

Review Article

Canonical Sets of Best L_1 -Approximation

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In mathematics, the term *approximation* usually means either interpolation on a point set or approximation with respect to a given distance. There is a concept, which joins the two approaches together, and this is the concept of characterization of the best approximants via interpolation. It turns out that for some large classes of functions the best approximants with respect to a certain distance can be constructed by interpolation on a point set that does not depend on the choice of the function to be approximated. Such point sets are called *canonical sets of best approximation*. The present paper summarizes results on canonical sets of best L_1 -approximation with emphasis on multivariate interpolation and best L_1 -approximation by blending functions. The best L_1 -approximants are characterized as transfinite interpolants on canonical sets. The notion of a Haar-Chebyshev system in the multivariate case is discussed also. In this context, it is shown that some multivariate interpolation spaces share properties of univariate Haar-Chebyshev systems. We study also the problem of best one-sided multivariate L_1 -approximation by sums of univariate functions. Explicit constructions of best one-sided L_1 -approximants give rise to well-known and new inequalities.

1. Canonical Interpolation Sets of Best L_1 -Approximation

We start with the notion of a canonical set of best L_1 -approximation. Let μ be a positive Borel measure defined on the compact set $K \subset \mathbf{R}^d$ such that the quantity $\|f\|_1 := \int_K |f| d\mu$ represents a norm in the linear space $C(K)$ of functions that are continuous on K . Let U be a linear subspace of $C(K)$.

1.1. The Problem of Best $L_1(\mu)$ -Approximation

Given a function $f \in C(K)$, then we have the following.

(A) Best $L_1(\mu)$ -approximation of f by elements of U : find a function $u_f \in U$ such that

$$\|f - u_f\|_1 \leq \|f - u\|_1 \quad \forall u \in U. \quad (1.1)$$

(B) Best one-sided from above $L_1(\mu)$ -approximation by elements of U : find a function $u_f^* \in U$, $u_f^* \geq f$ on K , such that

$$\|f - u_f^*\|_1 \leq \|f - u\|_1 \quad \forall u \in U, u \geq f \text{ on } K. \quad (1.2)$$

(C) Best one-sided from below $L_1(\mu)$ -approximation by elements of U : find a function $u_{*f} \in U$, $u_{*f} \leq f$ on K , such that

$$\|f - u_{*f}\|_1 \leq \|f - u\|_1 \quad \forall u \in U, u \leq f \text{ on } K. \quad (1.3)$$

Any solution u_f of the approximation problem (A) is called a best $L_1(\mu)$ -approximant to f from U . Any solution of the approximation problem (B) is called a best one-sided from above $L_1(\mu)$ -approximant to f from U ; respectively from below for the approximation problem (C). When μ is the usual Lebesgue measure, we use the notation L_1 -approximation, respectively, L_1 -approximant.

For qualitative results on best $L_1(\mu)$ -approximation from finite-dimensional subspaces see [1].

1.2. Lagrange Interpolation Problem

Given a function $f \in C(K)$. We call a subset $X \subset K$ *unisolvant* for U if the interpolation problem

$$u(x) = f(x), \quad x \in X \quad (1.4)$$

possesses a unique solution $u \in U$ for every $f \in C(K)$.

1.3. Canonical Sets of Best L_1 -Approximation

Let $\mathcal{C} \subset C(K)$ be a class of functions. We say that a set $X \subset K$, which is unisolvant for U , is a *canonical set of best $L_1(\mu)$ -approximation* to \mathcal{C} , if for all $f \in \mathcal{C}$ the solution of the interpolation problem (1.4) is a best $L_1(\mu)$ -approximant to f from U . Analogously, a unisolvant set $X^* \subset K$ for U (resp., $X_* \subset K$) is called a *canonical set of best one-sided from above* (resp., from below) $L_1(\mu)$ -approximation to \mathcal{C} , if for all $f \in \mathcal{C}$ the solution of the interpolation problem (1.4) with X replaced by X^* (resp., X_*) is a best one-sided from above (resp., from below) $L_1(\mu)$ -approximant to f from U . In the case of one-sided $L_1(\mu)$ -approximation the interpolation

problem (1.4) is considered in a broader *Lagrange-Hermite interpolation sense*: for all points $x \in X$ which are interior for K we require matching not only the values of u and f but also the values of their first (partial) derivatives.

2. Canonical Sets of Univariate Best L_1 -Approximation

For details on the results in the present section see [1–6]. In the univariate case, when the set K is an interval $[a, b]$ and the subspace U is *finite-dimensional*, the problem of existence and characterization of a canonical set is well studied and is closely related to the notion of a *Haar-Chebyshev system* (see [1] for details).

2.1. Haar-Chebyshev System of Order n

A set of n functions $\{u_1, u_2, \dots, u_n\} \subset C[a, b]$ is a Haar-Chebyshev system (T -system) of order n on the interval $[a, b]$ if each nontrivial linear combination of u_1, u_2, \dots, u_n has at most $n - 1$ zeros in $[a, b]$. In other words, a set of n functions $\{u_1, u_2, \dots, u_n\}$ is a Haar-Chebyshev system of order n on the interval $[a, b]$ if it is linearly independent with respect to an arbitrary chosen set of n distinct points in $[a, b]$. We say that an n -dimensional subspace U of $C[a, b]$ is a *Haar-Chebyshev space* (T -space) of order n on $[a, b]$ if every $u \in U$, $u \neq 0$, has no more than $n - 1$ distinct zeros in $[a, b]$. Equivalently, U is a T -space of order n on $[a, b]$, if for every basis u_1, u_2, \dots, u_n of U there exists an $\varepsilon \in \{-1, 1\}$ such that $\varepsilon \det \{u_i(x_j)\}_{i,j=1}^n > 0$ for all $a \leq x_1 < x_2 < \dots < x_n \leq b$. It is also often said that the basis functions u_1, u_2, \dots, u_n constitute a T -space of order n on $[a, b]$.

2.2. Characterization of Best $L_1(\mu)$ -Approximation by Canonical Interpolation Sets

Following [1], throughout this section we suppose that μ is a finite positive nonatomic measure. The problem of best $L_1(\mu)$ -approximation by a Haar-Chebyshev space has an elegant solution via interpolation on a canonical set. Let $U \subset C[a, b]$ be a Haar-Chebyshev space of order n and let u_1, u_2, \dots, u_n be a basis of U . Consider the convex cone $\mathcal{C}^+(U)$ defined as

$$\mathcal{C}^+(U) := \left\{ u_{n+1} : \det \{u_i(x_j)\}_{i,j=1}^{n+1} \geq 0 \right\} \quad (2.1)$$

for all $a \leq x_1 < x_2 < \dots < x_n < x_{n+1} \leq b$.

Theorem 2.1. *Let $U \subset C[a, b]$ be a Haar-Chebyshev space of order n on $[a, b]$. Then the following holds true.*

- (a) *Uniqueness of the Canonical Set. There is a unique set of points $a = x_0 < x_1 < \dots < x_{n+1} = b$ such that*

$$\sum_{j=1}^{n+1} (-1)^j \int_{x_{j-1}}^{x_j} u \, d\mu = 0, \quad \forall u \in U. \quad (2.2)$$

(b) *Best $L_1(\mu)$ -Approximation via Interpolation.* Let $f \in C^+(U)$. Then the unique solution $u_f \in U$ of the interpolation problem

$$u_f(x_j) = f(x_j), \quad j = 1, 2, \dots, n \quad (2.3)$$

is the unique best $L_1(\mu)$ -approximant to f from U .

Remark 2.2. Following Theorem 2.1, the set $\{x_k\}_{k=1}^n$ is a canonical set of best $L_1(\mu)$ -approximation to the convex cone $C^+(U)$ of functions from the T -space U . In many cases the approximating subspace U can be a kernel of a certain differential operator \mathfrak{D} , that is, $U = \text{Ker } \mathfrak{D}$ and the convex cone $C_{\mathfrak{D}}^+(U)$ of functions to be approximated can be defined by

$$C_{\mathfrak{D}}^+(U) := \{f \in C[a, b] : \mathfrak{D}f(x) \geq 0 \text{ for } x \in [a, b]\}. \quad (2.4)$$

Remark 2.3. Note that if a canonical set of best L_1 -approximation exists for a functional set, then the nonlinear problem of best L_1 -approximation becomes a linear interpolation problem in this functional set.

2.3. Best L_1 -Approximation by Algebraic Polynomials

Denote by π_{n-1} the linear space of polynomials of degree $\leq n-1$. Taking into account that the polynomial basis $u_k = x^{k-1}$, $k = 1, \dots, n$ of π_{n-1} is a T -system of order n on any interval $[a, b]$, the best polynomial L_1 -approximant to a given function f from π_{n-1} can be characterized as a polynomial interpolant to f with respect to a canonical set if f belongs to an appropriately chosen convex cone of functions. In this direction we formulate a result due to S. Bernstein [2, 3].

Theorem 2.4. *Let $f \in C^n[-1, 1]$ and $f^{(n)}(x) \geq 0$ for $x \in [-1, 1]$. Then, the polynomial p_f of degree $\leq n-1$ is the unique polynomial of best L_1 -approximation to f from π_{n-1} on $[-1, 1]$ if and only if*

$$p_f(x_k) = f(x_k), \quad 1 \leq k \leq n, \quad (2.5)$$

where the interpolation nodes $x_k = \cos[k\pi/(n+1)]$, $k = 1, \dots, n$ are the zeros of the n -degree Chebyshev polynomial of second kind

$$U_n(x) = \frac{\sin[(n+1)\arccos(x)]}{\sqrt{1-x^2}}. \quad (2.6)$$

Remark 2.5. By Theorem 2.4, the interpolation points x_k , $k = 1, \dots, n$ form a canonical set of best L_1 -approximation from π_{n-1} to the convex cone

$$\left\{ f \in C^n[-1, 1] : f^{(n)}(x) \geq 0, \quad x \in [-1, 1] \right\}. \quad (2.7)$$

Note that a canonical set may change with respect to the choice of the approximating space. For example, following Theorem 2.4, the canonical set of best polynomial L_1 -approximation depends on the degree of the polynomial approximant.

Corollary 2.6. *The n -degree polynomial*

$$\frac{U_n(x)}{2^n} = \frac{\sin[(n+1)\arccos(x)]}{2^n\sqrt{1-x^2}} \quad (2.8)$$

is the unique monic (with leading coefficient 1) polynomial of degree n which has a minimal L_1 -norm (minimal L_1 -deviation) on $[-1, 1]$.

We formulate a result on best L_1 -approximation by algebraic polynomials according to A. Markov (see [2, 3] for details) that is based on the notion of a Haar-Chebyshev system.

Theorem 2.7. *Let $f \in C[-1, 1]$ be such that the set*

$$\{1, x, x^2, \dots, x^{n-1}, f\} \quad (2.9)$$

is a Haar-Chebyshev system of order $n+1$ on $[-1, 1]$. Then the unique polynomial Lagrange interpolant p_f to f from π_{n-1} with respect to the zeros x_k , $k = 1, \dots, n$ of the n -degree Chebyshev polynomial of second kind is the unique best L_1 -approximant to f from π_{n-1} .

Remark 2.8. In the particular case $f \in C^{(n)}[-1, 1]$ and $f^{(n)}(x) > 0$, $x \in [-1, 1]$ of Theorem 2.4, the set of functions $\{1, x, x^2, \dots, x^{n-1}, f\}$ is a Haar-Chebyshev system of order $n+1$ in $[-1, 1]$. Hence in this particular case, Theorem 2.4 is a corollary by Theorem 2.7. The notion of Haar-Chebyshev system implies that $f-p$ cannot have more than n zeros on $[-1, 1]$, where $p \in \pi_{n-1}$.

2.4. Best L_1 -Approximation by Trigonometric Polynomials

Let T_n be the linear space of trigonometric polynomials of degree $\leq n$. Consider the normed linear space $L_1[0, 2\pi]$ of 2π -periodic functions whose absolute value has a finite integral on $[0, 2\pi]$ and equipped with the L_1 -norm

$$\|f\|_1 := \int_0^{2\pi} |f(\theta)| d\theta. \quad (2.10)$$

The linear space T_n is a finite dimensional subspace of $L_1[0, 2\pi]$. For a given $f \in L_1[0, 2\pi]$, there exists a best L_1 -approximant $t \in T_n$ (see [2] for details). Note that T_n is a Haar-Chebyshev space of order $2n+1$ on $[0, 2\pi]$. Let $\tilde{C}^{2n+1}[0, 2\pi]$ denote the linear space of 2π -periodic functions having continuous derivatives of order $2n+1$ on $[0, 2\pi]$. The next theorem [7] is a canonical set characterization of the best trigonometric L_1 -approximants from T_n to functions from the convex cone

$$\tilde{C}_+^{2n+1} := \left\{ f : f \in \tilde{C}^{2n+1}[0, 2\pi], \mathfrak{D}^{(2n+1)} f(\theta) \geq 0 \text{ for } \theta \in [0, 2\pi) \right\}, \quad (2.11)$$

where the differential operator $\mathfrak{D}^{(2n+1)}$ is defined as

$$\mathfrak{D}^{(2n+1)} := D \prod_{k=1}^n (D^2 + k^2), \quad D = \frac{d}{dx}. \quad (2.12)$$

Theorem 2.9. *Let $f \in \tilde{C}_+^{2n+1}$. Then, the unique Lagrange trigonometric interpolant t_f from T_n to f at the interpolation nodes $k\pi/(n+1)$, $k = 1, \dots, 2n+1$ is the unique best L_1 -approximant to f from T_n .*

2.5. Characterization of Best One-Sided $L_1(\mu)$ -Approximation

Best one-sided $L_1(\mu)$ -approximation is related to the principal representations of the measure μ , that is, to the so-called *quadrature formulae of Gaussian, Lobatto, and Radau type* (for details see [1, 3–6]).

2.5.1. Quadrature Formulae of Gaussian, Lobatto, and Radau Type

Let U be a Haar-Chebyshev space of order n .

- (a) If $n = 2m$, then there exist unique sets of points $a < x_1^G < x_2^G < \dots < x_m^G < b$ and $a = x_0^L < x_1^L < \dots < x_m^L = b$, and unique sets of positive numbers $\{a_j^G\}_{j=1}^m$ and $\{a_j^L\}_{j=0}^m$ such that the quadrature formulae

$$\int_a^b f d\mu \approx Q^G[f] := \sum_{j=1}^m a_j^G f(x_j^G) \quad (\text{Gaussian type quadrature formula}), \quad (2.13)$$

$$\int_a^b f d\mu \approx Q^L[f] := \sum_{j=0}^m a_j^L f(x_j^L) \quad (\text{Lobatto type quadrature formula})$$

are exact for all $u \in U$.

- (b) If $n = 2m + 1$, then there exist unique sets of points $a = x_0^{R-} < x_1^{R-} < \dots < x_m^{R-} < b$ and $a < x_0^{R+} < x_1^{R+} < \dots < x_m^{R+} = b$, and unique sets of positive numbers $\{a_j^{R-}\}_{j=0}^m$ and $\{a_j^{R+}\}_{j=0}^m$ such that the quadrature formulae

$$\int_a^b f d\mu \approx Q^{R-}[f] := \sum_{j=0}^m a_j^{R-} f(x_j^{R-}) \quad (\text{left Radau type quadrature formula}), \quad (2.14)$$

$$\int_a^b f d\mu \approx Q^{R+}[f] := \sum_{j=0}^m a_j^{R+} f(x_j^{R+}) \quad (\text{right Radau type quadrature formula})$$

are exact for all $u \in U$.

Let u_1, u_2, \dots, u_n be a basis of U such that $\det \{u_i(x_j)\}_{i,j=1}^n > 0$ for $a \leq x_1 < x_2 < \dots < x_n \leq b$. Hence, the set of functions $\{u_1, u_2, \dots, u_n\}$ is a Haar-Chebyshev system (T -system)

of order n on $[a, b]$ or in other words U is an n -order T -space. The following theorem holds true.

Theorem 2.10. *Let $U \subset C[a, b]$ be a T -space of order n . Then, for every $f \in C^+(U)$ there exist elements of best one-sided $L_1(\mu)$ -approximation from below $u_{*f} \in U$ and from above $u_f^* \in U$. Moreover, the best one-sided approximants $u_{*f} \in U, u_f^* \in U$ to f and the function f satisfy*

$$(a) \quad Q^G[u_{*f}] = Q^G[f] \text{ and } Q^L[u_f^*] = Q^L[f], \text{ if } n = 2m, \text{ and}$$

$$(b) \quad Q^{R-}[u_{*f}] = Q^{R-}[f] \text{ and } Q^{R+}[u_f^*] = Q^{R+}[f], \text{ if } n = 2m + 1.$$

In order to guarantee the uniqueness of the best one-sided $L_1(\mu)$ -approximant to a given function we need the following restriction on U .

2.5.2. Extended Haar-Chebyshev System of Multiplicity 2

A set of n functions $\{u_1, u_2, \dots, u_n\} \subset C^1[a, b]$ is an n -order extended Haar-Chebyshev system of multiplicity 2 (n -order ET-system of multiplicity 2) on the interval $[a, b]$ if each nontrivial linear combination $u \neq 0$ of u_1, u_2, \dots, u_n has at most $n - 1$ zeros in $[a, b]$, provided that the common zeros of u and u' are counted twice (counting multiplicities 2). In other words, U is an n -order extended Haar-Chebyshev space of multiplicity 2 (n -order ET-space of multiplicity 2) on the interval $[a, b]$, if $U \subset C^1[a, b]$, $\dim(U) = n$ and each nontrivial $u \in U, u \neq 0$ has at most $n - 1$ zeros in $[a, b]$, provided that the common zeros of u and u' are counted twice (counting multiplicities 2).

2.5.3. Construction of Best One-Sided $L_1(\mu)$ -Approximants via Interpolation on Canonical Sets

For details on the results presented here, see [1, 3, 4] and the references given there.

Theorem 2.11. *Let $U \subset C^1[a, b]$ be an n -order ET-space of multiplicity 2 on the interval $[a, b]$. Then, for every $f \in C^+(U) \cap C^1[a, b]$ one has the following.*

(a) *If $n = 2m$, then the unique solutions u_{*f} and u_f^* of the Lagrange-Hermite interpolation problems*

$$\begin{aligned} u_{*f}(x_j^G) &= f(x_j^G), \quad u'_{*f}(x_j^G) = f'(x_j^G), \quad j = 1, 2, \dots, m, \quad u_{*f} \in U, \\ u_f^*(x_j^L) &= f(x_j^L), \quad (u_f^*)'(x_j^L) = f'(x_j^L), \quad j = 1, 2, \dots, m-1 \\ u_f^*(x_0^L) &= f(x_0^L), \quad u_f^*(x_m^L) = f(x_m^L), \quad u_f^* \in U, \end{aligned} \quad (2.15)$$

are the unique best one-sided from below, respectively, from above $L_1(\mu)$ -approximants to f from U .

(b) If $n = 2m + 1$, then the unique solutions u_{*f} and u_f^* of the Lagrange-Hermite interpolation problems

$$\begin{aligned}
 u_{*f}(x_j^{R-}) &= f(x_j^{R-}), & u'_{*f}(x_j^{R-}) &= f'(x_j^{R-}), & j &= 1, 2, \dots, m, \\
 u_{*f}(x_0^{R-}) &= f(x_0^{R-}), & u_{*f} &\in \mathcal{U}, \\
 u_f^*(x_j^{R+}) &= f(x_j^{R+}), & (u_f^*)'(x_j^{R+}) &= f'(x_j^{R+}), & j &= 0, 1, \dots, m-1, \\
 u_f^*(x_m^{R+}) &= f(x_m^{R+}), & u_f^* &\in \mathcal{U},
 \end{aligned} \tag{2.16}$$

are the unique best one-sided from below, respectively, from above $L_1(\mu)$ -approximants to f from \mathcal{U} .

Best one-sided L_1 -approximant to a given function exists and is unique under some conditions. Next theorem (for details and similar results see [3, 4, 8]) is an example in this direction.

Theorem 2.12. *Let f be a differentiable, 2π -periodic function and let T_n denote the linear space of all trigonometric polynomials of degree at most n . Then, the best one-sided L_1 -approximant from above, respectively, from below to f from T_n exists and is unique.*

Remark 2.13. The set of trigonometric polynomials $t_*(\theta) = \alpha \cos(n\theta)$, where $-1 \leq \alpha \leq 1$ are best one-sided from below L_1 -approximants to the continuous 2π -periodic function $f(\theta) = |\cos(n\theta)|$ from T_n . Hence, continuity is not enough for uniqueness of the best one-sided L_1 -approximant to be claimed.

The following theorem (see [7] for details) gives a characterization of the best one-sided L_1 -approximants via canonical sets. It can be considered as a refinement of Theorem 2.12 for functions from the convex cone $\tilde{\mathcal{C}}_+^{2n+1}$.

Theorem 2.14. *Let $f \in \tilde{\mathcal{C}}_+^{2n+1}$. Let $t_{*f} \in T_n$ be the unique interpolant from T_n to f at the interpolation nodes $2k\pi/(n+1)$, $k = 1, 2, \dots, n$ with multiplicities 2 and $t_{*f}(0) = f(0)$. The interpolant t_{*f} to f from T_n is the unique best one-sided from below L_1 -approximant to f from T_n .*

Let $t_f^ \in T_n$ be the unique interpolant from T_n to f at the interpolation nodes $2k\pi/(n+1)$, $k = 1, 2, \dots, n$ with multiplicities 2 and $t_f^*(2\pi) = f(2\pi^-)$. The interpolant t_f^* to f from T_n is the unique best one-sided from above L_1 -approximant to f from T_n .*

In order to proceed with the multivariate case we remark that the above-stated univariate results are constituted by the following prerequisites.

- (1) The domain K where the functions are defined is an interval.
- (2) The approximating space \mathcal{U} in most of the cases is a kernel of a linear ordinary differential operator \mathfrak{D} , that is, $\mathcal{U} = \text{Ker } \mathfrak{D}$.
- (3) The set on which certain interpolation problem is unisolvent for \mathcal{U} consists of a finite number of points.

(4) The convex cone $C_{\mathfrak{D}}^+(U)$ of functions to be approximated is defined by

$$C_{\mathfrak{D}}^+(U) := \{f : \mathfrak{D}f \text{ exists and } \mathfrak{D}f \geq 0 \text{ on } [a, b]\}. \quad (2.17)$$

(5) The best (one-sided) L_1 -approximants are characterized as Lagrange-Hermite interpolants on canonical sets.

Conditions (3) – (5) are due to the fact that U is a T -space or ET -space of multiplicity 2. It is known that there are no point-wise Haar-Chebyshev systems in the multivariate case [9, 10] provided that the interpolation set is of finite number of points and the interpolating space U is finite-dimensional. This fact is not surprising taking into account that the kernels of linear partial differential operators are *infinite-dimensional linear spaces*. Therefore, the natural interpolation sets in the multivariate case $K \subset \mathbf{R}^d$ should be nondenumerable point sets, and more precisely, $(d-1)$ -dimensional manifolds. Hence, an appropriate transfinite interpolation on lower dimensional manifolds can be a basis for a canonical set characterization of the best L_1 -approximants in the multivariate case. However, unlike the finite number of points in an interval $[a, b]$ which are topologically equivalent, there is a countless variety of possible interpolation sets (interpolation grids) in the multivariate case and one cannot expect that there would be a simple characterization corresponding to that in the univariate case. Instead, one should consider *pairs* (U, X) consisting of an approximation space U and a set $X \subset K$ such that X is unisolvent for U . Such sets X are candidates for canonical sets in the multivariate L_1 -approximation. There is no consistent general treatment of the canonical sets of best approximation in \mathbf{R}^d and each known result is a solution of a particular problem. In the next sections we will discuss some of these results and we will focus on multivariate interpolation and best L_1 -approximation by blending functions.

3. Algebraic Blending Functions

The linear space π_{m-1} of univariate polynomials of degree $\leq m-1$ can be defined as the general solution of the homogeneous, m -order, linear differential equation $p^{(m)}(x) = 0$, $x \in [-1, 1]$. Hence,

$$\pi_{m-1} = \left\{ p \in C^m(I) : \frac{d^m p}{dx^m} = 0, I := [-1, 1] \right\}, \quad (3.1)$$

where by $C^m(I)$ we denote the linear space of functions having continuous m -order derivative on $[-1, 1]$. The linear space π_{m-1} is of finite dimension m .

Let $I^2 = [-1, 1]^2$ and

$$C^{m,n} := \left\{ f(x, y) : D^{i,j} f(x, y) \in C(I^2), 0 \leq i \leq m, 0 \leq j \leq n \right\}, \quad (3.2)$$

where $D^{i,j} f(x, y) := (\partial^{i+j} / \partial x^i \partial y^j) f(x, y)$ is the partial derivative of f of order (i, j) . $C^{m,n}$ denotes the linear space of all real valued functions having continuous partial derivatives up to order (m, n) on I^2 .

The classical univariate polynomials have a natural multivariate extension by the so-called *algebraic blending functions*. Let m, n be nonnegative integer such that $m + n \geq 1$. The space $B^{m,n}$ of algebraic blending functions of order (m, n) on the unit square I^2 is defined as

$$B^{m,n} := \left\{ h \in C^{m,n} : D^{m,n}h(x, y) = 0, (x, y) \in I^2 \right\}. \quad (3.3)$$

In other words, the linear space $B^{m,n}$ is the general solution of the homogeneous, (m, n) -order, linear partial differential equation $D^{m,n}h(x, y) = 0$ on I^2 or, saying it differently, the linear space $B^{m,n}$ is the kernel of the linear partial differential operator $D^{m,n}$.

Each blending function $h \in B^{m,n}$ of order (m, n) can be represented in the form

$$h(x, y) = \sum_{k=0}^{m-1} a_k(y)x^k + \sum_{l=0}^{n-1} b_l(x)y^l, \quad a_k(y) \in C^n(I), \quad b_l(x) \in C^m(I) \quad (3.4)$$

and obviously, each such function belongs to $B^{m,n}$. Hence,

$$B^{m,n} = \left\{ h : h(x, y) = \sum_{k=0}^{m-1} a_k(y)x^k + \sum_{l=0}^{n-1} b_l(x)y^l, \quad a_k \in C^n(I), \quad b_l \in C^m(I) \right\}. \quad (3.5)$$

Note that the above representation of a function from $B^{m,n}$ is not unique and the linear space $B^{m,n}$ is of infinite dimension.

4. Transfinite Interpolation by Algebraic Blending Functions

In two-dimensional case, transfinite interpolation (beyond the finite interpolation) is to construct a simple interpolation function (blending interpolation function for example) over a planar domain in such a way that it matches (interpolates) a given function and its partial derivatives on curves. Transfinite interpolation is an approximate recovery of functions via interpolation with variety of applications, for example, in geometric modeling and finite element methods. The notion can be extended in a natural way for higher dimensions. The Dirichlet problem is an example of a transfinite interpolation scheme in the linear space of harmonic functions.

In contrast to the transfinite interpolation schemes, the classical interpolation schemes are restricted to a finite or denumerable number of interpolation points. For example, approximate recovery of a univariate function can be obtained by a univariate Lagrange interpolation polynomial: let $f \in C[-1, 1]$ and x_1, x_2, \dots, x_m be m distinct points in $[-1, 1]$. Then, there exists a unique polynomial interpolant $p_f \in \pi_{m-1}$ to f of degree $\leq m - 1$ such that

$$p_f(x_k) = f(x_k), \quad k = 1, 2, \dots, m \quad (4.1)$$

with an error representation formula

$$f(x) - p_f(x) = \frac{f^{(m)}(\xi)}{m!} (x - x_1) \cdots (x - x_m), \quad x \in [-1, 1], \quad \xi \in (-1, 1). \quad (4.2)$$

Existence and uniqueness of univariate Lagrange interpolants are closely connected with the notion of Haar-Chebyshev system in the univariate case.

4.1. Haar-Chebyshev Systems in the Multivariate Case

Let us remind that a set of m univariate functions $\{u_1, u_2, \dots, u_m\} \subset C[-1, 1]$ is a Haar-Chebyshev system (T -system) of order m in $[-1, 1]$ if each nontrivial linear combination of u_1, u_2, \dots, u_m has at most $m - 1$ zeros in $[-1, 1]$. In other words, a set of m functions is a Haar-Chebyshev system of order m in $[-1, 1]$ if it is linearly independent with respect to an arbitrary chosen set of m distinct points in $[-1, 1]$. The concept of a Haar-Chebyshev system (T -system) is based on counting the zeros and it is essentially restricted to the univariate case. It is known [9, 10] that there is no universal Haar-Chebyshev system on any set in \mathbf{R}^d that contains an interior point, in particular, on I^2 . Hence, the point-wise Lagrange univariate interpolation scheme cannot be extended to the point-wise interpolation scheme in the multivariate case.

However, we can extend the notion of a Haar-Chebyshev system in the multivariate case with respect to transfinite interpolation by blending functions on grids (curves) with a prescribed geometry. Being defined as the kernel of the differential operator $D^{m,n}$, the linear space $B^{m,n}$ of algebraic blending functions of order (m, n) shares interpolation properties of the algebraic polynomials. In particular, Lagrange interpolation by algebraic polynomials has a multivariate extension to transfinite Lagrange interpolation by blending functions.

4.2. Transfinite Interpolation Grid

Let

$$\omega_{\mathbf{x}_m, \mathbf{y}_n}(x, y) := \prod_{s=1}^m (x - x_s) \prod_{k=1}^n (y - y_k). \quad (4.3)$$

We define an (m, n) interpolation grid $\mathbf{G}_{m,n}$ associated with the 2 point sets of distinct in each set points $\mathbf{x}_m := \{x_s, 1 \leq s \leq m\}$ and $\mathbf{y}_n := \{y_k, 1 \leq k \leq n\}$:

$$\mathbf{G}_{m,n} := \left\{ (x, y) \in I^2 : \omega_{\mathbf{x}_m, \mathbf{y}_n}(x, y) = 0 \right\}. \quad (4.4)$$

The interpolation grid $\mathbf{G}_{m,n}$ is a set of m vertical and n horizontal line-segments in I^2 .

4.3. Transfinite Lagrange Interpolation by Algebraic Blending Functions

Let $f \in C^{m,n}$. Let $\mathbf{x}_m := \{x_s, 1 \leq s \leq m\}$ and $\mathbf{y}_n := \{y_k, 1 \leq k \leq n\}$ be sets of distinct points (in each case) in I . Then there exists a unique blending interpolant $h_f \in B^{m,n}$ to f , satisfying the following transfinite interpolation conditions:

$$(h_f)|_{\mathbf{G}_{m,n}} = f|_{\mathbf{G}_{m,n}}. \quad (4.5)$$

By using the univariate fundamental Lagrange interpolating polynomials $l_{s,m}(x)$, $s = 1, \dots, m$ and $l_{k,n}(y)$, $k = 1, \dots, n$ we give the following explicit construction of the unique transfinite Lagrange interpolant to f on the grid $\mathbf{G}_{m,n}$:

$$\begin{aligned} h_f(x, y) &= \sum_{s=1}^m f(x_s, y) l_{s,m}(x) + \sum_{k=1}^n f(x, y_k) l_{k,n}(y) \\ &\quad - \sum_{s=1}^m \sum_{k=1}^n f(x_s, y_k) l_{s,m}(x) l_{k,n}(y) \end{aligned} \quad (4.6)$$

with the error representation formula

$$f(x, y) - h_f(x, y) = \frac{D^{m,n} f(\xi, \eta)}{m!n!} \omega_{x_m, y_n}(x, y), \quad (x, y) \in I^2, \quad (\xi, \eta) \in I^2. \quad (4.7)$$

From the above error representation formula, for each $g \in B^{m,n}$ we have $h_g = g$ to conclude that for a fixed (m, n) interpolation grid $\mathbf{G}_{m,n}$ there exists a unique transfinite interpolant h_f from $B^{m,n}$ to a given function $f \in C^{m,n}$.

In addition, the univariate concept of a Haar-Chebyshev system can be extended in the infinite dimensional linear space of two-variable blending functions and interpolation grids consisting of vertical and horizontal line-segments as follows: *if $h \in B^{m,n}$ satisfies $h|_{\mathbf{G}_{m,n}} = 0$, then $h = 0$ on I^2 . In other words, an (m, n) blending grid cannot be a zero set of $h \in B^{m,n}$, h being nonidentically zero on I^2 .*

4.4. Haar-Chebyshev Pair of a Linear Space and a Set of Grids in the Multivariate Case

In the one-dimensional Lagrange interpolation problem we have a finite set of points which are similar in the sense they obey the natural ordering on the real line. In the bivariate case we have set of grid-lines which are also similar to each other in the sense that m of them are parallel to the one axis, and n of them, to the other. The points where the lines intersect the coordinate axes are also naturally ordered. Each blending grid $\mathbf{G}_{m,n}$ is unisolvent in transfinite Lagrange interpolation sense for $B^{m,n}$. On the other hand, the interpolation sets in the bivariate (multivariate) case may have diverse geometry and we can not talk about Haar-Chebyshev systems in general. However, we can argue that the pair of the linear space $B^{m,n}$ and the set of blending grids $\mathcal{G}_{m,n}$

$$\{B^{m,n}, \mathcal{G}_{m,n}\} \text{ is a Haar-Chebyshev pair on } I^2, \quad (4.8)$$

where $\mathcal{G}_{m,n}$ is the set of all blending grids of order (m, n) in I^2 , consisting of m vertical and n horizontal line-segments.

4.5. Transfinite Lagrange-Hermite Interpolation by Algebraic Blending Functions

Let

$$\mathbf{G}_{m,n} := \left\{ (x, y) \in I^2 : \omega_{x_m, y_n}(x, y) = 0 \right\} \quad (4.9)$$

be (m, n) interpolation grid associated with the point sets $\mathbf{x}_m := \{x_\mu, 1 \leq \mu \leq m\}$ and $\mathbf{y}_n := \{y_\nu, 1 \leq \nu \leq n\}$. The interpolation grid $\mathbf{G}_{m,n}$ consists of m vertical and n horizontal line segments on I^2 . We associate with each point x_μ a multiplicity s_μ and with each point y_ν a multiplicity j_ν . Let $\sum_{\mu=1}^m s_\mu = M$ and $\sum_{\nu=1}^n j_\nu = N$. Then for a given function $f \in C^{M,N}$, there exists a unique transfinite Lagrange-Hermite blending interpolant h_f from $B^{M,N}$ to f , satisfying the transfinite interpolation conditions

$$\begin{aligned} D^{s,0} h_f(x_\mu, y) &= D^{s,0} f(x_\mu, y), \quad s = 0, \dots, s_\mu - 1, \quad \mu = 1, \dots, m, \quad y \in [-1, 1], \\ D^{0,j} h_f(x, y_\nu) &= D^{0,j} f(x, y_\nu), \quad j = 0, \dots, j_\nu - 1, \quad \nu = 1, \dots, n, \quad x \in [-1, 1] \end{aligned} \quad (4.10)$$

with respect to the given (m, n) blending grid $\mathbf{G}_{m,n}$, where as usual $D^{0,0} f = f$.

Explicit construction of the transfinite Lagrange-Hermite interpolant h_f to f from $B^{M,N}$ is

$$\begin{aligned} h_f(x, y) &= \sum_{\mu=1}^m \sum_{s=0}^{s_\mu-1} D^{s,0} f(x_\mu, y) l_{\mu,M}^s(x) \\ &\quad + \sum_{\nu=1}^n \sum_{j=0}^{j_\nu-1} D^{0,j} f(x, y_\nu) l_{\nu,N}^j(y) \\ &\quad - \sum_{\mu=1}^m \sum_{s=0}^{s_\mu-1} \sum_{\nu=1}^n \sum_{j=0}^{j_\nu-1} D^{s,j} f(x_\mu, y_\nu) l_{\mu,M}^s(x) l_{\nu,N}^j(y), \end{aligned} \quad (4.11)$$

where the fundamental Lagrange-Hermite interpolation polynomials $l_{\mu,M}^s(x)$, $\mu = 1, \dots, m$ and $l_{\nu,N}^j(y)$, $\nu = 1, \dots, n$ satisfy the interpolation conditions

$$\begin{aligned} D^{r,0} l_{\mu,M}^s(x_\kappa) &= \delta_{\mu,\kappa} \delta_{s,r}, \quad \kappa = 1, \dots, m, \quad r = 0, \dots, s_\mu - 1, \\ D^{0,r} l_{\nu,N}^j(y_\kappa) &= \delta_{\nu,\kappa} \delta_{j,r}, \quad \kappa = 1, \dots, n, \quad r = 0, \dots, j_\nu - 1. \end{aligned} \quad (4.12)$$

4.6. Error Representation of Transfinite Lagrange-Hermite Interpolation by Algebraic Blending Functions in Terms of B-Splines

Let

$$\Omega_{(\mathbf{x}_m, \mathbf{s}_m); (\mathbf{y}_n, \mathbf{j}_n)}(x, y) := \prod_{\mu=1}^m (x - x_\mu)^{s_\mu} \prod_{\nu=1}^n (y - y_\nu)^{j_\nu}, \quad (4.13)$$

where $\mathbf{s}_m := (s_1, \dots, s_m)$ and $\mathbf{j}_n := (j_1, \dots, j_n)$. Then the following error representation formula holds

$$\begin{aligned} f(x, y) - h_f(x, y) &= \Omega_{(\mathbf{x}_m, \mathbf{s}_m); (\mathbf{y}_n, \mathbf{j}_n)}(x, y) \\ &\times \int_{-1}^1 \int_{-1}^1 B_{M-1; [(\mathbf{x}_m, \mathbf{s}_m), x]}(v) B_{N-1; [(\mathbf{y}_n, \mathbf{j}_n), y]}(w) D^{M,N} f(v, w) dv dw, \end{aligned} \quad (4.14)$$

where $B_{M-1; [(\mathbf{x}_m, \mathbf{s}_m), x]}$ is the normalized B-spline of degree $M - 1$ with knots (\mathbf{x}_m, x) of multiplicities $(\mathbf{s}_m, 1)$ and $B_{N-1; [(\mathbf{y}_n, \mathbf{j}_n), y]}$ is the normalized B-spline of degree $N - 1$ with knots (\mathbf{y}_n, y) of multiplicities $(\mathbf{j}_n, 1)$. For details on B-splines see [11–13]. As a corollary we obtain Cauchy error representation for transfinite Lagrange-Hermite interpolation by blending functions

$$f(x, y) - h_f(x, y) = \frac{D^{M,N} f(\xi, \eta)}{M!N!} \Omega_{(\mathbf{x}_m, \mathbf{s}_m); (\mathbf{y}_n, \mathbf{j}_n)}(x, y), \quad (x, y) \in I^2, \quad (\xi, \eta) \in I^2. \quad (4.15)$$

Note that $h_g = g$ for a blending function $g \in B^{M,N}$. Therefore, each blending function from $B^{M,N}$ can be represented as transfinite Lagrange-Hermite interpolant on a grid consisting of horizontal and vertical line-segments.

Remark 4.1. The multivariate results which resemble the broadest extent the univariate theory concern transfinite interpolation and best L_1 -approximation by algebraic blending functions. For simplicity of the notations the results on transfinite interpolation and best L_1 -approximation by algebraic blending functions are stated in the bivariate case $d = 2$ although they are entirely valid in higher dimensions.

5. Best L_1 -Approximation by Algebraic Blending Functions

Let $(\mathcal{F}, \|\cdot\|)$ be a normed linear space. Let U be a subspace of \mathcal{F} . Then $u_f \in U$ is called best approximant to $f \in \mathcal{F}$ from U if for all $u \in U$

$$\|f - u_f\| \leq \|f - u\|. \quad (5.1)$$

Let f satisfy $D^{m,n} f(x, y) > 0$ on I^2 and let $h \in B^{m,n}$. Then an (m, n) blending grid is a maximal set of zeros for $f - h$ in a sense that if $f(x, y) - h(x, y) = 0$ on (m, n) blending grid $\mathbf{G}_{m,n}$, then $f(x, y) - h(x, y) \neq 0$ for $(x, y) \in I^2 \setminus \mathbf{G}_{m,n}$. In other words, $f - h$ cannot vanish on $\mathbf{G}_{m,n} \cup (x_0, y_0)$, where $(x_0, y_0) \in I^2 \setminus \mathbf{G}_{m,n}$.

Approximation by blending functions is useful in problems of improving the efficiency of data transfer systems, image processing, reducing the size of the table of a function of many variables, cubature formulae, and numerical solution of differential and integral equations. For results on existence of best algebraic blending L_1 -approximants see [14, 15]. The above notion of a Haar-Chebyshev system in the multivariate case is essential in the proof of the next theorem (see [16] for details).

Theorem 5.1. *Let $f \in C^{m,n}$ satisfy $D^{m,n}f(x, y) \geq 0$ for $(x, y) \in I^2$. Then f possesses a unique best L_1 -approximant from $B^{m,n}$ that is the unique transfinite Lagrange interpolant h_f to f from $B^{m,n}$ with respect to the blending grid*

$$\bar{G}_{m,n} = \left\{ (x, y) : \prod_{i=1}^m \prod_{j=1}^n (x - x_i)(y - y_j) = 0 \right\}, \quad (5.2)$$

where $x_k, 1 \leq k \leq m$ and $y_j, 1 \leq j \leq n$ are the zeros of the Chebyshev polynomials of second kind $U_m(x)$ and $U_n(y)$, respectively.

Remark 5.2. Following Theorem 5.1, we conclude that the blending grid $\bar{G}_{m,n}$ is the canonical set of best L_1 -approximation from $B^{m,n}$ to the convex cone

$$\left\{ f \in C^{m,n}(I^2) : D^{m,n}f(x, y) \geq 0 \text{ for } (x, y) \in I^2 \right\}. \quad (5.3)$$

Note that in the above convex cone, the non linear problem of best L_1 -approximation becomes a linear one.

Corollary by Theorem 5.1 is the interesting fact that the best L_1 -approximant to the polynomial $x^m y^n$ from $B^{m,n}$ is also a polynomial, namely,

$$\left\| \frac{U_m(x)U_n(y)}{2^{m+n}} \right\|_{L_1(I^2)} \leq \|x^m y^n - h(x, y)\|_{L_1(I^2)}, \quad h \in B^{m,n} \quad (5.4)$$

and in particular the unique polynomial of minimal L_1 -norm (minimal L_1 -deviation) on I^2 from the class of monic polynomials $\{\sum_{i=0}^m \sum_{j=0}^n a_{ij} x^i y^j : a_{mn} = 1\}$ is $2^{-m-n} U_m(x) U_n(y)$.

Remark 5.3. Note that in contrast to the result in Theorem 5.1, the best uniform (Chebyshev) approximant from $B^{1,1}$ to $f \in C^{1,1}$ satisfying $D^{1,1}f(x, y) \geq 0$ for $(x, y) \in I^2$, is *never unique* unless $f \in B^{1,1}$ (see [17] for details).

6. Lagrange-Hermite Transfinite Interpolation by Trigonometric Blending Functions

In bivariate case, transfinite interpolation by trigonometric blending functions is to construct a blending trigonometric interpolant over a planar domain in such a way that matches (interpolates) a given function and its partial derivatives on curves that constitute the interpolation set. For details on the results in the present section see [7].

Let $\tilde{C}^{2m+1,2n+1}[0,2\pi]^2$ be the vector space of two-variable functions $f(\theta, \eta)$ which are 2π -periodic in each variable and

$$\mathfrak{D}^{(2m+1,2n+1)} f(\theta, \eta) := \mathfrak{D}_\theta^{(2m+1)} \mathfrak{D}_\eta^{(2n+1)} f(\theta, \eta) \quad (6.1)$$

is continuous on $[0,2\pi]^2$, where $\mathfrak{D}_\theta^{(2m+1)} = \mathfrak{D}_\theta \prod_{k=1}^m (\mathfrak{D}_\theta^2 + k^2)$ and $\mathfrak{D}_\theta = \partial/\partial\theta$ is the partial derivative with respect to θ . The vector space $B_{m,n}^t$ of *trigonometric blending functions of order (m, n)* is defined as

$$B_{m,n}^t := \left\{ h_t(\theta, \eta) \in \tilde{C}^{2m+1,2n+1}[0,2\pi]^2 : \mathfrak{D}^{(2m+1,2n+1)} h_t(\theta, \eta) = 0 \right\}. \quad (6.2)$$

Each *trigonometric blending function* $h_t \in B_{m,n}^t$ can be represented in the form

$$h_t(\theta, \eta) = \sum_{k=0}^m [a_k(\eta) \cos(k\theta) + b_k(\eta) \sin(k\theta)] + \sum_{p=0}^n [c_p(\theta) \cos(p\eta) + d_p(\theta) \sin(p\eta)], \quad (6.3)$$

where a_k, b_k, c_p , and d_p are sufficiently smooth 2π -periodic functions.

Remark 6.1. Note that the above representation of $h_t \in B_{m,n}^t$ is not unique in contrast to the uniqueness of the corresponding representation in T_n .

Let $\bar{\theta}_{l_1} := (\theta_0, \theta_1, \dots, \theta_{l_1})$, $0 \leq \theta_0 < \theta_1 < \dots < \theta_{l_1} < 2\pi$ be $l_1 + 1$ interpolation nodes with positive integer multiplicities $\bar{\lambda}_{l_1} := (\lambda_0, \lambda_1, \dots, \lambda_{l_1})$ and let $\bar{\eta}_{l_2} := (\eta_0, \eta_1, \dots, \eta_{l_2})$, $0 \leq \eta_0 < \eta_1 < \dots < \eta_{l_2} < 2\pi$ be $l_2 + 1$ interpolation nodes with positive integer multiplicities $\bar{\beta}_{l_2} := (\beta_0, \beta_1, \dots, \beta_{l_2})$.

6.1. Construction of Lagrange-Hermite Transfinite Interpolation by Trigonometric Blending Functions

Let $f(\theta, \eta) \in \tilde{C}^{2m+1,2n+1}[0,2\pi]^2$. Given the interpolation nodes with corresponding multiplicities $(\bar{\theta}_{l_1}, \bar{\lambda}_{l_1})$ and $(\bar{\eta}_{l_2}, \bar{\beta}_{l_2})$, $\sum_{i=0}^{l_1} \lambda_i = 2m + 1$ and $\sum_{j=0}^{l_2} \beta_j = 2n + 1$. Find a trigonometric blending function $h_{t,f} \in B_{m,n}^t$ of order (m, n) , satisfying the following Lagrange-Hermite interpolation conditions $(f^{(i,j)})$ denoting the partial derivative $\partial^{i+j} f / \partial \theta^i \partial \eta^j$:

$$\begin{aligned} h_{t,f}^{(i,0)}(\theta_{s_1}, \eta) &= f^{(i,0)}(\theta_{s_1}, \eta), \quad i = 0, 1, \dots, \lambda_{s_1} - 1; \quad s_1 = 0, 1, \dots, l_1; \\ h_{t,f}^{(0,j)}(\theta, \eta_{s_2}) &= f^{(0,j)}(\theta, \eta_{s_2}), \quad j = 0, 1, \dots, \beta_{s_2} - 1; \quad s_2 = 0, 1, \dots, l_2. \end{aligned} \quad (6.4)$$

Theorem 6.2. Let $f(\theta, \eta) \in \tilde{C}^{2m+1,2n+1}[0,2\pi]^2$. Given the interpolation nodes with corresponding multiplicities $(\bar{\theta}_{l_1}, \bar{\lambda}_{l_1})$ and $(\bar{\eta}_{l_2}, \bar{\beta}_{l_2})$ such that $\sum_{i=0}^{l_1} \lambda_i = 2m + 1$ and $\sum_{j=0}^{l_2} \beta_j = 2n + 1$, then there exists a unique trigonometric blending interpolant $h_{t,f}$ from $B_{m,n}^t$ (of order (m, n)) to f satisfying the

transfinite interpolation conditions (6.4). In addition, the following point-wise error representation holds:

$$f(\theta, \eta) - h_{t,f}(\theta, \eta) = \frac{\mathfrak{D}^{(2m+1, 2n+1)} f(\xi, \zeta)}{[m!]^2 [n!]^2} [B_1(\theta, \eta) - h_{t, B_1}(\theta, \eta)], \quad (6.5)$$

where $(\xi(\theta, \eta), \zeta(\theta, \eta)) \in (0, 2\pi)^2$, $B_1(\theta, \eta) := B_1(\theta)B_1(\eta)$ and $B_1(\theta) = \theta - \pi$, $\theta \in [0, 2\pi)$ is the first univariate Bernoulli function.

6.2. Uniqueness of the Lagrange-Hermite Trigonometric Blending Interpolant

Suppose that there is another trigonometric blending interpolant $\tilde{h}_{t,f} \in B_{m,n}^t$ to f satisfying the interpolation conditions (6.4). Evidently, $h_{t, \tilde{h}_{t,f}} = h_{t,f}$. Therefore, by using (6.5)

$$\begin{aligned} \tilde{h}_{t,f} - h_{t, \tilde{h}_{t,f}} &= \frac{\mathfrak{D}^{(2m+1, 2n+1)} \tilde{h}_{t,f}(\xi, \zeta)}{[m!]^2 [n!]^2} [B_1(\theta, \eta) - h_{t, B_1}(\theta, \eta)] = 0 \\ &\implies \tilde{h}_{t,f} = h_{t, \tilde{h}_{t,f}} = h_{t,f}. \end{aligned} \quad (6.6)$$

Lemma 6.3. Let $h_{t, B_1}(\theta, \eta)$ be the unique trigonometric blending interpolant from $B_{m,n}^t$ to $B_1(\theta, \eta) = B_1(\theta)B_1(\eta)$ satisfying the transfinite interpolation conditions (6.4) on the interpolating grid

$$\mathbf{G}_{1,1_2} := \left\{ (\theta, \eta) \in [0, 2\pi]^2 : \prod_{s_1=0, s_2=0}^{1,1_2} (\theta - \theta_{s_1})(\eta - \eta_{s_2}) = 0 \right\}. \quad (6.7)$$

Then, $[B_1(\theta, \eta) - h_{t, B_1}(\theta, \eta)] \neq 0$ for each point $(\theta, \eta) \notin \mathbf{G}_{1,1_2}$.

Corollary 6.4. Let $h_{t,f}$ be the unique trigonometric blending interpolant from $B_{m,n}^t$ to $f \in \tilde{\mathcal{C}}^{2m+1, 2n+1}[0, 2\pi]^2$ satisfying the transfinite interpolation conditions (6.4). If $\mathfrak{D}^{(2m+1, 2n+1)} f(\xi, \zeta) \neq 0$ for $(\xi, \zeta) \in [0, 2\pi]^2$, then $f(\theta, \eta) - h_{t,f}(\theta, \eta) \neq 0$ for $(\theta, \eta) \notin \mathbf{G}_{1,1_2}$.

7. Best L_1 -Approximation by Trigonometric Blending Functions

Consider the normed vector space $L_1[0, 2\pi]^2$ of functions that are 2π -periodic in each variable, whose absolute value has a finite integral on $[0, 2\pi]^2$ and equipped with the norm

$$\|f\|_{L_1[0, 2\pi]^2} := \iint_{[0, 2\pi]^2} |f(\theta, \eta)| dA = \left(\text{Fubini's Theorem} \right) \iint_0^{2\pi} |f(\theta, \eta)| d\theta d\eta, \quad (7.1)$$

where A is the area measure. Denote for simplicity $\|f\|_1 := \|f\|_{L_1[0, 2\pi]^2}$. We restrict $B_{m,n}^t$ to $B_{m,n}^t \subset L_1[0, 2\pi]^2$.

Let us consider the convex cone

$$\tilde{\mathcal{C}}_+^{2m+1,2n+1} := \left\{ f \in \tilde{\mathcal{C}}^{2m+1,2n+1}[0, 2\pi]^2 : \mathfrak{D}^{(2m+1,2n+1)} f(\theta, \eta) \geq 0 \text{ for } (\theta, \eta) \in [0, 2\pi]^2 \right\}. \quad (7.2)$$

We construct best L_1 -approximants from $B_{m,n}^t$ to functions in the convex cone $\tilde{\mathcal{C}}_+^{2m+1,2n+1}$. Following the interpolation problem (6.4) let $\bar{\theta} = \{i\pi/(m+1), i = 1, \dots, 2m+1\}$ and $\bar{\eta} = \{j\pi/(n+1), j = 1, \dots, 2n+1\}$, where all interpolation nodes are of multiplicity 1. By Theorem 6.2, there exists a unique trigonometric blending interpolant $h_{t,f} \in B_{m,n}^t$ to $f \in \tilde{\mathcal{C}}_+^{2m+1,2n+1}$ on the blending grid

$$\tilde{\mathbf{G}}_{m,n} := \left\{ (\theta, \eta) \in [0, 2\pi]^2 : \prod_{i=1, j=1}^{2m+1, 2n+1} \left[\theta - \frac{i\pi}{m+1} \right] \left[\eta - \frac{j\pi}{n+1} \right] = 0 \right\}, \quad (7.3)$$

consisting of $2m+1$ vertical and $2n+1$ horizontal line segments. The next theorem (see [7] for details) demonstrates that the interpolation grid $\tilde{\mathbf{G}}_{m,n}$ is the canonical sets of best L_1 -approximation from $B_{m,n}^t$ to the convex cone $\tilde{\mathcal{C}}_+^{2m+1,2n+1}$.

Theorem 7.1. *The unique trigonometric blending interpolant $h_{t,f} \in B_{m,n}^t$ to $f \in \tilde{\mathcal{C}}_+^{2m+1,2n+1}$ on the blending grid $\tilde{\mathbf{G}}_{m,n}$ is the unique best L_1 -approximant to f from $B_{m,n}^t$.*

8. Canonical Set of Best L_1 -Approximation on a Triangle

Another canonical set result is presented in [18], where the domain is a triangle $\Delta \in \mathbf{R}^2$. The approximating space consists of all functions that are sums of functions of the barycentric coordinates $\{\lambda_i\}_{i=1}^3$ with respect to the vertices of Δ :

$$B(\Delta) := \left\{ h \in C^3(\Delta) : h(\lambda_1, \lambda_2) = h_1(\lambda_1) + h_2(\lambda_2) + h_3(\lambda_3), \lambda_1 + \lambda_2 + \lambda_3 = 1, \lambda_i \geq 0 \right\}. \quad (8.1)$$

$B(\Delta)$ is the kernel of the differential operator ∂^3 which acts on the linear space $C^3(\Delta)$ and is defined by

$$\partial^3 f(\lambda_1, \lambda_2) := \frac{\partial}{\partial \lambda_1} \frac{\partial}{\partial \lambda_2} \left(\frac{\partial}{\partial \lambda_1} - \frac{\partial}{\partial \lambda_2} \right) f(\lambda_1, \lambda_2). \quad (8.2)$$

Let $\mathcal{M} := \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3$ be the union of the medians $\{\mathcal{M}_i\}_{i=1}^3$ in Δ . The following interpolation theorem holds true.

Theorem 8.1. *Let $f \in C^3(\Delta)$. Then there exists a unique transfinite interpolant $h_f \in B(\Delta)$ to f such that*

$$(h_f)|_{\mathcal{M}} = f|_{\mathcal{M}}. \quad (8.3)$$

Moreover, if $f \in C^m(\Delta)$ for some $m > 3$, then $h_f \in C^m(\Delta)$ as well.

On the basis of Theorem 8.1 and an appropriate error representation formula, the corresponding best $L_1(\Delta)$ -approximation is characterized in terms of a canonical point set. The convex cone here is defined by another differential operator $\tilde{\mathfrak{D}}$ (see [18, page 456] for details) that is represented in terms of ∂^3 in a bit intricate way involving infinite series. The canonical result of best L_1 -approximation reads as follows.

Theorem 8.2. *Let $f \in C^3(\Delta)$ satisfy $\tilde{\mathfrak{D}}f \geq 0$ on Δ . Then, the unique transfinite interpolant h_f from Theorem 8.1 is the unique best L_1 -approximant to the function f from $B(\Delta)$.*

Remark 8.3. By Theorem 8.2, the point set \mathcal{M} is a canonical set of best L_1 -approximation on a triangle.

9. Best L_1 -Approximation by Harmonic Functions

Here, we discuss results on best L_1 -approximation of subharmonic functions by harmonic functions. Let $B(r)$ denote the open ball centered at $\mathbf{0}$ of radius $r > 0$ in the d -dimensional space \mathbf{R}^d and let $B := B(1)$ be the unit ball centered at $\mathbf{0}$ in \mathbf{R}^d and $B_0 := B(2^{-1/d})$. Denote by $\mathcal{H}(B)$ the linear space of harmonic functions on B and let $\mathcal{S}(B)$ denote the convex cone of subharmonic functions on B . Let $f \in C(\bar{B})$ be continuous on the closed unit ball and let $f \in \mathcal{S}(B)$. A function $h_f \in C(\bar{B}) \cap \mathcal{H}(B)$ is called a best harmonic L_1 -approximant to f on B if

$$\|f - h_f\|_{L_1(B)} \leq \|f - h\|_{L_1(B)} \quad (9.1)$$

for all $h \in C(\bar{B}) \cap \mathcal{H}(B)$. The next theorem gives a canonical set characterization of the best harmonic L_1 -approximants to the convex cone $C(\bar{B}) \cap \mathcal{S}(B)$ (see [19] for details).

Theorem 9.1. *Let $f \in C(\bar{B}) \cap \mathcal{S}(B)$. Then the harmonic function h_f is a best harmonic L_1 -approximant from $C(\bar{B}) \cap \mathcal{H}(B)$ to f if and only if*

- (i) $h_f = f$ on ∂B_0 (that is, h_f is a transfinite harmonic interpolant to f on ∂B_0);
- (ii) $h_f \leq f$ on $\bar{B} \setminus B_0$.

Remark 9.2. If a best harmonic L_1 -approximant to $f \in C(\bar{B}) \cap \mathcal{S}(B)$ exists, then it is unique. However, a best harmonic L_1 -approximant to an arbitrary function $f \in C(\bar{B}) \cap \mathcal{S}(B)$ need not exist. For example, $g(x, y) := x^4 y^4$ is a subharmonic polynomial on the unit disk in \mathbf{R}^2 which does not possess a best L_1 -approximant from $C(\bar{B}) \cap \mathcal{H}(B)$. However obviously, $D^{2,2}g(x, y) \geq 0$ and according to Theorem 5.1, a unique best L_1 -approximant on $[-1, 1]^2$ to g from $B^{2,2}$ exists. For other results on best harmonic L_1 -approximation see [20, 21].

Remark 9.3. The constructive characterization of Theorem 9.1 is in terms of the Dirichlet problem on ∂B_0 . It can be considered as a transfinite interpolation problem by harmonic functions on the spherical interpolation grid ∂B_0 . The canonical set of best harmonic L_1 -approximation to the convex cone $C(\bar{B}) \cap \mathcal{S}(B)$ on the unit ball is ∂B_0 .

10. Best One-Sided L_1 -Approximation by Algebraic Blending Functions

Let $f \in C(I^2)$. A function $h^* \in B^{m,n}$ is called best one-sided from above L_1 -approximant to f from the linear space $B^{m,n}$ if

$$h^*(x, y) \geq f(x, y), \quad (x, y) \in I^2, \quad \int_{I^2} (h^* - f) \leq \int_{I^2} (h - f) \quad (10.1)$$

for all $h \in B^{m,n}(I^2)$ such that $h(x, y) \geq f(x, y)$, $(x, y) \in I^2$. Analogously, we define best one-sided from below L_1 -approximant to f from $B^{m,n}$.

This type of one-sided L_1 -approximation (with respect to a convex set rather than a subspace) has been a subject of much research activity (see [1] for details). The results in this area have mainly dealt with the case when the best approximant is from a finite-dimensional linear space. However, as we have mentioned above, the linear space $B^{m,n}$ of algebraic blending functions of order (m, n) is of infinite dimension.

First we reformulate a general result in the particular case of $B^{m,n}$ approximating linear space (see [22, 23] for details). It shows that the canonical sets must satisfy certain conditions.

Theorem 10.1. *Let $f \in C(I^2)$. Let*

$$\mathcal{U}(f) := \left\{ h \in B^{m,n} : h(x, y) \geq f(x, y) \text{ on } I^2 \right\}. \quad (10.2)$$

Let $h^* \in \mathcal{U}(f)$ and let $Z := Z(h^* - f)$ be the zero set of $(h^* - f)$ in I^2 . Then, the following are equivalent.

- (a) *The blending function h^* is a best one-sided from above L_1 -approximant to f from $B^{m,n}$ on I^2 .*
- (b) *Gaussian property for the zero set Z holds*

$$\int_{I^2} h \geq 0 \text{ whenever } h \in B^{m,n}, h \geq 0 \text{ on } Z. \quad (10.3)$$

- (c) *The domain I^2 is a quadrature domain with respect to the point set Z and the linear space $B^{m,n}$; that is, there is a positive measure μ with support in Z such that*

$$\int_{I^2} h = \int_Z h d\mu \quad (10.4)$$

for all $h \in B^{m,n}$.

By analogy with the canonical sets in the best L_1 -approximation by blending functions one can expect that the best one-sided L_1 -approximants by algebraic blending functions are Lagrange-Hermite transfinite interpolants on grids consisting of vertical and horizontal lines. However, Theorem 10.1 and the next result (see [24] for details) show that this is not the case: *transfinite interpolation grids consisting of vertical and horizontal line-segments in I^2 cannot be canonical sets of best one-sided L_1 -approximation by blending functions.*

Theorem 10.2. *Let $m \geq 1$ and $n \geq 1$. Then there exists an algebraic blending function $h \in B^{2m,2n}$ such that it is positive on the Legendre (m, n) grid*

$$\mathbf{G}_{m,n,L} := \left\{ (x, y) \in I^2 : L_m(x)L_n(y) = 0 \right\}, \quad (10.5)$$

where L_m and L_n are the Legendre polynomials of degree m , respectively n , that is,

$$h|_{\mathbf{G}_{m,n,L}} > 0, \quad (10.6)$$

but its integral on I^2 is negative:

$$\int_{I^2} h < 0. \quad (10.7)$$

Analogous result concerning best approximation by trigonometric blending functions is published in [7].

10.1. Best One-Sided L_1 -Approximation by Algebraic $B^{1,1}$ -Blending Functions

We give a constructive characterization in terms of canonical sets for the best one-sided L_1 -approximant to a function $f \in C^{1,1}$ satisfying $D^{1,1}f \geq 0$ on I^2 from the infinite-dimensional linear space $B^{1,1}$. The best L_1 -approximants are transfinite Lagrange-Hermite interpolants on the diagonals of I^2 as canonical sets. *The occurrence of the diagonals as canonical sets of best one-sided L_1 -approximation (not interpolation grids consisting of vertical and horizontal line-segments in I^2 !) is a fact which, to the best of our knowledge, has been first observed in [25].*

Note that

$$B^{1,1} = \left\{ h(x, y) : h(x, y) = a(x) + b(y), a \in C^1(I), b \in C^1(I) \right\}. \quad (10.8)$$

In other words, we approximate two-variable functions by sums of univariate functions on I^2 .

The proof of existence, uniqueness and explicit construction of the best one-sided L_1 -approximant from above and from below consists of three main steps (see [25, 26] for details).

(A) *Transfinite Lagrange-Hermite Interpolation Formula on an Appropriate Grid with an Error Remainder Term*

Theorem 10.3. *Let $f \in C^{1,1}$.*

(a) *The $B^{1,1}$ blending function*

$$h_f^*(x, y) := f(-1, -1) + \int_{-1}^x D^{1,0}f(t, t)dt + \int_{-1}^y D^{0,1}f(t, t)dt \quad (10.9)$$

is the unique Lagrange-Hermite transfinite interpolant to f from $B^{1,1}$ satisfying the transfinite interpolation conditions

$$\left(h_f^*\right)_{|\Delta^*} = f_{|\Delta^*}, \quad \left(\text{grad } h_f^*\right)_{|\Delta^*} = \left(\text{grad } f\right)_{|\Delta^*} \quad (10.10)$$

on the main diagonal Δ^* of I^2 . Moreover, the error representation formula holds

$$f(x, y) - h_f^*(x, y) = -\frac{D^{1,1}f(\xi, \eta)}{2}(x - y)^2, \quad (x, y) \in I^2, \quad (\xi, \eta) \in I^2. \quad (10.11)$$

(b) The $B^{1,1}$ blending function

$$h_{*f}(x, y) := f(-1, 1) + \int_{-1}^x D^{1,0}f(t, -t)dt - \int_{-1}^{-y} D^{0,1}f(t, -t)dt \quad (10.12)$$

is the unique Lagrange-Hermite transfinite interpolant to f from $B^{1,1}$ satisfying the transfinite interpolation conditions

$$\left(h_{*f}\right)_{|\Delta_*} = f_{|\Delta_*}, \quad \left(\text{grad } h_{*f}\right)_{|\Delta_*} = \left(\text{grad } f\right)_{|\Delta_*} \quad (10.13)$$

on the antidiagonal Δ_* of I^2 . Moreover, the following error representation formula holds:

$$f(x, y) - h_{*f}(x, y) = \frac{D^{1,1}f(\rho, \sigma)}{2}(x + y)^2, \quad (x, y) \in I^2, \quad (\rho, \sigma) \in I^2. \quad (10.14)$$

The following observation is an essential fact in the proof of the next theorem. Let f satisfy $D^{1,1}f(x, y) > 0$ on I^2 and let $h \in B^{1,1}(I^2)$. If $(f - h) = 0$ on Δ^* and $(\text{grad } (f - h))_{|\Delta^*} = (0, 0)$, then $(f - h) \neq 0$ on $I^2 \setminus \Delta^*$. In other words, $(f - h)$ can not vanish on $\Delta^* \cup (x_0, y_0)$, where $(x_0, y_0) \in I^2 \setminus \Delta^*$.

(B) Transfinite Cubature Formulae on Δ^* and Δ_* with $B^{1,1}$ Blending Degree of Precision

Let $f \in C^{1,1}$. Then

$$\int_{I^2} f - 2 \int_I f(x, x)dx = -\frac{4}{3}D^{1,1}f(\alpha, \beta), \quad (\alpha, \beta) \in I^2. \quad (10.15)$$

In particular

$$\int_{I^2} h = 2 \int_I h(x, x)dx, \quad h \in B^{1,1}. \quad (10.16)$$

Analogously,

$$\int_{I^2} f - 2 \int_I f(x, -x)dx = \frac{4}{3}D^{1,1}f(\gamma, \delta), \quad (\gamma, \delta) \in I^2. \quad (10.17)$$

In particular

$$\int_{I^2} h = 2 \int_I h(x, -x) dx, \quad h \in B^{1,1}. \quad (10.18)$$

(C) *Constructive Characterization of the Best One-Sided L_1 -Approximants from $B^{1,1}$.*

Theorem 10.4. *Let $f \in C^{1,1}$ satisfy $D^{1,1}f(x, y) \geq 0$ for $(x, y) \in I^2$. Then we have the following.*

- (a) *The function f has a unique best one-sided from above L_1 -approximant from $B^{1,1}$. The unique best one-sided from above L_1 -approximant h_f^* is characterized by the Lagrange-Hermite type transfinite interpolation conditions on the main diagonal Δ^* of I^2 :*

$$\left(h_f^*\right)_{|\Delta^*} = f_{|\Delta^*}, \quad \left(\text{grad } h_f^*\right)_{|\Delta^*} = \left(\text{grad } f\right)_{|\Delta^*}. \quad (10.19)$$

- (b) *The function f has a unique best one-sided from below L_1 -approximant from $B^{1,1}$. The unique best one-sided from below L_1 -approximant h_{*f} is characterized by the Lagrange-Hermite type transfinite interpolation conditions on the antidiagonal Δ_* of I^2 :*

$$\left(h_{*f}\right)_{|\Delta_*} = f_{|\Delta_*}, \quad \left(\text{grad } h_{*f}\right)_{|\Delta_*} = \left(\text{grad } f\right)_{|\Delta_*}. \quad (10.20)$$

Proof of Theorem 10.4. We sketch the proof of (a). The proof of (b) follows the same steps. Given $f \in C^{1,1}$ such that $D^{1,1}f(x, y) \geq 0$ for $(x, y) \in I^2$. By Theorem 10.3, $h_f^* \geq f$ on I^2 . Consider an arbitrary blending function $h \in B^{1,1}$ such that $h \geq f$ on I^2 . By using the transfinite cubature formula from (B), we have

$$\begin{aligned} \int_{I^2} (h - f) &= \int_{I^2} (h - h_f^*) + \int_{I^2} (h_f^* - f) \\ &= 2 \int_{-1}^1 (h - h_f^*)(x, x) dx + \int_{I^2} (h_f^* - f) \\ &\geq \int_{I^2} (h_f^* - f), \end{aligned} \quad (10.21)$$

taking into account that $\int_{-1}^1 (h - h_f^*)(x, x) dx \geq 0$ which follows from $h \geq f$ on I^2 and $(h_f^*)_{|\Delta^*} = f_{|\Delta^*}$. Hence,

$$\int_{I^2} (h - f) \geq \int_{I^2} (h_f^* - f), \quad (10.22)$$

for each function $h \in B^{1,1}$ satisfying $h \geq f$ on I^2 . □

Uniqueness of the Best One-Sided from above L_1 -Approximant from $B^{1,1}$ to f in the Convex Cone $\{f \in C^{1,1} : D^{1,1}f(x, y) \geq 0, (x, y) \in I^2\}$.

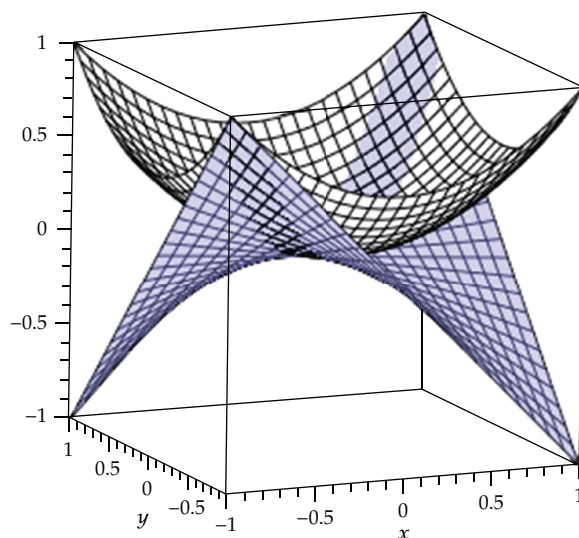


Figure 1: $f(x, y) = xy$ and $h_f^*(x, y)$.

Suppose that $h \in B^{1,1}$ is another best one-sided L_1 -approximant to f from $B^{1,1}$. It follows from (10.21) that $f|_{\Delta^*} = (h_f^*)|_{\Delta^*} = h|_{\Delta^*}$. From $h \geq f$ on I^2 we conclude that $(\text{grad } h)|_{\Delta^*} = (\text{grad } f)|_{\Delta^*}$. By Theorem 10.3, $h = h_f^*$. The proof is completed.

Example 10.5. Consider $f(x, y) = xy$, $D^{1,1}f(x, y) = 1 > 0$. According to Theorem 10.4, the transfinite Lagrange-Hermite interpolant to f from $B^{1,1}$

$$\begin{aligned} h_f^*(x, y) &= f(-1, -1) + \int_{-1}^x D^{1,0}f(t, t)dt + \int_{-1}^y D^{0,1}f(t, t)dt \\ &= 1 + \int_{-1}^x t dt + \int_{-1}^y t dt = \frac{x^2 + y^2}{2} \end{aligned} \quad (10.23)$$

is the unique best one-sided from above L_1 -approximant from $B^{1,1}$ to $f(x, y) = xy$ (see Figure 1).

Example 10.6. Let $f(x, y) = x^p y^q$, $x \geq 0$, $p > 0$, $y \geq 0$, $q > 0$, $p + q = 1$. We compute $D^{1,0}f(x, y) = px^{p-1}y^q$, $D^{0,1}f(x, y) = qx^p y^{q-1}$, $D^{1,1}f(x, y) = pqx^{p-1}y^{q-1} > 0$ for $x, y > 0$. According to Theorem 10.4, the transfinite Lagrange-Hermite interpolant to f from $B^{1,1}$

$$\begin{aligned} h_f^*(x, y) &= f(0, 0) + \int_0^x D^{1,0}f(t, t)dt + \int_0^y D^{0,1}f(t, t)dt \\ &= \int_0^x p dt + \int_0^y q dt = px + qy \end{aligned} \quad (10.24)$$

is the unique best one-sided from above L_1 -approximant from $B^{1,1}$ to $f(x, y) = x^p y^q$ on $[0, 1]^2$.

10.2. Best One-Sided L_1 -Approximation by Algebraic $B^{2,1}$ -Blending Functions

Next theorem (see [27] for details) gives characterization of the best one-sided L_1 -approximants by algebraic $B^{2,1}$ -blending functions in terms of transfinite Lagrange-Hermite interpolation on canonical sets.

Theorem 10.7. *Let $f \in C^{2,1}$ and $D^{2,1}f(x, y) \geq 0$ for $(x, y) \in I^2$.*

(a) *Let*

$$\mathbf{v}^* := \{(x, 2|x| - 1) : x \in [-1, 1]\}. \quad (10.25)$$

The function f possesses a unique best one-sided from above L_1 -approximant h^ from $B^{2,1}$. The unique best one-sided from above L_1 -approximant h_f^* to f is characterized by the simultaneous Lagrange-Hermite type transfinite interpolation conditions:*

$$(h_f^*)|_{\mathbf{v}^*} = f|_{\mathbf{v}^*}, \quad (\text{grad } h_f^*)|_{\mathbf{v}^*} = (\text{grad } f)|_{\mathbf{v}^*}. \quad (10.26)$$

(b) *Let*

$$\mathbf{v}_* := \{(x, 1 - 2|x|) : x \in [-1, 1]\}. \quad (10.27)$$

*The function f possesses a unique best one-sided from below L_1 -approximant h_{*f} from $B^{2,1}$. The unique best one-sided from below L_1 -approximant h_{*f} to f is characterized by the simultaneous Lagrange-Hermite type transfinite interpolation condition:*

$$(h_{*f})|_{\mathbf{v}_*} = f|_{\mathbf{v}_*}, \quad (\text{grad } h_{*f})|_{\mathbf{v}_*} = (\text{grad } f)|_{\mathbf{v}_*}. \quad (10.28)$$

The following two steps (see [27] for details) are essential in the proof of Theorem 10.7.

(A) *Transfinite Interpolation Formulas with Remainder Term*

Let $f \in C^{2,1}$ and let h_f^* and h_{*f} be the transfinite interpolants to f from Theorem 5.1. Then

$$f(x, y) - h_f^*(x, y) = -\frac{D^{2,1}f(\xi_1, \eta_1)}{24} (y - 2|x| + 1)^2 (y + 4|x| + 1), \quad (10.29)$$

where $(x, y) \in I^2$ and $(\xi_1, \eta_1) \in I^2$ and

$$f(x, y) - h_{*f}(x, y) = \frac{D^{2,1}f(\xi_2, \eta_2)}{24} (y - 2|x| + 1)^2 (-y + 4|x| + 1), \quad (10.30)$$

where $(x, y) \in I^2$ and $(\xi_2, \eta_2) \in I^2$.

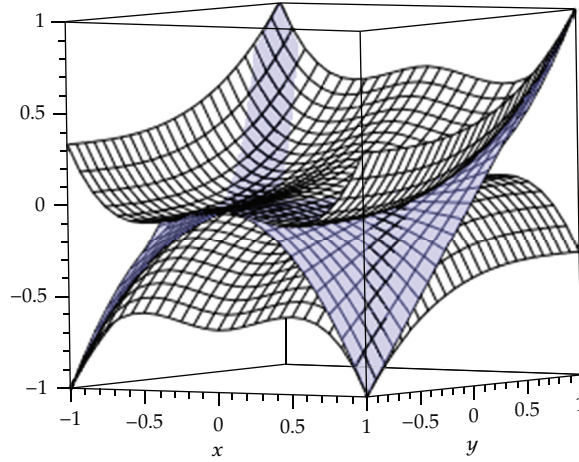


Figure 2: $f(x, y) = x^2 y$, $h_f^*(x, y)$ and $h_{*f}(x, y)$.

(B) Transfinite Cubature Formulae on \mathbf{v}^* and \mathbf{v}_* with $B^{2,1}$ Blending Degree of Precision

Let $f \in C^{2,1}$. Then

$$\int_{I^2} f = 2 \int_{-1}^1 f(x, 2|x| - 1) dx - \frac{1}{3} D^{2,1} f(\rho_1, \sigma_1), (\rho_1, \sigma_1) \in I^2, \quad (10.31)$$

$$\int_{I^2} f = 2 \int_{-1}^1 f(x, 1 - 2|x|) dx + \frac{1}{3} D^{2,1} f(\rho_2, \sigma_2), (\rho_2, \sigma_2) \in I^2.$$

Example 10.8. Consider $f(x, y) = x^2 y$, $D^{2,1} f(x, y) = 2 > 0$ on I^2 . Then, by Theorem 10.7, the $B^{2,1}$ transfinite blending interpolant

$$h_f^*(x, y) := \frac{4}{3} x^2 |x| - x^2 + \frac{1}{12} (1 + y)^3 \quad (10.32)$$

to f is the unique best one-sided from above L_1 -approximant from $B^{2,1}$ to $f(x, y) = x^2 y$. Analogously, the $B^{2,1}$ transfinite interpolant

$$h_{*f}(x, y) := -\frac{4}{3} x^2 |x| + x^2 - \frac{1}{12} (1 + y)^3 \quad (10.33)$$

to f is the unique best one-sided from below L_1 -approximant from $B^{2,1}$ to $f(x, y) = x^2 y$ (see Figure 2). Note that the best one-sided approximants to the polynomial $f(x, y) = x^2 y$ are not polynomials. Moreover, they have a limited $C^{2,1}$ -smoothness, contrary to the unconstrained best L_1 -approximation by blending functions (see Theorem 5.1).

10.3. Best One-Sided L_1 -Approximation by $B^{2,2}$ -Blending Functions

Let $f \in C^{2,2}$ and $D^{2,2}f(x, y) \geq 0$ for $(x, y) \in I^2$. Then (see [28] for details), the unique best one-sided from above L_1 -approximant h_f^* to f from $B^{2,2}$ is the unique transfinite Lagrange-Hermite interpolant to f on the canonical grid

$$\mathbf{x} := \{(x, y) \in I^2 : |x| = |y|\} \quad (10.34)$$

satisfying the transfinite interpolation conditions

$$\left(h_f^*\right)_{|\mathbf{x}} = f_{|\mathbf{x}}, \quad \left(\text{grad } h_f^*\right)_{|\mathbf{x}} = (\text{grad } f)_{|\mathbf{x}}. \quad (10.35)$$

For $f(x, y)$ even with respect to one of the variables x or y , the unique best one-sided from below L_1 -approximant h_{*f} to f from $B^{2,2}$ is the unique transfinite Lagrange-Hermite interpolant to f on the canonical grid

$$\diamond := \{(x, y) \in I^2 : |x| + |y| = 1\} \quad (10.36)$$

satisfying the transfinite interpolation conditions

$$\left(h_{*f}\right)_{|\diamond} = f_{|\diamond}, \quad \left(\text{grad } h_{*f}\right)_{|\diamond} = (\text{grad } f)_{|\diamond}. \quad (10.37)$$

Surprisingly, there is no universal canonical grid for the entire convex cone

$$\left\{f \in C^{2,2}, D^{2,2}f(x, y) \geq 0, (x, y) \in I^2\right\} \quad (10.38)$$

concerning the best one-sided from below L_1 -approximation from $B^{2,2}$ (see [28] for details).

The best one-sided from above L_1 -approximant to f has the smoothness of f . The best one-sided from below L_1 -approximant to f (if it exists) is a *blending transfinite spline function with two line-segment knots* $\{(x, 0) : x \in I\}$ and $\{(0, y) : y \in I\}$.

Example 10.9. Consider $f(x, y) = x^2y^2$, $D^{2,2}f(x, y) = 4 > 0$. Then $h_f^*(x, y) := (x^4 + y^4)/2$ is the unique best one-sided from above L_1 -approximant to $f(x, y) = x^2y^2$ from $B^{2,2}$ and

$$h_{*f}(x, y) := \frac{1}{2}(x^4 + y^4) - \frac{4}{3}(|x|^3 + |y|^3) + x^2 + y^2 - \frac{1}{6} \quad (10.39)$$

is the unique best one-sided from below L_1 -approximant to $f(x, y) = x^2y^2$ from $B^{2,2}$ (see Figure 3). Note that $h_{*f} \notin C^{3,0} \cup C^{0,3}$. Hence, the best one-sided from below L_1 -approximant from $B^{2,2}$ to f does not inherit the smoothness of f .

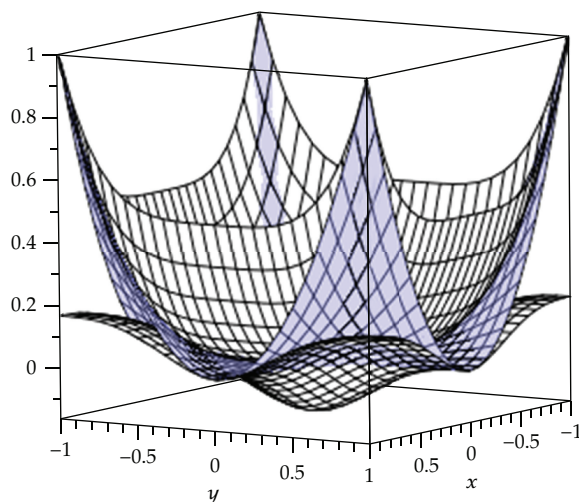


Figure 3: $f(x, y) = x^2 y^2$, $h_f^*(x, y)$ and $h_{*f}(x, y)$.

11. Best One-Sided L_1 -Approximation by Quasi-Blending Functions

The characterization result of best one-sided L_1 -approximation by algebraic $B^{2,1}$ -blending functions (see Theorem 10.7) has a natural extension to approximation by quasi-blending functions (see [29] for details). Define the space $QB^{m,1}$ of all quasi-blending functions of order $(m, 1)$ by

$$QB^{m,1} := \left\{ h \in C^{0,1}(I^2) : D^{0,1}h(\cdot, y) \in \pi_{m-1} \text{ for a fixed } y \in I \right\}, \quad (11.1)$$

where π_{m-1} is the space of all univariate algebraic polynomials of degree not exceeding $m-1$. Any $h \in QB^{m,1}$ can be represented in the form

$$h(x, y) = b(x) + \sum_{\mu=0}^{m-1} a_{\mu}(y) x^{\mu}, \quad (x, y) \in I^2, \quad (11.2)$$

with $a_0, a_1, \dots, a_{m-1} \in C^1(I)$ and $b \in C^0(I)$. Obviously, this representation is not unique. The canonical sets \mathbf{v}_* and \mathbf{v}^* for the best L_1 -approximation by $B^{2,1}$ -blending functions (see Theorem 10.7) are examples of the so-called m -oscillating point sets.

Given $m+1$ points $-1 = x_0 < x_1 < \dots < x_m = 1$. Let $\eta \in \{-1, 1\}$. A continuous function $\varphi : I \rightarrow I$ is called m -oscillating if

- (i) $\varphi(x_{\mu}) = \eta(-1)^{\mu}$, $0 \leq \mu \leq m$;
- (ii) The restriction $\varphi_{\mu} := \varphi|_{[x_{\mu-1}, x_{\mu}]}$ is a homeomorphism between $[x_{\mu-1}, x_{\mu}]$ and I .

A point set $\Gamma \in I^2$ is called m -oscillating if it is the graph of an m -oscillating function. The m -oscillating point sets are unisolvent for the following transfinite interpolation by quasi-blending functions.

Let $m \in \mathbf{N}$, $f \in C^{m,1}(I^2)$, and Γ be an m -oscillating point set associated with a given m -oscillating function φ . Then, there exists a unique quasi-blending function $h \in QB^{m,1}$ satisfying the following transfinite interpolation conditions:

$$h|_{\Gamma} = f|_{\Gamma}, \quad \left(D^{0,1}h\right)|_{\Gamma} = \left(D^{0,1}f\right)|_{\Gamma}. \quad (11.3)$$

We define two m -oscillating point sets which turn out to be canonical sets of best L_1 -approximation by quasi-blending functions from $QB^{m,1}$. Given $m \in \mathbf{N}$ and $x, s \in (-1, 1)$, consider the functions

$$q_m(x, s) := \begin{cases} P_k^{((1+s)/2, -(1+s)/2)}(x)P_k^{((1+s)/2, -(1+s)/2)}(-x), & \text{if } m = 2k, \\ P_k^{(-(1-s)/2, -(1+s)/2)}(x)P_k^{((1-s)/2, (1+s)/2)}(x), & \text{if } m = 2k + 1, \end{cases} \quad (11.4)$$

where $P_n^{(\alpha, \beta)}$ denote the corresponding Jacobi polynomials of degree n . Each polynomial $q_m(\cdot, s)$ has m distinct zeros $-1 < \varphi_1(s) < \dots < \varphi_m(s) < 1$. Moreover, it can be shown (see [29] for details) that the inverse functions $\varphi_\mu := \varphi_\mu^{-1} : [x_{\mu-1}, x_\mu] \rightarrow I$, $1 \leq \mu \leq m$ exist, are continuously differentiable, and join together to an m -oscillating function $\varphi : I \rightarrow I$ such that the quadrature formulae

$$\int_{I^2} f(x, y) \, dx \, dy \approx 2 \int_{-1}^1 f(s, \varphi(s)) \, ds; \quad \int_{I^2} f(x, y) \, dx \, dy \approx 2 \int_{I^2} f(s, -\varphi(s)) \, ds \quad (11.5)$$

are exact in the space $QB^{m,1}$. Let the m -oscillating point sets Γ_m and L_m be given by

$$\begin{aligned} \Gamma_m &:= \left\{ (x, (-1)^m \varphi(x)) \in I^2 : x \in I \right\}, \\ L_m &:= \left\{ (x, (-1)^{m+1} \varphi(x)) \in I^2 : x \in I \right\}. \end{aligned} \quad (11.6)$$

The next theorem shows that the sets Γ_m and L_m are canonical sets of best one-sided L_1 -approximation from $QB^{m,1}$.

Theorem 11.1. *Let $f \in C^{m,1}(I^2)$ satisfy $D^{m,1}f \geq 0$ on I^2 . Then the following holds.*

- (a) *The function f possesses a unique best one-sided L_1 -approximant from above from $QB^{m,1}$. The best one-sided from above approximant h_f^* is characterized by the following Lagrange-Hermite transfinite interpolation conditions:*

$$(i) \quad \left(h_f^*\right)|_{\Gamma_m} = f|_{\Gamma_m}, \quad (ii) \quad \left(D^{0,1}h_f^*\right)|_{\Gamma_m} = \left(D^{0,1}f\right)|_{\Gamma_m}. \quad (11.7)$$

- (b) *The unique best one-sided from below L_1 -approximant h_{*f} to f from $QB^{m,1}$ is characterized by the following Lagrange-Hermite transfinite interpolation conditions:*

$$(i) \quad \left(h_{*f}\right)|_{L_m} = f|_{L_m}, \quad (ii) \quad \left(D^{0,1}h_{*f}\right)|_{L_m} = \left(D^{0,1}f\right)|_{L_m}. \quad (11.8)$$

12. Best One-Sided L_1 -Approximation by Sums of Univariate Functions

Here we present a multivariate extension of Theorem 10.4 (see [30] for details). Let $d \geq 2$ and $C^{1,d}$ be the linear space of all differentiable functions f , defined on the d -dimensional cube $I^d := [-1, 1]^d$ and having continuous mixed derivatives

$$D_{i,j}^{1,1} f := \frac{\partial^2 f}{\partial x_i \partial x_j}, \quad 1 \leq i < j \leq d. \quad (12.1)$$

We denote by $B^{1,d}$ the subspace of all d -variable functions h which are sums of univariate ones, that is,

$$B^{1,d} := \left\{ h \in C^{1,d} : h(\mathbf{x}) = h(x_1, \dots, x_d) := \sum_{i=1}^d h_i(x_i) \right\}. \quad (12.2)$$

or equivalently

$$B^{1,d} = \left\{ h \in C^{1,d} : D_{i,j}^{1,1} h = 0, \quad 1 \leq i < j \leq d \right\}. \quad (12.3)$$

Let $\Delta^* := \{(t, \dots, t) \in I^d : t \in [-1, 1]\}$ be the *main* diagonal of the d -dimensional cube I^d . Let

$$\text{grad } f := \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_d} \right) \quad (12.4)$$

denote the gradient of the d -variable function f .

12.1. Transfinite Lagrange-Hermite Interpolation to a Function $f \in C^{1,d}$ from $B^{1,d}$ on the Diagonal Δ^* of I^d

Theorem 12.1. *Let $f \in C^{1,d}$. Then we have the following*

(a) *The function $h_f^* \in B^{1,d}$, where*

$$h_f^*(x_1, \dots, x_d) := f(-1, \dots, -1) + \sum_{i=1}^d \int_{-1}^{x_i} \frac{\partial f}{\partial x_i}(t, \dots, t) dt \quad (12.5)$$

is the unique transfinite interpolant to f from $B^{1,d}$ satisfying the following Lagrange-Hermite transfinite interpolation conditions:

$$\left(h_f^* \right)_{|\Delta^*} = f_{|\Delta^*}, \quad \left(\text{grad } h_f^* \right)_{|\Delta^*} = \left(\text{grad } f \right)_{|\Delta^*}. \quad (12.6)$$

(b) The following error representation formula holds

$$f(\mathbf{x}) - h_f^*(\mathbf{x}) = - \sum_{1 \leq i < j \leq d} \frac{\partial^2 f}{\partial x_i \partial x_j}(\xi_{i,j}) \frac{(x_i - x_j)^2}{2}, \quad (12.7)$$

where $\mathbf{x} = (x_1, \dots, x_d) \in I^d$ and $\xi_{i,j} \in I^d$.

Proof. Proof of (a). Let $\mathbf{x} = (x, \dots, x) \in \Delta^*$. Then $(\partial h_f^* / \partial x_i)(\mathbf{x}) = (\partial f / \partial x_i)(x, \dots, x) = (\partial f / \partial x_i)(x)$ and, from here, $(\text{grad } h_f^*)|_{\Delta^*} = (\text{grad } f)|_{\Delta^*}$.

Proof of (b). Let $\mathbf{x} = (x_1, \dots, x_d) \in I^d$. Without any restriction (with a permutation of the variables if necessary) we suppose $x_1 \leq x_2 \leq \dots \leq x_d$. Consider the auxiliary function $g(\mathbf{x}) = f(\mathbf{x}) - h_f^*(\mathbf{x})$. Then by (12.5), $g|_{\Delta^*} = 0$ and $(\text{grad } g)|_{\Delta^*} = 0$, and in view of this, for each i , $1 \leq i \leq d$

$$\begin{aligned} \frac{\partial g}{\partial x_i}(\mathbf{x}) &= \frac{\partial g}{\partial x_i}(x_1, \dots, x_d) - \frac{\partial g}{\partial x_i}(x_i, \dots, x_i) \\ &= - \sum_{j=1, j \neq i}^d (x_i - x_j) \int_0^1 \frac{\partial^2 g}{\partial x_i \partial x_j} [x_1 + s(x_i - x_1), \dots, x_d + s(x_i - x_d)] ds. \end{aligned} \quad (12.8)$$

Denote $\mathbf{x}_k := (x_k, \dots, x_k, x_{k+1}, \dots, x_d)$, $k = 1, \dots, d$, $\mathbf{x}_1 = \mathbf{x} = (x_1, \dots, x_d)$, $\mathbf{x}_d = (x_d, \dots, x_d)$. By using the representation (12.8) for $\partial g / \partial x_i$ we obtain

$$\begin{aligned} g(\mathbf{x}) &= \sum_{k=1}^{d-1} [g(\mathbf{x}_k) - g(\mathbf{x}_{k+1})] = - \sum_{k=1}^{d-1} \int_{x_k}^{x_{k+1}} \frac{d}{dt} g(t, \dots, t, x_{k+1}, \dots, x_d) dt \\ &= - \sum_{k=1}^{d-1} \int_{x_k}^{x_{k+1}} \sum_{i=1}^k \frac{\partial g}{\partial x_i}(t, \dots, t, x_{k+1}, \dots, x_d) dt \\ &= \sum_{k=1}^{d-1} \sum_{i=1}^k \sum_{j=k+1}^d \int_{x_k}^{x_{k+1}} (t - x_j) \int_0^1 \frac{\partial^2 g}{\partial x_i \partial x_j}(\mathbf{x}_{s,t,k}) ds dt, \end{aligned} \quad (12.9)$$

where $\mathbf{x}_{s,t,k} := (t, \dots, t, x_{k+1} + s(t - x_{k+1}), \dots, x_d + s(t - x_d))$. Changing twice the order of summation in (12.9) and applying the integral mean value theorem we obtain

$$\begin{aligned}
g(\mathbf{x}) &= \sum_{i=1}^{d-1} \sum_{k=i}^{d-1} \sum_{j=k+1}^d \int_{x_k}^{x_{k+1}} (t - x_j) \int_0^1 \frac{\partial^2 g}{\partial x_i \partial x_j}(\mathbf{x}_{s,t,k}) ds dt \\
&= \sum_{i=1}^{d-1} \sum_{j=i+1}^d \sum_{k=i}^{j-1} \int_{x_k}^{x_{k+1}} (t - x_j) \int_0^1 \frac{\partial^2 g}{\partial x_i \partial x_j}(\mathbf{x}_{s,t,k}) ds dt \\
&= \sum_{i=1}^{d-1} \sum_{j=i+1}^d \sum_{k=i}^{j-1} \left[\frac{(x_{k+1} - x_j)^2}{2} - \frac{(x_k - x_j)^2}{2} \right] D_{i,j}^{1,1} g(\mathbf{x}_{k,j}) \\
&= - \sum_{i=1}^{d-1} \sum_{j=i+1}^d D_{i,j}^{1,1} g(\xi_{i,j}) \frac{(x_\xi - x_j)^2}{2}, \quad \xi_{i,j} \in I^d, \quad 1 \leq i < j \leq d.
\end{aligned} \tag{12.10}$$

This completes the proof. \square

12.2. Transfinite Cubature Formula, Exact in the Linear Space $B^{1,d}$

Let $h \in B^{1,d}$. Then the following transfinite cubature formula holds:

$$\int \cdots \int_{I^d} h(x_1, \dots, x_d) dx_1 \cdots dx_d = 2^{d-1} \int_{\Delta^*} h(t, \dots, t) dt. \tag{12.11}$$

Proof. Integrate (12.7) on I^d for $f = h \in B^{1,d}$, taking into account that $h_h^* = h$ if $h \in B^{1,d}$ and the representation (12.5) of h_h^* . \square

The next theorem gives a canonical set characterization of the best one-sided from above L_1 -approximant from $B^{1,d}$ to a function, belonging to the convex cone

$$\mathcal{C}^{+,d} := \left\{ f \in C^{1,d} : D_{i,j}^{1,1} f \geq 0 \text{ on } I^2; \quad 1 \leq i < j \leq d \right\}. \tag{12.12}$$

Theorem 12.2. *Let $f \in \mathcal{C}^{+,d}$. Then the unique transfinite Lagrange-Hermite interpolant h_f^* from $B^{1,d}$ to f satisfying the transfinite interpolation conditions*

$$\left(h_f^* \right)_{|\Delta^*} = f_{|\Delta^*}, \quad \left(\text{grad } h_f^* \right)_{|\Delta^*} = \left(\text{grad } f \right)_{|\Delta^*} \tag{12.13}$$

is the unique best one-sided from above L_1 -approximant to f from $B^{1,d}$.

Proof. By (12.7) we conclude that $f(\mathbf{x}) \leq h_f^*(\mathbf{x})$, $\mathbf{x} \in I^d$. Taking into account the cubature (12.11) we conclude by Theorem 10.1 that the transfinite interpolant h_f^* to f is the best one-sided from above L_1 -approximant to $f \in \mathcal{C}^{+,d}$ from $B^{1,d}$.

Uniqueness of the Best One-Sided from above L_1 -Approximant from $B^{1,d}$ to $\mathcal{C}^{+,d}$.

Suppose $h \in B^{1,d}$, $h \geq f$ on I^d is another best one-sided from above L_1 -approximant to $f \in C^{+,d}$ from $B^{1,d}$. Then, by using (12.6) and (12.11) we obtain

$$\begin{aligned} 0 &= \int_{I^d} (h_f^* - f) - \int_{I^d} (h - f) = \int_{I^d} (h_f^* - h) \\ &= 2^{d-1} \int_{\Delta^*} (h_f^* - h) = 2^{d-1} \int_{\Delta^*} (f - h) \leq 0. \end{aligned} \tag{12.14}$$

Hence, $h|_{\Delta^*} = f|_{\Delta^*}$. However, $h \geq f$ on I^d and from here, $(grad\ h)|_{\Delta^*} = (grad\ f)|_{\Delta^*}$. By Theorem 12.1 we conclude that $h = h_f^*$. The proof is completed. \square

Remark 12.3. According to Theorem 12.2, the set Δ^* is the canonical point set of best one-sided from above L_1 -approximation from $B^{1,d}$ to the convex cone $C^{+,d}$.

Denote by E^d the set

$$E^d := \{\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_d), \varepsilon_i = \pm 1, i = 1, \dots, d\} \tag{12.15}$$

of all d -dimensional vectors with entries ± 1 . For a fixed $\boldsymbol{\varepsilon} \in E^d$ consider the $\boldsymbol{\varepsilon}$ -diagonal of I^d : $\Delta_{\boldsymbol{\varepsilon}} := \{(\varepsilon_1 t, \dots, \varepsilon_d t) : t \in [-1, 1]\}$. By using the linear transformation $u_i = \varepsilon_i x_i, i = 1, \dots, d$, the following corollary by Theorems 12.1 and 12.2 holds.

Corollary 12.4. *Let, for a fixed vector $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_d) \in E^d$, the function $f \in C^{1,d}$ satisfy*

$$\varepsilon_i \varepsilon_j D_{i,j}^{1,1} f \geq 0 \text{ on } I^d, \quad 1 \leq i < j \leq d. \tag{12.16}$$

Then the unique Lagrange-Hermite interpolant

$$h_{\boldsymbol{\varepsilon},f}^*(x_1, \dots, x_d) = f(-\varepsilon_1, \dots, -\varepsilon_d) + \sum_{i=1}^d \varepsilon_i \int_{-1}^{\varepsilon_i x_i} \frac{\partial f}{\partial x_i}(\varepsilon_1 t, \dots, \varepsilon_d t) dt \tag{12.17}$$

from $B^{1,d}$ to f , satisfying the transfinite interpolation conditions

$$\left(h_{\boldsymbol{\varepsilon},f}^*\right)_{|\Delta_{\boldsymbol{\varepsilon}}} = f|_{\Delta_{\boldsymbol{\varepsilon}}}, \quad \left(\text{grad } h_{\boldsymbol{\varepsilon},f}^*\right)_{|\Delta_{\boldsymbol{\varepsilon}}} = (\text{grad } f)_{|\Delta_{\boldsymbol{\varepsilon}}}, \tag{12.18}$$

is the unique best one-sided from above L_1 -approximant to f from $B^{1,d}$.

Remark 12.5. Theorems 12.1 and 12.2 are common basis for well-known classical and new inequalities. For example, the inequalities 9, 13, 16, 25, and 61 published in [31] are corollaries from the explicit constructions (12.5) and (12.7) of the best one-sided from above L_1 -approximants to appropriately chosen functions (see [30] for details). More precisely, the right-hand side expressions of these inequalities are best one-sided from above L_1 -approximants to the left-hand side ones in the sense of Theorems 12.1 and 12.2. We give some examples.

For a set of d numbers $\{x_1, \dots, x_d\}$ denote $x_* := \min\{x_1, \dots, x_d\}$, $x^* := \max\{x_1, \dots, x_d\}$.

Example 12.6. Consider $f(x_1, \dots, x_d) = x_1^{p_1} \cdots x_d^{p_d}$, where $x_i > 0, p_i > 0, i = 1, \dots, d$ and $\sum_{i=1}^d p_i = 1$. We compute

$$\frac{\partial f}{\partial x_i}(\mathbf{x}) = p_i x_i^{p_i-1} \prod_{j=1, j \neq i}^d x_j^{p_j}; \quad \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}) = p_i x_i^{p_i-1} p_j x_j^{p_j-1} \prod_{k=1, k \neq i, j}^d x_k^{p_k} > 0 \quad (12.19)$$

for $x_i > 0, i = 1, \dots, d$. Then

$$\begin{aligned} h_f^*(x_1, \dots, x_d) &= f(x_*, \dots, x_*) + \sum_{i=1}^d \int_{x_*}^{x_i} \frac{\partial f}{\partial x_i}(t, \dots, t) dt \\ &= \sum_{i=1}^d \int_0^{x_i} p_i t^{p_i-1} \prod_{j=1, j \neq i}^d t^{p_j} dt \\ &= \sum_{i=1}^d \int_0^{x_i} p_i dt = \sum_{i=1}^d p_i x_i \end{aligned} \quad (12.20)$$

is the unique best one-sided from above L_1 -approximant to f from $B^{1,d}$ on $[x_*, x^*]^d$. Hence,

$$x_1^{p_1} \cdots x_d^{p_d} \leq p_1 x_1 + \cdots + p_d x_d \quad (12.21)$$

with the case of equality only for $x_1 = \cdots = x_d$. The inequality is easily extended to $x_i \geq 0, i = 1, \dots, d$. This is the well-known inequality between the geometric mean and the arithmetic mean of a set of nonnegative numbers.

Example 12.7. Let $f(t)$ be a univariate function satisfying $f'(t) + t f''(t) > 0$ for $t > 0$. Then, for the d -variable function $f(x_1 \cdots x_d), x_i > 0, i = 1, \dots, d$ we compute

$$\begin{aligned} \frac{\partial f}{\partial x_i}(\mathbf{x}) &= \left[\prod_{j=1, j \neq i}^d x_j \right] f'(x_1 \cdots x_d) \\ \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}) &= \left[\prod_{k=1, k \neq i, j}^d x_k \right] f'(x_1 \cdots x_d) + \left[\prod_{k=1, k \neq i}^d x_k \right] \left[\prod_{s=1, s \neq j}^d x_s \right] f''(x_1 \cdots x_d) \\ &= \left[\prod_{k=1, k \neq i, j}^d x_k \right] (f'(x_1 \cdots x_d) + (x_1 \cdots x_d) f''(x_1 \cdots x_d)) > 0 \end{aligned} \quad (12.22)$$

for $x_i > 0$, $i = 1, \dots, d$. In view of Theorems 12.1 and 12.2 the unique transfinite interpolant to f from $B^{1,d}$ on $[x_*, x^*]^2$

$$\begin{aligned} h_f^*(x_1, \dots, x_d) &= f(x_*) + \sum_{i=1}^d \int_{x^*}^{x_i} \frac{\partial f}{\partial x_i}(t, \dots, t) dt \\ &= f(x_*) + \sum_{i=1}^d \int_{x^*}^{x_i} t^{d-1} f'(t^d) dt = \frac{1}{d} \sum_{i=1}^d f(x_i^d) \end{aligned} \quad (12.23)$$

is the unique best one-sided from above L_1 -approximant to f from $B^{1,d}$ on $[x_*, x^*]^d$. Hence,

$$f(x_1 \cdots x_d) \leq \frac{1}{d} \sum_{i=1}^d f(x_i^d) \quad (12.24)$$

with the case of equality only for $x_1 = \cdots = x_d$.

Example 12.8. Let $f(t) = (\beta + t)^\alpha$, $\beta \geq 0$, $\alpha > 0$, $t > 0$. We compute

$$f'(t) + t f''(t) = (\beta + t)^{\alpha-2} \alpha (\beta + \alpha t) > 0. \quad (12.25)$$

Then as a corollary by Example 12.7, we obtain the inequality

$$(\beta + x_1 \cdots x_d)^\alpha \leq \frac{1}{d} \sum_{i=1}^d (\beta + x_i^d)^\alpha, \quad x_i > 0, \quad i = 1, \dots, d \quad (12.26)$$

with the case of equality only for $x_1 = \cdots = x_d$. Obviously, the inequality holds for $x_i \geq 0$, $i = 1, \dots, d$.

Example 12.9. Let $f(t) = \sin(t)$, $t \geq 0$. We compute $f'(t) + t f''(t) = \cos(t) - t \sin(t) > 0$ for $t \in (0, t_*)$, where $t_* \approx 0.8603336$ is the unique solution of the nonlinear equation $\cot(t) = t$, $t \in [0, \pi/2]$. Then, following Example 12.7, we obtain the inequality

$$\sin(x_1 \cdots x_d) \leq \frac{1}{d} \sum_{i=1}^d \sin(x_i^d), \quad 0 < x_i < t_*^{1/d}, \quad 1 \leq i \leq d \quad (12.27)$$

with the case of equality only for $x_1 = \cdots = x_d$. Obviously the inequality holds for $0 \leq x_i \leq t_*^{1/d}$, $1 \leq i \leq d$.

Example 12.10. Let the univariate function $f(t)$ satisfy $2f'(t) + t f''(t) > 0$ for $t > 0$. Consider the d -variable function

$$\tilde{f}(x_1, \dots, x_d) := \sum_{1 \leq i < j \leq d} f\left(\frac{1}{x_i + x_j}\right), \quad x_i > 0, \quad 1 \leq i \leq d. \quad (12.28)$$

Computing partial derivatives of \tilde{f} we obtain

$$\begin{aligned}\frac{\partial \tilde{f}}{\partial x_i} &= - \sum_{j=1, j \neq i}^d \frac{1}{(x_i + x_j)^2} f' \left(\frac{1}{x_i + x_j} \right) \\ \frac{\partial^2 \tilde{f}}{\partial x_i \partial x_j} &= \frac{2}{(x_i + x_j)^3} f' \left(\frac{1}{x_i + x_j} \right) + \frac{1}{(x_i + x_j)^4} f'' \left(\frac{1}{x_i + x_j} \right) \\ &= \frac{1}{(x_i + x_j)^3} \left[2f' \left(\frac{1}{x_i + x_j} \right) + \frac{1}{(x_i + x_j)} f'' \left(\frac{1}{x_i + x_j} \right) \right] > 0,\end{aligned}\tag{12.29}$$

where $1 \leq i < j \leq d$. In view of Theorems 12.1 and 12.2, the transfinite interpolant to \tilde{f} from $B^{1,d}$ on $[x_*, x_*]^d$

$$\begin{aligned}h_{\tilde{f}}^*(x_1, \dots, x_d) &= \tilde{f}(x_*, \dots, x_*) + \sum_{i=1}^d \frac{\partial \tilde{f}}{\partial x_i}(t, \dots, t) dt \\ &= \frac{d(d-1)}{2} f \left(\frac{1}{2x_*} \right) - (d-1) \sum_{i=1}^d \int_{x_*}^{x_i} \frac{1}{4t^2} f' \left(\frac{1}{2t} \right) dt \\ &= \frac{d-1}{2} \sum_{i=1}^d f \left(\frac{1}{2x_i} \right)\end{aligned}\tag{12.30}$$

is the unique best one-sided from above L_1 -approximant to \tilde{f} from $B^{1,d}$. Hence, the following inequality holds:

$$\sum_{1 \leq i < j \leq d} f \left(\frac{1}{x_i + x_j} \right) \leq \frac{d-1}{2} \sum_{i=1}^d f \left(\frac{1}{2x_i} \right)\tag{12.31}$$

with the case of equality only for $x_1 = \dots = x_d$. In the particular cases $f(t) = t$ and $f(t) = \ln(t)$ we obtain the inequalities

$$\sum_{1 \leq i < j \leq d} \frac{1}{x_i + x_j} \leq \frac{d-1}{4} \sum_{i=1}^d \frac{1}{x_i}; \quad \sum_{1 \leq i < j \leq d} \ln \frac{1}{x_i + x_j} \leq \frac{d-1}{2} \sum_{i=1}^d \ln \frac{1}{2x_i}\tag{12.32}$$

with the case of equalities only for $x_1 = \dots = x_d$.

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