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Multivariate Splines and Algebraic Geometry

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Abstract. Multivariate splines are effective tools in numerical analysis and approximation theory. Despite an extensive literature on the subject, there remain open questions in finding their dimension, constructing local bases, and determining their approximation power. Much of what is currently known was developed by numerical analysts, using classical methods, in particular the so-called Bernstein-Bézier techniques. Due to their many interesting structural properties, splines have become of keen interest to researchers in commutative and homological algebra and algebraic geometry. Unfortunately, these communities have not collaborated much. The purpose of the half-size workshop is to intensify the interaction between the different groups by bringing them together. This could lead to essential breakthroughs on several of the above problems

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Introduction by the Organisers

The workshop *Multivariate Splines and Algebraic Geometry*, was attended by 26 researchers interested in multivariate splines, polynomial approximation and algebraic geometry. A key problem in both pure and applied mathematics is to construct finite dimensional spaces of functions that are capable of approximating complicated or unknown functions well. Such spaces are especially important for scientific computing, where they are used in computer-aided geometric design, data fitting, and the solution of partial differential equations by the finite-element method. Historically, polynomials have played the central role, but more recently it has been recognized that spaces of piecewise polynomials are much more efficient and effective. A C^r-differentiable piecewise polynomial function on a d-dimensional simplicial complex $\Delta \subseteq \mathbb{R}^d$ is called a *spline*. Let $S^r_k(\Delta)$ denote the vector space of C^r splines on a fixed Δ, where each individual polynomial has degree at most k. But before we can use spline spaces, we need to solve several basic problems such as finding their dimension, constructing local bases, and determining their approximation power. Despite an extensive literature on the subject, there remain open questions in all of these areas. Much of what is currently known was developed by approximation theorists, using methods of classical analysis, in particular the so-called Bernstein-Bézier techniques. However, due to their many interesting structural properties, splines have also become of keen interest to researchers in commutative and homological algebra, geometry, combinatorics, and topology. Unfortunately, these various communities had not collaborated much. The main purpose of the workshop was to intensify the interaction between the different groups. We believe that the workshop brought together the two communities and fostered fruitful collaborations between individual researchers. We expect that such collaborations and the combined use of tools from the various mathematical fields will lead to essential breakthroughs on several of the above problems.

The workshop began with a pair of introductory lecture series: *Algebraic geometry for approximators* and *Approximation theory for geometers*. These lectures established a firm grounding in common language and tools. We also held two open problems sessions that took place in the evenings. They highlighted the key conjectures as well. Several exciting new areas were discussed, such as T-Splines and the study of splines on polyhedral (rather than simplicial) complexes. We believe that the workshop generated strong ties between the two communities, and also emphasized to the younger participants the need for interdisciplinary techniques.

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Workshop: Multivariate Splines and Algebraic Geometry

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Abstracts

Multivariate splines and the Bernstein-Bézier form of a polynomial

Peter Alfeld

The book [1] by Lai and Schumaker provides a comprehensive statement of the state of the art of the subject addressed in this talk. The reader should refer to that book for a detailed discussion of specific results, techniques, and concepts, the history of the subject, and an extensive list of references. I will omit most details in this abstract, and in particular to save space, and to avoid offense by omission, I will not attribute specific results to specific people.

Splines are smooth (at least continuous, usually at least once differentiable) piecewise polynomial functions defined on the partition of an underlying domain. The domain of a *univariate spline* is an interval, the domain of a *multivariate spline* is a (usually compact and simply connected) subset of $\mathbb{R}^k$ where $k > 1$. In most applications $k = 2$ (*bivariate splines*), sometimes $k = 3$ (*trivariate splines*), and only rarely is $k > 3$.

In one variable, an interval $\Omega = [a, b]$ is partitioned into subintervals $I_i = [x_{i-1}, x_i]$, $i = 1, \ldots N$, where $a = x_0 < x_1 < \ldots < x_N = b$ and the relevant spline space is

$$S_d^r(\Omega) = \left\{ s \in C^r(\Omega) : s\big|_{I_i} \in P_d, \quad i = 1, \ldots, N \right\}$$

where P_d is the set of polynomials of degree d in one variable.

In what follows, r will always denote the degree of smoothness, d the polynomial degree, k the dimension of the domain, and N the number of regions in the partition.

In two or more variables, $(k > 1)$, various types of partitions are possible and have been investigated. The focus in this talk is on splines defined on *triangulations* of a polygonal region in $\mathbb{R}^2$. Let Δ_i, $i = 1, 2, \ldots, N$ denote the triangles in the triangulation of Ω. Then the relevant spline space is defined similarly as in the univariate case:

$$S_d^r(\Omega) = \left\{ s \in C^r(\Omega) : s\big|_{\Delta_i} \in P_d, \quad i = 1, \ldots, N \right\}$$

where P_d is now the space of polynomials of degree d in two variables.

The sets $S_d^r(\Omega)$ are linear spaces, and as such have a (finite) dimension. In the case $k = 1$, the dimension of S_d^r can be computed using only first semester Calculus and is given by

$$\dim S_d^r = (d+1) + (N-1)(d-r) = N(d-r) + r + 1. \qquad (k = 1)$$

Note that in particular the dimension of S_d^r depends only on the **number** of subintervals in the partition, and not on the **lengths** of those subintervals. For multivariate splines, the situation is very different. The dimension of S_d^r (and a number of other properties such as the solvability of certain interpolation problems) depends not only on the combinatorics and topology of the underlying triangulation, but also on its **geometry**, i.e., the precise location of the vertices.

This fact is the source of the vastly increased complexity of multivariate splines as compared to the simplicity of univariate splines.

To understand the issues better we need to introduce a new tool. Progress in bivariate spline spaces for the past 40 years or so has benefited in a central and essential way from expressing a bivariate polynomial in its Bernstein-Bézier form. Suppose Δ is a triangle with non-zero area and vertices v_1, v_2 and v_3, and x is a point in $\mathbb{R}^2$. Then the *barycentric coordinates* b_1, b_2, b_3 of x with respect to Δ are defined by the equations

$$x = b_1 v_1 + b_2 v_2 + b_3 v_3 \quad \text{where} \quad b_1 + b_2 + b_3 = 1.$$

Then it is easy to see that any bivariate polynomial p of degree d can be written uniquely in its *Bernstein-Bézier-form* as

$$p(x) = \sum_{i+j+k=d} \frac{d!}{i!j!k!} c_{ijk} b_1^i b_2^j b_3^k.$$

The c_{ijk} are the *Bézier ordinates*, or simply *coefficients*, of p. The integers $\frac{d!}{i!j!k!}$ are convenient normalizing constants. Associated with each coefficient c_{ijk} is a *domain point*

$$P_{ijk} = \frac{iv_1 + jv_2 + kv_3}{d} \in \Delta$$

and a *control point*

$$C_{ijk} = (P_{ijk}, c_{ijk}) \in \mathbb{R}^3.$$

The source of the power of the Bernstein-Bézier-form stems from its ability to express algebraic issues, such as differentiability across a common edge of two triangles, geometrically, e.g., as the requirement that certain quadrilaterals of control points be contained in a plane. Space constraints prohibit a detailed description and illustration of the connection between algebra and geometry in this abstract. Instead I give a list of the points that were made, and argued in terms of the Bernstein-Bézier-form, in the talk, and that will be discussed more fully in a forthcoming survey paper.

1. The smoothness conditions in terms of the Bernstein-Bézier-form, and hence for example the dimension of S_d^r, are affinely invariant.

2. The control points at the vertices of a triangle lie on the graph of the polynomial.

3. A control point at a vertex and its two neighbors on the two attached edges define the tangent plane at the vertex.

4. For continuity across a common edge the two sets of control points along the edge, from the two triangles, must coincide.

5. For first order differentiability across a common edge, the d quadrilaterals along edge, formed by the control points on the edge, and the rows adjacent to the edge, must each be planar. Similar criteria apply for higher order smoothness

6. A set of domain points is a *determining set* if setting the corresponding coefficients to zero forces all other coefficients to be zero. A determining set is a *minimal determining set* if it contains no smaller determining set. The cardinality of a minimal determining set is unique and equals the dimension of S_d^r.

7. The unique minimal determining set for S_d^0 is the set of all domain points. For $d > 1$ the dimension of S_d^0 equals $V + (d-1)E + (d-2)(d-3)N/2$ where V is the number of vertices, E is the number of edges, and N is the number of triangles. (Of course, for $d = 1$ the dimension is V, and for $d = 0$ the dimension is 1.)
8. A major technique in spline analysis consists of thinking of a spline space as a subspace of a larger but simpler space, such as S_d^0, and analyzing the smoothness conditions that define the subspace.
9. The simplest case where the dimension of S_d^r depends on the geometry is the case of an interior vertex of degree 4. If the four interior edges form two parallel pairs then (and only then) the vertex is called a singular vertex, and the dimension of S_2^1 is 8. If the vertex is not singular then the dimension is only 7.
10. The dimension of S_d^1, including its dependence on the geometry, is completely understood on all stars of interior vertices, for all values of r and all values of d.
11. However, there are triangulations with more than one interior vertex where the dimension changes with the geometry, in such a way that the change disappears if the size of the triangulation is reduced. Let us call such a triangulation *sensitive* (with respect to r and d). The earliest known, and smallest, example of a sensitive triangulation is the so-called **Morgan-Scott split**. This is a particular triangulation of a triangular domain, with 7 triangles. If that triangulation is "symmetric", the dimension of S_2^1 is 7, otherwise it is only 6. Larger sensitive triangulations are known, and I conjecture that the size of sensitive triangulations is unbounded.
12. Consider a vertex star of degree 6. The dimension of S_2^1 on that space is 9. However, one cannot interpolate to function values at the 7 vertices if the union of the triangles forms a regular hexagon, or a projective transformation of a regular hexagon. One can interpolate otherwise. The issue was discussed in complete detail in Alexei Kolesnikov's talk.
13. Consider a sensitive triangulation and move the vertices, without changing the way the triangles are connected. The dimension of S_d^r depends on the precise location of the vertices, and there is a minimum value of the dimension. This value is referred to as the *generic* dimension. If the dimension of S_d^r is greater than that generic value then there is an arbitrarily small perturbation of the location of the vertices that will cause the dimension to assume the generic value.
14. The dimension is known in the case that $r = 1$ and $d = 4$, and, for all r if $d \geq 3r + 2$. It is (of course) also known for all d if $r = 0$, or when $d \leq r$. The generic dimension is known for all $r > 0$ if $d = 3r + 1$.
15. It is known that $\dim S_3^1 \geq 3V_B + 2V_I + 1 + \sigma$ where V_B is the number of boundary vertices, V_I is the number of interior vertices, and σ is the number of singular vertices. No case is know where the dimension exceeds the lower bound. I (and everybody in the field whose opinion I know) conjectures that the lower bound always equals the dimension. However, so far a proof of the conjecture has been elusive. The problem has been recognized for about 40 years, and constitutes the best known open problem in multivariate splines. It is very challenging.

16. It is unknown whether it is always possible to interpolate to the function values at the vertices of the triangulation with a spline in S^1_3. I conjecture it is.
17. The dimension of S^1_4 in general equals $6V - 3 + \sigma$. The proof of this fact is quite complicated. However, one can see very easily that the dimension equals $6V - 3$ if there are no parallel pairs of edges meeting at any vertex. In that case it is also true that one can interpolate to function and gradient values at the vertices.
18. In general, however, it is unknown if such interpolation is always possible. I conjecture that it is not.
19. The dimension analysis becomes easier as the polynomial degree increases relative to r because the domain points can be divided into sets that can be analyzed locally and independently. Full separation occurs when $d > 4r$.
20. Similar issues as those discussed here occur for trivariate splines. There is, however, one additional major fact. In two variables one can fully analyze the dimension problem if d is sufficiently large relative to r. In the case of trivariate splines defined on general tetrahedral decompositions, however, one can analyze the dimension problem for arbitrarily large values of d only after analyzing the corresponding **bivariate** problem for **all** values of d. This can be shown by a process called *coning* which for any planar triangulation constructs a particular trivariate space that is the direct sum of a set of bivariate spaces on the given triangulation. In other words, no general dimension statement for trivariate spaces can be given until the bivariate space S^1_3—and the even more difficult space S^1_2—is understood.
21. It is clear from the preceding discussion that the original spline spaces S^r_d are quite difficult to use for applications, although there have been many attempts in that direction. A major technique to make them more usable consists of considering subspaces of S^r_d defined by adding specific additional smoothness conditions, or superspaces obtained by subdividing each triangle (or tetrahedron, or simplex). In this way one obtains what are called *macro-elements* in the approximation theory community, and *finite elements* in the numerical differential equation community. This is a large area of past and current research.

References

[1] Ming-Jun Lai and Larry L. Schumaker, *Spline Functions on Triangulations*, Cambridge University Press, 2007, ISBN 0-521-87592-7.

Splines with prescribed values along algebraic curves

Oleg Davydov
(joint work with Abid Saeed)

Motivation. Spaces of multivariate piecewise polynomial splines are usually defined on triangulated polyhedral domains without imposing any boundary conditions. However, applications such as the finite element method require at least the ability to prescribe zero values on parts of the boundary. Fitting data with curved discontinuities of the derivatives is another situation where the interpolation of

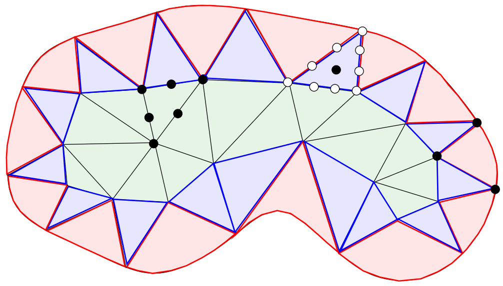

FIGURE 1. A triangulation of a curved domain with ordinary triangles (green), pie-shaped triangles (pink) and buffer triangles (blue), with the sets of domain points $D_{d,T}$, $D^*_{d-1,T}$, $D^0_{d+1,T}$ for $d = 2$ shown on one triangle of each type.

prescribed values along an algebraic surface is highly desirable. It turns out that such conditions make the otherwise well understood spaces of e.g. bivariate C^1 macro-elements on triangulations significantly more complex. Even in the simplest case of polygonal domain, the dimension of the space of splines vanishing on the boundary is dependent on its geometry, with consequences for the construction of stable bases (or stable minimal determining sets) [1, 2]. We report on recent and ongoing research on bivariate splines with prescribed values along a piecewise conic boundary.

Continuous splines. Let $\Omega \subset \mathbb{R}^2$ be a bounded curvilinear polygonal domain with $\Gamma = \partial\Omega = \bigcup_{j=1}^m \overline{\Gamma}_j$, where each Γ_j is an open arc of an algebraic curve $q_j(x) = 0$ of at most second order (i.e., either a straight line or a conic). Let $\triangle = \triangle_0 \cup \triangle_1$ be a *triangulation* of Ω, where each triangle $T \in \triangle_1 = \triangle_B \cup \triangle_P$ either has one edge replaced with a curved segment of the boundary (*pie-shaped* triangle in $\triangle_P$), or has a common edge with a pie-shaped triangle (*buffer* triangle in $\triangle_B$), while the remaining *ordinary* triangles $T \in \triangle_0$ have all straight edges, see Figure 1. We assume that there is at least one triangle $T \in \triangle_B$ attached to the common boundary vertex z of Γ_j, Γ_{j+1} if $q_j/q_{j-1} \neq$ const and at least one of q_j, q_{j-1} is of order 2.

For any $d \geq 1$ we set $S_d := \{s \in C^0(\Omega) s|_T \in \mathbb{P}_{d+i},\ T \in \triangle_i,\ i = 0, 1\}$, $S_{d,0} := \{s \in S_d(\triangle) s|_\Gamma = 0\}$, where $\mathbb{P}_n$ denotes the space of bivariate polynomials of degree at most n. By Bézout theorem $S_{d,0}|_T = \mathbb{P}_d q_j$, for each $T \in \triangle_P$ with a curved side Γ_j. Therefore, in addition to the standard sets of domain points $D_{d,T}$ for all $T \in \triangle_0$, we consider for any $T \in \triangle_P$ the set $D^*_{d-1,T} := D_{d-1,T^*}$, where T^* denotes the triangle obtained by joining the boundary vertices of T by a straight line, and each $\xi \in D^*_{d-1,T}$ represents a dual functional for $\mathbb{P}_{d-1}q$ which picks the coefficient c_ξ in the expansion

$$p = \sum_{\xi \in D^*_{d-1,T}} c_\xi B_\xi^{d-1} q, \qquad p \in \mathbb{P}_{d-1}q.$$

Finally, for all $T \in \triangle_B$ we consider $D^0_{d+1,T} := D_{d+1,T} \setminus \partial T$.

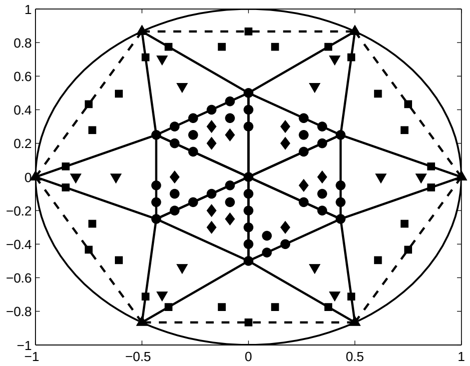

FIGURE 2. MDS for C^1 quintic macro-elements vanishing on a curved boundary. Circles, diamonds: usual MDS points for C^1 quintics on ordinary triangles. Squares: points of degree 4 on curved triangles. Triangles: points of degree 6 on buffer triangles.

Theorem 1. *Let*

$$M_0 := \bigcup_{T \in \triangle_0} D_{d,T} \cup \bigcup_{T \in \triangle_P} D^*_{d-1,T} \cup \bigcup_{T \in \triangle_B} D^0_{d+1,T}.$$

Then M_0 is a stable local minimal determining set for the space $S_{d,0}$.

Numerical experiments in [3, 5] confirm the effectiveness of the high order finite element method based on these MDS for the approximation of elliptic boundary value problems on domains enclosed by piecewise conics.

C^1 **splines.** A similar construction is suggested in [4, 5] for the spaces of C^1 quintic macro-elements that vanish on a piecewise conic boundary. Figure 2 gives an example of a stable MDS in this case. Note that in contrast to the standard parametric patching approach to finite elements on curved domains, our splines are C^1 everywhere in the domain rather than only inside the patches. This allows to employ our spaces in Bhmer's finite element method for elliptic fully nonlinear equations [6, 2], which is confirmed by the numerical experiments in [4, 5].

References

[1] O. Davydov and A. Saeed, *Stable splitting of bivariate spline spaces by Bernstein-Bézier methods, in* "Curves and Surfaces - 7th International Conference, Avignon, France, June 24-30, 2010" (J.-D. Boissonnat et al, Eds.), LNCS 6920, Springer-Verlag, 2012, pp. 220–235. doi:10.1007/978-3-642-27413-8_14

[2] O. Davydov and A. Saeed, *Numerical solution of fully nonlinear elliptic equations by Böhmer's method*, J. Comput. Appl. Math., **254** (2013), 43–54. doi:10.1016/j.cam.2013.03.009

[3] O. Davydov, G. Kostin and A. Saeed, *Polynomial finite element method for domains enclosed by piecewise conics*, in preparation.

[4] O. Davydov and A. Saeed, C^1 *piecewise polynomial splines on curved domains and numerical solution of fully nonlinear elliptic equations*, in preparation.

[5] A. Saeed, *Bivariate Piecewise Polynomials on Curved Domains, with Applications to Fully Nonlinear PDE's*, PhD Thesis, University of Strathclyde, Glasgow, 2012.
[6] K. Böhmer, *On finite element methods for fully nonlinear elliptic equations of second order*, SIAM J. Numer. Anal., **46 (3)** (2008), 1212–1249.

Piecewise polynomials and regularity

Michael DiPasquale

Let $\mathcal{P}$ be a subdivision of a region in $\mathbb{R}^n$ by convex polytopes. $C^r(\mathcal{P})$ denotes the set of piecewise polynomial functions (splines) on $\mathcal{P}$ that are continuously differentiable of order r. Splines are a fundamental tool in approximation theory and numerical analysis [6]; recently they have also been recognized in a geometric context as the equivariant cohomology ring of algebraic varieties with a torus action [17]. Practical applications include surface modelling, computer-aided design, and computer graphics [6].

One of the fundamental questions in spline theory is to determine the dimension of the vector space $C^r_d(\mathcal{P})$ of splines of degree at most d. In the bivariate, simplicial case, a signature result of Alfeld-Schumaker using Bernstein-Bezier techniques is a formula for $\dim C^r_d(\Delta)$ when $d \geq 3r+1$ and $\Delta \subset \mathbb{R}^2$ is a generic simplicial complex [2]. For $\Delta \subset \mathbb{R}^2$ simplicial and nongeneric, Hong [13] and Ibrahim-Schumaker [14] derive a formula for $\dim C^r_d(\Delta)$ when $d \geq 3r+2$ as a byproduct of constructing local bases for these spaces.

An algebraic approach to the dimension question was pioneered by Billera [4] using homological and commutative algebra. In [5], Billera-Rose show that $C^r_d(\mathcal{P}) \cong C^r(\widehat{\mathcal{P}})_d$, the dth graded piece of the algebra $C^r(\widehat{\mathcal{P}})$ of splines on the cone $\widehat{\mathcal{P}}$ over $\mathcal{P}$. The function $HF(C^r(\widehat{\mathcal{P}}),d) = \dim_{\mathbb{R}} C^r(\widehat{\mathcal{P}})_d$ is known as the *Hilbert function* of $C^r(\widehat{\mathcal{P}})$ in commutative algebra, and a standard result is that the values of the Hilbert function eventually agree with the *Hilbert polynomial* $HP(C^r(\widehat{\mathcal{P}}),d)$ of $C^r(\widehat{\mathcal{P}})$. An important invariant of $C^r(\widehat{\mathcal{P}})$ is the *postulation number* $\wp(C^r(\widehat{\mathcal{P}}))$, which is the largest integer d so that $HP(C^r(\widehat{\mathcal{P}}),d) \neq HF(C^r(\widehat{\mathcal{P}}),d)$.

This talk is on the paper [9], where we provide upper bounds on the postulation number $\wp(C^\alpha(\mathcal{P}))$ for *central* polytopal complexes $\mathcal{P} \subset \mathbb{R}^{n+1}$, $C^\alpha(\mathcal{P})$ being the algebra of *mixed splines* over $\mathcal{P}$. A *central* polytopal complex is one in which the intersection of all interior faces is nonempty; if $\mathcal{P}$ is central then splines on $\mathcal{P}$ are a graded algebra. *Mixed splines* are splines in which different smoothness conditions are imposed across codimension one faces.

The main reason for bounding $\wp(C^\alpha(\mathcal{P}))$ is that the Hilbert polynomial of $C^\alpha(\mathcal{P})$ has been computed in situations where there are no known bounds on $\wp(C^\alpha(\mathcal{P}))$, rendering these formulas impractical. Currently, bounds which do not make heavy restrictions on the complex $\mathcal{P}$ are known only in the simplicial case. These bounds are recorded in Table 1. In particular classes of complexes (e.g. cross-cut partitions) better and sometimes exact bounds are known. For brevity, we denote $\wp(C^r(\widehat{\Delta}))$ by $\wp_r$ in Table 1.

Analytic Methods

Bound	Context	Computed by
$\wp_r \leq 3r$	*generic* simplicial $\Delta \subset \mathbb{R}^2$	Alfeld-Schumaker [2]
$\wp_r \leq 3r+1$	*all* simplicial $\Delta \subset \mathbb{R}^2$	Hong [13]
		Ibrahim-Schumaker [14]
$\wp_1 \leq 3$	*all* simplicial $\Delta \subset \mathbb{R}^2$	Alfeld-Piper-Schumaker [1]
$\wp_1 \leq 7$	*generic* simplicial $\Delta \subset \mathbb{R}^3$	Alfeld-Schumaker-Whiteley [3]

Homological Methods

Bound	Context	Computed by
$\wp_r \leq 4r$	*all* simplicial $\Delta \subset \mathbb{R}^2$	Mourrain-Villamizar [16]
$\wp_1 \leq 1$	*generic* simplicial $\Delta \subset \mathbb{R}^2$	Billera [4]

Table 1. Bounds on $\wp_r = \wp(C^r(\widehat{\Delta}))$

In contrast, the Hilbert polynomial $HP(C^\alpha(\mathcal{P}), d)$ has been computed for *all* central polytopal complexes $\mathcal{P} \subset \mathbb{R}^3$. This is done in the simplicial case with mixed smoothness by Schenck-Geramita [10], in the polytopal case with uniform smoothness by Schenck-McDonald [15], and in the polytopal case with mixed smoothness and boundary conditions in [8]. In this paper we provide the first bound on $\wp(C^\alpha(\mathcal{P}))$ for all central polytopal complexes $\mathcal{P} \subset \mathbb{R}^3$. Specifically, given *smoothness parameters* $\alpha(\tau)$ associated to each codimension one face $\tau \in \mathcal{P}$, our first result is the following.

Theorem 1 Let $\mathcal{P} \subset \mathbb{R}^3$ be a central, pure, hereditary three-dimensional polytopal complex. Set

$$e(\mathcal{P}) = \max_{\tau \in \mathcal{P}_2^0} \{ \sum_{\gamma \in (\mathrm{st}(\tau))_2} (\alpha(\gamma)+1) \},$$

where $\mathrm{st}(\tau)$ denotes the star of τ and $(\mathrm{st}(\tau))_2$ denotes the 2-faces of $\mathrm{st}(\tau)$. Then

$$\wp(C^\alpha(\mathcal{P})) \leq e(\mathcal{P}) - 3.$$

In particular, $HP(C^\alpha(\mathcal{P}), d) = \dim_{\mathbb{R}} C^\alpha(\mathcal{P})_d$ for $d \geq e(\mathcal{P}) - 2$.

From an algebraic perspective, another reason for bounding $\wp(C^\alpha(\mathcal{P}))$ is that almost all existing bounds, including most in Table 1, have been computed using analytic techniques. There are a few instances where algebraic techniques are applied to bound $\wp(C^\alpha(\mathcal{P}))$. In [4], Billera proves $\wp(C^1(\widehat{\Delta})) \leq 1$ for *generic* simplicial complexes. The most general bound produced by homological techniques to date is by Mourrain-Villamizar [16]; building on work of Schenck-Stillman [19] they prove that $\wp(C^r(\widehat{\Delta})) \leq 4r$ for Δ a planar simplicial complex, recovering an earlier result of Alfeld-Schumaker. Our second result is the following.

Theorem 2 Let $\Delta \subset \mathbb{R}^3$ be a central, pure, hereditary three-dimensional simplicial complex. For a 2-face $\tau \in \Delta_2^0$, set

$$M(\tau) = (\alpha(\tau)+1) + \max\{(\alpha(\gamma_1)+1) + (\alpha(\gamma_2)+1) | \gamma_1 \neq \gamma_2 \in (\mathrm{st}(\tau))_2\}.$$

Then

$$\wp(C^\alpha(\Delta)) \le \max_{\tau\in\Delta_2^0}\{M(\tau)\} - 2.$$

In particular, $HP(C^\alpha(\Delta), d) = \dim_{\mathbb{R}} C^r(\Delta)_d$ for $d \ge \max_{\tau\in\Delta_2^0}\{M(\tau)\} - 1$.

Setting $\alpha(\tau) = r$ for all $\tau \in \Delta_2^0$ in Theorem 2, we recover that $HP(C^r(\widehat{\Delta}), d) = \dim C^r(\widehat{\Delta})_d$ for $d \ge 3r+2$. This was originally proved via constructing local bases by Hong [13] and Ibrahim-Schumaker [14], and is the best bound valid for all planar simplicial complexes recorded in Table 1.

A key tool we use to prove these results is the *Castelnuovo-Mumford regularity* of $C^\alpha(\mathcal{P})$, denoted *reg*$(C^\alpha(\mathcal{P}))$. This invariant is also used by Schenck-Stiller in [18]. Our particular way of using regularity is inspired by an observation used in the Gruson-Lazarsfeld-Peskine theorem bounding the regularity of curves in projective space [11]. In the context of splines this observation is roughly that, if we are lucky, we can bound *reg*$(C^\alpha(\mathcal{P}))$ by the regularity of a 'bad' approximation. We take as our approximation certain locally-supported subalgebras of splines introduced in [7]. This could be viewed as an algebraic analogue of locally-supported bases used in [13, 14].

References

[1] P. Alfeld, B. Piper, L. Schumaker, *An Explicit Basis for C^1 Quartic Bivariate Splines*, SIAM J. Num. Anal. **24** (1987), 891-911.

[2] P. Alfeld, L. Schumaker, *On the dimension of bivariate spline spaces of smoothness r and degree $d = 3r+1$*, Numer. Math. **57** (1990), 651-661.

[3] P. Alfeld, L. Schumaker, W. Whiteley, *The generic dimension of the space of C^1 splines of degree $d \ge 8$ on tetrahedral decompositions.* SIAM J. Numer. Anal. **30** (1993), 889-920.

[4] L. Billera, *Homology of smooth splines: generic triangulations and a conjecture of Strang*, Trans. Amer. Math. Soc. **310** (1988), 325-340.

[5] L. Billera, L. Rose, *A Dimension Series for Multivariate Splines*, Discrete Comput. Geom. **6** (1991), 107-128.

[6] C. de Boor, *A practical guide to splines*, Springer-Verlag, New York, 2001.

[7] M. DiPasquale, *Lattice-Supported Splines on Polytopal Complexes*, Adv. in Appl. Math. **55** (2014), 1-21.

[8] M. DiPasquale, *Associated Primes of Spline Complexes*, submitted, http://arxiv.org/abs/1410.6894

[9] M. DiPasquale, *Regularity of Mixed Spline Spaces*, submitted, http://arxiv.org/abs/1411.2176

[10] A. Geramita, H. Schenck, *Fat points, inverse systems, and piecewise polynomial functions*, J. Algebra **204** (1998), 116-128.

[11] L. Gruson, R. Lazarsfeld, C. Peskine, *On a theorem of Castelnuovo, and the equations defining space curves*, Invent. Math. **72** (1983), 491–506.

[12] D. Grayson, M. Stillman, Macaulay2, available at http://www.math.uiuc.edu/Macaulay2.

[13] D. Hong, *Spaces of bivariate spline functions over triangulation, Approx.* Theory Appl. **7** (1991), 56-75.

[14] A. Ibrahim, L. Schumaker, *Super spline spaces of smoothness r and degree $d \ge 3r+2$*, Constr. Approx. **7** (1991), 401-423.

[15] T. McDonald, H. Schenck, *Piecewise Polynomials on Polyhedral Complexes*, Adv. in Appl. Math. **42** (2009), 82-93.

[16] B. Mourrain, N. Villamizar, *Homological techniques for the analysis of the dimension of triangular spline spaces*, J. Symbolic Comput. **50** (2013), 564-577.
[17] S. Payne, *Equivariant Chow cohomology of toric varieties*, Math. Res. Lett. **13** (2006), 29-41.
[18] H. Schenck, P. Stiller, *Cohomology vanishing and a problem in approximation theory*, Manuscripta Math. **107** (2002), 43-58.
[19] H. Schenck, M. Stillman, *Local Cohomology of Bivariate Splines*, J. Pure Appl. Algebra **117, 118** (1997), 535-548.

Three topics in multivariate spline theory

Simon Foucart

This report highlights some interactions between multivariate spline theory and other areas of mathematics. Firstly, we show that a concept from spline theory is pertinent in polytope geometry by revealing that Dehn–Sommerville equations are nothing but a count of domain points. Secondly, we point out that the concept of Hilbert series can be exploited in spline theory to generate dimension formulas for fixed simplicial partitions. Thirdly, we show how a conjecture emanating in spline theory can be reformulated in a language close to Algebraic Geometry in the hope to stimulate an attack from this direction.

A low-tech approach to Dehn–Sommerville equations: Given a polytope P in $\mathbb{R}^n$, let $f_i(P)$ be the number of i-dimensional faces of P, $i \in \{0, 1, \dots, n-1\}$. Euler–Poincaré equation states that

$$\sum_{i=0}^{n-1} (-1)^i f_i(P) = 1 + (-1)^{n-1}. \tag{1}$$

In the case of a simplicial polytope S in $\mathbb{R}^n$, this relation is complemented by the Dehn–Sommerville equations, namely

$$\sum_{i=k}^{n-1} (-1)^i \binom{i+1}{k+1} f_i(S) = (-1)^{n-1} f_k(S), \qquad k \in \{0, 1, \dots, n-1\}. \tag{2}$$

We reveal a connection with multivariate spline theory that leads to an elementary proof of Dehn–Sommerville equations. It shares a similarity with the proof of [4] by making an indirect use of a generalization

$$\sum_{i=0}^{n-1} (-1)^i f_i(P, F) = (-1)^{n-1} \tag{3}$$

of the Euler–Poincaré equation in which F is an arbitrary face of a polytope P in $\mathbb{R}^n$ and $f_i(P, F)$ denote the number of i-dimensional faces of P containing F, see [4, Theorem 8.3.1]. Spline theory brings to the table the concept of d-domain points relative to an n-simplex. There are $\binom{d+n}{n}$ such d-domain points. If k faces of the n-simplex are removed, then $\binom{d-k+n}{n}$ of the d-domain points are left. In particular, the number of interior d-domain points is $\binom{d-1}{n}$. We equip each facet of S with its d-domain points. Dehn–Sommerville equations simply follow

from counting the domain points in two different ways (in fact, the equations (2) are equivalent to the twofold count below because the subsequent arguments are completely reversible). Precisely, we have
Fact: For $d \geq 1$, the total number of domain points is

$$u_d = \sum_{k=0}^{n-1} f_k(S) \binom{d-1}{k} = \sum_{\ell=1}^{n} (-1)^{\ell-1} f_{n-\ell}(S) \binom{d+n-\ell}{n-\ell}.$$

The justification of (2) follows from this fact by calculating the generating function $G(z) = \sum_{d \geq 1} u_d z^d$ in two different ways to obtain

$$G(z) = \sum_{k=0}^{n-1} f_k \frac{z^{k+1}}{(1-z)^{k+1}} = \sum_{\ell=1}^{n} (-1)^{\ell-1} f_{n-\ell} \frac{1}{(1-z)^{n-\ell+1}} - (1+(-1)^{n-1}) f_{-1}.$$

Rearranging, setting $x = z/(z-1)$, and changing a summation index, we arrive at

$$\sum_{k=-1}^{n-1} (-1)^{k+1} f_k x^{k+1} = (-1)^n \sum_{i=-1}^{n-1} f_i (x-1)^{i+1}.$$

We now simply look at the coefficient of x^{k+1} to deduce (2).

The technique just outlined allows us to also retrieve the connection with the shelling parameters $(\alpha_0, \alpha_1, \ldots, \alpha_n)$ of S (aka its h-vector when $\alpha_0 = 1$). It starts from a third way of counting the domain points, namely

$$u_d = \sum_{k=0}^{n} \alpha_k \binom{d-k+n-1}{n-1}.$$

Generating dimension formulas for spline spaces: Let Δ_n represent a simplicial partition of a domain $\Omega \subseteq \mathbb{R}^n$. The space of $\mathcal{C}^r$ splines of degree $\leq d$ over Δ_n is denoted by

$$\mathcal{S}_d^r(\Delta_n) := \left\{ s \in \mathcal{C}^r(\Omega) : s_{|T} \ n\text{-variate polynomial of degree} \leq d, \text{ all } T \in \Delta_n \right\}.$$

We are interested in the dimension of this space. Although finding a general formula is a notoriously difficult problem, we take a different viewpoint, in that we specify a fixed simplicial partition Δ_n and we want to produce a formula for $\dim \mathcal{S}_d^r(\Delta_n)$ in a somewhat automated fashion. We recall the method of [5], which was based on the use of Alfeld's applet [1] to compute $\dim \mathcal{S}_d^r(\Delta_n)$ for fixed d and r and on a fundamental fact from Algebraic Geometry to infer a special form of the general formula. We also explain how the fundamental fact (i.e., the form of the Hilbert series) can be obtained from Bernstein–Bézier techniques alone, and also how the additional information from [3] can be derived from the results of [2]. We highlight some of the formulas conjectured in [5] and pinpoint the ones for octahedra (regular and generic) in the case $n = 3$. As an aside, we draw attention to a method proposed by P. Clarke based on some conversion to Commutative Algebra and implemented in SAGE as a package called SplineDim (downloadable on the author's webpage).

Reformulations of Schumaker's partial interpolation conjecture: Given a triangle T with vertices $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^2$ and given an integer $d \geq 1$, let

$$\mathcal{D} := \left\{ \xi_{i,j,k} := \frac{i}{d}\mathbf{v}_1 + \frac{j}{d}\mathbf{v}_2 + \frac{k}{d}\mathbf{v}_3 : i \geq 0, j \geq 0, k \geq 0, i+j+k = d \right\}$$

$$\mathcal{D}' := \left\{ \xi_{i,j,k} := \frac{i}{d}\mathbf{v}_1 + \frac{j}{d}\mathbf{v}_2 + \frac{k}{d}\mathbf{v}_3 : i \geq 1, j \geq 1, k \geq 1, i+j+k = d \right\}$$

be the set of $\binom{d+2}{2}$ domain points and of $\binom{d-1}{2}$ interior domain points, respectively, relative to T and d. Associated to each domain point, there is a bivariate Bernstein polynomial defined for $\mathbf{v} \in T$ by

$$B_{\xi_{i,j,k}}(\mathbf{v}) = \frac{d!}{i!j!k!} x^i y^j z^k,$$

where (x, y, z) are the barycentric coordinates of $\mathbf{v}$, so that $\mathbf{v} = x\,\mathbf{v}_1 + y\,\mathbf{v}_2 + z\,\mathbf{v}_3$ with $x \geq 0$, $y \geq 0$, $z \geq 0$, and $x + y + z = 1$. Schumaker's conjecture (see [7, Conjecture 2.22]) states, in its weak form:

$$\text{Is } \mathbf{B}_{\Gamma,\Gamma} := \left[B_\xi(\eta)\right]_{\eta,\xi\in\Gamma} \text{ an invertible matrix for any } \Gamma \subseteq \mathcal{D}? \tag{4}$$

and, in its strong form:

$$\text{Is } \det\left(\mathbf{B}_{\Gamma,\Gamma}\right) > 0 \text{ for any } \Gamma \subseteq \mathcal{D}? \tag{5}$$

We observe numerically that the strong form holds at least up to $d = 17$ (this observation was also made in [6]), that its trivariate version holds at least up to $d = 16$, and that its quadrivariate version holds at least up to $d = 14$ (see the MATLAB reproducible file available on the author's webpage). Finally, we show that Schumaker's conjecture has the following equivalent formulations.

- The strong version (5) can be phrased as a linear complementarity problem:
Does there exist, for every $\mathbf{q} \in \mathbb{R}^n$, an $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{x} \geq \mathbf{0}$, $\mathbf{Mx} + \mathbf{q} \geq \mathbf{0}$, and $\langle \mathbf{Mx} + \mathbf{q}, \mathbf{x} \rangle = 0$? Here $\mathbf{M} = \mathbf{B}_{\mathcal{D},\mathcal{D}}$, $n = \binom{d+2}{2}$, or $\mathbf{M} = \mathbf{B}_{\mathcal{D}',\mathcal{D}'}$, $n = \binom{d-1}{2}$.
- The weak version (4) can be phrased in terms of the space $\mathcal{P}_d$ of polynomials of degree d in two variables u and v:
Does $\left\{[d - i(1-u) - j(1-v)]^d, (i,j) \in \Lambda\right\} \cup \left\{u^i v^j, (i,j) \in \mathcal{I} \setminus \Lambda\right\}$ form a basis for the space $\mathcal{P}_d$ whatever the subset Λ of $\mathcal{I} := \{(i,j) : i \geq 0, j \geq 0, i+j \leq d\}$?
- The strong version (5) can be phrased in terms of a bivariate Vandermonde matrix at points $(x_{i,j}, y_{i,j}) \in \mathbb{R}^2$, $(i,j) \in \mathcal{I}$, located at the intersections of three families of $d+1$ lines:
Is $\det\left[x_{i,j}^\mu y_{i,j}^\nu\right]_{(i,j),(\mu,\nu)\in\Lambda} > 0$ for any $\Lambda \subseteq \mathcal{I}$, where $x_{i,j} := i/(d-i-j)$ and $y_{i,j} := j/(d-i-j)$?

We hope that these three reformulations can stimulate new attempts to settle Schumaker's conjecture by offering new angles of attack.

References

[1] P. Alfeld, `http://www.math.utah.edu/~pa/3DMDS/`.
[2] P. Alfeld, *Upper and lower bounds on the dimension of multivariate spline spaces*, SIAM J. Numer. Anal. **33** (1996), 571–588.
[3] L. J. Billera and L. L. Rose, *A dimension series for multivariate splines*, Discrete and Computational Geometry **6** (1991), 107–128.
[4] B. Grünbaum, Convex polytopes. Springer, Graduate Texts in Mathematics, vol. 221.
[5] S. Foucart and T. Sorokina, *Generating dimension formulas for multivariate splines*, Albanian J. Math **7** (2013), 25–35.
[6] G. Jaklič and T. Kanduč, *On positivity of principal minors of bivariate Bézier collocation matrix*, Appl. Math. Comput. **227** (2014), 320–328.
[7] M.-J. Lai and L. L. Schumaker, Spline functions on triangulations. Cambridge University Press, Encyclopedia of Mathematics and its Applications, vol. 110.

Multivariate toric Bézier patches

Luis David Garcia-Puente

Bézier patches and multi-sided patches. Bézier curves, triangular and rectangular Bézier patches are the fundamental units in geometric modeling. While the classical Bézier patches are widely used, special applications require more flexible multi-sided patches. In the last two decades, several control point schemes for multi-sided patches have been introduced and studied in the geometric modeling community. These include the S-patches of Loop and DeRose [6], Warren's hexagon [7], and the toric Bézier patches of Krasauskas [5]. The latter patches have received significant attention since they provide a natural generalization of Bézier patches to arbitrary lattice polygons.

The widespread adoption of Bézier patches is due in part to their possessing many useful mathematical properties. Some, such as affine invariance, end-point interpolation and the convex hull property, are built into their definitions and also hold for toric Bézier patches. In this talk, we will discuss some advances in a long-term project aimed at using and developing methods in computational algebraic geometry to elucidate which other properties are satisfied by toric Bézier patches [2, 1, 3]. In particular, we will focus on linear precision for toric Bézier patches.

Toric Bézier patches. Toric Bézier patches are based upon toric varieties from algebraic geometry and their shape may be any polytope with integer vertices. Toric Bézier patches begin with the finite set $\mathcal{A} = \{\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{Z}^2$ of lattice points inside a given polygon Δ and a vector of nonnegative *weights* $w = (w_0, \ldots, w_n) \in \mathbb{R}^{n+1}_{>}$. This data defines a map $\varphi_{\Delta,w} \colon (\mathbb{C}^\times)^2 \to \mathbb{P}^n$ via

$$\varphi_{\Delta,w}(x, y) \ := \ [w_0 x^{a_{01}} y^{a_{02}},\, w_1 x^{a_{11}} y^{a_{12}},\, \ldots,\, w_n x^{a_{n1}} y^{a_{n2}}] \,.$$

The closure of the image $\varphi_{\Delta,w}((\mathbb{C}^\times)^2)$ is called the *(translated) toric variety* $X_{\Delta,w}$. The closure of $\varphi_{\Delta,w}(\mathbb{R}^2_>)$ is the *positive part* $X^+_{\Delta,w}$ of $X_{\Delta,w}$ (the points of $X_{\Delta,w}$ with nonnegative coordinates). A toric patch of shape Δ, denoted by $Y_{\Delta,w,\mathcal{B}}$, is the image of a parametrization

$$(1) \qquad F(\mathbf{x}) : \Delta \longrightarrow \mathbb{R}^3, \qquad \mathbf{x} \longmapsto \sum_{\mathbf{a}\in\mathcal{A}} w_{\mathbf{a}}\beta_{\mathbf{a}}(\mathbf{x})\mathbf{b}_{\mathbf{a}},$$

obtained by composing an arbitrary parametrization $\Delta \to X^+_{\mathcal{A},w}$ given by a set of *basis functions* $\{\beta_{\mathbf{a}}(\mathbf{x}) : \Delta \to \mathbb{R} \mid \mathbf{a} \in \mathcal{A}\}$ with a linear projection π_b given by *control points* $\mathcal{B} = \{\mathbf{b}_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\} \subset \mathbb{R}^3$ (Figure 1).

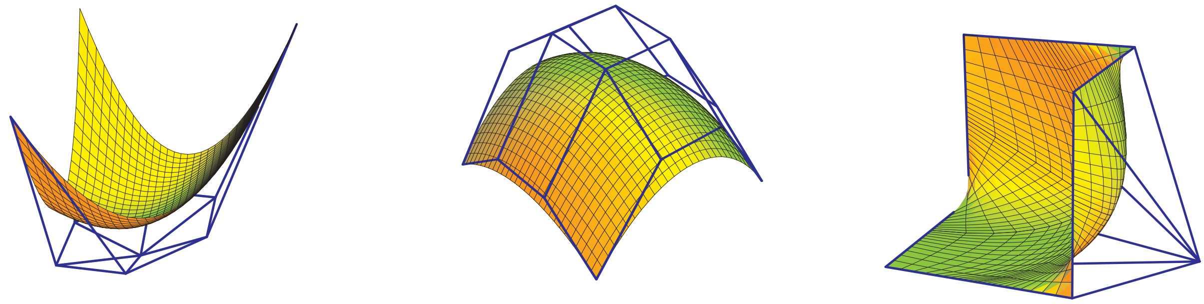

FIGURE 1. Toric Bézier patches

A canonical parametrization of a toric Bézier patch can be obtained by the following procedure. An r-sided polygon Δ with integer vertices is defined by its *facet inequalities*.

$$\Delta = \left\{(s,t) \in \mathbb{R}^2 \mid h_i(s,t) \geq 0,\ i = 1,\ldots,r\right\},$$

where $h_i(s,t) = \mathbf{v}_i \cdot (s,t) + c_i$ with inward pointing primitive normal vector $\mathbf{v}_i$. For each lattice point $\mathbf{a} \in \mathcal{A} := \Delta \cap \mathbb{Z}^2$, there is a *toric Bézier basis function*

$$\beta_{\mathbf{a}}(s,t) := h_1(s,t)^{h_1(\mathbf{a})}h_2(s,t)^{h_2(\mathbf{a})}\cdots h_r(s,t)^{h_r(\mathbf{a})}.$$

Let $w = \{w_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\}$ be a set of positive weights and $\mathcal{B} = \{\mathbf{b}_{\mathbf{a}} \in \mathbb{R}^3 \mid \mathbf{a} \in \mathcal{A}\}$ be a set of *control points* in $\mathbb{R}^3$ indexed by $\mathcal{A}$. The toric Bézier patch $Y_{\Delta,w,\mathcal{B}}$ of shape Δ is parametrized by

$$\frac{\sum_{\mathbf{a}\in\mathcal{A}} w_{\mathbf{a}}\beta_{\mathbf{a}}(s,t)\mathbf{b}_{\mathbf{a}}}{\sum_{\mathbf{a}\in\mathcal{A}} w_{\mathbf{a}}\beta_{\mathbf{a}}(s,t)} : \Delta \longrightarrow \mathbb{R}^3.$$

In particular, given the polygons in Figure 2, and a fixed set of weights and control points, we obtain the following parametrizations of a triangular toric Bézier surface patch and a rectangular toric Bézier surface patch.

$$F(s,t) = \sum_{k,l} \frac{\binom{n}{kl} s^k t^l (n-s-t)^{n-k-l}}{n^n}\mathbf{b}_{kl},$$

$$F(s,t) = \sum_{k,l} \frac{\binom{m}{k}\binom{n}{l} s^k (m-s)^{m-k} t^l (n-t)^{n-l}}{m^m n^n}\mathbf{b}_{kl}.$$

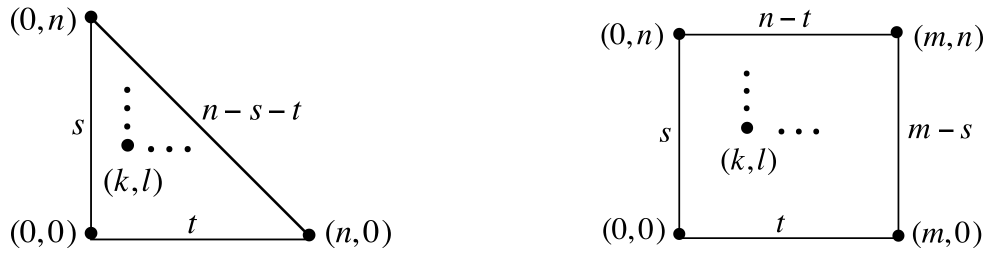

FIGURE 2. Polygons with three and four sides.

These formulas are linear reparametrizations of the usual *rational Bézier triangle* and *rational Bézier rectangle* whose images are depicted in Figure 1, along with an hexagonal toric patch.

Linear Precision. The ability of a patch to replicate affine functions is an important property of Bézier patches known as *linear precision.* More precisely, a patch parametrized by Equation 1 has *linear precision* if the *tautological map* $\tau := \sum_{\mathbf{a}\in\mathcal{A}} w_{\mathbf{a}}\beta_{\mathbf{a}}(\mathbf{x})\mathbf{a}$ is the identity function on Δ.

An important problem is to classify which toric Bézier patches have linear precision. In [2], we showed that every toric patch has a unique reparametrization with linear precision, namely the inverse of the moment map μ_Δ^{-1}. The *moment map* $\mu_\Delta\colon X_{\Delta,w}\to\mathbb{C}^2$ is defined by

$$X_{\Delta,w}\ni z=[z_0,z_1,\ldots,z_n]\ \longmapsto\ \mu_\Delta(z)\ :=\ \frac{\sum_{i=0}^n \mathbf{a}_i z_i}{\sum_{i=0}^n z_i}\ .$$

Of practical and theoretical interest is the classification of the toric surface patches whose unique reparametrization with linear precision is a rational function. The following result, proved in [2], is the cornerstone in the classification of the toric surface patches with *rational* linear precision obtained in [4].

Theorem 1. *Given a toric patch of shape* (Δ, w)*, let* $f = f_{\Delta,w}$ *be the homogeneous polynomial whose dehomogenization is the sum of monomials with exponents* $\mathcal{A}$ *and coefficients* w*. The toric patch admits a rational reparametrization with linear precision if and only if the toric differential*

$$D_{\rm toric}f\ :=\ \Bigl[x\tfrac{\partial}{\partial x}f\ :\ y\tfrac{\partial}{\partial y}f\ :\ z\tfrac{\partial}{\partial z}f\Bigr]\ :\ \mathbb{CP}^2\ \dashrightarrow\ \mathbb{CP}^2$$

defines a birational isomorphism.

Theorem 2 (Graf Von Bothmer-Ranestad-Sottile)**.** *There are only three shapes of toric surface patches with rational linear precision: Bézier triangles, Bézier rectangles, and toric surface patches of trapezoidal shape defined by the integer lattice points in the trapezoid with corners* $(0,0),(0,n),(m,n)$*, and* $(m+dn,0)$*.*

Future Work. It is an interesting and difficult problem to classify the toric Bézier volumes that have rational linear precision. The result about surface patches used the classification of *birational maps* of the projective plane $\mathbb{P}^2$. Very little is known in general about birational maps of projective space $\mathbb{P}^3$, despite the fact that this is a central problem in classical algebraic geometry. So treating this case, which is important for high-dimensional modeling, will require the difficult task of advancing the theory about such birational maps.

References

[1] Gheorghe Craciun, Luis David García-Puente, and Frank Sottile, *Some geometrical aspects of control points for toric patches*, Mathematical Methods for Curves and Surfaces 2008. Lecture Notes in Computer Science, **5862**, Springer (2010) 111–135.

[2] Luis David Garcia-Puente and Frank Sottile, *Linear precision for parametric patches*, Advances in Computational Mathematics, **33**, no. 2 (2010) 191–214.

[3] Luis David Garcia-Puente, Frank Sottile, and Chungang Zhu, *Toric degenerations of Bezier patches*, ACM Transactions on Graphics, **30**, no. 5, article 110 (2011).

[4] H.-C. Graf van Bothmer, K. Ranestad, and F. Sottile, *Linear precision for toric surface patches*, Foundations of Computational Mathematics, **10**, no. 1 (2010) 37–66.

[5] R. Krasauskas, *Toric surface patches*, Advances in Computational Mathematics, **17**, no. 1-2 (2002) 89–113.

[6] Charles T. Loop and Tony D. DeRose, *A multisided generalization of Bézier surfaces*, ACM Trans. Graph. **8** (1989), no. 3, 204–234.

[7] Joe Warren, *Creating multisided rational Bézier surfaces using base points*, ACM Trans. Graph. **11** (1992), no. 2, 127–139.

Blossoming approach for determining the dimension of the bivariate spline space $S_n^1(\triangle)$

Gašper Jaklič
(joint work with Jernej Kozak)

In the last 30 years the problem of determining the dimension of the bivariate spline space has received a considerable attention. For a given triangulation $\triangle$ of a polygonal region $\Omega \subset \mathbb{R}^2$ with N triangles Ω_i, *the bivariate spline space* of degree n and smoothness r is defined as

$$S_n^r(\triangle) := \{f \in C^r(\Omega); \quad f|_{\Omega_i} \in \Pi_n(\mathbb{R}^2), \quad i = 1, 2, \dots, N\},$$

where $\Pi_n(\mathbb{R}^2)$ denotes the space of bivariate polynomials of total degree $\leq n$. In contrast to the univariate case, the bivariate spline space has a much more complex structure and even such basic problems as determining its dimension or construction of its basis are surprisingly hard to tackle. Even more surprising is the fact that the "simplest" spaces of splines of the lowest degrees are the most complex. For example, for the most interesting case - the space of cubic C^1 splines $S_3^1(\triangle)$, quite frequently used in practical applications, the dimension is still unknown in general, even though a great deal of research has been done on the topic. But it is essential that the dimension is known in advance in some important applications, in particular for Lagrange interpolation by bivariate splines.

In general, the problem has been solved for a spline space of degree n and smoothness r over a regular triangulation $\triangle$, $S_n^1(\triangle)$, where the degree n is large in comparison to the smoothness r ($n \geq 3r + 2$ ([5]), $n = 4, r = 1$ ([1])). Recall that a triangulation is *regular*, if two adjacent triangles Ω_i, Ω_j can have only one vertex or the whole edge in common.

The dimension of the spline space $S_3^1(\triangle)$ is known for particular classes of triangulations only [8, 9, 4, 6, 2, 12], etc. It has been conjectured that the dimension is equal to Schumaker's lower bound ([8, 9])

$$\dim S_3^1(\triangle) \geq 3V_B(\triangle) + 2V_I(\triangle) + \sigma(\triangle) + 1, \tag{1}$$

where $V_B(\triangle)$ denotes the number of boundary vertices, $V_I(\triangle)$ the number of internal vertices, and $\sigma(\triangle) = \sum_{i=1}^{V_I(\triangle)} \sigma_i$,

$$\sigma_i = \begin{cases} 1, & \text{if vertex is singular,} \\ 0, & \text{otherwise.} \end{cases}$$

A vertex is *singular* if it is obtained as an intersection of exactly two lines.

Suppose that a triangulation $\triangle$ consists of a set of triangles that all have one common vertex v. Suppose every triangle in $\triangle$ has at least one neighbour with which it shares a common edge. Then we call $\triangle$ *a cell*. If v is an interior vertex, then $\triangle$ is *an interior cell*, otherwise it is *a boundary cell* (see [10]).

The main obstacle in the study of the dimension problem is the fact that the dimension depends not only on the topology of the triangulation $\triangle$ but also on its geometry. It has been conjectured (see [11]) that the dimension is equal to Schumaker's lower bound for $n \geq 2r+1$ and that the dimension jump occurs only for singular vertices.

In this paper, the blossoming approach is used (see [3, 6]). The idea is to study the smoothness conditions between polynomial patches, written as their blossoms ([7]). This is a dual approach to the well known classical approach (see [10], e.g.) and brings a new insight to the dimension problem. An overview of cell reduction at the boundary of the triangulation is given. Thus sufficient conditions for an inductive approach for determining whether the dimension of $S_n^1(\triangle), n \geq 3$ is equal to Schumaker's lower bound for a large class of triangulations $\triangle$ are obtained. It is shown that inner cells of degrees $k = 4, 5, \ldots, 8$ can be tackled, but the reduction can be applied only in the case $k = 4$ and for special cases for $k = 5$. For $k = 6, 7, 8$, a negative result is proven. Furthermore, inner cells with more than 2 free boundary edges are studied. Since it is possible to reduce most of the cases by methods for boundary cells, we focus the study to the cases with collinearities. It is proven that a cell of degree 4 with 1 common edge and a cell of degree 5 with 2 common edges with the rest of the triangulation can be reduced. An algorithm that extends the results of [6] is presented, and it is proven that the results can be generalized to $S_n^1(\triangle)$.

References

[1] P. Alfeld, B. Piper, L. L. Schumaker, *An explicit basis for C^1 quartic bivariate splines*, SIAM J. Numer. Anal. **24(4)** (1987), 891–911.

[2] L. J. Billera, *Homology of smooth splines: generic triangulations and a conjecture of Strang*, Trans. Am. Math. Soc., **310(1)** (1988), 325–340.

[3] Z. B. Chen, Y. Y. Feng, J. Kozak, *The blossom approach to the dimension of the bivariate spline space*, J. Comput. Math. **18(2)** (2000), 183–199.

[4] O. Davydov, G. Nürnberger, and F. Zeilfelder, *Cubic spline interpolation on nested polygon triangulations*, in Curve and Surface Fitting: Saint-Malo 1999, A. Cohen, C. Rabut, and L. L. Schumaker (Eds.), Vanderbilt University Press, 2000, pp. 161–170.

[5] A. K. Ibrahim, L. L. Schumaker, *Super spline spaces of smoothness r and degree $d \geq 3r+2$*, Constr. Approx. **7** (1991), 401–423.

[6] G. Jaklič, *On the dimension of the bivariate spline space $S_3^1(\triangle)$*, Int. J. Comput. Math. **82(11)** (2005), 1355–1369.

[7] M.-J. Lai, *A characterization theorem of multivariate splines in blossoming form*, Computer Aided Geometric Design **8** (1991), 513–521.

[8] L. L. Schumaker, *On the dimension of the space of piecewise polynomials in two variables*, in: Multivariate Approximation Theory, W. Schempp and K. Zeller (Eds.), Birkhauser, Basel, 1979, pp. 251–264.

[9] L. L. Schumaker, *Bounds on the dimension of spaces of multivariate piecewise polynomials*, Rocky Mountain J. of Math. **14** (1984), 251–265.

[10] L. L. Schumaker, M-J. Lai, *Spline functions on triangulations*, Encyclopedia of mathematics and its applications, Cambridge University Press, 2007.

[11] S. O. Tohaneanu, *Smooth planar r-splines of degree $2r$*, J. Approx. Theory **132** (2005), 72–76.

[12] W. Whiteley, *A matrix for splines*, in: Progress in Approximation Theory, P. Nevai and A. Pinkus (Eds.), Academic Press, Boston, 1991, pp. 821–828.

Completeness of hierarchical Zwart-Powell box splines

Bert Jüttler

(joint work with Dominik Mokriš and Urška Zore)

1. Introduction

Hierarchical splines [2, 4] have been introduced in order to eliminate the limitations caused by tensor-product constructions for multivariate splines with respect to adaptive refinement. They have been successfully used, e.g., in isogeometric analysis [7] and reverse engineering [3]. Recently, the completeness question, i.e., whether they span the full space of piecewise polynomials, has attracted some attention ([5], [6]). Sufficient conditions for completeness have been identified.

Zwart-Powell box splines are a well-studied type of bivariate box splines (see [1]) on type II-triangulations. In order to perform adaptive refinement, it appears to be a promising idea to introduce a hierarchy of finer grids, analogously to the case of hierarchical tensor-product splines. The aim of this talk is to analyze the completeness in this case. Additional complications, which are not present in the tensor-product case, are introduced by the presence of linear dependencies. We modify the resulting hierarchical generating system by introducing additional functions in order to obtain a convenient system of linear dependency relations. The resulting system spans the full space of quadratic C^1-smooth polynomials on the hierarchical criss-cross grid if the domain hierarchy satisfies certain criteria.

2. Completeness on multicell domains

Firstly we extend the existing completeness results to the class of multicell domains. We consider a type II-triangulation on the integer grid $\mathbb{Z}^2$, see Fig. 1, left.

A union of *cells* (i.e., of unit squares in the grid) is called a *multicell domain* and will be denoted by a calligraphic letter. Its partition into the *elementary cells* (triangles of the grid) shall be denoted by the same letter in the straight font.

The Zwart-Powell box splines are piecewise quadratic C^1-smooth functions defined by the Bernstein-Bézier coefficients as shown in Figure 1, left.

For the partition M of a multicell domain $\mathcal{M}$ into elementary cells, we denote with

$$\mathbb{S}^r_d(M) = \{s \in C^r(\mathcal{M}) \mid \forall e \in M : s|_e \in \mathbb{P}_d|_e\}$$

the space of all C^r–smooth spline functions of degree d, where $\mathbb{P}_d|_e$ is the space of polynomials of degree d restricted to the elementary cell e.

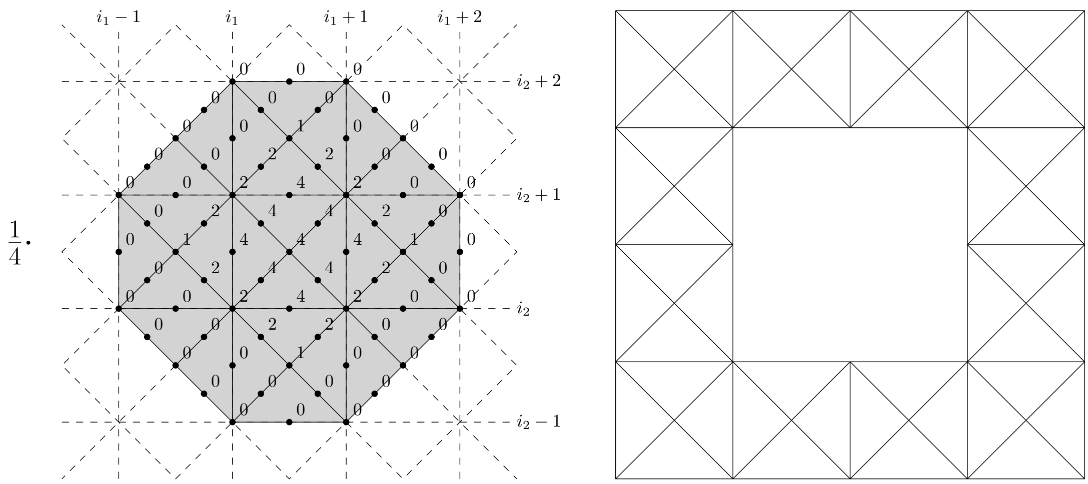

Figure 1. Left: Bernstein-Bézier coefficients defining a Zwart-Powell element. Right: The multicell domain from Example 1.

Proposition 1. *Any multicell domain $\mathcal{M}$ that is a union of mutually disjoint simply connected components is* ZP-complete, *i.e.,*

$$\operatorname{span}\{\zeta_{\boldsymbol{i}}|_{\mathcal{M}} \mid \boldsymbol{i} \in I_{\mathcal{M}}\} = \mathbb{S}^1_2(M),$$

if its intersections with the supports of all Zwart-Powell box spline are connected.

Note that the functions $\zeta_{\mathbf{i}}|_{\mathcal{M}}$ with $\mathbf{i} \in I_{\mathcal{M}}$ do not form a basis, as they are linearly dependent. Also the assumption of simply connected domains is necessary, as the next example shows.

Example 1. *The dimension of the spline space $\mathbb{S}^1_2$ on the multicell domain in Figure 1, right, is equal to 36. However, there are only 36 Zwart-Powell box splines with supports intersecting the multicell domain. They do not span the entire spline space due to the built-in linear dependency.*

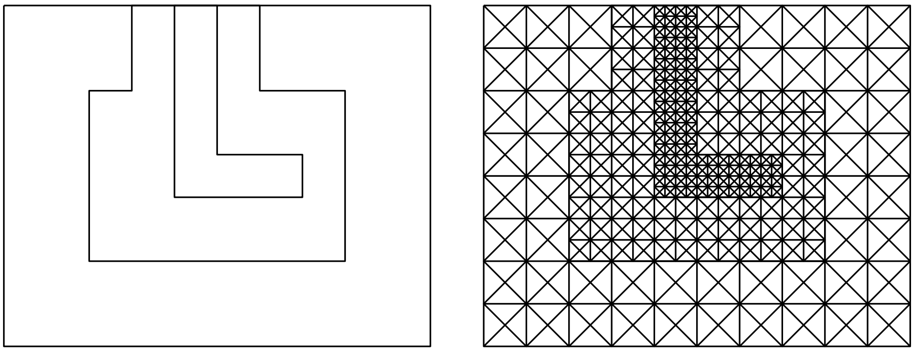

FIGURE 2. Hierarchy of nested domains (left) and the hierarchical grid defined by it (right).

3. Hierarchical generating systems

Now we proceed to the case of hierarchical generating systems. We consider Zwart-Powell box splines of level ℓ, which are defined on the scaled grid with vertices $2^{-\ell}\mathbb{Z}^2$. The set of these box splines is denoted by Z^ℓ. The spaces spanned by these box splines are nested.

In addition, we consider a given hierarchy of nested domains

$$\Omega^0 \supseteq \cdots \supseteq \Omega^{N-1} \supseteq \Omega^N = \emptyset.$$

with the property that each of them is a multicell domain with respect to the grid of level $\ell - 1$ (except for Ω^0, which is a multicell domain with respect to the grid of level 0). Furthermore, we use the notation $\mathcal{M}^\ell = \overline{\Omega^0 \setminus \Omega^{\ell+1}}$. Each $\mathcal{M}^\ell$ is a multicell domain with respect to the grid of level ℓ, see Fig. 2.

Now we recall an analogue to Kraft's selection mechanism [4], which was introduced in [8].

Definition 2. *The* Kraft generating system *is defined by*

$$\mathcal{K} = \bigcup_{\ell=0}^{N-1} \{\zeta_i^\ell \in Z^\ell \mid \operatorname{supp} \zeta_i^\ell \cap \mathcal{M}^\ell \neq \emptyset \text{ and } \operatorname{supp} \zeta_i^\ell \cap \mathcal{M}^{\ell-1} = \emptyset\}. \tag{1}$$

Furthermore, we define the *partial chessboard pattern*

$$\pi_k^\ell(\mathbf{x}) = \sum_{\substack{\mathbf{i}\in\mathbb{Z}^2,\\ \operatorname{supp} \zeta_\mathbf{i}^\ell \cap \mathcal{M}_k^{\ell-1} \neq \emptyset}} \chi_\mathbf{i} \zeta_\mathbf{i}^\ell(\mathbf{x}),$$

where $\mathcal{M}_k^{\ell-1}$ is the k-th connected component of $\mathcal{M}^{\ell-1}$ and $\chi_\mathbf{i}$ is the chessboard pattern, $\chi_\mathbf{i} = (-1)^{i_1+i_2}$. If we add these functions to the Kraft generating system, we obtain the *enriched Kraft generating system*.

We may also introduce *decoupling* (see [5]) in order to relax the assumption required for our proofs of completeness. Each function $\zeta_\mathbf{i}^\ell \in Z^\ell$ can be expressed as a linear combination of functions from $Z^{\ell+1}$; for each connected component of $\operatorname{supp} \zeta \cap \mathcal{M}^\ell$ we sum all the refined functions multiplied with their coefficients and thus obtain the *decoupled functions* from $\zeta_\mathbf{i}^\ell$.

The *enriched decoupled Kraft generating system* $\mathcal{D}$ is formed similarly to the Kraft generating system but the functions $\zeta_{\mathbf{i}}^{\ell}$ in (1) are replaced by their decoupled versions.

4. Completeness of hierarchical generating systems

The *multilevel spline space* $\mathbb{H}$ is given by

$$\mathbb{H} = \{s : \Omega^0 \to \mathbb{R} \mid \forall \ell : s|_{\mathcal{M}^\ell} \in \mathbb{S}_2^1(M^\ell)\}.$$

It is the space of all C^1 smooth quadratic spline functions on the hierarchical grid that is defined by the nested subdomains, cf. Fig. 2. We derive conditions which imply that the enriched (decoupled) Kraft generating system is complete, i.e., it spans the entire multilevel spline space.

Theorem 3. *The enriched Kraft generating system spans* $\mathbb{H}$ *if each* $\mathcal{M}^\ell$ *is a union of mutually disjoint simply connected domains with the property that their offsets in distance* $2^{-\ell-1}$ *do neither intersect each other nor possess self-intersections.*

The enriched decoupled hierarchical Kraft generating system spans $\mathbb{H}$ *if each* $\mathcal{M}^\ell$ *is a union of mutually disjoint simply connected domains.*

In addition to these results it is possible to control the number of linear dependencies, and this leads to the following result: If the assumptions of the previous theorem are satisfied, then

$$\dim \mathbb{H} = |\mathcal{D}| - n,$$

where n is the number of connected components of Ω^0.

5. Closure

We presented several generating systems for the space of piecewise quadratic C^1–smooth splines on a hierarchical criss-cross grid. Their completeness is guaranteed under certain assumptions on the domain configuration. The enriched decoupled Kraft generating system requires very few assumptions and covers a wide range of hierarchical meshes. The number of linear dependencies, which are present in it, can be analyzed.

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References

[1] C. de Boor, K. Höllig and S. Riemenschneider, *Box Splines*, Springer, 1993.

[2] D. R. Forsey and R. H. Bartels, *Hierarchical B-spline Refinement*, SIGGRAPH Comput. Graph., **22** (1988), 205–212.

[3] G. Kiss, C. Giannelli, U. Zore, B. Jüttler, D. Großmann, and J. Barner, *Adaptive CAD model (re-) construction with THB-splines*, Graph. Models, **76** (2014), 273–288.

[4] R. Kraft. *Adaptive and linearly independent multilevel B–splines*, In A. Le Méhauté, C. Rabut, and L. L. Schumaker, editors, Surface Fitting and Multiresolution Methods. Vanderbilt University Press, Nashville, (1997), 209–218.

[5] D. Mokriš, B. Jüttler, C. Giannelli, *On the completeness of hierarchical tensor-product B-splines*, J. Comput. Appl. Math., **271** (2014), 53–70.
[6] D. Mokriš, B. Jüttler, *TDHB-splines: the truncated decoupled basis of hierarchical tensor-product splines*, Comput. Aided Geom. Design, **31** (2014), 531–544.
[7] A.-V. Vuong, C. Giannelli, B. Jüttler, and B. Simeon, *A hierarchical approach to adaptive local refinement in isogeometric analysis*, Comput. Methods Appl. Mech. Engrg., **200** (2011), 3554–3567.
[8] U. Zore, B. Jüttler, *Adaptively refined multilevel spline spaces from generating systems*, Comput. Aided Geom. Design, **31** (2014), 545–566.

Interpolation properties of C^1 quadratic splines on hexagonal cells

Alexei Kolesnikov
(joint work with L. Allen, K. Borst, B. Claiborne, K. Pilewski)

Let Δ be a hexagonal cell with the interior vertex v_0 and exterior vertices $v_1, \ldots, v_6$. Let $S_2^1(\Delta)$ be the vector space of all C^1 quadratic splines on Δ. The dimension of the vector space is 9. However, it was observed by Alfeld, Piper, and Schumaker in [1] that, for some hexagonal cells Δ, every spline in $S_2^1(\Delta)$ that vanishes on 5 of the 6 exterior vertices will vanish at the 6th exterior vertex as well. This means that the exterior vertices cannot be contained in any interpolation set for $S_2^1(\Delta)$ for such cells Δ; we say in this case that $S_2^1(\Delta)$ *does not interpolate at the exterior vertices*. The authors of [1] asked for a characterization of all the cells Δ such that $S_2^1(\Delta)$ does not interpolate at the exterior vertices. The question was posed again by Peter Alfeld during a seminar talk at Towson University. This talk presents the results of a student research project that addressed the question. The project was directed by the speaker during 2014 Summer Undergraduate Applied Mathematics Institute at Carnegie Mellon University.

We start by obtaining an explicit basis for the vector space $S_2^1(\Delta)$ (in fact, the bases are obtained not only for hexagonal cells, but for all cells with at least 5 exterior vertices). After a basis is fixed, for any $s \in S_2^1(\Delta)$ and any $v \in \mathbb{R}^2$, the condition $s(v) = 0$ gives a linear equation for the coefficients of s in the given basis. Thus, the question whether $S_2^1(\Delta)$ interpolates at the exterior vertices reduces to the problem of whether the conditions $s(v_i) = 0$, $i = 1, \ldots, 6$ result in linearly independent equations. We note that, for polynomial functions, such questions have been classically studied (see, for example [2]).

A key observation is that, for the chosen basis, the linear independence of the above equations can be reformulated in purely geometric terms. In the case when the diagonals of Δ intersect at the interior vertex, the resulting characterization is particularly easy to state: $S_2^1(\Delta)$ does not interpolate at the exterior vertices if and only if the hexagon formed by the exterior vertices is regular up to a *projective* transformation. Previously, the only known example of a hexagonal cell Δ such that $S_2^1(\Delta)$ does not interpolate at the exterior vertices was that of a regular, up to an affine transformation, cell. Thus, the above characterization results in a wider class "non-interpolation" cells. We do obtain a geometric characterization (in terms of certain cross-ratios) of the cells Δ such that $S_2^1(\Delta)$ does not interpolate at exterior vertices, but a complete description of the class of all such cells still remains out of reach.

References

[1] P. Alfeld, B. Piper, L. Schumaker, *An Explicit Basis for $\mathcal{C}^1$ Quartic Bivariate Splines*, SIAM Journal on Numerical Analysis, **24**, (1987), 891–911.

[2] D. Eisenbud, M. Green, J. Harris, *Cayley–Bacharach Theorems and Conjectures*, Bulletin of the American Mathematical Society, **33**, (1996), 295–310.

Polygonal spline spaces, dimensions, and their numerical solution of the Poisson equation

Ming-Jun Lai

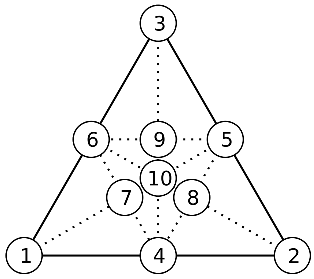
(joint work with Michael Floater)

My talk is based on a recent joint work with Michael S. Floater. I shall use generalized barycentric coordinates (GBCs) to form Bernstein-Bézierpolynomial-like functions over a polygon with any number of sides and then explain the dimension of the linear span of these Bernstein-Bézierfunctions for any fixed polygon. Next I shall explain how to use these functions to form a continuous polygonal spline space of order d over a partition consisting of polygons. An algebraic geometry approach to define such a polygonal space is presented. In particular, I shall explain a polygonal spline space which is able to reproduce all polynomials of degree d. This spline space consists of serendipity elements. Dimension of this polygonal spline space is given. Locally supported basis functions (polygonal finite elements) for the space are constructed for order $d = 2$ using any GBC and for $d \geq 3$ using Wachspress GBC. Our quadratic elements are simpler than the 'serendipity' elements that have appeared in the recent literature. Also, the basis functions are fewer than those of the virtual element method for $d \geq 2$. We use these polygonal splines to solve Poisson equations. Mainly we implement them for the numerical solution of the Poisson equation on two special types of non-triangular partitions to present a proof of concept for solving the equation over mixed partitions. Numerical solutions based on quadrangulations and pentagonal partitions are demonstrated to show the efficiency and effectiveness of these polygonal splines. They appear to better (using less degrees of freedom to find a more accurate solution) than the traditional continuous polynomial finite element method. Many open problems are proposed. After my presentation, a few audience pointed out that the first result I reported was proved in an earlier literature using algebraic geometry method. We discussed how to interpret the notation and see the dimension formula reported in my talk agrees with the formula in the published work. I am very happy to know a part of my research on approximation theory has also been studied in algebraic geometry. During this week I have learned a lot of basic concepts and ideas in algebraic geometry and geometry/topology. This made me to be able to explain my work using algebraic geometry language. The ideas of super splines and serendipity elements from approximation theory may challenge algebraic geometers how to interpret and introduce new concepts for algebraic geometry study. Anyway, I find this workshop is extremely fruitful and useful.

Simplex spline bases on the Powell-Sabin 12-split: part I

Tom Lyche and Georg Muntingh
(joint work with Elaine Cohen, Richard Riesenfeld)

Piecewise polynomials or splines defined over triangulations form an indispensable tool in the sciences, with applications ranging from scattered data fitting to finding numerical solutions to partial differential equations. In applications like geometric modeling and solving PDEs by isogeometric methods one often desires a low degree spline with C^1, C^2 or C^3 smoothness. For a general triangulation, it is known that the minimal degree of a triangular C^r element is $4r+1$, e.g., degrees 5, 9, 13 for the classes C^1, C^2 or C^3. To obtain smooth splines of lower degree one can split each triangle in the triangulation into several subtriangles. One such split that we consider here is the Powell-Sabin 12-split of a triangle.

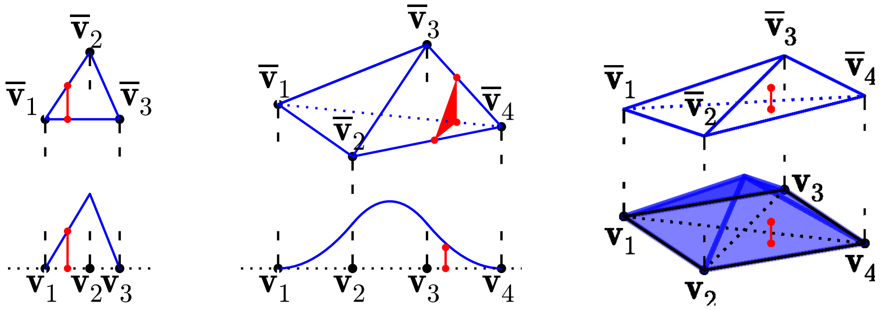


The 12-split with numbering of vertices.

Once a space is chosen one determines its dimension. The spaces $\mathcal{S}_2^1(\triangle)$ and $\mathcal{S}_5^3(\triangle)$ of C^1 quadratics and C^3 quintics on the 12-split $\triangle$ of a single triangle have dimension 12 and 39, respectively. Over a general triangulation $\mathcal{T}$ of a polygonal domain we can 12-split each triangle in $\mathcal{T}$ to obtain a triangulation $\mathcal{T}_{12}$. The dimensions of the corresponding C^1 quadratic and C^2 quintic spaces (the latter with C^3 supersmoothness at the vertices and the interior edges of each macro triangle) are $3|\mathcal{V}|+|\mathcal{E}|$ and $10|\mathcal{V}|+3|\mathcal{E}|$, respectively, where $|\mathcal{V}|$ and $|\mathcal{E}|$ are the number of vertices and edges in $\mathcal{T}$. Moreover, in addition to giving C^1 and C^2 spaces on any triangulation these spaces are suitable for multiresolution analysis, see for example [2].

To compute with these spaces one needs a suitable basis. In the univariate case the B-spline basis is an obvious choice. In this talk we consider a bivariate generalization known as simplex splines. We review the construction and a few properties shown in [1] of the S-basis consisting of C^1 quadratic simplex splines in $\mathcal{S}_2^1(\triangle)$. We also introduce some concepts needed in Part II of this talk given by Georg Muntingh.

A short background on simplex splines. Let $\boldsymbol{K} = \{\boldsymbol{v}_1 \cdots \boldsymbol{v}_{d+s+1}\} \subset \mathbb{R}^s$ be a finite multiset. Consider a simplex $\sigma = [\overline{\boldsymbol{v}}_1, \ldots, \overline{\boldsymbol{v}}_{d+s+1}] \subset \mathbb{R}^{d+s}$ together with a projection $\pi : \sigma \longrightarrow \mathbb{R}^s$ satisfying $\pi(\overline{\boldsymbol{v}}_i) = \boldsymbol{v}_i$. We define the *simplex spline* $B[\boldsymbol{K}](\boldsymbol{x}) = \mathrm{vol}_d\big(\sigma \cap \pi^{-1}(\boldsymbol{x})\big)/\mathrm{vol}_{d+s}(\sigma)$. For instance, three knots in $\mathbb{R}^1$ define a linear B-spline, four knots in $\mathbb{R}^1$ define a quadratic B-spline, and four knots in $\mathbb{R}^2$ define a linear bivariate simplex spline:



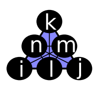
Simplex splines have all the usual properties of univariate B-splines. This includes continuity which can be controlled locally, a recurrence relation, and differentiation and knot insertion formulas. The support of a simplex spline is the convex hull of its knots, and in $\mathbb{R}^2$ the collection of knotlines is obtained by connecting each knot to all other knots (the complete graph). A simplex spline with $d+3$ knots in $\mathbb{R}^2$ has $d-m+1$ continuous derivatives across a knot line containing m knots counting multiplicites.

Simplex splines on the 12-split. Since the knotlines form a complete graph the simplex splines are natural candidates for a C^r basis on this split. A simplex spline on the 12-split will have a knotset of the form $\boldsymbol{K} = \{\boldsymbol{v}_1^{m_1} \cdots \boldsymbol{v}_{10}^{m_{10}}\}$, where $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{10}$ are the vertices numbered as above, and $m_i \geq 0$ is the multiplicity of $\boldsymbol{v}_i$, i.e., the number of repetitions of $\boldsymbol{v}_i$ in the multiset. A convenient scaling is the *(area normalized) simplex spline* $Q[\boldsymbol{K}] : \mathbb{R}^2 \longrightarrow \mathbb{R}$, recursively defined by

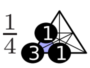
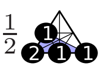
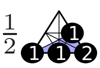
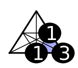
$$Q[\boldsymbol{K}](\boldsymbol{x}) := \begin{cases} 0 & \text{if } \mathrm{area}([\boldsymbol{K}]) = 0, \\ \mathbf{1}_{[\boldsymbol{K})}(\boldsymbol{x}) \frac{\mathrm{area}(\triangle)}{\mathrm{area}([\boldsymbol{K}])} & \text{if } \mathrm{area}([\boldsymbol{K}]) \neq 0 \text{ and } |\boldsymbol{K}| = 3, \\ \sum_{j=1}^{10} \beta_j Q[\boldsymbol{K} \backslash \boldsymbol{v}_j](\boldsymbol{x}) & \text{if } \mathrm{area}([\boldsymbol{K}]) \neq 0 \text{ and } |\boldsymbol{K}| > 3, \end{cases}$$

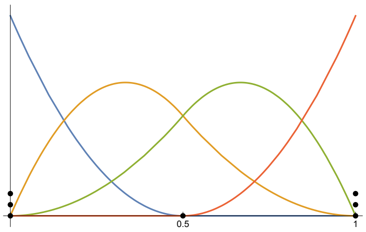
with $\boldsymbol{x} = \beta_1 \boldsymbol{v}_1 + \cdots + \beta_{10}\boldsymbol{v}_{10}, \beta_1 + \cdots + \beta_{10} = 1$, and $\beta_i = 0$ whenever $m_i = 0$.

By Theorem 4 in [4] this definition is independent of the choice of the β_j. Whenever $m_7 = m_8 = m_9 = m_{10} = 0$, we use the graphical notation



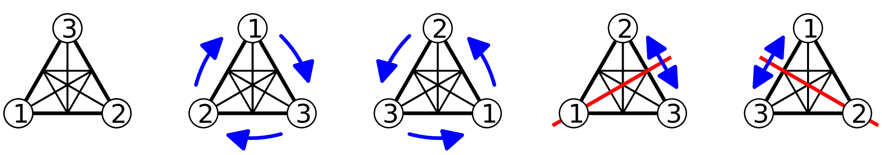
$$:= Q[\boldsymbol{v}_1^i \boldsymbol{v}_2^j \boldsymbol{v}_3^k \boldsymbol{v}_4^l \boldsymbol{v}_5^m \boldsymbol{v}_6^n].$$

B-splines on the boundary. It is useful for the simplex splines to restrict to consecutive univariate B-splines on the boundary. For example, on $[\boldsymbol{v}_1, \boldsymbol{v}_2]$ the quadratic simplex splines $\frac{1}{4}$  , $\frac{1}{2}$  , $\frac{1}{2}$  , $\frac{1}{4}$  restrict to:



$B[\boldsymbol{v}_1^3, \boldsymbol{v}_4]$, $B[\boldsymbol{v}_1^2, \boldsymbol{v}_4, \boldsymbol{v}_2]$, $B[\boldsymbol{v}_1, \boldsymbol{v}_4, \boldsymbol{v}_2^2]$, $B[\boldsymbol{v}_4, \boldsymbol{v}_2^3]$

Symmetries. Identifying a triangle with an equilateral triangle, its symmetries



form a group S_3 that acts on the simplex splines by permuting knots. We write

$$[\mathcal{B}]_{S_3} := \{Q[\sigma(\boldsymbol{K})] \,:\, Q[\boldsymbol{K}] \in \mathcal{B},\ \sigma \in S_3\}$$

for the set of simplex splines related to $\mathcal{B}$ by a symmetry in S_3. Let

$$c_4 := \frac{c_1 + c_2}{2}, \quad c_5 := \frac{c_2 + c_3}{2}, \quad c_6 := \frac{c_1 + c_3}{2},$$
$$c_7 := \frac{c_4 + c_6}{2}, \quad c_8 := \frac{c_4 + c_5}{2}, \quad c_9 := \frac{c_5 + c_6}{2}, \quad c_{10} := \frac{c_1 + c_2 + c_3}{3}.$$

Via the identification $c_i \leftrightarrow \boldsymbol{v}_i$ with the vertices of , the group S_3 acts on polynomials in $c_1, \ldots, c_{10}$ and simplex splines, or combinations of these, e.g.,

$$\left[c_4 c_{10}\right]_{S_3} = \left\{c_4 c_{10}, c_5 c_{10}, c_6 c_{10}\right\}.$$

The quadratic S-basis. It is given by

$$\left[\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right]_{S_3} = \left\{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \ldots, \frac{3}{4}, \frac{3}{4}\right\}$$

and is the unique simplex spline basis for $\mathcal{S}_2^1(\ )$ with local linear independence. Moreover, it is symmetric, reduces to B-splines on the boundary, can be computed by a pyramidal scheme, and has Bézier-like smoothness conditions across adjacent macro triangles. Furthermore, it has a barycentric Marsden identity

$$\left(\sum \left[c_1\right]_{S_3}\right)^2 = \sum \left[\frac{1}{4}c_1^2\right]_{S_3} \cup \left[\frac{3}{4}c_4 c_{10}\right]_{S_3} \cup \left[\frac{1}{2}c_1 c_4\right]_{S_3},$$

which yields polynomial reproduction, explicit dual functionals and a simple quasi-interpolant. These show that the S-basis is stable independently of the geometry, which implies an h^2 bound on the distance between a spline and its control surface.

References

[1] Elaine Cohen, Tom Lyche, Richard Riesenfeld, *A B-spline-like basis for the Powell-Sabin 12-split based on simplex splines*, Math. Comp. **82** (2013), no. 283, 1667–1707.

[2] Tom Lyche and Georg Muntingh, *A Hermite interpolatory subdivision scheme for C^2-quintics on the Powell-Sabin 12-split*, Comput. Aided Geom. Design **31** (2014), no. 7–8, 464–474.

[3] Tom Lyche and Georg Muntingh, *Stable simplex spline bases for C^3 quintics on the Powell-Sabin 12-split*, Available at `http://arxiv.org/abs/1504.02628`.

[4] Charles A. Micchelli, *On a numerically efficient method for computing multivariate B-splines*, in "Multivariate approximation theory", Walter Schempp and Karl Zeller (eds.), International Series of Numerical Mathematics Vol. 51, Birkhäuser Verlag, Basel, Boston, Stuttgart, 1979, 211–248.

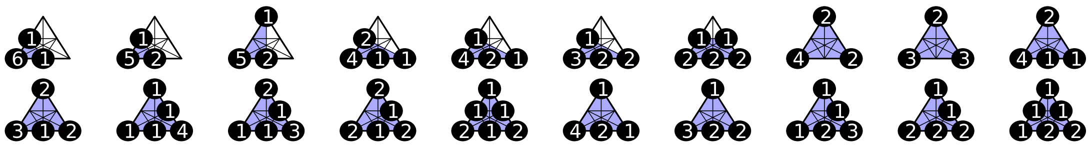
Simplex spline bases on the Powell-Sabin 12-split: part II

Tom Lyche and Georg Muntingh

Analogous to the C^1 quadratic simplex spline basis from [1], we derive C^3 quintic simplex spline bases on the Powell-Sabin 12-split  of a triangle $\triangle$ [3]. The resulting computations are implemented in a `Sage` worksheet, which can be downloaded and tried out online in `SageMathCloud` [4]. We follow the notation in Part I.

A case-by-case analysis of the possible knot multiplicities yields:

Theorem 1. *With one representative for each S_3 equivalence class, these are the C^3 quintic simplex splines on*  *that reduce to a B-spline on the boundary of $\triangle$:*

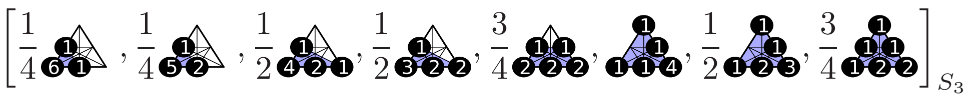


We first create a large list of potential bases for the space $\mathcal{S}_5^3($  $)$ of C^3 quintics on the 12-split. Using the macro-element from [2], we then narrow this down to a short list with good properties:

Theorem 2. *There are precisely six sets $\mathcal{B} = \mathcal{B}_a, \mathcal{B}_b, \mathcal{B}_c, \mathcal{B}_d, \mathcal{B}_e, \mathcal{B}_f$ satisfying:*

(1) *$\mathcal{B}$ is a basis of $\mathcal{S}_5^3($*  *$)$ consisting of simplex splines.*
(2) *$\mathcal{B}$ is S_3-invariant.*
(3) *$\mathcal{B}$ reduces to a B-spline basis on the boundary.*
(4) *$\mathcal{B}$ has a positive partition of unity and a Marsden identity, for which the dual polynomials have only real linear factors.*
(5) *$\mathcal{B}$ has all its domain points inside the macro triangle $\triangle$, with precisely 8 domain points on each edge of $\triangle$.*

For instance, the basis $\mathcal{B}_c = \{S_j\}_{j=1}^{39}$ is

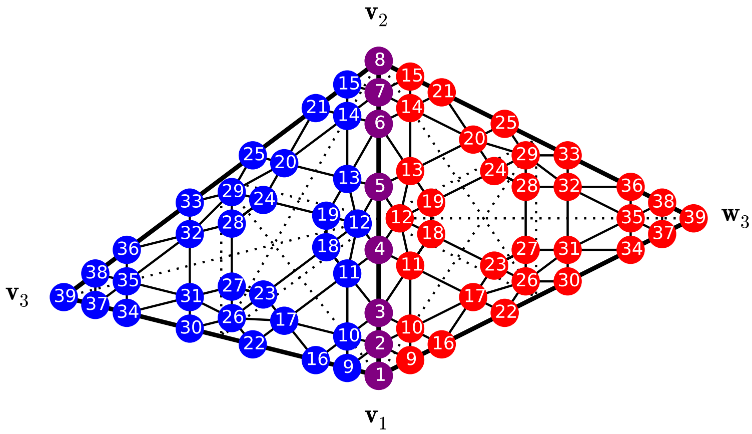


and satisfies the barycentric Marsden identity

$$\left(c_1 \; + c_2 \; + c_3 \;\right)^5 =$$

$$\sum \left[\frac{1}{4}c_1^5\right]_{S_3} \cup \left[\frac{1}{4}c_1^4c_4\right]_{S_3} \cup \left[\frac{1}{2}c_1^2c_2c_4^2\right]_{S_3} \cup \left[\frac{3}{4}c_1c_2c_4c_5c_{10}\right]_{S_3}$$
$$\cup \left[\frac{1}{2}c_1^3c_4^2\right]_{S_3} \cup \left[\frac{1}{2}c_1c_2^2c_4c_5\right]_{S_3} \cup \left[\frac{3}{4}c_1c_2c_4^2c_{10}\right]_{S_3} \cup \left[c_2^3c_4c_5\right]_{S_3}$$

Factoring the dual polynomials and replacing 'c_i' by '$\boldsymbol{v}_i$', one obtains 39 sets $\{\boldsymbol{p}^*_{j,r}\}_{r=1}^5$, $j = 1, \ldots, 39$, of dual points. Taking the average of each set one arrives at the domain points $\{\boldsymbol{\xi}_j\}_{j=1}^{39}$. To preserve the symmetry of , the domain points are forced to form a hybrid mesh with triangles, quadrilaterals, and a hexagon in the center. This mesh is shown below on two adjacent macro triangles, together with an ordering of the domain points.



The collocation matrix $\{S_j(\boldsymbol{\xi}_i\}_{i,j=1}^{39}$ is nonsingular, showing that $\{\boldsymbol{\xi}_i\}_{i=1}^{39}$ is unisolvent for $\mathcal{S}_5^3(\;)$, i.e., there is a unique Lagrange interpolant at the domain points. Moreover, it was previously shown that there is a unique Hermite interpolant for the space $\mathcal{S}_5^3(\;)$ based on values and derivatives at the corners, midpoints, and quarterpoints [2]. Finally, the Marsden identity yields that

$$Q(f) := \sum_{j=1}^{39} S_j \sum_{k=1}^{5} \frac{1}{5!}k^5(-1)^{k-1} \sum_{1\le r_1<\cdots<r_k\le 5} f\left(\frac{\boldsymbol{p}^*_{j,r_1} + \cdots + \boldsymbol{p}^*_{j,r_k}}{k}\right)$$

is a quasi-interpolant that reproduces all polynomials up to degree 5 and has approximation order 6. Moreover, using the Lagrange interpolant we show that the six bases are stable in the L_∞ norm with a condition number bounded independently of the geometry. As a consequence we obtain an h^2 bound of the distance between the Bézier ordinates and the values of the spline at the corresponding domain points.

As in the above figure, let $\triangle := [\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3]$ and $\tilde{\triangle} := [\boldsymbol{v}_1, \boldsymbol{v}_2, \tilde{\boldsymbol{v}}_3]$ be triangles sharing the edge $[\boldsymbol{v}_1, \boldsymbol{v}_2]$. Imposing a smooth join along $[\boldsymbol{v}_1, \boldsymbol{v}_2]$ of

$$f(\boldsymbol{v}) := \sum_{i=1}^{39} c_i S_i(\boldsymbol{v}),\ \boldsymbol{v} \in \triangle, \qquad \tilde{f}(\boldsymbol{v}) := \sum_{i=1}^{39} \tilde{c}_i \tilde{S}_i(\boldsymbol{v}),\ \boldsymbol{v} \in \tilde{\triangle}$$

translates into linear relations among the Bézier ordinates c_i and $\tilde{c}_i$.

Theorem 3. *Let $(\beta_1, \beta_2, \beta_3)$ be the barycentric coordinates of $\tilde{\boldsymbol{v}}_3$ with respect to the triangle $\triangle$. Then f and $\tilde{f}$ meet with*
C^0 smoothness if and only if $\tilde{c}_i = c_i$, for $i = 1, \ldots, 8$;
C^1 smoothness if and only if in addition

$$\tilde{c}_9 = \beta_1 c_1 + \beta_2 c_2 + \beta_3 c_9, \qquad \tilde{c}_{11} = \beta_1(2c_3 - c_2) + \beta_2 c_4 + \beta_3 c_{11},$$
$$\tilde{c}_{10} = \beta_1 c_2 + \beta_2 c_3 + \beta_3 c_{10}, \qquad \tilde{c}_{12} = \beta_1 \tfrac{2c_4+c_5}{3} + \beta_2 \tfrac{c_4+2c_5}{3} + \beta_3 c_{12},$$

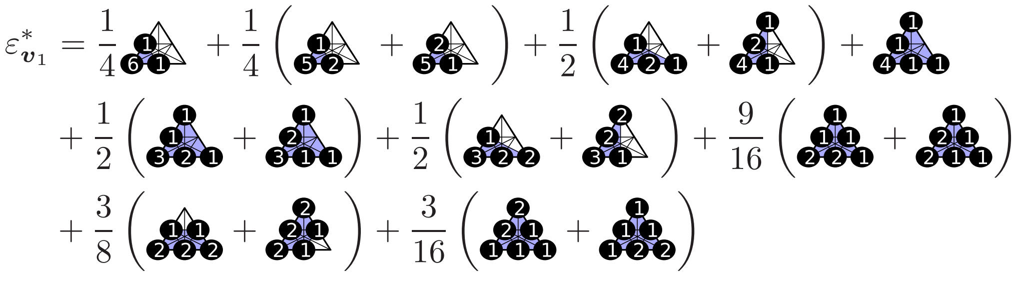
and analogous conditions for $\tilde{c}_{13}, \tilde{c}_{14}$, and $\tilde{c}_{15}$;
C^2 smoothness if and only if in addition

$$\tilde{c}_{16} = \beta_1^2 c_1 + 2\beta_1\beta_2 c_2 + \beta_2^2 c_3 + 2\beta_1\beta_3 c_9 + 2\beta_2\beta_3 c_{10} + \beta_3^2 c_{16},$$
$$\tilde{c}_{17} = \beta_1^2 c_2 + \beta_2^2 c_4 + \beta_3^2 c_{17} + 2\beta_1\beta_2 \tfrac{3c_3-c_2}{2} + 2\beta_1\beta_3 \tfrac{3c_{10}-c_2}{2} + 2\beta_2\beta_3 \tfrac{c_{10}+2c_{11}-c_3}{2},$$
$$\tilde{c}_{18} = \beta_1^2 \tfrac{2c_3+2c_4-c_2}{3} + \beta_2^2 \tfrac{c_4+2c_5}{3} + \beta_3^2 c_{18} + 2\beta_1\beta_2 \tfrac{c_2-2c_3+6c_4+c_5}{6}$$
$$+ 2\beta_1\beta_3 \tfrac{c_2-2c_3+2c_4-c_5+3c_{11}+3c_{12}}{6} + 2\beta_2\beta_3 \tfrac{9c_{12}-2c_5-c_{11}}{6},$$

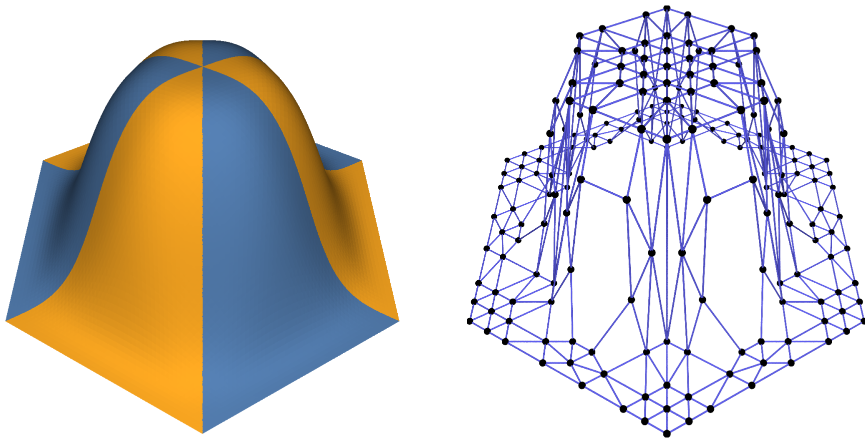
and analogous conditions for $\tilde{c}_{19}, \tilde{c}_{20}$, and $\tilde{c}_{21}$.

Whenever the domain points follow the shape of the macro triangles, we recover the classical Bézier conditions. All conditions are valid for the domain points as well, so that they also hold for the control points. Although conditions for C^3 smoothness can also be derived, one of these involves only $(\beta_1, \beta_2, \beta_3)$ and the Bézier ordinates on one triangle, showing that this element cannot be used to obtain C^3 smoothness on a general triangulation.

One can easily convert between $\mathcal{B}_c$ and the Hermite nodal basis from [2]. For instance, the nodal function corresponding to the point evaluation at $\boldsymbol{v}_1$ is



which, on a regular hexagon split at its barycenter, has the graph and wireframe



References

[1] Elaine Cohen, Tom Lyche, Richard Riesenfeld, *A B-spline-like basis for the Powell-Sabin 12-split based on simplex splines*, Math. Comp. **82** (2013), no. 283, 1667–1707.

[2] Tom Lyche and Georg Muntingh, *A Hermite interpolatory subdivision scheme for C^2-quintics on the Powell-Sabin 12-split*, Comput. Aided Geom. Design **31** (2014), no. 7–8, 464–474.

[3] Tom Lyche and Georg Muntingh, *Stable simplex spline bases for C^3 quintics on the Powell-Sabin 12-split*, Available at `http://arxiv.org/abs/1504.02628`.

[4] Georg Muntingh, *Personal Website*, `https://sites.google.com/site/georgmuntingh/academics/software`.

Almost polynomial splines over planar T-meshes

Carla Manni

(joint work with Cesare Bracco, Tom Lyche, Fabio Roman, Hendrik Speleers)

Almost polynomial (generalized) splines are smooth piecewise functions with sections in spaces of the form (see [4]):

$$\mathbb{P}_p^{U,V} := \langle 1, t, \ldots, t^{p-2}, U(t), V(t) \rangle, \quad t \in [a,b], \quad 2 \leq p \in \mathbb{N}.$$

Classical polynomial splines are obtained by taking the functions U, V equal to t^{p-1}, t^p. In such a case, the space $\mathbb{P}_p^{U,V}$ is the space of algebraic polynomials of degree p, denoted by $\mathbb{P}_p$. Other interesting examples are trigonometric or exponential generalized splines for which U, V are taken as $\cos(\alpha t), \sin(\alpha t)$, or $\cosh(\alpha t), \sinh(\alpha t)$, respectively.

Under suitable conditions on U, V, the space $\mathbb{P}_p^{U,V}$ has the same structural properties as $\mathbb{P}_p$. Similarly, generalized splines possess all the desirable properties of polynomial splines. In particular, they admit a representation in terms of basis functions that are a natural extension of the polynomial B-splines. Moreover, classical algorithms (like degree elevation, knot insertion, differentiation formulas, etc.) can be explicitly rephrased for them. Such basis functions are referred to as generalized B-splines (GB-splines).

Generalized splines are popular tools in the computer aided geometric design community. Besides their theoretical interest, generalized spline spaces offer the possibility of controlling the shape of their elements by means of some shape parameters (the value α in the case of trigonometric and exponential generalized splines mentioned above), see [8]. Moreover, they are an interesting alternative to non-uniform rational B-splines (NURBS), see [4, 10] and references therein. In particular, trigonometric and exponential generalized splines allow for an exact representation of conic sections as well as some transcendental curves (helix, cycloid, etc.) and are attractive from the geometrical point of view. For example, in contrast with NURBS, they are able to provide parameterizations of conic sections with respect to the arc length so that equally spaced points in the parameter domain correspond to equally spaced points on the described curve.

Thanks to the above properties, tensor-products of generalized B-splines are also an interesting problem-dependent alternative to tensor-product (polynomial) B-splines and NURBS.

Adaptive local refinement is fundamental in applications. Unfortunately, any tensor-product structure lacks adequate local refinement, and this drawback triggered the interest in alternative spline structures. Confining the discussion to local tensor-product structures, we mention T-splines [15], hierarchical splines [6, 7], and locally refined (LR-) splines [5].

T-splines, hierarchical splines and LR-splines can be seen as special instances of splines over T-meshes, see [13, 14]. A complete understanding of these spline spaces requires the knowledge of the dimension of the spline space defined on a prescribed T-mesh for a given degree and smoothness, see [9, 13] and references therein. Among the various techniques to tackle this difficult problem, one can use the homological approach proposed in [12], where the technique presented in [1] for splines on triangulations has been fine-tuned for splines on planar T-meshes.

As mentioned above, generalized splines enjoy the fundamental properties of polynomial splines, including the behavior with respect to local refinement. In particular, GB-splines support (locally refined) hierarchical structures in the same way as (polynomial) B-splines, see [11]. T-spline structures based on trigonometric GB-splines have been addressed in [2]. Results on the dimension of generalized spline spaces over T-meshes have been provided in [3] by extending the approach based on so-called determining sets, see [13].

In this talk we deepen the parallelism between polynomial splines and generalized splines over planar T-meshes. In particular, we extend the homological approach of [12] to generalized splines, in order to address the problem of determining the dimension of a generalized spline space on a prescribed T-mesh for a given degree and smoothness.

References

[1] L.J. Billera, *Homology of smooth splines: Generic triangulations and a conjecture of Strang*, Trans. Amer. Math. Soc. **310** (1988), 325–340.

[2] C. Bracco, D. Berdinsky, D. Cho, M.-J. Oh, and T.-W. Kim, *Trigonometric generalized T-splines*, Comput. Methods Appl. Mech. Engrg. **268** (2014), 540–556.

[3] C. Bracco and F. Roman, *Spaces of generalized splines over T-meshes*, Tech. report available at http://arxiv.org/pdf/1401.7911.pdf.

[4] P. Costantini, T. Lyche, and C. Manni, *On a class of weak Tchebycheff systems*, Numer. Math. **101** (2005), 333–354.
[5] T. Dokken, T. Lyche, and K.F. Pettersen, *Polynomial splines over locally refined box-partitions*, Comput. Aided Geom. Design **30** (2013), 331–356.
[6] D.R. Forsey and R.H. Bartels, *Hierarchical B-spline refinement*, Comput. Graph. **22** (1988), 205–212.
[7] C. Giannelli, B. Jüttler, and H. Speleers, *THB-splines: The truncated basis for hierarchical splines*, Comput. Aided Geom. Design **29** (2012), 485–498.
[8] B.I. Kvasov and P. Sattayatham, *GB-splines of arbitrary order*, J. Comput. Appl. Math. **104** (1999), 63–88.
[9] X. Li and F. Chen, *On the instability in the dimension of spline spaces over T-meshes*, Comput. Aided Geom. Design **28** (2011), 420–426.
[10] E. Mainar, J.M. Peña, and J. Sánchez-Reyes, *Shape preserving alternatives to the rational Bézier model*, Comput. Aided Geom. Design **18** (2001), 37–60.
[11] C. Manni, F. Pelosi, and H. Speleers, *Local hierarchical h-refinements in IgA based on generalized B-splines*, In: M. Floater et al. (eds.), Mathematical Methods for Curves and Surfaces 2012, LNCS **8177**, pp. 341–363. Springer-Verlag (2014).
[12] B. Mourrain, *On the dimension of spline spaces on planar T-meshes*, Math. Comp. **83** (2014), 847–871.
[13] L.L. Schumaker and L. Wang, *Approximation power of polynomial splines on T-meshes*, Comput. Aided Geom. Design **29** (2012), 599–612.
[14] L.L. Schumaker and L. Wang, *On Hermite interpolation with polynomial splines on T-meshes*, J. Comput. Appl. Math. **240** (2013), 42–50.
[15] T. Sederberg, D.L. Cardon, G.T. Finnigan, J. Zheng, and T. Lyche, *T-spline simplification and local refinement*, ACM Trans. Graphics **23** (2004), 276–283.

Spline spaces on T-meshes

Bernard Mourrain

Standard parametrisations of surfaces in CAGD are based on tensor product B-spline functions, defined from a grid of nodes over a rectangular domain. These models are easy to control but their refinment has some drawback. Inserting a node in one direction implies the insertion of several control points in the other directions of the parameter space. If for instance, regions along the diagonal of the parameter domain need to be refined, this will create a fine grid with a significant part of the parameter domain which is refined for no reason. To avoid this problem, while extending the standard tensor product representation of CAGD, functions attached to subdivisions with T-junctions instead a grid, have recently been analyzed. Such a T-subdivision is a partition of an axis-aligned box Ω (e.g., the unit square) into smaller axis–aligned boxes, called the cells of the subdivision.

A first family of T-splines has been introduced in [6]. Their construction involves functions which are piecewise rational functions, defined as blending functions over the T-mesh. There is no proof linear independency of these functions. More recently other types of splines on T-meshes have been proposed [2], [3]. They are piecewise polynomial functions of given regularity and bi-degree on the T-subdivision.

Computing the dimension and bases of these vector spaces of spline functions on T-subdivisions is an important but non-trivial issue. It has a direct impact in approximation problems such as surface reconstruction or isogeometric analysis, where controlling the space of functions used to approximate a solution is critical. In geometric design, it can also have some importance by providing more freedom to control a shape.

In this talk, we give a formula [4] for the dimension of the space $\mathcal{S}^{r,r'}_{m,m'}(\mathcal{T})$ of bivariate functions that are piecewise polynomial of bidegree (m, m') and class $C^{r,r'}$ over a planar T-subdivision $\mathcal{T}$. We exploit the homological techniques developed in [1] and [5]. By extending them to T-subdivisions, we show how to relate this dimension to the number of nodes on the maximal interior segments of the subdivision. We show that for $m \geq 2r + 1$ and $m' \geq 2r' + 1$, the dimension depends directly on the number of faces, interior edges and interior points, It yields lower and upper bounds on the dimension of these spline spaces for general T-subdivisions.

References

[1] L. Billera, *Homology theory of smooth splines: generic triangulations and a conjecture of Strang*. Trans. Amer. Math. Soc., **310** (1988), 325–340.

[2] Jiansong Deng, Falai Chen, and Yuyu Feng, *Dimensions of spline spaces over T-meshes*, J. Comput. Appl. Math. **194(2)** (2006), 267–283.

[3] Tor Dokken, Fredrik Kjell, and Tom Lyche, *Polynomial splines over locally refined box-partitions*, Computer Aided Geometric Design, **30(3)** (2013), 331–356.

[4] Bernard Mourrain, *On the dimension of spline spaces on planar T-meshes*, Mathematics of Computation, **83** (2014), 847–871.

[5] Hal Schenck and Mike Stillman, *Local cohomology of bivariate splines*, Journal of Pure and Applied Algebra, **117-118** (1997), 535 – 548.

[6] Thomas W. Sederberg, Jianmin Zheng, Almaz Bakenov, and Ahmad Nasri, *T-splines and T-nurccs*, ACM Trans. Graph., **22(3)** (2003), 477–484.

Multivariate interpolation and approximation with polynomials and splines

Ulrich Reif

Understanding the approximation properties of polynomial and piecewise polynomial functions is a basic task in approximation theory. Amazingly, some fundamental questions in that respect could only be answered recently, and others are still open today. In this talk, we report on recent progress and identify some current challenges.

While univariate interpolation with polynomials is easily understood, the multivariate case is much more complicated and leads to some unexpected phenomena, even in the simplest case of interpolation by tensor product polynomials on a tensor product grid of nodes. The following results can be found in [5]. Let $\mathbb{P}_{\mathbf{n}}$ denote the space of d-variate polynomials of coordinate order $\mathbf{n} = [n_1, \ldots, n_d]$, and let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_d \subset \Omega := [0, 1]^d$ be a regular grid of interpolation nodes of according size. Given a continuous function $f : \Omega \to \mathbb{R}$, there exists a unique polynomial $p = If \in \mathbb{P}_{\mathbf{n}}$ with $f(\Gamma) = p(\Gamma)$. Writing the error operator $E = \mathrm{Id} - I$ as

$$E = -\sum_{\alpha_i \in \{0,1\}} (-E_d)^{\alpha_d} \cdots (-E_1)^{\alpha_1},$$

it can be shown that the interpolation error is bounded by

$$\|f - p\|_{\infty,\Omega} \leq \sum_{\alpha_i \in \{0,n_i\}} \frac{1}{\alpha!} \|\partial^\alpha f\|_{\infty,\Omega}.$$

Estimates in terms of the pure partial derivatives alone can be obtained if the spacing of nodes is taken into account. To this end, let

$$\delta[\mathbf{m}] := [\delta[m_1], \ldots, \delta[m_d]], \quad \delta[m_k] := \min_i \gamma^k_{i+m_k+1} - \gamma^k_i,$$

denote the vector of least distances when skipping m_k nodes in the kth coordinate direction. Then, as a consequence of an embedding theorem in anisotropic Sobolev spaces, one obtains the estimate

$$\|f - p\|_{\infty,\Omega} \leq C(\delta[\mathbf{m}]) \sum_{k=1}^{d} \|\partial_k^{n_k} f\|_{\infty,\Omega}$$

for vectors $\mathbf{m}$ satisfying $\sum_k m_k/n_k < 1$. Examples show that the dependence of the constant C on $\delta[\mathbf{m}]$ is unavoidable.

An even more fundamental question concerns the approximability of functions by polynomials in Sobolev spaces. With $\mathbb{P}_n$ the space of polynomials of total order n, the famous Bramble-Hilbert Lemma [1] states that

$$\inf_{\pi \in \mathbb{P}_n} \|f - \pi\|_{p,\Omega} \leq C \sum_{|\alpha|=n} \|\partial^\alpha f\|_{p,\Omega}$$

if the domain Ω is bounded by Lipschitz graphs. Subsequent research aimed at enlarging the set of domains, and at providing specific values for the constant C, see, for instance, [2, 3, 4, 8]. For the approximation by tensor product polynomials in anisotropic Sobolev spaces, it was shown in [6] that

$$\inf_{\pi \in \mathbb{P}_{\mathbf{n}}} \|f - \pi\|_{p,\Omega} \leq C(\Omega, n) \sum_{k=1}^{d} \|\partial_k^{n_k} f\|_{p,\Omega}$$

if Ω is bounded by axis-aligned graphs. The dependence of the constant on the shape of the domain is given explicitly. When admitting also mixed partial derivatives on the right hand side, the estimate can be generalized to domains which are bounded by a finite set of diffeomorphic images of graphs of continuous functions, thus enlarging the set of domains significantly.

When approximating functions by tensor product splines of coordinate order $\mathbf{n}$ with knots $T = T_1 \times \cdots \times T_d$, known results concerning approximation in anisotropic Sobolev spaces contain constants which depend on the aspect ratio of grid cells. Examples show that in dimensions $d \geq 3$ this phenomenon is unavoidable, but the bivariate case is different: Here, the dependence if the constant on the aspect ratio is only due to B-splines whose support in Ω is not connected. The concept of *diversification*, as introduced in [7], enlarges the spline space by providing a separate copy of each B-spline for every connected component of $\operatorname{supp} b_i \cap \Omega$. For the resulting space $\mathbb{S}_{\mathbf{n}}$ it can be shown that

$$\min_{s \in \mathbb{S}_{\mathbf{n}}} \|f - s\|_{p,\Omega} \le C(\Omega, n) \sum_{k=1}^{d} \|\partial_k^{n_k} f\|_{p,\Omega}$$

if Ω is bounded by a finite set of axis-aligned Lipschitz graphs. Here, the constant C depends on the shape of the domain and the order, but not on the aspect ratio.

A major challenge for future research concerns estimates in higher-dimensional cases. In particular, the dependence of the constant on the shape of the domain and the aspect ratio needs to be clarified.

References

[1] J.H. Bramble and S.R. Hilbert, *Estimation of linear functionals on Sobolev spaces with application to Fourier transforms and spline interpolation*, SIAM J. Numer. Anal. **7** (1970), 112-124.

[2] L.T. Dechevski and E. Quak, *On the Bramble-Hilbert Lemma*, Numer. Funct. Anal. Optim. **11** (1990), 485-495.

[3] T. Dupont and R. Scott, *Polynomial approximation of functions in Sobolev spaces*, Math. Comp. **34** (1980), 441-463.

[4] R.G. Durán, *On polynomial approximation in Sobolev spaces*, SIAM J. Numer. Anal. **20** (1983), 985-988.

[5] B. Mößner and U. Reif, *Error bounds for polynomial tensor product interpolation*, Computing **86** (2009), 187-197.

[6] U. Reif, *Polynomial Approximation on Domains Bounded by Diffeomorphic Images of Graphs*, Journal of Approximation Theory **164** (2012), 954–970.

[7] U. Reif and N. Sissouno, *Approximation with Diversified B-Splines*, CAGD **31** (2014), 510-520.

[8] R. Verfürth, *A note on polynomial approximation in Sobolev spaces*, Math. Model. Numer. Anal. **33** (1999), 715-719.

Prony's method in several variables

Tomas Sauer

In 1795, R. Prony [3] gave an ingenious trick to recover a function of the form

$$f(x) = \sum_{\omega \in \Omega} f_\omega e^{\omega^T x}, \qquad \Omega \subset \mathbb{C}^s,\ \#\Omega < \infty, \tag{1}$$

from integer samples by noting that if $p = \sum_\alpha p_\alpha (\cdot)^\alpha$ is a polynomial, then

$$\sum_{\beta \in \mathbb{Z}^s} f(\alpha + \beta)\, p_\beta = \sum_{\omega \in \Omega} f_\omega\, e^{\omega^T \alpha}\, p(e^\omega), \qquad \alpha \in \mathbb{N}_0^s.$$

This shows that any polynomial from the zero dimensional ideal

$$I_\Omega := \{ f \in \Pi : f(e^\omega) = 0,\ \omega \in \Omega \}$$

of total degree $\le n$ belongs to the kernel of the Hankel matrix

$$F_n = [f(\alpha + \beta) : |\alpha|, |\beta| \le n]$$

for large enough n. Indeed, if we use $\Pi_n \subset \mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_s]$ to denote the polynomials of *total degree* $\leq n$ in s variables, we have the following result.

Theorem 1. *If n is sufficiently large, then a polynomial $p = \sum p_\alpha(\cdot)^\alpha$ belongs to $I_\Omega \cap \Pi_n$ if and only if the vector $p = [p_\alpha : |\alpha| \leq n]$ satisfies $F_n p = 0$.*

This observation is the starting point to recover the *frequencies* Ω and the coefficients f_ω, $\omega \in \Omega$. With the *a priori* knowledge of $\#\Omega$, one could build the matrix F_n for $n = \#\Omega$ whose kernel defines a system of polynomial system whose solutions are exactly the common zeros of I_Ω, hence $e^\Omega = \{e^\omega : \omega \in \Omega\}$.

Unfortunately, this naive approach is not feasible as the size of the matrix F_n grows exponentially in the number of variables. A more suitable approach uses homogeneous H–bases. Recall that an H–basis H of an ideal I is a finite subset of I such that

$$f \in I \qquad \Leftrightarrow \qquad f = \sum_{h \in H} f_h\, h, \quad \deg f \geq \deg f_h + \deg h,\ h \in H.$$

Although any Gröbner basis with respect to a *graded term order*, i.e., a term order "$\prec$" such that $\alpha \prec \beta$ whenever $|\alpha| < |\beta|$, is also an H–basis, such bases can be constructed by means of orthogonal projections in a way that does not use any term order at all and only works on homogeneous components.

As shown in [4], such an H–basis allows for an algorithmically computable decomposition

$$f = \sum_{h \in H} f_h\, h + r \tag{2}$$

where the *remainder* r depends only on the ideal $\langle H \rangle$ and an inner product on Π and thus forms a *normal form* modulo ideal. The image of the normal form map $\nu : f \to r$ with the decomposition in (2) gives the finite dimensional degree reducing interpolation space $\nu(\Pi)$ whose maximal degree,

$$\deg \nu(\Pi) := \max\{\deg p : p \in \nu(\Pi)\}$$

is the lower bound for F_n. More precisely: Theorem 1 holds for $n > \deg \deg \nu(\Pi)$.

With a proper guess of $n > \deg \deg \nu(\Pi)$ which replaces the $n = \#\Omega$ from the univariate case, one can successively build the matrices

$$F_{n,k} = \left[f(\alpha + \beta) : \begin{array}{l} |\alpha| \leq n \\ |\beta| \leq k \end{array} \right], \qquad k = 0, 1, 2, \ldots$$

from which it is possible to construct orthonormal bases for $\nu(\Pi) \cap \Pi_k$ and $I_\Omega \cap \Pi_k$ with the positive side effect that as soon as $k \geq \deg \deg \nu(\Pi)$, the ideal basis is an H–basis for I_Ω. Knowing this basis and the basis of the normal form space, one can use *reduction*, again by means of Linear Algebra, to compute *s multiplication tables* of (modest) size $\#\Omega \times \#\Omega$ whose eigenvalues are the components of the zeros e^Ω, cf. [5]. Once the frequencies are known, the coefficients f_ω can be determined by solving a linear system.

The advantage of using homogeneous, term order free H–bases lies in the fact that all computations are based on well–understood and well–implemented methods of numerical Linear Algebra: in addition to a singular value decomposition for numerical rank computations, some QR decompositions are needed to determine orthogonal projections and complements. All this can be very easily realized in `Matlab` or `Octave`, [1], and it turns out that the method works very well for generic configurations where the points e^Ω do not lie on a low degree algebraic variety. It works especially well in the *trigonometric* situation when all the frequencies are purely imaginary, i.e., when $\Omega \subset i\mathbb{R}^s$. Moreover, as is not uncommon in multivariate polynomials, the numerical behavior depends mostly on the *total degree* of the polynomials to be considered and therefore the performance of the method even improves if the number of variables increases while the number of frequencies remains constant.

In addition, the method can be easily extended to the recovery of *oligonomials* or *fewnomials*, i.e., sparse (multivariate) polynomials of the form

$$f(x) = \sum_{\alpha \in A} f_\alpha x^\alpha, \qquad A \subset \mathbb{N}_0^s, \tag{3}$$

where A is again a set of small cardinality whose elements can nevertheless be quite large. By considering the matrix

$$\left[f\left(e^{\Xi(\alpha+\beta)}\right) : |\alpha|, |\beta| \le n \right], \qquad \Xi \in \mathbb{R}^{s \times s}$$

with an arbitrary nonsingular matrix Ξ, the problem (3) is easily reduced to (1) with $\Omega = \Xi A$.

References

[1] John W. Eaton, David Bateman, and Søren Hauberg, *GNU Octave version 3.0.1 manual: a high-level interactive language for numerical computations*, CreateSpace Independent Publishing Platform, 2009, ISBN 1441413006.

[2] G. Plonka and M. Tasche, *Prony methods for recovery of structured functions*, GAMM–Mitt. **37** (2014), 239–258.

[3] C. Prony, *Essai expérimental et analytique sur les lois de la dilabilité des fluides élastiques, et sur celles de la force expansive de la vapeur de l'eau et de la vapeur de l'alkool, à différentes températures*, J. de l'École polytechnique **2** (1795), 24–77.

[4] T. Sauer, *Gröbner bases, H–bases and interpolation*, Trans. Amer. Math. Soc. **353** (2001), 2293–2308.

[5] H. J. Stetter, *Matrix eigenproblems at the heart of polynomial system solving*, SIGSAM Bull. **30** (1995), no. 4, 22–25.

Algebraic methods in approximation theory

Hal Schenck

This survey talk gave an overview of several fundamental algebraic constructions which arise in the study of splines. Splines play a key role in approximation theory, geometric modeling, and numerical analysis; their properties depend on combinatorics, topology, and geometry of a simplicial or polyhedral subdivision of a region in $\mathbb{R}^k$, and are often quite subtle. We describe four algebraic techniques which are useful in the study of splines: homology, graded algebra, localization, and inverse systems. Our goal is to give a hands-on introduction to the methods, and illustrate them with concrete examples in the context of splines. We highlight progress made with these methods, such as a formula for the third coefficient of the polynomial giving the dimension of the spline space in high degree. A talk by Stillman later in the conference showed how to use a package in the `Macaulay2` software system to compute the algebraic objects described in the talk.

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1. Introduction
2. Homology and chain complexes
2.1. Algebraic setting
2.2. Topological motivation
2.3. Splines and homology
3. Graded algebra
3.1. Rings and modules, Hilbert polynomial and series
3.2. Free resolutions
3.3. Grading in the spline setting
4. Localization
4.1. Basics of the construction
4.2. Application: polyhedral splines
4.3. Application: vanishing of homology
5. Inverse systems and powers of linear forms
5.1. Powers of linear forms and fatpoints
5.2. Application: planar splines of mixed smoothness
5.3. A conjecture in algebraic geometry
6. Open questions
6.1. Higher dimensions
6.2. Polyhedral complexes
6.3. Supersmoothness
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References

[1] P. Alfeld, *Upper and lower bounds on the dimension of multivariate spline spaces*, SIAM J. Numer. Anal., **33** (1996), 571–588.

[2] P. Alfeld, L. Schumaker, *The dimension of bivariate spline spaces of smoothness r for degree* $d \geq 4r+1$, Constr. Approx. **3** (1987), 189-197.

[3] P. Alfeld, L. Schumaker, *On the dimension of bivariate spline spaces of smoothness r and degree* $d = 3r+1$, Numer. Math. **57** (1990), 651-661.

[4] P. Alfeld, L. Schumaker, *Bounds on the dimensions of trivariate spline spaces*, Adv. Comput. Math. **29** (2008), 315–335.

[5] P. Alfeld, L. Schumaker, and W. Whiteley. *The generic dimension of the space of* C^1 *splines of degree* $d \geq 8$ *on tetrahedral decompositions*, SIAM J. Numer. Anal. **30** (1993) 889-920.

[6] L. Billera, *Homology of smooth splines: generic triangulations and a conjecture of Strang*, Trans. Amer. Math. Soc. **310** (1988), 325-340.

[7] L. Billera, L. Rose, *A dimension series for multivariate splines*, Discrete Comput. Geom. **6** (1991), 107-128.

[8] L. Billera, L. Rose, *Modules of piecewise polynomials and their freeness*, Math. Zeit. **209** (1992), 485-497.

[9] C. Chui, R. Wang, *Multivariate spline spaces*, Journal of Mathematical Analysis and Applications **94** (1983), 197-221.

[10] R. Courant, *Variational methods for the solution of problems of equilibrium and vibration*, Bulletin of the A.M.S. **49**, 1-23, 1943.

[11] J. Dalbec, H. Schenck, *On a conjecture of Rose*, J. Pure Appl. Algebra **165** (2001), 151-154.

[12] C. de Boor, *A practical guide to splines*, 2nd edition, Appl. Math. Sciences, vol. 27, Springer, Berlin-Heidelberg-New York, 2001.

[13] D. Diener, *Instability in the spaces of bivariate piecewise polynomials of degree $2r$ and smoothness order r*, SIAM J. Numer. Anal. **27** (1990), 543–551.

[14] M. Dipasquale, *Shellability and freeness of continuous splines*, J. Pure Appl. Algebra **216** (2012), 2519–2523.

[15] M. Dipasquale, *Lattice supported polyhedral splines*, Adv. in Appl. Math. **55** (2014), 1-21.

[16] M. Dipasquale, *Regularity of Mixed Spline Spaces*, preprint, `arXiv 1411.2176`, (2015).

[17] D. Eisenbud, *Commutative Algebra with a view towards Algebraic Geometry*, Graduate Texts in Mathematics, vol. 150, Springer, Berlin-Heidelberg-New York, 1995.

[18] J. Ensalem and A. Iarrobino, *Inverse system of a symbolic power I*, Journal of Algebra **174** (1995), 1080-1090.

[19] A. Geramita, H. Schenck, *Fat points, inverse systems, and piecewise polynomial functions*, J. Algebra **204** (1998), 116-128.

[20] D. Grayson, M. Stillman, *Macaulay 2: a software system for algebraic geometry and commutative algebra*, `http://www.math.uiuc.edu/Macaulay2`

[21] D. Hong, *Spaces of bivariate spline functions over triangulation*, Approx. Theory Appl. **7** (1991), 56–75.

[22] A. Iarrobino, *Inverse system of a symbolic power III: thin algebras and fat points*, Compositio Math. **108** (1997), 319-356.

[23] M. Lai, L. Schumaker, *Spline functions on triangulations*, Cambridge University Press, London/New York, 2007.

[24] F.H.S. Macaulay, *The algebraic theory of modular systems*, Cambridge University Press, London/New York, 1916.

[25] T. McDonald, H. Schenck, *Piecewise polynomials on polyhedral complexes*, Advances in Applied Mathematics **42** (2009), 82-93.

[26] R. Miranda, *Linear systems of plane curves*, Notices Amer. Math. Soc. **46** (1999), 192–201.

[27] B. Mourrain, N. Villamizar, *Homological techniques for the analysis of the dimension of triangular spline spaces*, J. Symbolic Comput.**50** (2013), 564–577.

[28] L. Rose, *Combinatorial and topological invariants of modules of piecewise polynomials*, Adv. in Math. **116** (1995), 34-45.

[29] L. Rose, *Graphs, syzygies, and multivariate splines*, Discrete Comp. Geom. **32** (2004), 623-637.

[30] H. Schenck, *Homological methods in the theory of splines*, Ph.D. thesis, Cornell University, May 1997.

[31] H. Schenck, *A spectral sequence for splines*, Adv. in Appl. Math. **19** (1997), 183-199.

[32] H. Schenck, *Computational Algebraic Geometry*, Cambridge Univ. Press, Cambridge, 2003.

[33] H. Schenck, *Equivariant Chow cohomology of nonsimplicial toric varieties*, Transactions of the A.M.S. **364** (2012), 4041-4051.

[34] H. Schenck, *Splines on the Alfeld split of a simplex and type A root systems*, Journal of Approximation Theory **182** (2014), 1-6.

[35] H. Schenck, P. Stiller, *Cohomology vanishing and a problem in approximation theory*, Manuscripta Math. **107** (2002), 43-58.

[36] H. Schenck, M. Stillman, *A family of ideals of minimal regularity and the Hilbert series of* $\mathcal{C}^r(\hat{\Delta})$, Adv. in Appl. Math. **19** (1997), 169-182.

[37] H. Schenck, M. Stillman, *Local cohomology of bivariate splines*, J. Pure Appl. Algebra **117, 118** (1997), 535-548.

[38] L. Schumaker, *On the dimension of spaces of piecewise polynomials in two variables*, Multivariate Approx. Theory, Birkhaüser (1979), 396-412.
[39] L. Schumaker, *Bounds on the dimension of spaces of multivariate piecewise polynomials*, Rocky Mountain J. Math. **14** (1984), 251-264.
[40] P. Stiller, *Certain reflexive sheaves on* $\mathbb{P}^n$ *and a problem in approximation theory*, Transactions of the A.M.S. **279** (1983), 125-142.
[41] G. Strang, *The dimension of piecewise polynomial spaces and one sided approximation*, Springer Lecture Notes in Computer Science **365** (1974), 144-152.
[42] S. Tohaneanu, *Smooth planar r-splines of degree* $2r$, J. Approximation Theory **132** (2005), 72–76.
[43] W. Whiteley, *A matrix for splines*, in *Progress in Approximation Theory*, Academic Press, Boston, 1991.
[44] S. Yuzvinsky, *Modules of splines on polyhedral complexes*, Math. Zeit. **210** (1992), 245-254.

On one class of ideal projectors

Boris Shekhtman

Ideal projectors where introduced by G. Birkhoff (cf. [1]) as an elegant generalization of Hermite interpolation to several variables:

Definition 1. *A linear idempotent operator* P *on* $\mathbb{C}[x_1,\ldots,x_d]$ *is called an ideal projector if* $\ker P$ *is an ideal in* $\mathbb{C}[x_1,\ldots,x_d]$.

In one variable the set of non-trivial ideal projectors coincides with the set of all Hermite interpolation projectors and contains the set of all Lagrange interpolation projectors. The latter can be characterized as projectors whose kernels are radical ideals.

It is well known and easy to see that for any ideal projector P from $\mathbb{C}[x]$ onto an N-dimensional subspace $G \subset \mathbb{C}[x]$ there exist a sequence of Lagrange projectors P_n onto G such that $P_n f \to Pf$, for all $f \in \mathbb{C}[x]$ i.e., P is a limit of Lagrange projectors. Here I will address the extension of this property to ideal projectors in several variables and introduce a few classes of ideal projectors where such extension is and is not possible.

Definition 2. *An ideal projector* P *on* $\mathbb{C}[x_1,\ldots,x_d]$ *is called Hermite projector if there exists a sequence* P_n *of Lagrange projectors onto ran* P *such* $P_n f \to Pf$ *for all* $f \in \mathbb{C}[x_1,\ldots,x_d]$.

Problem 1 (C. de Boor, [2])**.** *What finite-dimensional ideal projectors are Hermite projectors?*

Due to the nature of this question we will reserve the term "ideal projectors" only to the projectors with finite-dimensional range, i.e., to the projectors with zero-dimensional kernel. Let us mention that the property of being Hermite has nothing to do with the range of ideal projector and depends entirely on its kernel.

Theorem 3. *Let* P *be an ideal projector onto an* N*-dimensional subspace* $G \subset \mathbb{C}[x_1,\ldots,x_d]$*. The following are equivalent:*

(i) *P is Hermite*

(ii) *The commuting sequence* $(M_j)_{j=1,\dots,d}$ *of operators*

$$M_j : \begin{array}{ccc} \mathbb{C}[x_1,\dots,x_d]/\ker P & \to & \mathbb{C}[x_1,\dots,x_d]/\ker P \\ [f] & \to & [x_j f] \end{array} \tag{1}$$

is a limit of simultaneously diagonalizable sequence of commuting operators on $\mathbb{C}[x_1,\dots,x_d]/\ker P$.

(iii) *The ideal* $\ker P$ *is smoothable.*

The equivalence of (i) and (ii) was shown in [3] while the equivalence of (i) and (iii) was shown in [7]. More on smoothable ideals can be found in [4].

In one and two variables every ideal projector is Hermite (cf. [6]). In three or more variables there are non-Hermite ideal projectors (cf. [5, 6]).

So what classes of ideal projectors are Hermite:

Theorem 4. *Let* P *be an ideal projector on onto an* N*-dimensional subspace* $G \subset \mathbb{C}[x_1,\dots,x_d]$. *Then* P *is Hermite if*

(i) $\ker P$ *is curvilinear, i.e.,* $\mathbb{C}[x_1,\dots,x_d]/\ker P \simeq \mathbb{C}[s]/J$ *for some ideal* $J \subset \mathbb{C}[s]$.
(ii) $\ker P$ *is curvisurfaced, i.e.,* $\mathbb{C}[x_1,\dots,x_d]/\ker P \simeq \mathbb{C}[s,t]/J$ *for some ideal* $J \subset \mathbb{C}[s,t]$.
(iii) $\ker P$ *is a monomial ideal.*
(iv) $\ker P$ *is Gorenstein and* $d = 3$.
(v) $N \le 7$.

A complete description of Hermite projectors for $d = 4$ and $N = 8$ is given in [4]. An algorithmic description of Hermite projectors for arbitrary d is given in [7].

The main object of this talk is to investigate the properties of ideal projectors whose kernel is a complete intersection. There are several definitions of the term "complete intersection" in the literature. We will stick to the following one:

Definition 5. *A zero-dimensional ideal* $J \subset \mathbb{C}[x_1,\dots,x_d]$ *is a complete intersection if* J *can be generated by precisely* d *polynomials.*

Conjecture 1. *If* P *is an ideal projector whose kernel is a complete intersection then* P *is Hermite.*

Here is a partial result supporting this conjecture (cf. [8]):

Theorem 6. *Let* P *be an ideal projector,* $\ker P = \langle f_1,\dots,f_d\rangle$ *and* $\mathbf{f} := (f_1,\dots,f_d) : \mathbb{C}^d \to \mathbb{C}^d$ *has no roots at infinity. Then* P *is Hermite.*

An answer to the next question may help resolve the conjecture:

Problem 2. *What is a characterization of complete intersection in terms of commuting matrices defined by* (1)*?*

References

[1] G. Birkhoff, *The algebra of multivariate interpolation*, in *Constructive Approaches to Mathematical Models*, C. V. Coffman and G. J. Fix (eds.), Academic Press, New York, 1979, pp. 345–363.

[2] C. de Boor, *What are the limits of Lagrange projectors?*, in *Constructive Theory of Functions (Varna 2005)*, B. Bojanov (ed.), Marin Drinov Academic Publishing House, Sofia, (2006), pp. 51–63.

[3] C. de Boor and B. Shekhtman, *On the pointwise limits of bivariate Lagrange projectors*, Linear Algebra Appl. **429** (2008), 311–325.

[4] Dustin A. Cartwright, Daniel Erman, Mauricio Velasco and Bianca Viray, *Hilbert schemes of* 8 *points in* $\mathbb{A}^d$, *Algebra and Number Theory* **3** (2009), 763–795.

[5] A. Iarrobino, *Reducibility of the families of* 0*-dimensional schemes on a variety*, Inventiones Math. **15** (1972), 72–77.

[6] B. Shekhtman, *On a conjecture of Carl de Boor regarding the limits of Lagrange interpolants*, Constr. Approx. **24** (3) (2006), 365–370.

[7] B. Shekhtman, *On the limits of Lagrange projectors*, Constr. Approx. **29** (3) (2009), 293–301.

[8] B. Shekhtman, *On one class of ideal projectors*, Constr. Approx., submitted.

Dimension of trivariate C^1 splines on bipyramid cells

Tatyana Sorokina
(joint work with Julien Colvin, Devan DiMatteo)

We study dimension of trivariate C^1 splines on bipyramid cells, that is, cells with $n+2$ boundary vertices, n of which are coplanar with the interior vertex. We fix a terminological error in the earlier lower bound on the dimension given by J. Shan. Moreover, we derive a new upper bound which in many cases is equal to the known lower bound. In the remaining cases, where our upper bound differs from the known lower bound, we conjecture that the dimension coincides with the upper bound. We use tools from both algebraic geometry and Bernstein-Bézier analysis.

A *bipyramid cell* is a tetrahedral partition Δ such that:

- there is exactly one interior vertex v_0;
- n boundary vertices v_1 through v_n are coplanar along with v_0, and form a polygon surrounding v_0 in the *base plane* $B := [v_1, \dots, v_n]$;
- each vertex v_i, $i = 1, \dots, n$, is connected to v_0 by the interior edges $[v_0, v_i]$;
- two boundary vertices v_{n+1} and v_{n+2} lie outside the *base plane* B, are on opposite sides of B, and are connected to v_0 by the interior edges $[v_0, v_{n+1}]$ and $[v_0, v_{n+2}]$; and
- vertices v_{n+1} and v_{n+2} each connect to the boundary vertices v_i, $i = 1, \dots, n$.

Our goal is to find the dimension of the spline space $S^1_d(\Delta)$. The next lemma summarizes several immediately obvious facts about Δ.

Proposition 1. *In a bipyramid cell the following holds:*

(1) *there are* $2n$ *tetrahedra,* $3n$ *interior triangular faces, and* $n+2$ *interior edges;*

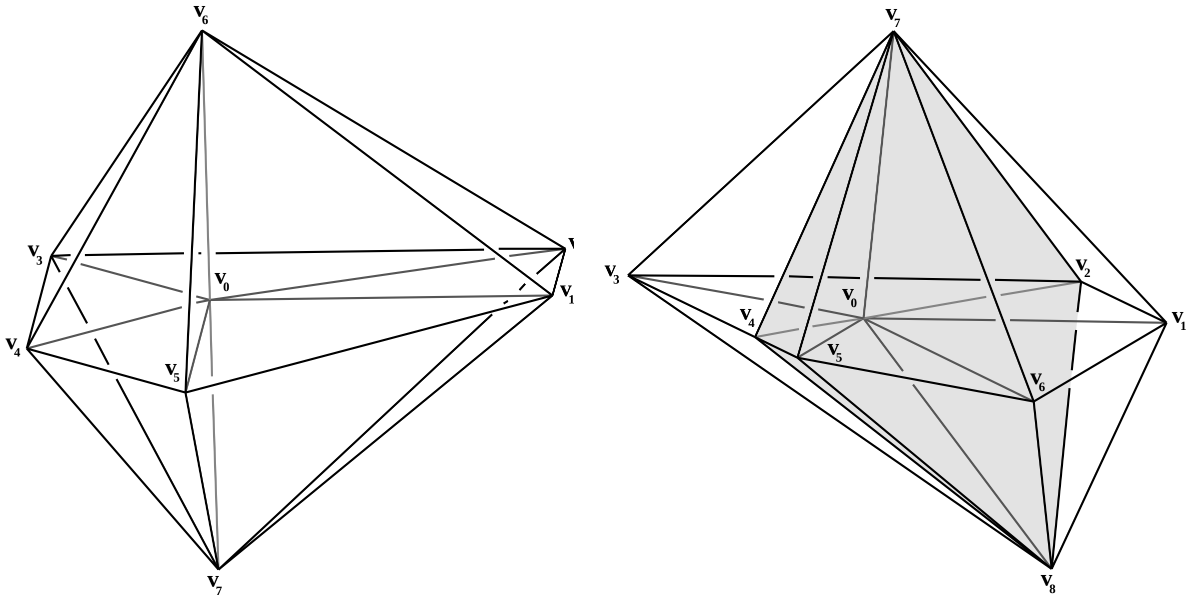

FIGURE 1. Collinear (left) and coplanar II (right) cases

(2) *the number of different slopes m formed by the coplanar interior edges $[v_0, v_i]$, $i = 1, \ldots, n$, in the base plane satisfies $2 \leq \lceil \frac{n}{2} \rceil \leq m \leq n$; and*
(3) *the number of distinct planes h containing the interior triangular faces is as follows:*
Case 1 (Generic): v_{n+1} and v_{n+2} are not collinear with v_0, and the plane containing v_{n+1}, v_{n+2} and v_0 does not contain any other v_i. Then $h = 2m + 1$;
Case 2 (Coplanar): v_{n+1} and v_{n+2} are not collinear with v_0, and the plane containing v_{n+1}, v_{n+2} and v_0 contains at least one other v_i, $i = 1, \ldots, n$. Then $h = 2m$;
Case 3 (Collinear): v_{n+1} and v_{n+2} are collinear with v_0. Then $h = m + 1$.

For further investigation, we split the coplanar case into two subcases as follows:

- *Coplanar Case I*: v_{n+1} and v_{n+2} are not collinear with v_0, and the plane containing v_{n+1}, v_{n+2} and v_0 contains exactly one other v_i, $i = 1, \ldots, n$.
- *Coplanar Case II*: v_{n+1} and v_{n+2} are not collinear with v_0, and the plane containing v_{n+1}, v_{n+2} and v_0 contains exactly two other v_i, $i = 1, \ldots, n$.

Figure (left) shows an example of the collinear case; Figure (right) shows an example of the coplanar case II, where vertices v_0, v_2, v_4, v_7, v_8 lie in the same plane (shaded). Using results in [1] and [2], we compute an improved lower bound on the dimension of $S^1_d(\Delta)$. We show that there is a notational error in [2] and we fix it. We prove that splines on bypyramid cells have additional smoothness across certain faces, and using this fact combined with the results in [3], we prove a new upper bound on the dimension of $S_d^1(\Delta)$. Comparing the bounds yields the following dimension and bounds.

Theorem 1. *Let Δ be a bipyramid cell. In the collinear case, for $d \geq 2$, the following holds*

$$\dim S_d^1(\Delta) = \begin{cases} 8\binom{d}{3} + 12(d-1) + (3-d)_+, & \text{if } m = 2, \\ 2n\binom{d}{3} + n(d-1) + 6d - 4, & \text{if } m \geq 3. \end{cases}$$

Theorem 2. *Let Δ be a bipyramid cell. In the coplanar case I, for $d \geq 2$, the following holds*

$$\dim S_d^1(\Delta) = 2n\binom{d}{3} + 7d - 3.$$

Theorem 3. *Let Δ be a bipyramid cell. In the coplanar case II, for $d \geq 2$, the following holds*

$$\dim S_d^1(\Delta) = \begin{cases} 8\binom{d}{3} + 10d - 8, & \text{if } m = 2, \\ 2n\binom{d}{3} + 8d - 4, & \text{if } m \geq 3. \end{cases}$$

Theorem 4. *Let Δ be a bipyramid cell. In the generic case, for $d \geq 2$, and $m \geq 3$, the following bounds hold*

$$2n\binom{d}{3} + 6d - 2 \leq \dim S_d^1(\Delta) \leq 2n\binom{d}{3} + 6d - 1 + (4-m)_+.$$

Moreover, for $m = 2$, dim $S_2^1(\Delta) = 11$, and for $d \geq 3$, the following bounds hold

$$8\binom{d}{3} + 8d - 5 \leq \dim S_d^1(\Delta) \leq 8\binom{d}{3} + 8d - 3 - (4-d)_+.$$

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References

[1] B. Mourrain and N. Villamizar, *Dimension of spline spaces on tetrahedral partitions: a homological approach, Mathematics in Computer Sciences* **8(2)** (2014), 157–174.

[2] J. Shan, *Lower Bound on the Dimension of Trivariate Splines on cells*, Approximation Theory XIV: San Antonio 2013, Springer Proceedings in Mathematics, Gregory E. Fasshauer and Larry L. Schumaker (eds), 309–333.

[3] T. Sorokina, *Intrinsic supersmoothness of multivariate splines*, Numerische Mathematik **116** (2010), 421–434.

Smooth Powell-Sabin splines: from the construction of B-splines to quasi-interpolation

Hendrik Speleers

Smooth (finite element) spline spaces defined over triangulations have been studied extensively and applied in different contexts (see, e.g., [2] and references quoted therein). Typically, such spline spaces provide good approximation properties and possess a small dimension which can be expressed in terms of geometrically interesting characteristics of the triangulation (like the number of vertices, edges and/or triangles). In addition, a stable basis representation is often required for practical purposes.

For the construction of smooth splines with a low polynomial degree, one often considers triangulations with a particular macro-structure. Each triangle in the triangulation is then split into a number of subtriangles. The Clough-Tocher split (into three subtriangles) and the Powell-Sabin 6-split (into six subtriangles) are commonly used splits; see [2].

In this talk we focus on a specific family of bivariate spline spaces $\mathbb{S}^r(\Delta_{PS})$, $r \geq 1$, defined on any given triangulation Δ endowed with a Powell-Sabin 6-split Δ_{PS}. They have polynomial degree $d = 3r - 1$ for a given global smoothness of order r and local supersmoothness of order $\rho = 2r - 1$ around certain points and edges. We refer to [5] for details on the spaces $\mathbb{S}^r(\Delta_{PS})$. They have a simple dimension formula, namely

$$\dim \mathbb{S}^r(\Delta_{PS}) = n_v\, N(2r-1) + n_t\, N(r-2),$$

with n_v and n_t the number of vertices and triangles in Δ, and $N(k) := (k+1)(k+2)/2$. The most known member of this family is the C^1 quadratic space [3].

In the first part, we show how a suitable B-spline representation can be constructed for this family of spline spaces. Dierckx presented in [1] a geometric approach to construct a normalized B-spline basis for the C^1 quadratic space. Recently, a suitable normalized B-spline basis for the C^2 quintic space was proposed in [4], and the approach was generalized for the family of spline spaces $\mathbb{S}^r(\Delta_{PS})$ in [5]. The presented basis functions have a local support, they are nonnegative, and they form a partition of unity. The B-spline representation allows for a natural definition of control points, which can be useful for geometric modelling of smooth surfaces. A spline in such a representation can also be evaluated in a stable way using a sequence of simple convex combinations.

In the second part, we discuss a general recipe [6] to construct quasi-interpolants of arbitrary smoothness based on such Powell-Sabin B-splines. This will enable us to produce various local approximation schemes that can be tailored to special requests by a given data set or function. We first derive a Marsden-like identity representing polynomials of at most degree $3r - 1$ in terms of Powell-Sabin B-splines of smoothness r. We provide an elegant construction and proof based on blossoming. We then use this identity to develop quasi-interpolants with good approximation properties. More precisely, the coefficients of such quasi-interpolating splines are computed by evaluating the blossom values at some particular points of a chosen local operator that approximates local portions of the data.

References

[1] P. Dierckx, *On calculating normalized Powell-Sabin B-splines*, Comput. Aided Geom. Design **12** (1997), 61–78.
[2] M.J. Lai and L.L. Schumaker, *Spline Functions on Triangulations*, Cambridge University Press (2007).
[3] M. Powell and M. Sabin, *Piecewise quadratic approximations on triangles*, ACM Trans. Math. Softw. **3** (1977), 316–325.
[4] H. Speleers, *A normalized basis for quintic Powell-Sabin splines*, Comput. Aided Geom. Design **27** (2010), 438–457.
[5] H. Speleers, *Construction of normalized B-splines for a family of smooth spline spaces over Powell-Sabin triangulations*, Constr. Approx. **37** (2013), 41–72.
[6] H. Speleers, *A family of smooth quasi-interpolants defined over Powell-Sabin triangulations*, Constr. Approx. **41** (2015), 297–324.

Splines in geometry and topology

Julianna Tymoczko

The goal of this survey is to describe how splines arise in geometry and topology and to discuss what kinds of questions matter in the geometric and topological context.

Cohomology is a ring associated to a geometric object X that encodes geometric properties of X like the dimension, the number of connected components, whether X is singular, and so on. Equivariant cohomology is an enhanced version of cohomology defined in the case when X carries an "appropriate" action of a torus $T = \mathbb{C}^* \times \mathbb{C}^* \times \cdots \mathbb{C}^*$. On the one hand, equivariant cohomology carries strictly more information than ordinary cohomology. On the other hand, Goresky, Kottwitz, and MacPherson established the principle that in many cases equivariant cohomology is actually easier to compute than ordinary cohomology [8].

There are three conditions that Goresky, Kottwitz, and MacPherson require the T-action on X to satisfy:

(1) X must have a finite number of T-fixed points.
(2) X must have a finite number of one-dimensional T-orbits.
(3) X must be *equivariantly formal* with respect to the T-action.

Equivariant formality is a technical condition that is typically verified by one of many easier conditions that imply it. The first two conditions can be encoded more concisely in an edge-labeled graph G_X that is often called either the *moment graph* or the *GKM graph* of X.

- The vertices of G_X correspond bijectively to the T-fixed points in X.
- The edges uv of G_X correspond bijectively to the one-dimensional T-orbits in X, each of which rather miraculously has precisely two T-fixed points

u and v in its closure. (In fact the closure of each one-dimensional T-orbit is homeomorphic to a 2-dimensional sphere and the T-fixed points correspond to the north and south poles.)

- Each edge uv in G_X is labeled by the weight of the T-action on the corresponding one-dimensional orbit in X. This weight is a homogenous linear form and essentially records the direction of the one-dimensional orbit.

Goresky, Kottwitz, and MacPherson's result says the following [8].

Theorem 1 (Goresky, Kottwitz, MacPherson)**.** *If X is a compact complex manifold that carries the action of a torus T satisfying Conditions (1)–(3) above, then the equivariant cohomology of X can be described as*

$$H_T^*(X) \cong \left\{ p \in \mathbb{C}[t_1, \ldots, t_n]^k : \begin{array}{l} \text{for each edge } uv \text{ in } G_X \text{ the difference } p_u - p_v \\ \text{is a multiple of the label on the edge } uv \text{ in } G_X \end{array} \right\}$$

where n is the number of copies of $\mathbb{C}^*$ in T and k is the number of vertices in G_X.

Goresky, Kottwitz, and MacPherson's theorem developed out of a considerable amount of earlier work on localizing equivariant cohomology at T-fixed points, including results of Atiyah and Bott [2], Guillemin and Sternberg [9], Kirwan [10], Cheng-Skjelbred [6], and many others. Later and somewhat independently, algebraic geometers and topologists like Payne [11] and Bahri, Franz, and Ray [3] gave a similar construction of equivariant cohomology except they described it in terms of piecewise polynomials on a polytope dual to the graph G_X.

The point is that these constructions of equivariant cohomology in fact give the ring of splines $S_\infty^0(G_X)$ though none of the previous authors use that name. Goresky, Kottwitz, and MacPherson's perspective is dual to the classical analytic perspective: if we started with a triangulation then G_X would be its dual graph, an approach used first by Billera and Rose [4]. Other differences are smaller, for instance that the polynomials are taken with complex coefficients rather than real coefficients. Geometers/topologists have not yet used other parameters d and r in the splines $S_r^d(G_X)$, nor is there an immediate cohomological interpretation for d—though this would be interesting to pursue.

In the remainder of the talk we elaborate further on the following topics:

- We describe classical results on the equivariant cohomology of specific families that are important in geometric applications, including projective space, the Grassmannian of k-dimensional subspaces of a fixed vector space $\mathbb{C}^n$, and flag varieties, and say where the Alfeld split fits into these examples.
- We sketch existing formulae for equivariant cohomology bases, especially the formula of Andersen-Jantzen-Soergel [1] and Billey [5] for bases when X is a partial flag variety, and sketch the most common geometric approaches to constructing bases (especially *flow-up* bases and *symmetrized* bases).

- We discuss applications to geometric representation theory, which arise especially when graph automorphisms of G_X also induce actions on the collection of splines over G_X.
- Gilbert, Polster, and Tymoczko generalized the definition of splines to arbitrary rings R and graphs $G = (V, E)$ as follows [7]. Choose a function $\alpha : E \to \{$ ideals in $R\}$. Then the ring of generalized splines $R_{G,\alpha}$ is defined as

$$R_{G,\alpha} = \left\{ p \in R^{|V|} : \text{ for each edge } uv \text{ in } E \text{ the difference } p_u - p_v \in \alpha(uv) \right\}$$

  We summarize recent work of various authors when the ring of coefficients R is the integers or the integers mod m rather than a polynomial ring.
- The key difference between how geometers and topologists think of splines and how analysts think of splines is that (equivariant) cohomology forms a *ring*: in other words, we can multiply cohomology classes and the products carry important geometric information. For splines with a fixed, known basis, we ask for multiplication tables with respect to this basis. We summarize some of what is known and not known in classical geometric examples and for generalized splines when R is the integers or the integers mod m.

References

[1] H. Andersen, J. Jantzen, and W. Soergel, *Representations of quantum groups at a pth root of unity and of semisimple groups in characteristic p: independence of p*, *Astrisque* No. **220** (1994).

[2] M. F. Atiyah and R. Bott, *The moment map and equivariant cohomology*, Topology **23** (1984), 1–28.

[3] A. Bahri, M. Franz, and N. Ray, *The equivariant cohomology ring of weighted projective space*, Math. Proc. Cambridge Philos. Soc. **146** (2009), 395–405.

[4] L. Billera and L. Rose, *A dimension series for multivariate splines*, Discrete Comput. Geom. **6** (1991), 107–128.

[5] S. Billey, *Kostant polynomials and the cohomology ring for G/B*, Duke Math. J. **96** (1999), 205–224.

[6] T. Chang and T. Skjelbred, *The topological Schur lemma and related results*, Ann. Math. **100** (1974), 307–321.

[7] S. Gilbert, S. Polster, and J. Tymoczko, *Generalized splines on arbitrary graphs*, `arXiv:1306.0801`.

[8] M. Goresky, R. Kottwitz, and R. MacPherson, *Equivariant cohomology, Koszul duality, and the localization theorem*, Invent. Math. **131** (1998), no. 1, 25–83. MR 1489894 (99c:55009)

[9] V. W. Guillemin and S. Sternberg, *Supersymmetry and equivariant de Rham theory*, Mathematics Past and Present, Springer-Verlag, Berlin, 1999.

[10] F. Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, Princeton UP, Princeton, NJ, 1984.

[11] S. Payne, *Equivariant Chow cohomology of toric varieties*, Math. Res. Lett. **13** (2006), 29–41.

Bounds on the dimension of spline spaces on triangulations

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(joint work with Bernard Mourrain)

The problem of finding the dimension of a spline space on a given triangulation, or on a simplicial partition in $\mathbb{R}^n$, was first formally formulated by Strang [8]. Serious difficulties already begin to arise in the planar case, and depending on the embedding of the triangulation in $\mathbb{R}^2$, the dimension of the space is usually larger than the formula conjectured by Strang. The classical methods to compute the dimension of a spline space include the construction of nodal bases and the Bernstein–Bézier representation of the polynomials [9].

In 1988, L. Billera introduced the use of homological algebra and some algebraic machinery to study splines [2]. By means of this approach, he was able to prove the dimension for the space of C^1 bivariate splines for triangulations whose edges are in sufficiently general position, for any fixed polynomial degree. The homological construction was continued by Schenck and Stillman in [13], and studied in [6, 12, 13, 14]. We follow this approach and prove a formula for an upper bound on the dimension of bivariate spline spaces, and new lower and upper bounds for trivariate spline spaces. The formulas we present include terms that take into account the geometry of the faces surrounding the interior faces of the partition and, having no restriction on the orderings of the faces, these bounds improve previous results [15, 9]. Furthermore, the approach leads to connections between the dimension problem on spline spaces and classical problems in algebraic geometry.

The construction is as follows. For a connected, finite n-dimensional simplicial complex, supported on on $|\Delta| \subset \mathbb{R}^n$, let us denote by $S_d^r(\Delta)$ the space of splines of degree less than or equal to d defined on Δ, with global order of smoothness r (≥ 0). We assume that Δ and all its links are pseudomanifolds, we could think of Δ as the triangulation of a (topological) n-ball. It was proved that $S^r(\hat{\Delta})_d \cong S_d^r(\Delta)$ as $\mathbb{R}$-vector spaces [3], where $\hat{\Delta}$ is the cone with vertex at the origen obtained by embedding Δ in the hyperplane $\{x_{n+1} = 1\} \subset \mathbb{R}^{n+1}$. Thus, in particular, for a fixed d, $\dim S_d^r(\Delta) = \dim S^r(\hat{\Delta})_d$, which are the splines on $\hat{\Delta}$ of degree exactly d.

We denote by Δ^0 the set of interior faces of Δ, and for $i = 0, \ldots, d-1$ let Δ_i^0 be the set of i-dimensional interior faces of Δ whose support is not contained in the boundary $\partial\Delta$ of $|\Delta|$. With Δ_d^0 we denote the set of all maximal d-faces of Δ, and f_i^0 will be the cardinality of Δ_i^0, for $i = 0, \ldots, d$.

If $R := \mathbb{R}[x_1, \ldots, x_{d+1}]$ is the polynomial ring in $d+1$ variables, let us consider the constant (chain) complex $\mathcal{R}$ on Δ i.e., $\mathcal{R}_i = R^{f_i^0}$ for $i = 0, \ldots, d$, where the boundary maps ∂_i are induced by the usual simplicial boundary maps $\bar{\partial}_i$ used to compute the relative homology of $(\Delta, \partial\Delta)$ with coefficients in R. For the facets $\tau \in \Delta_{n-1}^0$, let ℓ_τ be the linear form that vanishes on $\hat{\tau}$, and for every interior face $\beta \in \Delta^0$ we consider the ideal $\mathcal{J}(\beta) = (\ell_\tau^{r+1})_{\tau \ni \beta}$. These ideals define a complex $\mathcal{J}$

of ideals, and the quotient $\mathcal{R}/\mathcal{J}$, given by

$$0 \xrightarrow{\partial_{d+1}} \bigoplus_{\sigma\in\Delta_d^0} \mathcal{R} \xrightarrow{\partial_d} \bigoplus_{\tau\in\Delta_{d-1}^0} \mathcal{R}/\mathcal{J}(\tau) \xrightarrow{\partial_{d-1}} \cdots \xrightarrow{\partial_1} \bigoplus_{\beta\in\Delta_0^0} \mathcal{R}/\mathcal{J}(\gamma) \xrightarrow{\partial_0} 0$$

where ∂_i are the induced relative (module $\partial\Delta$) simplicial boundary maps, is such that its top homology module is isomorphic to the spline space $S^r(\hat{\Delta})$ [2]. Therefore, the Euler characteristic equation applied to $\mathcal{R}/\mathcal{J}$, leads us to the formula

$$(1) \quad \dim S_d^r(\Delta) = \sum_{i=0}^{n}(-1)^i \sum_{\beta\in\Delta_{n-i}^0} \dim \mathcal{R}/\mathcal{J}(\beta)_d - \sum_{i=1}^{n}(-1)^i \dim\, H_{n-i}(\mathcal{R}/\mathcal{J})_d,$$

where the subindex d indicates the d-th part of the graded module. Thus, in order to study the dimension of a spline space we need to analyze ideals generated by powers of linear forms in two, three,... and n-variables.

Since $\mathcal{J}(\beta) = 0$ for all maximal faces β of Δ, then

$$\bigoplus_{\beta\in\Delta_d^0} \mathcal{R}/\mathcal{J}(\beta)_k = \bigoplus_{\beta\in\Delta_d^0} \mathcal{R}_k \quad \text{and hence} \quad \dim \bigoplus_{\beta\in\Delta_d^0} \mathcal{R}_k = f_d^0 \cdot \binom{k+d}{d}.$$

Similarly, for the facets since $\mathcal{J}(\tau)$ is generated by only one power of a linear form, namely ℓ_τ, then a formula for $\dim \mathcal{R}/\mathcal{J}(\tau)_d$ can be easily deduced.

For a specific face $\beta \in \Delta_i^0$ for some $0 \le i < d-1$, let us observe that we may make an affine change of coordinates and assume that the linear forms in $\mathcal{J}(\beta)$ involve only the variables $x_1, \dots, x_{d-i}$,

$$\mathcal{R}/\mathcal{J}(\beta) \cong \mathbb{R}[x_{n+1-i}, \dots, x_{d+1}] \otimes_{\mathbb{R}} \mathbb{R}[x_1, \dots, x_{n-i}]/\mathcal{J}(\beta).$$

In the case of a triangulation of a region in the plane, the formula (1) reduces to

$$\dim S_d^r(\Delta) = \sum_{\sigma\in\Delta_2^0} \dim \mathcal{R}_d - \sum_{\tau\in\Delta_1^0} \dim \mathcal{R}/\mathcal{J}(\tau)_d + \sum_{\gamma\in\Delta_0^0} \dim \mathcal{R}/\mathcal{J}(\gamma)_d + \dim H_0(\mathcal{J})_d.$$

In order to have an explicit formula for the dimension, the only terms that remain to be computed are the dimension of the ideals $\mathcal{J}(\gamma)$, which are generated by linear forms in two variables, and the homology term. A formula for $\dim \mathcal{J}(\gamma)_d$ was proved in [13], and in [6] by using *inverse systems of fat points* this result was extended so that the powers can be different; this in terms of splines corresponds to have allow different order of smoothness across the different edges. Since $\dim H_0(\mathcal{J})_d \ge 0$, then a lower bound formula for $\dim S_d^r(\Delta)$ can be directly derived. By establishing a numbering on the interior vertices, we prove an upper bound on $\dim S_d^r(\Delta)$ by finding the dimension of a spline space on a partition constructed from the initial triangulation, and whose homology term is zero.

More precisely, we have the following result. Given a numbering on Δ_0^0, let $\tilde{t}_i$ be the number of edges with different slopes attaching the vertex γ_i to vertices on the boundary or of lower index. Consider the ideals generated by these forms, the socle degree of such an ideal is $\tilde{\Omega}_i - 1$, with $\tilde{\Omega}_i = \lfloor \tilde{t}_i\, r/\tilde{t}_i - 1 \rfloor + 1$, and the multiplicity of the syzygies are $\tilde{a}_i = \tilde{t}_i\,(r+1) + (1-\tilde{t}_i)\,\tilde{\Omega}_i$, and $\tilde{b}_i = \tilde{t}_i - 1 - \tilde{a}_i$. Then, the dimension of the spline space on the triangulation Δ is bounded by

$$\dim S_d^r(\Delta) \leq \binom{d+2}{2} + f_1^0 \binom{d+2-(r+1)}{2} - \sum_{i,\,\tilde{t}_i=1}^{f_0^0} \binom{d+2-(r+1)}{2}$$
$$- \sum_{i=1,\tilde{t}_i\geq 2}^{f_0^0} \left[\tilde{t}_i \binom{d+2-(r+1)}{2} - \tilde{b}_i \binom{d+2-\tilde{\Omega}_i}{2} - \tilde{a}_i \binom{d+2-(\tilde{\Omega}_i+1)}{2} \right].$$

We adopt the convention that integers $m < u$ the binomial coefficient $\binom{m}{u} = 0$. In the upper bound from [15], the vertices must be numbered in such a way that the vertex $i+1$ is in a triangle having as a corner at least one the previous i vertices. For example, in Fig. 1, the numbering in (1) or (2), but not the one in (3).

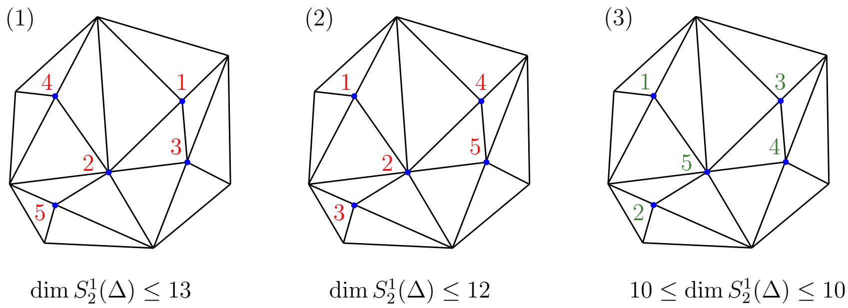

FIGURE 1. Effect of the numbering on the upper bound.

On the other hand, the formula for the upper bound given above can be applied to any numbering, and that leads to find the dimension of the space in many cases, as it is the case for $S_2^1(\Delta)$ in the one in the example.

The computation of the dimension of splines spaces on simplicial complexes in higher dimension involves the study of ideals generated by powers of linear forms in three or more variables. In general, the dimension of such ideals is an open problem in algebraic geometry, known as the Froberg's conjecture [5]. This conjecture has been proved true for several cases, in particular for ideals in three variables [1], which is the case we need for studying trivariate splines. Thus, for splines on tetrahedral partitions, by using Froberg's formula we get a bound on $\dim \mathcal{R}/\mathcal{J}(\gamma_i)$, for $\gamma_i \in \Delta_0^0$. In the case of $H_1(\mathcal{J}) = H_0(\mathcal{J}) = 0$, this directly yields an upper bound on the dimension of the spline space. For general partitions, analogously as in the bivariate case, by fixing a numbering on the edges and on the vertices we prove a lower and an upper bound for the spline space, respectively.

The proofs of these results and additional references can be found in [10, 11]. The further study of ideals generated by powers of linear forms in three variables, and related topics, such as inverse systems of fat points, the Segre-Harbourne-Gimigliano-Hirschowitz's conjecture [4], or the Weak Lefschetz Property, would yield better bounds on the dimension of spaces of trivariate splines. We plan to continue exploring this connections and improve the bounds we have presented.

References

[1] D. Anick, *Thin algebras of embedding dimension three*, J. Algebra **100** 1 (1986), 235–259.

[2] L. Billera, *Homology of smooth splines: generic triangulations and a conjecture of Strang*, Trans. Amer. Math. Soc. **310(1)** (1988), 325–340.

[3] L. Billera and L. Rose, *A dimension series for multivariate splines*, Discrete Comput. Geom. **6(2)** (1991), 107–128.

[4] C. Ciliberto, *Geometric aspects of polynomial interpolation in more variables and of Waring's problem*, European Congress of Mathematics, **1** (Barcelona, 2000), Progr. Math., **201**, Birkhauser, Basel, (2001), 289–316.

[5] R. Froberg, *An inequality for Hilbert series of graded algebras*, Math. Scand. **56(2)** (1985), 117–144.

[6] A. Geramita and H. Schenck, *Fat points, inverse systems, and piecewise polynomial functions*, J. Algebra **204(1)** (1998), 116–128.

[7] A. Iarrobino, *Inverse system of a symbolic power. III. Thin algebras and fat points*, Compositio Math. **108(3)** (1997), 319–356.

[8] G. Strang, *Piecewise polynomials and the finite element method*, Bull. Amer. Math. Soc. **79** (1973), 1128–1137.

[9] M.-J. Lai and L. Schumaker. *Spline functions on triangulations*, Encyclopedia of Mathematics and its Applications, **110**, Cambridge University Press, 2007.

[10] B. Mourrain and N. Villamizar. *Homological techniques for the analysis of the dimension of triangular spline spaces*, J. Symbolic Comput. **50** (2013), 564–577.

[11] B. Mourrain and N. Villamizar. *Bounds on the dimension of trivariate spline spaces: a homological approach*, Mathematics in Computer Sciences **8(2)** (2014), 157–174.

[12] H. Schenck, *A spectral sequence for splines*, Adv. in Appl. Math. **19(2)** (1997), 183–199.

[13] H. Schenck and M. Stillman, *A family of ideals of minimal regularity and the Hilbert series of $C^r(\hat{\Delta})$*, Adv. in Appl. Math. **19(2)** (1997), 169–182.

[14] H. Schenck and M. Stillman, *Local cohomology of bivariate splines*, J. Pure Appl. Algebra **117/118** (1997), 535–548.

[15] L. Schumaker, *Bounds on the dimension of spaces of multivariate piecewise polynomials*, Rocky Mountain J. Math. **14** (1984), 251–264.

Rees Algebras and Powers of Ideals

Gwyneth R. Whieldon

1. Bigraded algebra of spline modules

Let Δ be a subdivided domain of $\mathbb{R}^n$, and $S^r_d(\Delta)$ be spline space of smoothness r and degree d as in [5],

$$S^r_d(\Delta) := \{f \in C^r : f|_\sigma \text{ is polynomial on } \sigma \in \Delta, \deg(f|_\sigma) \leq d\}.$$

One possible approach for calculating the dimension of these spline spaces,

$$H_\Delta(d, r) := \dim_{\Bbbk}\left(S^r_d(\Delta)\right),$$

is to construct a module that somehow includes the algebraic data for all S^r_d. One way to do this is to create a semigroup graded algebra with homogeneous components corresponding to each spline space.

Definition 1 (Semigroup Graded Modules, [6]). *Let A be a semigroup and M a module. M is said to be A-graded if it permits a direct sum decomposition $M = \oplus_{a \in A} M_a$ where each M_a is an additive subgroup of M such that $M_{a_1} M_{a_2} \subseteq M_{a_1 a_2}$ for all $a_1, a_2 \in A$.*

We may create a semigroup-graded module of all spline spaces of Δ over semigroup $A = \mathbb{Z}_{min} \times \mathbb{Z}$, with addition given by $(r, d) + (r', d') := (\min(r, r'), d + d')$, via

$$\mathcal{R}(\Delta) := \bigoplus_{(r,d) \in A} S_d^r(\Delta) t^r s^d.$$

The semigroup A provides a natural grading for $\mathcal{R}(\Delta)$, as a product of two splines $f \in S_d^r(\Delta)$ and $g \in S_{d'}^{r'}$ will be in $S_{d+d'}^{\min(r,r')}$.

Computing a presentation of $\mathcal{R}(\Delta)$ is nontrivial. However, if we were able to obtain such a presentation, finding asymptotic behaviors of $H_\Delta(d, r)$. In the remainder of this abstract, we show the utility of finding a presentation of this module with an application to powers of ideals. In particular, a presentation of (and free resolution for) $\mathcal{R}(\Delta)$ would provide a bound on how large d must be for the dimension of $S_d^r(\Delta)$ to stabilize.

2. Rees Algebras of Equigenerated Ideals

2.1. **Rees Algebras and Degree Restrictions.** Let $I \subseteq R = \mathbb{k}[x_1, ..., x_n]$ be a homogeneous ideal generated by forms of degree r. To calculate many invariants of powers $I^n$of an ideal I, we pass to the Rees algebra of I. The Rees algebra $\mathcal{R}(I)$ of an ideal I is a bigraded ($\mathbb{Z}^2$-graded) module which contains the ideal I and all of its powers:

Definition 2. *Let $I = (f_0, f_1, ..., f_k) \subseteq R = \mathbb{k}[x_1, ..., x_N]$. The* Rees algebra $\mathcal{R}(I)$ of I *is*

$$\mathcal{R}(I) = R \oplus It \oplus I^2t^2 \oplus I^3t^3 \oplus \cdots \oplus I^nt^n \oplus \cdots .$$

Here we present $\mathcal{R}(I)$ as a quotient module of the ring $S = R[w_0, w_1, ..., w_k] = \mathbb{k}[x_1, ..., x_N, w_0, w_1, ..., w_k]$.

Proposition 1 (Presentation of Rees algebras, [10]). *Let $I = (f_1, ..., f_k) \subseteq R = \mathbb{k}[x_1, ..., x_N]$ and let $\mathcal{R}(I)$ be its Rees algebra. Then $\mathcal{R}(I) = R[w_1, ..., w_k]/L = \mathbb{k}[x_1, ..., x_N, w_0, w_1, ..., w_k]/L$, with presentation ideal*

$$L = (f_i - w_i t : 1 \leq i \leq k) S[t] \cap S.$$

If $S = \mathbb{k}[x_1, ..., x_N, w_1, ..., w_k]$, and $\mathcal{R}(I) = S/L$, then L is the Rees ideal *of I.*

2.2. **Resolutions and Bigradings of Rees Algebras.** Taking a free resolution $\mathcal{F}$ (with an appropriately chosen bigrading) of L gives resolutions of all powers of L, and can be used to bound or explicitly compute Betti numbers

$$\beta_{i,j}(I^n) = \dim \left(\operatorname{Tor}_i^R(\mathbb{k}, I^n)\right)_j$$

for all n.

We will assume throughout this abstract that $I = (f_0, f_1, ..., f_k)$ is an equigenerated ideal of degree r in $R = \mathbb{k}[x_1, ..., x_N]$. We set $\mathcal{R}(I) = S/L$ with L the Rees ideal of I and $S = \mathbb{k}[x_1, ..., x_N, w_0, w_1, ..., w_k]$ as our notation throughout.

We bigrade $\mathcal{R}(I)$ by $\deg(x_i) = (1, 0)$ and $\deg(w_i) = (0, 1)$ and take the minimal graded free resolution of $\mathcal{R}(I)$ with respect to this grading.

$$\begin{aligned}\mathcal{F}: \quad \mathcal{R}(I) \leftarrow S \leftarrow \bigoplus_{(j,m)} S(-j,-m)^{\beta_{1,(j,m)}} \leftarrow \cdots \\ \cdots \leftarrow \bigoplus_{(j,m)} S(-j,-m)^{\beta_{p,(j,m)}} \leftarrow 0.\end{aligned}$$

Restricting to the strand $(*, d)$, we obtain a (possibly nonminimal) resolution of I^d:

$$\begin{aligned}\mathcal{F}_d: \quad I^d \leftarrow S_{(*,d)} \leftarrow \bigoplus_{(j,m)} S(-j,-m)^{\beta_{1,(j,m)}}_{(*,d)} \leftarrow \cdots \\ \cdots \leftarrow \bigoplus_{(j,m)} S(-j,m)^{\beta_{p,(j,m)}}_{(*,d)} \leftarrow 0.\end{aligned}$$

Tensoring this resolution with $\mathbb{k}$ and taking the homology of the maps gives us $\dim \operatorname{Tor}_i^R(\mathbb{k}, I^d)_{j+rd} = \beta_{i,j+rd}(I^d)$. This shift in the indices of $\beta_{i,j+rd}(I^d)$ accounts for the shift in grading to agree with that of R while viewing I^d as an R module.

Alternately, we could have first tensored with S/M for $M = (x_1, ..., x_N)$, taken homology of our maps, then restricted in degrees. This will give us modules $\operatorname{Tor}_i^S(S/M, \mathcal{R}(I))_j$, and as these two actions commute, we have that

$$\begin{aligned}\operatorname{Tor}_i^S(S/M, \mathcal{R}(I))_{(j,d)} &= \operatorname{Tor}_i^R(S/M, I^d)_{j+rd} \\ &= \operatorname{Tor}_i^R(\mathbb{k}, I^d)_{j+rd},\end{aligned}$$

where the second equality follows from $S/M \cong \mathbb{k}$ as an R-module.

Hence we have that all Betti numbers of higher powers can be written in terms of the dimensions of the bigraded modules $\operatorname{Tor}_i^S(S/M, \mathcal{R}(I))$, as

$$\beta_{i,j+rd}(I^d) = \dim \operatorname{Tor}_i^S(S/M, \mathcal{R}(I))_{(j,d)}.$$

In particular, using this we can be show that the shapes of the Betti tables of the ideals I^d stabilize, in the sense that there exists some D such that for all $d \geq D$, $\beta_{i,j+rd}(I^d) \neq 0 \Leftrightarrow \beta_{i,j+rD}(I^D) \neq 0$. Using x-regularity of the Rees algebra, presented as a quotient module, bigraded by $\mathbb{Z}^2$, we can find an upper bound for this degree D. [11]

Not much is known about equivalent stabilization results or regularity for modules graded over monoids other than $\mathbb{Z}$ or $\mathbb{Z}^n$, so identifying similar regularity tools as those used here (the x-regularity of a Rees algebra) may provide insight into the behavior of the dimensions of spline spaces $S_d^r(\Delta)$, given by $H_\Delta(d, r)) = \dim_{\mathbb{k}} \mathcal{R}(\Delta)_{(d,r)}$.

References

[1] K. Borna, *On linear resolution of powers of an ideal*, Osaka J. Math. **46(4)** (2009) 1047–1058.

[2] A. Conca, *Regularity jumps for powers of ideals*, In Commutative algebra **244** of Lect. Notes Pure Appl. Math., 21–32. Chapman & Hall/CRC, Boca Raton, FL, 2006.

[3] S. Dale Cutkosky, Jürgen Herzog, and Ngô Viêt Trung. *Asymptotic behaviour of the Castelnuovo-Mumford regularity*, Compositio Math. **118(3)** (1999), 243–261.

[4] J. Herzog, T. Hibi, and X. Zheng, *Monomial ideals whose powers have a linear resolution.* ArXiv Mathematics e-prints, July 2003.

[5] A. Kolesnikov, and T. Sorokina, *Multivariate C^1-continuous splines on the Alfeld split of a simplex*, Approximation theory XIV: San Antonio 2013, Springer Proc. Math. Stat., **83** (2014), 283–294.

[6] C. Năstăsescu, and F. van Oystaeyen, *Graded ring theory*, **28** of North-Holland Mathematical Library. North-Holland Publishing Co., Amsterdam-New York, 1982.

[7] T. Römer, *Homological properties of bigraded algebras*, Illinois J. Math. **45(4)** (2001), 1361–1376.

[8] B. Sturmfels, *Four counterexamples in combinatorial algebraic geometry*, Journal of Algebra **230** (2000), 282–294.

[9] I. Swanson, *Powers of ideals, Primary decompositions, Artin-Rees lemma and regularity*, Math. Ann. **307(2)** (1997) 299–313.

[10] W. V. Vasconcelos, *Arithmetic of blowup algebras*, **195** of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1994.

[11] G. Whieldon, *Stabilization of Betti tables*, Journal of Commutative Algebra **6(1)** (2014) 113-126.

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