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Abstract

An universal modulation instability is subject to Lighthill criterion: nonlinearity and dispersion should make opposite contributions to the wave frequency. Recent studies of wave instabilities in optical fibers with the minimum chromatic dispersion revealed situations in which the criterion is violated and fast unstable modulations appear due to the four wave mixing process. We derive a generalized criterion, it applies to an arbitrary dispersion and to both slow and fast unstable modulations. Since the fast modulations depend on nonlinear dispersion, we also demonstrate how to describe them in the framework of a single generalized nonlinear Schrödinger equation.

1 Introduction

Monochromatic nonlinear waves are important special solutions of weakly nonlinear hyperbolic systems [40, 62]. Their stability with respect to self-modulations is governed by the Lighthill criterion. A wave experiences modulation instability (MI) if contribution of the *group velocity dispersion* (GVD) to the wave frequency is opposite to that of the *nonlinearity*. MI discovery [41, 61, 9, 8, 44, 30, 66, 26] is related to the discovery of the nonlinear Schrödinger equation (NLSE), an universal model for slow wave modulations. NLSE is integrable [70] and provides a striking correlation between, e.g., water waves and fiber optics [16].

MI manifests itself by appearance of Stokes and anti-Stokes sideband waves, which grow at the expense of the seed carrier wave. They take part in nonlinear interactions, generate a cascade, and can in turn be destroyed by modulations [28, 14, 56, 34]. As MI develops, system's spectrum on a logarithmic scale takes a typical $\wedge$-shape with the seed carrier wave frequency in the middle. Finally, the system enters into a chaotic wave-turbulent state [68] with the addition of solitons and spontaneous rogue waves or breathers [20, 22, 67, 45].

MI in optical fibers was first observed in [55]. Single-mode fibers [43], which are in the focus of this work, offer important advantages for studies of nonlinear wave interactions. To derive NLSE for fibers, one rigorously eliminates two radial space coordinates [2]. The resulting equation is truly one-dimensional, as opposed by the NLSE for deep-water waves. With the generalized NLSE (GNLSE) one can get rid of the slowly varying envelope approximation [11, 13, 24]. Attenuation is extremely small in the fiber transparency window. One can generate millions of pulses per second and collect their statistics to study optical supercontinuum [19], wave turbulence [58], and rare extreme events [54, 36]. Optical pulses are well suited to envelope equations like NLSE. For instance, the pioneer work [55] deals with pulses containing $\approx 20\,000$ field oscillations, such wave packets are unavailable in water channels. Moreover, envelope and few-cycle pulses readily coexist in optical fibers, non-envelope effects are then described by the GNLSE.

Lighthill criterion requires a negative GVD for MI in a focusing nonlinear fiber [2]. An important feature of optical fibers is that their dispersion law can be manipulated [71]. Microstructured fibers may have several zero-dispersion frequencies (ZDF) at which GVD vanishes making the standard Lighthill criterion degenerate. Such spectral regions have minimum chromatic dispersion and are of interest for optical communication. This triggered studies of MI for a small or zero GVD.

GVD in fibers is quantified by the second-order derivative of the wave vector with respect to frequency [2]. The third-order dispersion has no influence on MI domain and increment [59, 48]. To describe MI at ZDF one needs GNLSE with the fourth-order dispersion. MI requires then a negative fourth-order dispersion in a focusing fiber [15]. A combined effect of a small GVD and the fourth-order dispersion may result in a new instability mode

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with the growing “fast modulation” of the carrier wave [1]. The classical Lighthill criterion is unsuitable for such situations.

Both slow and fast wave modulations result from the resonance wave interactions in the form of the four-wave mixing [2] (FWM). The FWM conditions provide the possible MI bands for any dispersion law and does not require GNLSE [38, 29]. Wave instability due to the fast modulation was first observed in [60] and related to FWM in [42]. Such FWM instability [65] requires a small positive GVD and negative fourth-order dispersion or vice versa [23]. It clearly violates the classical Lighthill criterion. The FWM instability was further observed in several experiments and quantified using the resonance wave interactions and GNLSE [47, 25, 49, 10, 17, 64].

Following [65, 33], one can distinguish between MI and FWM instability, where Lighthill criterion applies only to MI. On the other hand, both instabilities appear due to literally the same resonance interaction of four waves. A natural question is whether we can generalize the standard Lighthill criterion. The generalization is given in Section 4, after we discuss NLSE (Section 2) and GNLSE (Section 3). Dispersion of nonlinearity, which is important for the FWM instability, is further accounted for in Sections 5 and 6.

2 Background

A useful way to study a physical system is to choose some direction Ox and to look for a monochromatic small-amplitude plane wave solution, $Ae^{i(kx-\omega t)}$. Each system variable has its own complex amplitude A, but all variables share the same wave vector k and frequency ω. The shared parameters are connected by the dispersion relation, $\omega = f(k)$, which ensures existence of the assumed waveform [62, 39]. General system states are constructed as combinations of such linear waves, e.g., using Fourier transform or replacing $A =$ const by a slow-varying complex envelope $\Psi(x,t)$.

Continuation of plane wave solutions for nonlinear systems is an important problem [31]. One can look for *simple waves*, in which all variables explicitly depend on phase $\phi = kx - \omega t$ with the period 2π and thus implicitly on space and time. A less restrictive approach is based on the fact that the monochromatic plane waves survive in a generic weakly nonlinear wave-system as long as one can neglect small higher-order nonlinear terms oscillating at multiple frequencies. Yet the dispersion relation starts to include wave amplitude, $\omega = f_{\mathsf{nl}}(k,|A|^2)$, where $f(k) = f_{\mathsf{nl}}(k,0)$. The difference $\delta\omega = f_{\mathsf{nl}}(k,|A|^2) - f(k)$ is the nonlinear frequency shift. Stability is crucial for such nonlinear waves: in one-dimensional geometry a wave with the wavenumber k is unstable if

$$\delta\omega(k,|A|^2)\cdot f''(k) < 0, \tag{1}$$

which is Lighthill criterion [69]. A fundamental fact is that, roughly speaking, a half of weakly nonlinear waves is unstable with respect to modulations.

The complex envelope $\Psi(x,t)$ of a carrier wave with the wave vector k is described by the following NLSE

$$i(\partial_t\Psi + v_{\mathsf{gr}}\partial_x\Psi) + \frac{\mu}{2}\partial_x^2\Psi + \Gamma|\Psi|^2\Psi = 0, \tag{2}$$

where the group velocity $v_{\mathsf{gr}} = f'(k)$ and the dispersion parameter $\mu = f''(k)$. Parameter Γ is positive (negative) for focusing (defocusing) nonlinearity. The NLSE solution $\Psi = Ae^{i\Gamma|A|^2t}$ provides the nonlinear correction to the carrier wave frequency

$$f_{\mathsf{nl}}(k,|A|^2) = f(k) - \Gamma|A|^2. \tag{3}$$

According to Eq. (1) the currier wave is unstable when $\mu\Gamma > 0$.

NLSE (2) is a time-propagated equation in which one starts from $\Psi(x,t)|_{t=0}$ to calculate $\Psi(x,t)|_{t>0}$. In some cases, especially in optical fibers, it is convenient to solve an alternative space-propagated equation. To distinguish between time- and space-propagated problems we now denote the space variable by z and envelope by $\psi(z,t)$. Consequently, one starts from $\psi(z,t)|_{z=0}$ to calculate $\psi(z,t)|_{z>0}$, e.g., when the input is known at the beginning of the fiber and the output should be calculated at the end of it.

Well-posed z-propagated equations assume unidirectionality. The dispersion relation is used in the form $k = \beta(\omega)$ or $k = \beta_{\text{nl}}(\omega, |A|^2)$, with $\beta(\omega) = \beta_{\text{nl}}(\omega, 0)$. The nonlinear frequency shift is replaced by the nonlinear wave vector shift, $\delta k = \beta_{\text{nl}}(\omega, |A|^2) - \beta(\omega)$. A monochromatic wave, which propagates along Oz with the carrier frequency ω, is unstable if

$$\delta k(\omega, |A|^2) \cdot \beta''(\omega) < 0, \tag{4}$$

because $\delta\omega$ in Eq. (1) is replaced by $-\delta k$ and positive $f''(k)$ with positive v_{gr} correspond to negative $\beta''(\omega)$.

Equation (4) is z-propagated version of the Lighthill criterion (1). Wave evolution is described by the z-propagated NLSE

$$i(\partial_z\psi + \beta_1\partial_t\psi) - \frac{\beta_2}{2}\partial_t^2\psi + \gamma|\psi|^2\psi = 0, \tag{5}$$

where the inverse group velocity $\beta_1 = \beta'(\omega)$ and the GVD parameter $\beta_2 = \beta''(\omega)$ are calculated at the wave carrier frequency. Coefficients in (2) and (5) are connected by $\beta_1 = 1/v_{\text{gr}}$, $\beta_2 = -\mu/v_{\text{gr}}^3$, and $\gamma = \Gamma/v_{\text{gr}}$. The solution $\psi = Ae^{i\gamma|A|^2 z}$ of the NLSE (5) describes the carrier wave with

$$\beta_{\text{nl}}(\omega, |A|^2) = \beta(\omega) + \gamma|A|^2, \tag{6}$$

cf., Eq. (3). The carrier wave is destroyed by modulations when $\beta_2\gamma < 0$.

3 Higher-order dispersion

The space-propagated framework is used in the rest of the manuscript. We abbreviate the seed monochromatic carrier wave by *pump*, when appropriate. Pump frequency is denoted by ω_0, its wave vector is $\beta_0 = \beta(\omega_0)$. Optical materials are mostly focusing [12], for simplicity we take $\gamma > 0$. MI is expected for a negative GVD, $\beta_2 = \beta''(\omega_0) < 0$. If it is the case, the spectral interval of growing perturbations reads [2]

$$(\omega - \omega_0)^2 < -\frac{4\gamma|A|^2}{\beta_2}. \tag{7}$$

The interval diverges if (negative) $\beta_2 \to 0$. A better MI description is then required, it is provided by GNLSE.

The simplest GNLSE reads [12, 2]

$$i\partial_z\psi + \sum_{j=1}^{J}\frac{\beta_j}{j!}(i\partial_t)^j\psi + \gamma|\psi|^2\psi = 0, \tag{8}$$

where NLSE (5) corresponds to $J = 2$. By analogy, β_j should be j-th derivative of $\beta(\omega)$ at $\omega = \omega_0$. *Attenuation is neglected*, such that $\beta(\omega)$ and all dispersion parameters β_j are real. Complex-valued dispersion parameters are discussed in Sections 5 and 6.

To get a better understanding of Eq. (8), we consider a monochromatic modulation $\psi \propto e^{i(\kappa z - \Omega t)}$ which corresponds to a shifted carrier wave $e^{i(\beta_0+\kappa)z - i(\omega_0+\Omega)t}$. One expects that for a small-amplitude (linear) modulation

$$\kappa = \beta(\omega_0 + \Omega) - \beta(\omega_0),$$

whereas linearized Eq. (8) provides

$$\kappa = \sum_{j=1}^{J}\frac{\beta_j}{j!}\Omega^j, \qquad \beta_j \in \mathbb{R}.$$

We then approximate a real part of the dispersion law by a polynomial of order J. The approximation is usually associated with the Taylor expansion, but it can actually be any fitting polynomial for $\beta(\omega)$ in the transparency window. Non-polynomial fitting functions are also possible [51, 53, 4, 46].

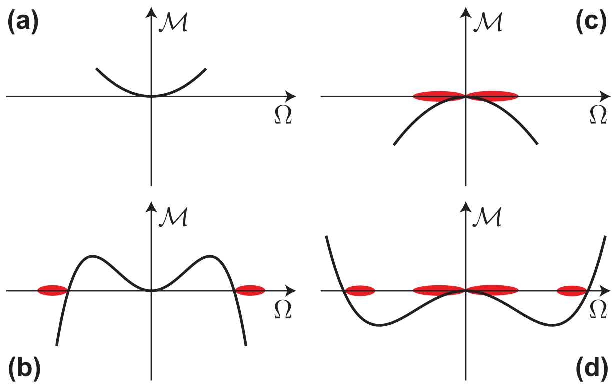

Figure 1: Different shapes of $\mathcal{M}(\Omega)$ and frequency bands of the unstable offsets (red). (a,c) classical applications of the Lighthill criterion. (b,d) new unstable Ω-bands appear for a more complex mismatch function.

In what follows we use the following real-valued dispersion function

$$\mathcal{D}(\Omega) = \beta(\omega_0 + \Omega) - \beta_0 - \beta_1 \Omega \approx \sum_{j=2}^{J} \frac{\beta_j}{j!} \Omega^j,$$

and write the GNLSE (8) in the form

$$i\partial_z \psi + \mathcal{D}(i\partial_\tau)\psi + \gamma |\psi|^2 \psi = 0, \tag{9}$$

where $\psi = \psi(z, \tau)$. The delay $\tau = t - \beta_1 z$ is introduced to remove β_1-term from Eq. (8). Note that real-valued dispersion coefficients β_j yield a self-adjoint operator $\mathcal{D}(i\partial_\tau)$. Equation (9) is then a Hamiltonian one

$$i\partial_z \psi + \frac{\delta}{\delta\psi^*} \int_{-\infty}^{\infty} \left[\psi^* \mathcal{D}(i\partial_\tau)\psi + \frac{\gamma}{2} |\psi|^4 \right] d\tau = 0.$$

The pump solution of Eq. (9) will be used in the form

$$\psi = \sqrt{P_0} e^{i\gamma P_0 z}, \qquad P_0 = \text{const} \tag{10}$$

where P_0 replaces $|A|^2$ and describes power of the pump. The z-propagated Lighthill criterion (4) predicts pump instability if

$$\beta_2 = \mathcal{D}''(0) < 0. \tag{11}$$

Possible violation of this criterion has been discussed in the Introduction. Generalization of the criterion (11) is given in the next Section.

4 Generalized Lighthill criterion

Nonlinear process that is responsible for MI is FWM. Two "input" and two "output" waves are involved in such mixing if the following conditions are satisfied [68]

$$\omega_1 + \omega_2 = \omega_3 + \omega_4, \tag{12}$$

$$\beta(\omega_1) + \beta(\omega_2) = \beta(\omega_3) + \beta(\omega_4). \tag{13}$$

We set $\omega_{1,2} = \omega_0$ for the seed pump wave, and we get $\omega_{3,4} = \omega_0 \mp \Omega$ for the Stokes and anti-Stokes daughter waves. The offset Ω is the modulation frequency when (and only when) $\Omega \ll \omega_0$, yet GNLSE makes it possible to study any offsets. It is profitable to introduce the wave vector mismatch function [2]

$$\mathcal{M}(\Omega) = \frac{\beta(\omega_0 + \Omega) - 2\beta(\omega_0) + \beta(\omega_0 - \Omega)}{2}, \tag{14}$$

which measures to which extent the resonance condition (13) is violated. The factor $\frac{1}{2}$ is introduced to enjoy a more direct relation between FWM and GNLSE in what follows.

The generalized Lighthill criterion (GLC) for a focusing nonlinearity claims:

GLC The pump is unstable if for some offset Ω_s both the mismatch vanishes, $\mathcal{M}(\Omega_s) = 0$, and $\mathcal{M}(\Omega)$ is either monotone or has a local maximum at $\Omega = \Omega_s$ (Fig. 1b,c,d).

Frequency offsets of the growing perturbations compose a subset of the domain $\mathcal{M}(\Omega) < 0$ (red bands in Fig. 1).

A slow modulation yields $\mathcal{M}(\Omega) \approx \frac{1}{2}\beta_2\Omega^2$, GLC reduces then to the classical condition (11). The general formulation is valid both for vanishing GVD [38, 29, 15, 1] and for fast unstable modulations observed in [60, 42, 65, 23, 47, 25, 10, 49, 17, 64].

GLC is quantified by specifying the first non-vanishing derivative

$$\mathcal{M}^{(m)}(\Omega_s) \neq 0 \quad \text{for some} \quad m \in \mathbb{N} \quad \text{such that}$$
$$\mathcal{M}^{(m-1)}(\Omega_s) = \mathcal{M}^{(m-2)}(\Omega_s) = \cdots = \mathcal{M}(\Omega_s) = 0.$$

The pump is always unstable for an odd m. For an even m, the pump is unstable if $\mathcal{M}^{(m)}(\Omega_s) < 0$, i.e., when $\mathcal{M}(\Omega)$ has a local maximum at Ω_s. Frequency bands of the growing perturbations increase with the increase of the pump power P_0, they scale as $\sqrt[m]{\gamma P_0/|\mathcal{M}^{(m)}(\Omega_s)|}$.

One should always try $\Omega_s = 0$ when using GLC, because by construction $\mathcal{M}(0) = 0$. Equation (14) provides [38]

$$\mathcal{M}^{(2n-1)}(0) = 0, \quad \mathcal{M}^{(2n)}(0) = \beta_{2n}, \quad \forall n \in \mathbb{N}. \tag{15}$$

GLC requires $\beta_2 < 0$, which is the standard Lighthill criterion. If $\beta_2 = 0$ (ZDF), MI occurs for $\beta_4 < 0$ and so on in accord with [38, 29, 15, 1]. Frequency bands of growing modulations scale as $\sqrt[2n]{-\gamma P_0/\beta_{2n}}$.

It is important to stress that non-zero offsets with $\mathcal{M}(\Omega_s) = 0$ and $\mathcal{M}'(\Omega_s) \neq 0$ are also possible and yield instability of the pump [60, 42, 65, 23, 17, 64, 47, 25, 10, 49]. Such offsets appear in pairs because $\mathcal{M}(\Omega)$ is an even function. We then deal with the FWM instability [65] or, informally speaking, with the growing fast modulations. To the best of our knowledge such possibility was first predicted in [66].

Figure 1 illustrates two typical situations in which non-zero roots of $\mathcal{M}(\Omega)$ come into play. The mismatch shown in Fig. 1a should not lead to MI, yet considering larger offsets (Fig. 1b) we see that the pump is unstable. The mismatch in Fig. 1c leads to the classical MI, yet considering larger offsets (Fig. 1d) we see that two additional unstable bands appear. The non-MI regimes of pump instability provide a temporal analog of the Turing instability [57, 46]

GLC resulted from the analysis of the papers cited above. We provide an independent derivation in the rest of this Section for completeness. The special solution (10) of Eq. (9) describes an unperturbed pump wave at frequency ω_0 and wave vector $\beta_0 + \gamma P_0$. Considering a perturbation imposed by two sideband waves with the frequencies $\omega_0 \pm \Omega$ we set

$$\psi(z,\tau) = \left[\sqrt{P_0} + u(z)e^{-i\Omega\tau} + v^*(z)e^{i\Omega\tau}\right] e^{i\gamma P_0 z}, \tag{16}$$

where Ω is real, v^* denotes complex conjugation, and the yet unknown behavior of $u(z)$ and $v(z)$ for $z \to \infty$ along the fiber is of interest. Inserting the above $\psi(z,\tau)$ in GNLSE (9), we obtain the following linearized equations for $u(z)$ and $v(z)$

$$[i\partial_z + \mathcal{D}(\Omega) + \gamma P_0]u + \gamma P_0 v = 0, \tag{17}$$
$$[-i\partial_z + \mathcal{D}(-\Omega) + \gamma P_0]v + \gamma P_0 u = 0. \tag{18}$$

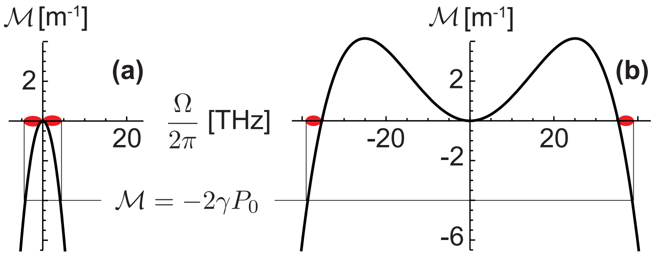

Figure 2: Mismatch $\mathcal{M}$ is plotted vs offset Ω to obtain unstable Ω-bands (red) from Eq.(21) for the bulk silica dispersion, $\gamma = 50\,\mathrm{W}^{-1}\mathrm{km}^{-1}$, and $P_0 = 40\,\mathrm{W}$. (a) pump wavelength $\lambda = 1.4\,\mu$m yields $\beta_2 < 0$ and two standard MI bands. (b) $\lambda = 1.265\,\mu$m yields $\beta_2 > 0$ and two non-MI bands in agreement with GLC.

Recall that $\mathcal{D}(\Omega)$ is taken real-valued and that $\mathcal{D}(\Omega)$ and $\mathcal{D}(-\Omega)$ are independent of each other, as opposed by the general relation $\beta(-\omega) = -\beta(\omega)$ for lossless media [18]. To proceed we split $\mathcal{D}(\Omega)$ into even and odd parts [11]

$$\mathcal{M}(\Omega) = \frac{\mathcal{D}(\Omega) + \mathcal{D}(-\Omega)}{2}, \quad \mathcal{N}(\Omega) = \frac{\mathcal{D}(\Omega) - \mathcal{D}(-\Omega)}{2},$$

where the even part is precisely the mismatch function in Eq. (14). With both $u(z)$ and $v(z)$ proportional to $e^{i\kappa z}$, the system (17–18) yields the dispersion relation

$$[\kappa - \mathcal{N}(\Omega)]^2 = \mathcal{M}(\Omega)[\mathcal{M}(\Omega) + 2\gamma P_0], \tag{19}$$

which describes two branches of $\kappa(\Omega)$. The pump is unstable if a real Ω yields a complex κ, because $\mathrm{Im}[\kappa] < 0$ for one of the branches.

For a generic Ω the mismatch $\mathcal{M}(\Omega)$ in Eq. (19) is much larger than γP_0 and $\mathrm{Im}[\kappa] = 0$. The branches of $\kappa(\Omega)$ are then given by

$$\kappa_{1,2} = \mathcal{N}(\Omega) \pm [\mathcal{M}(\Omega) + \gamma P_0], \tag{20}$$

where $O(\gamma^2 P_0^2/\mathcal{M})$ terms are neglected. Unstable modulations may come into play for a special choice of Ω. Recall, that we consider a focusing $\gamma > 0$ medium. The instability appears if $\mathcal{M}(\Omega)$ is negative and small enough such that

$$-2\gamma P_0 < \mathcal{M}(\Omega) < 0, \tag{21}$$

as illustrated in Fig. 2. The fastest growth rate $e^{\gamma P_0 z}$ is reached for the perturbation with

$$\mathcal{M}(\Omega) = -\gamma P_0 \quad \Rightarrow \quad \mathrm{Im}[\kappa] = \pm\gamma P_0. \tag{22}$$

An unstable Ω-domain appears each time when the mismatch $\mathcal{M}(\Omega)$ approaches zero from the negative side, which explains GLC. Such domains always appear in pairs (Fig. 1). Using the leading term of the Taylor expansion

$$\mathcal{M}(\Omega) \approx \frac{\mathcal{M}^{(m)}(\Omega_s)}{m!}(\Omega - \Omega_s)^m,$$

together with the inequality (21), we see that width of the unstable Ω-band is approximated by

$$\Delta\Omega \approx \sqrt[m]{2m!\gamma P_0/|\mathcal{M}^{(m)}(\Omega_s)|}.$$

For $m = 2$ we return to the classical relation (7) due to Eq. (15). Figure 2a,b illustrates two relevant shapes of $\mathcal{M}(\Omega)$ leading to the MI regime ($\Delta\Omega \propto \sqrt{P_0}$) and non-MI regime ($\Delta\Omega \propto P_0$) of the pump instability for the fused silica dispersion.

To conclude this Section we stress that the FWM conditions (12–13) adopt the linear dispersion law [32]. In general, it is problematic to account for the nonlinear frequency or wave vector shifts because amplitudes of the interacting waves are unknown and changing. An initial stage of MI is an exception, one can generalize Eq. (14) by introducing a nonlinear mismatch $\mathcal{M}_{\mathrm{nl}}(\Omega)$.

To define $\mathcal{M}_{\mathrm{nl}}(\Omega)$ one should replace pump's β_0 with $\beta_{\mathrm{nl}}(\omega_0, P_0) = \beta_0 + \gamma P_0$ in accord with Eq. (6). The sideband waves are small, yet they get their nonlinear wave vector shifts due to the presence of the pump. By linearizing GNLSE (9) with respect to $\psi = \psi_0 + \delta\psi$, we obtain the pump-induced wave vector shift $2\gamma P_0$. Altogether, the linear mismatch (14) is replaced by the nonlinear one [38, 29]

$$\mathcal{M}_{\mathrm{nl}}(\Omega) = \mathcal{M}(\Omega) + \gamma P_0, \tag{23}$$

where Eq. (19) yields that MI domain is determined by $|\mathcal{M}_{\mathrm{nl}}(\Omega)| < \gamma P_0$ and that the largest increment is achieved for the sidebands with $\mathcal{M}_{\mathrm{nl}}(\Omega) = 0$. Although the nonlinear mismatch (23) offers a more symmetric MI description, in what follows we will stay with the more common $\mathcal{M}(\Omega)$.

5 Nonlinear dispersion

The standard MI band (7) is close to the carrier frequency (Fig. 2a) such that both the carrier wave and the growing daughter waves are characterized by the same nonlinear coefficient γ in the NLSE (5) or GNLSE (9). The generalized MI condition (21) may involve perturbations that are separated from the carrier frequency (Fig. 2b). The frequency-independent γ, which was used in all GNLSE-based studies of MI, is then problematic. To address this problem we now consider GNLSE with the general dispersive nonlinear term

$$i\partial_z\psi + \mathcal{D}(i\partial_\tau)\psi + \gamma(|\psi|^2 + \mathcal{I})\psi = 0, \tag{24}$$

where

$$\mathcal{I} = \psi^*\mathcal{A}(i\partial_\tau)\psi + \psi\mathcal{B}(i\partial_\tau)\psi^*, \quad \mathcal{A}(0) = \mathcal{B}(0) = 0.$$

Here γ refers to $\gamma(\omega_0)$ and it is safe to assume that both $\mathcal{A}(0)$, $\mathcal{B}(0)$ are zero.

The new operators $\mathcal{A}(i\partial_\tau)$ and $\mathcal{B}(i\partial_\tau)$ depend on individual application and, among other things, describe dispersion of the nonlinearity

$$\frac{\gamma(\omega_0 + \Omega)}{\gamma(\omega_0)} = 1 + \mathcal{A}(\Omega) + \mathcal{B}(-\Omega).$$

In most cases $\mathcal{A}(i\partial_\tau)$ and $\mathcal{B}(i\partial_\tau)$ will be polynomials with respect to $i\partial_\tau$ in a full analogy with $\mathcal{D}(i\partial_\tau)$. Nonlocal operators are also possible, their action is calculated in the frequency domain by multiplication with operator symbols $\mathcal{A}(\Omega)$, $\mathcal{B}(\Omega)$ which are analytic for $\operatorname{Im}\Omega > 0$ due to causality. Examples are as follows.

(A) in general, $\mathcal{A}(\Omega)$ and $\mathcal{B}(\Omega)$ are different and provide a complex-valued $\mathcal{I}$, e.g.,

$$\begin{array}{l}\mathcal{A}(\Omega) = 2\Omega/\omega_0,\\ \mathcal{B}(\Omega) = \Omega/\omega_0,\end{array} \quad\Rightarrow\quad \mathcal{I}\psi = i\omega_0^{-1}\partial_\tau(|\psi|^2\psi), \tag{25}$$

for the self-steepening nonlinearity [2, 12]. In a similar way one can obtain the derivative NLSE [35].

(B) another possible choice is

$$\begin{array}{l}\mathcal{A}(\Omega) = T_H\Omega,\\ \mathcal{B}(\Omega) = 0,\end{array} \quad\Rightarrow\quad \mathcal{I}\psi = iT_H|\psi|^2\partial_\tau\psi.$$

With the time scale $T_H = \beta_3/\beta_2$ and $J = 3$, this leads to the Hirota equation [27]

$$i\partial_z\psi + \sum_{j=2,3}\frac{\beta_j}{j!}(i\partial_\tau)^j\psi + \gamma|\psi|^2\left(\psi + i\frac{\beta_3}{\beta_2}\partial_\tau\psi\right) = 0,$$

in physical units. One can also obtain a more sophisticated Sasa-Satsuma equation [50].

(C) the so-called *interpulse Raman scattering* is described by

$$\begin{aligned} \mathcal{A}(\Omega) &= iT_R\Omega, \\ \mathcal{B}(\Omega) &= iT_R\Omega, \end{aligned} \quad \Rightarrow \quad \mathcal{I} = -T_R\partial_\tau|\psi|^2, \tag{26}$$

where $T_R \approx 3$ fs for fused silica fibers [2].

(D) physical mechanisms that lead to second-order polynomials $\mathcal{A}(\Omega)$ and $\mathcal{B}(\Omega)$ were discussed in [37, 3].

(E) to give a more complicated example, consider a general expression that accounts for the Raman effect [11]

$$\mathcal{I} = f_R\left[\int_0^\infty h(\tau')|\psi(z,\tau-\tau')|^2 d\tau' - |\psi(z,\tau)|^2\right]. \tag{27}$$

Here f_R and $1 - f_R$ quantify relative contributions of the Raman and Kerr interactions in GNLSE (24), while $h(\tau)$ is a causal response function with $\int_0^\infty h(\tau)d\tau = 1$.

By changing to the frequency domain in Eq. (27), one can derive that

$$\mathcal{I} = \mathcal{R}(i\partial_\tau)\Big(|\psi(z,\tau)|^2\Big), \tag{28}$$

where we now deal with a non-polynomial symbol

$$\mathcal{R}(\Omega) = f_R\left[\int_0^\infty h(\tau)e^{i\Omega\tau}d\tau - 1\right], \quad \mathcal{R}(0) = 0. \tag{29}$$

For instance, for fused silica fibers one can take [2]

$$\begin{gathered} f_R = 0.18, \quad h(t) = \frac{\nu_1^2+\nu_2^2}{\nu_1}e^{-\nu_2 t}\sin(\nu_1 t), \\ \nu_1^{-1} = 12.2\,\text{fs}, \quad \nu_2^{-1} = 32\,\text{fs}. \end{gathered}$$

To return to Eq. (26) one should neglect the real part of $\mathcal{R}(\Omega)$ and approximate the imaginary part by $iT_R\Omega$. The approximation works reasonably well for slow modulations with $T_R\Omega \lesssim 0.3$.

To link Eq. (28) to the nonlinear term in GNLSE (24), recall that we are interested in small perturbations of the monochromatic pump wave (10)

$$\psi(z,\tau) = \sqrt{P_0}e^{i\gamma P_0 z} + \delta\psi,$$

in which case Eq. (28) yields

$$\mathcal{I} = \psi^*\mathcal{R}(i\partial_\tau)\psi + \psi\mathcal{R}(i\partial_\tau)\psi^* + O(\delta\psi^2).$$

Therefore to address MI with the full integral Raman term (27), it is sufficient to use GNLSE (24) and set

$$\mathcal{A}(i\partial_\tau) = \mathcal{B}(i\partial_\tau) = \mathcal{R}(i\partial_\tau). \tag{30}$$

The above examples demonstrate that GNLSE (24) is an adequate framework for studies of MI with the dispersive nonlinearity. We are in a good position to recognize the pump wave (10) as an exact solution of Eq. (24). There is no need to account for the further nonlinear terms like $(\partial_\tau\psi)^2\psi^*$ or $|\partial_\tau\psi|^2\psi$, which appear, e.g., in the Lakshmanan-Porsezian-Daniel equation [5]. Such terms vanish when GNLSE is linearized around the pump wave, they do not affect MI.

Another important fact is that the pump wave (10) is an exact solution of Eq. (24) even for a complex-valued $\mathcal{D}(\Omega)$. In what follows we allow for the complex-valued dispersion parameters

$$\beta^{(j)}(\omega_0) = \beta_j + i\alpha_j \quad \text{for} \quad j \geq 2,$$

to trace the effect of linear wave dissipation. In the case of interest, a complex GVD with $\alpha_2 > 0$ accounts for the diffusion. However, we ignore both α_0 and α_1. In other words, gain terms in a driven dissipative wave system should be organized in such a way that

$$\mathrm{Im}[\beta(\omega_0 + \Omega)] \approx \frac{1}{2}\alpha_2\Omega^2. \tag{31}$$

A full treatment of the interplay between dispersion, dissipation, nonlinearity, and gain requires use of the Ginsburg-Landau type models (see, e.g., Refs. [6, 63]), which is out of scope of this work. We now turn to MI for the dispersive nonlinearity (24).

6 Beyond MI

To summarize results of the previous Section, the pump wave (10) solves Eq. (24) both for the real- and complex-valued $\mathcal{A}(\Omega)$, $\mathcal{B}(\Omega)$, and $\mathcal{D}(\Omega)$, which includes many if not all relevant envelope equations. The pump stability problem is addressed by substituting (16) into (24) and linearizing with respect to $u(z)$ and $v(z)$ which provides

$$\begin{bmatrix} i\partial_z + \mathcal{D}(\Omega) & 0 \\ 0 & -i\partial_z + \mathcal{D}^*(-\Omega) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \gamma P_0 \begin{bmatrix} 1+\mathcal{A}(\Omega) & 1+\mathcal{B}(\Omega) \\ 1+\mathcal{B}^*(-\Omega) & 1+\mathcal{A}^*(-\Omega) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0.$$

The previous system (17–18) is a special case of the above equation for $\mathcal{A}(\Omega) = \mathcal{B}(\Omega) = 0$ and real-valued $\mathcal{D}(\Omega)$.

In what follows it is convenient to split $\mathcal{D}(\Omega)$, $\mathcal{A}(\Omega)$, and $\mathcal{B}(\Omega)$ into symmetric "s" and asymmetric "a" components following the pattern

$$\mathcal{D}_{s,a}(\Omega) = \frac{\mathcal{D}(\Omega) \pm \mathcal{D}^*(-\Omega)}{2}.$$

Thus, we generalize even $\mathcal{M}(\Omega)$ and odd $\mathcal{N}(\Omega)$ dispersion components, which were used in the Section 4 for a real-valued $\mathcal{D}(\Omega) = \mathcal{M}(\Omega) + \mathcal{N}(\Omega)$. Recall that the even component is the mismatch function (14). For a complex-valued $\mathcal{D}(\Omega) = \mathcal{D}_s(\Omega) + \mathcal{D}_a(\Omega)$ we have

$$\mathcal{D}_s(-\Omega) = \mathcal{D}_s^*(\Omega), \quad \mathcal{D}_a(-\Omega) = -\mathcal{D}_a^*(\Omega).$$

Informally speaking, $\mathcal{D}_s(\Omega)$ is the complex mismatch.

Inserting $u, v \propto e^{i\kappa z}$ into the above matrix equation we derive the dispersion relation

$$(\kappa - \mathcal{D}_a - \gamma P_0 \mathcal{A}_a)^2 = \mathcal{D}_s[\mathcal{D}_s + 2\gamma P_0(1+\mathcal{A}_s)] + \gamma^2 P_0^2[(1+\mathcal{A}_s)^2 - (1+\mathcal{B}_s)^2 + \mathcal{B}_a^2], \tag{32}$$

cf., Eq. (19). For each Ω the latter equation provides two complex-valued solutions for κ, the pump wave is unstable if $\mathrm{Im}\,\kappa < 0$. Note that Eq. (32) is invariant under replacement

$$\Omega \mapsto -\Omega, \qquad \kappa \mapsto -\kappa^*,$$

followed by the complex conjugation. Unstable Ω-bands are therefore symmetric with respect to the carrier frequency, Stokes and anti-Stokes sideband waves always grow with the same rate.

Equation (32) is the main result of this Section. Adding to previous studies, it accounts for the dispersion of nonlinearity, which is relevant when pump instability involves a wide range of frequencies. Let us consider several examples.

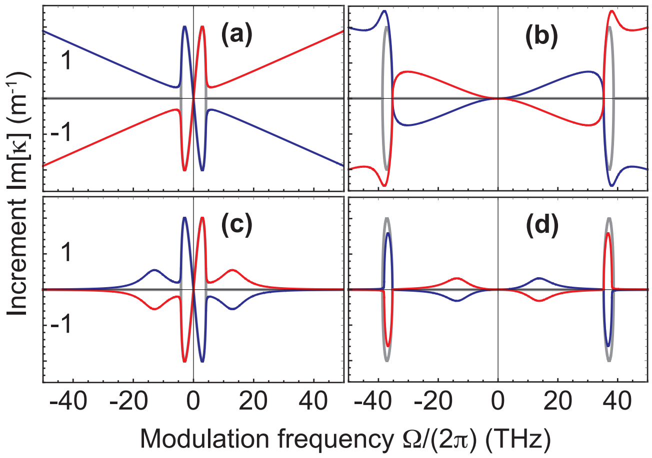

Figure 3: (a,b) instability increments (color lines) are calculated from Eq. (33) for the MI regime (as in Fig. 2a) and FWM regime (Fig. 2b). Solutions of the Raman-free Eq. (19) (grey lines) are shown for comparison. (c,d) instability increments are calculated from the more accurate Eq. (34).

6.1 Self-steepening effect on MI

Here we discuss how the MI is modified by the self-steepening effect [52, 48, 1, 7]. The dispersion relation (32) is then combined with the shock nonlinearity (25) and a real-valued $\mathcal{D}(\Omega) = \mathcal{M}(\Omega) + \mathcal{N}(\Omega)$, i.e.,

$$\left[\kappa - \mathcal{N}(\Omega) - 2\gamma P_0 \frac{\Omega}{\omega_0}\right]^2 = \mathcal{M}(\Omega)[\mathcal{M}(\Omega) + 2\gamma P_0] + \gamma^2 P_0^2 \frac{\Omega^2}{\omega_0^2}.$$

Without the self-steepening term the fastest growth rate is achieved for $\mathcal{M}(\Omega) = -\gamma P_0$. Taking the self-steepening term into account, we obtain from the above equation

$$\mathcal{M}(\Omega) = -\gamma P_0 \quad \Rightarrow \quad \mathrm{Im}[\kappa] = \pm\gamma P_0 \sqrt{1 - \frac{\Omega^2}{\omega_0^2}},$$

cf., Eq. (22). The correction is unimportant for the classical MI regime (Fig. 2a) with $\Omega \ll \omega_0$. As to the situation shown in Fig. 2b, the instability is inhibited but does not disappear completely. The reason is that any GNLSE at the carrier frequency ω_0 is suitable in the spectral window $[0, 2\omega_0]$ at most, such that $|\Omega| < \omega_0$.

6.2 Raman effect on MI

Here we consider how the MI description in Section 3 is modified by the Raman effect [48, 65, 10, 7]. The dispersion relation (32) is then combined with a real-valued $\mathcal{D}(\Omega) = \mathcal{M}(\Omega) + \mathcal{N}(\Omega)$ and Raman nonlinearity (26), which yields

$$[\kappa - \mathcal{N}(\Omega)]^2 = \mathcal{M}(\Omega)[\mathcal{M}(\Omega) + 2\gamma P_0(1 + i\Omega T_R)]. \tag{33}$$

The right hand side of Eq. (33) is complex-valued, cf., Eq. (19). Therefore $\mathrm{Im}[\kappa]$ is nonzero for a generic Ω and is negative for one of two branches. This leads to a striking conclusion: *the pump-wave is unstable for all possible modulations.* Moreover, the increment increases without limit with the increase of Ω (Fig. 3a,b). This indicates that Eq. (26) is a very bad approximation even for the classical MI regime [21].

A better description of the Raman effect is clearly required, it is provided by Eq. (27) in place of Eq. (26). The dispersion relation (32) should then be used with Eq. (30), where

$$\mathcal{A}_s(\Omega) = \mathcal{B}_s(\Omega) = \mathcal{R}(\Omega), \quad \mathcal{A}_a(\Omega) = \mathcal{B}_a(\Omega) = 0,$$

because Eq. (29) yields that $\mathcal{R}(\Omega) = \mathcal{R}^*(-\Omega)$. The result reads

$$[\kappa - \mathcal{N}(\Omega)]^2 = \mathcal{M}(\Omega)[\mathcal{M}(\Omega) + 2\gamma P_0(1 + \mathcal{R}(\Omega))], \tag{34}$$

where Eq. (33) is recovered for $\Omega \ll 1/T_R$. Note that $\mathcal{R}(\Omega)$ vanishes for $\Omega \to \infty$ and there is no growth of the increment (Fig. 3c,d). Solutions of Eq. (34) for a generic Ω outside MI and FWM bands, i.e., with $\mathcal{M}(\Omega) \gg \gamma P_0$, are given by

$$\kappa_{1,2} = \mathcal{N}(\Omega) \pm \mathcal{M}(\Omega) \pm \gamma P_0[1 + \mathcal{R}(\Omega)], \tag{35}$$

cf., Eq. (20). Raman peaks are clearly observed in Fig. 3c,d. The pump-wave is formally unstable for all possible modulations [10], this is consequence of the Raman gain [11]. Unstable frequency bands are then limited by dissipation, as explained in the next Section.

6.3 Diffusion effect on MI

In this Section we use complex-valued $\beta(\omega)$ from Eq. (31) to quantify how MI and FWM are affected by diffusion. We start with GNLSE (9) in which we set

$$\mathcal{D}(\Omega) = \mathcal{M}(\Omega) + \mathcal{N}(\Omega) + \frac{i\alpha_2\Omega^2}{2},$$

such that the general dispersion relation (32) takes the form

$$\left[\kappa - \mathcal{N}(\Omega) - \frac{i\alpha_2\Omega^2}{2}\right]^2 = \mathcal{M}(\Omega)[\mathcal{M}(\Omega) + 2\gamma P_0],$$

cf., Eq. (19). MI increments, which can be calculated with and without the diffusion term, are related by

$$\mathrm{Im}[\kappa] = \mathrm{Im}[\kappa]_{\alpha_2=0} + \frac{\alpha_2\Omega^2}{2}.$$

MI condition $\mathrm{Im}[\kappa] < 0$ is then always inhibited by the downhill diffusion with $\alpha_2 > 0$. Note, that diffusion does not provide gain, as opposed by the Raman effect.

In the classical MI regime with $\mathcal{M}(\Omega) = \frac{1}{2}\beta_2\Omega^2$ and $\beta_2 < 0$, the instability, being suppressed by diffusion, never disappears completely because

$$\mathrm{Im}[\kappa]_{\Omega\to 0} = \frac{\alpha_2\Omega^2}{2} \pm C\Omega, \quad C = \sqrt{-\beta_2\gamma P_0},$$

where we neglected $O(\Omega^3)$ terms to observe that $\mathrm{Im}[\kappa]$ is negative for slow enough modulations. On the contrary, MI at ZDF with $\mathcal{M}(\Omega) = \frac{1}{24}\beta_4\Omega^4$ and $\beta_4 < 0$, is eliminated if $\alpha_2 > \sqrt{-\beta_4\gamma P_0/3}$. In a similar way FWM instability, which involves fast modulations, may be completely eliminated by diffusion.

Finally, Eq. (35) for the Raman gain takes the form

$$\kappa_{1,2} = \mathcal{N}(\Omega) + \frac{i\alpha_2\Omega^2}{2} \pm \mathcal{M}(\Omega) \pm \gamma P_0[1 + \mathcal{R}(\Omega)].$$

We conclude that diffusion completely eliminates Raman gain for the fast modulations for which $\mathcal{R}(\Omega) \to 0$.

7 Conclusions

In summary, modulation instability is an universal phenomenon and it is reasonable to expect that it is governed by a simple universal criterion. However, many studies indicated that the commonly accepted Lighthill criterion is limited even in the most simple case of a single-mode nonlinear fiber. On one hand, the criterion does not apply to waves with the vanishing group velocity dispersion. On the other hand, Lighthill criterion does not apply to situations where slow modulations coexist with the fast ones. The latter appear and may grow because nonlinear waves can mix in many different ways.

We have found that Lighthill criterion can be generalized. The generalized criterion successfully applies to the vanishing dispersion and to the both slow and fast modulations. It is equivalent to a simple geometric property of the mismatch function, as summarized in Fig. 1.

Furthermore, the recently found instability regimes involve growing daughter waves with the frequencies which are considerably different from each other and from that of the seed wave. This makes use of a simple envelope equation with an arbitrary dispersion but just one nonlinear coefficient questionable. To address this problem we introduced a general dispersive nonlinearity and studied how it affects both classical and new regimes of the modulation instability.

Last but not least, special attention is given to interplay of linear and nonlinear dissipative effects such as Raman gain and diffusion. We demonstrate how recently found instability regimes, in which all modulations seem to grow, are regularized by diffusion.

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