Snapshots of modern mathematics
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Invitation to quiver representation and Catalan combinatorics

Baptiste Rognerud

Representation theory is an area of mathematics that deals with abstract algebraic structures and has numerous applications across disciplines. In this snapshot, we will talk about the representation theory of a class of objects called quivers and relate them to the fantastic combinatorics of the Catalan numbers.

1 Quiver representation and a Danish game

Representation theory is based on the idea of taking your favorite mathematical object and looking at how it acts on a simpler object. In fact, it allows you to choose from a wide range of objects and play around with them.[1] Usually, the goal is to understand your favorite object better using the mathematics of the simpler object. But you can also do the opposite: you can use your favorite object to solve a problem related to the simpler object instead. For example, the famous Rubik's cube can be solved using the action of finite groups (of size $2^{27}3^{14}5^{3}7^{2}11$) consisting of all the moves on the cube preserving it.

The representation theory of finite groups was invented at the end of the 19th century and was heavily studied in the 20th century. This culminated in deep, (yet!) unsolved problems and has played a huge part in the classification of finite simple groups.

[1] Historically, the first examples had a finite group as the first object and a vector space or a finite set as the second object. We refer to the snapshot by Eugenio Giannelli and Jay Taylor [3] for an introduction to this setting.

Since then, mathematicians have looked at the representation theory of many different objects. In this article, we will look at "finite quivers". Roughly speaking, a finite quiver is a collection of a finite set of vertices and a finite set of arrows between the vertices. (See Figure 1 for illuminating examples.)

$$A_n = \; 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n \qquad J = \begin{tikzcd} \bullet \arrow[loop right] \end{tikzcd} \qquad Kr = \; \bullet \rightrightarrows \bullet$$

$$\mathrm{C} = \begin{tikzcd} & 2 \arrow[d] & \\ 1 \arrow[ur] \arrow[ur, bend left] \arrow[r] & 5 & 3 \arrow[l] \arrow[ul] \arrow[dl] \arrow[loop right] \\ 6 \arrow[ur] & 4 \arrow[u] & \end{tikzcd}$$

Figure 1: Examples of quivers.

While group representation theory is quite old, the work in quiver representation theory became serious only in the 1970s, mostly under the influence of Pierre Gabriel (1933–2015). He proved that quivers provide a useful combinatorial tool for understanding the representation theory of more general objects.[2] In addition to being a useful tool, quiver representation theory is also a deep and wonderful mathematical theory with very interesting connections to other areas like combinatorics, geometry, topology, and physics.

The definition of a quiver representation is slightly technical and it would take a lot of time and effort to start from the definition and reach the interesting parts of the theory. This is why we will explore the representation theory of quivers without worrying about what a representation is. The hope is that the analogy we use will be pertinent enough to keep the reader interested, and if the reader has had some exposure to these ideas, they will be able to make the ideas in the first section rigorous.[3]

The naivest problem in representation theory is to understand *all* the representations (whatever representations are) of our favorite object. But if you think about it, "understand all the" is *almost never an interesting problem* in mathematics!

[2] Gabriel proved that over an algebraically closed field, the representation theory of any finite-dimensional algebra is equivalent to the representation theory of a finite quiver modulo some relations.

[3] If the reader knows linear algebra, we encourage them to look at [1], which is not only an excellent introduction to the subject but also goes much further than what we will discuss here.

Say we have at our disposal a bunch of Lego bricks (plastic bricks from the famous Danish toy production company) we want to build a tree with. Turns out that we have enough bricks to construct the three trees in Figure 2. Being rather proud of our constructions, we want to write the building instructions to share with our friends. Unfortunately, we have only one wooden pencil and nothing to distinguish between the colors of the bricks in the instructions. This means that we cannot distinguish between the first and the second tree. Since these two only differ in a detail which is insignificant[4] to us (the color), we want to consider them as equal, even though they are not. We say that they are "isomorphic". Now, the last tree has the same global shape as the first two. But if we disassemble it, we end up with a different set of bricks. This property will be reflected in our instructions, so we will call it "not isomorphic" to the first two.

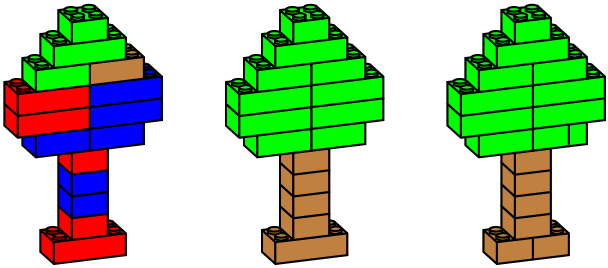

Figure 2: Three similar trees made of plastic bricks.

In representation theory, you can always "paint in red or blue" a representation by, say, relabelling the elements. This would be as meaningless as the colors of the bricks in our example. Therefore, it is much more natural to try to understand all the representations up to a suitable notion of isomorphism.

We can do even better: we can build a larger construction just by placing smaller constructions (like the three trees in Figure 2) next to each other. A connected subset of bricks (one not obtained by placing two constructions next to each other) will be called *indecomposable*. We can almost completely recover the figure by knowing only its indecomposable parts. So, the only missing information is the relative position of these parts. We have a similar notion in representation theory: any representation can be decomposed as a sum of *indecomposable representations*. This decomposition is *unique* up to reordering of its indecomposable components. This idea – that comes from the Krull–Remak–Schmidt theorem – is the reason why we are mainly interested in

[4] What one means by "insignificant detail" depends on the situation and the mathematical object under consideration.

indecomposable representations. The *ultimate goal* of a representation theorist is to understand *all the isomorphism classes of indecomposable representations* of their favorite object.

In our analogy, the ultimate goal would be to understand all the connected constructions that we can build with our given set of bricks (without caring about the colors). The "bricks" in representation theory are called simple representations. A representation is *simple* if the only representations that it contains are the zero representation[5] and itself. Any indecomposable representation is made of simple representations. This is called the Jordan–Hölder theorem. We can disassemble a representation like we disassemble our construction made of plastic bricks. In more technical terms, we say that we pass to the "Grothendieck group" of the representation. During this process, we may lose a lot of information in exactly the same way that we lose information if we disassemble the two constructions in Figure 3.

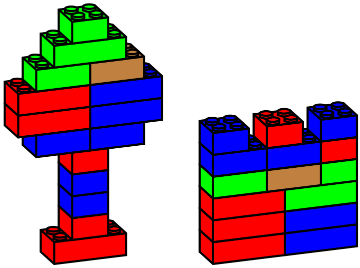

Figure 3: Two different figures which are equal in the Grothendieck group.

We have talked about disassembling a representation. But to reach our goal, we also need to understand the ways in which we can "interlock" the different bricks. Is it easy? In general, no. And this is where our analogy stops! As discovered by Gabriel, most quivers have infinitely many isomorphism classes of indecomposable representations, even if they only have a finite number of simple representations. In our analogy, this would mean that there are infinitely many ways of building the Lego tree from a finite set of Lego bricks – and we know that this cannot be true!

An important theorem that comes from Gabriel's pioneering work says that there are finitely many isomorphism classes of indecomposable representations of a finite quiver Q if and only if Q is an orientation of what is called a "Dynkin

[5] Loosely speaking, the zero representation is simply a map that sends everything to zero.

diagram”[6], in particular a Dynkin diagram of type A, D or E. (See Figure 4 for what these diagrams look like.)

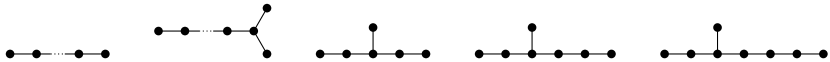

Figure 4: Dynkin diagrams of type A_n, D_n, E_6, E_7 and E_8.

One can be more precise about the indecomposable representations of a finite quiver. There are three possibilities: a quiver can be of “finite” representation type (an orientation of a Dynkin diagram), of “tame” representation type, or of “wild” representation type. A quiver is of *tame* representation type if it has infinitely many isomorphism classes of indecomposable representations but one can still classify them. On the other hand, it is *wild* if its representation theory includes, in a precise sense, the representation theory of the quiver  .[7]

The quivers of tame representation type are known and correspond to certain extensions of the Dynkin diagrams. All the others, that is most quivers, are of wild representation type.. Wild quivers are *really wild*: their representation theory is *undecidable* in a precise sense. But this also means that the indecomposable representations of these quivers are not classifiable and that we cannot achieve our original goal!

It is therefore reasonable to try to answer partial and easier questions. For example, can we find a special family of representations which is small enough to be classified and large enough to be useful? Or can we use the knowledge of the representation theory of finite or tame quivers to get a partial understanding of some wild quivers?

To answer these two questions, Brenner and Butler introduced tilting representations (defined in the next section) in 1979. Indeed, these give an interesting family that one can hope to classify. A tilting representation produces a weak form of equivalence, which can be used to translate partial information of the representation theory of a quiver to the representation theory of another quiver. To stay elementary, we cannot be more precise about this weak notion of equivalence, but we can and will classify tilting representations for the simple case of a quiver of type A_n. (See Figure 1.) We will see that tilting representations are related to extremely classical objects directly coming from the famous Swiss mathematician Leonhard Euler (1707–1783). We still don’t want to dive into

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[6] Dynkin diagrams are ubiquitous in mathematics. They can be used to classify a number of objects like the Platonic solids, the finite subgroups of the special orthogonal group $SO(3)$, and the complex semisimple Lie algebras. We refer to [10, 5] for more details.

[7] Wild quivers are indeed the quivers that are neither finite nor tame, but that discussion would be too involved to deal with here.

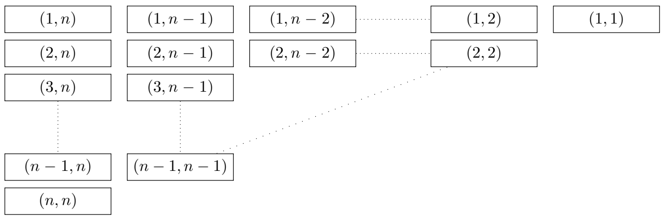
the mathematical details of representation theory, but it is time to do some mathematics, to state precise results and to prove them. For that, we introduce a combinatorial model for the representation theory of this particular quiver.

2 Counting tilting representations

We will look at the quiver:

$$A_n = 1 \to 2 \to \ldots \to n.$$

Our model is based on the diagram $\mathcal{T}_n$ which is shaped like a staircase:



A representation of the quiver A_n is an arbitrary way of coloring some of the boxes of the diagram $\mathcal{T}_n$ black. If you look at $\mathcal{T}_n$, you will see that the coordinates of the boxes are of the form (i, j) with $1 \leq i \leq j \leq n$. So, we can identify the box (i, j) with the usual interval $[i, j] := \{z \in \mathbb{Z} \ : \ i \leq z \leq j\}$. A representation of A_n is nothing but a finite set of intervals with boundaries in $\{1, 2, \cdots, n\}$. Like in our analogy from section 1, each box represents an indecomposable representation. A representation is given by deciding which indecomposable representation appears in it. (If this is the case, we color the box black. Otherwise, we leave the box empty.) The simple representations are given by intervals of size 1.

To understand our model, we should start playing with it. If $n = 1$, the diagram $\mathcal{T}_1$ contains only one box and there are only two possible representations. If $n = 2$, the diagram $\mathcal{T}_2$ has three boxes and there are 8 possible representations. Here is the list of all the representations for $n = 1$ and 2.

We can try figuring out what the number of representations of a quiver A_n for any natural number n would be. The diagram $\mathcal{T}_n$ has $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ boxes. In order to give a representation, we have to independently choose whether any given box is empty or not. Since this gives us two choices for each box, the total number of representations would simply be $2^{\frac{n(n+1)}{2}}$.

If R is a representation of A_n, then its size, denoted by $|R|$, is the number of black squares that it contains. The number of representations of A_n of size k is equal to the binomial coefficient $\binom{\frac{n(n+1)}{2}}{k}$. Using the formula

$$\sum_{k=0}^{\frac{n(n+1)}{2}} \binom{\frac{n(n+1)}{2}}{k} = 2^{\frac{n(n+1)}{2}}$$

gives us the very same answer once again.

We will now talk about what are called "tilting representations". A *tilting* representation[8] for A_n is a representation T such that

1. $|T| = n$, and
2. any two intervals of T are either contained in each other or are disjoint by at least one integer.

For $n = 1$, there is only one tilting representation of A_1 which is given by $\{[1, 1]\}$. For $n = 2$, we have to consider representations consisting of two intervals. There are only 3 such representations:

1. $R_1 = \{[1, 2], [2, 2]\}$,
2. $R_2 = \{[1, 2], [1, 1]\}$, and
3. $R_3 = \{[1, 1], [2, 2]\}$.

For R_1, we have $[2, 2] \subset [1, 2]$, and for R_2 we have $[1, 1] \subset [1, 2]$. So, both of them satisfy the two conditions for qualifying as a tilting representation. However, for R_3, we have $[1, 1] \not\subset [2, 2]$ and $[2, 2] \not\subset [1, 1]$ and they are not disjoint by at least one integer. So, R_3 is not a tilting representation. For $n = 3$, there are $\binom{6}{3} = 20$ representations of size 3. It starts getting tedious but one can still check that only five of them satisfy the conditions we have laid down above.

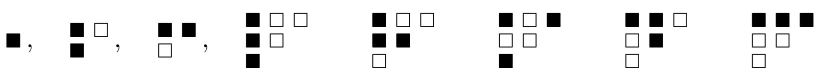


Figure 5: Tilting representations for $n = 1, 2$ and 3.

Time for a question. If you are given a natural number n, can you find a closed-form expression for t_n, the number of tilting representations of A_n?

[8] A tilting representation of A_n is a representation which has no self-extension and which has exactly n non-isomorphic indecomposable components.

We encourage the reader to stop reading the article and try answering this question. The first step is to determine t_4. You may then try to come up with a method which can be generalized to t_5, t_6 and so on.

From our experiments, we have found that $t_1 = 1, t_2 = 2, t_3 = 5$. Note that for $n = 0$, the empty filling of the empty diagram $\mathcal{T}_0$ is a tilting representation, so we can add $t_0 = 1$ to our list. This is probably not enough to guess the formula for t_n.[9] Since $\binom{10}{4} = 210$ and $\binom{15}{5} = 3003$, we will not look for t_4 and t_5 by enumerating all the possibilities.

Taking a glance at Figure 5, we observe that in a tilting representation, there is always a ■ at the top left of $\mathcal{T}_n$. In representation-theoretic language, the interval $[1, n]$ is the only "projective-injective" representation of A_n and it can be proved that any projective-injective representation must appear in a tilting representation.

It is also useful to generalize our problem by considering the set of partial tilting representations.[10] They are the representations satisfying the second condition for being a tilting representation but not necessarily the first condition. It can also be proved that a partial tilting representation is always of size less than or equal to n. (This is Bongartz's Lemma.) In other words, tilting representations are the partial tilting representations of maximal size.

While this is usually proved in great generality using algebraic arguments, it is also accessible in our combinatorial model. We leave this as a challenging exercise for the reader.

We can now write down a recursive formula for our sequence $(t_n)_{n\in\mathbb{N}}$:

$$\begin{cases} t_0 &= 1, \\ t_{n+1} &= \sum_{k=0}^{n} t_k t_{n-k} \text{ for } n \in \mathbb{N}. \end{cases} \tag{1}$$

We will work out why this is correct. For a tilting representation T of A_{n+1}, look for the left-most black square $I = [1, j]$ in the first row of T which is not $[1, n+1]$. If the first row only contains $[1, n+1]$, then we set $I = \emptyset$ and $j = 0$. Since T is a tilting representation, our second condition in the definition implies that an interval J of T is either equal to $[1, n+1]$ or $J \subseteq I$ or $J \subseteq [j+2, n+1]$. Hence the interval I splits the representation T into three parts:

1. a root $[1, n+1]$,
2. the right part: the intervals contained in I, and

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[9] Actually, if you use the wonderful On-line Encyclopedia of Integer Sequences [8] and enter the sequence $1, 1, 2, 5$, you will find a very good candidate for our sequence. At the time of writing this article, there are 2219 registered sequences involving our numbers, so it is only one of at least 2219 good candidates.

[10] Partial tilting representations are counted by what are called "large Schröder numbers" which start as $1, 2, 6, 22, 90, \ldots$

3. the left part: the intervals contained in $[j+2, n+1]$.

The black squares of the right part are below and to the right of $[1, j]$, so we can erase all the columns to the left of $[1, j]$ without losing any information. Hence, we can identify the right part with a tilting representation of A_j. Similarly, we can identify the left part with a tilting representation of A_{n-j} by erasing the top $j+1$ rows. Hence, a tilting representation of A_{n+1} produces the data of a root, a right tilting representation, and a left tilting representation. Conversely, given a tilting representation of A_j and a tilting representation of A_{n-j}, we can construct a tilting representation of A_{n+1} by placing in the diagram $\mathcal{T}_{n+1}$ the representation of A_j below and to the right of $[1, j]$, and the representation of A_{n-j} below and to the right of $[j+2, n+1]$ while not forgetting to add the root $[1, n+1]$ as well.

We denote by Tilt_n the set of tilting representations of A_n and by Tilt_{n+1}^j the set of tilting representations with $I = [1, j]$ as defined above. Then, we have just proved that there is a bijection between the sets Tilt_{n+1}^j and $\mathrm{Tilt}_j \times \mathrm{Tilt}_{n-j}$. This implies that the cardinality of Tilt_{n+1}^j is $t_j t_{n-j}$, which in turn means that Formula (1) is correct.

With this recursive formula, it is much easier to compute the values of t_n for small values of n. For example, we have $t_4 = t_0t_3 + t_1t_2 + t_2t_1 + t_3t_0 = 5 + 2 + 2 + 5 = 14$. Similarly we can compute $t_5 = 42$, $t_6 = 132$, $t_7 = 429$ etc. One can also obtain a closed-form expression for t_n for any natural number n:

$$t_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!\,n!} \tag{2}$$

Since this is slightly more advanced, we prefer to skip the proof and refer to Sections 1.3 and 1.4 of [7] for the details.

3 Catalan numbers

The Catalan numbers are one of the most famous sequences of integers in mathematics. It seems that this sequence was first discovered by Euler in 1751 when he tried to count the number of possible triangulations of a convex regular polygon. Eugène Catalan (1814–1894) was a Belgian mathematician who published several articles on this sequence of numbers (which he called Segner's numbers) during his life. Maybe Catalan's contribution was not the most important one, but his name has been attached to the sequence which is now so famous that it is impossible to change it. We refer to Appendix B of [7] for more historical information, including on the choice of the name for the sequence.

The Catalan numbers are mostly famous because they count many different objects. (In [7], you can find 214 kinds of such objects. There are many more in nature to look for.)

In order to show why our recursive formula for t_n is correct, we noticed that a tilting representation of size $n+1$ is the data of a root $[1, n+1]$, a left tilting representation of size k and a right tilting representation of size $n-k$.

Something similar appears in real life when we look at the family tree of a person. This person can be thought as the root of the tree with two parents each with a family tree of their own. The family tree of the first person is obtained by attaching the family tree of the first parent (say on the left) and the one of the second parent (say on the right) to the root. In other words, a family tree is the data of a root, a left family tree, and a right family tree.

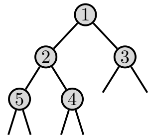

Figure 6: The start of the family tree of “1”.

In mathematics, family trees are more commonly known as binary trees. The number of people appearing in a family tree is called its size. We denote by $\mathcal{B}_n$ the set of all possible family trees of size n. By our argument, we see that

$$|\mathcal{B}_{n+1}| = \sum_{k=0}^{n} |\mathcal{B}_k| \cdot |\mathcal{B}_{n-k}|, \tag{3}$$

and that the empty tree is the only family tree of size 0. In conclusion, we have proven that there is a bijection between the tilting representations for A_n and the family trees of size n. This follows from Equations (1) and (3). More generally, the Catalan numbers can count any family of objects which satisfies the fundamental decomposition into a root, a left object, and a right object.

4 Further references

If the reader wants to learn more about quiver representation theory, we recommend looking at [6]. If you enjoyed the combinatorial model of section 2, you will be happy to learn that this can be generalized to other quivers as well. Indeed, the representations of any quiver of finite representation type can be described in terms of a finite directed graph called the Auslander–Reiten quiver. We also refer to [6] for more details. In particular, there is a nice algorithm, called the knitting algorithm, which can be used to construct it. If you are interested in the interaction between Catalan numbers and representation theory, you will be pleased to know that there is much more to it. We refer to [9] and [4] for more information. If you can read French, you should also look at Gabriel's note on Catalan numbers in representation theory [2]. To our knowledge, this article, written for a general audience, actually marks the first appearance of the Catalan numbers in quiver representation theory. As another follow-up, it may be interesting to learn more about Dynkin diagrams and finite Coxeter groups since they play a very important role not only in quiver representation theory but also in various areas of modern mathematics and physics.

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Baptiste Rognerud *is a maître de conférences at the Université de Paris.*

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Mathematisches Forschungsinstitut
Oberwolfach gGmbH
Schwarzwaldstr. 9 – 11
77709 Oberwolfach
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