational problems and applications*, Advances in Design and Control, **15**, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2008.

[18] N. Kenmochi, Y. Mizuta, T. Nagai, Projections onto convex sets, convex functions and their subdifferentials, Bull. Fac. Edu., Chiba Univ., **29**(1980), 11–22.

[19] N. Kenmochi and K. Shirakawa, Stability for a parabolic variational inequality associated with total variation functional, Funkcial. Ekvac., **44**(2001), 119–137.

[20] T. Ohtsuka, Motion of interfaces by an Allen-Cahn type equation with multiple-well potentials. Asymptot. Anal., **56**(2008), 87–123.

[21] T. Ohtsuka, K. Shirakawa and N. Yamazaki, Optimal control problems of singular diffusion equation with constraint, Adv. Math. Sci. Appl., **18**(2008), 1–28.

[22] T. Ohtsuka, K. Shirakawa and N. Yamazaki, Optimal control problem for Allen-Cahn type equation associated with total variation energy, Discrete Contin. Dyn. Syst. Ser. S, **5**(2012), 159–181.

[23] M. Ôtani, Nonmonotone perturbations for nonlinear parabolic equations associated with subdifferential operators, Cauchy problems, J. Differential Equations, **46**(1982), 268–299.

[24] K. Shirakawa and M. Kimura, Stability analysis for Allen-Cahn type equation associated with the total variation energy, Nonlinear Anal., **60**(2005), 257–282.

[25] K. Švadlenka and S. Omata, Mathematical modelling of surface vibration with volume constraint and its analysis, Nonlinear Anal., **69**(2008), 3202–3212.

[26] Y. Tonegawa, Integrality of varifolds in the singular limit of reaction-diffusion equations, Hiroshima Math. J., **33**(2003), 323–341.