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Low rank surrogates for polymorphic fields with application to fuzzy-stochastic partial differential equations

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Abstract

We consider a general form of fuzzy-stochastic PDEs depending on the interaction of probabilistic and non-probabilistic ("possibilistic") influences. Such a combined modelling of aleatoric and epistemic uncertainties for instance can be applied beneficially in an engineering context for real-world applications, where probabilistic modelling and expert knowledge has to be accounted for. We examine existence and well-definedness of polymorphic PDEs in appropriate function spaces. The fuzzy-stochastic dependence is described in a high-dimensional parameter space, thus easily leading to an exponential complexity in practical computations.

To aleviate this severe obstacle in practise, a compressed low-rank approximation of the problem formulation and the solution is derived. This is based on the Hierarchical Tucker format which is constructed with solution samples by a non-intrusive tensor reconstruction algorithm. The performance of the proposed model order reduction approach is demonstrated with two examples. One of these is the ubiquitous groundwater flow model with Karhunen-Loève coefficient field which is generalized by a fuzzy correlation length.

1 Introduction

Mathematical models for real-world problems often depend on uncertain parameters, for instance describing material properties, forces, boundaries or the geometry of an object. The uncertainty is called irreducible or aleatoric when it stems from an inherently random physical process. The respective parameters are typically represented in a probabilistic framework as random variables, fields, or processes. Opposite to this, reducible uncertainties originating in a lack of knowledge, imprecision, or vagueness regarding the described property are called epistemic [69, 63]. These uncertainties are quite often modelled within a non-probabilistic framework. Examples for such frameworks are the evidence theory by Dempster and Shafer [77], the random set theory [58] and possibility theory [82]. Furhermore, over the last decades there have been attempts to develop a general theory of uncertainty, see e.g. the theory of imprecise probabilities [79]. It is still a matter of debate in different scientific communities if the probability framework is sufficient for the treatment of epistemic uncertainties as discussed in [74, 28]. In addition to the debate about an adequate description of (different forms of) uncertainties, another challenge in the thriving field of Uncertainty Quantification (UQ) is how to cope with the often prohibitive computational cost in numerical simulations. It is well-known that the "curse of dimensionality" is introduced when the number of uncertain parameters gets large, which is complicated further in case that different uncertainties interact. When both probabilistic and non-probabilistic uncertainties are present, we denote this as *polymorphic uncertainties*. These challenges usually lead to the necessity to evaluate a huge number of solutions (*i.e.* samples) of the underlying mathematical model. As an alternative, given sufficient regularity of the parameter to solution map, surrogate models (exploiting regularity, sparsity or low-rank approximability) can be employed. These enable the efficient evaluation of samples as well as quantities of interest (QoI) depending on the (parametric) solution.

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In this paper, we are concerned with a combined possibilistic-probabilistic modelling in the context of PDEs. Prior work in this direction can be found in the fuzzy-stochastic uncertainty model presented in [83] for a diffusion equation, which is also analysed in [66] based on pure sampling propagation of the uncertainties. In [35] and [25], a surrogate response based on Artificial Neural Networks was used for fuzzy and fuzzy-stochastic propagation, respectively. A sparse grid response surface for pure fuzzy dependence was developed in [51].

As a model order reduction technique, we use the Hierarchical Tucker format in conjunction with a sample-based reconstruction algorithm, see [45] for an introduction to tensor calculus. With this non-intrusive approach, we investigate surrogate models based on separability properties of the mapping from paramater space to the QoI. Let $p_1, \dots p_d$ be the d parameters and imagine that the QoI φ is approximated well by

$$\tilde{\varphi}\left(p_1, \dots, p_d\right) = \prod_{k=1}^{d} f_k(p_k).$$

In this fully separated form, the computation of the absolute maximum is reduced to finding the absolute maximum of the d functions $\{f_k\}_{k\in\{1,\dots,d\}}$. Similarly, under the assumption that the parameters are independently distributed, the computation of the expectation simplifies to the multiplication of the d means of $\{f_k\}_{k\in\{1,\dots,d\}}$. Obviously, in practice it is not always possible to find this particular separability structure. Nonetheless, other forms of separability are found in various tensor formats, e.g., the Canonical Polyadic format [48], the Tucker format [21], the Tensor Train format [70] or the Hierarchical Tucker format [46, 42, 41]. These types of representations are called low-rank approximations, we refer to [52, 45] for a broader overview. The properties of (hierarchical) tensor formats are an active field of research. For numerical computations, they were shown to exhibit favourable properties in different application areas and can in particular be a viable approach to mitigate the curse of dimensionality in high-dimensional problems. Recent examples in UQ can *e.g.* be found in [60, 24, 18].

This paper is structured as follows: In Section 2, a brief introduction into possibility theory is given. Moreover, the challenge of interaction between probabilistic and possibilistic parameters is addressed for a simple mapping. The examination is extended to (parametric) polymorphic PDEs in Section 3. In Section 4, the hierarchical $\mathcal{H}$ tensor is introduced as a main tool to circumvent the curse of dimensionality for the representation of the parameter to solution map. Numerical examples illustrating the performance of the proposed approach are presented in Section 5.

2 Polymorphic uncertainty propagation

Uncertainty encompasses all situations in which the precise prediction is not possible. This may be due to the lack of knowledge which may be reducible by more measurements, or due to inherent properties like in quantum mechanics, where the uncertainty is irreducible since the measurement of an object changes the object itself. For problems in the engineering sciences, the sources of uncertainty may stem from inexact models, vagueness in linguistic descriptions of parameters, from insufficient data and natural phenomena such as weather.

A classical way to handle uncertainties is the probability theory build upon Kolmogorov's axioms [53] which define a probability measure. Assume we would like to describe the uncertainty of an parameter p with classical probability theory. The only knowledge about this parameter is that it lies in an interval $[a, b]$. Following Laplace's principle of insufficient reason, one would model the own belief by a uniform distribution $p \sim \mathcal{U}(a,b)$. This model of uncertainty has the advantage of an underlying mature theoretical and practical framework. As such it allows to determine statistical moments or to incorporate new information via Bayesian updating [12]. Therefore, this modelling of uncertainty is advisable in most cases und thus quite common in the engineering sciences [61, 56].

Nontheless, the uniform distribution is not an accurate model for *total ignorance*. If the only knowledge are the upper and lower bounds for the parameter p, the only valid statement about the mean value should be that it lies in this interval. In other words, any probability density bounded to the interval should be possible. If more evidence is provided the set of possible probability densities is further restricted. This may be achieved by e.g. applying evidence theory [22] or possibility theory [27] to provide upper and lower probability bounds. In fact, this is not the only way to characterise the set of admissible probability densities, see the theory of imprecise probabilities [79]. This kind of modelling of uncertainties is also found in engineering sciences in [11, 40, 49].

In the next section, we briefly introduce possibility theory and probability theory with the focus on uncertainty propagation.

2.1 Possibility and probability

In this section we present a type of uncertainity description based on probability or possibility theory. The latter may be used to describe uncertain events that are not of stochastic nature, see [28]. We denote a mixture of possibilistic and probabilistic uncertainty descriptions and their propagation as *polymorphic uncertainty propagation*. Here, the probabilistic and possibilistic frameworks will be realised based on *random variables* (Section 2.2) and *fuzzy sets* (Section 2.3).

In the probabilistic framework we consider some complete probability space $(\Omega, \Sigma, \mathbb{P})$ with sample domain Ω, sigma algebra $\Sigma \subset \mathcal{P}(\Omega)$ and probability measure $\mathbb{P}\colon \Sigma \to [0,1]$ that satisfies the *probability axioms*

1 $\mathbb{P}(\emptyset) = 0,\ \mathbb{P}(\Omega) = 1$

2 $\mathbb{P}(\cup_i A_i) = \sum_i \mathbb{P}(A_i)$ for mutually disjoint $A_i \in \Sigma$ for $i \in \mathbb{N}$.

In the possibilistic framework we consider a possibility space $(Z, \mathcal{A}, \mathrm{Pos})$. Here, $Z \neq \emptyset$ is the universal set, $\mathcal{A} = \mathcal{A}(Z) \subset \mathcal{P}(Z)$ is an *ample field* [19], in particular $\mathcal{A}$ is closed under arbitrary unions and intersections and under complementation in Z and $\mathrm{Pos}\colon \mathcal{A} \to [0,1]$ is called a (normal) *possibility measure* satisfying the *possibility axioms*

1 $\mathrm{Pos}(\emptyset) = 0,\ \mathrm{Pos}(Z) = 1$

2 $\mathrm{Pos}(\cup_i Z_i) = \sup_i \mathrm{Pos}(Z_i)$ for (uncountable many) $Z_i \in \mathcal{A}$.

The second condition marks a main difference of both concepts and results in several consequences. *E.g.* opposite to the probability measure in general, the possiblity measure is not self-dual since it only holds that

$$\begin{array}{rcrcll} \mathrm{Pos}(A) & + & \mathrm{Pos}(Z \setminus A) & \geq & 1, & \text{for all } A \in \mathcal{A}, \\ \mathbb{P}(A) & + & \mathbb{P}(\Omega \setminus A) & = & 1, & \text{for all } A \in \Sigma. \end{array} \tag{2.1}$$

A dual measure $\mathrm{Nec}\colon \mathcal{A} \to [0,1]$ called *necessity* is defined as $\mathrm{Nec}(\cdot) := 1 - \mathrm{Pos}(Z \setminus \cdot)$. This definition for example implies

$$\mathrm{Nec}(A) + \mathrm{Nec}(Z \setminus A) \leq 1 \quad \text{and} \quad \mathrm{Pos}(A) + \mathrm{Pos}(Z \setminus A) \geq 1,$$

instead of $P(A)+P(Z\setminus A) = 1$ in the probability framework. General concepts of possibility measure und necessity measures can be found in [26]. In [54] a self-dual measure called credibility measure $\mathrm{Cred}(\cdot) := 1/2(\mathrm{Pos}(\cdot) + \mathrm{Nec}(\cdot))$ was introduced and used for fuzzy optimization.

The possibility and necessity can be viewed as bounds for imprecise probabilities, see *e.g.* [27, 20]. Let $\Sigma \subset \mathcal{A}$ be a sigma algebra and $\mathcal{A}$ be an ample field. Then given a possibility measure $\mathrm{Pos}\colon \mathcal{A} \to [0,1]$, a set of measures $\mathcal{M}$ on Σ can be defined as

$$\mathcal{M} := \{\mathbb{P} \,|\, \forall A \in \Sigma : \mathrm{Nec}(A) \leq \mathbb{P}(A) \leq \mathrm{Pos}(A)\}. \tag{2.2}$$

Conversely, given a set $\mathcal{M}_1$ of probability measures on a sigma algebra Σ and a set $\mathcal{M}_2$ of possibility measures on an ample field $\mathcal{A}$ s.t. $\Sigma \subset \mathcal{A}$, then

$$\{\mathrm{Pos} \,|\, \forall A \in \Sigma : \mathrm{Pos}(A) = \sup_{\mathbb{P}\in\mathcal{M}_1} \mathbb{P}(A)\} \tag{2.3}$$

defines a non-empty subset of possibility measures in $\mathcal{M}_2$. The connection of possibility theory and the subsequent fuzzy arithmetic is stated in (2.8).

2.2 Random variables

Based on the complete probability space $(\Omega, \Sigma, \mathbb{P})$ in the probabilistic framework, uncertainties may be described by some abstract random variable

$$\boldsymbol{\xi}\colon \Omega \mapsto (E, \mathcal{B}(E)) \tag{2.4}$$

with measurable space $(E, \mathcal{B}(E))$ with the special case $E = \mathbb{R}^{M\leq\infty}$. Furthermore define $\Gamma := \operatorname{img}\boldsymbol{\xi}$, which we will utilize in the framework of parametric random PDEs in Section 3.2. Denote by $\mu := \boldsymbol{\xi}_{\#}\mathbb{P}$ the push-forward probability measure w.r.t. $\boldsymbol{\xi}$. Instead of the abstract probability space $(\Omega, \Sigma, \mathbb{P})$ we work in the image space $(\Gamma, \mathcal{B}(\Gamma), \mu)$ and identity $\boldsymbol{\xi}(\omega)$ with $\boldsymbol{y} = \boldsymbol{\xi}(\omega) \in \Gamma$. In the case that $\boldsymbol{\xi} = (\xi_i)$ consists of stochastic independent random variables ξ_i with $\Gamma_i := \operatorname{img}\xi_i$ and push-forward measure μ_i w.r.t. ξ_i, we can rewrite the image space based on

$$\Gamma = \bigtimes_{i=1}^{M} \Gamma_i, \quad \mu = \bigotimes_{i=1}^{M} \mu_i. \tag{2.5}$$

Then the propagation of $\boldsymbol{\xi}$ under a measurable map $f\colon \Gamma \to V$ for some set V defines a V-valued random variable $f_V(\omega) = f(\boldsymbol{\xi}(\omega))$ in a $\mathbb{P}$-*a.e* pathwise manner. If in addition $V \subset \mathbb{R}^N$ for $N \in \mathbb{N}$, we may characterise the propagation by means of the cumulative distribution function (CDF) of f_V, see Figure 1.

2.3 Fuzzy sets

We now introduce a possibilistic framework based on fuzzy sets, *e.g.* [81] or more recently [65, 62, 64]. This section will give an overview of the basic definitions of fuzzy sets and their propagation in Theorems 2.4 and 2.5.

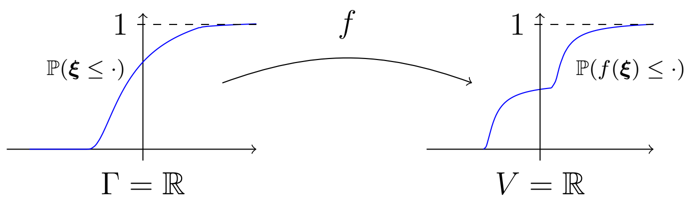


Figure 1: CDF propagation of a random variable.

Definition 2.1. (Fuzzy set/variable, α-cuts and interactivity)
Let $Z \neq \emptyset$ be a set and $\mu_{\tilde{z}}\colon Z \mapsto [0,1]$ be a map s.t. there exists $z \in Z$ with $\mu_{\tilde{z}}(z) = 1$. The map $\mu_{\tilde{z}}$ is called (normalized) membership function. We define a (normalised) fuzzy set $\tilde{z}$ on Z by

$$\tilde{z} := \{(z, \mu_{\tilde{z}}(z)) \mid z \in Z\}. \tag{2.6}$$

If $\mu_{\tilde{z}}(Z) = \{0,1\}$ with uniqe $z^* \in Z$ with $\mu_Z(z^*) = 1$ then $\tilde{z}$ is called a crisp set. Furthermore, we denote by $\mathcal{F}(Z)$ the set of all fuzzy sets on Z. Thus, we simply write $\tilde{z} \in \mathcal{F}(Z)$. If $Z \subset \mathbb{R}^n$ for $n \in \mathbb{N}$ we call $\tilde{z} \in \mathcal{F}(Z)$ a fuzzy variable $(n = 1)$ or vectorial fuzzy variable $(n > 1)$ described by a (joint) membership function $\mu_{\tilde{z}}(n > 1)$. Let $\alpha \in [0,1]$ then the α-cut C_α of $\mu_{\tilde{z}}$ is defined as

$$C_\alpha[\tilde{z}] := \{z \in Z : \mu_{\tilde{z}}(z) \geq \alpha\}. \tag{2.7}$$

The 0-cut $C_0[\tilde{z}]$ defines the so-called support of $\mu_{\tilde{z}}$. Let $\tilde{z}_i \in \mathcal{F}(Z_i)$ for sets Z_i with $i = 1, \ldots, M < \infty$, $Z := \bigtimes_{i=1}^M Z_i$ and $\tilde{z} = (\tilde{z}_i)_i$. If the joint membership function associated with $\tilde{z}$ has the form $\mu_{\tilde{z}} = \min_i \mu_{\tilde{z}_i}$ then $\tilde{z}$ is called non-interactive and interactive otherwise.

Given a membership function $\mu_{\tilde{z}}$, a possibility measure can be derived via

$$\operatorname{Pos}(A) := \sup_{z \in A} \mu_{\tilde{z}}(z), \quad \forall A \in \mathcal{A} = \mathcal{A}(Z). \tag{2.8}$$

This lays the grounds of possibility in terms of fuzzy sets in the spirit of [82].

For the numerical treatment of fuzzy sets it is convenient to require some properties which lead to a beneficial algebraic structure. We denote a function μ as *quasi-concave* on Z if for all $z_1, z_2 \in Z$ and all $\lambda \in [0,1]$ s.t. $\lambda z_1 + (1-\lambda) z_2 \in Z$ it holds

$$\mu(\lambda z_1 + (1-\lambda) z_2) \geq \min(\mu(z_1), \mu(z_2)). \tag{2.9}$$

Moreover, we call a function $\mu\colon Z \to [0,1]$ to be *upper semicontinuous* if

$$\limsup_{z \mapsto z_0} \leq \mu(z_0), \quad \forall z_0 \in Z. \tag{2.10}$$

Definition 2.2. (Fuzzy number/vector/interval/area)
Let $\tilde{z} \in \mathcal{F}(Z)$ with $Z \subset \mathbb{R}^n$ for some $n \in \mathbb{N}$ s.t. Z is bounded and convex and the (joint) membership function $\mu_{\tilde{z}}$ is semi-upper continuous and quasi-concave. If there exists a unique $z^* \in Z$ s.t. $\mu_{\tilde{z}}(z^*) = 1$ then we call $\tilde{z}$ a fuzzy number resp. fuzzy vector for $n = 1$ resp. $n > 1$. Otherwise, $\tilde{z}$ is called a fuzzy interval resp. fuzzy area for $n = 1$ resp. $n > 1$ if there exists more than one $z \in Z$ s.t. $\mu_{\tilde{z}}(z) = 1$.

For most practical cases a restriction to these particular fuzzy sets does not represent an impairment in modelling the epistemic uncertainties. We point out the nestedness property of α-cuts of the fuzzy structures from Definition 2.2, that is

$$C_\alpha[\tilde{z}] \subset C_\beta[\tilde{z}], \quad \forall \alpha \geq \beta. \tag{2.11}$$

This property is beneficial for the computation of propagation of these fuzzy structures, see Theorem 2.5.

Example 2.3. *A particular class of fuzzy numbers is given by so called LR-fuzzy numbers $\tilde{z}$ forming a subset of $\mathcal{F}(Z)$ with $Z \subset \mathbb{R}$. Here the membership function $\mu_{\tilde{z}}$ is described by two upper semicontinous functions $f_L\colon (-\infty, 0] \to [0,1]$ and $f_R\colon [0,\infty) \to [0,1]$ with $f_L(0) = f_R(0) = 1$ with f_L (f_R) being monotonously increasing (decreasing) and $\lim_{z\to-\infty} f_L(z) = 0$ ($\lim_{z\to\infty} f_R(z) = 0$) s.t. there exists $z^* \in Z$ with*

$$\mu_{\tilde{z}}(z) = \begin{cases} f_L(z^* - z), & z \le z^*, \\ f_R(z - z^*), & z \ge z^*, \end{cases} \tag{2.12}$$

and $\mu_{\tilde{z}}(z^*) = 1$. The most popular LR-fuzzy number is the triangle fuzzy number $\tilde{z} = \langle l, z^*, r\rangle$ specified by left and right limit l, r and peak position z^*, see Figure 2.

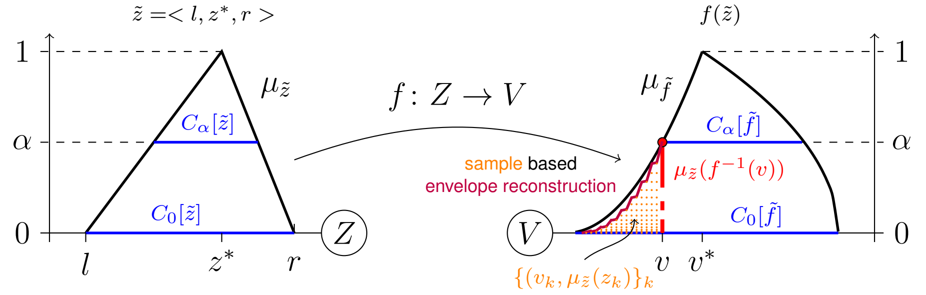


Figure 2: Fuzzy propagation via α-cuts or full-sampling and membership reconstruction with $v_k = f(z_k)$, for $Z = V = \mathbb{R}$.

The propagation of fuzzy sets through mappings can be realised with ZADEH's extension principle.

Theorem 2.4. *(ZADEH's extension principle [81])*
Consider a function $f\colon Z \mapsto V$ with a non-empty set V. Let $\tilde{z} \in \mathcal{F}(Z)$ with membership function $\mu_{\tilde{z}}$. Define

$$\tilde{f} := f(\tilde{z}) := \{(f(z), \mu_{\tilde{f}}(f(z)) \in V \times [0,1], z \in Z\} \tag{2.13}$$

with membership function $\mu_{\tilde{f}}$ defined as

$$\mu_{\tilde{f}}(v) := \begin{cases} \sup\limits_{z \in f^{-1}(v)} \mu_{\tilde{z}}(z) & f^{-1}(v) \neq \emptyset, \\ 0 & f^{-1} = \emptyset, \end{cases} \quad \text{for all } v \in V. \tag{2.14}$$

Then $\tilde{f} \in \mathcal{F}(V)$ with membership function $\mu_{\tilde{f}}$.

Within this framework the function f can be interpreted as map $f\colon \mathcal{F}(Z) \mapsto \mathcal{F}(V)$. If more underlying structure is assumed, the extension principle can be equivalently formulated based on constrained optimization of the map f itself. This approach follows from the following theorem.

Theorem 2.5. *(α-cut propagation [67])*
Let $f\colon Z \mapsto V$ be continuous between metric spaces (Z, d_1) and (V, d_2). Let $\tilde{z} \in \mathcal{F}(Z)$ with support $C_0[\tilde{z}] \subset K \subset Z$ for a compact set K with convex Z. Furthermore, let the membership function $\mu_{\tilde{z}}$ be quasi-concave and semi-upper continous. Then, $\mu_{\tilde{f}}$ can be characterised via α-cuts as

$$C_\alpha[\tilde{f}] = f(C_\alpha[\tilde{z}]). \tag{2.15}$$

Moreover, if $V = \mathbb{R}$ then

$$C_\alpha[\tilde{f}] = \left[\min_{z \in C_\alpha[\tilde{z}]} f(z), \max_{z \in C_\alpha[\tilde{z}]} f(z) \right]. \tag{2.16}$$

We note that (2.15) follows from the assumption that $C_\alpha[\tilde{z}]$ is closed and consequently compact as it is a closed subset of a compact set K in a metric space. Since f is assumed to be continous, then compact sets are mapped to compact sets.

The optimization required due to the extension principle in (2.14) easily becomes a computationally involved task. In this work *three* main concepts to realise the propagation are considered:

- *Semi sampling in V*: Directly solve the constrained optimization problem with a global optimizer. For a given sequence $(v_k)_k \subset V$, compute the supremum over $Z_k := \{z \in Z : f(z) = v_k\}$, see red line in Figure 2.
- *Full sampling approach*: Choose a sequence $(z_k)_k \subset Z$ and compute $(f_k)_k = [f(z_k)]_k$ and $(\mu_k)_k = [\mu_{\tilde{z}}(z_k)]_k$. Use the data sample pairs (v_k, μ_k) and reconstruct $\mu_{\tilde{f}}$, *e.g.* by convex hull or an envelope approach, see orange/purple graphics in Figure 2.
- *α-cut optimization*: Based on (2.16) in Theorem 2.5 with $V = \mathbb{R}$ for a given discretization $\alpha \in \{\alpha_1, \ldots, \alpha_l\} \subset [0, 1]$, compute $C_\alpha[\tilde{f}]$ and build $\mu_{\tilde{f}}$ based on interpolation between the obtained points, see blue graphics in Figure 2.

Each of these approaches may still be challenging, *e.g.* if f is a complex function with high evaluation cost, which is e.g. the case if the evaluation of f depends on a finite element solution of a complex physical model. It becomes mandatory to construct a less costly surrogate $f_h \approx f$ which is then used to perform the propagation. In this work, the hierachical tensor decompositions introduced in Section 4 are employed for this.

2.4 Polymorphic parametric maps

When both different kinds of uncertainty are present we denote this as *polymorphy*. In this section the potential interactions of possibility (via fuzzy sets) and probability are investigated from a parameteric view. For this, we separate the data parameters into random and fuzzy parameters. For a more general scenario involving fuzzy-fuzzy or random-random dependence we refer to Remark 2.11. The goal of this section is to define a fuzzy random variable denoted as $\tilde{\boldsymbol{p}}$ with realizations both in fuzzy and random coordinate space denoted as $\boldsymbol{p}$.

Let $Z = Z_1 \times Z_2$ with $Z_i \subset \mathbb{R}^{N_i}$, $N = N_1 + N_2$ and a fuzzy vector

$$\tilde{\boldsymbol{z}} = [\tilde{\boldsymbol{z}}_1, \tilde{\boldsymbol{z}}_2] \in \mathcal{F}(Z), \quad \mu_{\tilde{\boldsymbol{z}}} = \min(\mu_{\tilde{\boldsymbol{z}}_1}, \mu_{\tilde{\boldsymbol{z}}_2}). \tag{2.17}$$

In particular, we assume a non-interacting structure of $\tilde{\boldsymbol{z}}_1$ and $\tilde{\boldsymbol{z}}_2$. For $M \in \mathbb{N}$ with $M = M_1 + M_2$, consider a (realisation of a) fuzzy random variable of the form

$$\boldsymbol{\xi}(\boldsymbol{z}, \cdot) = [\boldsymbol{\xi}_1(\cdot), \boldsymbol{\xi}_2(\boldsymbol{z_2}, \cdot)] \colon (\Omega, \Sigma, \mathbb{P}) \to (E, \mathcal{B}(E)), \quad E = \mathbb{R}^M, \tag{2.18}$$

pathwise to be a random variable for $\boldsymbol{z} = [\boldsymbol{z}_1, \boldsymbol{z}_2] \in C_0[\tilde{\boldsymbol{z}}] \subset Z$. We assume $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}(\boldsymbol{z}_2)$ to be independent for all $\boldsymbol{z}_2 \in Z_2$. Then the pathwise image of $\boldsymbol{\xi}(\boldsymbol{z})$ is given by $\Gamma(\boldsymbol{z}) := \Gamma_1 \times \Gamma_2(\boldsymbol{z}_2)$ with

$$\Gamma_1 := \operatorname{img} \boldsymbol{\xi}_1 \subset \mathbb{R}^{M_1}, \quad \Gamma_2(\boldsymbol{z}_2) := \operatorname{img} \boldsymbol{\xi}_2(\boldsymbol{z}_2) \subset \mathbb{R}^{M_2}, \quad \forall \boldsymbol{z}_2 \in C_0[\tilde{\boldsymbol{z}}_2], \tag{2.19}$$

yielding an image identification $\boldsymbol{y}(\boldsymbol{z}) = [\boldsymbol{y}_1, \boldsymbol{y}_2(\boldsymbol{z}_2)] = \boldsymbol{\xi}(\boldsymbol{z}, \omega)$. We define an overall image Γ_2 and a graph W as

$$\Gamma_2 := \bigcup_{\boldsymbol{z}_2 \in C_0[\tilde{\boldsymbol{z}}_2]} \Gamma_2(\boldsymbol{z}_2), \tag{2.20}$$

$$W := \{(\boldsymbol{y}_2(\boldsymbol{z}_2), \boldsymbol{z}_2) \,|\, \boldsymbol{z}_2 \in Z_2\} \subset W^{\#} := \Gamma_2 \times C_0[\tilde{\boldsymbol{z}}_2]. \tag{2.21}$$

With this, either $\boldsymbol{p} = [\boldsymbol{y}_1, \boldsymbol{y}_2(\boldsymbol{z}_2), \boldsymbol{z}_1] \in \Gamma_1 \times \Gamma_2 \times Z_2$ or $\boldsymbol{p} = [\boldsymbol{y}_1, (\boldsymbol{y}_2(\boldsymbol{z}_2), \boldsymbol{z}_2), \boldsymbol{z}_1] \in \Gamma_1 \times W \times Z_2 \subset \Gamma_1 \times W^{\#} \times Z_2$. Either structure yields the definition of a slightly different polymorphic dependence map.

Definition 2.6. (V-valued fuzzy-random fields, Γ_2-case)
For a set V and measurable mapping

$$f \colon \Gamma_1 \times \Gamma_2 \times Z_1 \to V, \quad (\boldsymbol{y}_1, \overline{\boldsymbol{y}}_2, \boldsymbol{z}_1) \mapsto f(\boldsymbol{y}_1, \overline{\boldsymbol{y}}_2, \boldsymbol{z}_1), \tag{2.22}$$

a V-valued polymorphic field $g \colon \Omega \times \mathcal{F}(Z) \to \mathcal{F}(V)$ can be defined as

$$g(\omega, \tilde{\boldsymbol{z}}) = f(\boldsymbol{\xi}_1(\omega), \boldsymbol{\xi}_2(\tilde{\boldsymbol{z}}_2, \omega), \tilde{\boldsymbol{z}}_1). \tag{2.23}$$

Definition 2.7. (V-valued fuzzy-random fields, W-case)
For a set V and measurable mapping

$$f_W \colon \Gamma_1 \times W \times Z_1 \to V, \quad (\boldsymbol{y}_1, \boldsymbol{w}, \boldsymbol{z}_1) \mapsto f_W(\boldsymbol{y}_1, \boldsymbol{w}, \boldsymbol{z}_1), \tag{2.24}$$

a V-valued polymorphic field $g \colon \Omega \times \mathcal{F}(Z) \to \mathcal{F}(V)$ can be defined as

$$g(\omega, \tilde{\boldsymbol{z}}) = f(\boldsymbol{\xi}_1(\omega), (\boldsymbol{\xi}_2(\tilde{\boldsymbol{z}}_2, \omega), \tilde{\boldsymbol{z}}_2), \tilde{\boldsymbol{z}}_1). \tag{2.25}$$

Note that Definition 2.6 is a special case of Definition 2.7, considering a substructure given by a map $h \colon W \mapsto \Gamma_2$, such that $f_W(\boldsymbol{y}_1, \boldsymbol{w}, \boldsymbol{z}_1) = f(\boldsymbol{y}_1, h(\boldsymbol{w}), \boldsymbol{z}_1)$.

Remark 2.8. *In the spirit of the pure fuzzy case we can also interpret $h(\tilde{z}) = g(\cdot, \tilde{z})$ as a fuzzy set in $\mathcal{F}(V^{\Omega})$.*

Both dependency structures require their own surrogate design of the mapping f. In the Γ_2 case we can define the respective parameter independent of $\boldsymbol{z}_2$. However, there is a more involved parameter dependence in the W-case, see Figure 3 for an illustration of both cases. Additionally, the geometry of W can be arbitrarily complex.

In view of the embedding of W into the tensor domain $W^{\#}$, we may also consider a measurable map

$$f^{\#} \colon \Gamma_1 \times \Gamma_2 \times Z_2 \times Z_1 \to V, \quad (\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{z}_2, \boldsymbol{z}_1) \mapsto f^{\#}(\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{z}_2, \boldsymbol{z}_1), \tag{2.26}$$

such that

$$f \equiv f^{\#}, \quad \text{in } \Gamma_1 \times W \times Z_2. \tag{2.27}$$

Instead of generating a surrogate of f with respect to W, if possible it might be easier to construct $f^{\#}$ in the tensor domain instead. A special case occurs if W is a tensor domain, *i.e.* $W = W^{\#}$, as in the case for random variables with unbounded image, like normal or lognormal distributed fuzzy-random variables, see Figure 3.

An important class of fuzzy dependent random variables as in (2.18) is given in Example 2.9.

Example 2.9. Let a random vector $\boldsymbol{\xi}_2$ for a given parameter $\boldsymbol{z}_2$ be distributed as $\boldsymbol{\xi} \sim \mathcal{D}(\boldsymbol{z}_2)$. Such a random vector may be characterised by a density ρ, which itself depends on a parameter $\boldsymbol{z}_2 \in Z_2 \subset \mathbb{R}^K$ for $K \in \mathbb{N}$. Let $\tilde{\boldsymbol{z}}_2$ be a fuzzy set pathwise given as $\boldsymbol{z}_2 \in C_0[\tilde{\boldsymbol{z}}_2]$. Then $\boldsymbol{\xi}_2 = \boldsymbol{\xi}_2(\tilde{\boldsymbol{z}}_2)$ is a fuzzy random vector of the form (2.18). In Figure 3 two particular examples of a (bounded) uniform and lognormal scalar fuzzy random variable are illustrated.

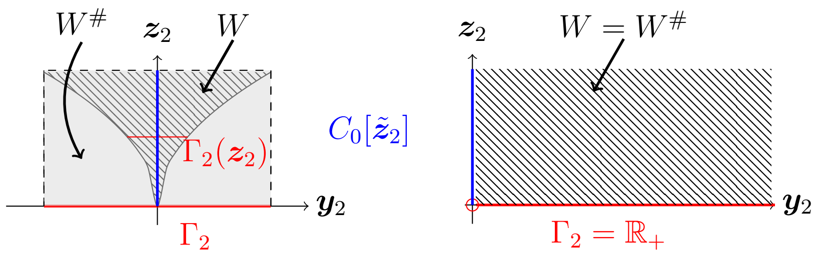


Figure 3: RVs with fuzzy dependence $\boldsymbol{z}_2 \in C_0[\tilde{\boldsymbol{z}}_2] = [0,1]$. Left: non-tensor image range for $\boldsymbol{\xi}_2 \sim \mathcal{U}(-\boldsymbol{z}_2^2, \boldsymbol{z}_2^2)$, Right: tensor domain image range for $\boldsymbol{\xi}_2 \sim \mathcal{LN}(\boldsymbol{z}_2, 1)$

In the case of $N_2 = M_2 = 0$ and recalling the special case that $\boldsymbol{\xi}_1 = (\xi_{1,i})_i$ are independent with images $\Gamma_{1,i}$ and $\tilde{\boldsymbol{z}}_1 = (\tilde{z}_{1,j})_j$ are non-interacting with supports in $Z_{1,j}$ for $i = 1, \ldots, M_1$, $j = 1, \ldots, N_1$, we are in the full tensor scenario

$$f \colon \left(\bigtimes_{i=1}^{M_1} \Gamma_{1,i} \right) \times \left(\bigtimes_{j=1}^{N_1} Z_{1,j} \right) \to V. \tag{2.28}$$

Example 2.10. *Let $\tilde{\boldsymbol{z}} \in \mathcal{F}(Z)$ with $Z \subset \mathbb{R}^n$ be a fuzzy vector and $D(\boldsymbol{z}) \subset \mathbb{R}^d$ be a domain indexed with $\boldsymbol{z} \in C_0 \subset Z$. Then $D(\tilde{z})$ defines a fuzzy set denoted as fuzzy domain, via* ZADEH*'s extension principle* (2.14)*.*

Remark 2.11. *We note that the general case of fuzzy sets depending on {random variables, fuzzy sets} and random variables depending on {random variables} is not discussed. The latter may occur in maximum-likelihood parameter estimations of random variables in a Bayesian framework. In this general case, let $\tilde{\boldsymbol{\theta}} = [\boldsymbol{\theta}_\Omega, \tilde{\boldsymbol{\theta}}_\mathcal{F}, c]$ be an array of random vector $\boldsymbol{\theta}_\Omega$, fuzzy set $\tilde{\boldsymbol{\theta}}_\mathcal{F}$ and crisp parameter c with pathwise realisation denoted as $\boldsymbol{\theta}$. Then, in the spirit of* (2.21) *we may write*

$$f \colon W \to V, \quad \boldsymbol{w} = (\boldsymbol{y}(\boldsymbol{\theta}), \boldsymbol{z}(\boldsymbol{\theta}), \boldsymbol{\theta}) \in W. \tag{2.29}$$

This describes the pathwise image $\boldsymbol{y}(\boldsymbol{\theta}) = \boldsymbol{\xi}(\boldsymbol{\theta}, \omega)$ with $\boldsymbol{\theta}$-dependent random variable $\boldsymbol{\xi}$ and a pathwise realisation $\boldsymbol{z}(\boldsymbol{\theta})$ of a $\boldsymbol{\theta}$-dependent random set $\tilde{z}(\boldsymbol{\theta})$. The definition of W is essential due to the different propagation structure introduced by possibilistic and probabilistic arithmetic. Since W can exhibit an arbitrary structure, a pathwise approach might be the only viable option. In particular, the construction of surrogates is not straightforward.

3 Polymorphic partial differential equations on polymorphic domains

We now apply the polymorphic framework developed in the preceding section to the modelling of uncertain data in differential equations. As common in the field of Uncertainty Quantification, we focus on

PDEs with uncertain data as the most relevant setting for engineering problems. While PDEs depending on a countable (infinite) sequence of random variables have been examined extensively, mainly due to the popular representation of random fields in terms of the Karhunen-Loève expansion, more general models such as polymorphic uncertainties are encountered far more scarcely. A recent work on fuzzy-stochastic PDEs is [66]. Polymorphic uncertainty models with many examples of practical interest were also described in [73, 25, 83]. A complementing discussion of PDEs with fuzzyness can be found in [17].

In what follows, we first define an abstract setting of parametric PDEs. Different possible settings with (a combination of) random variables and fuzzy sets are developed subsequently, leading to a rather generic notion of polymorphic PDEs.

3.1 Abstract parameteric partial differential equation model

In this section we present *parametric PDEs* as the foundation of *polymorphic PDEs*. Starting from an abstract equation (3.2) with pathwise solution $u(\boldsymbol{p})$, we successively assume more structure w.r.t. to the parameter itself and the parameter dependence of the operator, right-hand side, domain and solution space. This allows for an interpretation of a solution map $\boldsymbol{p} \to u(\boldsymbol{p})$ in fuzzy sets on BOCHNER spaces.

Let either $P := \Gamma_1 \times W \times Z_1$ or $P := \Gamma_1 \times \Gamma_2 \times Z_2$ be the parameter domain used for a polymorphic input description motivated by Section 2.4. Fix $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{y}(\boldsymbol{z}), \boldsymbol{z}) \in P$,

$$\boldsymbol{p} = [\boldsymbol{y}_1, (\boldsymbol{y}_2(\boldsymbol{z}_2), \boldsymbol{z}_2), \boldsymbol{z}_1] \quad \text{or} \quad \boldsymbol{p} = [\boldsymbol{y}_1, \boldsymbol{y}_2(\boldsymbol{z}_{\boldsymbol{2}}), \boldsymbol{z}_{\boldsymbol{1}})], \tag{3.1}$$

in its image representing a realisation of $[\boldsymbol{\xi}_1, (\boldsymbol{\xi}_2(\boldsymbol{z}_2), \boldsymbol{z}_2), \boldsymbol{z}_1]$ or $[\boldsymbol{\xi}_1, \boldsymbol{\xi}_2(\boldsymbol{z}_2), \boldsymbol{z}_1]$. By this identification and an abuse of notation, we henceforth let $\boldsymbol{p}$ be both the parameter and a $\boldsymbol{z}$-dependent random vector. In cases of either no dependence on $\boldsymbol{z}$ or $\boldsymbol{\xi}$, that is $\boldsymbol{p} = \boldsymbol{\xi}_1$ or $\boldsymbol{p} = \boldsymbol{z}_2$, we neglect the subindices and write $\boldsymbol{p} = \boldsymbol{\xi}$ or $\boldsymbol{p} = \boldsymbol{z}$.

Let $D(\boldsymbol{p}) \subset \mathbb{R}^d$ be a LIPSCHITZ domain with $d \in \mathbb{N}$. We consider parameter dependent PDEs which are pathwise of the form

$$\mathcal{L}(\boldsymbol{p})u(\boldsymbol{p}) \quad = \quad f(\boldsymbol{p}), \quad \text{in } \mathcal{V}(\boldsymbol{p})^*, \tag{3.2}$$

for an bijective linear operator $\mathcal{L}(\boldsymbol{p})\colon \mathcal{V}(\boldsymbol{p}) \to \mathcal{V}(\boldsymbol{p})^*$ with BANACH space $\mathcal{V}(\boldsymbol{p}) = \mathcal{V}(D(\boldsymbol{p}), \boldsymbol{p})$ and its dual $\mathcal{V}(\boldsymbol{p})^*$ such that $f \in \mathcal{V}(\boldsymbol{p})^*$. In the special case $\mathcal{V}(\boldsymbol{p}) = H(\boldsymbol{p})$ is a HILBERT space, it is assumed that

$$(w, v)_{L(\boldsymbol{p})} := \langle \mathcal{L}(\boldsymbol{p})w, v \rangle_{H(\boldsymbol{p})^*, H(\boldsymbol{p})} \tag{3.3}$$

defines a scalar product on $H(\boldsymbol{p})$ such that the LAX-MILGRAM Theorem [1] can be applied to ensure existence und uniqueness of the solution $u(\boldsymbol{p}) \in H(\boldsymbol{p})$ of (3.2). It in particular is the solution of the variational problem: Seek $u(\boldsymbol{p}) \in H(\boldsymbol{p})$ such that

$$(u(\boldsymbol{p}), v)_{L(\boldsymbol{p})} = \langle f(\boldsymbol{p}), v \rangle_{H(\boldsymbol{p})^*, H(\boldsymbol{p})}, \quad \forall v \in H(\boldsymbol{p}).$$

This is the concept of pathwise solutions.

Example 3.1. *The* SOBOLEV *space* $H^1_{\Gamma_0(\boldsymbol{p})}(D(\boldsymbol{p}))$ *with a parameter dependent* DIRICHLET *boundary segment* $\Gamma_0(\boldsymbol{p})$ *is an example for the space* $H(\boldsymbol{p}) = H(D(\boldsymbol{p}), \boldsymbol{p})$.

Since we assume pathwise that the solution space itself can be parameter dependent, a non-pathwise solution concept in terms of BOCHNER type solution spaces might not be straightforward. The case of pure random dependence is rather well understood, *e.g.* [55] and references in the subsequent sections. We recall some results from an abstract prespective. This allows us to define a solution concept on fuzzy sets.

3.1.1 Pure random case

Based on the notation in section 2.2 let $\boldsymbol{p} = \boldsymbol{\xi}(\omega)$ be a random vector with image Γ and push-forward measure $\mu = \boldsymbol{\xi}_{\#}\mathbb{P}$ of $\mathbb{P}$ under $\boldsymbol{\xi}$. We define the (image) probability space $L^P(\Gamma, \mu) := L^P(\Gamma, \mathcal{B}(\Gamma), \mu)$ with norm $\|.\|_{L^P(\Gamma,\mu)}$,

$$\|v\|^2_{L^P(\Gamma,\mu)} := \int_\Gamma |v(\boldsymbol{y})|^P \mathrm{d}\mu(\boldsymbol{y}) = \int_\Omega |v(\boldsymbol{\xi}(\omega))|^P \mathrm{d}\mathbb{P}.$$

Assume that $D \neq D(\boldsymbol{p}) \subset \mathbb{R}^d_1$ and also the HILBERT space $H = H(D) \neq H(\boldsymbol{p})$ be independent of $\boldsymbol{p}$ with the norm $\|.\|_H$. The operator $\mathcal{L}(\boldsymbol{p})$ is assumed to have the form

$$\mathcal{L}(\boldsymbol{p}) = \mathcal{L}(C_{[]}(\cdot, \boldsymbol{\xi}(\omega))), \tag{3.4}$$

with an array of random fields $C_{[]} = [C_1, \ldots, C_K]$ for $K \in \mathbb{N}$, see Example 3.2.

Example 3.2.

1 $C_{[]} = [A, c]$ *with random field diffusion matrix* A *and reaction field* c*, the operator* $\mathcal{L}$ *can be written as*

$$\langle \mathcal{L}(C_{[]}(\cdot, \boldsymbol{\xi}(\omega))])w, v\rangle = \int_D A(x, \boldsymbol{\xi}(\omega))\nabla w \nabla v + c(x, \boldsymbol{\xi}(\omega))wv \, \mathrm{d}x,$$

describing a random reaction-diffusion problem with $d_2 = 1$.

2 $C_{[]} = [\boldsymbol{C}]$ *with random field tensor* $\boldsymbol{C}$ *describing an anisotropic random material (such as a composite material) in linear elasticity for* $d_2 = 2, 3$*. Here,* $\mathcal{L}$ *takes the form*

$$\langle \mathcal{L}(C_{[]}(\cdot, \boldsymbol{\xi}(\omega)))w, v\rangle = \int_D \boldsymbol{C}(x, \boldsymbol{\xi}(\omega))\boldsymbol{\epsilon}(w) \colon \boldsymbol{\epsilon}(v) \, \mathrm{d}x,$$

with $\boldsymbol{\epsilon}(w) = \nabla^{\mathrm{sym}} w = 1/2(\nabla w + \nabla^T w)$*, see e.g. [14] for the deterministc case.*

Given the operator in (3.4), the abstract equation (3.2) becomes

$$\mathcal{L}(C_{[]}(\cdot, \boldsymbol{\xi}(\omega)))u(\boldsymbol{\xi}(\omega)) = f(\boldsymbol{\xi}(\omega)), \quad \text{in } H(D)^*. \tag{3.5}$$

An analytical framework for this type of problem may be stated in a BOCHNER space $L^P(\Gamma,\mu;H)$, *e.g.* [38, 75, 78, 10, 29] and references therein with

$$\|v\|_{L^P(\Gamma,\mu;H)} := \begin{cases} \int\limits_\Gamma \|v(\boldsymbol{y})\|_H^P \mathrm{d}\mu(\boldsymbol{y}) & P \in [1,\infty), \\ \operatorname*{ess\,sup}_{\boldsymbol{y}\in\Gamma} \|v(\boldsymbol{y})\|_H & P = \infty. \end{cases}$$

With the identification $\boldsymbol{y} = \xi(\omega)$ for some $P \geq 1$, it can be shown that

$$\boldsymbol{y} \to u(\boldsymbol{y}) \in L^P(\Gamma,\mu;H),$$

assuming enough regularity with $f \in L^Q(\Gamma,\mu;H^*)$ and $C_k \in L^{S_k}(\Gamma,\mu;L_k)$ with $L_k \subset H^l$ for some $l \in \mathbb{N}$ and $Q, S_k \geq 1$, $k = 1,\dots,K$. See *e.g.* [10, 29] for the case $H = H_0^1(D)$ and $C_{[]} = [C_1]$ with $C_1 = A \in L_1 = L^\infty(D)^{d\times d}$ for the diffusion problem with $c \equiv 0$ from Example 3.2.

An important special case is the HILBERTian case $P = 2$ in the space $V := L^2(\Gamma,\mu;H)$, see *e.g.* [38, 3, 4, 59, 33] and the references therein, enabling the framework of GALERKIN projections based on a weak formulation: seek $u \in L^2(\Gamma,\mu;H)$

$$a(u,v) := \int\limits_\Gamma (u(\boldsymbol{y}), v(\boldsymbol{y}))_{\mathcal{L}(\boldsymbol{y})} \,\mathrm{d}\mu = \int\limits_\Gamma \langle f(\boldsymbol{y}), v(\boldsymbol{y})\rangle_{H(\boldsymbol{p})^*,H(\boldsymbol{p})} \mathrm{d}\mu := \ell(v). \tag{3.6}$$

Existence and uniqueness may be shown in the framework of the LAX-MILGRAM theorem based on the bilinear form $a\colon V\times V \to \mathbb{R}$ and the linear form $\ell\colon V \to \mathbb{R}$.

3.1.2 Probability spaces of Bochner type and parameteric domains

As before, let $\boldsymbol{p} \in P$ of the form (3.1) and consider the case

$$D = D(\boldsymbol{p}), \quad H(\boldsymbol{p}) = H(D(\boldsymbol{p})).$$

This means that we only consider a parameteric domain but no additional parametric description of the pathwise solution space. Within this framework in order to remain in a BOCHNER setting pathwise in $\boldsymbol{z} = [\boldsymbol{z}_1, \boldsymbol{z}_2]$, one can utilise a *parameter domain mapping approach*. We refer to [47, 72] and references therein for the random domain mapping approach.

Recall $Z = Z_1 \times Z_2$ and $\boldsymbol{z} = [\boldsymbol{z}_1, \boldsymbol{z}_2] \in Z$ and the idenfication of $\boldsymbol{p}$ in (3.1). We now consider two types of reference mappings, either completly mapping to a non-parameter dependend domain D^{ref} or keeping the parameter $\boldsymbol{z}$ in $D^{\mathrm{ref}}(\boldsymbol{z})$. We will restrict the discussion to the first case, the second one is sketched in Remark 3.3. Assume there exists a sufficiently smooth invertible map

$$\Phi\colon D^{\mathrm{ref}} \times Z \times \Omega \to \mathbb{R}^{d_1}, \quad \Phi(D^{\mathrm{ref}}, \boldsymbol{z}, \omega) = \Phi(D^{\mathrm{ref}}, \boldsymbol{p}) = D(\boldsymbol{p}),$$

that transforms the problem given pathwise as

$$\mathcal{L}(\boldsymbol{p})u(\boldsymbol{p}) = f(\boldsymbol{p}) \quad \text{in } H(D(\boldsymbol{p}))$$

to

$$\mathcal{L}^{\mathrm{ref}}(\boldsymbol{p})u^{\mathrm{ref}}(\boldsymbol{p}) = f^{\mathrm{ref}}(\boldsymbol{p}) \quad \text{in } H(D^{\mathrm{ref}}). \tag{3.7}$$

Note that the mapping Φ shifts the parameter dependence of the domain into the transformed operator. Now again assume bijectivity of the operator $\mathcal{L}^{\mathrm{ref}}(\boldsymbol{p})\colon H(D^{\mathrm{ref}}) \to H(D^{\mathrm{ref}})^*$. If $\mathcal{L}(\boldsymbol{p}) = \mathcal{L}(C_{[]}(\cdot,\boldsymbol{p}))$ with an array of parameter dependent fields analogue to (3.4), then $\mathcal{L}^{\mathrm{ref}}(\boldsymbol{p})$ again might become of the form $\mathcal{L}^{\mathrm{ref}}(C_{[]}^{\mathrm{ref}}(\cdot,\boldsymbol{p}))$ with a transformed array of parameter dependent fields. This change might be based on an associated integral based weak formulation, such that Φ acts as a change of variables in a weak sense. We observe that in this situation, (3.7) is pointwise (*i.e.* for fixed $\boldsymbol{z}$) of type (3.5). For fixed $\boldsymbol{z}$ the parameter $\boldsymbol{p}$ only depends on (the image of) a random vector

$$\boldsymbol{\xi}(\boldsymbol{z}) := [\boldsymbol{\xi}_1, \boldsymbol{\xi}_2(\boldsymbol{z}_2)], \quad \Gamma(\boldsymbol{z}) := \operatorname{img}\boldsymbol{\xi}(\boldsymbol{z}),$$

only keeping the full parameter dependence of u on $\boldsymbol{p}$ from (3.1). For fixed $\boldsymbol{z}$, let $\mu(\boldsymbol{z}) = \boldsymbol{\xi}(\boldsymbol{z})_{\#}\mathbb{P}$ denote the push-forward measure of $\boldsymbol{P}$ under $\boldsymbol{\xi}(\boldsymbol{z})$. We define the $\boldsymbol{z}$-dependent BOCHNER space $L^P(\Gamma(\boldsymbol{z}),\mu(\boldsymbol{z});H(D))$ as a solution space such that, for fixed $\boldsymbol{z}$,

$$\boldsymbol{y}(\boldsymbol{z}) \mapsto u(\boldsymbol{p}(\boldsymbol{y}(\boldsymbol{z}),\boldsymbol{z})) \in L^P(\Gamma(\boldsymbol{z}),\mu(\boldsymbol{z});H(D)), \tag{3.8}$$

with the image identification $\boldsymbol{\xi}(\boldsymbol{z},\omega) = \boldsymbol{y}(\boldsymbol{z})$ given $\boldsymbol{z} \in Z$ using the notation $u(\boldsymbol{p}(\boldsymbol{y}(\boldsymbol{z}),\boldsymbol{z}))$ to emphasise the additional dependence of u on $\boldsymbol{z}$.

Remark 3.3. *Due to the pathwise concept in $\boldsymbol{z}$ one might consider a transformation $\Phi[\boldsymbol{z}]$ with*

$$\Phi[\boldsymbol{z}]\colon D^{\mathrm{ref}}(\boldsymbol{z}) \times \Omega \to \mathbb{R}^{d_1}, \quad \Phi[\boldsymbol{z}](D^{\mathrm{ref}}(\boldsymbol{z}),\omega) = \Phi[\boldsymbol{z}](D^{\mathrm{ref}}(\boldsymbol{z}),\boldsymbol{\xi}_1(\omega),\boldsymbol{\xi}_2(\boldsymbol{z}_2(\omega))) = D(\boldsymbol{p}).$$

The underlying solution space then is a parameterized BOCHNER *space* $L^P\left(\Gamma(\boldsymbol{z}),\mu(\boldsymbol{z});H(D^{\mathrm{ref}}(\boldsymbol{z})\right)$.

As in (3.6) in the HILBERTian case $P = 2$ for the space $V[\boldsymbol{z}] := L^2(\Gamma(\boldsymbol{z}),\mu(\boldsymbol{z});H)$ with $H = H(D)$ or $H = H(D^{\mathrm{ref}}(\boldsymbol{z}))$, we consider a weak formulation pathwise in $\boldsymbol{z} \in Z$: seek $u(\boldsymbol{z}) \in V[\boldsymbol{z}]$ such that

$$a[\boldsymbol{z}](u(\boldsymbol{z}),v) = \ell(v) \quad \forall v \in V[\boldsymbol{z}], \tag{3.9}$$

with

$$a[\boldsymbol{z}](w,v) := \int_{\Gamma(\boldsymbol{z})} (w(\boldsymbol{y}(\boldsymbol{z}),v(\boldsymbol{y}(\boldsymbol{z})))_{\mathcal{L}(\boldsymbol{y}(\boldsymbol{z})}\,\mathrm{d}\mu(\boldsymbol{z}),$$
$$\ell[\boldsymbol{z}](v) := \int_{\Gamma(\boldsymbol{z})} \langle f(\boldsymbol{y}(\boldsymbol{z})),v(\boldsymbol{y}(\boldsymbol{z}))\rangle_{H(\boldsymbol{p})^*,H(\boldsymbol{p})}\mathrm{d}\mu(\boldsymbol{z}).$$

Existence and uniqueness follow pathwise in $\boldsymbol{z}$ by the LAX-MILGRAM theorem applied to the bilinear form $a[\boldsymbol{z}]\colon V[\boldsymbol{z}] \times V[\boldsymbol{z}] \to \mathbb{R}$ and the linear form $\ell[\boldsymbol{z}]\colon V[\boldsymbol{z}] \to \mathbb{R}$.

3.1.3 Pure fuzzy case

In this setting, assume that $\boldsymbol{p} = \boldsymbol{z} \in Z$ with $\boldsymbol{z} \in C_0[\tilde{\boldsymbol{z}}]$. We then remain in the pathwise setting and $u(\boldsymbol{z}) \in H(D(\boldsymbol{z}),\boldsymbol{z})$.

3.1.4 Fully separated case

We now assume the simpler structure $\boldsymbol{p} = (\boldsymbol{\xi}(\omega), \boldsymbol{z}) = (\boldsymbol{y}, \boldsymbol{z}) \in \Gamma \times Z$. This setting was discussed in [66] for the diffusion problem. Furthermore, let $D \neq D(\boldsymbol{p})$ and $H = H(D)$ be non-parametric. Recalling (3.5), the abstract equation (3.2) has the form

$$\mathcal{L}\left(C_{[]}(\cdot, \boldsymbol{\xi}(\omega), \boldsymbol{z})\right) u(\boldsymbol{\xi}(\omega), \boldsymbol{z}) = f(\boldsymbol{\xi}(\omega), \boldsymbol{z}), \quad \text{in } H(D)^*.$$

Analogously to Section 3.1.1 and in particular with (3.6) and the scalar product $(\cdot,\cdot)_{\mathcal{L}(\boldsymbol{y},\boldsymbol{z})}$ from (3.3), assume that

$$a[\boldsymbol{z}](u,v) := \int_\Gamma (u(\boldsymbol{y}), v(\boldsymbol{y}))_{\mathcal{L}(\boldsymbol{y},\boldsymbol{z})} \,\mathrm{d}\mu(\boldsymbol{y}), \tag{3.10}$$

$$\ell[\boldsymbol{z}](v) := \int_\Gamma \langle f(\boldsymbol{y},\boldsymbol{z}), v(\boldsymbol{y})\rangle_{H^*,H} \mathrm{d}\mu(\boldsymbol{y}) \tag{3.11}$$

define a V-elliptic bilinear form $a[\boldsymbol{z}]\colon V \times V \to \mathbb{R}$ and a continuous linear form $\ell[\boldsymbol{z}]\colon V \to \mathbb{R}$ with $V = L^2(\Gamma, \mu; H)$. Here again, μ is the push-forward of $\mathbb{P}$ under $\boldsymbol{\xi}$ and $\Gamma = \operatorname{img} \boldsymbol{\xi}$. Existence and uniqueness of a $u(\boldsymbol{z}) \in V$ are ensured by the LAX-MILGRAM theorem. Note that V is defined independently of $\boldsymbol{z}$ opposite to the setting in Section 3.1.2. Hence, we can write

$$u = u(x, \boldsymbol{y}, \boldsymbol{z}), \quad x \in D, \boldsymbol{y} \in \Gamma, \boldsymbol{z} \in Z. \tag{3.12}$$

We now consider the setting of (2.5) with $\boldsymbol{\xi} = (\xi_i)_i$ consisting of independent random variables ξ_i with image Γ_i and push-forwards measure μ_i for $i = 1, \dots, M \leq \infty$. Let H be separabel, then one can indentify V with a tensor space, such that

$$V = L^P(\Gamma, \mu, H) = L^P(\Gamma, \mu) \otimes H = \left(\bigotimes_{i=1}^M L^P(\Gamma_i, \mu_i)\right) \otimes H. \tag{3.13}$$

This structure of a tensorised solution space is discussed in many works, see *e.g.* [75, 15, 6, 5] including a regularity analysis allowing for tensor based approximation schemes like sparse grid interpolation [68, 36], polynomial chaos expansions [80, 33], stochastic collocation [2] and low rank format representation in tensor train format [23, 32, 31].

Remark 3.4. *In the case $H = H(D^{\mathrm{ref}}(\boldsymbol{z})$, one works with $V = V[\boldsymbol{z}] = L^2(\Gamma, \mu; H(D^{\mathrm{ref}}(\boldsymbol{z}))$ in a pathwise manner.*

3.2 Polymorphic PDEs

With the pathwise interpretation of $\boldsymbol{z} \in C_0[\tilde{z}]$ being an element of the support of a fuzzy number $\tilde{\boldsymbol{z}}$, we can transfer the results of Section 3.1 to fuzzy sets. To summarise the preceding section, we arrived at several solution space concepts,

$$\begin{aligned} V[\boldsymbol{z}] \in \big\{ & H(\boldsymbol{p}), L^P(\Gamma(\boldsymbol{z}), \mu(\boldsymbol{z}); H(D(\boldsymbol{z}))), && \text{(full pathwise)} \\ & L^P(\Gamma(\boldsymbol{z}), \mu(\boldsymbol{z}); H(D)), L^P(\Gamma, \mu; H(D^{\mathrm{ref}}(\boldsymbol{z})), && \text{(semi pathwise)} \\ & L^P(\Gamma, \mu; H(D))\big\}, && \text{(constant pathwise)} \end{aligned}$$

each assuming more substructure. All these concepts lift to fuzzy sets in different manner. For ease of presentation we will discuss the fuzzy set interpretation for only one of the 3 structured concepts of full, semi and constant pathwise design.

Let $\tilde{\boldsymbol{z}}$ be a fuzzy set on Z with support $C_0[\tilde{\boldsymbol{z}}] \subset Z$ and fix $\boldsymbol{z} \in C_0[\tilde{\boldsymbol{z}}]$.

1 Let $(\boldsymbol{y}(\boldsymbol{z}), \boldsymbol{z})$ be a realisation of $(\boldsymbol{\xi}(\boldsymbol{z}), \boldsymbol{z})$. In case that

$$u(\boldsymbol{y}(\boldsymbol{z}), \boldsymbol{z}) \in V[\boldsymbol{z}] = H((\boldsymbol{y}(\boldsymbol{z}), \boldsymbol{z})),$$

by ZADEH's extension principle $u(\boldsymbol{y}(\boldsymbol{z}), \boldsymbol{z}) \in C_0[u(\boldsymbol{y}(\tilde{\boldsymbol{z}}), \tilde{\boldsymbol{z}})]$. Note that $\boldsymbol{\xi}(\tilde{\boldsymbol{z}})$ is a fuzzy set on the set of random variables. Then,

$$u(\boldsymbol{y}(\tilde{\boldsymbol{z}}), \tilde{\boldsymbol{z}}) \in \mathcal{F}(V), \quad V := \bigcup_{z \in C_0[\tilde{\boldsymbol{z}}]} \bigcup_{y(z) \in \operatorname{img} \boldsymbol{\xi}(z)} H((\boldsymbol{y}(\boldsymbol{z}), \boldsymbol{z})).$$

2 Let $\Gamma(\boldsymbol{z}) = \operatorname{img} \boldsymbol{\xi}(\boldsymbol{z})$ and assume that

$$u[\boldsymbol{z}] \colon \Gamma(\boldsymbol{z}) \to H, \quad \boldsymbol{y}(\boldsymbol{z}) \mapsto u[\boldsymbol{z}](\boldsymbol{y}(\boldsymbol{z})) := u(\boldsymbol{y}(\boldsymbol{z}), \boldsymbol{z})$$

is an element of $L^P(\Gamma(\boldsymbol{z}), \mu(\boldsymbol{z}); H)$. Then,

$$u[\tilde{\boldsymbol{z}}] \in \mathcal{F}(V), \quad V := \bigcup_{z \in C_0[\tilde{\boldsymbol{z}}]} L^P(\Gamma(\boldsymbol{z}), \mu(\boldsymbol{z}); H).$$

3 In the simple case that $\boldsymbol{p} = (\boldsymbol{\xi}, \boldsymbol{z})$ and $\Gamma = \operatorname{img} \boldsymbol{\xi}$ such that the solution map

$$u[\boldsymbol{z}] \colon \Gamma \to H, \quad \boldsymbol{y} \mapsto u[\boldsymbol{z}] := u(\boldsymbol{y}, \boldsymbol{z})$$

is in $L^P(\Gamma, \mu; H)$, it simply follows that

$$u[\tilde{\boldsymbol{z}}] \in \mathcal{F}(V), \quad V = L^P(\Gamma, \mu; H).$$

3.3 Polymorphic quantities of interest

In this section we consider quantities of interests (QoIs) $\mathcal{Q}$ depending on the structure of the defined solution concepts. As in section 3.1 let $\boldsymbol{p} \in P$ represent an arbitrary realisation of a polymorphic variable and denote by $\tilde{\boldsymbol{p}}$ the polymorphic variable itself. For fixed $\boldsymbol{p}$ and a HILBERT space $H(\boldsymbol{p})$ define a map

$$q[\boldsymbol{p}] \colon H(\boldsymbol{p}) \to \mathbb{R}^d, \quad v \mapsto q[\boldsymbol{p}][v]. \tag{3.14}$$

Consider a family of functions $\{u(\boldsymbol{p}) \in H(\boldsymbol{p}), \boldsymbol{p} \in P\}$. Then a map

$$\mathcal{Q} \colon P \to \mathbb{R}^d, \quad \boldsymbol{p} \mapsto \mathcal{Q}[\boldsymbol{p}] := q[\boldsymbol{p}](u(\boldsymbol{p})) \tag{3.15}$$

defines a quantity of interest being a polymorphic field $\mathcal{Q}(\tilde{\boldsymbol{p}})$. In the case that $\tilde{\boldsymbol{p}} = \boldsymbol{\xi}/\ \tilde{\boldsymbol{p}} = \tilde{\boldsymbol{z}}$ is a random variable/ a fuzzy set, then $\mathcal{Q}(\tilde{\boldsymbol{p}})$ is a random variable/fuzzy set. For an example of such a polymorphic quantity of interest we refer to the numerical section 5.1 and Example 3.5.

Another type of QoIs is given for the underlying structure of BOCHNER SPACES $V = L^P(\Gamma, \mu; H)$ with $H \neq H(\boldsymbol{p}), \Gamma \neq \Gamma(\boldsymbol{p}), \mu \neq \mu(\boldsymbol{p})$. For a given fuzzy set $\tilde{\boldsymbol{z}}$ let $u(\boldsymbol{z}) \in V$ for all $\boldsymbol{z} \in C_0[\tilde{\boldsymbol{z}}]$. Moreover define a map

$$q\colon V \to \mathbb{R}^d \quad \text{or} \quad q\colon V \to H, \tag{3.16}$$

defining a fuzzy QoI $\tilde{\mathcal{Q}} := \mathcal{Q}(\tilde{\boldsymbol{z}}) = q(u(\tilde{\boldsymbol{z}}))$.

Example 3.5. [66] *Important examples of fuzzy QoIs $\tilde{\mathcal{Q}}$ include:*

- *k-th fuzzy moments of $q(u(\tilde{\boldsymbol{z}}))$ given as*

$$\tilde{\mathcal{Q}} := \mathbb{E}\left[v^k(\tilde{\boldsymbol{z}})\right] \in \mathcal{F}(H), \quad k \leq P. \tag{3.17}$$

  This also includes the fuzzy variance $\tilde{\mathbb{V}} = \mathbb{E}\left[v^2(\tilde{\boldsymbol{z}})\right] - \mathbb{E}\left[v(\tilde{\boldsymbol{z}})\right]^2 \in \mathcal{F}(H)$ if $P \geq 2$.

- *fuzzy probabilities of failure, see e.g. [71], given as*

$$\tilde{\mathcal{Q}} = \mathbb{P}\left(g(u(\tilde{\boldsymbol{z}})) \geq 0\right), \tag{3.18}$$

  for a limit state function $g\colon H \to \mathbb{R}$.

- *fuzzy cdfs, in the case of QoI of type* (3.15) *with image in $\mathbb{R}^d, d \geq 1$ given as*

$$\mathbb{P}(\mathcal{Q}(\tilde{\boldsymbol{p}}) \leq \boldsymbol{t}), \quad \textit{or} \quad \mathbb{P}(q(u(\tilde{\boldsymbol{p}}))), \quad \boldsymbol{t} \in \mathbb{R}^d, \tag{3.19}$$

  where $\mathbb{P}$ acts on the stochastic part of $\tilde{\mathbf{p}}$ defined pointwise in $\boldsymbol{z}$.

4 The $\mathcal{H}$-rank format

To introduce the hierarchical Tucker (HT) decomposition and thus the hierarchical Tucker approximation (HTA), we need two founding concepts. First, the *dimension tree* which represents the structure of the decomposition, defined in Section 4.1. Second, the notion of a matricisation of a tensor array A which is a rearrangement of its entries, defined in Section 4.2. For an accesible introduction to these concepts, see [42, 43].

Definition 4.1. Let $D := \{1, \dots, d\}$. A tree T_D is called a *dimension tree* if the following three conditions hold:

- the index set D is the root of the tree T_D,
- all vertices $t \in T_D$ are non-empty subsets $t \subset D$,
- every vertex $t \in T_D$ with $\#t \geq 2$ has two sons $t_1, t_2 \in T_D$ with the property

$$t = t_1 \cup t_2, \qquad \varnothing = t_1 \cap t_2.$$

Furthermore, we define the set of leaves of T_D by $\mathcal{L}\left(T_D\right) := \{t \in T_D : \#t = 1\}$ and the set of sons of t by $\operatorname{sons}(t)$.

Definition 4.2. Let $D := \{1, \dots, d\}$ and $\mathcal{I}_1, \dots, \mathcal{I}_d$ be finite index sets with $\mathcal{I} := \mathcal{I}_1 \times \dots \times \mathcal{I}_d$. Given a subset $t \subset D$ with complement $[t] := D \setminus t$, the *matricisation*

$$\mathcal{M}_t : \mathbb{R}^{\mathcal{I}} \to \mathbb{R}^{\mathcal{I}_t} \otimes \mathbb{R}^{\mathcal{I}_{[t]}}, \qquad \mathcal{I}_t := \bigtimes_{\mu \in t} \mathcal{I}_\mu, \quad \mathcal{I}_{[t]} := \bigtimes_{\mu \in [t]} \mathcal{I}_\mu$$

of a tensor $A \in \mathbb{R}^{\mathcal{I}}$ is defined by its entries

$$\mathcal{M}\left(A\right)_{(i_\mu)_{\mu\in t}, (i_\mu)_{\mu \in [t]}} := A_{(i_1,\dots,i_d)}, \quad i_\mu \in \mathcal{I}_\mu, \mu \in D.$$

Using this concept, we assign a matricisation $\mathcal{M}_t$ of the tensor to the vertex t in the dimension tree. This association yields the definition of the *hierarchical rank*

$$\mathbf{k} = \left(k_t\right)_{t \in T_D}, \quad k_t := \operatorname{rank}\left(\mathcal{M}_t\left(A\right)\right), \quad t \in T_D.$$

The set of tensors with a certain hierarchical rank $\mathbf{k}$ is denoted as

$$\mathcal{H}_{\mathbf{k}} := \left\{ A \in \mathbb{R}^{\mathcal{I}} : \operatorname{rank}\left(\mathcal{M}_t\left(A\right)\right) \le k_t, t \in T_D \right\}.$$

Using the following lemma, a tensor in $\mathcal{H}_{\mathbf{k}}$ can be represented in a recursive fashion.

Lemma 4.3. *Let $A \in \mathcal{H}_{\mathbf{k}}$. Then $A = (U_D)_{\cdot,1}$ can be represented by the recursive relation*

$$\left(U_t\right)_{\cdot,j} = \sum_{j_1=1}^{k_{t_1}} \sum_{j_2=1}^{k_{t_2}} \left(B_t\right)_{j,j_1,j_2} \left(U_{t_1}\right)_{\cdot,j_1} \otimes \left(U_{t_2}\right)_{\cdot,j_2}, \quad j = 1, \dots, k_t, \tag{4.1}$$

for all $t \in T_D \setminus \mathcal{L}(T_D)$ with $\operatorname{sons}(t) = \{t_1, t_2\}$, where $B_t \in \mathbb{R}^{k_t \times k_{t_1} \times k_{t_2}}$ and $U_t \in \mathbb{R}^{\mathcal{I}_t \times k_t}$, $\mathcal{I} := \times_{\mu \in t} \mathcal{I}_\mu$, for all $t \in T_D$.

This is the key to the hierarchical Tucker decomposition. It reduces the number of stored entries from n^d to $\mathcal{O}\left(d \cdot k^3 + dkn\right)$ with $k = \max_{t \in T_D} k_t$ and $n = \max_{\mu \in D} \#\mathcal{I}_\mu$ since only the transfer tensors $B_t \in \mathbb{R}^{k_t} \times \mathbb{R}^{k_{t_1}} \times \mathbb{R}^{k_{t_2}}$ and the *frames* $U_{\{\mu\}} \in \mathbb{R}^{\mathcal{I}_\mu} \times \mathbb{R}^{k_t}$ have to be stored. In Figure 4, we find an example of a balanced dimension tree for $d = 4$ and the relation of the transfer tensors to each other and the frames.

Once an accurate approximation $\tilde{A} \in \mathcal{H}_{\mathbf{k}}$ of the tensor $A \in \mathbb{R}^{\mathcal{I}}$ is found, the evaluation of a single entry requires only $\mathcal{O}\left(dk^3\right)$ arithmetic operations. In [41], a quasi-optimal algorithm was developed based on singular value decomposition. For the resulting tensor $\mathcal{B} \in \mathcal{H}_{\mathbf{k}}$, it holds

$$\|A - \mathcal{B}\|_F \le \sqrt{2d-3} \min_{\mathcal{B} \in \mathcal{H}_{\mathbf{k}}} \|\mathcal{A} - \mathcal{B}^*\|_F. \tag{4.2}$$

The use of a SVD is expensive in the sense that all entries of a matricisation are accessed. Another method to find a low rank approximation of a matrix is the skeleton or cross approximation [39]. The basic idea is to directly use the rows and columns of a matrix. Let $M \in \mathbb{R}^{\mathcal{I}_1 \times \mathcal{I}_2}$, $\mathcal{P} \subset \mathcal{I}_1$ and $\mathcal{Q} \subset \mathcal{I}_2$. The cross approximation then reads

$$M^\times := M|_{\mathcal{I}_1 \times \mathcal{Q}} \cdot S^{-1} \cdot M|_{\mathcal{P} \times \mathcal{I}_2},$$

where S is the submatrix that arises at the intersections of the chosen rows and columns. This method only employs the entries of the rows and columns. Thus, the computation of most of the entries in the

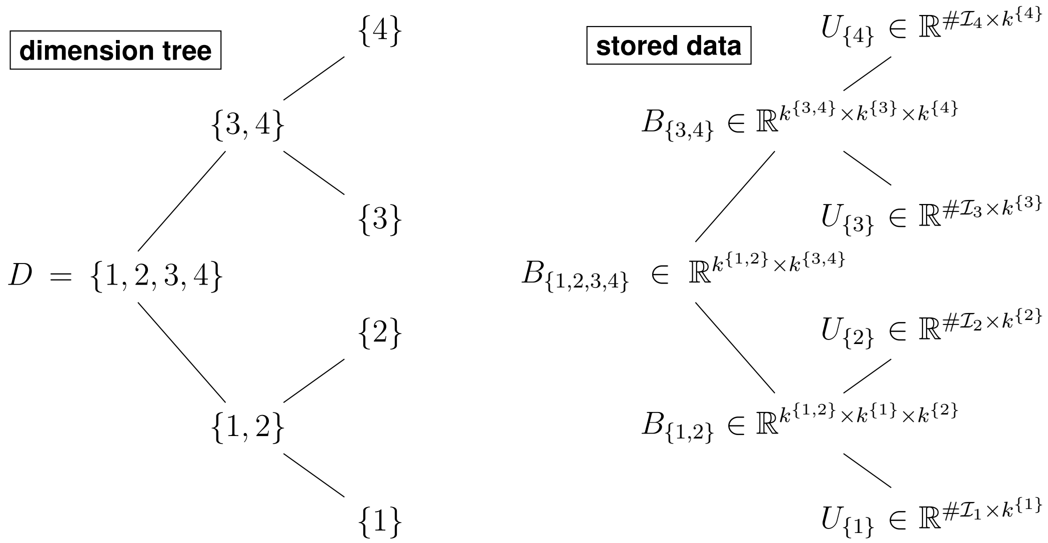


Figure 4: A balanced dimension tree on the left and the arrangement of data representing the HT decomposition on the right.

matrix are skipped. In [9], an adaptive cross approximation algorithm is employed to construct a hierarchical Tucker decomposition in a non-intrusive fashion. This reduces the number of evaluated tensor entries significantly. Additionally, the hierarchical rank is determined adaptively. Numerical results in [8] show the practicability of this approach.

The most simple way to use the $\mathcal{H}$-rank tensor is its evaluation for arbitrary indices of the underlying tensor array, which requires only $\mathcal{O}(dk^3)$ operations, cf. [45]. To evaluate the $\mathcal{H}$-rank approximation at arbitrary points, the format has to be extended by a set of functions $\left(\psi_k^{(i)}(p_i)\right)_{k\in\{1,\ldots,N\}}$ for each direction i, where

$$\psi_j^{(i)}(p) = \begin{cases} 1, & p = p_{i,k} \text{ with } k = j, \\ 0, & p = p_{i,k} \text{ with } k \neq j. \end{cases}$$

In between the grid points, the function maps to arbitrary real values. For instance, when taking triangular functions, the resulting approximation is the product of piecewise linear functions in each direction. More sophisticated basis functions like wavelets or splines may also be implemented in principle.

Given a basis $\left(\psi_k^{(i)}(p_i)\right)_{k\in\{1,\ldots,N\}}$ and a $\mathcal{H}$-rank tensor with an underlying index set $\mathcal{I}$, the evaluation for an arbitrary point in $\mathbf{p} = (p_k)_{k=1}^d \in P$ with the structure $P = \bigtimes_{k=1}^d P_k$ corresponds to

$$\begin{aligned} \Phi(\mathbf{p}) &= \sum_{\underline{i}\in\mathcal{I}} \mathcal{H}\left(p_{\underline{i}}\right) \prod_{k=1}^{d} \psi_{i_k}^{(k)}(p_k) \\ &= \sum_{\underline{i}\in\mathcal{I}} \mathcal{H}\left(p_{\underline{i}}\right) \left(W_1 \otimes \ldots \otimes W_d\right)_{\underline{i}} = \left\langle \mathcal{H}, W_1 \otimes \ldots \otimes W_d \right\rangle, \end{aligned}$$

with $W_k = \left(\psi_1^{(k)}(p_k) \cdots \psi_d^{(k)}(p_k)^T\right)$. This is just a contraction between a $\mathcal{H}$ tensor and the rank 1 tensor $\mathcal{W} = (W_1 \otimes \ldots \otimes W_d)$. The evaluation costs are constituted from the evaluation costs of each basis function and the contraction.

In addition to cutting the costs of a simple evaluation, a function given in the extended $\mathcal{H}$-rank format allows for simplified usage of tensorized linear operators, since it holds

$$\mathcal{L}\left(\Phi\right) = \sum_{\underline{i}\in\mathcal{I}} \mathcal{H}\left(p_{\underline{i}}\right) \prod_{k=1}^{d} \mathcal{L}_k \left(\psi_{i_k}^{(k)}\right).$$

With slightly more effort it is also possible to work with bilinear forms. Let $(\cdot,\cdot)_{\mathcal{B}}$ be a bilinear form and $\tilde{\Phi}$ another extended $\mathcal{H}$-rank tensor. The evaluation reads

$$\begin{aligned}\left(\Phi, \tilde{\Phi}\right)_{\mathcal{B}} &= \sum_{\underline{i}\in\mathcal{I}} \mathcal{H}_{\underline{i}} \left(\prod_{k=1}^{d} \psi_{i_k}^{(k)}, \tilde{\Phi}\right)_{\mathcal{B}} \\ &= \sum_{\underline{i}\in\mathcal{I}} \sum_{\underline{j}\in\mathcal{J}} \mathcal{H}_{\underline{i}} \tilde{\mathcal{H}}_{\underline{j}} \left(\prod_{k=1}^{d} \psi_{i_k}^{(k)}, \prod_{l=1}^{d} \tilde{\psi}_{j_l}^{(l)}\right)_{\mathcal{B}},\end{aligned}$$

which is a two-sided contraction of two $\mathcal{H}$-rank tensors.

If a form of *separability* holds for the bilinear form, namely

$$(\cdot,\cdot)_{\mathcal{B}} = \bigotimes_{k=1}^{d} (\cdot,\cdot)_{\mathcal{B}^{(i)}},$$

the computation becomes feasible by exploiting the Kronecker structure, see Section 4.3.

4.1 Tensor interpolation

Let $v \in V = \bigotimes_{j=1}^{d} V_j$ a tensor space with an *uniform crossnorm*, cf. [45], and let U be a finite function space of $\mathcal{H}$-rank $\mathbf{k}$, where each function $u \in U$ is constructed accordingly to Eq. 4.1. The space itself is hence constructed from a set of d finite dimensional subspaces $U_j \subset V_j$. Each subspace U_j is spanned by a set of basis functions $\left\{\psi_j^{(k)}\right\}_{j\in\{1,\dots,r_j\}}$, with $\dim(U_j) < \infty$. Consequently, each function u is represented by a suitable coefficient tensor $\mathcal{H}^u$ and the linear combination

$$u = \sum_{\underline{i}\in\mathcal{I}} \mathcal{H}_{\underline{i}}^u \bigotimes_{k=1}^{d} \psi_{i_k}^{(k)}.$$

Let $\Lambda_i^j \in V^*$ for $i = 1,\dots,\dim\left(U_j\right)$ be linear independent linear functionals on U_j and $L_i^j \in U_j$ be the Lagrange functions defined such that

$$\Lambda_\mu^j \left(L_\nu^j\right) = \delta_{\mu\nu} \quad \text{with } 1 \le \nu, \mu \le \dim\left(U_j\right).$$

For each subspace U_j we introduce the interpolation operator

$$\mathcal{I}_j\left(f\right) = \sum_{i=1}^{d} \Lambda_i^j L_i^j(f).$$

The operator $\mathcal{I}_j$ is a projection from V_j to U_j and the operatornorm

$C_j = \|\mathcal{I}_j\|_{V_j \leftarrow V_j}$ is denoted as *stability constant*. Taking the infimum over $g \in U_j$ yields

$$\|f - \mathcal{I}_j(f)\|_{V_j} = \|f - \mathcal{I}_j(g) + \mathcal{I}_j(g) - \mathcal{I}_j(f)\|_{V_j} \tag{4.3}$$

$$\leq \|f - g\|_{V_j} + \|\mathcal{I}_j(g - f)\|_{V_j} \tag{4.4}$$

$$\leq (1 + C_j) \inf\left\{\|f - g\|_{V_j} : g \in U_j\right\}. \tag{4.5}$$

Utilizing the tensor product, we define an interpolation operator from V to U by

$$\mathcal{I} = \bigotimes_{j=1}^{d} \mathcal{I}_j : V \to U. \tag{4.6}$$

For each direction, the univariate error is given by

$$\epsilon_j(f) = \inf\left\{\|f - g\| : g \in \left[\bigotimes_{k=1}^{j-1} V_k\right] \otimes U_j \otimes \left[\bigotimes_{k=j+1}^{d} V_k\right]\right\}.$$

Using (4.5) and the uniform crossnorm property recursively, the following proposition can be shown.

Theorem 4.4. *([45]) Let the norm of V be a uniform crossnorm and let f be a multivariate function defined on a bounded product domain $\mathbf{I} = \bigtimes_{k=1}^{d} I_k$. With $\epsilon_j(f)$ and C_j from above, the interpolation error of $\mathcal{I}$ from* (4.6) *can be bounded by*

$$\|f - \mathcal{I}(f)\| \leq \sum_{j=1}^{d} \left[\prod_{k=1}^{j-1} C_k\right] (1 + C_j)\epsilon_j(f).$$

This relates the multivariate interpolation error to the univariate ones. In this work, the univariate spaces V_j are interpolated by CHEBYSHEV polynomials, meaning that the subspaces U_j are polynomial spaces $\mathcal{P}_{r_j - 1}$. We define the grid points modulo affine transformation as CHEBYSHEV quadrature points in a reference domain given by

$$\xi_i^r = \cos\left(\frac{2i + 1}{2r_j + 2}\pi\right) \in [-1, 1] \qquad (i = 0, \ldots, r_j - 1).$$

The linear functionals Λ_i^j become Dirac functionals and the Lagrange functions become polynomials such that

$$\Lambda_i^j(f) = f(\xi_i^j) \quad \text{and} \quad L_i^j(x) = \prod_{k \in \{0, \ldots, r_j - 1\} \setminus \{i\}} \frac{x - \xi_k^j}{\xi_i - \xi_k^j}.$$

A CHEBYSHEV interpolation of polynomial degree p leads to a stability constant

$$C_{\text{stab}} \leq 1 + \frac{2}{\pi} \log(p + 1),$$

see [57]. For the multivariate product of the CHEBYSHEV interpolation, Theorem 4.4 yields

$$\|f - \mathcal{I}(f)\| \leq \mathcal{O}(\log(p + 1)^d) \cdot \max_{1 \leq j \leq d} \epsilon_j(f), \quad \text{with } p = \max_{1 \leq j \leq d} r_j - 1.$$

The Interpolation error depends heavily on the approximation error and only logarithmically on the order of polynomials.

4.2 Error analysis for the HTA as surrogate model

Our goal is the computation of an accurate HTA surrogate model for some quantity of interest $q : P \to \mathbb{R}$ with a compact parameter space P of tensorised form. We denote its numerical approximation by $q_{\mathrm{a}} : P \to \mathbb{R}$. In our case, this approximation, *e.g.* will be the result of an evaluation of a FEM discretization or numerical quadrature. We denote the pointwise absolute approximation error as

$$\varepsilon_{\mathrm{a}} = \sup_{p \in P} |q(p) - q_{\mathrm{a}}(p)|.$$

The investigation of this error is a topic in its own right and not subject of this work. Since only q_{a} is available, the HTA is constructed with this basis.

A second source of error comes from the question if q_{a} has a certain low-rank structure. Let $\mathcal{G}$ denote a tensorized grid in P and the distance of the resulting tensor array $q(\mathcal{G})$ to a hierarchical low-rank manifold $\mathcal{H}_{\mathbf{k}}$ be given by

$$\varepsilon_{\mathrm{lr}}(\mathcal{G}) = \min_{|\mathbf{k}|<K} \min_{\mathcal{B} \in \mathcal{H}_{\mathbf{k}}} \|q(\mathcal{G}) - \mathcal{B}\|,$$

where K is an upper limit to the $\mathcal{H}$ rank. It should be emphasized here that this error is difficult to assess and in most cases it only may be determined by testing various tensor tree topologies. If the mapping exhibits a low-rank structure, (4.2) indicates that it is possible to find a quasi-optimal hierachical tensor representation using a SVD-based construction method which yields a mapping $q_{\mathrm{H}} : \mathcal{G} \to \mathbb{R}$. An *extension* by an interpolation basis leads to a *surrogate model* $q_{\mathrm{ext}} : P \to \mathbb{R}$. As described in Section 4.1, the interpolation and the univariate approximation remain as additional error sources of this surrogate.

Let $\mathcal{I}_{\mathcal{G}}$ be the polynomial interpolation operator with interpolation points on the tensorized grid $\mathcal{G}$. In order to provide a rough estimate for the total error, we introduce

$$\varepsilon_{\mathrm{int}}(q) = \sup_{p \in P} |q(p) - (\mathcal{I}_{\mathcal{G}} q)(p)| \quad \text{and} \quad \varepsilon_{\mathrm{grid}}(u, v) = \sup_{p \in \mathcal{G}} |u(p) - v(p)|.$$

By application of the triangle inequality and with the stability constant C of the interpolation, it holds

$$\begin{aligned} |q(p) - q_{\mathrm{ext}}| &\leq |q(p) - q_a(p)| + |q_a(p) - (\mathcal{I}_{\mathcal{G}} q_{\mathrm{a}})(p)| + |(\mathcal{I}_{\mathcal{G}} q_{\mathrm{a}})(p) - (\mathcal{I}_{\mathcal{G}} q_{\mathrm{H}})(p)| \\ &\leq \varepsilon_{\mathrm{a}} + \varepsilon_{\mathrm{int}}(q_{\mathrm{a}}) + C \cdot \varepsilon_{\mathrm{grid}}(q_{\mathrm{a}}, q_{\mathrm{H}}). \end{aligned}$$

Note that $\varepsilon_{\mathrm{grid}}$ is strongly connected to the low-rank error $\varepsilon_{\mathrm{lr}}$. We will omit the detailed investigation of each error for the sake of focus in this paper.

4.3 The $\mathcal{H}$-rank approximation for the Galerkin method

As it was described in Section 2.4, there are manifold ways how fuzzy and stochastic variables may interact. In this Section only *fully seperated interaction* is investigated, see Section 3.1.4. Meaning, that in terms of Section 2.4 the set W is empty. The goal of this section is to provide a Galerkin formalism for a fuzzy-stochastic weak formulation of a PDE, where the discretized solution to this weak formulation is an extended hierarchical tensor.

Let $\boldsymbol{p} = (\boldsymbol{\xi}(\omega), \boldsymbol{z}) = (\boldsymbol{y}, \boldsymbol{z}) \in \Gamma \times Z$ with $\boldsymbol{\xi} = (\xi_i)_i$ consisting of independent random variables ξ_i with image Γ_i and push-forwards measure ρ_i for $i = 1, \ldots, d_{\mathrm{p}} < \infty$ and let Z be a product domain with non-interactive fuzzy variables $\tilde{z}_i$. Analogously to Eq. 3.13 we define pathwise for fixed $\boldsymbol{z}$

$$V(z) = L^2(\Gamma, \mu, H(D)) = H(D) \otimes L^2(\Gamma, \mu).$$

It is hence possible to perform a stochastic Galerkin projection for a fixed $\boldsymbol{z}$. For the treatment of the fuzzy parameters, we introduce the space $L^2(Z, w)$ with CHEBYSHEV weights $w(\boldsymbol{z})$. Assuming full seperability for the Bochner space setting yields the solution space

$$\begin{aligned} L^2(Z, w, V(z)) &= H(D) \otimes L^2(\Gamma, \mu) \otimes L^2(Z, w) \\ &= H(D) \otimes \left(\bigotimes_{i=1}^{d_{\mathrm{p}}} L^2(\Gamma_i, \mu_i) \right) \otimes \left(\bigotimes_{i=1}^{d_{\mathrm{z}}} L^2(Z_i, w_i) \right). \end{aligned}$$

Accordingly, let $a(\cdot, \cdot) : H \times H \mapsto \mathbb{R}$ be a bilinear form describing a parameter dependend weak formulation of a PDE with fixed $\mathbf{y}$ and $\mathbf{z}$. Then, the fuzzy-stochastic weak formulation reads: Find $u(x, \boldsymbol{y}, \boldsymbol{z}) \in H(D) \otimes L^2(\Gamma, \mu) \otimes L^2(Z, w)$ such that for all test functions v

$$\int_\Gamma \int_Z a\left(u(x, \boldsymbol{y}, \boldsymbol{z}), v(x, \boldsymbol{y}, \boldsymbol{z})\right) w(\boldsymbol{z}) \mathrm{d}\mu \mathrm{d}\boldsymbol{z} = \int_\Gamma \int_Z \langle f(x), v(x, \boldsymbol{y}, \boldsymbol{z}) \rangle_{H^*, H} \, w(\boldsymbol{z}) \mathrm{d}\mu \mathrm{d}\boldsymbol{z} \tag{4.7}$$

holds. Here the parameter space is split into the non-probabilistic space $L^2(Z, w)$ and the probabilistic parameter space $L^2(\Gamma, \mu)$. The parameters drawn from these spaces are assumed to be independend of each other. Since we assume that the random variables are i.i.d and the fuzzy variables are non-interactive it holds

$$\mu = \prod_{j=1}^{d_{\mathrm{p}}} \mu_j \text{ and } w = \prod_{j=1}^{d_{\mathrm{z}}} w_j.$$

For $L^2(\Gamma_i, \mu_i)$ and $L^2(Z_j, w_j)$ we select a finite subspace, *e.g.* spanned by polynomials, orthogonal with respect to the bilinear forms

$$\begin{aligned} (u, v)_{\mu_j} &: L^2(\Gamma_j, \mu_j) \times L^2(\Gamma_j, \mu_j) \to \mathbb{R}, \ (u, v) \mapsto \int_{\Gamma_j} uv \mathrm{d}\mu_j \\ \text{and} \quad (u, v)_{w_j} &: L^2(Z_j, w_j) \times L^2(Z_j, w_j) \to \mathbb{R}, \ (u, v) \mapsto \int_{Z_j} uv w_j \mathrm{d}z_j. \end{aligned}$$

We denote these polynomial spaces as $\mathcal{P}_{\mathrm{P}_j}$ and $\mathcal{P}_{\mathrm{Z}_j}$, with dimension n_{P_j} and n_{Z_j} respectively. For $H(D)$ we select a suitable FEM space $V_h(D)$, with dimension N. For these finite spaces, we define the index set

$$\begin{aligned} \Lambda = \Lambda_V \times \Lambda_P \times \Lambda_Z = \big\{ (\alpha, \nu_1, \ldots, \nu_{d_{\mathrm{p}}}, \mu_1, \ldots, \nu_{d_{\mathrm{z}}}) \in \mathbb{N}^{1+d_{\mathrm{p}}+d_{\mathrm{z}}} \mid \\ 0 \le \alpha \le N, 0 \le \nu_j \le n_{\mathrm{P}_j} - 1, 0 \le \mu_j \le n_{\mathrm{Z}_j} - 1 \big\}, \end{aligned} \tag{4.8}$$

and the fully discrete space

$$\mathcal{V} = \left\{ u(x, \boldsymbol{y}, \boldsymbol{z}) = \sum_{(\alpha, \mu, \nu) \in \Lambda} \mathcal{X}^{\mathrm{u}}_{(\alpha, \mu, \nu)} h_\alpha(x) p_\mu(\boldsymbol{y}) q_\nu(\boldsymbol{z}) \;\middle|\; h_\alpha \in V_h(D), p_\mu \in \mathcal{P}_{\mathrm{P}_j}, q_\nu \in \mathcal{P}_{\mathrm{Z}_j} \right\}, \tag{4.9}$$

with a tensor array $\mathcal{X}^{\mathrm{u}}$. This function space is spanned by $N \cdot \prod_{j=1}^{d_{\mathrm{p}}} n_{\mathrm{P}_j} \cdot \prod_{j=1}^{d_z} n_{\mathrm{Z}_j}$ basis functions. One approach to break the curse of dimensionality is to reduce the number of basis functions, cf. [16]. Another approach is to use a nested multi-level function basis, cf. [13]. In this paper, the curse of dimensionality is tackled with a low-rank approximation of $\mathcal{X}^{\mathrm{u}}$.

Since in our case the bilinear form $a(\cdot,\cdot)$ only acts on V_h, for $u, v \in \mathcal{V}$ it yields

$$\int_\Gamma \int_Z a(u,v)\, w(\boldsymbol{z}) \mathrm{d}\mu \mathrm{d}\boldsymbol{z} = \sum_{(\alpha,\mu,\nu)\in\Lambda} \sum_{(\alpha',\mu',\nu')\in\Lambda} \mathcal{X}^{\mathrm{u}}_{(\alpha,\mu,\nu)} \mathcal{X}^{\mathrm{v}}_{(\alpha',\mu',\nu')} A^V_{\alpha,\alpha'} A^P_{\nu,\nu'} A^Z_{\mu,\mu'},$$

where

$$\begin{aligned} A^V_{\alpha,\alpha'} &= a\left(h_\alpha(x), h_{\alpha'}(x)\right) \\ A^P_{\nu,\nu'} &= \int_\Gamma p_\nu(\boldsymbol{y}) p_{\nu'}(\boldsymbol{y}) \mathrm{d}\mu \\ A^Z_{\mu,\mu'} &= \int_Z q_\nu(\boldsymbol{z}) q_{\nu'}(\boldsymbol{z}) w(\boldsymbol{z}) \mathrm{d}\boldsymbol{z}. \end{aligned}$$

For the right-hand side, it holds

$$\begin{aligned} &\int_\Gamma \int_Z \langle f(x), \sum_{(\alpha,\mu,\nu)\in\Lambda} \mathcal{X}^{\mathrm{u}}_{(\alpha,\mu,\nu)} h_\alpha(x) p_\mu(\boldsymbol{y}) q_\nu(\boldsymbol{z}) \rangle_{H^*,H} w(\boldsymbol{z}) \mathrm{d}\mu \mathrm{d}\boldsymbol{z} \\ &\quad = \sum_{(\alpha,\mu,\nu)\in\Lambda} \mathcal{X}^{\mathrm{u}}_{(\alpha,\mu,\nu)} \langle f(x), h_\alpha(x) \rangle \int_Z q_\nu(\boldsymbol{z}) w(\boldsymbol{z}) \mathrm{d}\boldsymbol{z} \int_\Gamma p_\mu(\boldsymbol{y}) \mathrm{d}\mu \\ &\quad = \sum_{(\alpha,\mu,\nu)\in\Lambda} \mathcal{X}^{\mathrm{u}}_{(\alpha,\mu,\nu)} b^V_\alpha b^P_\nu b^Z_\mu. \end{aligned}$$

To summarize, this yields a linear system of equations of the form

$$\mathbf{A} \cdot \mathcal{X}^{\mathrm{u}} = \mathbf{b}, \text{ with } \mathbf{A} = A^V \otimes A^P \otimes A^Z \text{ and } \mathbf{b} = b^V \otimes b^P \otimes b^Z. \tag{4.10}$$

The matrices A^P and A^Z, as well as the vectors b^P and b^Z are constructed by KRONECKER products.

Due to the curse of dimensionality, this linear system is difficult to solve directly since even the solution $\mathcal{X}^{\mathrm{u}}$ easily exhibits a prohibitively large number of entries. As a remedy, a hierarchical tensor format is employed to represent $\mathcal{X}^{\mathrm{u}}$. The term $\mathbf{A} \cdot \mathcal{X}^{\mathrm{u}}$ is a matrix-matrix multiplication of the leafs with the respective matrices A^V_α, $A^P_{\mu_j}$ or $A^Z_{\nu_i}$ which yields another hierarchical tensor. To solve the linear system, linear iterations of the basic form

$$\mathcal{X}^{\mathrm{u}}_{j+1} = \mathcal{X}^{\mathrm{u}}_j - \mathbf{C} \cdot \left(\mathbf{A} \cdot \mathcal{X}^{\mathrm{u}}_j - \mathbf{b}\right) \tag{4.11}$$

may be applied. Here, $\mathbf{C}$ is some preconditioning matrix and $\mathcal{X}^{\mathrm{u}}_0$ is any initial Hierarchical tensor. When for the spectral radius $\rho(\mathbf{C} \cdot \mathbf{A}) < 1$ holds, the convergence of (4.11) is assured. See [50] on the choice of $\mathbf{C}$. In each step the ranks of $\mathcal{X}^{\mathrm{u}}_{j+1}$ are increased. Therefore, a truncation algorithm is used to keep the number of ranks of $\mathcal{X}^{\mathrm{u}}_j$ at a feasible number. Accelerated versions of (4.11) like GMRES are also possible. For example in [7] a projection method, similar to the Krylov subspace method where introduced.

5 Numerical Experiments

In this section, two examples are presented to illustrate polymorphic uncertainties in partial differential equations. The first example is a 2D orthotropic elasticity problem defined on a domain with an ellipsoidal cavity. This model can be employed with adhesive films with air containments, where the durability (or failure) of these films is of interest.

The second example is the diffusion equation with a random coefficient field based on a GAUSSian kernel, which is ubiquitous in the field of uncertainty quantification. We decompose this random field by a Karhune-Loève expansion in M terms, thus introducing M probabilistic parameters. Additionaly, the correlation length of the GAUSSian kernel is modelled as a fuzzy number. This modification of the problem presents a generalisation of the typical examples as for instance found in [76] with fixed correlation length.

The numerical simulations were performed with `FEniCS` [34] for the FE simulation based on meshes generated with `gmsh` [37], `ALEA` [30] for the UQ part involving fuzzy propagation and for the tensor reconstruction the code basis used in [44].

5.1 Fuzzy-stochastic linear elasticity with orthotropic media

Let $D = [-2.5, 2.5]^2$ and consider an ellipsoid $\mathcal{E}$ specified by midpoint $M = (m_1, m_2)^T \in D$, radii range a, b possibly rotated by $\alpha \in [0, 2\pi]$, see Figure 5. The geometrical parameters are collected in $\boldsymbol{p}_{\text{geo}} := (m_1, m_2, a, b, \alpha)$. Then,

$$\mathcal{E}(\boldsymbol{p}_{\text{geo}}) = \left\{ \boldsymbol{x} = (x_1, x_2) \in D : \frac{(x_1 - m_1)^2}{a^2} + \frac{(x_2 - m_2)^2}{b^2} = 1 \right\}. \tag{5.1}$$

The perforated domain of interest is denoted as $D(\boldsymbol{p}_{\text{geo}}) := D \setminus \overline{\mathcal{E}(\boldsymbol{p}_{\text{geo}})}$. Furthermore, consider an orthotropic elastic material described by a compliance matrix

$$S = \begin{pmatrix} \frac{1}{E_{11}} & -\frac{\nu 1,2}{E_{11}} & 0 \\ -\frac{\nu 1,2}{E_{11}} & \frac{1}{E_{22}} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{pmatrix} \tag{5.2}$$

with directional elastic modulus E_{11} and E_{22}, shear modulus G_{12} and POISSON ratio $\nu_{1,2}$. Furthermore, denote by $\boldsymbol{C} := S^{-1}$ the stiffness tensor of the material. Let R be the conversion from tensor strain to engineering strain and for $\beta \in [0, 2\pi]$ let $Q_3[\beta]$ be a rotation operator for arbitrary second order tensors through an angle β with

$$\boldsymbol{R} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \boldsymbol{Q}_3[\beta] := \begin{pmatrix} \sin^2\beta & \cos^2\beta & 2\sin\beta\cos\beta \\ \sin^2\beta & \cos^2\beta & -2\sin\beta\cos\beta \\ -\sin\beta\cos\beta & \sin\beta\cos\beta & \cos^2\beta - \sin^2\beta \end{pmatrix}. \tag{5.3}$$

This results in the stress $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_{12})^T$ to strain $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \epsilon_{12})^T$ relation of the orthotropic material rotated at angle β given by

$$\boldsymbol{\sigma} = \boldsymbol{C}[\beta]\boldsymbol{\epsilon}, \quad \boldsymbol{C}[\beta] := \boldsymbol{Q}_3[\beta]^{-1}\boldsymbol{C}\boldsymbol{R}\boldsymbol{Q}_3[\beta]\boldsymbol{R}^{-1}, \tag{5.4}$$

with a parametric stiffness matrix $\boldsymbol{C}[\beta]$.

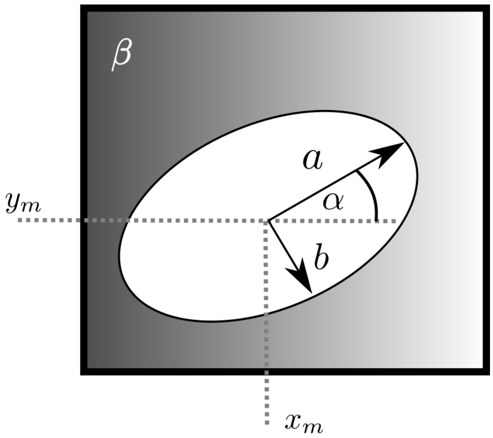


Figure 5: Illustration of parameters for the ellipse.

The parameteric linear elastic problem in $\boldsymbol{p} = (\boldsymbol{p}_{\text{geo}}, \beta)$ in strong form reads

$$\begin{cases} \left.\begin{aligned} \mathbf{f} &= -\operatorname{div}\boldsymbol{\sigma} && \text{equilibrium eq.} \\ \boldsymbol{\epsilon} &= \tfrac{1}{2}\left[\nabla\mathbf{u} + \nabla^T\mathbf{u}\right] && \text{strain-displacement eq.} \\ \boldsymbol{\sigma} &= \mathbf{C}[\beta] : \boldsymbol{\epsilon} && \text{constitutive eq.} \end{aligned}\right\} & \text{on } D(\boldsymbol{p}_{\text{geo}}), \\ \quad \mathbf{u} = \mathbf{0} \qquad \text{Dirichlet b.c.} & \text{on } \Gamma_{\mathbf{0}}, \\ \quad \boldsymbol{\sigma}\cdot\mathbf{n} = \mathbf{h} \qquad \text{Neumann b.c.} & \text{on } \Gamma_{\boldsymbol{\sigma}} := \partial D \setminus \Gamma_{\mathbf{0}}, \\ \quad \boldsymbol{\sigma}\cdot\mathbf{n} = \mathbf{0} \qquad \text{Neumann b.c.} & \text{on } \Gamma_{\boldsymbol{\sigma},\mathbf{0}} := \partial\mathcal{E}(\boldsymbol{p}_{\text{geo}}), \end{cases} \tag{5.5}$$

with $\Gamma_{\mathbf{0}} := [-2.5, 2.5] \times \{-2.5\}$ denoting the bottom facet of $D(\boldsymbol{p}_{\text{geo}})$. Moreover, write $\Gamma_{\boldsymbol{\sigma}} = \Gamma_R \cup \Gamma_T \cup \Gamma_L$ with right, top and left facets of D. For simplicity, we choose $\boldsymbol{f} = \mathbf{0}$ and define $h \equiv -\boldsymbol{e}_1$ on Γ_R, $h \equiv -\boldsymbol{e}_2$ on Γ_T and $h \equiv \boldsymbol{e}_1$ on Γ_L. The (weak) solution of (5.5) is ensured by the LAX-MILGRAM theorem with $\boldsymbol{u}(\boldsymbol{p}) \in H^1_{\Gamma_{\mathbf{0}}}(D(\boldsymbol{p}_{\text{geo}}))^2$. For given parameter $\boldsymbol{p} = (\boldsymbol{p}_{\text{geo}}, \beta)$, consider the free HELMHOLTZ energy

$$\Phi(\boldsymbol{p}) := \frac{1}{2} \int\limits_{D(\boldsymbol{p}_{\text{geo}})} \boldsymbol{\epsilon} : \boldsymbol{C}[\beta] : \boldsymbol{\epsilon} \mathrm{d}\boldsymbol{x} \in \mathbb{R}, \tag{5.6}$$

as the quantity of interest.

For the considered model, the parameters are assumed not be be coupled *s.t.* we can write Φ as mapping

$$\Phi\colon \bigtimes_{d=1}^{M} I_d \mapsto \mathbb{R}, \tag{5.7}$$

with closed intervals $I_d \subset \mathbb{R}$, see Table 1.

Table 1: Polymorphic parameter description for (5.5).

parameter	x_m	y_m	a	b	α	β
model	$\langle -\frac{3}{2}, 0, \frac{3}{2} \rangle$	$\langle -\frac{3}{2}, 0, \frac{3}{2} \rangle$	$\langle \frac{1}{10}, \frac{5}{10}, \frac{8}{10} \rangle$	$\langle \frac{1}{10}, \frac{5}{10}, \frac{8}{10} \rangle$	$\mathcal{U}(0, 2\pi)$	$\mathcal{U}(0, 2\pi)$

All results presented here are computed with a surrogate model based on hierarchical tensors which is extended by a high-dimensional interpolation with CHEBYSHEV polynomials, see Section 4. For each direction 10 CHEBYSHEV points are used and the $\mathcal{H}$-rank $\mathbf{k}$ is constrained s.t. $\|\mathbf{k}\|_\infty \leq 10$. From

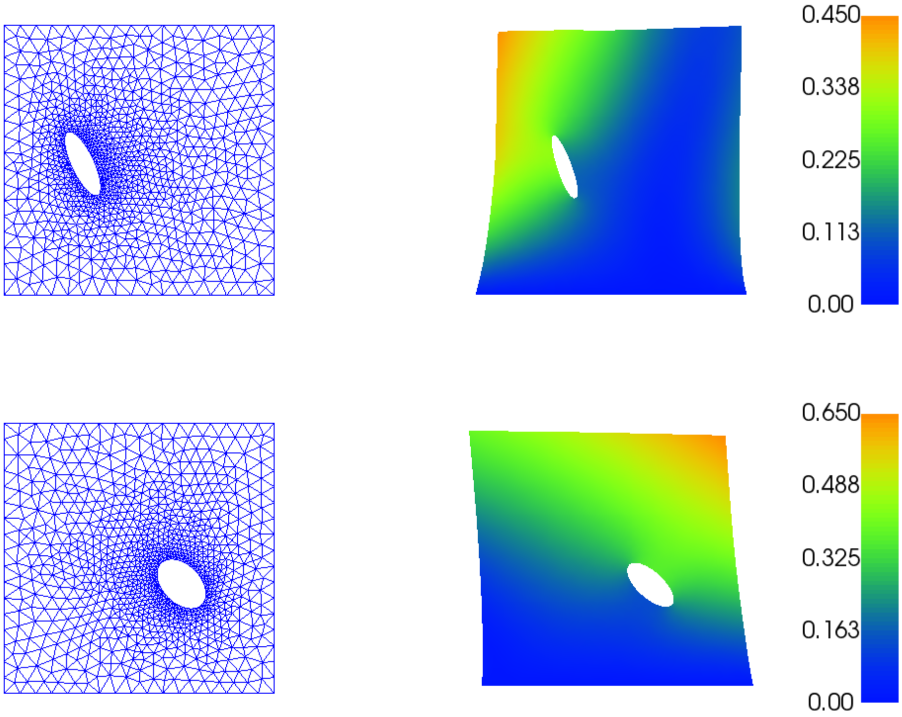
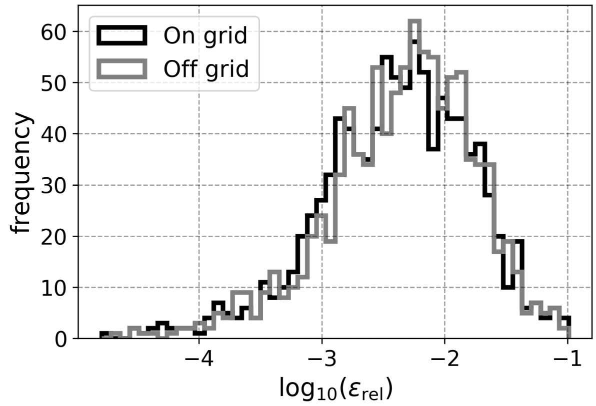


Figure 6: Illustration of displacement of (5.5) obtained by FE-simulation based on weak-formulation for realisations of $\boldsymbol{p}$.

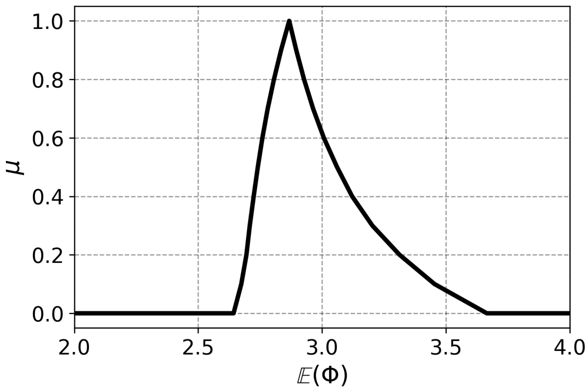
each matricization, a random submatrix of size 20×20 is chosen in order to find the pivot index sets for the construction of the $\mathcal{H}$ tensor. With this setup, 20.180 evaluations of the actual model result in a $\mathcal{H}$-rank with $k_i = 10$ for all entries. The error histogram in Figure 7 shows two distributions. The first one for random samples on the tensor grid $\mathcal{G}$ which contains the low-rank error. The second one for random samples on the whole domain which additionally contains the interpolation error. Both look very similar. Thus, it can be assumed that the low-rank error is dominating. The high ranks and the error histograms are indications that the quantity of interest at hand does not necessarily hold a low-rank structure. It is possible that a change of basis, a simple transformation of the tree structure or more ranks on the transfer nodes are a remedy. We postpone accuracy improvements to further work.

Employing this surrogate, it becomes feasible to investigate the polymorphic nature of the QoI (5.6). Depending on the interaction and level of nestedness of the probabilistic and non-probabilistic parameters, different characteristic values may be deduced. For random variables, the mean and variance are characteristic values, for fuzzy variables the propagated membership function is characteristic. In this example, the parameters are explicitly independent and only random and fuzzy variables are present. Therefore, the *randomness* of the QoI simply transfers to the propagated membership function, *i.e.* each alpha cut is a random variable. The same holds for the *fuzzyness* of the QoI, which *e.g.* has the effect that the mean value becomes a fuzzy variable.

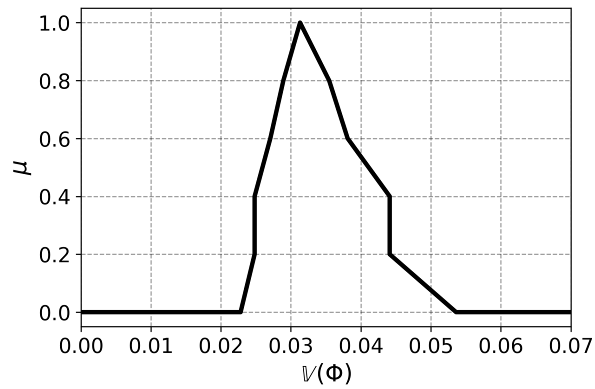
In Figure 7 the membership function of the mean and variance as well as the cumulative distribution function are shown. The computation of the membership function of the mean and variance involves a minimization and maximization via POWELL's method where each function evaluation is the computation of the mean respectively variance for one set of fixed non-probabilistic parameters. Since the extended $\mathcal{H}$ tensor has a polynomial representation, the computation of the mean is similar to a regular evaluation with an integrated polynomial basis. The computation of the variance involves a contraction of the extendet tensor with itself. For the mean, 10 equally distributed α-cuts are used. This results in appproximately 3.500 evaluations. For the variance, only 5 α-cuts are used. The tensor structure is not employed and only the empirical variance is computed via random sampling. This already leads to 101.000 evaluations. The fuzzy cumulative distribution function is produced by computing 5 α-cuts for each of the 500 random samples of the random input variables. In total, around $1.600.000$ evaluations of the $\mathcal{H}$ tensor are needed.



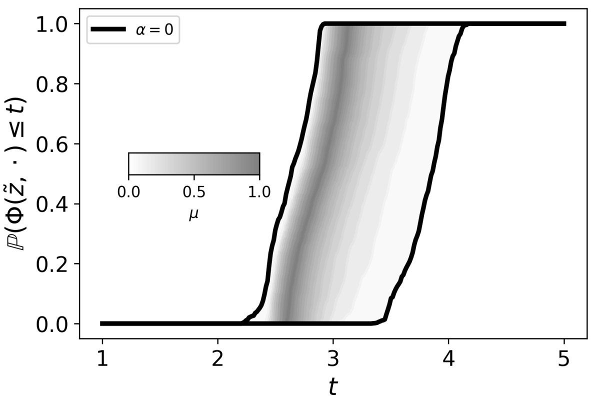
(a) Tensor approximation quality on and off the Chebyshef grid.



(b) Membership function of the mean $\mathbb{E}(\Phi)$.



(c) Membership function of the variance $\mathbb{V}(\Phi)$.



(d) Fuzzy cumulative distribution function.

Figure 7: Different views on the polymorphic properties of the QoI Φ for the elasticity problem in (5.5).

5.2 KLE type problems

Let $D = (0,1)^d$ with $d = 2$ and denote by $\|\cdot\|_2$ the EUCLIDEAN norm. Motivated by the representation of random fields through Karuhn-Loève expansions, denote by $c\colon D \times D \mapsto \mathbb{R}$ a covariance kernel. As a classical example, consider the isotropic and stationary GAUSSIAN kernel c given by

$$c(\boldsymbol{x}_1, \boldsymbol{x}_2, \rho^2) = c(\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|, \rho^2) := \sigma^2 \exp -\frac{\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|_2^2}{\rho^2 \Lambda^2}, \quad \boldsymbol{x}_1, \boldsymbol{x}_2 \in D, \tag{5.8}$$

with a *correlation length* parameter $\rho \in Z = \mathbb{R}^+$ and a scaling parameter $\sigma^2 > 0$ and diameter $\Lambda = \operatorname{diam}(D) = 1$. Motivated by a possibly unknown correlation length in practise, we model $z := \rho^2$

as a triangular fuzzy number $\tilde{z} = \langle 0.5, 0.75, 1\rangle$ and choose $\sigma = 1$. For fixed $z \in C_0[\tilde{z}] = [0.5, 1]$, describe a random field $\kappa\colon D \times \Omega \mapsto \mathbb{R}$ as

$$\kappa(\boldsymbol{x}, \omega, z) = \kappa(\boldsymbol{x}, \boldsymbol{\xi}(\omega), z) = \kappa_0(\boldsymbol{x}, z) + \sum_{m=1}^{\infty} \kappa_m(\boldsymbol{x}, z)\xi_m(\omega), \tag{5.9}$$

with mean $\kappa_0(\cdot, z) \equiv 2.5$ and *iid* random variables $\xi_m \sim \mathcal{U}(-1, 1)$. Here,

$$\kappa_m(\cdot, z) = \sqrt{\lambda_m}(z)\phi_m(\cdot, z), \tag{5.10}$$

where $(\lambda_m(z), \phi_m(\cdot, z))$ denotes the m-th eigenpair of the associated covariance operator C w.r.t. c given as

$$C\colon L^2(D) \mapsto L^2(D), \quad u \mapsto C[v](\cdot) := \int_D c(\boldsymbol{x}_1, \cdot, z)v(\boldsymbol{x}_1)\mathrm{d}\boldsymbol{x}_1. \tag{5.11}$$

By a push-forward argument [75] with $\mu = \boldsymbol{\xi}_\#\mathbb{P}$ and image identification $\boldsymbol{y} = \boldsymbol{\xi}(\omega) \in \Xi := [-1, 1]^\infty$, $y_m = \xi_m(\omega) \in [-1, 1]$, we henceforth shall work in a parametric framework. Denote by $\kappa^M(\cdot, z)$ the truncation of (5.9) after M terms. We have convergence of $\kappa^M(\cdot, z)$ to $\kappa(\cdot, z)$ in $L^2(\Xi, \mathcal{B}(\Xi), \mu; L^\infty(D))$ [55], where $\mathcal{B}(\Xi)$ denotes the BOREL sigma algebra on Ξ omitted in the following notations. The accuracy can be characterised by the decay of the eigenvalues $(\lambda_m(z))_m$. The GAUSSIAN kernels is the smooth limit of the *Matérn* kernel class with regularity parameter $\nu \to \infty$. The decay of the eigenvalues depends on the correlation length ρ, *e.g.* [76] given as

$$\lambda_m(\rho) \in \mathcal{O}\left(\sigma^2 \frac{(1/\rho)^{m^{1/d}+2}}{\Gamma(0.5m^{1/d})}\right), \quad \forall m \geq 1, \tag{5.12}$$

with Gamma function Γ. For $z \in C_0[\tilde{z}]$ representing a fixed realisation of ρ^2 and $\boldsymbol{y} \in \Xi$, consider the elliptic model problem

$$\begin{aligned} -\operatorname{div}\kappa(\boldsymbol{x}, \boldsymbol{y}, z)\nabla u(\boldsymbol{x}, \boldsymbol{y}, z) &= f(\boldsymbol{x}) && \text{in } D, \\ \gamma_0 u(\boldsymbol{x}, \boldsymbol{y}, z) &= 0 && \text{on } \Gamma_0 := [0, 1] \times \{0\}, \\ \gamma_1^\kappa[z]u(\boldsymbol{x}, \boldsymbol{y}, z) &= g && \text{on } \Gamma_g := \partial D \setminus \Gamma_0, \end{aligned} \tag{5.13}$$

with DIRICHLET trace operator γ_0 and conormal trace operator $\gamma_1^\kappa[z]$. The conormal trace g is defined for $\boldsymbol{x} = (x_1, x_2)$ as

$$g(\boldsymbol{x}) = \begin{cases} \sin(5\pi x_1), & \boldsymbol{x} \in \Gamma_{\mathrm{t}} := \{1\} \times [0, 1], \\ 0, & \boldsymbol{x} \in \Gamma_{\mathrm{br}} := \partial D \setminus (\Gamma_0 \cup \Gamma_{\mathrm{t}}). \end{cases} \tag{5.14}$$

The source term is deterministic and given by $f(\boldsymbol{x}) = 10\exp(-20(x_1 - 0.5)^2 + (x_2 - 0.5)^2))$. The field κ is uniformly bounded and strictly positiv, *i.e.* there exists positive constants $\underline{\kappa}, \overline{\kappa}$ s.t.

$$0 < \underline{\kappa} < \kappa(z; \boldsymbol{x}, \omega) < \overline{\kappa} < \infty, \quad \textit{a.e. in } D \times \Xi \times C_0[\tilde{z}]. \tag{5.15}$$

Since $g \in H_{00}^{1/2}(\Gamma_g)$ and $f \in L^2(D)$ due to the LAX-MILGRAM theorem for fixed $z \in C_0[\tilde{z}]$, there exists a unique solution $u(\cdot, z) \in V := L^2(\Xi, \mu; \mathbb{P}; H^1_{\Gamma_0}(D))$. We note that V is independent of Z. Hence, $u(\cdot, \tilde{z}) \in \mathcal{F}(V)$ defines a fuzzy set on the HILBERT space V.

For a given prescribed tolerance $\delta > 0$, let $M^\delta \in \mathbb{N}$ denote the minimal number needed to uniformly bound the truncation expansion error in (5.9) w.r.t. the range of z, *i.e.*

$$M^\delta := \inf_{M\in\mathbb{N}} \left\{ \sup_{z\in C_0[\tilde{z}]} \|\kappa(\cdot,z) - \kappa^M(\cdot,z)\|_{L^2(\Xi,\mu;L^\infty(D))} < \delta \right\} \tag{5.16}$$

The uniform bound is motivated for fixing the parametric dimension for the considered range of the correlation length and associated eigenvalue decay. Let $u^M(\cdot,z) \in V^M$ denote the weak solution of (5.13) with truncated field κ^M instead of κ. Here $V^M = L^2\left(\Xi^M, \mu^M; H^1_{\Gamma_0}(D)\right)$ denotes the solution space associated to the truncation with $\Xi^M = \operatorname{img} \boldsymbol{\xi}^M$, $\mu^M = \boldsymbol{\xi}^M_\#$ and $\boldsymbol{\xi}^M := (\xi_m)_{m=1}^M$.

Based on the underlying V-elliptic bilinear form associated to the weak formulation of (5.13), the truncation error can be estimated like

$$\|u(\cdot,z) - u^M(\cdot,z)\|_V \le C\|\kappa(\cdot,z) - \kappa^M(\cdot,z)\|_{L^2(\Xi,\mu;L^\infty(D))}. \tag{5.17}$$

Consequently, $\|u(\cdot,z) - u^{M^\delta}(\cdot,z)\|_V \in O(\delta)$ uniform in $z \in C_0[\tilde{z}]$. In our experiments we choose $M^\delta = 10$ with a uniform 50×50 triangular mesh with global polynomial degree of 2. The eigenvalue problem is approximated on a 30×30 grid instead. The effect of additional approximation of eigenfunctions is studied in [10].

Next a physical quantity of interest $\mathcal{Q}$ is defined by

$$q\colon H^1_{\Gamma_0}(D) \to \mathbb{R}, v \mapsto q(v) := \int_{\Gamma_{\mathrm{br}}} v^2 \mathrm{d}S, \tag{5.18}$$

defining our polymorphic quantity of interest as

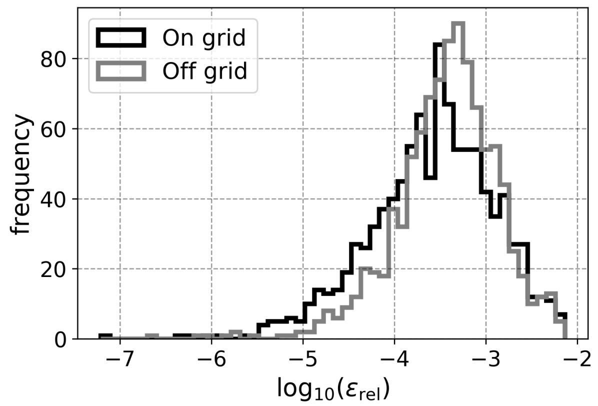
$$q(u(\cdot,\tilde{\boldsymbol{p}})), \quad \tilde{\boldsymbol{p}} = (\boldsymbol{y}, \tilde{z}) = (\boldsymbol{\xi}(\omega), \tilde{z}). \tag{5.19}$$

The results for the diffusion problem are generated in a similar way, as in the preceeding section with slight changes. To account for the decrease of influence of each additional KLE term, the number of CHEBYSHEV points used is decreased by one for every two additional expansion terms. For the fuzzy parameter 15 points are used, for the first KLE term 6 points and for the 10-th expansion term only 2 points. The maximal rank was set to 15, but was not exhausted by the cross approximation at each transfer node. Together with the error distribution in Figure 8, this is a strong indication for an underlying low-rank structure. For the construction of the $\mathcal{H}$ tensor 20100 evaluations of the actual model were used. It can be assumed that this number may be decreased by using a tree structure which considers that the correlation lenghts ρ influences every KL term equally.

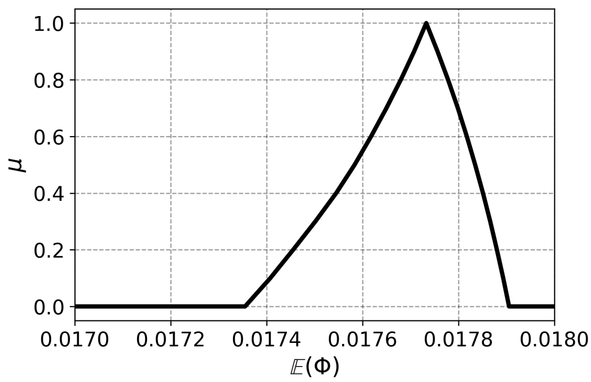
The number of α-cuts und random samples used in the graphs (b)-(d) in Figure 8 are similar to the preceeding section. Since a optimization method is involved, the number of evaluations differs. For (b) 601, for (c) 101.000 and for (d) 415.029 evaluations were used.

6 Conclusion

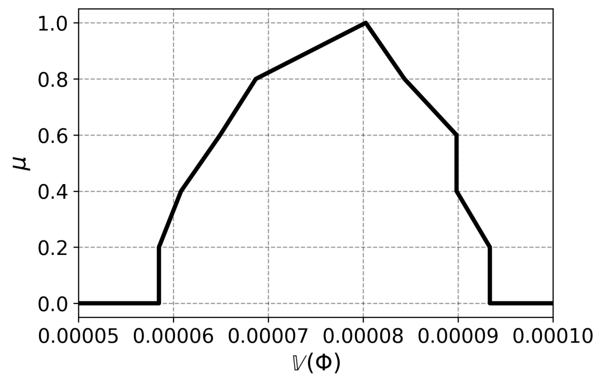
We presented a hybrid uncertainty description approach stated in a probabilistic and possibilistic modeling framework. The approach entered into the setting of general linear elliptic partial differential equations. From a parametric point of view we unified both frameworks yielding to possible high-dimensional



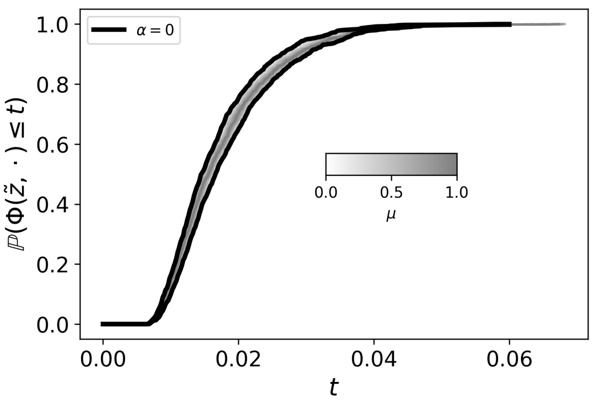
(a) Tensor approximation quality on and off the Chebyshef grid.



(b) Membership function of the mean $\mathbb{E}(\Phi)$.



(c) Membership function of the variance $\mathbb{V}(\Phi)$.



(d) Fuzzy cumulative distribution function.

Figure 8: Different views on the polymorphic properties of the QoI q for the diffusion problem in (5.13).

parameter dependent partial differential equations. Based on the special case of a separated parameteric dependence structure low-rank formats in terms of hierachical tensor formats we used as a surrogate response, allowing for very fast propagation of fuzzy-stochastic input realisations for desired quantities of interest. A $\mathcal{H}$-rank approximation for a GALERKIN scheme was presented, allowing for the representation of the whole map $\boldsymbol{p} \to u(\cdot, \boldsymbol{p})$. Based on this several quantities of interests can be computed directly without rebuilding surrogates for each of them. Numerical examples for a linear elastic -and a diffusion problem with polymorphic dependency in geometry, material or through the modeling of a KLE with underlying fuzzy GAUSSIAN kernel were demonstrated.

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