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Well-posedness of Hibler's dynamical sea-ice model

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Abstract

This paper establishes the local-in-time well-posedness of solutions to an approximating system constructed by mildly regularizing the dynamical sea ice model of *W.D. Hibler, Journal of Physical Oceanography, 1979*. Our choice of regularization has been carefully designed, prompted by physical considerations, to retain the original coupled hyperbolic-parabolic character of Hibler's model. Various regularized versions of this model have been used widely for the numerical simulation of the circulation and thickness of the Arctic ice cover. However, due to the singularity in the ice rheology, the notion of solutions to the original model is unclear. Instead, an approximating system, which captures current numerical study, is proposed. The well-posedness theory of such a system provides a first-step groundwork in both numerical study and future analytical study.

1 Introduction

1.1 The sea-ice dynamic-thermodynamic model

Global climate changes, especially global warming, have large impact on the Arctic sea-ice, which has, in return, determining effects on not only global climate but also the local and global ecosystem, human activities etc. (see e.g., [14]). However, the theory concerning the mechanic property of sea-ice remains immature and primitive, as pointed out by [10], and thus remains mostly open. If the problem is statically determinate, as pointed out in [13], a sea-ice dynamical model based on the viscous-plastic rheology was introduced in [5], where the thickness of ice plays an essential role in the thermodynamics, and characterizes the strength of the ice interaction (i.e., ice rheology). The velocity of sea-ice $\mathbf{u}$ is described by two-dimensional momentum balance equations, where the viscosity effect is characterized by a viscous-plastic rheology, and the strength of viscosity depends on the thickness of ice. The mean ice thickness h and the compactness of ice A are described by two continuity equations with thermodynamic source terms. That is, with a simplified ice rheology (see (1.11a), below), the above quantities are governed by the coupled system,

$$m(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) + \nabla p = \operatorname{div} \mathbb{S} + \mathcal{F}, \tag{1.1a}$$

$$\partial_t h + \operatorname{div}(h\mathbf{u}) = \mathcal{S}_h, \tag{1.1b}$$

$$\partial_t A + \operatorname{div}(A\mathbf{u}) = \mathcal{S}_A + A \operatorname{div} \mathbf{u} \cdot \chi_{\{A \geq 1\}}, \tag{1.1c}$$

with

$$\text{ice mass } m := \rho_{\mathrm{ice}} h, \tag{1.2a}$$

$$\text{pressure } p := c_p h \exp(c_a A), \tag{1.2b}$$

$$\text{viscoplastic stress } \mathbb{S} := p \frac{\nabla u + \nabla u^\top}{\left|\nabla u + \nabla u^\top\right|} + p \frac{\operatorname{div} \mathbf{u} \mathbb{I}_2}{\left|\operatorname{div} \mathbf{u}\right|}, \tag{1.2c}$$

$$\mathcal{F} := -m\eta \mathbf{u}^\perp + \tau_{\mathrm{a}} + \tau_{\mathrm{w}}, \tag{1.2d}$$

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$$\text{air flow stress } \tau_{\mathrm{a}} := \rho_{\mathrm{a}} C_{\mathrm{a}} \big|\mathbf{U}_{\mathrm{g}}\big| (\mathbf{U}_{\mathrm{g}} \cos\phi + \mathbf{U}_{\mathrm{g}}^{\perp} \sin\phi), \tag{1.2e}$$

$$\begin{aligned}\text{water flow stress } \tau_{\mathrm{w}} := \rho_{\mathrm{w}} C_{\mathrm{w}} \big|\mathbf{U}_{\mathrm{w}} - \mathbf{u}\big| [(\mathbf{U}_{\mathrm{w}} - \mathbf{u}) \cos\theta \\ + (\mathbf{U}_{\mathrm{w}} - \mathbf{u})^{\perp} \sin\theta],\end{aligned} \tag{1.2f}$$

$$\mathcal{S}_h := \big[f(h/A)A + (1-A)f(0)\big] \cdot \chi_{\{h>0\}}, \tag{1.2g}$$

$$\mathcal{S}_A := \big(\big(f(0)\big)^{+}/h_0\big)(1-A) + (-A/(2h)) \cdot (\mathcal{S}_h)^{-}. \tag{1.2h}$$

Here $\chi_{\{h>0\}}, \chi_{\{A\geq 1\}}$ are the characteristic functions of sets $\{h>0\}, \{A \geq 1\}$, defined by

$$\chi_{\{h>0\}} = \begin{cases} 1 & h > 0, \\ 0 & h \leq 0, \end{cases} \quad \chi_{\{A\geq 1\}} = \begin{cases} 1 & A \geq 1, \\ 0 & A < 1, \end{cases} \tag{1.2i}$$

respectively. In addition, $\mathbf{v}^{\perp} = (-v_2, v_1)^{\top}$ for any vector $\mathbf{v} = (v_1, v_2)^{\top}$; $\rho_{\mathrm{ice}}, \rho_{\mathrm{a}}, \rho_{\mathrm{w}}$ represent the density of ice, air, and water, respectively; $c_p, c_a, C_{\mathrm{a}}, C_{\mathrm{w}}$ are the thermodynamic constants; and $\mathbf{U}_{\mathrm{g}}, \mathbf{U}_{\mathrm{w}}, \phi, \theta$ denote the velocity and stress angle of the air and the water, which, for simplicity of presentation, are assumed to be constant in this paper.

System (1.1) is used to simulate the evolution of sea-ice in numerical study. For instance, the model successfully reproduces many of the observed features of the circulation and thickness of the Arctic ice cover in [5]. See [6, 9, 11] and the references therein for further model development and computational investigation. In particular, see [7] for a review of an elastic-viscous-plastic sea-ice dynamics model, and [12] for a summary of popular models.

Despite the high involvement of system (1.1) in applications, the fundamental problem of well-posedness of solutions is widely open, which is related to the validity of the model as pointed out by [13]. In [4], a linear well-posedness theory is developed for an approximating model. We would like to point out that the main challenge in establishing the well-posedness theory is the singularity arising in the stress tensor (1.2c) when $|\nabla \mathbf{u}| \to 0^{+}$. In fact, among the numerical investigations, such singularity is usually truncated, i.e. regularized, by replacing it with its strictly positive approximation (e.g., $\sqrt{|\nabla \mathbf{u}|^2 + \varepsilon^2}$).

Notably, we would like to point out an investigation of very singular diffusion equations in [3, 2], where the authors discuss the notion of solutions to

$$\partial_t u = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right).$$

Similarly, the positive one-homogeneity of the potential related to (1.2c) calls for a subdifferential formulation of the problem, however set in the Eulerian frame. We leave such investigation to our future study.

In this paper, due to the obstacles mentioned above, we propose to study the following regularized approximating problem of (1.1): for $\varepsilon, \omega \in (0,1)$,

$$m(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) + \nabla p = \operatorname{div} \mathbb{S}_{\varepsilon} + \mathcal{F}, \tag{1.3a}$$

$$\partial_t h + \operatorname{div}(h\mathbf{u}) = \mathcal{S}_{h,\omega}, \tag{1.3b}$$

$$\partial_t A + \operatorname{div}(A\mathbf{u}) = \mathcal{S}_{A,\omega} + A \operatorname{div} \mathbf{u} \cdot \chi_A^{\omega}, \tag{1.3c}$$

where m, p, and $\mathcal{F}$ are as in (1.2a), (1.2b), and (1.2d), respectively, and

$$\mathbb{S}_{\varepsilon} = \mathbb{S}_{\varepsilon}(p, \nabla \mathbf{u}) := p \frac{\nabla \mathbf{u} + \nabla \mathbf{u}^{\top}}{\sqrt{\big|\nabla \mathbf{u} + \nabla \mathbf{u}^{\top}\big|^2 + \varepsilon^2}} + p \frac{\operatorname{div} \mathbf{u} \mathbb{I}_2}{\sqrt{\big|\operatorname{div} \mathbf{u}\big|^2 + \varepsilon^2}}, \tag{1.4a}$$

$$\mathcal{S}_{h,\omega} := \big[f(h/(A+\omega))A + (1-A)f(0)\big] \chi_{\{h>0\}}, \tag{1.4b}$$

$$\mathcal{S}_{A,\omega} := \frac{\big(f(0)\big)^+}{h_0}(1-A) - \frac{A}{2h} \cdot \frac{\sqrt{|\mathcal{S}_{h,\omega}|^2+\omega^2} - \mathcal{S}_{h,\omega}}{2}, \tag{1.4c}$$

$$\chi_A^\omega := 1 - \frac{(1-A)^+}{(1-A)^+ + \omega}. \tag{1.4d}$$

To be more precise, we will establish the local in time well-posedness of strong solutions to (1.3) in domain $\Omega := \mathbb{T}^2 \subset \mathbb{R}^2$:

Theorem 1.1. *Consider initial data*

$$(\mathbf{u}, h, A)|_{t=0} = (\mathbf{u}_{\rm in}, h_{\rm in}, A_{\rm in}) \in (H^3(\Omega))^3 \tag{1.5}$$

to system (1.3)*, satisfying*

$$0 < \underline{h} \le h_{\rm in} \le \overline{h} < \infty, \quad \textit{and} \quad 0 \le A_{\rm in} \le 1. \tag{1.6}$$

In addition, we assume that

$$\begin{gathered} \underline{f} \le f \le \overline{f}, \\ \left|f'\right| + \left|f''\right| + \left|f'''\right| \le M_f, \end{gathered} \tag{1.7}$$

for some constants $\underline{f}, \overline{f} \in \mathbb{R}$, $M_f \in (0,\infty)$. Then there exists a unique strong solution $(\mathbf{u}, h, A)$ to system (1.3) *in $[0,T]\times\Omega$, for some $T \in (0,\infty)$ depending on initial data, with*

$$\begin{gathered} \mathbf{u} \in L^\infty(0,T;H^3(\Omega)) \cap L^2(0,T;H^4(\Omega)), \\ h, A \in L^\infty(0,T;H^3(\Omega)), \\ \partial_t \mathbf{u}, \partial_t h, \partial_t A \in L^\infty(0,T;L^2(\Omega)), \end{gathered} \tag{1.8}$$

and

$$\begin{gathered} \|\mathbf{u}, h, A\|_{L^\infty(0,T;H^3(\Omega))} + \|\mathbf{u}\|_{L^2(0,T;H^4(\Omega))} \\ + \|\partial_t \mathbf{u}, \partial_t h, \partial_t A\|_{L^\infty(0,T;L^2(\Omega))} \le \mathfrak{C}_{\rm in}, \\ 0 \le A \le 1, \qquad 0 < \frac{1}{4}\underline{h} \le h \le 4\overline{h}, \end{gathered} \tag{1.9}$$

where $\mathfrak{C}_{\rm in} \in (0,\infty)$ is some positive constant depending only on initial data. Moreover, the solution is stable with respect to perturbation of initial data.

Now, let us explain our strategy. Instead of directly constructing solutions to system (1.3), we consider another regularized system, parametrized by $(\mu, \lambda, \iota, \nu) \in (0,1)^4$:

$$m(\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) + \nabla p = \operatorname{div} \mathbb{S}_{\varepsilon,\mu,\lambda} - \iota\Delta^2 \mathbf{u} + \mathcal{F}, \tag{1.10a}$$

$$\partial_t h + \operatorname{div}(h\mathbf{u}) = \mathcal{S}_{h,\omega,\nu}, \tag{1.10b}$$

$$\partial_t A + \operatorname{div}(A\mathbf{u}) = \mathcal{S}_{A,\omega,\nu} + A\operatorname{div}\mathbf{u}\cdot\chi_A^\omega, \tag{1.10c}$$

where m, p, $\mathcal{F}$, and χ_A^ω are as in (1.3) and (1.4), and

$$\mathbb{S}_{\varepsilon,\mu,\lambda} := \mathbb{S}_\varepsilon + \mathbb{S}_{\mu,\lambda}, \tag{1.11a}$$

$$\mathcal{S}_{h,\omega,\nu} := \big[f(h^+/(A^+ + \omega))A + (1-A)f(0)\big]\cdot\chi_h^\nu, \tag{1.11b}$$

$$\mathcal{S}_{A,\omega,\nu} := \frac{\big(f(0)\big)^+}{h_0+\nu}(1-A) - \frac{A}{2h^+ + \nu} \cdot \frac{\sqrt{|\mathcal{S}_{h,\omega,\nu}|^2+\omega^2} - \mathcal{S}_{h,\omega,\nu}}{2}, \tag{1.11c}$$

with $\mathbb{S}_\varepsilon$ as in (1.3) and

$$\mathbb{S}_{\mu,\lambda} := \mu(\nabla \mathbf{u} + \nabla \mathbf{u}^\top) + \lambda \mathrm{div}\, \mathbf{u}\mathbb{I}_2, \tag{1.11d}$$

$$\chi_h^\nu := \frac{h^+}{h^+ + \nu}, . \tag{1.11e}$$

We will construct solutions to system (1.10) through a contraction mapping argument. That is, we consider a “linearizationöf (1.10), and establish a contraction mapping with respect to L^2 topology with bounds in a smooth function space. Then with a uniform-in-$(\mu, \lambda, \iota, \nu)$ estimate, we will be able to pass the limit $(\mu, \lambda, \iota, \nu) \to (0^+, 0^+, 0^+, 0^+)$, and eventually construct the strong solution to (1.3). The proof of Theorem 1.1 is then finished by showing the uniqueness and continuous dependency on the initial data. We would like to mention that the key ingredient in establishing the well-posedness of solutions involves showing the monotonicity of $\mathbb{S}_\varepsilon(\cdot)$ in $\nabla \mathbf{u}$, which is not trivially obvious due to the fact that $\mathbb{S}_\varepsilon(\cdot)$ is nonlinear in $\nabla \mathbf{u}$. In particular, we will require the inequality of the type

$$\big(\mathbb{S}_\varepsilon(p_1, \nabla \mathbf{u}_1) - \mathbb{S}_\varepsilon(p_2, \nabla \mathbf{u}_2)\big) : \big(\nabla \mathbf{u}_1 - \nabla \mathbf{u}_2\big) \gtrsim |\nabla(\mathbf{u}_1 - \mathbf{u}_2)|^2 + \cdots .$$

We successfully establish this inequality by writing $\mathbb{S}_\varepsilon(p_1, \nabla \mathbf{u}_1) - \mathbb{S}_\varepsilon(p_2, \nabla \mathbf{u}_2)$ in a symmetric form (see (4.29), below).

We would like to make some remarks before going into details of the proof. Our ice rheology (1.4a) is a simplified version of the one from [5]. For some technical reasons, we are not sure whether Theorem 1.1 will apply to the original ice rheology from [5]. We have not successfully established a proper uniform-in-ε estimates of the solutions to (1.3). Therefore, we have not yet been able to establish a proper notion of solutions to the original system (1.1). However, our approximation (1.3) agrees with the most common numerical approaches to (1.1), which, as we explain before, is restricted to a truncated ice rheology. Thus, in this sense, our analytical results provide a solid ground for current numerical schemes of (1.1). Another issue is that we only consider the case when $h_{\mathrm{in}} \geq \underline{h} > 0$, i.e., there is no absence of ice in the domain of study. To carry out the limit $\underline{h} \to 0^+$, more comprehensive *a priori* estimates are required. We leave this to future study.

Recently we have learnt an independent study [1] by Brandt, Disser, Haller-Dintelmann, and Hieber. It is worth pointing out that in this paper, system (1.3) remains hyperbolic in the equations of h and A, and therefore it remains a mixed type system, while in [1], the system investigated is parabolic in all components. In particular, due to the hyperbolicity, system (1.3) is expected to have a completely different long time dynamics than those investigated in [1].

This paper is organized as follows. In the next subsection, we will summarize some notations used in this paper. In Section 2, we will detail the approximation scheme to (1.10). In Section 3, we establish the well-posedness of solutions to (1.10) via a contraction mapping argument. Finally in Section 4, we establish the uniform-in-$(\mu, \lambda, \iota, \nu)$ estimates, and pass to the limit $(\mu, \lambda, \iota, \nu) \to (0^+, 0^+, 0^+, 0^+)$ to show the existence of solutions to (1.3). The well-posedness of solutions is then established in Section 4.3

1.2 Notations

We use $L^p(\cdot)$ and $H^s(\cdot)$ to denote the standard Lebesgue and Sobolev spaces, respectively. For any functional space $\mathcal{X}$ and functions $\psi, \phi, \cdots$, we denote by

$$\|\psi, \phi, \cdots\|_{\mathcal{X}} := \|\psi\|_{\mathcal{X}} + \|\phi\|_{\mathcal{X}} + \cdots .$$

In addition,

$$\psi^+ := \begin{cases} \psi & \text{if } \psi \geq 0, \\ 0 & \text{if } \psi < 0, \end{cases} \qquad \psi^- = \psi^+ - \psi.$$

Let $\partial \in \{\partial_x, \partial_y\}$. For any multi-index $(\alpha_1, \alpha_2) \in (\mathbb{Z}^+)^2$, denote by $\partial^\alpha := \partial_x^{\alpha_1}\partial_y^{\alpha_2}$ with $\alpha = \alpha_1 + \alpha_2$. Throughout this paper, we use the notation $X \lesssim Y$ to represent $X \leq CY$ for some generic constant $C \in (0, \infty)$, which may be different from line to line. We use $C_{a,b,\cdots}$ to emphasize the dependency on the quantities $a, b, \cdots$. In addition, by $\mathcal{H}(\cdots)$, it represents a generic bounded function of the arguments.

2 An approximation scheme to solve (1.10)

2.1 A "linearization" of (1.10)

Given $\mathbf{u}^o$, assumed to be smooth enough, we consider first the following coupled hyperbolic system

$$\partial_t h_{\mathrm{m}} + \operatorname{div}(h_{\mathrm{m}}\mathbf{u}^o) = \mathcal{S}_{h_{\mathrm{m}},\omega,\nu}, \tag{2.1a}$$

$$\partial_t A_{\mathrm{m}} + \operatorname{div}(A_{\mathrm{m}}\mathbf{u}^o) = \mathcal{S}_{A_{\mathrm{m}},\omega,\nu} + A_{\mathrm{m}}\operatorname{div}\mathbf{u}^o \cdot \chi^\omega_{A_{\mathrm{m}}}, \tag{2.1b}$$

where $\mathcal{S}_{h_{\mathrm{m}},\omega,\nu}$, $\mathcal{S}_{A_{\mathrm{m}},\omega,\nu}$, $\chi^\omega_{h_{\mathrm{m}}}$, and $\chi^\omega_{A_{\mathrm{m}}}$ are defined as in (1.11b), (1.11c), (1.11e), and (1.4d), with h and A replaced by h_{m} and A_{m}, respectively. Here we use the subscript m (short for 'mapping') and the superscript o (short for 'origin') to label outputs and inputs in our contraction mapping.

We claim that, at least locally in time, there exists a unique solution $(h_{\mathrm{m}}, A_{\mathrm{m}})$ to (2.1a) and (2.1b) with proper initial data, for smooth enough $\mathbf{u}^o$. $(h_{\mathrm{m}}, A_{\mathrm{m}})$ can be arbitrarily regular, provided that $\mathbf{u}^o$ and initial data are regular enough. We leave the investigation of the regularity of $(h_{\mathrm{m}}, A_{\mathrm{m}})$ in the subsequent sections.

We remark that such claims follow from the standard well-posedness theory of hyperbolic equations (see, e.g., [8]). Hence the proof is omitted.

Let $(h_{\mathrm{m}}, A_{\mathrm{m}})$ be the solution to (2.1a) and (2.1b) as above, and consider the following equation:

$$\rho_{\mathrm{ice}} h_{\mathrm{m}} \partial_t \mathbf{u}_{\mathrm{m}} + \iota\Delta^2 \mathbf{u}_{\mathrm{m}} = -\rho_{\mathrm{ice}} h_{\mathrm{m}} \mathbf{u}^o \cdot \nabla \mathbf{u}^o - \nabla p_{\mathrm{m}} + \operatorname{div}\mathbb{S}_{\varepsilon,\mu,\lambda,\mathrm{m}} + \mathcal{F}_{\mathrm{m}}, \tag{2.1c}$$

where p_{m}, $S_{\varepsilon,\mu,\lambda,\mathrm{m}}$, and $\mathcal{F}_{\mathrm{m}}$ are defined as in (1.2b), (1.11a), and (1.2d), with h, A, and u replaced by h_{m}, A_{m}, and $\mathbf{u}^o$, respectively.

To solve the linear equation (2.1c) by, e.g., a Galerkin method, one will need to deal with the possible degeneracy of h_{m}. For this, we subsequently show that for $\mathbf{u}^o$ smooth enough, with appropriate initial data, h_{m} and A_{m} satisfy certain non-degeneracy property.

2.2 Non-negativity and uniform bound of A_{m}: $0 \leq A_{\mathrm{m}} \leq 1$

In this subsection, we show that $0 \leq A_{\mathrm{m}} \leq 1$ for a smooth enough $\mathbf{u}^o$. In fact, we only require that

$$\operatorname{div}\mathbf{u}^o \in L^1(0, T; L^\infty(\Omega)), \tag{2.2}$$

for some $T > 0$.

Non-negativity of A_{m}:

Taking the L^2-inner product of (2.1b) with $\left(-A_{\mathrm{m}}^-\right)$ leads to, after applying integration by parts in the resultant

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|A_{\mathrm{m}}^-\|_{L^2(\Omega)}^2 = \int\left(\frac{1}{2} - \frac{(1-A_{\mathrm{m}})^+}{(1-A_{\mathrm{m}})^+ + \omega}\right)\operatorname{div}\mathbf{u}^o\left|A_{\mathrm{m}}^-\right|^2 dx \\ + \int \underbrace{\mathcal{S}_{A_{\mathrm{m}},\omega,\nu}(-A_{\mathrm{m}}^-)}_{\leq 0}\, dx \lesssim \|\operatorname{div}\mathbf{u}^o\|_{L^\infty(\Omega)}\|A_{\mathrm{m}}^-\|_{L^2(\Omega)}^2.\end{aligned} \tag{2.3}$$

Therefore, applying Grönwall's inequality to (2.3) yields

$$\|A_{\mathrm{m}}^-\|_{L^2(\Omega)}^2 \leq e^{C\int_0^t \|\operatorname{div}\mathbf{u}^o(s)\|_{L^\infty(\Omega)}\, ds}\|A_{\mathrm{in}}^-\|_{L^2(\Omega)}^2 = 0,$$

which implies

$$A_{\mathrm{m}} \geq 0.$$

Non-negativity of $1 - A_{\mathrm{m}}$:

Consider the following equation for $1 - A_{\mathrm{m}}$, derived from (2.1b):

$$\partial_t(1-A_{\mathrm{m}}) = -\mathcal{S}_{A_{\mathrm{m}},\omega,\nu} - \mathbf{u}^o\cdot\nabla(1-A_{\mathrm{m}}) + A_{\mathrm{m}}\operatorname{div}\mathbf{u}^o\frac{(1-A_{\mathrm{m}})^+}{(1-A_{\mathrm{m}})^+ + \omega}. \tag{2.4}$$

As before, taking the L^2-inner product of (2.4) with $\left[-(1-A_{\mathrm{m}})^-\right]$, after applying integration by parts, leads to

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|(1-A_{\mathrm{m}})^-\|_{L^2(\Omega)}^2 = \int\left(\frac{1}{2}\operatorname{div}\mathbf{u}^o\left|(1-A_{\mathrm{m}})^-\right|^2 + \underbrace{\mathcal{S}_{A_{\mathrm{m}},\omega,\mu}(1-A_{\mathrm{m}})^-}_{\leq 0}\right) dx \\ - \int \underbrace{A_{\mathrm{m}}\operatorname{div}\mathbf{u}^o\frac{(1-A_{\mathrm{m}})^+}{(1-A_{\mathrm{m}})^+ + \omega}(1-A_{\mathrm{m}})^-}_{=0}\, dx,\end{aligned}$$

which yields

$$\frac{d}{dt}\|(1-A_{\mathrm{m}})^-\|_{L^2(\Omega)}^2 \lesssim \|\operatorname{div}\mathbf{u}^o\|_{L^\infty(\Omega)}\|(1-A_{\mathrm{m}})^-\|_{L^2(\Omega)}^2. \tag{2.5}$$

Then as before, after applying Grönwall's inequality to (2.5), one can conclude

$$A_{\mathrm{m}} \leq 1.$$

2.3 Non-negativity, lower and upper bounds of h_{m}

Let $\underline{h}, \overline{h} \in [0,\infty)$ be the lower and upper bounds of h_{in}, respectively, i.e., $-0 \leq \underline{h} \leq h_{\mathrm{in}} \leq \overline{h} < \infty$ (see (1.6)). In this section, we will show that

$$\frac{1}{4}\underline{h} \leq h_{\mathrm{m}} \leq 4\overline{h}$$

locally in time. Again we assume that $\mathbf{u}^o$ has the regularity (2.2).

Non-negativity of h_{m}:

After applying the L^2-inner product of (2.1a) with $\left(-h_{\mathrm{m}}^{-}\right)$ and applying integration by parts in the resultant, one has

$$\frac{1}{2}\frac{d}{dt}\|h_{\mathrm{m}}^{-}\|_{L^2(\Omega)}^2 = -\frac{1}{2}\int \left|h_{\mathrm{m}}^{-}\right|^2 \operatorname{div}\mathbf{u}^o\,dx \lesssim \|\operatorname{div}\mathbf{u}^o\|_{L^\infty(\Omega)}\|h_{\mathrm{m}}^{-}\|_{L^2(\Omega)}^2, \tag{2.6}$$

since the term $\mathcal{S}_{h_{\mathrm{m}},\omega,\nu}(-h_{\mathrm{m}}^{-})$ vanishes. Therefore, applying Grönwall's inequality to (2.6), as before in (2.3), eventually implies

$$h_{\mathrm{m}} \geq 0.$$

Lower and upper bounds of h_{m}:

Since $A_{\mathrm{m}} \in [0,1]$, one has $\left|\mathcal{S}_{h_{\mathrm{m}},\omega,\nu}\right| \leq 3(\left|\overline{f}\right| + \left|\underline{f}\right|)$. Then following the characteristic method, since $h_{\mathrm{m}} \geq 0$, one has

$$\begin{aligned}\partial_t(e^{-\int_0^t \|\operatorname{div}\mathbf{u}^o\|_{L^\infty(\Omega)}(s)\,ds}h_{\mathrm{m}}) + \mathbf{u}^o\cdot\nabla(e^{-\int_0^t \|\operatorname{div}\mathbf{u}^o\|_{L^\infty(\Omega)}(s)\,ds}h_{\mathrm{m}})\\ \leq 3(\left|\overline{f}\right| + \left|\underline{f}\right|)e^{-\int_0^t \|\operatorname{div}\mathbf{u}^o\|_{L^\infty(\Omega)}(s)\,ds}.\end{aligned}$$

Thus, integrating in the above inequation along the characteristic path given by $\mathbf{u}^o$ yields

$$h_{\mathrm{m}}(\mathbf{x},t) \leq \left(\overline{h} + 3(\left|\overline{f}\right| + \left|\underline{f}\right|)t\right) \times e^{\int_0^t \|\operatorname{div}\mathbf{u}^o(s)\|_{L^\infty(\Omega)}\,ds}. \tag{2.7}$$

Similarly, one can show that

$$h_{\mathrm{m}}(\mathbf{x},t) \geq \left(\underline{h} - 3(\left|\overline{f}\right| + \left|\underline{f}\right|)t\right) \times e^{-\int_0^t \|\operatorname{div}\mathbf{u}^o(s)\|_{L^\infty(\Omega)}\,ds}. \tag{2.8}$$

Then it immediately follows that $\frac{1}{4}\underline{h} \leq h_{\mathrm{m}} \leq 4\overline{h}$ provided that the following conditions are satisfied:

$$0 < t \leq \begin{cases} \dfrac{\underline{h}}{6(\left|\overline{f}\right| + \left|\underline{f}\right|)} & \text{if } \underline{h} > 0,\\[2ex] \dfrac{\overline{h}}{3(\left|\overline{f}\right| + \left|\underline{f}\right|)} & \text{if } \underline{h} = 0,\end{cases} \tag{2.9}$$

$$\text{and}\qquad e^{\int_0^t \|\operatorname{div}\mathbf{u}^o(s)\|_{L^\infty(\Omega)}\,ds} \leq e^{t^{1/2}\left(\int_0^t \|\mathbf{u}^o(s)\|_{H^3(\Omega)}^2\,ds\right)^{1/2}} \leq 2.$$

2.4 Non-vanishing total ice mass

Due to the fact that $\left|\mathcal{S}_{h_{\mathrm{m}},\omega,\nu}\right| \leq 3(\left|\overline{f}\right| + \left|\underline{f}\right|)$, one can show immediately after integrating (2.1a), that

$$\frac{d}{dt}\int h_{\mathrm{m}}\,dx \leq 3(\left|\overline{f}\right| + \left|\underline{f}\right|)\left|\Omega\right|.$$

Therefore,

$$\frac{1}{2}\int h_{\mathrm{in}}\,dx \leq \int h_{\mathrm{m}}\,dx \leq 2\int h_{\mathrm{in}}\,dx, \tag{2.10}$$

provided

$$t \leq \frac{\int h_{\mathrm{in}}\,dx}{6(\left|\overline{f}\right| + \left|\underline{f}\right|)\left|\Omega\right|}. \tag{2.11}$$

2.5 Well-posedness of (2.1c) with strictly positive ice mass

Consider $h_{\mathrm{in}} \geq \underline{h} > 0$. Then we have shown in section 2.3 that $h_{\mathrm{m}} \geq \underline{h}/4 > 0$ locally in time. Then during this local time, (2.1c) is a non-degenerate biharmonic evolutionary equation. Then following the standard Galerkin method, one can establish the well-posedness of strong solutions to (2.1c), provided that $\mathbf{u}^o$ is sufficiently smooth. We omit the details here.

3 Well-posedness of solutions to (1.10) with $\underline{h} > 0$ and $\iota > 0$ fixed

In this section, we aim at showing that the map defined by

$$\mathfrak{M} : \mathbf{u}^o \mapsto \mathbf{u}_{\mathrm{m}}, \tag{3.1}$$

where $\mathbf{u}_{\mathrm{m}}$ is the unique solution to (2.1c) with h_{m} and A_{m} being solutions to (2.1a) and (2.1b), respectively, is bounded in $\mathfrak{X}_{T^*}$ and contracting with contraction constant $1/2$ in $L^\infty(0,T^*;L^2(\Omega)) \cap L^2(0,T^*;H^2(\Omega))$, where

$$\begin{aligned} \mathfrak{X}_{T^*} := \big\{\mathbf{u} | \mathbf{u} \in L^\infty(0,T^*;H^2(\Omega)) \cap L^2(0,T^*;H^3(\Omega)), \\ \partial_t \mathbf{u}, \nabla^4 \mathbf{u} \in L^2(0,T^*;L^2(\Omega))\big\}, \end{aligned} \tag{3.2}$$

for some $T^* \in (0,\infty)$ to be determined. Throughout this section, unless stated otherwise, the initial data for $\mathbf{u}_{\mathrm{m}}, \mathbf{u}^o, h,$ and A are assumed to be $\mathbf{u}_{\mathrm{in}}, \mathbf{u}_{\mathrm{in}}, h_{\mathrm{in}},$ and A_{in}, given in Theorem 1.1, respectively.

Consequently, one can apply the Banach fixed-point theorem, i.e., the contraction mapping theorem, to show the existence of solutions to system (1.10).

Let $\mathfrak{c}_{\mathrm{in}} \in (0,\infty)$ be the bound of the initial data defined by

$$\|\nabla h_{\mathrm{in}}, \nabla A_{\mathrm{in}}\|_{L^4(\Omega)} + \|\mathbf{u}_{\mathrm{in}}\|_{H^2(\Omega)} \leq \mathfrak{c}_{\mathrm{in}} \tag{3.3}$$

3.1 Uniform bounds

Let $\mathbf{u}^o \in \mathfrak{X}_{T^*}$ satisfy

$$\sup_{0 \leq s \leq t} \|\mathbf{u}^o(s)\|^2_{H^2(\Omega)} + \int_0^t \big(\|\partial_t \mathbf{u}^o(s)\|^2_{L^2(\Omega)} + \|\mathbf{u}^o(s)\|^2_{H^3(\Omega)}\big)\, ds \leq \mathfrak{c}_o, \tag{3.4}$$

with $t \in [0,T^*]$, for some $\mathfrak{c}_o \in (0,\infty)$ to be determined later.

Estimates for h_{m} and A_{m} Aside from the point-wise estimates deduced in Sections 2.2 and 2.3, we shall need a uniform H^1-estimate for A_{m} and h_{m}.

We record the equation after applying $\partial \in \{\partial_x, \partial_y\}$ to (2.1a), as follows:

$$\partial_t \partial h_{\mathrm{m}} + \mathbf{u}^o \cdot \nabla \partial h_{\mathrm{m}} + \partial \mathbf{u}^o \cdot \nabla h_{\mathrm{m}} + \partial h_{\mathrm{m}} \mathrm{div}\, \mathbf{u}^o + h_{\mathrm{m}} \mathrm{div}\, \partial \mathbf{u}^o = \partial \mathcal{S}_{h_{\mathrm{m}},\omega,\nu}. \tag{3.5}$$

Then taking the L^2-inner product of (3.5) with $4\big|\partial h_{\mathrm{m}}\big|^2\partial h_{\mathrm{m}}$ leads to, after applying integration by parts,

$$
\begin{aligned}
&\frac{d}{dt}\|\partial h_{\mathrm{m}}\|_{L^4(\Omega)}^4 = -3\int \operatorname{div}\mathbf{u}^o\big|\partial h_{\mathrm{m}}\big|^4\,dx\\
&-4\int\big(\partial\mathbf{u}^o\cdot\nabla h_{\mathrm{m}}+h_{\mathrm{m}}\operatorname{div}\partial\mathbf{u}^o\big)\big|\partial h_{\mathrm{m}}\big|^2\partial h_{\mathrm{m}}\,dx+4\int\partial\mathcal{S}_{h_{\mathrm{m}},\omega,\nu}\big|\partial h_{\mathrm{m}}\big|^2\partial h_{\mathrm{m}}\,dx\\
&\lesssim\|\nabla\mathbf{u}^o\|_{L^\infty(\Omega)}\|\nabla h_{\mathrm{m}}\|_{L^4(\Omega)}^4+\|h_{\mathrm{m}}\|_{L^\infty(\Omega)}\|\nabla^2\mathbf{u}^o\|_{L^4(\Omega)}\|\nabla h_{\mathrm{m}}\|_{L^4(\Omega)}^3\\
&\quad+\int\big|\partial\mathcal{S}_{h_{\mathrm{m}},\omega,\nu}\big|\big|\partial h_{\mathrm{m}}\big|^2\partial h_{\mathrm{m}}\,dx.
\end{aligned}
\tag{3.6}
$$

Meanwhile, simple calculation shows that

$$
\big|\partial\mathcal{S}_{h_{\mathrm{m}},\omega,\nu}\big|\lesssim\big(\frac{1}{\omega}+\frac{1}{\nu^{1/2}}\big)\big|\partial h_{\mathrm{m}}\big|+\big(1+\frac{|h_{\mathrm{m}}|}{\omega^2}\big)\big|\partial A_{\mathrm{m}}\big|,
$$

where we have used (1.7). Consequently, one concludes from (3.6) that

$$
\begin{aligned}
&\frac{d}{dt}\|\nabla h_{\mathrm{m}}\|_{L^4(\Omega)}^4\lesssim\big(\|\nabla\mathbf{u}^o\|_{L^\infty(\Omega)}+\frac{1}{\omega}+\frac{1}{\nu}\big)\|\nabla h_{\mathrm{m}}\|_{L^4(\Omega)}^4\\
&+\big(1+\frac{\|h_{\mathrm{m}}\|_{L^\infty(\Omega)}}{\omega^2}\big)\|\nabla A_{\mathrm{m}}\|_{L^4(\Omega)}\|\nabla h_{\mathrm{m}}\|_{L^4(\Omega)}^3\\
&+\|h_{\mathrm{m}}\|_{L^\infty(\Omega)}\|\nabla^2\mathbf{u}^o\|_{L^4(\Omega)}\|\nabla h_{\mathrm{m}}\|_{L^4(\Omega)}^3.
\end{aligned}
\tag{3.7}
$$

The estimate for ∇A_{m} is obtained from (2.1b) in a similar fashion, we record it here:

$$
\begin{aligned}
&\frac{d}{dt}\|\nabla A_{\mathrm{m}}\|_{L^4(\Omega)}^4\lesssim\bigg(\|\nabla\mathbf{u}^o\|_{L^\infty(\Omega)}+\frac{\|\nabla\mathbf{u}^o\|_{L^\infty(\Omega)}}{\omega}\\
&\qquad+1+\frac{1}{\nu}+\frac{\|h_{\mathrm{m}}\|_{L^\infty(\Omega)}}{\omega^2}\bigg)\|\nabla A_{\mathrm{m}}\|_{L^4(\Omega)}^4\\
&+\big(\frac{1}{\nu^2}+\frac{1}{\omega^2}\big)\|\nabla h_{\mathrm{m}}\|_{L^4(\Omega)}\|\nabla A_{\mathrm{m}}\|_{L^4(\Omega)}^3\\
&+\|\nabla^2\mathbf{u}^o\|_{L^4(\Omega)}\|\nabla A_{\mathrm{m}}\|_{L^4(\Omega)}^3,
\end{aligned}
\tag{3.8}
$$

where we have used the fact that $A_{\mathrm{m}}\in[0,1]$.

After combining (3.7) and (3.8) and applying Grönwall's inequality, one can derive that

$$
\sup_{0\le s\le t}\|\nabla h_{\mathrm{m}}(s),\nabla A_{\mathrm{m}}(s)\|_{L^4(\Omega)}^4\le e^{H_{h,A,1}(t)}\big(\|\nabla h_{\mathrm{in}},\nabla A_{\mathrm{in}}\|_{L^4(\Omega)}^4+G_{h,A,1}(t)\big),
\tag{3.9}
$$

where

$$
\begin{aligned}
H_{h,A,1}(t)&:=C_{\omega,\nu}\int_0^t\big(1+\|\nabla\mathbf{u}^o(s)\|_{L^\infty(\Omega)}+\|h_{\mathrm{m}}(s)\|_{L^\infty(\Omega)}\\
&\quad+\|\nabla^2\mathbf{u}^o(s)\|_{L^4(\Omega)}+\|h_{\mathrm{m}}(s)\|_{L^\infty(\Omega)}\|\nabla^2\mathbf{u}^o(s)\|_{L^4(\Omega)}\big)\,ds,
\end{aligned}
\tag{3.10}
$$

$$
G_{h,A,1}(t):=\int_0^t\big(1+\|h_{\mathrm{m}}(s)\|_{L^\infty(\Omega)}+\|\nabla^2\mathbf{u}^o(s)\|_{L^4(\Omega)}\big)\,ds.
\tag{3.11}
$$

On the other hand, in direct consequence of equations (2.1a) and (2.1b), one has

$$
\begin{aligned}
&\|\partial_t h_{\mathrm{m}},\partial_t A_{\mathrm{m}}\|_{L^4(\Omega)}\le C\big(1+1/\nu+\|\nabla\mathbf{u}^o\|_{L^4(\Omega)}\\
&+\|h_{\mathrm{m}}\|_{L^\infty(\Omega)}\|\nabla\mathbf{u}^o\|_{L^4(\Omega)}+\|\mathbf{u}^o\|_{L^\infty(\Omega)}\|\nabla h_{\mathrm{m}},\nabla A_{\mathrm{m}}\|_{L^4(\Omega)}\big),
\end{aligned}
\tag{3.12}
$$

where we have used the fact that $0 \le A_{\rm m} \le 1$ and (1.7).

Estimates for $\mathbf{u}_{\rm m}$

Taking the L^2-inner product of (2.1c) with $2\mathbf{u}_{\rm m} + 2\partial_t \mathbf{u}_{\rm m} - 2\Delta \mathbf{u}_{\rm m}$ leads to, after applying integration by parts,

$$\begin{aligned}
&\frac{d}{dt}\|\rho_{\rm ice}^{1/2} h_{\rm m}^{1/2}\mathbf{u}_{\rm m}, \iota^{1/2}\nabla^2\mathbf{u}_{\rm m}, \rho_{\rm ice}^{1/2} h_{\rm m}^{1/2}\nabla\mathbf{u}_{\rm m}\|_{L^2(\Omega)}^2 \\
&\qquad + 2\|\rho_{\rm ice}^{1/2} h_{\rm m}^{1/2}\partial_t\mathbf{u}_{\rm m}, \iota^{1/2}\nabla^2\mathbf{u}_{\rm m}, \iota^{1/2}\nabla^3\mathbf{u}_{\rm m}\|_{L^2(\Omega)}^2 \\
&= \underbrace{\int \rho_{\rm ice}\partial_t h_{\rm m}|\mathbf{u}_{\rm m}|^2\,dx}_{\mathcal{R}_1} + \underbrace{2\int \rho_{\rm ice}(\nabla h_{\rm m}\cdot\nabla)\mathbf{u}_{\rm m}\cdot\partial_t\mathbf{u}_{\rm m}\,dx}_{\mathcal{R}_2} \\
&\underbrace{-\int \rho_{\rm ice}\partial_t h_{\rm m}|\nabla\mathbf{u}_{\rm m}|^2\,dx}_{\mathcal{R}_3} \underbrace{-2\int \rho_{\rm ice} h_{\rm m}(\mathbf{u}^o\cdot\nabla)\mathbf{u}^o\cdot(\mathbf{u}_{\rm m}+\partial_t\mathbf{u}_{\rm m}-\Delta\mathbf{u}_{\rm m})\,dx}_{\mathcal{R}_4} \\
&\underbrace{-2\int \nabla p_{\rm m}\cdot(\mathbf{u}_{\rm m}+\partial_t\mathbf{u}_{\rm m}-\Delta\mathbf{u}_{\rm m})\,dx}_{\mathcal{R}_5} + \underbrace{2\int \mathcal{F}_{\rm m}\cdot(\mathbf{u}_{\rm m}+\partial_t\mathbf{u}_{\rm m}-\Delta\mathbf{u}_{\rm m})\,dx}_{\mathcal{R}_6} \\
&+ \underbrace{2\int \operatorname{div}\mathbb{S}_{\varepsilon,\mu,\lambda,{\rm m}}\cdot(\mathbf{u}_{\rm m}+\partial_t\mathbf{u}_{\rm m}-\Delta\mathbf{u}_{\rm m})\,dx}_{\mathcal{R}_7}.
\end{aligned} \tag{3.13}$$

We obtain the following estimates for the $\mathcal{R}_j$ terms by applying Hölder's inequality and the Sobolev embedding inequality:

$$\begin{aligned}
&\mathcal{R}_1 \lesssim \|\partial_t h_{\rm m}\|_{L^2(\Omega)}\|\mathbf{u}_{\rm m}\|_{L^4(\Omega)}^2, \\
&\mathcal{R}_2 \lesssim \|\partial_t \mathbf{u}_{\rm m}\|_{L^2(\Omega)}\|\nabla\mathbf{u}_{\rm m}\|_{L^4(\Omega)}\|\nabla h_{\rm m}\|_{L^4(\Omega)}, \\
&\mathcal{R}_3 \lesssim \|\partial_t h_{\rm m}\|_{L^2(\Omega)}\|\nabla\mathbf{u}_{\rm m}\|_{L^4(\Omega)}^2, \\
&\mathcal{R}_4 \lesssim \|h_{\rm m}\|_{L^\infty(\Omega)}\|\mathbf{u}^o\|_{L^4(\Omega)}\|\nabla\mathbf{u}^o\|_{L^4(\Omega)}\|\partial_t\mathbf{u}_{\rm m}, \mathbf{u}_{\rm m}, \nabla^2\mathbf{u}_{\rm m}\|_{L^2(\Omega)}, \\
&\mathcal{R}_5 \lesssim \|\nabla p_{\rm m}\|_{L^2(\Omega)}\|\partial_t\mathbf{u}_{\rm m}, \mathbf{u}_{\rm m}, \nabla^2\mathbf{u}_{\rm m}\|_{L^2(\Omega)}, \\
&\mathcal{R}_6 \lesssim \big(1 + \|\mathbf{u}^o\|_{L^2(\Omega)} + \|h_{\rm m}\|_{L^\infty(\Omega)}\|\mathbf{u}^o\|_{L^2(\Omega)}\big)\|\partial_t\mathbf{u}_{\rm m}, \mathbf{u}_{\rm m}, \nabla^2\mathbf{u}_{\rm m}\|_{L^2(\Omega)}, \\
&\mathcal{R}_7 \lesssim \big(\frac{1}{\varepsilon}\|p_{\rm m}\|_{L^\infty(\Omega)}\|\nabla^2\mathbf{u}^o\|_{L^2(\Omega)} + (\mu+\lambda)\|\nabla^2\mathbf{u}^o\|_{L^2(\Omega)} + \|\nabla p_{\rm m}\|_{L^2(\Omega)}\big) \\
&\qquad\qquad \times \|\partial_t\mathbf{u}_{\rm m}, \mathbf{u}_{\rm m}, \nabla^2\mathbf{u}_{\rm m}\|_{L^2(\Omega)}.
\end{aligned}$$

To deduce the above estimates, consider $\underline{h} > 0$ and let t satisfy (2.9) and (2.11). Therefore, the estimates in Section 2.3 guarantee that $0 < 1/4\underline{h} \le h_{\rm m} \le 4\overline{h} < \infty$. Consequently, (3.13) yields, after applying the Sobolve embedding inequality and Hölder's inequality,

$$\begin{aligned}
&\frac{d}{dt}\|\rho_{\rm ice}^{1/2} h_{\rm m}^{1/2}\mathbf{u}_{\rm m}, \rho_{\rm ice}^{1/2} h_{\rm m}^{1/2}\nabla\mathbf{u}_{\rm m}, \iota^{1/2}\nabla^2\mathbf{u}_{\rm m}\|_{L^2(\Omega)}^2 \\
&\qquad + 2\|\rho_{\rm ice}^{1/2} h_{\rm m}^{1/2}\partial_t\mathbf{u}_{\rm m}, \iota^{1/2}\nabla^2\mathbf{u}_{\rm m}, \iota^{1/2}\nabla^3\mathbf{u}_{\rm m}\|_{L^2(\Omega)}^2 \\
&\le C_{\varepsilon,\mu,\lambda,\underline{h},\overline{h}}\big(\|\partial_t h_{\rm m}\|_{L^4(\Omega)} + \|\nabla h_{\rm m}, \nabla A_{\rm m}\|_{L^4(\Omega)}^2\big) \\
&\quad \times \big(\|\rho_{\rm ice}^{1/2} h_{\rm m}^{1/2}\mathbf{u}_{\rm m}, \rho_{\rm ice}^{1/2} h_{\rm m}^{1/2}\nabla\mathbf{u}_{\rm m}, \iota^{1/2}\nabla^2\mathbf{u}_{\rm m}\|_{L^2(\Omega)}^4 + 1\big).
\end{aligned} \tag{3.14}$$

Furthermore, consider t small enough such that

$$
\begin{aligned}
H_{h,A,1}(t) + G_{h,A,1}(t) &\leq C_{\omega,\nu,\overline{h}} t^{1/2} \big(t^{1/2} + \big(\int_0^t \|\nabla \mathbf{u}^o(s)\|^2_{H^2(\Omega)} \, ds \big)^{1/2} \big) \\
&\leq C_{\omega,\nu,\overline{h}} t^{1/2} \big(t^{1/2} + \mathfrak{c}_o^{1/2} \big) \leq 1,
\end{aligned}
\tag{3.15}
$$

where we have applied Hölder's inequality. Then (3.9) and (3.12) imply that, after applying the Sobolev embedding inequality,

$$
\|\nabla h_{\mathrm{m}}, \nabla A_{\mathrm{m}}, \partial_t h_{\mathrm{m}}, \partial_t A_{\mathrm{m}}\|_{L^4(\Omega)} \leq C_{\omega,\nu,\overline{h},\mathfrak{c}_{\mathrm{in}}} \big(1 + \mathfrak{c}_o^{1/2} \big). \tag{3.16}
$$

Consequently, (3.14) yields the following estimate:

$$
\begin{aligned}
&\sup_{0 \leq s \leq t} \|\mathbf{u}_{\mathrm{m}}(s)\|^2_{H^2(\Omega)} + \int_0^t \big(\|\partial_t \mathbf{u}_{\mathrm{m}}(s)\|^2_{L^2(\Omega)} + \|\mathbf{u}_{\mathrm{m}}(s)\|^2_{H^3(\Omega)} \big) \, ds \\
&\leq C_{\varepsilon,\iota,\mu,\lambda,\omega,\nu,\underline{h},\overline{h},\mathfrak{c}_{\mathrm{in}}} \Big[\big(\frac{C_{\varepsilon,\mu,\lambda,\omega,\nu,\underline{h},\overline{h},\mathfrak{c}_{\mathrm{in}},1}}{C_{\varepsilon,\mu,\lambda,\omega,\nu,\underline{h},\overline{h},\mathfrak{c}_{\mathrm{in}},2} - (1+\mathfrak{c}_o)t} - 1 \big)^2 \big(1 + (1+\mathfrak{c}_o)t \big) + 1 \Big] \\
&\leq C_{\varepsilon,\iota,\mu,\lambda,\omega,\nu,\underline{h},\overline{h},\mathfrak{c}_{\mathrm{in}}} \Big[2 \big(2 \frac{C_{\varepsilon,\mu,\lambda,\omega,\nu,\underline{h},\overline{h},\mathfrak{c}_{\mathrm{in}},1}}{C_{\varepsilon,\mu,\lambda,\omega,\nu,\underline{h},\overline{h},\mathfrak{c}_{\mathrm{in}},2}} - 1 \big)^2 + 1 \Big],
\end{aligned}
\tag{3.17}
$$

provided that t is small enough and where we have made the choice

$$
\mathfrak{c}_o := C_{\varepsilon,\iota,\mu,\lambda,\omega,\nu,\underline{h},\overline{h},\mathfrak{c}_{\mathrm{in}}} \Big[2 \big(2 \frac{C_{\varepsilon,\mu,\lambda,\omega,\nu,\underline{h},\overline{h},\mathfrak{c}_{\mathrm{in}},1}}{C_{\varepsilon,\mu,\lambda,\omega,\nu,\underline{h},\overline{h},\mathfrak{c}_{\mathrm{in}},2}} - 1 \big)^2 + 1 \Big], \tag{3.18}
$$

where the right-hand side is as in (3.17). Then (2.9), (2.11), (3.15), and (3.17) imply that, there exists $T^* \in (0,\infty)$ such that

$$
\sup_{0 \leq s \leq t} \|\mathbf{u}_{\mathrm{m}}(s)\|^2_{H^2(\Omega)} + \int_0^t \big(\|\partial_t \mathbf{u}_{\mathrm{m}}(s)\|^2_{L^2(\Omega)} + \|\mathbf{u}_{\mathrm{m}}(s)\|^2_{H^3(\Omega)} \big) \, ds \leq \mathfrak{c}_o, \tag{3.19a}
$$

and

$$
\frac{1}{4}\underline{h} \leq h_{\mathrm{m}} \leq 4\overline{h}, \ \ \frac{1}{2} \int h_{\mathrm{in}} \, dx \leq \int h_{\mathrm{m}} \, dx \leq 2 \int h_{\mathrm{in}} \, dx, \tag{3.19b}
$$

for $t \in [0, T^*]$. In addition, using equation (1.10), it is easy to obtain

$$
\int_0^t \|\Delta^2 \mathbf{u}_{\mathrm{m}}(s)\|^2_{L^2(\Omega)} \, ds \leq C_{\overline{h},\mu,\lambda,\varepsilon} \mathfrak{c}_o. \tag{3.20}
$$

Therefore, $\mathfrak{M}$, defined in (3.1), maps $\mathfrak{X}_{T^*}$ into itself for such choices of T^* and $\mathfrak{c}_o$.

We remark here that, $\mathfrak{c}_0 \to \infty$ as $\iota \to 0^+$, i.e., the estimates we obtain here depend on $\iota > 0$. We will remove the dependency of ι in Section 4.

3.2 Contraction mapping and well-posedness

For $j = 1, 2$, consider $\mathbf{u}_j^o \in \mathfrak{X}_{T^*}$ satisfying (3.4), and let $h_{\mathrm{m},j}, A_{\mathrm{m},j}$, and $\mathbf{u}_{\mathrm{m},j} = \mathfrak{M}(\mathbf{u}_j^o)$, be the solutions to (2.1a), (2.1b), and (2.1c), respectively, with $\mathbf{u}^o$ replaced by $\mathbf{u}_j^o$ and with the same initial data. Then we have the estimates of $h_{\mathrm{m},j}, A_{\mathrm{m},j}$, and $\mathbf{u}_{\mathrm{m},j}$ as in Sections 2.2 and 2.3, as well as (3.16) and (3.19a).

In the following, let $\sigma \in (0,1)$ be a constant to be determined later. Denote by

$$\begin{aligned} &\delta h_{\mathrm{m}} := h_{\mathrm{m},1} - h_{\mathrm{m},2}, \quad \delta A_{\mathrm{m}} := A_{\mathrm{m},1} - A_{\mathrm{m},2}, \\ &\delta \mathbf{u}_{\mathrm{m}} := \mathbf{u}_{\mathrm{m},1} - \mathbf{u}_{\mathrm{m},2}, \quad \delta \mathbf{u}^o := \mathbf{u}^o_1 - \mathbf{u}^o_2. \end{aligned} \tag{3.21}$$

The notations

$$\delta p_{\mathrm{m}}, \delta \mathbb{S}_{\varepsilon,\mu,\lambda,\mathrm{m}}, \delta \mathcal{F}_{\mathrm{m}}, \delta \mathcal{S}_{h_{\mathrm{m}},\omega,\nu}, \delta \mathcal{S}_{A_{\mathrm{m}},\omega,\nu}, \delta \chi^{\omega}_{A_{\mathrm{m}}},$$

have similar meanings. Then $\delta h_{\mathrm{m}}, \delta A_{\mathrm{m}}, \delta \mathbf{u}_{\mathrm{m}}$ satisfy

$$\partial_t \delta h_{\mathrm{m}} + \operatorname{div}(\delta h_{\mathrm{m}} \mathbf{u}^o_1) + \operatorname{div}(h_{\mathrm{m},2} \delta \mathbf{u}^o) = \delta \mathcal{S}_{h_{\mathrm{m}},\omega,\nu}, \tag{3.22a}$$

$$\begin{aligned} &\partial_t \delta A_{\mathrm{m}} + \operatorname{div}(\delta A_{\mathrm{m}} \mathbf{u}^o_1) + \operatorname{div}(A_{\mathrm{m},2} \delta \mathbf{u}^o) = \delta \mathcal{S}_{A_{\mathrm{m}},\omega,\nu} \\ &+ \delta A_{\mathrm{m}} \operatorname{div} \mathbf{u}^o_1 \cdot \chi^{\omega}_{A_{\mathrm{m},1}} + A_{\mathrm{m},2} \operatorname{div} \delta \mathbf{u}^o \cdot \chi^{\omega}_{A_{\mathrm{m},1}} + A_{\mathrm{m},2} \operatorname{div} \mathbf{u}^o_2 \cdot \delta \chi^{\omega}_{A_{\mathrm{m}}}, \end{aligned} \tag{3.22b}$$

$$\begin{aligned} &\rho_{\mathrm{ice}} h_{\mathrm{m},1} \partial_t \delta \mathbf{u}_{\mathrm{m}} + \rho_{\mathrm{ice}} \delta h_{\mathrm{m}} \partial_t \mathbf{u}_{\mathrm{m},2} + \iota \Delta^2 \delta \mathbf{u}_{\mathrm{m}} = -\rho_{\mathrm{ice}} h_{\mathrm{m},1} \mathbf{u}^o_1 \cdot \nabla \delta \mathbf{u}^o \\ &- \rho_{\mathrm{ice}} h_{\mathrm{m},1} \delta \mathbf{u}^o \cdot \nabla \mathbf{u}^o_2 - \rho_{\mathrm{ice}} \delta h_{\mathrm{m}} \mathbf{u}^o_2 \cdot \nabla \mathbf{u}^o_2 - \nabla \delta p_{\mathrm{m}} + \operatorname{div} \delta \mathbb{S}_{\varepsilon,\mu,\lambda,\mathrm{m}} + \delta \mathcal{F}_{\mathrm{m}}. \end{aligned} \tag{3.22c}$$

After taking the L^2-inner product of (3.22a) and (3.22b) with $4|\delta h_{\mathrm{m}}|^2 \delta h_{\mathrm{m}}$ and $4|\delta A_{\mathrm{m}}|^2 \delta A_{\mathrm{m}}$, respectively, and applying integration by parts in the resultant, one has

$$\begin{aligned} \frac{d}{dt} \|\delta h_{\mathrm{m}}, \delta A_{\mathrm{m}}\|^4_{L^4(\Omega)} &= \underbrace{-3 \int \operatorname{div} \mathbf{u}^o_1 (|\delta h_{\mathrm{m}}|^4 + |\delta A_{\mathrm{m}}|^4)\, dx}_{\mathcal{R}_8} \\ & \underbrace{-4 \int (\delta \mathbf{u}^o \cdot \nabla h_{\mathrm{m},2} |\delta h_{\mathrm{m}}|^2 \delta h_{\mathrm{m}} + \delta \mathbf{u}^o \cdot \nabla A_{\mathrm{m},2} |\delta A_{\mathrm{m}}|^2 \delta A_{\mathrm{m}})\, dx}_{\mathcal{R}_9} \\ & \underbrace{-4 \int (h_{\mathrm{m},2} \operatorname{div} \delta \mathbf{u}^o |\delta h_{\mathrm{m}}|^2 \delta h_{\mathrm{m}} + A_{\mathrm{m},2} \operatorname{div} \delta \mathbf{u}^o |\delta A_{\mathrm{m}}|^2 \delta A_{\mathrm{m}})\, dx}_{\mathcal{R}_{10}} \\ & + \underbrace{4 \int \operatorname{div} \mathbf{u}^o_1 |\delta A_{\mathrm{m}}|^4 \chi^{\omega}_{A_{\mathrm{m},1}}\, dx}_{\mathcal{R}_{11}} + \underbrace{4 \int A_{\mathrm{m},2} \operatorname{div} \delta \mathbf{u}^o |\delta A_{\mathrm{m}}|^2 \delta A_{\mathrm{m}} \chi^{\omega}_{A_{\mathrm{m},1}}\, dx}_{\mathcal{R}_{12}} \\ & + \underbrace{4 \int A_{\mathrm{m},2} \operatorname{div} \mathbf{u}^o_2 |\delta A_{\mathrm{m}}|^2 \delta A_{\mathrm{m}} \delta \chi^{\omega}_{A_{\mathrm{m}}}\, dx}_{\mathcal{R}_{13}} + \underbrace{4 \int \delta \mathcal{S}_{h_{\mathrm{m}},\omega,\nu} |\delta h_{\mathrm{m}}|^2 \delta h_{\mathrm{m}}\, dx}_{\mathcal{R}_{14}} \\ & + \underbrace{4 \int \delta \mathcal{S}_{A_{\mathrm{m}},\omega,\nu} |\delta A_{\mathrm{m}}|^2 \delta A_{\mathrm{m}}\, dx}_{\mathcal{R}_{15}}. \end{aligned} \tag{3.23}$$

In the following, we sketch the estimates of the $\mathcal{R}_j$ terms by applying Hölder's inequality and the Sobolev embedding inequality:

$$\begin{aligned} &\mathcal{R}_8 + \mathcal{R}_{11} + \mathcal{R}_{13} \lesssim \left(\|\operatorname{div} \mathbf{u}^o_1\|_{L^\infty(\Omega)} + (\frac{1}{\omega} + \frac{1}{\omega^2}) \|\operatorname{div} \mathbf{u}^o_2\|_{L^\infty(\Omega)} \right) \\ &\qquad \times \|\delta h_{\mathrm{m}}, \delta A_{\mathrm{m}}\|^4_{L^4(\Omega)}, \\ &\mathcal{R}_9 \lesssim \|\delta \mathbf{u}^o\|_{L^\infty(\Omega)} \|\nabla h_{\mathrm{m},2}, \nabla A_{\mathrm{m},2}\|_{L^4(\Omega)} \|\delta h_{\mathrm{m}}, \delta A_{\mathrm{m}}\|^3_{L^4(\Omega)}, \\ &\mathcal{R}_{10} + \mathcal{R}_{12} \lesssim (\overline{h} + 1) \|\operatorname{div} \delta \mathbf{u}^o\|_{L^4(\Omega)} \|\delta h_{\mathrm{m}}, \delta A_{\mathrm{m}}\|^3_{L^4(\Omega)}, \\ &\mathcal{R}_{14} + \mathcal{R}_{15} \lesssim C_{\overline{h},\omega,\nu} \|\delta h_{\mathrm{m}}, \delta A_{\mathrm{m}}\|^4_{L^4(\Omega)}, \end{aligned} \tag{3.24}$$

where we have used the identity

$$\delta\left(\frac{g}{g+\varepsilon}\right) = \frac{\delta g}{g_1+\varepsilon} - \frac{g_2\delta g}{(g_1+\varepsilon)(g_2+\varepsilon)}$$

for $g=(1-A_{\mathrm{m}})^+ = 1-A_{\mathrm{m}}$ in the estimate of $\delta\chi^{\omega}_{A_{\mathrm{m}}}$ in $\mathcal{R}_{13}$. In view of (3.23) and (3.24), one has

$$\frac{d}{dt}\|\delta h_{\mathrm{m}},\delta A_{\mathrm{m}}\|^2_{L^4(\Omega)} \le C_{\sigma,\varepsilon,\omega,\nu,\overline{h},\mathfrak{c}_o,\mathfrak{c}_{\mathrm{in}}}\|\delta h_{\mathrm{m}},\delta A_{\mathrm{m}}\|^2_{L^4(\Omega)} + \sigma\|\delta\mathbf{u}^o\|^2_{H^2(\Omega)}, \tag{3.25}$$

where we have used (3.4) and (3.9). Consequently, applying Grönwall's inequality to (3.25) yields

$$\begin{aligned}&\sup_{0\le s\le t}\|\delta h_{\mathrm{m}}(s),\delta A_{\mathrm{m}}(s)\|^2_{L^4(\Omega)}\\ &\le \sigma\left(\int_0^t \|\delta\mathbf{u}^o(s)\|^2_{H^2(\Omega)}\,ds\right)e^{C_{\sigma,\varepsilon,\omega,\nu,\overline{h},\mathfrak{c}_o,\mathfrak{c}_{\mathrm{in}}}(t+t^{1/2})},\end{aligned} \tag{3.26}$$

where we have also employed Young's inequality.

Taking the L^2-inner product of (3.22c) with $2\delta\mathbf{u}_{\mathrm{m}}$ and applying integration by parts in the resultant yields

$$\begin{aligned}\rho_{\mathrm{ice}}\frac{d}{dt}\|h^{1/2}_{\mathrm{m},1}\delta\mathbf{u}_{\mathrm{m}}\|^2_{L^2(\Omega)} + 2\iota\|\nabla^2\delta\mathbf{u}_{\mathrm{m}}\|^2_{L^2(\Omega)} &= \underbrace{\rho_{\mathrm{ice}}\int\partial_t h_{\mathrm{m},1}\left|\delta\mathbf{u}_{\mathrm{m}}\right|^2dx}_{\mathcal{R}_{16}}\\ &\underbrace{-2\int\rho_{\mathrm{ice}}\delta h_{\mathrm{m}}\partial_t\mathbf{u}_{\mathrm{m},2}\cdot\delta\mathbf{u}_{\mathrm{m}}\,dx}_{\mathcal{R}_{17}}\underbrace{-2\int\rho_{\mathrm{ice}}h_{\mathrm{m},1}(\mathbf{u}^o_1\cdot\nabla)\delta\mathbf{u}^o\cdot\delta\mathbf{u}_{\mathrm{m}}\,dx}_{\mathcal{R}_{18}}\\ &\underbrace{-2\int\rho_{\mathrm{ice}}h_{\mathrm{m},1}(\delta\mathbf{u}^o\cdot\nabla)\mathbf{u}^o_2\cdot\delta\mathbf{u}_{\mathrm{m}}\,dx}_{\mathcal{R}_{19}}\underbrace{-2\int\rho_{\mathrm{ice}}\delta h_{\mathrm{m}}(\mathbf{u}^o_2\cdot\nabla)\mathbf{u}^o_2\cdot\delta\mathbf{u}_{\mathrm{m}}\,dx}_{\mathcal{R}_{20}}\\ &+\underbrace{2\int\delta p_{\mathrm{m}}\mathrm{div}\,\delta\mathbf{u}_{\mathrm{m}}\,dx}_{\mathcal{R}_{21}}+\underbrace{2\int\mathrm{div}\,\delta\mathbb{S}_{\varepsilon,\mu,\lambda,\mathrm{m}}\cdot\delta\mathbf{u}_{\mathrm{m}}\,dx}_{\mathcal{R}_{22}}\\ &+\underbrace{2\int\delta\mathcal{F}_{\mathrm{m}}\cdot\delta\mathbf{u}_{\mathrm{m}}\,dx}_{\mathcal{R}_{23}}.\end{aligned} \tag{3.27}$$

In the following, again, we sketch the estimates for the terms $\mathcal{R}_j$ by applying Hölder's inequality, the Gagliardo-Nirenberg inequality, and the Sobolev embedding inequality:

$$\begin{aligned}\mathcal{R}_{16} &\lesssim \|\partial_t h_{\mathrm{m},1}\|_{L^4(\Omega)}\|\delta\mathbf{u}_{\mathrm{m}}\|^{3/2}_{L^2(\Omega)}\|\delta\mathbf{u}_{\mathrm{m}}\|^{1/2}_{H^1(\Omega)},\\ \mathcal{R}_{17} &\lesssim \|\partial_t\mathbf{u}_{\mathrm{m},2}\|_{L^2(\Omega)}\|\delta h_{\mathrm{m}}\|_{L^4(\Omega)}\|\delta\mathbf{u}_{\mathrm{m}}\|^{1/2}_{L^2(\Omega)}\|\delta\mathbf{u}_{\mathrm{m}}\|^{1/2}_{H^1(\Omega)},\\ \mathcal{R}_{18} &\lesssim \overline{h}\|\mathbf{u}^o_1\|_{H^2(\Omega)}\|\nabla\delta\mathbf{u}^o\|_{L^2(\Omega)}\|\delta\mathbf{u}_{\mathrm{m}}\|_{L^2(\Omega)},\\ \mathcal{R}_{19} &\lesssim \overline{h}\|\delta\mathbf{u}^o\|^{1/2}_{L^2(\Omega)}\|\delta\mathbf{u}^o\|^{1/2}_{H^1(\Omega)}\|\nabla\mathbf{u}^o_2\|_{L^4(\Omega)}\|\delta\mathbf{u}_{\mathrm{m}}\|_{L^2(\Omega)},\\ \mathcal{R}_{20} &\lesssim \|\delta h_{\mathrm{m}}\|_{L^4(\Omega)}\|\mathbf{u}^o_2\|_{H^2(\Omega)}\|\nabla\mathbf{u}^o_2\|_{L^4(\Omega)}\|\delta\mathbf{u}_{\mathrm{m}}\|_{L^2(\Omega)},\\ \mathcal{R}_{21} &\lesssim (1+\overline{h})\|\delta h_{\mathrm{m}},\delta A_{\mathrm{m}}\|_{L^2(\Omega)}\|\nabla\delta\mathbf{u}_{\mathrm{m}}\|_{L^2(\Omega)},\end{aligned}$$

$$\mathcal{R}_{23} \lesssim (1 + \overline{h} + \sum_{j=1}^{2} \|\mathbf{u}_j^o\|_{H^2(\Omega)})(\|\delta \mathbf{u}^o\|_{L^2(\Omega)} + \|\delta h_{\mathrm{m}}\|_{L^2(\Omega)})\|\delta \mathbf{u}_{\mathrm{m}}\|_{L^2(\Omega)}.$$

To estimate $\mathcal{R}_{22}$, we rewrite it as

$$\begin{aligned}
\mathcal{R}_{22} &= 2\int \operatorname{div}\left[\mu(\nabla \delta \mathbf{u}^o + (\nabla \delta \mathbf{u}^o)^\top) + \lambda \operatorname{div} \delta \mathbf{u}^o \mathbb{I}_2\right] \cdot \delta \mathbf{u}_{\mathrm{m}}\, dx \\
&+ 2\int \operatorname{div}\left[p_{\mathrm{m},1}\delta\left(\frac{\nabla \mathbf{u}^o + (\nabla \mathbf{u}^o)^\top}{\sqrt{|\nabla \mathbf{u}^o + (\nabla \mathbf{u}^o)^\top|^2 + \varepsilon^2}}\right) + p_{\mathrm{m},1}\delta\left(\frac{\operatorname{div} \mathbf{u}^o \mathbb{I}_2}{\sqrt{|\operatorname{div} \mathbf{u}^o|^2 + \varepsilon^2}}\right)\right] \cdot \delta \mathbf{u}_{\mathrm{m}}\, dx \\
&+ 2\int \delta p_{\mathrm{m}}\left[\frac{\nabla \mathbf{u}_2^o + (\nabla \mathbf{u}_2^o)^\top}{\sqrt{|\nabla \mathbf{u}_2^o + (\nabla \mathbf{u}_2^o)^\top|^2 + \varepsilon^2}} + \frac{\operatorname{div} \mathbf{u}_2^o \mathbb{I}_2}{\sqrt{|\operatorname{div} \mathbf{u}_2^o|^2 + \varepsilon^2}}\right] : \nabla \delta \mathbf{u}_{\mathrm{m}}\, dx.
\end{aligned}$$

Therefore, applying Hölder's inequality and the Sobolev embedding inequality implies

$$\begin{aligned}
\mathcal{R}_{22} &\lesssim C_{\varepsilon,\mu,\lambda}(1 + \overline{h} + \|\nabla h_{\mathrm{m},1}, \nabla A_{\mathrm{m},1}\|_{L^4(\Omega)})\|\delta \mathbf{u}^o\|_{H^2(\Omega)}\|\delta \mathbf{u}_m\|_{L^2(\Omega)} \\
&+ C_\varepsilon \overline{h} \sum_{j=1}^{2} \|\nabla^2 \mathbf{u}_j^o\|_{L^2(\Omega)}\|\nabla \delta \mathbf{u}^o\|_{L^4(\Omega)}\|\delta \mathbf{u}_m\|_{L^2(\Omega)}^{1/2}\|\delta \mathbf{u}_m\|_{H^1(\Omega)}^{1/2} \\
&+ (1 + \overline{h})\|\delta h_{\mathrm{m}}, \delta A_{\mathrm{m}}\|_{L^2(\Omega)}\|\nabla \delta \mathbf{u}_{\mathrm{m}}\|_{L^2(\Omega)},
\end{aligned}$$

where we have used the identity

$$\begin{aligned}
\delta\left(\frac{g}{\sqrt{|g|^2 + \varepsilon^2}}\right) &= \frac{\delta g}{\sqrt{|g_1|^2 + \varepsilon^2}} \\
&- \frac{g_2 \delta |g|^2}{\sqrt{|g_1|^2 + \varepsilon^2}\sqrt{|g_2|^2 + \varepsilon^2}(\sqrt{|g_1|^2 + \varepsilon^2} + \sqrt{|g_2|^2 + \varepsilon^2})}
\end{aligned}$$

for $g = \nabla \mathbf{u}^o + (\nabla \mathbf{u}^o)^\top$ and $\operatorname{div} \mathbf{u}^o \mathbb{I}_2$, respectively.

Then, after substituting the bounds in (3.16) and (3.19a) and applying interpolation inequalities, one can obtain from (3.27) that

$$\begin{aligned}
\rho_{\mathrm{ice}} \frac{d}{dt}\|h_{\mathrm{m},1}^{1/2}\delta \mathbf{u}_{\mathrm{m}}\|_{L^2(\Omega)}^2 + \iota\|\delta \mathbf{u}_{\mathrm{m}}\|_{H^2(\Omega)}^2 &\le C_{\sigma,\varepsilon,\mu,\lambda,\mathfrak{c}_o,\mathfrak{c}_{\mathrm{in}}}\|\delta \mathbf{u}_{\mathrm{m}}\|_{L^2(\Omega)}^2 \\
+ C_{\overline{h}}(1 + \|\partial_t \mathbf{u}_{\mathrm{m},2}\|_{L^2(\Omega)}^2)(\|\delta h_{\mathrm{m}}\|_{L^4(\Omega)}^2 + \|\delta h_{\mathrm{m}}, \delta A_{\mathrm{m}}\|_{L^2(\Omega)}^2) \\
+ \sigma\|\delta \mathbf{u}^o\|_{H^2(\Omega)}^2,
\end{aligned} \tag{3.28}$$

where Young's inequality is applied.

Thus, after substituting (3.26) into (3.28) and applying Grönwall's inequality to the resultant, one has

$$\begin{aligned}
\sup_{0\le s\le t}\|\delta \mathbf{u}_{\mathrm{m}}(s)\|_{L^2(\Omega)}^2 + \int_0^t \|\delta \mathbf{u}_{\mathrm{m}}(s)\|_{H^2(\Omega)}^2\, ds &\le \sigma C_{\iota,\underline{h},\overline{h},\varepsilon,\omega,\nu,\mathfrak{c}_o,\mathfrak{c}_{\mathrm{in}}} \\
\times \exp\left[C_{\sigma,\iota,\mu,\lambda,\underline{h},\overline{h},\varepsilon,\omega,\nu,\mathfrak{c}_o,\mathfrak{c}_{\mathrm{in}}}(t + t^2)\right] \int_0^t \|\delta \mathbf{u}^o\|_{H^2(\Omega)}^2\, ds.
\end{aligned}$$

Therefore, after choosing σ and t small enough, one can conclude that

$$\begin{aligned}
&\sup_{0\le s\le t}\|\delta \mathbf{u}_{\mathrm{m}}(s)\|_{L^2(\Omega)}^2 + \int_0^t \|\delta \mathbf{u}_{\mathrm{m}}(s)\|_{H^2(\Omega)}^2\, ds \\
&\quad \le \frac{1}{2}\left(\sup_{0\le s\le t}\|\delta \mathbf{u}^o(s)\|_{L^2(\Omega)}^2 + \int_0^t \|\delta \mathbf{u}^o(s)\|_{H^2(\Omega)}^2\, ds\right).
\end{aligned} \tag{3.29}$$

Now we update the smallness of T^*, so that (3.29) holds true for $t \in (0, T^*]$. Then the map $\mathfrak{M}$, defined in (3.1), is contracting with constant $1/2$. By means of Banach's fixed point theorem, we conclude that there exists a unique solution to (1.10) in $\mathfrak{X}_{\mathrm{T}^*}$.

What is left is to show that such solutions are stable. Namely, they continuously depend on the initial data. Let $(\mathbf{u}_j, h_j, A_j)$ be two solutions to (1.10), associated with initial data $(\mathbf{u}_{\mathrm{in},j}, h_{\mathrm{in},j}, A_{\mathrm{in},j})$, $j = 1, 2$, satisfying (3.3). Then it is easy to check that (3.26) and (3.29) still hold true with $\delta\mathbf{u}^o, \delta\mathbf{u}_m, \delta h_{\mathrm{m}}, \delta A_{\mathrm{m}}$ replaced by $\delta\mathbf{u} := \mathbf{u}_1 - \mathbf{u}_2, \delta h := h_1 - h_2, \delta A := A_1 - A_2$, with additional initial data on the righthand side, i.e.,

$$\begin{aligned} &\sup_{0\leq s\leq t}\big(\|\delta h(s), \delta A(s)\|_{L^4(\Omega)}^2 + \|\delta\mathbf{u}(s)\|_{L^2(\Omega)}^2\big) + \int_0^t \|\delta\mathbf{u}(s)\|_{H^2(\Omega)}^2\,ds \\ &\quad\leq C_{\varepsilon,\omega,\nu,\underline{h},\overline{h},\mathfrak{c}_o,\mathfrak{c}_{\mathrm{in}}}\big(\|h_{\mathrm{in},1} - h_{\mathrm{in},2}, A_{\mathrm{in},1} - A_{\mathrm{in},2}\|_{L^4(\Omega)}^2 + \|\mathbf{u}_{\mathrm{in},1} - \mathbf{u}_{\mathrm{in},2}\|_{L^2(\Omega)}^2\big). \end{aligned} \tag{3.30}$$

Hence, we have established the local-in-time well-posedness of strong solutions to system (1.10). We would like to remind readers that the estimates obtained in this section depend on $(\mu, \lambda, \iota, \nu)$. In the next section, we aim at removing such dependency.

4 Well-posedness of solutions to (1.3) with $\underline{h} > 0$

4.1 $(\mu, \lambda, \iota, \nu)$-independent estimates of solutions to (1.10)

We shall only present the uniform-in-$(\mu, \lambda, \iota, \nu)$ *a priori* estimate in this subsection, based on which the standard different quotient argument can be established.

Throughout this section, we use the notation $X \lesssim Y$ to represent $X \leq CY$ for some generic constant $C \in (0, \infty)$, which may be different from line to line, and depend on $\varepsilon, \omega, \underline{h}, \overline{h}$, but is independent of $(\mu, \lambda, \iota, \nu)$.

To begin with, let

$$\mathcal{E}(t) := \sup_{0\leq s\leq t}\|\mathbf{u}(s), h(s), A(s)\|_{H^3(\Omega)}^2 + \int_0^t \|\mathbf{u}(s)\|_{H^4(\Omega)}^2\,ds, \tag{4.1}$$

and

$$\begin{aligned} \mathfrak{E}(t) &:= \sup_{0\leq s\leq t}\|\mathbf{u}(s), h(s), A(s)\|_{H^3(\Omega)}^2 \\ &\quad+ \int_0^t \int \bigg(\frac{|\nabla^3(\nabla\mathbf{u}(s) + \nabla\mathbf{u}^\top(s))|^2}{(|\nabla\mathbf{u}(s) + \nabla\mathbf{u}^\top(s)|^2 + \varepsilon^2)^{3/2}} + \frac{|\nabla^3 \mathrm{div}\,\mathbf{u}(s)|^2}{(|\mathrm{div}\,\mathbf{u}(s)|^2 + \varepsilon^2)^{3/2}}\bigg)\,dx\,ds. \end{aligned} \tag{4.2}$$

One can easily check that $\mathcal{E}$ and $\mathfrak{E}$ are essentially equivalent in the sense that estimates on one imply estimates on the other. Indeed, it is trivial that $\mathfrak{E} \lesssim \mathcal{E}$. On the other hand, applying integration by parts yields that

$$\begin{aligned} &\int |\nabla^4\mathbf{u}|^2\,dx = \frac{1}{2}\int |\nabla^3(\nabla\mathbf{u} + \nabla\mathbf{u}^\top)|^2\,dx - \int |\nabla^3\mathrm{div}\,\mathbf{u}|^2\,dx \\ &\quad\lesssim (\varepsilon^3 + \|\mathbf{u}\|_{H^3(\Omega)}^3)\int\bigg(\frac{|\nabla^3(\nabla\mathbf{u} + \nabla\mathbf{u}^\top)|^2}{(|\nabla\mathbf{u} + \nabla\mathbf{u}^\top|^2 + \varepsilon^2)^{3/2}} + \frac{|\nabla^3\mathrm{div}\,\mathbf{u}|^2}{(|\mathrm{div}\,\mathbf{u}|^2 + \varepsilon^2)^{3/2}}\bigg)\,dx. \end{aligned} \tag{4.3}$$

Therefore, we have

$$\mathfrak{E}(t) \lesssim \mathcal{E}(t) \lesssim (1 + t + \mathfrak{E}^2(t))\mathfrak{E}(t). \tag{4.4}$$

Estimates for h and A

It is easy to check that (2.3), (2.5), (2.6), (2.7), and (2.8) also hold true with $A_{\mathrm{m}}, h_{\mathrm{m}}, \mathbf{u}^o$ replaced by $A, h, \mathbf{u}$, respectively. Therefore, for $s \in (0,t)$ with t satisfying (2.9), with $\mathbf{u}^o$ replaced by $\mathbf{u}$, we have

$$0 \leq A \leq 1, \qquad 0 < \frac{1}{4}\underline{h} \leq h \leq 4\overline{h}. \tag{4.5}$$

Notice that the smallness of t here is independent of $(\mu, \lambda, \iota, \nu)$.

Next, we shall establish the regularity estimates of A and h. Indeed, after applying ∂^3 to (1.10b) and (1.10c), one can obtain the following equations:

$$\begin{aligned}\partial_t \partial^3 h + \mathbf{u}\cdot\nabla\partial^3 h = \partial^3 \mathcal{S}_{h,\mu,\nu} - \partial^3(h \operatorname{div}\mathbf{u})\\ + \big(\mathbf{u}\cdot\nabla\partial^3 h - \partial^3(\mathbf{u}\cdot\nabla h)\big),\end{aligned} \tag{4.6a}$$

$$\begin{aligned}\partial_t \partial^3 A + \mathbf{u}\cdot\nabla\partial^3 A = \partial^3 \mathcal{S}_{A,\omega,\nu} + \partial^3(A\operatorname{div}\mathbf{u}\cdot\chi_A^\omega)\\ - \partial^3(A\operatorname{div}\mathbf{u}) + \big(\mathbf{u}\cdot\nabla\partial^3 A - \partial^3(\mathbf{u}\cdot\nabla A)\big).\end{aligned} \tag{4.6b}$$

Then, applying the L^2-inner product of (4.6a) and (4.6b) with $2\partial^3 h$ and $\partial^3 A$, respectively, and integration by parts in the resultant leads to

$$\begin{aligned}\frac{d}{dt}\|\partial^3 h\|_{L^2(\Omega)}^2 = \underbrace{\int \big(\operatorname{div}\mathbf{u}|\partial^3 h|^2 - 2\partial^3(h\operatorname{div}\mathbf{u})\partial^3 h\big)\,dx}_{\mathcal{I}_1}\\ + \underbrace{2\int \big(\mathbf{u}\cdot\nabla\partial^3 h - \partial^3(\mathbf{u}\cdot\nabla h)\big)\partial^3 h\,dx}_{\mathcal{I}_2} + \underbrace{2\int \partial^3 \mathcal{S}_{h,\mu,\nu}\partial^3 h\,dx}_{\mathcal{I}_3},\end{aligned} \tag{4.7a}$$

$$\begin{aligned}\frac{d}{dt}\|\partial^3 A\|_{L^2(\Omega)}^2 = \underbrace{\int \big(\operatorname{div}\mathbf{u}|\partial^3 A|^2 - 2\partial^3(A\operatorname{div}\mathbf{u})\partial^3 A\big)\,dx}_{\mathcal{I}_4}\\ + \underbrace{2\int \big(\mathbf{u}\cdot\nabla\partial^3 A - \partial^3(\mathbf{u}\cdot\nabla A)\big)\,dx}_{\mathcal{I}_5} + \underbrace{2\int \partial^3 \mathcal{S}_{A,\mu,\nu}\partial^3 A\,dx}_{\mathcal{I}_6}\\ + \underbrace{2\int \partial^3(A\operatorname{div}\mathbf{u}\cdot\chi_A^\omega)\partial^3 A\,dx}_{\mathcal{I}_7}.\end{aligned} \tag{4.7b}$$

Directly applying Hölder's inequality and the Sobolev embedding inequality leads to the following estimates:

$$\begin{aligned}\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_4 + \mathcal{I}_5 + \mathcal{I}_7 \lesssim \mathcal{H}(\|\mathbf{u}, h, A\|_{H^3(\Omega)})\\ + \|\mathbf{u}\|_{H^4(\Omega)}\|h, A\|_{H^3(\Omega)}^2.\end{aligned} \tag{4.8}$$

Similarly,

$$\mathcal{I}_3 + \mathcal{I}_6 \lesssim \mathcal{H}(\|h, A\|_{H^3(\Omega)}). \tag{4.9}$$

Therefore, after substituting estimates (4.8) and (4.9) into (4.7a) and (4.7b), one can derive that

$$\frac{d}{dt}\|\partial^3 h, \partial^3 A\|_{L^2(\Omega)}^2 \lesssim \mathcal{H}(\|\mathbf{u}, h, A\|_{H^3(\Omega)}) + \|\mathbf{u}\|_{H^4(\Omega)}\|h, A\|_{H^3(\Omega)}^2.$$

Similar estimates also hold for lower order derivatives. Hence we have shown that

$$\frac{d}{dt}\|h, A\|^2_{H^3(\Omega)} \leq \mathcal{H}(\|\mathbf{u}, h, A\|_{H^3(\Omega)}) + C_{\omega,\underline{h},\overline{h}}\|\mathbf{u}\|_{H^4(\Omega)}\|h, A\|^2_{H^3(\Omega)},$$

for some constant $C_{\omega,\underline{h},\overline{h}} \in (0,\infty)$, independent of ι and ν. Consequently, applying Grönwall's inequality concludes that

$$\begin{aligned}\sup_{0\leq s\leq t}\|h(s), A(s)\|^2_{H^3(\Omega)} \leq e^{C_{\omega,\underline{h},\overline{h}}\int_0^t \|\mathbf{u}(s)\|_{H^4(\Omega)}\,ds} \\ \times\Big(\|h_{\mathrm{in}}, A_{\mathrm{in}}\|^2_{H^3(\Omega)} + \int_0^t \mathcal{H}(\|\mathbf{u}(s), h(s), A(s)\|_{H^3(\Omega)})\,ds\Big).\end{aligned} \tag{4.10}$$

Estimates for $\mathbf{u}$

After applying ∂^3 to (1.10a), one can obtain the following equation:

$$\begin{aligned} m(\partial_t\partial^3\mathbf{u} + \mathbf{u}\cdot\nabla\partial^3\mathbf{u}) + \nabla\partial^3 p = \operatorname{div}\partial^3\mathbb{S}_\varepsilon + \operatorname{div}\partial^3\mathbb{S}_{\mu,\lambda} \\ -\iota\Delta^2\partial^3\mathbf{u} + \partial^3\mathcal{F} + [m\partial_t\partial^3\mathbf{u} - \partial^3(m\partial_t\mathbf{u})] \\ +[m\mathbf{u}\cdot\nabla\partial^3\mathbf{u} - \partial^3(m\mathbf{u}\cdot\nabla\mathbf{u})]. \end{aligned} \tag{4.11}$$

Then, applying the L^2-inner product of (4.11) with $2\partial^3\mathbf{u}$ and integration by parts in the resultant leads to

$$\begin{aligned} &\frac{d}{dt}\|\rho_{\mathrm{ice}}^{1/2}h^{1/2}\partial^3\mathbf{u}\|^2_{L^2(\Omega)} + 2\mu\|\nabla\partial^3\mathbf{u}\|^2_{L^2(\Omega)} + 2(\mu+\lambda)\|\operatorname{div}\partial^3\mathbf{u}\|^2_{L^2(\Omega)} \\ &+ 2\iota\|\nabla^2\partial^3\mathbf{u}\|^2_{L^2(\Omega)} = \underbrace{\int[\rho_{\mathrm{ice}}\partial_t h + \operatorname{div}(\rho_{\mathrm{ice}}h\mathbf{u})]\big|\partial^3 u\big|^2\,dx}_{\mathcal{I}_8} \\ &\underbrace{-2\int\partial^3\mathbb{S}_\varepsilon : \nabla\partial^3\mathbf{u}\,dx}_{\mathcal{I}_9} + \underbrace{2\int[m\partial_t\partial^3\mathbf{u} - \partial^3(m\partial_t\mathbf{u})]\cdot\partial^3\mathbf{u}\,dx}_{\mathcal{I}_{10}} \\ &+ \underbrace{2\int[\rho_{\mathrm{ice}}h\mathbf{u}\cdot\nabla\partial^3\mathbf{u} - \partial^3(\rho_{\mathrm{ice}}h\mathbf{u}\cdot\nabla\mathbf{u})]\cdot\partial^3\mathbf{u}\,dx}_{\mathcal{I}_{11}} \\ &+ \underbrace{2\int\partial^3 p\operatorname{div}\partial^3\mathbf{u}\,dx}_{\mathcal{I}_{12}} \underbrace{-2\int\partial^2\mathcal{F}\cdot\partial^4\mathbf{u}\,dx}_{\mathcal{I}_{13}}. \end{aligned} \tag{4.12}$$

The estimates of $\mathcal{I}_{\mathrm{j}}$, $\mathrm{j}\in\{8,11,12\}$, are standard, which we will record below. Applying Hölder's inequality and the Sobolev embedding inequality yields that

$$\begin{aligned} \mathcal{I}_8 &\lesssim \big(\|\partial_t h\|_{L^2(\Omega)} + \|\operatorname{div}(h\mathbf{u})\|_{L^2(\Omega)}\big)\|\partial^3\mathbf{u}\|_{L^2(\Omega)}\|\partial^3\mathbf{u}\|_{H^1(\Omega)} \\ &\lesssim \big(\|h\|_{L^\infty(\Omega)} + \|\nabla h\|_{L^4(\Omega)}\big)\|\mathbf{u}\|^2_{H^3(\Omega)}\|\mathbf{u}\|_{H^4(\Omega)}, \\ \mathcal{I}_{11} &\lesssim \|h\|_{H^3(\Omega)}\|\mathbf{u}\|^2_{H^3(\Omega)}\|\mathbf{u}\|_{H^4(\Omega)}, \\ \mathcal{I}_{12} &\lesssim \big(\|A\|^3_{H^3(\Omega)} + 1\big)\|h\|_{H^3(\Omega)}\|\mathbf{u}\|_{H^4(\Omega)}. \end{aligned} \tag{4.13}$$

To estimate $\mathcal{I}_{13}$, notice that

$$\|\partial^2\mathcal{F}\|_{L^2(\Omega)} \lesssim \|\partial^2\big(|\mathbf{U}_{\mathrm{w}} - \mathbf{u}|(\mathbf{U}_{\mathrm{w}} - \mathbf{u})\big)\|_{L^2(\Omega)} + \|h\|_{H^2(\Omega)}\|\mathbf{u}\|_{H^2(\Omega)} + \mathrm{l.o.t},$$

where $\mathrm{l.o.t}$ represents lower order terms of $\mathbf{u}$. Direct calculation yields that

$$\begin{aligned}\partial^2\big(|\mathbf{U}_{\mathrm{w}}-\mathbf{u}|(\mathbf{U}_{\mathrm{w}}-\mathbf{u})\big) &= |\mathbf{U}_{\mathrm{w}}-\mathbf{u}|\partial^2(\mathbf{U}_{\mathrm{w}}-\mathbf{u})\\ &+2\frac{(\mathbf{U}_{\mathrm{w}}-\mathbf{u})\cdot\partial(\mathbf{U}_{\mathrm{w}}-\mathbf{u})}{|\mathbf{U}_{\mathrm{w}}-\mathbf{u}|}\partial(\mathbf{U}_{\mathrm{w}}-\mathbf{u})\\ &+\Big(\frac{(\mathbf{U}_{\mathrm{w}}-\mathbf{u})\cdot\partial^2(\mathbf{U}_{\mathrm{w}}-\mathbf{u})+|\partial(\mathbf{U}_{\mathrm{w}}-\mathbf{u})|^2}{|\mathbf{U}_{\mathrm{w}}-\mathbf{u}|}\\ &-\frac{\big((\mathbf{U}_{\mathrm{w}}-\mathbf{u})\cdot\partial(\mathbf{U}_{\mathrm{w}}-\mathbf{u})\big)^2}{|\mathbf{U}_{\mathrm{w}}-\mathbf{u}|^3}\Big)(\mathbf{U}_{\mathrm{w}}-\mathbf{u}),\end{aligned}$$

which implies

$$\|\partial^2\big(|\mathbf{U}_{\mathrm{w}}-\mathbf{u}|(\mathbf{U}_{\mathrm{w}}-\mathbf{u})\big)\|_{L^2(\Omega)} \lesssim \|\mathbf{U}_{\mathrm{w}}-\mathbf{u}\|^2_{H^2(\Omega)}+\|\mathbf{U}_{\mathrm{w}}-\mathbf{u}\|^3_{H^2(\Omega)}.$$

Therefore, we have

$$\mathcal{I}_{13}\lesssim \|\partial^2\mathcal{F}\|_{L^2(\Omega)}\|\partial^4\mathbf{u}\|_{L^2(\Omega)}\lesssim \big(\|\mathbf{u}\|^3_{H^3(\Omega)}+\|h\|^2_{H^2(\Omega)}+1\big)\|u\|_{H^4(\Omega)}. \tag{4.14}$$

In order to estimate $\mathcal{I}_{10}$, we first rewrite $\mathcal{I}_{10}$ as follows,

$$\mathcal{I}_{10}=-2\int\partial^3 m\partial_t\mathbf{u}\cdot\partial^3\mathbf{u}\,dx+6\int\partial m\partial_t\partial\mathbf{u}\cdot\partial^4\mathbf{u}\,dx, \tag{4.15}$$

where we have applied integration by parts. Next, we will use equation (1.10a) to substitute $\partial_t\mathbf{u}$ and $\partial_t\partial\mathbf{u}$ in (4.15). Indeed, after rearranging (1.10a), it follows

$$\begin{aligned}\partial_t\mathbf{u} &= \frac{\operatorname{div}\mathbb{S}_{\varepsilon,\mu,\lambda}}{m}+\frac{\mathcal{F}}{m}-\frac{\nabla p}{m}-\mathbf{u}\cdot\nabla\mathbf{u}-\iota\frac{\Delta^2\mathbf{u}}{m},\\ \partial_t\partial\mathbf{u} &= \frac{\operatorname{div}\partial\mathbb{S}_{\varepsilon,\mu,\lambda}}{m}-\frac{\operatorname{div}\mathbb{S}_{\varepsilon,\mu,\lambda}}{m^2}\partial m+\frac{\partial\mathcal{F}}{m}-\frac{\mathcal{F}}{m^2}\partial m\\ &-\frac{\nabla\partial p}{m}+\frac{\nabla p}{m^2}\partial m-\partial\mathbf{u}\cdot\nabla\mathbf{u}-\mathbf{u}\cdot\nabla\partial\mathbf{u}\\ &-\iota\frac{\Delta^2\partial\mathbf{u}}{m}+\iota\frac{\Delta^2\mathbf{u}}{m^2}\partial m.\end{aligned}$$

Then similarly as before, directly applying Hölder's inequality and the Sobolev embedding inequality leads to,

$$\begin{aligned}\|\partial_t\mathbf{u}\|_{L^4(\Omega)}+\|\partial_t\partial\mathbf{u}\|_{L^2(\Omega)} &\lesssim \mathcal{H}(\|\mathbf{u}\|_{H^3(\Omega)},\|A\|_{H^2(\Omega)},\|h\|_{H^2(\Omega)})\\ &+\iota(1+\|h\|_{H^2(\Omega)})\|\mathbf{u}\|_{H^5(\Omega)}.\end{aligned}$$

Therefore, one can derive that,

$$\begin{aligned}\mathcal{I}_{10} \lesssim&\|\partial^3 m\|_{L^2(\Omega)}\|\partial_t\mathbf{u}\|_{L^4(\Omega)}\|\partial^3\mathbf{u}\|_{L^4(\Omega)}\\ &+\|\partial m\|_{L^\infty(\Omega)}\|\partial_t\partial\mathbf{u}\|_{L^2(\Omega)}\|\partial^4\mathbf{u}\|_{L^2(\Omega)}\\ \lesssim&\mathcal{H}(\|\mathbf{u}\|_{H^3(\Omega)},\|A\|_{H^2(\Omega)},\|h\|_{H^3(\Omega)})\|\mathbf{u}\|_{H^4(\Omega)}\\ &+\iota(\|h\|_{H^3(\Omega)}+\|h\|^2_{H^3(\Omega)})\|\mathbf{u}\|_{H^5(\Omega)}\|\mathbf{u}\|_{H^4(\Omega)}.\end{aligned} \tag{4.16}$$

Lastly, we will estimate $\mathcal{I}_9$. Notice that,

$$\mathcal{I}_9=-\int\partial^3\Big(p\frac{\nabla\mathbf{u}+\nabla\mathbf{u}^\top}{\sqrt{|\nabla\mathbf{u}+\nabla\mathbf{u}^\top|^2+\varepsilon^2}}\Big):\partial^3(\nabla\mathbf{u}+\nabla\mathbf{u}^\top)\,dx$$

$$-2\int \partial^3\Big(p\frac{\operatorname{div}\mathbf{u}}{\sqrt{|\operatorname{div}\mathbf{u}|^2+\varepsilon^2}}\Big)\partial^3\operatorname{div}\mathbf{u}\,dx.$$

Denote by $\mathrm{D}\mathbf{u} \in \{\nabla\mathbf{u}+\nabla\mathbf{u}^\top, \operatorname{div}\mathbf{u}\}$. In this notation, estimating $\mathcal{I}_9$ amounts to determining an estimate for

$$\int \partial^3\Big(p\frac{\mathrm{D}\mathbf{u}}{\sqrt{|\mathrm{D}\mathbf{u}|^2+\varepsilon^2}}\Big)\cdot\partial^3\mathrm{D}\mathbf{u}\,dx.$$

Direct calculation shows that

$$\begin{aligned}
\int \partial^3\Big(p\frac{\mathrm{D}\mathbf{u}}{\sqrt{|\mathrm{D}\mathbf{u}|^2+\varepsilon^2}}\Big)\cdot\partial^3\mathrm{D}\mathbf{u}\,dx &= \int p\Big(\frac{|\partial^3\mathrm{D}\mathbf{u}|^2}{\sqrt{|\mathrm{D}\mathbf{u}|^2+\varepsilon^2}} - \frac{(\mathrm{D}\mathbf{u}\cdot\partial^3\mathrm{D}\mathbf{u})^2}{(|\mathrm{D}\mathbf{u}|^2+\varepsilon^2)^{3/2}}\Big)\,dx \\
&\underbrace{-3\int p\frac{(\mathrm{D}\mathbf{u}\cdot\partial\mathrm{D}\mathbf{u})(\partial^2\mathrm{D}\mathbf{u}\cdot\partial^3\mathrm{D}\mathbf{u})+(\mathrm{D}\mathbf{u}\cdot\partial^2\mathrm{D}\mathbf{u})(\partial\mathrm{D}\mathbf{u}\cdot\partial^3\mathrm{D}\mathbf{u})}{(|\mathrm{D}\mathbf{u}|^2+\varepsilon^2)^{3/2}}\,dx}_{\mathcal{L}_1} \\
&\underbrace{-3\int p\frac{(\partial\mathrm{D}\mathbf{u}\cdot\partial^2\mathrm{D}\mathbf{u})(\mathrm{D}\mathbf{u}\cdot\partial^3\mathrm{D}\mathbf{u})}{(|\mathrm{D}\mathbf{u}|^2+\varepsilon^2)^{3/2}}\,dx}_{\mathcal{L}_2} \\
&\underbrace{+9\int p\frac{(\mathrm{D}\mathbf{u}\cdot\partial\mathrm{D}\mathbf{u})(\mathrm{D}\mathbf{u}\cdot\partial^2\mathrm{D}\mathbf{u})(\mathrm{D}\mathbf{u}\cdot\partial^3\mathrm{D}\mathbf{u})}{(|\mathrm{D}\mathbf{u}|^2+\varepsilon^2)^{5/2}}\,dx}_{\mathcal{L}_3} \\
&\underbrace{-3\int p\frac{|\partial\mathrm{D}\mathbf{u}|^2(\partial\mathrm{D}\mathbf{u}\cdot\partial^3\mathrm{D}\mathbf{u})}{(|\mathrm{D}\mathbf{u}|^2+\varepsilon^2)^{3/2}}\,dx}_{\mathcal{L}_4} \\
&\underbrace{+9\int p\frac{(\mathrm{D}\mathbf{u}\cdot\partial\mathrm{D}\mathbf{u})^2(\partial\mathrm{D}\mathbf{u}\cdot\partial^3\mathrm{D}\mathbf{u})+|\partial\mathrm{D}\mathbf{u}|^2(\mathrm{D}\mathbf{u}\cdot\partial\mathrm{D}\mathbf{u})(\mathrm{D}\mathbf{u}\cdot\partial^3\mathrm{D}\mathbf{u})}{(|\mathrm{D}\mathbf{u}|^2+\varepsilon^2)^{5/2}}\,dx}_{\mathcal{L}_5} \\
&\underbrace{-15\int p\frac{(\mathrm{D}\mathbf{u}\cdot\partial\mathrm{D}\mathbf{u})^3(\mathrm{D}\mathbf{u}\cdot\partial^3\mathrm{D}\mathbf{u})}{(|\mathrm{D}\mathbf{u}|^2+\varepsilon^2)^{7/2}}\,dx}_{\mathcal{L}_6} \\
&\underbrace{+3\int\Big[\partial p\partial^2\Big(\frac{\mathrm{D}\mathbf{u}}{\sqrt{|\mathrm{D}\mathbf{u}|^2+\varepsilon^2}}\Big)\cdot\partial^3\mathrm{D}\mathbf{u}+\partial^2 p\partial\Big(\frac{\mathrm{D}\mathbf{u}}{\sqrt{|\mathrm{D}\mathbf{u}|^2+\varepsilon^2}}\Big)\cdot\partial^3\mathrm{D}\mathbf{u}\Big]\,dx}_{\mathcal{L}_7} \\
&\underbrace{+\int\partial^3 p\frac{\mathrm{D}\mathbf{u}\cdot\partial^3\mathrm{D}\mathbf{u}}{\sqrt{|\mathrm{D}\mathbf{u}|^2+\varepsilon^2}}\,dx}_{\mathcal{L}_8}\,.
\end{aligned}$$

Notice that

$$\frac{|\partial^3\mathrm{D}\mathbf{u}|^2}{\sqrt{|\mathrm{D}\mathbf{u}|^2+\varepsilon^2}} - \frac{(\mathrm{D}\mathbf{u}\cdot\partial^3\mathrm{D}\mathbf{u})^2}{(|\mathrm{D}\mathbf{u}|^2+\varepsilon^2)^{3/2}} \geq \varepsilon^2\frac{|\partial^3\mathrm{D}\mathbf{u}|^2}{(|\mathrm{D}\mathbf{u}|^2+\varepsilon^2)^{3/2}}.$$

Therefore, applying Hölder's inequality and the Sobolev embedding inequality implies that,

$$|\mathcal{L}_4|+|\mathcal{L}_5|+|\mathcal{L}_6|+|\mathcal{L}_7|+|\mathcal{L}_8| \lesssim \|p\|_{H^3(\Omega)}(1+\|\mathrm{D}\mathbf{u}\|^3_{H^2(\Omega)})\|\partial^3\mathrm{D}\mathbf{u}\|_{L^2(\Omega)},$$

$$|\mathcal{L}_1| + |\mathcal{L}_2| + |\mathcal{L}_3| \lesssim \|p\|_{L^\infty(\Omega)} \|\mathrm{D}\mathbf{u}\|^{3/2}_{H^2(\Omega)} \|\mathrm{D}\mathbf{u}\|^{3/2}_{H^3(\Omega)}.$$

Therefore,

$$\begin{aligned} \int \partial^3 \Big(p \frac{\mathrm{D}\mathbf{u}}{\sqrt{|\mathrm{D}\mathbf{u}|^2 + \varepsilon^2}} \Big) \cdot \partial^3 \mathrm{D}\mathbf{u} \, dx \geq \varepsilon^2 \int \frac{p |\partial^3 \mathrm{D}\mathbf{u}|^2}{(|\mathrm{D}\mathbf{u}|^2 + \varepsilon^2)^{3/2}} \, dx \\ - \|p\|_{H^3(\Omega)} (1 + \|\mathrm{D}\mathbf{u}\|^3_{H^2(\Omega)}) \|\partial^3 \mathrm{D}\mathbf{u}\|_{L^2(\Omega)} \\ - \|p\|_{L^\infty(\Omega)} \|\mathrm{D}\mathbf{u}\|^{3/2}_{H^2(\Omega)} \|\mathrm{D}\mathbf{u}\|^{3/2}_{H^3(\Omega)}. \end{aligned}$$

Thus, we have shown that, thanks to the fact $p \geq c_p \underline{h}/4 > 0$,

$$\begin{aligned} \mathcal{I}_9 \leq - \frac{\varepsilon^2 c_p \underline{h}}{4} \int \Big(\frac{|\partial^3 (\nabla \mathbf{u} + \nabla \mathbf{u}^\top)|^2}{(|(\nabla \mathbf{u} + \nabla \mathbf{u}^\top)|^2 + \varepsilon^2)^{3/2}} + 2 \frac{|\partial^3 \mathrm{div}\, \mathbf{u}|^2}{(|\mathrm{div}\, \mathbf{u}|^2 + \varepsilon^2)^{3/2}} \Big) \, dx \\ + \mathcal{H}(\|\mathbf{u}, A, h\|_{H^3(\Omega)}) \|\mathbf{u}\|_{H^4(\Omega)} + \|\mathbf{u}\|^{3/2}_{H^3(\Omega)} \|\mathbf{u}\|^{3/2}_{H^4(\Omega)}. \end{aligned} \tag{4.17}$$

In addition, notice that, according to (4.3),

$$\begin{aligned} \|\mathbf{u}\|_{H^4(\Omega)} \lesssim \|\mathbf{u}\|_{H^3(\Omega)} + \|\nabla^4 \mathbf{u}\|_{L^2(\Omega)} \lesssim \|\mathbf{u}\|_{H^3(\Omega)} \\ + \Big[(\varepsilon^3 + \|\mathbf{u}\|^3_{H^3(\Omega)}) \int \Big(\frac{|\nabla^3 (\nabla \mathbf{u} + \nabla \mathbf{u}^\top)|^2}{(|\nabla \mathbf{u} + \nabla \mathbf{u}^\top|^2 + \varepsilon^2)^{3/2}} + \frac{|\nabla^3 \mathrm{div}\, \mathbf{u}|^2}{(|\mathrm{div}\, \mathbf{u}|^2 + \varepsilon^2)^{3/2}} \Big) \, dx \Big]^{1/2}. \end{aligned} \tag{4.18}$$

To sum up, after substituting estimates (4.13), (4.14), (4.16), (4.17),and (4.18) into (4.12), and applying Young's inequality, one can derive that

$$\begin{aligned} \frac{d}{dt} \|\rho_{\mathrm{in}}^{1/2} h^{1/2} \partial^3 \mathbf{u}\|^2_{L^2(\Omega)} + 2\iota \|\nabla^2 \partial^3 \mathbf{u}\|^2_{L^2(\Omega)} - 2\iota^2 \|\nabla^5 \mathbf{u}\|^2_{L^2(\Omega)} \\ + \frac{\varepsilon^2 c_p \underline{h}}{8} \int \Big(\frac{|\partial^3 (\nabla \mathbf{u} + \nabla \mathbf{u}^\top)|^2}{(|(\nabla \mathbf{u} + \nabla \mathbf{u}^\top)|^2 + \varepsilon^2)^{3/2}} + 2 \frac{|\partial^3 \mathrm{div}\, \mathbf{u}|^2}{(|\mathrm{div}\, \mathbf{u}|^2 + \varepsilon^2)^{3/2}} \Big) \, dx \\ \leq \mathcal{H}(\|\mathbf{u}, A, h\|_{H^3(\Omega)}, \iota), \end{aligned}$$

which implies, recalling $\iota \in (0,1)$,

$$\begin{aligned} \sup_{0 \leq s \leq t} \|\nabla^3 \mathbf{u}(s)\|^2_{L^2(\Omega)} + (\iota - \iota^2) \int_0^t \|\nabla^5 \mathbf{u}(s)\|^2_{L^2(\Omega)} \, ds \\ + \int_0^t \int \Big(\frac{|\partial^3 (\nabla \mathbf{u}(s) + \nabla \mathbf{u}^\top(s))|^2}{(|(\nabla \mathbf{u}(s) + \nabla \mathbf{u}^\top(s))|^2 + \varepsilon^2)^{3/2}} + 2 \frac{|\partial^3 \mathrm{div}\, \mathbf{u}(s)|^2}{(|\mathrm{div}\, \mathbf{u}(s)|^2 + \varepsilon^2)^{3/2}} \Big) \, dx \, ds \\ \leq C_{\varepsilon, \underline{h}, \overline{h}} \|\nabla^3 \mathbf{u}_{\mathrm{in}}\|^2_{L^2(\Omega)} + \int_0^t \mathcal{H}(\|\mathbf{u}(s), A(s), h(s)\|_{H^3(\Omega)}, \iota) \, ds, \end{aligned}$$

for some constant $C_{\varepsilon, \underline{h}, \overline{h}} \in (0, \infty)$, independent of μ, λ, ι, and ν.

Similar estimates also hold for lower order derivatives. Thus one can conclude that, for $\iota \ll 1$ small enough,

$$\begin{aligned} \sup_{0 \leq s \leq t} \|\mathbf{u}(s)\|^2_{H^3(\Omega)} \\ + \int_0^t \int \Big(\frac{|\partial^3 (\nabla \mathbf{u}(s) + \nabla \mathbf{u}^\top(s))|^2}{(|(\nabla \mathbf{u}(s) + \nabla \mathbf{u}^\top(s))|^2 + \varepsilon^2)^{3/2}} + 2 \frac{|\partial^3 \mathrm{div}\, \mathbf{u}(s)|^2}{(|\mathrm{div}\, \mathbf{u}(s)|^2 + \varepsilon^2)^{3/2}} \Big) \, dx \, ds \\ \leq C_{\varepsilon, \underline{h}, \overline{h}} \|\mathbf{u}_{\mathrm{in}}\|^2_{H^3(\Omega)} + \int_0^t \mathcal{H}(\|\mathbf{u}(s), A(s), h(s)\|_{H^3(\Omega)}) \, ds. \end{aligned} \tag{4.19}$$

Uniform estimates

The summation of (4.10) and (4.19) leads to

$$\begin{aligned}\mathfrak{E}(t) &\leq \left(e^{C_{\omega,\underline{h},\overline{h}} t^{1/2}\mathcal{E}^{1/2}(t)} + C_{\varepsilon,\underline{h},\overline{h}}\right)\\ &\quad \times \left(\|h_{\mathrm{in}}, A_{\mathrm{in}}, \mathbf{u}_{\mathrm{in}}\|^2_{H^3(\Omega)} + t \times \mathcal{H}(\mathfrak{E}(t))\right)\\ &\leq \left(e^{C_{\omega,\underline{h},\overline{h}} t^{1/2}\left[t^{3/2}+\mathfrak{E}(t)+\mathfrak{E}^3(t)\right]^{1/2}} + C_{\varepsilon,\underline{h},\overline{h}}\right)\\ &\quad \times \left(\|h_{\mathrm{in}}, A_{\mathrm{in}}, \mathbf{u}_{\mathrm{in}}\|^2_{H^3(\Omega)} + t \times \mathcal{H}(\mathfrak{E}(t))\right),\end{aligned}$$

where we have applied (4.3) and Young's inequality in the second inequality. Consequently, for t small enough, independent of μ, λ, ι, ν, one can conclude that

$$\mathfrak{E}(t) \leq C_{\varepsilon,\omega,\underline{h},\overline{h}} \times \|h_{\mathrm{in}}, A_{\mathrm{in}}, \mathbf{u}_{\mathrm{in}}\|^2_{H^3(\Omega)}, \tag{4.20}$$

and, thanks to (4.3),

$$\mathcal{E}(t) \leq \mathfrak{C}^2_{\mathrm{in}}, \tag{4.21}$$

for some constant $\mathfrak{C}_{\mathrm{in}} \in (0,\infty)$, depending only on $\varepsilon, \omega, \underline{h}, \overline{h}$, and

$$\|h_{\mathrm{in}}, A_{\mathrm{in}}, \mathbf{u}_{\mathrm{in}}\|_{H^3(\Omega)}.$$

Thus we have established the $(\mu, \lambda, \iota, \nu)$-independent estimates. Therefore, together with the well-posedness theory in Section 3 and continuity arguments, the existence time of solutions to (1.10) can be extended to some $T^{**} \in (0,\infty)$, independent of $(\mu, \lambda, \iota, \nu)$, which might be larger than T^*.

4.2 Limit as $(\mu, \lambda, \iota, \nu) \to (0^+, 0^+, 0^+, 0^+)$

Denote by $(\mathbf{u}_{\mu,\lambda,\iota,\nu}, h_{\mu,\lambda,\iota,\nu}, A_{\mu,\lambda,\iota,\nu})$, the solution constructed above to system (1.10). With (4.1), (4.21), and by comparison in system (1.10), it is easy to check that we have the following uniform-in-$(\mu, \lambda, \iota, \nu)$ estimates:

$$\begin{gathered}\|\mathbf{u}_{\mu,\lambda,\iota,\nu}, h_{\mu,\lambda,\iota,\nu}, A_{\mu,\lambda,\iota,\nu}\|_{L^\infty(0,T^{**};H^3(\Omega))} + \|\mathbf{u}_{\mu,\lambda,\iota,\nu}\|_{L^2(0,T^{**};H^4(\Omega))}\\ + \|\partial_t \mathbf{u}_{\mu,\lambda,\iota,\nu}, \partial_t h_{\mu,\lambda,\iota,\nu}, \partial_t A_{\mu,\lambda,\iota,\nu}\|_{L^\infty(0,T^{**};L^2(\Omega))} \leq \mathfrak{C}_{\mathrm{in}},\end{gathered} \tag{4.22}$$

for some constant $\mathfrak{C}_{\mathrm{in}} \in (0,\infty)$, and $T^{**} \in (0,\infty)$, independent of μ, λ, ι, and ν. Therefore, applying the Aubin-Lions lemma yields that there exists $(\mathbf{u}, h, A)$ satisfying (1.8) and (1.9), such that, as $(\mu, \lambda, \iota, \nu) \to (0^+, 0^+, 0^+, 0^+)$,

$$\begin{aligned} \mathbf{u}_{\mu,\lambda,\iota,\nu} &\to \mathbf{u} && \text{in } C(0,T^{**};H^3(\Omega)),\\ h_{\mu,\lambda,\iota,\nu} &\to h && \text{in } C(0,T^{**};H^2(\Omega)),\\ A_{\mu,\lambda,\iota,\nu} &\to A && \text{in } C(0,T^{**};H^2(\Omega)),\\ (\mathbf{u}_{\mu,\lambda,\iota,\nu}, h_{\mu,\lambda,\iota,\nu}, A_{\mu,\lambda,\iota,\nu}) &\overset{*}{\rightharpoonup} (\mathbf{u}, h, A) && \text{in } L^\infty(0,T^{**};H^3(\Omega)),\\ \mathbf{u}_{\mu,\lambda,\iota,\nu} &\rightharpoonup \mathbf{u} && \text{in } L^2(0,T^{**};H^4(\Omega)),\\ (\partial_t \mathbf{u}_{\mu,\lambda,\iota,\nu}, \partial_t h_{\mu,\lambda,\iota,\nu}, \partial_t A_{\mu,\lambda,\iota,\nu}) &\overset{*}{\rightharpoonup} (\partial_t \mathbf{u}, \partial_t h, \partial_t A) && \text{in } L^\infty(0,T^{**};L^2(\Omega)),\end{aligned} \tag{4.23}$$

and it is easy to verify that $(\mathbf{u}, h, A)$ satisfies system (1.3) in $(0, T^{**}]$.

4.3 Well-posedness of solutions for system (1.3)

To deduce the well-posedness of solutions to system (1.3), it remains to establish the uniqueness and the continuous dependency of solutions on initial data. Indeed, this can be done following similar arguments as in Section 3.2, which we will sketch below.

Denote by $(\mathbf{u}_{\mathrm{j}}, h_{\mathrm{j}}, A_{\mathrm{j}})$, $\mathrm{j} = 1, 2$, two solutions to system (1.3) with initial data $(\mathbf{u}_{\mathrm{in,j}}, h_{\mathrm{in,j}}, A_{\mathrm{in,j}})$ within $(0, T_{\mathrm{j}}^{**}]$, $\mathrm{j} = 1, 2$, as constructed above, respectively. In particular, (1.8) and (1.9) hold for $(\mathbf{u}_{\mathrm{j}}, h_{\mathrm{j}}, A_{\mathrm{j}})$, $\mathrm{j} = 1, 2$. Further, let $\delta\mathbf{u} := \mathbf{u}_1 - \mathbf{u}_2$, $\delta h := h_1 - h_2$, $\delta A := A_1 - A_2$, and $T_{12}^{**} := \min\{T_1^{**}, T_2^{**}\} \in (0, \infty)$. The triple $(\delta\mathbf{u}, \delta h, \delta A)$ satisfies the following equations:

$$\begin{gathered} \rho_{\mathrm{ice}} h_1 \partial_t \delta\mathbf{u} + \rho_{\mathrm{ice}} \delta h \partial_t \mathbf{u}_2 = \operatorname{div} \delta\mathbb{S}_\varepsilon - \nabla \delta p \\ - \rho_{\mathrm{ice}} h_1 \mathbf{u}_1 \cdot \nabla \delta\mathbf{u} - \rho_{\mathrm{ice}} h_1 \delta\mathbf{u} \cdot \nabla \mathbf{u}_2 - \rho_{\mathrm{ice}} \delta h \mathbf{u}_2 \cdot \nabla \mathbf{u}_2 + \delta\mathcal{F}, \end{gathered} \tag{4.24a}$$

$$\partial_t \delta h + \operatorname{div}(\delta h \mathbf{u}_1) + \operatorname{div}(h_2 \delta\mathbf{u}) = \delta\mathcal{S}_{h,\omega}, \tag{4.24b}$$

$$\begin{gathered} \partial_t \delta A + \operatorname{div}(\delta A \mathbf{u}_1) + \operatorname{div}(A_2 \delta\mathbf{u}) = \delta\mathcal{S}_{A,\omega} + \delta A \operatorname{div} \mathbf{u}_1 \cdot \chi_{A_1}^\omega \\ + A_2 \operatorname{div} \delta\mathbf{u} \cdot \chi_{A_1}^\omega + A_2 \operatorname{div} \mathbf{u}_2 \cdot \delta\chi_A^\omega. \end{gathered} \tag{4.24c}$$

After taking the L^2-inner product of (4.24a), (4.24b), and (4.24c) with $2\delta\mathbf{u}$, $2\delta h$, and $2\delta A$, respectively, and applying integration by parts in the resultant, one has

$$\begin{gathered} \frac{d}{dt} \|\rho_{\mathrm{ice}}^{1/2} h_1^{1/2} \delta\mathbf{u}\|_{L^2(\Omega)}^2 = \underbrace{-2 \int \delta\mathbb{S}_\varepsilon : \nabla \delta\mathbf{u} \, dx}_{\mathcal{I}_{14}} + \underbrace{\int \rho_{\mathrm{ice}} \partial_t h_1 |\delta\mathbf{u}|^2 \, dx}_{\mathcal{I}_{15}} \\ \underbrace{-2 \int \rho_{\mathrm{ice}} \delta h \partial_t \mathbf{u}_2 \cdot \delta\mathbf{u} \, dx}_{\mathcal{I}_{16}} + \underbrace{2 \int \delta p \operatorname{div} \delta\mathbf{u} \, dx}_{\mathcal{I}_{17}} + \underbrace{2 \int \delta\mathcal{F} \cdot \delta\mathbf{u} \, dx}_{\mathcal{I}_{18}} \\ \underbrace{-2 \int \rho_{\mathrm{ice}} \big(h_1 \mathbf{u}_1 \cdot \nabla \delta\mathbf{u} + h_1 \delta\mathbf{u} \cdot \nabla \mathbf{u}_2 + \delta h \mathbf{u}_2 \cdot \nabla \mathbf{u}_2 \big) \cdot \delta\mathbf{u} \, dx}_{\mathcal{I}_{19}} \end{gathered} \tag{4.25}$$

$$\begin{gathered} \frac{d}{dt} \|\delta h\|_{L^2(\Omega)}^2 = \underbrace{- \int \operatorname{div} \mathbf{u}_1 |\delta h|^2 \, dx}_{\mathcal{I}_{20}} \underbrace{-2 \int \operatorname{div}(h_2 \delta\mathbf{u}) \delta h \, dx}_{\mathcal{I}_{21}} \\ + \underbrace{2 \int \delta\mathcal{S}_{h,\omega} \delta h \, dx}_{\mathcal{I}_{22}}, \end{gathered} \tag{4.26}$$

$$\begin{gathered} \frac{d}{dt} \|\delta A\|_{L^2(\Omega)}^2 = \underbrace{- \int \operatorname{div} \mathbf{u}_1 |\delta A|^2 \, dx}_{\mathcal{I}_{23}} \underbrace{-2 \int \operatorname{div}(A_2 \delta\mathbf{u}) \delta A \, dx}_{\mathcal{I}_{24}} \\ + \underbrace{2 \int \delta\mathcal{S}_{A,\omega} \delta A \, dx}_{\mathcal{I}_{25}} + \underbrace{2 \int \operatorname{div} \mathbf{u}_1 \cdot \chi_{A_1}^\omega |\delta A|^2 \, dx}_{\mathcal{I}_{26}} \\ + \underbrace{2 \int A_2 \operatorname{div} \delta\mathbf{u} \cdot \chi_{A_1}^\omega \delta A \, dx}_{\mathcal{I}_{27}} + \underbrace{2 \int A_2 \operatorname{div} \mathbf{u}_2 \cdot \delta\chi_A^\omega \delta A \, dx}_{\mathcal{I}_{28}}. \end{gathered} \tag{4.27}$$

Then it is straightforward to check that, thanks to the uniform bounds in (1.9),

$$\sum_{15\leq \mathrm{j}\leq 28} \mathcal{I}_{\mathrm{j}} \lesssim \|\delta\mathbf{u}, \delta h, \delta A\|_{L^2(\Omega)}^2 + \|\delta\mathbf{u}, \delta h, \delta A\|_{L^2(\Omega)} \|\nabla\delta\mathbf{u}\|_{L^2(\Omega)}. \tag{4.28}$$

To estimate $\mathcal{I}_{14}$, we will have to investigate the monotonicity of $\mathbb{S}_\varepsilon$, which is an important ingredient in our proof. Notice that

$$\begin{aligned}2\delta\mathbb{S}_\varepsilon : \nabla\delta\mathbf{u} = \delta\Big(p\frac{\nabla\mathbf{u}+\nabla\mathbf{u}^\top}{\sqrt{|\nabla\mathbf{u}+\nabla\mathbf{u}^\top|^2+\varepsilon^2}}\Big) : \delta(\nabla\mathbf{u}+\nabla\mathbf{u}^\top)\\ +2\delta\Big(p\frac{\operatorname{div}\mathbf{u}}{\sqrt{|\operatorname{div}\mathbf{u}|^2+\varepsilon^2}}\Big)\delta\operatorname{div}\mathbf{u}.\end{aligned}$$

For $\mathrm{D}\mathbf{u} \in \{\nabla\mathbf{u}+\nabla\mathbf{u}^\top, \operatorname{div}\mathbf{u}\}$, direct calculation yields that

$$\begin{aligned}\delta\Big(p\frac{\mathrm{D}\mathbf{u}}{\sqrt{|\mathrm{D}\mathbf{u}|^2+\varepsilon^2}}\Big) = \frac{1}{2}\Big(\frac{p_1}{\sqrt{|\mathrm{D}\mathbf{u}_2|^2+\varepsilon^2}} + \frac{p_2}{\sqrt{|\mathrm{D}\mathbf{u}_1|^2+\varepsilon^2}}\Big)\delta\mathrm{D}\mathbf{u}\\ -\frac{1}{2}\frac{\big((\mathrm{D}\mathbf{u}_1+\mathrm{D}\mathbf{u}_2)\cdot\delta\mathrm{D}\mathbf{u}\big)\times(p_1\mathrm{D}\mathbf{u}_1+p_2\mathrm{D}\mathbf{u}_2)}{\sqrt{|\mathrm{D}\mathbf{u}_1|^2+\varepsilon^2}\sqrt{|\mathrm{D}\mathbf{u}_2|^2+\varepsilon^2}(\sqrt{|\mathrm{D}\mathbf{u}_1|^2+\varepsilon^2}+\sqrt{|\mathrm{D}\mathbf{u}_2|^2+\varepsilon^2})}\\ +\frac{\delta p}{2}\Big(\frac{\mathrm{D}\mathbf{u}_1}{\sqrt{|\mathrm{D}\mathbf{u}_1|^2+\varepsilon^2}} + \frac{\mathrm{D}\mathbf{u}_2}{\sqrt{|\mathrm{D}\mathbf{u}_2|^2+\varepsilon^2}}\Big).\end{aligned} \tag{4.29}$$

Therefore

$$\begin{aligned}\delta\Big(p\frac{\mathrm{D}\mathbf{u}}{\sqrt{|\mathrm{D}\mathbf{u}|^2+\varepsilon^2}}\Big)\cdot\delta\mathrm{D}\mathbf{u} = \frac{\delta p}{2}\Big(\frac{\mathrm{D}\mathbf{u}_1}{\sqrt{|\mathrm{D}\mathbf{u}_1|^2+\varepsilon^2}} + \frac{\mathrm{D}\mathbf{u}_2}{\sqrt{|\mathrm{D}\mathbf{u}_2|^2+\varepsilon^2}}\Big)\cdot\delta\mathrm{D}\mathbf{u}\\ +\frac{1}{2}\frac{\mathrm{M}}{\sqrt{|\mathrm{D}\mathbf{u}_1|^2+\varepsilon^2}\sqrt{|\mathrm{D}\mathbf{u}_2|^2+\varepsilon^2}(\sqrt{|\mathrm{D}\mathbf{u}_1|^2+\varepsilon^2}+\sqrt{|\mathrm{D}\mathbf{u}_2|^2+\varepsilon^2})},\end{aligned}$$

with

$$\begin{aligned}\mathrm{M} := \big(p_1\sqrt{|\mathrm{D}u_1|^2+\varepsilon^2}+p_2\sqrt{|\mathrm{D}u_2|^2+\varepsilon^2}\big)\big(\sqrt{|\mathrm{D}\mathbf{u}_1|^2+\varepsilon^2}+\sqrt{|\mathrm{D}\mathbf{u}_2|^2+\varepsilon^2}\big)\\ \times|\delta\mathrm{D}\mathbf{u}|^2 - \big((\mathrm{D}\mathbf{u}_1+\mathrm{D}\mathbf{u}_2)\cdot\delta\mathrm{D}\mathbf{u}\big)\times\big((p_1\mathrm{D}\mathbf{u}_1+p_2\mathrm{D}\mathbf{u}_2)\cdot\delta\mathrm{D}\mathbf{u}\big)\\ \geq C_{\underline{h},\mathfrak{C}_{\mathrm{in}}}\varepsilon|\delta\mathrm{D}\mathbf{u}|^2,\end{aligned}$$

for some constant $C_{\underline{h},\mathfrak{C}_{\mathrm{in}}} \in (0,\infty)$ depending on $\underline{h}$ and $\mathfrak{C}_{\mathrm{in}}$. Therefore, one can derive that

$$\begin{aligned}\mathcal{I}_{14} \lesssim -C_{\varepsilon,\underline{h},\mathfrak{C}_{\mathrm{in}}}\big(\|\nabla\delta\mathbf{u}+\nabla\delta\mathbf{u}^\top\|_{L^2(\Omega)}^2 + \|\operatorname{div}\delta\mathbf{u}\|_{L^2(\Omega)}^2\big)\\ +\|\delta h, \delta A\|_{L^2(\Omega)}\|\nabla\delta\mathbf{u}\|_{L^2(\Omega)},\end{aligned} \tag{4.30}$$

for some constant $C_{\varepsilon,\underline{h},\mathfrak{C}_{\mathrm{in}}} \in (0,\infty)$ depending on ε, $\underline{h}$, and $\mathfrak{C}_{\mathrm{in}}$. In addition, using integration by parts, one can derive that,

$$\|\nabla\delta\mathbf{u}\|_{L^2(\Omega)}^2 \lesssim \|\nabla\delta\mathbf{u}+\nabla\delta\mathbf{u}^\top\|_{L^2(\Omega)}^2 + \|\operatorname{div}\delta\mathbf{u}\|_{L^2(\Omega)}^2. \tag{4.31}$$

Consequently, after substituting (4.28), (4.30), and (4.31) into (4.25), (4.26), and (4.27), summing up the results, and applying Young's inequality, one can conclude that

$$\frac{d}{dt}\|\rho_{\mathrm{ice}}^{1/2}h_1^{1/2}\delta\mathbf{u}, \delta h, \delta A\|_{L^2(\Omega)}^2 \leq C_{\mathfrak{C}_{\mathrm{in}}}\|\rho_{\mathrm{ice}}^{1/2}h_1^{1/2}\delta\mathbf{u}, \delta h, \delta A\|_{L^2(\Omega)}^2,$$

which, after applying Grönwall's inequality, yields

$$\sup_{0\le s\le T_{12}^{**}} \|\delta\mathbf{u}(s), \delta h(s), \delta A(s)\|^2_{L^2(\Omega)} \le C_{\mathfrak{C}_{\rm in}} \|\delta u_{\rm in}, \delta h_{\rm in}, \delta A_{\rm in}\|^2_{L^2(\Omega)}, \tag{4.32}$$

with some constant $C_{\mathfrak{C}_{\rm in}} \in (0,\infty)$, depending on the initial data. The uniqueness and the continuous dependence on initial data of solutions to system (1.3) follow from (4.32).

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