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A tweezer for chimeras in small networks

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Abstract

We propose a control scheme which can stabilize and fix the position of chimera states in small networks. Chimeras consist of coexisting domains of spatially coherent and incoherent dynamics in systems of nonlocally coupled identical oscillators. Chimera states are generally difficult to observe in small networks due to their short lifetime and erratic drifting of the spatial position of the incoherent domain. The control scheme, like a tweezer, might be useful in experiments, where usually only small networks can be realized.

The study of coupled oscillator systems is a prominent field of research in nonlinear science with a wide range of applications in physics, chemistry, biology, and technology. An intriguing dynamical phenomenon in such systems are *chimera states* exhibiting a hybrid nature of coexisting coherent and incoherent domains [1, 2, 3, 4, 5, 6]. So far, chimera states have been theoretically investigated in a wide range of large-size networks [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31], where different kinds of coupling schemes varying from regular nonlocal to completely random topology have been considered.

The experimental verification of chimera states was first demonstrated in optical [32] and chemical [33, 34] systems. Further experiments involved mechanical [35], electronic [36, 37, 38] and electrochemical [39, 40] oscillator systems as well as Boolean networks [41].

Deeper analytical insight and bifurcation analysis of chimera states has been obtained in the framework of phase oscillator systems [42, 43, 44, 45]. However, most theoretical results refer to the continuum limit only, which explains the behavior of very large ensembles of coupled oscillators. In contrast, chimera states in small-size networks have attracted attention only recently [46, 47, 48, 49], although in lab experiments usually only small networks can be realized.

There are two principal difficulties preventing the observation of chimera states in small-size systems of nonlocally coupled oscillators. First, it is known that chimera states are usually chaotic transients which eventually collapse to the uniformly synchronized state [50]. Their mean lifetime decreases rapidly with decreasing system size such that one hardly observes chimeras already for $20-30$ coupled oscillators. Moreover, a clear distinction between initial conditions that lead to a chimera state, and those that go directly to the synchronized state is no more possible. Second, the position of the incoherent domain is not stationary but rather moves erratically along the oscillator array [51]. This motion has the statistical properties of a Brownian motion and its diffusion coefficient is inversely proportional to some power of the system size. Both effects, finite lifetime and random walk of the chimera position, are negligible in large size systems. However, they dominate the dynamics of small-size systems, making the observation of chimera states very difficult. To overcome these difficulties some control techniques have been suggested recently. It has been shown that the chimera lifetime as well as its basin of attraction can be effectively controlled by a special type of proportional control relying on the measurement of the global order parameter [52]. On the other hand, Bick and Martens [53] showed that

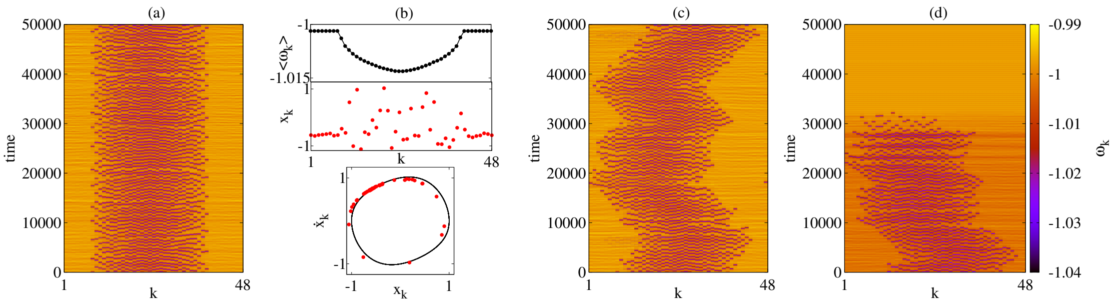

Figure 1: (Color online) Mean phase velocities for a system of $N = 48$ oscillators, and $R = 16$, $\varepsilon = 0.2$, $a = 0.02$; (a) stable chimera state, $K_s = 0.5$, $K_a = 2$; (b) mean phase velocity profile averaged over $\Delta T = 50000$ (top panel), snapshot of variables x_k (middle panel), and snapshot in the $(x_k, \dot{x}_k)$ phase space at time $t = 50000$ (bottom panel, limit cycle of the uncoupled unit shown in black), corresponding to chimera state shown in panel (a); (c) drifting chimera state, $K_s = 0.5$, $K_a = 0$; (d) collapse of chimera state, $K_s = K_a = 0$.

the chimera position can be stabilized by a feedback loop inducing a state-dependent asymmetry of the coupling topology. However, the latter control scheme relies on the evaluation of a finite difference derivative for some local mean field. This operation becomes ill-posed for small system sizes like 20–30 oscillators, therefore one needs to use a refined control in this case.

In this Letter, we propose an efficient control scheme which aims to stabilize chimera states in small networks. Like a tweezer, which helps to hold tiny objects, our control has two levers: the first one prevents the chimera collapse, whereas the second one stabilizes its lateral position. Our control strategy is universal and effective for large as well as for small networks. Although its justification relies on a phase-reduced model, the control works also for oscillators exhibiting both phase and amplitude dynamics.

We expect that our tweezer control can also be useful for theoretical studies. For example, recently Ashwin and Burylko [46] introduced the concept of 'weak chimeras'. This is a dynamical regime with an exact mathematical definition which exhibits many features of 'classical' chimera states. It turns out that systems with tweezer control fill the gap and provide examples of solution families which link classical and 'weak' chimeras.

We consider a system of N identical nonlocally coupled Van der Pol oscillators $x_k \in \mathbb{R}$ given by

$$\begin{aligned} \ddot{x}_k &= (\varepsilon - x_k^2)\dot{x}_k - x_k \\ &+ \frac{1}{R}\sum_{j=1}^{R}[a_-(x_{k-j} - x_k) + b_-(\dot{x}_{k-j} - \dot{x}_k)] \\ &+ \frac{1}{R}\sum_{j=1}^{R}\left[a_+(x_{k+j} - x_k) + b_+(\dot{x}_{k+j} - \dot{x}_k)\right]. \end{aligned} \tag{1}$$

Here, the scalar parameter $\varepsilon > 0$ determines the internal dynamics of all individual elements. For small ε the oscillation of the single element is sinusoidal, while for large ε it is a strongly

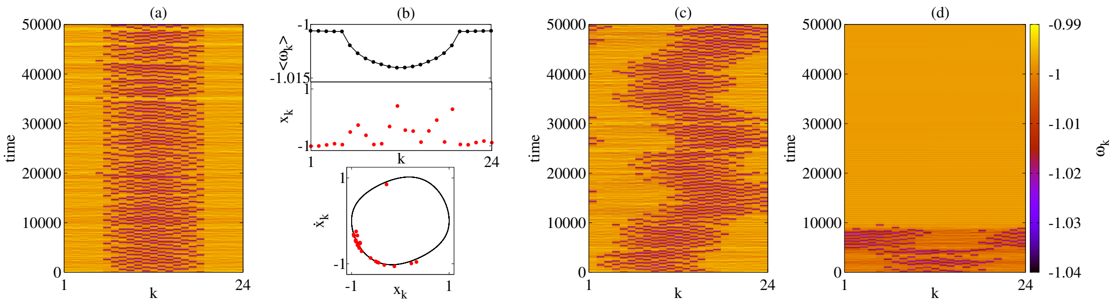

Figure 2: (Color online) Same as Fig. 1 for a system of $N = 24$ oscillators and $R = 8$.

nonlinear relaxation oscillation. Each element is coupled with R left and R right nearest neighbors. We assume that the oscillators are arranged on a ring (i.e., periodic boundary conditions) such that all indices in Eq. (1) are modulo N. The coupling constants in position and velocity to the right and to the left are denoted as a_-, a_+ and b_-, b_+, respectively. If left and right coupling constants are identical, i.e., $a_- = a_+$ and $b_- = b_+$, we call the coupling *symmetric*, otherwise we call it *asymmetric*. For the sake of simplicity we assume

$$a_- = a_+ = a, \qquad b_- = a\sigma_-, \qquad b_+ = a\sigma_+,$$

with rescaled coupling parameters a, σ_- and σ_+. Now we introduce a control scheme for σ_- and σ_+, with the aim to stabilize chimera states of Eq. (1) not only for large but also for small system sizes.

In order to motivate our control scheme, we first consider the case of small ε and a. Then, system (1) can be replaced by a simpler phase oscillator model (see details in the supplemental material). Roughly speaking, we substitute into Eq. (1) the ansatz

$$x_k(t) = \sqrt{\varepsilon} r_k \sin(t + \theta_k), \qquad \dot{x}_k(t) = \sqrt{\varepsilon} r_k \cos(t + \theta_k)$$

and average out the fast time t. In this way, we obtain a reduced system for the slowly varying amplitudes $r_k(t)$ and phases $\theta_k(t)$. If $a \ll \varepsilon$, the amplitudes remain nearly constant $r_k(t) \approx 2$ and we arrive at the Kuramoto-like system

$$\begin{aligned} \dot{\theta}_k &= a - \frac{a}{2R}\sqrt{1+\sigma_-^2}\sum_{j=1}^{R}\sin(\theta_k - \theta_{k-j} + \alpha_-) \\ &\quad - \frac{a}{2R}\sqrt{1+\sigma_+^2}\sum_{j=1}^{R}\sin(\theta_k - \theta_{k+j} + \alpha_+), \end{aligned} \tag{2}$$

where

$$\alpha_\pm = \operatorname{arccot}\sigma_\pm = \frac{\pi}{2} - \arctan\sigma_\pm.$$

Note that for $\sigma_- = \sigma_+$, equation (2) is equivalent to the system considered in [51, 50, 54]. This suggests a range of parameters $\sigma_\pm$ where chimera states should be expected, i.e., $\alpha_\pm \approx \pi/2$.

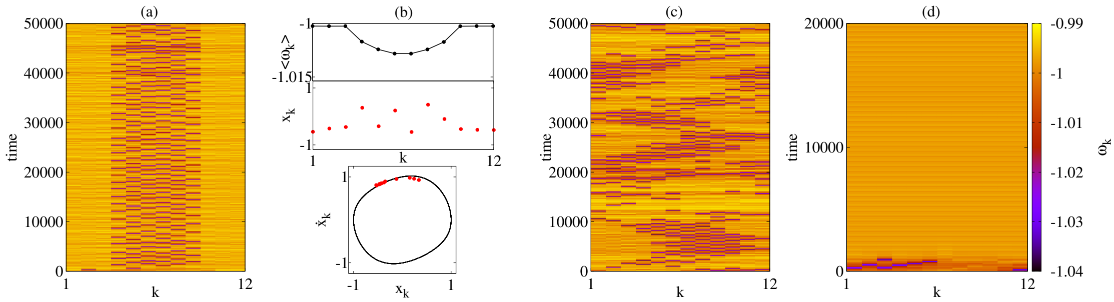

Figure 3: (Color online) Same as Fig. 1 for a system of $N = 12$ oscillators and $R = 4$.

Without loss of generality, we aim to pin the position of the incoherent domain to the center of the array $1, \ldots, N$. To this end, we define two complex order parameters

$$Z_1(t) = \frac{1}{[N/2]} \sum_{k=1}^{[N/2]} e^{i\phi_k(t)}$$

and

$$Z_2(t) = \frac{1}{[N/2]} \sum_{k=1}^{[N/2]} e^{i\phi_{N-k+1}(t)},$$

where $\phi_k(t)$ is the geometric phase of the k-th oscillator calculated by

$$\cos\phi_k(t) = \frac{x_k(t)}{\sqrt{x_k^2(t) + \dot{x}_k^2(t)}}, \ \sin\phi_k(t) = \frac{\dot{x}_k(t)}{\sqrt{x_k^2(t) + \dot{x}_k^2(t)}}.$$

Then we define a 'tweezer' feedback control of the form

$$\sigma_- = -K_s\left(\frac{|Z_1 + Z_2|}{2} - 1\right) + K_a(|Z_1| - |Z_2|),$$
$$\sigma_+ = -K_s\left(\frac{|Z_1 + Z_2|}{2} - 1\right) - K_a(|Z_1| - |Z_2|),$$

where K_s and K_a are gain constants. By construction, the quantity $Z = (Z_1 + Z_2)/2$ coincides with the global order parameter, therefore feedback terms proportional to K_s are analogous to the proportional control described in [52]. They suppress the collapse of small-size chimeras, but do not affect their wandering on the ring. The latter is the purpose of the terms proportional to K_a. Indeed, the difference $|Z_1| - |Z_2|$ measures a relative shift of the chimera's position with respect to the center of the array $1, \ldots, N$. On the other hand, a discrepancy between σ_- and σ_+ corresponds to an asymmetry of the coupling and therefore induces a translational motion of the chimera state. Thus, for non-zero K_a a centered configuration of the chimera state becomes energetically more preferable.

To demonstrate the action of our control scheme, we present numerical results for system (1) with a small number of elements. In all simulations we fix $a = 0.02$. To visualize the temporal

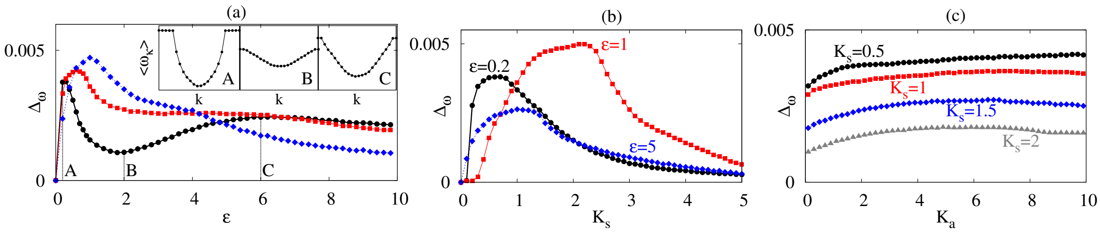

Figure 4: (Color online) Standard deviation of the mean phase velocity profiles for $N = 24$, $R = 8$, $\Delta T = 100000$, $a = 0.02$. (a) Effect of parameter ε for $K_s = 0.5$ (black circles), $K_s = 1$ (red squares), $K_s = 1.5$ (blue diamonds), and $K_a = 2$. Insets show examples of mean phase velocities profiles for (A) $\varepsilon = 0.2$, (B) $\varepsilon = 2$, (C) $\varepsilon = 6$. (b) Role of the proportional control strength K_s: $\varepsilon = 0.2$ (black circles), $\varepsilon = 1$ (red squares), $\varepsilon = 5$ (blue diamonds), and $K_a = 2$. (c) Role of the asymmetry control strength K_a: $K_s = 0.5$ (black circles), $K_s = 1$ (red squares), $K_s = 1.5$ (blue diamonds), $K_s = 2$ (grey triangles), and $\varepsilon = 0.2$.

dynamics we plot the mean phase velocity profiles

$$\omega_k(t) = \frac{1}{T_0}\int_0^{T_0} \dot{\phi}_k(t - t')dt', \quad k = 1, \ldots, N,$$

with averaging window $T_0 = 50$. Fig. 1(a) shows a space-time plot for the system of $N = 48$ coupled Van der Pol oscillators. When both the symmetric K_s and the asymmetric K_a control gains are switched on (Fig. 1(a)), the system develops a stable chimera state without any spatial motion of the coherent and incoherent domains. Fig. 1(b) depicts the mean phase velocity profiles averaged over the global time window $\Delta T = 50000$. It also shows a snapshot of the chimera for variables x_k as well as its projection on the phase plane$(x_k, \dot{x}_k)$. If we switch off the asymmetric part of the control $K_\mathrm{a} = 0$ and keep a positive symmetric gain $K_\mathrm{s} > 0$, we find that the chimera state starts to drift (Fig. 1(c)). Moreover, if we switch off also the symmetric part of the control $K_\mathrm{s} = 0$ we obtain a free chimera state, which collapses after some time (Fig. 1(d)). Note that the shape of the chimera state is almost unaffected by the control, which indicates that it is noninvasive on average, cf. [52].

Fig. 2 and Fig. 3 demonstrate that our control scheme remains effective for smaller networks with $N = 24$ and $N = 12$ oscillators, respectively. In the controlled system we find chimera states with the same shape of coherent and incoherent domains (Fig. 2(b), Fig. 3(b)). On the other hand, we observe an increasing difficulty for chimera states to survive in the uncontrolled system because of extremely fast wandering (Fig. 2(c), Fig. 3(c)) and short lifetimes (Fig. 2(d), Fig. 3(d)).

To analyse the influence of the system parameters on the controlled chimera states, we introduce the standard deviation of the mean phase velocity profile $\Delta_\omega = \sqrt{\frac{1}{N}\sum_{k=1}^{N}(\omega_k - \overline{\omega})^2}$, where $\overline{\omega} = \frac{1}{N}\sum_{k=1}^{N}\omega_k$. Larger values of Δ_ω correspond to a well pronounced arc-like mean phase velocity profile, characterizing chimera states. Fig. 4 depicts the influence of the parameter ε of the individual Van der Pol unit, and control parameters K_s, K_a on the chimera behavior in the network of $N = 24$ oscillators.

Increasing ε results in changing the dynamics of the individual elements from regular sinusoidal oscillations to relaxation oscillations. Fig. 4(a) shows that for small values of ε, the chimera states are well pronounced (inset A, black circles denoting $K_s = 0.5$), for intermediate values the difference between maximum and minimum phase velocity is very small (inset B), and increases again for even larger ε (inset C). When symmetric control becomes stronger ($K_s = 1$ or 1.5, shown by red squares and blue diamonds, respectively) for intermediate values of ε chimera states are better pronounced, although large nonlinearity results in a decrease of Δ_ω. The example shown in Fig. 2(a) corresponds to the point A here.

Fig. 4(b) depicts the influence of symmetric control K_s for three values of $\varepsilon = 0.2, 1, 5$. There exists a range of parameter K_s, where the symmetric control is most effective: further increase of K_s will result in the disappearance of the chimera state. In our examples, we have chosen $K_s = 0.5$ for $\varepsilon = 0.5$, close to the maximum of the black circles. Fig. 4(c) demonstrates the effect of changing the asymmetric control gain K_a for several fixed values of K_s. The standard deviation Δ_ω increases for small values of the control strength, and then stays approximately at the same value. Therefore, in our examples we choose $K_a = 2$. An appropriate choice of control strengths will lead to well pronounced chimera states, making them easily detectable in small networks.

To conclude, we have proposed an effective control scheme, which allows us to stabilize chimera states in large and in small-size networks. Our control is an interplay of two instruments, the symmetric control term suppresses the chimera collapse, and the asymmetric control effectively stabilizes the chimera's spatial position. We have demonstrated the effect of the control scheme in systems of $48, 24$, and 12 nonlocally coupled Van der Pol oscillators, and investigated the role of system parameters and control strengths for the most efficient stabilization of chimera states. Our proposed approach can be useful for the experimental realizations of chimera states, where usually small networks are studied, and it is very difficult to avoid chimera collapse and spatial drift.

Supplemental Material: Phase reduction

Let us denote

$$x_k(t) = \sqrt{\varepsilon} u_k(t), \qquad \dot{x}_k(t) = \sqrt{\varepsilon} v_k(t),$$

then the original system of N coupled Van der Pol oscillators Eq. (1) can be rewritten as a $2N$-dimensional dynamical system of the form

$$
\begin{aligned}
\dot{u}_k &= v_k, \\
\dot{v}_k &= \varepsilon(1-u_k^2)v_k - u_k \\
&+ \frac{a}{R}\sum_{j=1}^{R}\left[(u_{k-j}-u_k)+\sigma_-(\dot{u}_{k-j}-\dot{u}_k)\right] \\
&+ \frac{a}{R}\sum_{j=1}^{R}\left[(u_{k+j}-u_k)+\sigma_+(\dot{u}_{k+j}-\dot{u}_k)\right]. \qquad (0.1)
\end{aligned}
$$

We perform a phase reduction in order to determine a parameter set appropriate for observation of chimera states.

Assuming that ε and a are both small, we can apply the averaging procedure to Eqs. (0.1). To this end we substitute the ansatz

$$
u_k = r_k \sin(t+\theta_k), \qquad v_k = r_k \cos(t+\theta_k)
$$

into system (0.1) and average out the fast time t, assuming that $r_k(t)$ and $\theta_k(t)$ are slowly varying functions. As result we obtain the system

$$
\begin{aligned}
\dot{r}_k &= \frac{\varepsilon}{8} r_k(4-r_k^2) \\
&+ \frac{a}{2R}\sum_{j=1}^{R}\left[r_{k-j}\sin(\theta_{k-j}-\theta_k)+\ \sigma_-(r_{k-j}\cos(\theta_{k-j}-\theta_k)-r_k)\right] \\
&+ \frac{a}{2R}\sum_{j=1}^{R}\left[r_{k+j}\sin(\theta_{k+j}-\theta_k)+\ \sigma_+(r_{k+j}\cos(\theta_{k+j}-\theta_k)-r_k)\right],
\end{aligned}
$$

$$
\begin{aligned}
r_k\dot{\theta}_k &= \frac{a}{2R}\sum_{j=1}^{R}\left[-(r_{k-j}\cos(\theta_{k-j}-\theta_k)-r_k)-\ \sigma_- r_{k-j}\sin(\theta_k-\theta_{k-j})\right] \\
&+ \frac{a}{2R}\sum_{j=1}^{R}\left[-(r_{k+j}\cos(\theta_{k+j}-\theta_k)-r_k)-\ \sigma_+ r_{k+j}\sin(\theta_k-\theta_{k+j})\right],
\end{aligned}
$$

which can also be rewritten as follows

$$
\begin{aligned}
\dot{r}_k &= \frac{\varepsilon}{8} r_k\left(\left(4-\frac{4a}{\varepsilon}(\sigma_-+\sigma_+)\right)-r_k^2\right) \\
&+ \frac{a}{2R}\sqrt{1+\sigma_-^2}\sum_{j=1}^{R} r_{k-j}\cos(\theta_k-\theta_{k-j}+\alpha_-) \\
&+ \frac{a}{2R}\sqrt{1+\sigma_+^2}\sum_{j=1}^{R} r_{k+j}\cos(\theta_k-\theta_{k+j}+\alpha_+), \qquad (0.2)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\theta}_k &= a - \frac{a}{2R}\sqrt{1+\sigma_-^2}\sum_{j=1}^{R}\frac{r_{k-j}}{r_k}\sin(\theta_k - \theta_{k-j} + \alpha_-) \\
&\quad - \frac{a}{2R}\sqrt{1+\sigma_+^2}\sum_{j=1}^{R}\frac{r_{k+j}}{r_k}\sin(\theta_k - \theta_{k+j} + \alpha_+),
\end{aligned} \tag{0.3}
$$

where

$$
\alpha_\pm = \operatorname{arccot}\sigma_\pm = \frac{\pi}{2} - \arctan\sigma_\pm.
$$

If $0 < a \ll \varepsilon$, from Eq. (0.2) we find that $r_k \approx 2$ is a stable fixed point. Substituting this into the second equation we obtain a Kuramoto-like system

$$
\begin{aligned}
\dot{\theta}_k &= a - \frac{a}{2R}\sqrt{1+\sigma_-^2}\sum_{j=1}^{R}\sin(\theta_k - \theta_{k-j} + \alpha_-) \\
&\quad - \frac{a}{2R}\sqrt{1+\sigma_+^2}\sum_{j=1}^{R}\sin(\theta_k - \theta_{k+j} + \alpha_+).
\end{aligned}
$$

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