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Graph properties for nonlocal minimal surfaces

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Abstract. In this paper we show that a nonlocal minimal surface which is a graph outside a cylinder is in fact a graph in the whole of the space.

As a consequence, in dimension 3, we show that the graph is smooth.

The proofs rely on convolution techniques and appropriate integral estimates which show the pointwise validity of an Euler-Lagrange equation related to the nonlocal mean curvature.

1. Introduction

This paper deals with the geometric properties of the minimizers of a nonlocal perimeter functional.

More precisely, given $s \in (0, 1/2)$, and an open set $\Omega \subseteq \mathbb{R}^n$, the s-perimeter of a set $E \subseteq \mathbb{R}^n$ in Ω was defined in [7] as

$$\mathrm{Per}_s(E, \Omega) := L(E \cap \Omega, E^c) + L(\Omega \setminus E, E \setminus \Omega),$$

where $E^c := \mathbb{R}^n \setminus E$ and, for any disjoint sets F and G,

$$L(F, G) := \iint_{F \times G} \frac{dx\, dy}{|x - y|^{n+2s}}.$$

This nonlocal perimeter captures the global contributions between the set E and its complement and it is related to some models in geometry and physics, such as the motion by nonlocal mean curvature (see [8]) and the phase transitions in presence of long-range interactions (see [17]).

As customary in the calculus of variation literature, one says that E is s-minimal in Ω if $\mathrm{Per}_s(E, \Omega) < +\infty$ and $\mathrm{Per}_s(E, \Omega) \leqslant \mathrm{Per}_s(F, \Omega)$ among all the sets F which coincide with E outside Ω.

Several analytic and geometric properties of s-minimal sets have been recently investigated, in terms, for instance, of asymptotics [16, 3, 9, 1, 13], regularity [10, 18, 15] and classification [4, 11]. Some examples of s-minimal sets (or, more generally, of sets which possess vanishing nonlocal mean curvatures) have been given in [12, 14].

The main result of this paper establishes that an s-minimal set is an subgraph, if so are its exterior data:

Theorem 1.1. *Let Ω_o be an open and bounded subset of $\mathbb{R}^{n-1}$ with boundary of class $C^{1,1}$, and let $\Omega := \Omega_o \times \mathbb{R}$. Let E be an s-minimal set in Ω. Assume that*

$$E \setminus \Omega = \{x_n < u(x'),\ x' \in \mathbb{R}^{n-1} \setminus \Omega_o\}, \tag{1.1}$$

for some continuous function $u : \mathbb{R}^{n-1} \to \mathbb{R}$. Then

$$E \cap \Omega = \{x_n < v(x'),\ x' \in \Omega_o\},$$

for some $v : \mathbb{R}^{n-1} \to \mathbb{R}$.

The proof of Theorem 1.1 is based on a sliding method, but some (both technical and conceptual) modifications are needed to make the classical argument work, due to the contributions "coming from far". First of all, since the s-minimal set is not assumed to be smooth, some supconvolutions techniques are needed to take care of interior contact points. Moreover, a fine analysis of the possible contact points which lie on the boundary (and at infinity) is needed to complete the arguments.

We also mention that, in general s-minimal surfaces are not continuous up to the boundary of the domain (even if the datum outside is smooth), and indeed boundary stickiness phenomena occur (see [14] for concrete examples). The possible discontinuity at the boundary makes the proof of Theorem 1.1 quite delicate, since the graph property "almost fails" in a cylinder (see Theorem 1.2 in [14]), and, in general, the graph property cannot be deduced only from the outside data but it may also depend on the regularity of the domain.

As a matter of fact, we think that it is an interesting open problem to determine whether or not Theorem 1.1 holds true without the assumption that $\partial\Omega_o$ is of class $C^{1,1}$ (for instance, whether or not a similar statement holds by assuming only that $\partial\Omega_o$ is Lipschitz).

The results in Theorem 1.1 may be strengthen in the case of dimension 3, by proving that two-dimensional minimal graphs are smooth. Indeed, we have:

Theorem 1.2. *Let Ω_o be an open and bounded subset of $\mathbb{R}^2$ with boundary of class $C^{1,1}$, and let $\Omega := \Omega_o \times \mathbb{R}$. Let E be an s-minimal set in Ω. Assume that*

$$E \setminus \Omega = \{x_n < u(x'),\ x' \in \mathbb{R}^{n-1} \setminus \Omega_o\},$$

for some continuous function $u : \mathbb{R}^{n-1} \to \mathbb{R}$. Then

$$E \cap \Omega = \{x_3 < v(x'),\ x' \in \Omega_o\}, \tag{1.2}$$

for some $v \in C^\infty(\Omega_o)$.

The proof of Theorem 1.2 relies on Theorem 1.1 and on a Bernstein-type result of [15].

The rest of the paper is organized as follows. In Section 2 we discuss the notion of supconvolutions and subconvolutions for a nonlocal minimal surface, presenting the geometric and analytic properties that we need for the proof of Theorem 1.1.

In Section 3 we collect a series of auxiliary results needed to compute suitable integral contributions and obtain an appropriate fractional mean curvature equation in a pointwise sense (i.e., not only in the sense of viscosity, as done in the previous literature).

The proof of Theorem 1.1 is given in Section 4 and the proof of Theorem 1.2 is given in Section 5.

2. Supconvolution of a set

In this section, we introduce the notion of supconvolution and discuss its basic properties. This is the nonlocal modification of a technique developed in [5] for the local case.

Given $\delta > 0$, we define the supconvolution of the set $E \subseteq \mathbb{R}^n$ by

$$E^\sharp_\delta := \bigcup_{x\in E} \overline{B_\delta(x)}.$$

Lemma 2.1. *We have that*

$$E^\sharp_\delta = \bigcup_{\substack{v\in\mathbb{R}^n\\ |v|\leqslant\delta}} (E+v).$$

Proof. Let $y \in \overline{B_\delta(x)}$, with $x \in E$. Let $v := y - x$. Then $|v| \leqslant \delta$ and $y = x + v \in E + v$, and one inclusion is proved.

Viceversa, let now $y \in E + v$, with $|v| \leqslant \delta$. We set in this case $x := y - v$. Hence $|y - x| = |v| \leqslant \delta$, thus $y \in \overline{B_\delta(x)}$. In addition, $x \in (E + v) - v = E$, so the other inclusion is proved. □

Corollary 2.2. *If $p \in \partial E^\sharp_\delta$, then there exist $v \in \mathbb{R}^n$, with $|v| = \delta$, and $x_o \in \partial E$ such that $p = x_o + v$ and $B_\delta(x_o) \subseteq E^\sharp_\delta$.*

Also, if $E^\sharp_\delta$ is touched from the outside at p by a ball B, then E is touched from the outside at x_o by $B - v$.

Proof. Since $p \in \overline{E^\sharp_\delta}$, we have that there exists a sequence $p_j \in E^\sharp_\delta$ such that $p_j \to p$ as $j \to +\infty$. By Lemma 2.1, we have that $p_j \in E + v_j$, for some $v_j \in \mathbb{R}^n$ with $|v_j| \leqslant \delta$. That is, there exists $x_j \in E$ such that $p_j = x_j + v_j$. By compactness, up to a subsequence we may assume that $v_j \to v$ as $j \to +\infty$, for some $v \in \mathbb{R}^n$ with

$$|v| \leqslant \delta. \tag{2.1}$$

Therefore

$$x_j = p_j - v_j \to p - v =: x_o \tag{2.2}$$

as $j \to +\infty$. By construction,

$$x_o \in \overline{E} \tag{2.3}$$

and

$$p = x_o + v. \tag{2.4}$$

Now we show that

$$x_o \in \overline{E^c}. \tag{2.5}$$

For this, since $p \in \overline{\mathbb{R}^n \setminus E^\sharp_\delta}$, we have that there exists a sequence $q_j \in \mathbb{R}^n \setminus E^\sharp_\delta$ such that $q_j \to p$ as $j \to +\infty$.

Notice that

$$\overline{B_\delta(q_j)} \cap E = \varnothing. \tag{2.6}$$

Indeed, if not, we would have that there exists $z_j \in \overline{B_\delta(q_j)} \cap E$. So we can define $w_j := q_j - z_j$. We see that $|w_j| \leqslant \delta$ and therefore $q_j = z_j + w_j \in E + w_j \subseteq E^\sharp_\delta$, which is a contradiction.

Having established (2.6), we use it to deduce that $q_j - v_j \in E^c$. Thus passing to the limit

$$x_o = p - v = \lim_{j \to +\infty} q_j - v_j \in \overline{E^c}.$$

This proves (2.5).

From (2.3) and (2.5), we conclude that

$$x_o \in \partial E. \tag{2.7}$$

Now we show that

$$|v| = \delta. \tag{2.8}$$

To prove it, suppose not. Then, by (2.1), we have that $|v| < \delta$. That is, there exists $a \in (0, \delta)$ such that $|v| < \delta - a$. Then, by (2.2),

$$|x_j - p| \leqslant |x_j - x_o| + |x_o - p| = |x_j - x_o| + |v| < \delta - \frac{a}{2},$$

if j is large enough. Hence $B_{a/2}(p) \subseteq B_\delta(x_j) \subseteq E^\sharp_\delta$, that says that p lies in the interior of $E^\sharp_\delta$. This is in contradiction with the assumptions of Corollary 2.2, and so (2.8) is proved.

Now we claim that

$$B_\delta(x_o) \subseteq E^\sharp_\delta. \tag{2.9}$$

To prove this, let $z \in B_\delta(x_o)$. Then, $|z - x_o| \leqslant \delta - b$, for some $b \in (0, \delta)$. Accordingly, by (2.2), we have that $|z - x_j| \leqslant \delta - \frac{b}{2}$ if j is large enough. Hence $z \in B_\delta(x_j) \subseteq E^\sharp_\delta$. This proves (2.9).

Thanks to (2.4), (2.7), (2.8) and (2.9), we have completed the proof of the first claim in the statement of Corollary 2.2.

Now, to prove the second claim in the statement of Corollary 2.2, let us consider a ball B such that $B \subseteq \mathbb{R}^n \setminus E^\sharp_\delta$ and $p \in \partial B$. Then $x_o = p - v \in (\partial B) - v = \partial(B - v)$. Moreover,

$$B - v \subseteq (\mathbb{R}^n \setminus E^\sharp_\delta) - v = \mathbb{R}^n \setminus (E^\sharp_\delta - v).$$

Since $E \subseteq E^\sharp_\delta$, we have that

$$\mathbb{R}^n \setminus (E^\sharp_\delta - v) \subseteq \mathbb{R}^n \setminus (E - v).$$

Consequently, we obtain that $B - v \subseteq \mathbb{R}^n \setminus (E - v)$, which completes the proof of the second claim of Corollary 2.2. $\square$

The supconvolution has an important property with respect to the fractional mean curvature, as stated in the next result:

Lemma 2.3. *Let $p \in \partial E^\sharp_\delta$, $v \in \mathbb{R}^n$ with $|v| \leqslant \delta$ and $x_o \in \partial E$ such that $p = x_o + v$. Then*

$$\int_{\mathbb{R}^n} \frac{\chi_{E^\sharp_\delta}(y) - \chi_{\mathbb{R}^n \setminus E^\sharp_\delta}(y)}{|p-y|^{n+2s}}\, dy \geqslant \int_{\mathbb{R}^n} \frac{\chi_{E}(y) - \chi_{\mathbb{R}^n \setminus E}(y)}{|x_o-y|^{n+2s}}\, dy.$$

Proof. The claim follows simply by the fact that $E^\sharp_\delta \supseteq E+v$ and the translation invariance of the fractional mean curvature. $\square$

Corollary 2.4. *Let E be an s-minimal set in Ω. Let $p \in \partial E^\sharp_\delta$. Assume that $\overline{B_\delta(p)} \subseteq \Omega$ and that $E^\sharp_\delta$ is touched from the outside at p by a ball. Then*

$$\int_{\mathbb{R}^n} \frac{\chi_{E^\sharp_\delta}(y) - \chi_{\mathbb{R}^n \setminus E^\sharp_\delta}(y)}{|p-y|^{n+2s}}\, dy \geqslant 0.$$

Proof. By Corollary 2.2, we know that there exist $v \in \mathbb{R}^n$ with $|v| \leqslant \delta$ and $x_o \in \partial E$ such that $p = x_o + v$, and that E is touched by a ball from the outside at x_o.

We remark that $x_o \in \overline{B_\delta(p)} \subseteq \Omega$. So, we can use the Euler-Lagrange equation in the viscosity sense (see Theorem 5.1 in [7]) and obtain that

$$\int_{\mathbb{R}^n} \frac{\chi_{E}(y) - \chi_{\mathbb{R}^n \setminus E}(y)}{|p-y|^{n+2s}}\, dy \geqslant 0.$$

This and Lemma 2.3 give the desired result. $\square$

The counterpart of the notion of supconvolution is given by the notion of subconvolution. That is, we define

$$E^\flat_\delta := \mathbb{R}^n \setminus \big((\mathbb{R}^n \setminus E)^\sharp_\delta \big).$$

In this setting, we have:

Proposition 2.5. *Let E be an s-minimal set in Ω. Let $p \in \partial E^\sharp_\delta$. Assume that $\overline{B_\delta(p)} \subseteq \Omega$.*

Assume also that $E^\sharp_\delta$ is touched from above at p by a translation of $E^\flat_\delta$, i.e. there exists $\omega \in \mathbb{R}^n$ such that $E^\sharp_\delta \subseteq E^\flat_\delta + \omega$ and $p \in (\partial E^\sharp_\delta) \cap \big(\partial(E^\flat_\delta + \omega)\big)$.

Then $E^\sharp_\delta = E^\flat_\delta + \omega$.

Proof. Notice that

$$p \in \partial(E^\flat_\delta + \omega) = \partial E^\flat_\delta + \omega = \partial\big((\mathbb{R}^n \setminus E)^\sharp_\delta \big) + \omega.$$

Accordingly, by the first claim in Corollary 2.2 (applied to the set $\mathbb{R}^n \setminus E$ and to the point $p-\omega$), we see that there exist $\tilde v \in \mathbb{R}^n$, with $|\tilde v| = \delta$, and $\tilde x_o \in \partial(\mathbb{R}^n \setminus E) = \partial E$ such that $p - \omega = \tilde x_o + \tilde v$ and $B_\delta(\tilde x_o) \subseteq (\mathbb{R}^n \setminus E)^\sharp_\delta$. That is, the set $(\mathbb{R}^n \setminus E)^\sharp_\delta$ is touched from the inside at $p-\omega$ by a ball of radius δ. Taking the complementary set and translating by ω, we obtain that $E^\flat_\delta + \omega$ is touched from the outside at p by a ball of radius δ.

Then, since $E^\flat_\delta + \omega \supseteq E^\sharp_\delta$, we obtain that also $E^\sharp_\delta$ is touched from the outside at p by a ball of radius δ. Thus, making use of Corollary 2.4, we deduce that

$$\int_{\mathbb{R}^n} \frac{\chi_{E^\sharp_\delta}(y) - \chi_{\mathbb{R}^n \setminus E^\sharp_\delta}(y)}{|p-y|^{n+2s}}\, dy \geqslant 0. \tag{2.10}$$

Moreover, by Corollary 2.2, we know that $E^\sharp_\delta$ is touched from the inside at p by a ball of radius δ. By inclusion of sets, this gives that $E^\flat_\delta + \omega$ is touched from the inside at p by a ball of radius δ. Taking complementary sets, we obtain that $(\mathbb{R}^n \setminus E)^\sharp_\delta$ is touched from the outside at $p - \omega$ by a ball of radius δ. Therefore, we can use Corollary 2.4 (applied here to the set $(\mathbb{R}^n \setminus E)^\sharp_\delta$), and get that

$$\begin{aligned} 0 &\leqslant \int_{\mathbb{R}^n} \frac{\chi_{(\mathbb{R}^n \setminus E)^\sharp_\delta}(y) - \chi_{\mathbb{R}^n \setminus \left((\mathbb{R}^n \setminus E)^\sharp_\delta\right)}(y)}{|p-\omega-y|^{n+2s}}\, dy \\ &= \int_{\mathbb{R}^n} \frac{\chi_{\mathbb{R}^n \setminus E^\flat_\delta}(y) - \chi_{E^\flat_\delta}(y)}{|p-\omega-y|^{n+2s}}\, dy = -\int_{\mathbb{R}^n} \frac{\chi_{E^\flat_\delta+\omega}(y) - \chi_{\mathbb{R}^n \setminus (E^\flat_\delta+\omega)}(y)}{|p-y|^{n+2s}}\, dy. \end{aligned}$$

By comparing this estimate with the one in (2.10), we obtain that

$$\int_{\mathbb{R}^n} \frac{\chi_{E^\sharp_\delta}(y) - \chi_{\mathbb{R}^n\setminus E^\sharp_\delta}(y)}{|p-y|^{n+2s}}\,dy \geqslant 0 \geqslant \int_{\mathbb{R}^n} \frac{\chi_{E^\flat_\delta+\omega}(y) - \chi_{\mathbb{R}^n\setminus(E^\flat_\delta+\omega)}(y)}{|p-y|^{n+2s}}\,dy.$$

Since $E^\sharp_\delta$ lies in $E^\flat_\delta+\omega$, the inequality above implies that the two sets must coincide. $\square$

A useful variation of Proposition 2.5 consists in taking into account that the inclusion of the sets only occurs inside a cylinder

$$\mathscr{C}_R := \{x=(x',x_n)\in\mathbb{R}^n \text{ s.t. } |x'|<R\}. \tag{2.11}$$

Indeed, we have:

Proposition 2.6. *Let $R>4$ and $\delta\in(0,1)$. Let E be an s-minimal set in Ω. Let $p\in(\partial E^\sharp_\delta)\cap\mathscr{C}_1$. Assume that $\overline{B_\delta(p)}\subseteq\Omega$.*

Assume also that $E^\sharp_\delta$ is touched in $\mathscr{C}_R$ from above at p by a vertical translation of $E^\flat_\delta$, i.e. there exists $\omega=(\omega',0)\in\mathbb{R}^n$ such that $E^\sharp_\delta\cap\mathscr{C}_R\subseteq(E^\flat_\delta+\omega)\cap\mathscr{C}_R$ and $p\in(\partial E^\sharp_\delta)\cap\big(\partial(E^\flat_\delta+\omega)\big)$. Then

$$\int_{\mathscr{C}_R} \frac{\chi_{(E^\flat_\delta+\omega)\setminus E^\sharp_\delta}(y) - \chi_{E^\sharp_\delta\setminus(E^\flat_\delta+\omega)}(y)}{|p-y|^{n+2s}}\,dy \leqslant CR^{-2s},$$

for some $C>0$, independent of δ and R.

Proof. The proof is a measure theoretic version of the one in Proposition 2.5. We give the full details for the convenience of the reader.

Notice that

$$p\in\partial(E^\flat_\delta+\omega)=\partial E^\flat_\delta+\omega=\partial\big((\mathbb{R}^n\setminus E)^\sharp_\delta\big)+\omega.$$

Accordingly, by the first claim in Corollary 2.2 (applied to the set $\mathbb{R}^n\setminus E$ and to the point $p-\omega$), we see that there exist $\tilde v\in\mathbb{R}^n$, with $|\tilde v|=\delta$, and $\tilde x_o\in\partial(\mathbb{R}^n\setminus E)=\partial E$ such that $p-\omega=\tilde x_o+\tilde v$ and $B_\delta(\tilde x_o)\subseteq(\mathbb{R}^n\setminus E)^\sharp_\delta$. That is, the set $(\mathbb{R}^n\setminus E)^\sharp_\delta$ is touched from the inside at $p-\omega$ by a ball of radius δ. Taking the complementary set and translating by ω, we obtain that $E^\flat_\delta+\omega$ is touched from the outside at p by a ball of radius δ.

Then, since $(E^\flat_\delta+\omega)\cap\mathscr{C}_R\supseteq E^\sharp_\delta\cap\mathscr{C}_R$, we obtain that also $E^\sharp_\delta$ is touched from the outside at p by a ball of radius δ. Thus, making use of Corollary 2.4, we deduce that

$$\int_{\mathbb{R}^n} \frac{\chi_{E^\sharp_\delta}(y) - \chi_{\mathbb{R}^n\setminus E^\sharp_\delta}(y)}{|p-y|^{n+2s}}\,dy \geqslant 0. \tag{2.12}$$

Moreover, by Corollary 2.2, we know that $E^\sharp_\delta$ is touched from the inside at p by a ball of radius δ. By inclusion of sets, this gives that $E^\flat_\delta+\omega$ is touched from the inside at p by a ball of radius δ. Taking complementary sets, we obtain that $(\mathbb{R}^n\setminus E)^\sharp_\delta$ is touched from the outside at $p-\omega$ by a ball of radius δ. Therefore, we can use Corollary 2.4 (applied here to the set $(\mathbb{R}^n\setminus E)^\sharp_\delta$), and get that

$$\begin{aligned} 0 &\leqslant \int_{\mathbb{R}^n} \frac{\chi_{(\mathbb{R}^n\setminus E)^\sharp_\delta}(y) - \chi_{\mathbb{R}^n\setminus\left((\mathbb{R}^n\setminus E)^\sharp_\delta\right)}(y)}{|p-\omega-y|^{n+2s}}\,dy \\ &= \int_{\mathbb{R}^n} \frac{\chi_{\mathbb{R}^n\setminus E^\flat_\delta}(y) - \chi_{E^\flat_\delta}(y)}{|p-\omega-y|^{n+2s}}\,dy = -\int_{\mathbb{R}^n} \frac{\chi_{E^\flat_\delta+\omega}(y) - \chi_{\mathbb{R}^n\setminus(E^\flat_\delta+\omega)}(y)}{|p-y|^{n+2s}}\,dy. \end{aligned}$$

By comparing this estimate with the one in (2.12), we obtain that

$$\int_{\mathbb{R}^n} \frac{\chi_{E^\sharp_\delta}(y) - \chi_{\mathbb{R}^n\setminus E^\sharp_\delta}(y)}{|p-y|^{n+2s}}\,dy \geqslant 0 \geqslant \int_{\mathbb{R}^n} \frac{\chi_{E^\flat_\delta+\omega}(y) - \chi_{\mathbb{R}^n\setminus(E^\flat_\delta+\omega)}(y)}{|p-y|^{n+2s}}\,dy.$$

Since $E^\sharp_\delta\cap\mathscr{C}_R$ lies in $(E^\flat_\delta+\omega)\cap\mathscr{C}_R$, the inequality above implies that

$$\int_{\mathscr{C}_R} \frac{\chi_{(E^\flat_\delta+\omega)\setminus E^\sharp_\delta}(y) - \chi_{E^\sharp_\delta\setminus(E^\flat_\delta+\omega)}(y)}{|p-y|^{n+2s}}\,dy \leqslant 2\int_{\mathbb{R}^n\setminus\mathscr{C}_R} \frac{dy}{|p-y|^{n+2s}}.$$

Notice now that $|p-y| \geqslant |p'-y'| \geqslant |y'|-|p'| \geqslant R-1 \geqslant R/2$. Hence changing variable $\zeta := p-y$, we have

$$\int_{\mathscr{C}_R} \frac{\chi_{(E^\flat_\delta+\omega)\setminus E^\sharp_\delta}(y)-\chi_{E^\sharp_\delta\setminus(E^\flat_\delta+\omega)}(y)}{|p-y|^{n+2s}}\,dy \leqslant 2\int_{\mathbb{R}^n\setminus B_{R/2}} \frac{d\zeta}{|\zeta|^{n+2s}},$$

which gives the desired result. □

3. Auxiliary integral computations and a pointwise version of the Euler-Lagrange equation

We collect here some technical results, which are used during the proofs of the main results. First, we recall an explicit estimate on the weighted measure of a set trapped between two tangent balls.

Lemma 3.1. *For any $R>0$ and $\lambda\in(0,1]$, let*

$$P_{R,\lambda} := \left\{ x=(x',x_n)\in\mathbb{R}^n \text{ s.t. } |x'|\leqslant \lambda R \text{ and } |x_n|\leqslant R-\sqrt{R^2-|x'|^2}\right\}.$$

Then

$$\int_{P_{R,\lambda}} \frac{dx}{|x|^{n+2s}} \leqslant \frac{CR^{-2s}\lambda^{1-2s}}{1-2s},$$

for some $C>0$ only depending on n.

Proof. By scaling $y:=x/R$, we see that

$$\int_{P_{R,\lambda}} \frac{dx}{|x|^{n+2s}} = R^{-2s}\int_{P_{1,\lambda}} \frac{dy}{|y|^{n+2s}},$$

so it is enough to prove the desired claim for $R=1$.

To this goal, we observe that, if $\rho\in[0,1]$ then

$$1-\sqrt{1-\rho^2}\leqslant C\rho^2,$$

for some $C>0$ (independent of n and s). Therefore

$$\int_0^\lambda \frac{1-\sqrt{1-\rho^2}}{\rho^{2+2s}}\,d\rho \leqslant \frac{C\lambda^{1-2s}}{1-2s}, \tag{3.1}$$

up to renaming $C>0$.

In addition, using polar coordinates in $\mathbb{R}^{n-1}$ (and possibly renaming constants which only depend on n), we have

$$\begin{aligned}\int_{P_{1,\lambda}} \frac{dx}{|x|^{n+2s}} &\leqslant \int_{P_{1,\lambda}} \frac{dx}{|x'|^{n+2s}} = C\int_{\{|x'|\leqslant\lambda\}}\left(\int_0^{1-\sqrt{1-|x'|^2}} \frac{dx_n}{|x'|^{n+2s}}\right)dx' \\ &= C\int_{\{|x'|\leqslant\lambda\}} \frac{1-\sqrt{1-|x'|^2}}{|x'|^{n+2s}}\,dx' = C\int_0^\lambda \frac{1-\sqrt{1-\rho^2}}{\rho^{2+2s}}\,d\rho.\end{aligned}$$

This and (3.1) yield the desired result. □

A variation of Lemma 3.1 deals with the case of trapping between two hypersurfaces, as stated in the following result:

Lemma 3.2. *Let $C_o>0$ and $\alpha>2s$. For any $L>0$, let*

$$P_L := \left\{ x=(x',x_n)\in\mathbb{R}^n \text{ s.t. } |x'|\leqslant L \text{ and } |x_n|\leqslant C_o\,|x'|^{1+\alpha}\right\}.$$

Then

$$\int_{P_L} \frac{dx}{|x|^{n+2s}} \leqslant \frac{C\,C_o\,L^{\alpha-2s}}{\alpha-2s},$$

for some $C>0$ only depending on n.

Proof. Using polar coordinates in $\mathbb{R}^{n-1}$, we have

$$\int_{P_L} \frac{dx}{|x|^{n+2s}} \leqslant \int_{\{|x'|\leqslant L\}} \left(\int_{\{|x_n|\leqslant C_o\,|x'|^{1+\alpha}\}} \frac{dx_n}{|x'|^{n+2s}} \right) dx' = \int_{\{|x'|\leqslant L\}} \frac{2C_o\,|x'|^{1+\alpha}}{|x'|^{n+2s}}\, dx' = \frac{C\, C_o\, L^{\alpha-2s}}{\alpha-2s},$$

for some $C > 0$. □

Now we show that an s-minimal set does not have spikes going to infinity:

Lemma 3.3. *Let Ω_o be an open and bounded subset of $\mathbb{R}^{n-1}$ and let $\Omega := \Omega_o \times \mathbb{R}$. Let E be an s-minimal set in Ω.*

Assume that

$$E \setminus \Omega \subseteq \{x_n \leqslant v(x')\}, \tag{3.2}$$

for some $v : \mathbb{R}^{n-1} \to \mathbb{R}$, and that, for any $R > 0$,

$$M_R := \sup_{|x'|\leqslant R} v(x') < +\infty.$$

Then

$$E \cap \Omega \subseteq \{x_n \leqslant M\}$$

for some $M \in \mathbb{R}$ (which may depend on s, n, Ω_o and v).

Proof. Assume that $\Omega_o \subseteq \{|x'| < R_o\}$, for some R_o and let $R > R_o + 1$, to be chosen suitably large. We show that

$$E \subseteq \left\{ x_n \leqslant 2M_{5R} + \frac{3}{2}R \right\}. \tag{3.3}$$

For any $t \geqslant 2M_{5R} + 2R$ we slide a ball centered at $\{x_n = t\}$ of radius $R/2$ "from left to right". For this, we observe that

$$B_{R/2}(-2R, 0, \dots, 0, t) \subseteq E^c. \tag{3.4}$$

Indeed, if $x \in B_{R/2}(-2R, 0, \dots, 0, t)$, then

$$\big||x'| - 2R\big| = \big||x'| - |(-2R, 0, \dots, 0)|\big| \leqslant \big|x' - (-2R, 0, \dots, 0)\big| \leqslant \big|x - (-2R, 0, \dots, 0, t)\big| \leqslant \frac{R}{2}.$$

In particular,

$$|x'| \in (R,\, 3R).$$

In addition,

$$x_n \geqslant t - \frac{R}{2} \geqslant 2M_{5R} + 2R - \frac{R}{2} > 2M_{5R} \geqslant v(x').$$

These considerations and (3.2) imply that $x \in E^c$, thus establishing (3.4).

As a consequence of (3.4), we can slide the ball $B_{R/2}(-2R, 0, \dots, 0, t)$ in direction e_1 till it touches ∂E. Notice that if no touching occurs for any t, then (3.3) holds true and we are done. So we assume, by contradiction, that there exists $t \geqslant 2M_{5R} + 2R$ for which a touching occurs, namely there exists a ball $B := B_{R/2}(\rho, 0, \dots, 0, t)$ for some $\rho \in [-2R, 2R]$ such that

$$B \subset E^c \tag{3.5}$$

and there exists $p \in (\partial B) \cap (\partial E) \cap \overline{\Omega}$.

Let now B' be the ball symmetric to B with respect to p, and let K be the convex envelope of $B \cup B'$.

Notice that if $x \in B'$ then $x_n \geqslant t - \frac{3}{2}R \geqslant 2M_{5R} + \frac{R}{2} > 2M_{5R}$. That is, $B \cup B' \subseteq \{x_n > 2M_{5R}\}$ and so, by convexity

$$K \subseteq \{x_n > 2M_{5R}\}. \tag{3.6}$$

Now we claim that

$$K \subseteq \{x_n > v(x')\}. \tag{3.7}$$

Indeed, if $x \in K$ then $|x'| \leqslant \rho + 2R \leqslant 4R$, hence (3.7) follows from (3.6).

From (3.2) and (3.7) we conclude that

$$K \setminus \Omega \subseteq E^c. \tag{3.8}$$

Now define $B_\star := B_1\big(p + (2R_o + 2)e_1\big)$ and we observe that

$$B_\star \subseteq \Omega^c. \tag{3.9}$$

Indeed, if $x \in B_\star$, then

$$\begin{aligned} |x'| &\geqslant \big|\big(p' + (2R_o + 2)e_1\big)\big| - \big|x' - \big(p' + (2R_o + 2)e_1\big)\big| \\ &\geqslant 2R_o + 2 - |p'| - \big|x - \big(p + (2R_o + 2)e_1\big)\big| \geqslant 2R_o + 2 - R_o - 1 > R_o, \end{aligned}$$

which proves (3.9).

Now we check that

$$B_\star \subseteq K. \tag{3.10}$$

Indeed,

$$\text{if } x \in B_\star, \text{ then } |x - p| \leqslant 2R_o + 3, \tag{3.11}$$

and so in particular $|x - p| < \frac{R}{4}$ if R is large enough, and this proves (3.10).

In light of (3.8), (3.9) and (3.10), we have that

$$B_\star \subseteq K \cap \Omega^c = K \setminus \Omega \subseteq E^c. \tag{3.12}$$

Also, since we have slided the balls from left to right, we have that $B_\star$ is on the right of B and hence it lies outside B. Hence, (3.10) can be precised by saying that $B_\star \subseteq K \setminus B$.

Thus, as a consequence of (3.5) and (3.12),

$$\begin{aligned} \int_K \frac{\chi_{E^c}(y) - \chi_E(y)}{|p - y|^{n+2s}}\, dy &= \int_{K \setminus B_\star} \frac{\chi_{E^c}(y) - \chi_E(y)}{|p - y|^{n+2s}}\, dy + \int_{B_\star} \frac{dy}{|p - y|^{n+2s}} \\ &\geqslant \int_B \frac{dy}{|p - y|^{n+2s}} - \int_{K \setminus (B \cup B_\star)} \frac{dy}{|p - y|^{n+2s}} + \int_{B_\star} \frac{dy}{|p - y|^{n+2s}} \\ &\geqslant \int_B \frac{dy}{|p - y|^{n+2s}} - \int_{K \setminus B} \frac{dy}{|p - y|^{n+2s}} + \int_{B_\star} \frac{dy}{|p - y|^{n+2s}}, \end{aligned}$$

in the principal value sense. Hence, the contributions in B and B' cancel out by symmetry and, in virtue of Lemma 3.1 (used here with $\lambda := 1$), we obtain that

$$\int_K \frac{\chi_{E^c}(y) - \chi_E(y)}{|p - y|^{n+2s}}\, dy \geqslant -CR^{-2s} + \int_{B_\star} \frac{dy}{|p - y|^{n+2s}},$$

up to renaming $C > 0$. Now if $y \in B_\star$ we have that $|p - y| \leqslant 2R_o + 3 \leqslant C$, for some $C > 0$, thanks to (3.11). Also, if $y \in \mathbb{R}^n \setminus K$ then $|p - y| \geqslant R/4$. As a consequence, up to renaming $C > c > 0$ step by step,

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{\chi_{E^c}(y) - \chi_E(y)}{|p - y|^{n+2s}}\, dy &\geqslant \int_K \frac{\chi_{E^c}(y) - \chi_E(y)}{|p - y|^{n+2s}}\, dy - CR^{-2s} \\ &\geqslant -CR^{-2s} + \int_{B_\star} \frac{dy}{|p - y|^{n+2s}} \geqslant -CR^{-2s} + c\,|B_\star| \geqslant -CR^{-2s} + c, \end{aligned}$$

which is strictly positive if R is large enough. This is in contradiction with the Euler-Lagrange equation in the viscosity sense (see Theorem 5.1 in [7]) and so it proves (3.3). $\square$

Next result gives the continuity of the fractional mean curvature at the smooth points of the boundary:

Lemma 3.4. *Let*

$$\alpha \in (2s, 1]. \tag{3.13}$$

Let $E \subseteq \mathbb{R}^n$ and $x_o \in \partial E$. Assume that $(\partial E) \cap B_R(x_o)$ is of class $C^{1,\alpha}$, for some $R > 0$. Then

$$\lim_{\substack{x \to x_o \\ x \in \partial E}} \int_{\mathbb{R}^n} \frac{\chi_{E^c}(y) - \chi_E(y)}{|x-y|^{n+2s}}\, dy = \int_{\mathbb{R}^n} \frac{\chi_{E^c}(y) - \chi_E(y)}{|x_o-y|^{n+2s}}\, dy.$$

Proof. Up to a rigid motion, we suppose that $x_o = 0$ and that, in the vicinity of the origin, the set E is the subgraph of a function $u \in C^{1,\alpha}(\mathbb{R}^{n-1})$ with $u(0) = 0$ and $\nabla u(0) = 0$. By formulas (49) and (50) in [2], we can write the fractional mean curvature in terms of u, as long as $|x'|$ is small enough. More precisely, there exist an odd and smooth functions F, with $F(0) = 0$, $|F| + |F'| \leqslant C$, for some $C > 0$, a function $\Psi \in C^{1,\alpha}(\mathbb{R}^{n-1})$, and a smooth, radial and compactly supported function ζ such that, if $|x'|$ is small and $x_n = u(x')$,

$$\int_{\mathbb{R}^n} \frac{\chi_{E^c}(y) - \chi_E(y)}{|x-y|^{n+2s}}\, dy = \int_{\mathbb{R}^{n-1}} F\left(\frac{u(x'+y') - u(x')}{|y'|}\right) \frac{\zeta(y')}{|y'|^{n-1+2s}}\, dy' + \Psi(x'),$$

in the principal value sense. Since also, by symmetry,

$$\int_{\mathbb{R}^{n-1}} F\left(\frac{\nabla u(x') \cdot y'}{|y'|}\right) \frac{\zeta(y')}{|y'|^{n-1+2s}}\, dy' = 0$$

in the principal value sense, we write

$$\int_{\mathbb{R}^n} \frac{\chi_{E^c}(y) - \chi_E(y)}{|x-y|^{n+2s}}\, dy = \int_{\mathbb{R}^{n-1}} \left[F\left(\frac{u(x'+y') - u(x')}{|y'|}\right) - F\left(\frac{\nabla u(x') \cdot y'}{|y'|}\right)\right] \frac{\zeta(y')}{|y'|^{n-1+2s}}\, dy' + \Psi(x'). \tag{3.14}$$

So we define

$$G(x', y') := \left[F\left(\frac{u(x'+y') - u(x')}{|y'|}\right) - F\left(\frac{\nabla u(x') \cdot y'}{|y'|}\right)\right] \frac{\zeta(y')}{|y'|^{n-1+2s}}.$$

Notice that

$$\lim_{x' \to 0} G(x', y') = G(0, y').$$

Also, for any small $|x'|$ and bounded $|y'|$,

$$\left| F\left(\frac{u(x'+y') - u(x')}{|y'|}\right) - F\left(\frac{\nabla u(x') \cdot y'}{|y'|}\right)\right| \leqslant C\, \frac{|u(x'+y') - u(x') - \nabla u(x') \cdot y'|}{|y'|} \leqslant C\, |y'|^{\alpha}.$$

Therefore

$$|G(x', y')| \leqslant \frac{C}{|y'|^{n-1-\alpha+2s}} \in L^1_{\mathrm{loc}}(\mathbb{R}^{n-1}),$$

thanks to (3.13). Accordingly, by the Dominated Convergence Theorem,

$$\lim_{x' \to 0} \int_{\mathbb{R}^{n-1}} G(x', y')\, dy' = \int_{\mathbb{R}^{n-1}} G(0, y')\, dy'.$$

Consequently,

$$\begin{aligned} &\lim_{x' \to 0} \int_{\mathbb{R}^{n-1}} \left[F\left(\frac{u(x'+y') - u(x')}{|y'|}\right) - F\left(\frac{\nabla u(x') \cdot y'}{|y'|}\right)\right] \frac{\zeta(y')}{|y'|^{n-1+2s}}\, dy' + \Psi(x') \\ &= \int_{\mathbb{R}^{n-1}} \left[F\left(\frac{u(y') - u(0)}{|y'|}\right) - F\left(\frac{\nabla u(0) \cdot y'}{|y'|}\right)\right] \frac{\zeta(y')}{|y'|^{n-1+2s}}\, dy' + \Psi(x'), \end{aligned}$$

which, combined with (3.14), establishes the desired result. □

The result in Lemma 3.4 can be modified to take into account sets with lower regularity properties.

Lemma 3.5. *Let $R>0$, $E\subseteq\mathbb{R}^n$ and $x_o\in\partial E$. For any $k\in\mathbb{N}$, let $x_k\in\partial E$, with $x_k\to x_o$ as $k\to+\infty$, be such that E is touched from the inside at x_k by a ball of radius R, i.e. there exists $p_k\in\mathbb{R}^n$ such that*

$$B_R(p_k)\subseteq E \tag{3.15}$$

and $x_k\in\partial B_R(p_k)$.

Suppose that

$$\int_{\mathbb{R}^n}\frac{\chi_E(y)-\chi_{E^c}(y)}{|x_k-y|^{n+2s}}\,dy\leqslant 0.$$

Then

$$\int_{\mathbb{R}^n}\frac{\chi_E(y)-\chi_{E^c}(y)}{|x_o-y|^{n+2s}}\,dy\leqslant 0. \tag{3.16}$$

Proof. Fix $\lambda>0$, to be taken arbitrarily small in the sequel. Let $q_k:=p_k+2(x_k-p_k)$. We observe that the ball $B_R(q_k)$ is tangent to $B_R(p_k)$ at x_k. Therefore, by Lemma 3.1,

$$\int_{B_\lambda(x_k)\setminus\big(B_R(p_k)\cup B_R(q_k)\big)}\frac{dy}{|x_k-y|^{n+2s}}\leqslant CR^{-2s}\lambda^{1-2s}, \tag{3.17}$$

for some $C>0$. Also, using (3.15),

$$\int_{B_\lambda(x_k)}\frac{\chi_E(y)-\chi_{E^c}(y)}{|x_k-y|^{n+2s}}\,dy\geqslant\int_{B_\lambda(x_k)}\frac{\chi_{B_R(p_k)}(y)-\chi_{B^c_R(p_k)}(y)}{|x_k-y|^{n+2s}}\,dy. \tag{3.18}$$

Now we define T_k to be the half-space passing through x_k with normal parallel to x_k-p_k and containing $B_R(p_k)$. By symmetry,

$$\int_{B_\lambda(x_k)}\frac{\chi_{T_k}(y)-\chi_{T^c_k}(y)}{|x_k-y|^{n+2s}}\,dy=0.$$

Using this, (3.18) and (3.17), we obtain that

$$\begin{aligned}&\int_{B_\lambda(x_k)}\frac{\chi_E(y)-\chi_{E^c}(y)}{|x_k-y|^{n+2s}}\,dy\\ &\geqslant\int_{B_\lambda(x_k)}\frac{\chi_{B_R(p_k)}(y)-\chi_{B^c_R(p_k)}(y)}{|x_k-y|^{n+2s}}\,dy-\int_{B_\lambda(x_k)}\frac{\chi_{T_k}(y)-\chi_{T^c_k}(y)}{|x_k-y|^{n+2s}}\,dy\\ &=-2\int_{B_\lambda(x_k)\cap\big(T_k\setminus B_R(p_k)\big)}\frac{dy}{|x_k-y|^{n+2s}}\\ &\geqslant -CR^{-2s}\lambda^{1-2s}.\end{aligned} \tag{3.19}$$

Now we define

$$f_k(y):=\chi_{B^c_\lambda(x_k)}\cdot\frac{\chi_E(y)-\chi_{E^c}(y)}{|x_k-y|^{n+2s}}.$$

We observe that f_k vanishes in $B_\lambda(x_k)$. Also, if $y\in B_{2\lambda}(x_o)\setminus B_\lambda(x_k)$, we have that $|f_k(y)|\leqslant\frac{1}{\lambda^{n+2s}}$. Moreover, if $y\in\mathbb{R}^n\setminus B_{2\lambda}(x_o)$, we have that

$$|y-x_o|\leqslant|y-x_k|+|x_k-x_o|\leqslant|y-x_k|+\lambda\leqslant|y-x_k|+\frac{|y-x_o|}{2},$$

as long as k is large enough, and so $|y-x_k|\geqslant\frac{|y-x_o|}{2}$, which gives that $|f_k(y)|\leqslant\frac{1}{|x-x_o|^{n+2s}}$ for any $y\in\mathbb{R}^n\setminus B_{2\lambda}(x_o)$. As a consequence of these observations, we can use the Dominated Convergence Theorem and obtain that

$$\lim_{k\to+\infty}\int_{B^c_\lambda(x_k)}\frac{\chi_E(y)-\chi_{E^c}(y)}{|x_k-y|^{n+2s}}\,dy=\lim_{k\to+\infty}\int_{\mathbb{R}^n}f_k(y)\,dy=\int_{\mathbb{R}^n}\lim_{k\to+\infty}f_k(y)\,dy=\int_{B^c_\lambda(x_o)}\frac{\chi_E(y)-\chi_{E^c}(y)}{|x_o-y|^{n+2s}}\,dy.$$

Thus, if k is large enough,

$$\int_{B_\lambda^c(x_k)} \frac{\chi_E(y)-\chi_{E^c}(y)}{|x_k-y|^{n+2s}}\,dy \geqslant \int_{B_\lambda^c(x_o)} \frac{\chi_E(y)-\chi_{E^c}(y)}{|x_o-y|^{n+2s}}\,dy - R^{-2s}\lambda^{1-2s}.$$

Thus, recalling (3.19),

$$\begin{aligned} 0 &\geqslant \int_{\mathbb{R}^n} \frac{\chi_E(y)-\chi_{E^c}(y)}{|x_k-y|^{n+2s}}\,dy \\ &= \int_{B_\lambda(x_k)} \frac{\chi_E(y)-\chi_{E^c}(y)}{|x_k-y|^{n+2s}}\,dy + \int_{B_\lambda^c(x_k)} \frac{\chi_E(y)-\chi_{E^c}(y)}{|x_k-y|^{n+2s}}\,dy \\ &\geqslant \int_{B_\lambda^c(x_o)} \frac{\chi_E(y)-\chi_{E^c}(y)}{|x_o-y|^{n+2s}}\,dy - CR^{-2s}\lambda^{1-2s}, \end{aligned}$$

up to renaming $C>0$ line after line. Then, (3.16), in the principal value sense, follows by sending $\lambda \to 0$. □

A variation of Lemma 3.5 deals with the touching by sufficiently smooth hypersurfaces, instead of balls. In this sense, the result needed for our scope is the following:

Lemma 3.6. *Let $\Lambda > 0$. Let $E \subseteq \mathbb{R}^n$ and $x_o \in \partial E$. For any $k \in \mathbb{N}$, let $x_k \in \partial E$, with $x_k \to x_o$ as $k \to +\infty$, be such that E is touched from the inside in $B_\Lambda(x_k)$ at x_k by a surface of class $C^{1,\alpha}$, with $C^{1,\alpha}$-norm bounded independently of k and $\alpha \in (2s,1]$, Suppose that*

$$\int_{\mathbb{R}^n} \frac{\chi_E(y)-\chi_{E^c}(y)}{|x_k-y|^{n+2s}}\,dy \leqslant 0.$$

Then

$$\int_{\mathbb{R}^n} \frac{\chi_E(y)-\chi_{E^c}(y)}{|x_o-y|^{n+2s}}\,dy \leqslant 0.$$

Proof. The proof is similar to the one of Lemma 3.5. The only difference is that (3.17) is replaced here by

$$\int_{B_\lambda(x_k)\setminus(P_k^+\cup P_k^-)} \frac{dy}{|x_k-y|^{n+2s}} \leqslant C\,\lambda^{\alpha-2s}, \tag{3.20}$$

where $\lambda \in (0,\Lambda)$ can be taken arbitrarily small and P_k^+ is a region with $C^{1,\alpha}$-boundary that is contained in E and P_k^- is the even reflection of P_k^+ with respect to the tangent plane of P_k^+ at x_k. In this framework, (3.20) is a consequence of Lemma 3.2.

The rest of the proof follows the arguments given in the proof of Lemma 3.5, substituting $B_R(p_k)$ and $B_R(q_k)$ with P_k^+ and P_k^-. □

4. Graph properties of s-minimal sets and proof of Theorem 1.1

The goal of this section is to prove Theorem 1.1.

Proof of Theorem 1.1. The idea is to slide E from above till it touches itself. Namely, for any $t \geqslant 0$, we let $E_t := E + te_n$. By Lemma 3.3,

$$\Omega_o \times (-\infty,-M) \subseteq E\cap\Omega \subseteq \Omega_o \times (-\infty, M), \tag{4.1}$$

for some $M \geqslant 0$. Hence, if $t > 2M$, then $E_t \supseteq E$. So we take the smallest t for which such inclusion holds. Our goal is to show that $t = 0$.

Indeed, if we show that $t=0$, it means that $E + te_n \supseteq E$ for any $t \geqslant 0$, so we could define $v(x') := \inf\{\tau \text{ s.t. } (x',\tau) \in E^c\}$ and obtain that $E \cap \Omega_o$ is the subgraph of v (up to sets of zero measure).

To prove that $t = 0$, we argue by contradiction, assuming that

$$t > 0, \tag{4.2}$$

and so there is a contact point between ∂E and ∂E_t. We distinguish two cases, according to whether all the contact points are interior, or there are boundary contacts (no other possibilities occur, thanks to (1.1)). Namely, we have that either

$$\text{all the contact points lie in } \Omega_o \times \mathbb{R} \tag{4.3}$$

or

$$\text{there exists a contact point in } (\partial\Omega_o) \times \mathbb{R}. \tag{4.4}$$

The rest of the proof will take into account these two cases separately.

The case in which (4.3) *holds true.* Assume first that (4.3) is satisfied. Then we consider the subconvolution of E and we slide it from above till it touches the supconvolution of E (in the notation of Section 2). More explicitly, fixed $\delta > 0$, for any $\tau \in \mathbb{R}$, we consider $E^\flat_\delta + \tau e_n$. By (4.1), we have that if τ is large, then

$$(E^\flat_\delta + \tau e_n) \cap \overline{\Omega} \supseteq E^\sharp_\delta \cap \overline{\Omega}. \tag{4.5}$$

So we take the smallest $\tau = \tau_\delta$ for which such inclusion holds. From (4.2), we have that

$$\tau = \tau_\delta \geqslant \frac{t}{2} > 0, \tag{4.6}$$

for small δ. Also, by (4.3) (recall also the first statement in Corollary 2.2), if δ is small enough, we obtain that $\big(\partial(E^\flat_\delta + \tau_\delta e_n)\big) \cap \overline{\Omega}$ and $(\partial E^\sharp_\delta) \cap \overline{\Omega}$ possess a contact point p_δ in $\Omega_o \times \mathbb{R}$. Now we distinguish two subcases: either this is the first contact point in the whole of the space or not. In the first subcase, we have that (4.5) may be strengthen to $E^\flat_\delta + \tau e_n \supseteq E^\sharp_\delta$, and therefore we can apply Proposition 2.5, and we obtain that $E^\sharp_\delta = E^\flat_\delta + \tau_\delta e_n$. By taking δ arbitrarily small and using (4.6), we obtain that $E = E + \tau_o e_n$, with $\tau_o \geqslant t/2 > 0$, which is in contradiction with (1.1).

The second subcase is when the first contact point p_δ in $\overline{\Omega}$ does not prevent the sets to overlap outside $\overline{\Omega}$. In this case, we will show that this overlap only occurs at infinity and then we provide a contradiction arising from the contribution in bounded sets. Namely, first of all we recall the notation in (2.11) and we notice that for any $R > 0$ there exists $\delta_R > 0$ such that for any $\delta \in (0, \delta_R]$ we have that

$$(E^\flat_\delta + \tau e_n) \cap \mathscr{C}_R \supseteq E^\sharp_\delta \cap \mathscr{C}_R. \tag{4.7}$$

To prove (4.7), we argue by contradiction. If not, there exists some $R > 0$ and an infinitesimal sequence $\delta \to 0$ such that $\big(E^\sharp_\delta \setminus (E^\flat_\delta + \tau e_n)\big) \cap \mathscr{C}_R \neq \varnothing$. Then, let $q_\delta = (q'_\delta, q_{\delta,n})$ be a point in such set. By construction $|q_{\delta,n}| \leqslant 3M + 1$ and $|q'_\delta| \leqslant R$, therefore, up to subsequences, as $\delta \to 0$, we may suppose that $\tau_\delta \to \tau_\star$ and $q_\delta \to q_\star = (q'_\star, q_{\star,n}) \in \overline{\big(E \setminus (E + \tau_\star e_n)\big) \cap \mathscr{C}_R}$. Hence, by (4.5), $q_\star \in \mathbb{R}^n \setminus \Omega$ and so, by (1.1), we have that $u(q'_\star) + \tau_\star \leqslant q_{\star,n} \leqslant u(q'_\star)$. This gives that $\tau_\star \leqslant 0$, which is in contradiction with (4.6) and thus completes the proof of (4.7).

Thanks to (4.7), we can now use Proposition 2.6 and obtain that

$$\int_{\mathscr{C}_R} \frac{\chi_{(E^\flat_\delta + \tau e_n)\setminus E^\sharp_\delta}(y)}{|p-y|^{n+2s}}\,dy = \int_{\mathscr{C}_R} \frac{\chi_{(E^\flat_\delta + \tau e_n)\setminus E^\sharp_\delta}(y) - \chi_{E^\sharp_\delta \setminus (E^\flat_\delta + \tau e_n)}(y)}{|p-y|^{n+2s}}\,dy \leqslant C R^{-2s}, \tag{4.8}$$

for some $C > 0$, provided that $\delta > 0$ is small enough.

Now we fix $R_o > 0$ such that $\Omega \subset \mathscr{C}_{R_o}$. Since u is continuous in $\mathbb{R}^{n-1}$, it is uniformly continuous in compact sets and so we can define

$$\sigma_\delta := \sup_{\substack{|x'|,\,|y'| \leqslant R_o+3 \\ |x'-y'| \leqslant 2\delta}} |u(x') - u(y')|,$$

and we have that $\sigma_\delta \to 0$ as $\delta \to 0$.

We claim that, for small $\delta > 0$,

$$\begin{aligned} &\text{if } x = (x', x_n) \in \partial(E^\flat_\delta + \tau e_n),\ y = (y', y_n) \in \partial E^\sharp_\delta \text{ and } x' = y', \\ &\text{with } |x'| \in (R_o + 1, R_o + 2), \\ &\text{then } x_n \geqslant y_n + \tfrac{t}{4}. \end{aligned} \tag{4.9}$$

To prove it, we use the first statement in Corollary 2.2 to find $x_o \in (\partial E) + \tau e_n$ and $y_o \in \partial E$ such that

$$\max\{|x - x_o|, |y - y_o|\} \leqslant \delta.$$

Notice that $x_{o,n} = u(x'_o) + \tau$ and $y_{o,n} = u(y'_o)$. Moreover, $|x' - x'_o| \leqslant \delta$ and $|x' - y'_o| = |y' - y'_o| \leqslant \delta$, hence $|x'_o - y'_o| \leqslant 2\delta$. Therefore

$$\begin{aligned} x_n - y_n &= x_n - x_{o,n} + u(x'_o) + \tau - y_n + y_{o,n} - u(y'_o) \\ &\geqslant \tau - |x - x_o| - |y - y_o| - |u(x'_o) - u(y'_o)| \geqslant \tau - 2\delta - \sigma_\delta. \end{aligned}$$

This and (4.6) imply (4.9), as desired.

So we use (4.7) and (4.9) to deduce that, fixed $R > R_o + 4$ and $\delta > 0$ small enough (possibly in dependence of R),

$$\int_{\mathscr{C}_R} \frac{\chi_{(E^\flat_\delta + \tau e_n)\setminus E^\sharp_\delta}(y)}{|p - y|^{n+2s}}\, dy \geqslant \int_{\mathscr{C}_{R_o+2}\setminus \mathscr{C}_{R_o+1}} \frac{\chi_{(E^\flat_\delta + \tau e_n)\setminus E^\sharp_\delta}(y)}{|p - y|^{n+2s}}\, dy \geqslant c_o t,$$

for some $c_o > 0$ (possibly depending on the fixed R_o and M). From this and (4.8), we obtain that $t \leqslant \tilde{C} R^{-2s}$, for some $\tilde{C} > 0$ and so, by taking R as large as we wish, we conclude that $t = 0$. This is in contradiction with (4.2), and so we have completed the proof of Theorem 1.1 under assumption (4.3).

The case in which (4.4) *holds true.* Now we deal with the case in which (4.4) is satisfied. Hence, there exists a contact point $p = (p', p_n) \cap (\partial E_t) \cap (\partial E)$ with $p' \in \partial \Omega_o$.

Notice that

$$p \in \left(\overline{(\partial E_t) \cap \Omega}\right) \cap \left(\overline{(\partial E) \cap \Omega}\right). \tag{4.10}$$

Indeed, the graph property of $E \setminus \Omega$ and (4.2) imply that if $a_k \in \partial E_t$ and $b_k \in \partial E$ are such that $a_k \to p$ and $b_k \to p$ as $k \to +\infty$, then $a_k, b_k \in \Omega$. This proves (4.10).

Now, we observe that E is a variational subsolution in a neighborhood of p (according to Definition 2.3 in [7]): namely, if $A \subseteq E \cap \Omega$ and $p \in \overline{A}$, we have that

$$0 \geqslant \operatorname{Per}_s(E, \Omega) - \operatorname{Per}_s(E \setminus A, \Omega) = L(A, E^c) - L(A, E \setminus A).$$

Therefore (see Theorem 5.1 in [7]) we have that

$$\int_{\mathbb{R}^n} \frac{\chi_E(y) - \chi_{\mathbb{R}^n \setminus E}(y)}{|p - y|^{n+2s}}\, dy \geqslant 0. \tag{4.11}$$

in the viscosity sense (i.e. (4.11) holds true provided that E is touched by a ball from outside at p).

Our goal is now to establish fractional mean curvature estimates in the strong sense. For this, we define $p_t := p - te_n = (p', p_n - t) = (p'_t, p_{t,n})$. By (4.2), either

$$p_n \neq u(p') \tag{4.12}$$

or

$$p_{n,t} \neq u(p'_t). \tag{4.13}$$

We focus on the case in which (4.12) holds true (the case in (4.13) can be treated similarly, by exchanging the roles of p and p_t).

Then, either $B_r(p) \setminus \Omega \subseteq E$ or $B_r(p) \setminus \Omega \subseteq E^c$, for a small $r > 0$. In any case, by [6], we have that $(\partial E) \cap B_r(p)$ is a $C^{1,\frac{1}{2}+s}$-graph in the direction of the normal of Ω at p.

Let $\nu(p) = (\nu'(p), \nu_n(p))$ be such normal, say, in the interior direction. Since Ω is a cylinder, we have that $\nu_n(p) = 0$. Also, up to a rotation we can suppose that $\nu'(p) = e_1$. In this framework, we can write ∂E in the vicinity of p as a graph $G := \{x_1 = \Psi(x_2, \dots, x_n)\}$, for a suitable $\Psi \in C^{1,\frac{1}{2}+s}(\mathbb{R}^{n-1})$, with $\Psi(p_2, \dots, p_n) = p_1$.

We observe that

(4.14) there exists a sequence of points $p^{(k)} \in G$ such that $p^{(k)} \in \Omega$ and $p^{(k)} \to p$ as $k \to +\infty$.

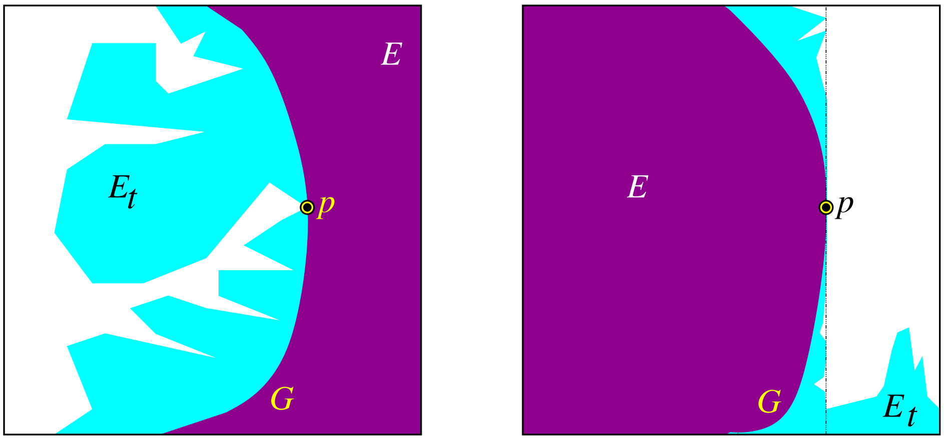

FIGURE 1. The alternative in (4.17) and (4.18).

Indeed, if not, we would have that ∂E in the vicinity of p lies in Ω^c. This is in contradiction with (4.10) and so it proves (4.14).

From (4.14), we obtain that there exists a sequence of points $p^{(k)} \to p$, such that

$$\partial E \text{ near } p^{(k)} \text{ is a graph of class } C^{1,\frac{1}{2}+s} \tag{4.15}$$

and

$$\int_{\mathbb{R}^n} \frac{\chi_E(y) - \chi_{E^c}(y)}{|p^{(k)} - y|^{n+2s}}\, dy = 0.$$

As a consequence of this, (4.15), and Lemma 3.2 we obtain that

$$\int_{\mathbb{R}^n} \frac{\chi_E(y) - \chi_{E^c}(y)}{|p - y|^{n+2s}}\, dy = 0.$$

Hence, since $E_t \supseteq E$ (and they are not equal, thanks to (4.2)),

$$\int_{\mathbb{R}^n} \frac{\chi_{E_t}(y) - \chi_{E_t^c}(y)}{|p - y|^{n+2s}}\, dy > 0. \tag{4.16}$$

Also, since $E_t \supseteq E$, we have that $(\partial E_t) \cap B_{\frac{r}{4}}(p)$ can only lie on one side of the graph G, i.e.

$$\text{either } E_t \cap B_{\frac{r}{4}}(p) \supseteq \{x_1 \geqslant \Psi(x_2, \dots, x_n)\} \tag{4.17}$$

$$\text{or } E_t \cap B_{\frac{r}{4}}(p) \subseteq \{x_1 \leqslant \Psi(x_2, \dots, x_n)\}, \tag{4.18}$$

see Figure 1.

In any case (recall (4.10)), we have that there exists a sequence of points $\tilde{p}^{(k)} \in (\partial E_t) \cap \Omega$ that can be touched by a surface of class class $C^{1,\frac{1}{2}+s}$ lying in E_t (indeed, for this we can either enlarge balls centered at G, or slide a translation of G, see Figure 2).

Then

$$\int_{\mathbb{R}^n} \frac{\chi_{E_t}(y) - \chi_{E_t^c}(y)}{|\tilde{p}^{(k)} - y|^{n+2s}}\, dy \leqslant 0.$$

Hence, by Lemma 3.2,

$$\int_{\mathbb{R}^n} \frac{\chi_{E_t}(y) - \chi_{E_t^c}(y)}{|p - y|^{n+2s}}\, dy \leqslant 0.$$

This is in contradiction with (4.16) and so the proof of Theorem 1.1 is complete. $\square$

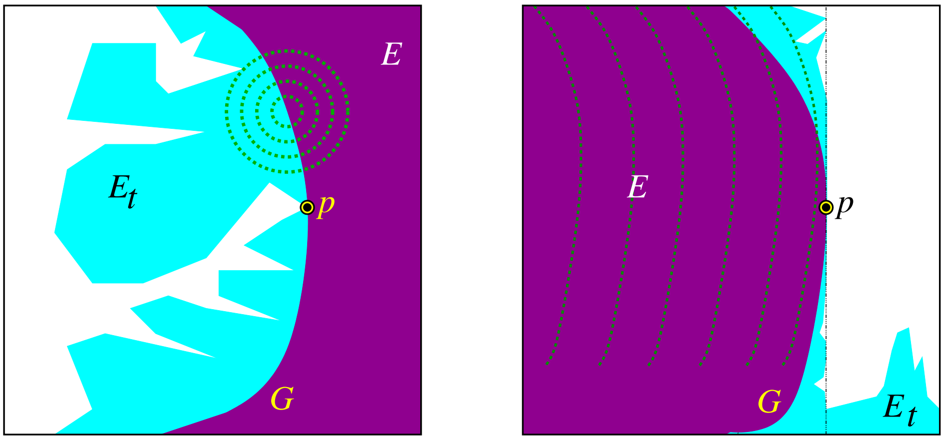

FIGURE 2. Touching ∂E_t, according to the alternative in (4.17) and (4.18).

5. Smoothness in dimension 3 and proof of Theorem 1.2

The goal of this section is to prove Theorem 1.2:

Proof of Theorem 1.2. By Theorem 1.1, we know that E is an epigraph, i.e. (1.2) holds true for some $v : \mathbb{R}^2 \to \mathbb{R}$. It remains to show that

$$v \in C^\infty(\Omega_o). \tag{5.1}$$

For this, we take $x_o \in (\partial E) \cap \Omega$ and we show that v is C^∞ in a neighborhood of x_o. Up to a translation, we suppose that x_o is the origin. Now we consider a blow-up E_0 of the set E, i.e., for any $r > 0$, we define $E_r := \frac{E}{r} := \{\frac{x}{r} \text{ s.t. } x \in E\}$ and E_0 to be a cluster point for E_r as $r \to 0$ (see Theorem 9.2 in [7]). In this way, we have that E_0 is an s-minimal set, and it is an epigraph (see e.g. (5.8) in [15]). Thus, by Corollary 1.3 in [15], we deduce that E_0 is a half-space.

Hence, by Theorem 9.4 in [7], we have that ∂E is a graph of class $C^{1,\alpha}$ in the vicinity of the origin – and, as a matter of fact, of class C^∞, thanks to Theorem 1 of [2]. $\square$

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