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# KILLING TENSORS ON TORI

KONSTANTIN HEIL, ANDREI MOROIANU, UWE SEMMELMANN

ABSTRACT. We show that Killing tensors on conformally flat  $n$ -dimensional tori whose conformal factor only depends on one variable, are polynomials in the metric and in the Killing vector fields. In other words, every first integral of the geodesic flow polynomial in the momenta on the sphere bundle of such a torus is linear in the momenta.

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## 1. INTRODUCTION

Killing tensors are symmetric  $p$ -tensors with vanishing symmetrized covariant derivative and correspond to Killing vector fields for  $p = 1$ . Originally, Killing tensors were studied in the physics literature since they define first integrals (polynomial in the momenta) of the equation of motion, and thus functions constant on geodesics. This property makes Killing tensors very important in the theory of integrable systems.

First integrals of the geodesic flow on the 2-dimensional torus is an intensively studied topic. The description of metrics with linear first integrals, i.e. Killing vector fields, is obvious. There also is a classification of metrics with quadratic first integrals, i.e. with Killing (non-parallel) 2-tensors. These metrics turn out to be of Liouville type, i.e. in conformal coordinates the metric can be written as  $\tilde{g} = (f(x)^2 + g(y)^2)(dx^2 + dy^2)$  cf. [4]. Surprisingly, the existence of first integrals of degree  $\geq 3$  independent of those of degree 1 and 2 on a 2-torus is a completely open problem. The conjecture is that there are no such first integrals cf. [2]. During the last thirty years many partial results were proved supporting this conjecture cf. [1], [6], [5].

It follows from the theory of integrable systems that besides the metric, it is not possible to have two other functional independent first integrals on the 2-torus cf. [7]. Indeed, in this situation the geodesic flow would be *superintegrable*. Then its trajectories would lie in the intersection of three level sets, one of which is the compact sphere bundle. One can conclude that all geodesics have to be closed. Then the manifold has the homology ring of a rank one symmetric space, which is not the case for the torus.

In particular this means that if the metric  $\tilde{g}$  on  $T^2$  carries a Killing vector field, then any Killing tensor of higher degree is functional dependent of it, i.e. it is expressible as polynomial

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in the Killing vector field and the metric. Since every metric  $\tilde{g}$  on  $T^2$  is conformal to the flat metric  $g$ , and has a Killing vector field if and only if the conformal factor only depends on one coordinate of  $T^2$ , the above fact can be equivalently stated as follows: *Every Killing tensor on the torus  $T^2$  equipped with a metric of the form  $\tilde{g} = e^{2f(y)}(dx^2 + dy^2)$  is a polynomial in the Killing vector field  $\frac{\partial}{\partial x}$  and in the metric tensor  $\tilde{g}$ .*

In our article we generalize this fact to the  $n$ -dimensional torus. Our main result is

**Theorem 1.1.** *Let  $K$  be a Killing tensor on the torus  $T^n$  equipped with a metric of the form  $\tilde{g} = e^{2f(x_n)}(dx_1^2 + \dots + dx_n^2)$ . Then  $K$  is a polynomial in the Killing vector fields  $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}}$  and in the metric tensor  $\tilde{g}$ .*

The idea of the proof is as follows. We first translate the Killing equation on the flat torus. Next, using the formalism developed in [3], we show that the components of any Killing tensor with respect to the flat metric are constant functions in  $x_1, \dots, x_{n-1}$ . The Killing equation then reduces to a system of ordinary differential equations with polynomial solutions, which translated back to the metric  $\tilde{g}$  yields the result.

Note that for dimensional reasons, the result above cannot be proved by the above arguments from the theory of integrable systems for  $n \geq 3$ . Indeed, assuming that  $K$  is a Killing tensor on  $(T^n, \tilde{g})$ , functionally independent on the Killing vector fields  $\xi_1, \dots, \xi_{n-1}$  and on the metric tensor  $\tilde{g}$ , then one would obtain  $n + 1$  first integrals of the geodesic flow on the tangent bundle of  $(T^n, \tilde{g})$ , but this no longer implies superintegrability since  $n + 1 < 2n - 1$  for  $n \geq 3$ .

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## 2. PRELIMINARIES

We will use the formalism introduced in our article [3]. For the convenience of the reader, we recall here the standard definitions and formulas which are relevant in the sequel.

Let  $(TM, g)$  be the tangent bundle of a  $n$ -dimensional Riemannian manifold  $(M, g)$ . We denote with  $\text{Sym}^p TM \subset TM^{\otimes p}$  the  $p$ -fold symmetric tensor product of  $TM$ . The elements of  $\text{Sym}^p TM$  are linear combinations of symmetrized tensor products

$$v_1 \cdot \dots \cdot v_p := \sum_{\sigma \in S_p} v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(p)},$$

where  $v_1, \dots, v_p$  are tangent vectors in  $TM$ .

Let  $\{e_i\}$  denote from now on a local orthonormal frame of  $(TM, g)$ . Using the metric  $g$ , we will identify  $TM$  with  $T^*M$  and thus  $\text{Sym}^2 T^*M \simeq \text{Sym}^2 TM$ . Under this identification we view the metric tensor as a symmetric 2-tensor  $L := 2g = \sum e_i \cdot e_i$ . The scalar product  $g$

induces a scalar product, also denoted by  $g$ , on  $\text{Sym}^p \text{TM}$  defined by

$$g(v_1 \cdots v_p, w_1 \cdots w_p) := \sum_{\sigma \in S_p} g(v_1, w_{\sigma(1)}) \cdots g(v_p, w_{\sigma(p)}) .$$

Using this scalar product, every element  $K$  of  $\text{Sym}^p \text{TM}$  can be identified with a polynomial map of degree  $p$  on  $\text{TM}$ , defined by the formula  $K(v_1, \dots, v_p) = g(K, v_1 \cdots v_p)$ . The metric adjoint of the bundle homomorphism  $v \cdot : \text{Sym}^p \text{TM} \rightarrow \text{Sym}^{p+1} \text{TM}$ ,  $K \mapsto v \cdot K$  is the contraction map  $v \lrcorner : \text{Sym}^{p+1} \text{TM} \rightarrow \text{Sym}^p \text{TM}$ ,  $K \mapsto v \lrcorner K$ , defined by

$$(v \lrcorner K)(v_1, \dots, v_{p-1}) := K(v, v_1, \dots, v_{p-1}) .$$

The metric adjoint of  $L \cdot : \text{Sym}^p \text{TM} \rightarrow \text{Sym}^{p+2} \text{TM}$  is the bundle homomorphism

$$\Lambda : \text{Sym}^{p+2} \text{TM} \rightarrow \text{Sym}^p \text{TM}, \quad K \mapsto \sum e_i \lrcorner e_i \lrcorner K .$$

The following commutator formulas are straightforward:

$$(1) \quad [\Lambda, v \cdot] = 2v \lrcorner, \quad [v \lrcorner, L \cdot] = 2v \cdot, \quad [\Lambda, v \lrcorner] = 0 = [L \cdot, v \cdot] .$$

For later use, let us state the following formula which holds for any vector  $v \in \text{TM}$  and symmetric tensor  $K \in \text{Sym}^q(\text{TM})$ :

$$(2) \quad (L \cdot K)(v, \dots, v) = (q+2)(q+1)K(v, \dots, v)|v|^2 .$$

Indeed, using (1) repeatedly we may write

$$(L \cdot K)(v, \dots, v) = \langle L \cdot K, v^{q+2} \rangle = \langle K, \Lambda v^{q+2} \rangle = (q+2)(q+1)\langle K, v^q \rangle |v|^2 .$$

We denote by  $\text{Sym}_0^p \text{TM} := \ker(\Lambda : \text{Sym}^p \text{TM} \rightarrow \text{Sym}^{p-2} \text{TM})$  the space of trace-free symmetric  $p$ -tensors. The bundle of symmetric tensors splits as

$$\text{Sym}^p \text{TM} \cong \text{Sym}_0^p \text{TM} \oplus \text{Sym}_0^{p-2} \text{TM} \oplus \dots ,$$

where the last summand in the decomposition is the trivial rank one bundle for  $p$  even and the tangent bundle  $\text{TM}$  for  $p$  odd. Correspondingly we have for any  $K \in \text{Sym}^p \text{TM}$  the decomposition

$$K = K_0 + LK_1 + L^2K_2 + \dots$$

with  $K_i \in \text{Sym}_0^{p-2i} \text{TM}$ , i.e.  $\Lambda K_i = 0$ , which is called the *standard decomposition* of  $K$ . For any  $v \in \text{TM}$  and  $K \in \text{Sym}_0^p \text{TM}$  the following projection formula holds (cf. Eq. (3) in [3]):

$$(3) \quad (v \cdot K)_0 = v \cdot K - \frac{1}{n+2(p-1)} L(v \lrcorner K) .$$

On sections of  $\text{Sym}^p \text{TM}$  we define two first order differential operators, the differential

$$d : \Gamma(\text{Sym}^p \text{TM}) \rightarrow \Gamma(\text{Sym}^{p+1} \text{TM}), \quad K \mapsto \sum e_i \cdot \nabla_{e_i} K ,$$

and the formal adjoint of  $d$ , the divergence operator  $\delta$ :

$$\delta : \Gamma(\text{Sym}^{p+1} \text{TM}) \rightarrow \Gamma(\text{Sym}^p \text{TM}), \quad K \mapsto - \sum e_i \lrcorner \nabla_{e_i} K ,$$

An important property of  $d$  is that it acts as derivation on symmetric products. Moreover,  $d$  commutes with  $L \cdot$  and for a section  $K$  of  $\text{Sym}_0^p TM$  the projection of  $dK$  onto the trace-free part is given by the following formula (cf. [3], Lemma 2.2):

$$(4) \quad (dK)_0 = dK + \frac{1}{n+2(p-1)} L \delta K .$$

A symmetric tensor  $K \in \Gamma(\text{Sym}^p TM)$  is called *conformal Killing tensor* if there exists some symmetric tensor  $k \in \Gamma(\text{Sym}^{p-1} TM)$  with  $dK = L \cdot k$ . It is called *Killing tensor* if  $dK = 0$  or equivalently if the symmetrized covariant derivative of  $K$  vanishes or if  $(\nabla_X K)(X, \dots, X) = 0$  holds for all vector fields  $X$ . A Killing tensor is in particular a conformal Killing tensor.

The defining equation for conformal Killing tensors is conformally invariant, i.e. a section  $K$  of  $\text{Sym}^p TM$  is a conformal Killing tensor with respect to the metric  $g$ , if and only if it is a conformal Killing tensor with respect to every conformally related metric  $\tilde{g} = e^{2f}g$ . Indeed (cf. [3], Lemma 3.3), the differential  $\tilde{d}$  with respect to the metric  $\tilde{g}$  is related to  $d$  by

$$(5) \quad \tilde{d}K = e^{-2f} (dK + L \cdot df \lrcorner K) .$$

### 3. PROOF OF THE THEOREM 1.1

We consider the  $n$ -dimensional torus  $T^n$  with the flat metric  $g := dx_1^2 + \dots + dx_n^2$  and a conformally related metric  $\tilde{g} := e^{2f}g$  such that  $f = f(x_n)$  only depends on the last variable. Let  $K \in \Gamma(\text{Sym}^p TM)$  be a Killing tensor on  $M := (T^n, \tilde{g})$ . By (5), the Killing equation  $\tilde{d}K = 0$  translates into the equation

$$(6) \quad dK = -L \cdot (df \lrcorner K)$$

with respect to the conformally equivalent flat metric  $g = e^{-2f}\tilde{g}$ .

We denote by  $\xi_j := \frac{\partial}{\partial x_j}$  the vector fields dual to  $dx_j$ , which form a global orthonormal frame on the flat torus  $(T^n, g)$ . In particular we have  $L = \sum_{j=1}^n \xi_j^2$ . For  $j = 1, \dots, n-1$ ,  $\xi_j$  are Killing vector fields with respect to  $\tilde{g}$  and parallel with respect to the flat metric  $g$ .

In (6) we use the standard decomposition  $K = \sum_{j \geq 0} L^j K_j$  with  $K_j \in \Gamma(\text{Sym}_0^{p-2j} TM)$ , together with the fact that  $d$  commutes with  $L$  and that  $df \lrcorner L = 2df$ , to obtain

$$\begin{aligned} \sum_{j \geq 0} L^j dK_j &= -L df \lrcorner \sum_{j \geq 0} L^j \cdot K_j = -L \sum_{j \geq 0} (2j df \cdot L^{j-1} \cdot K_j + L^j \cdot df \lrcorner K_j) \\ &= - \sum_{j \geq 0} (2j L^j \cdot df \cdot K_j + L^{j+1} \cdot df \lrcorner K_j) . \end{aligned}$$

Using (4) and (5) in the equation above and comparing the trace-free coefficients of  $L^j$  for every  $j$  yields the system

$$(7) \quad \begin{aligned} dK_j + \frac{1}{n+2(p-2j-1)} L \cdot \delta K_j + 2j (df \cdot K_j - \frac{1}{n+2(p-2j-1)} L \cdot df \lrcorner K_j) \\ = \frac{1}{n+2(p-2j+1)} \delta K_{j-1} - \left( 1 + \frac{2(j-1)}{n+2(p-2j+1)} \right) df \lrcorner K_{j-1} . \end{aligned}$$

Here  $j$  goes from 0 to  $\lfloor \frac{p}{2} \rfloor$ , where we set as usual  $K_{-1} = K_{\lfloor \frac{p}{2} \rfloor + 1} = 0$ . In particular, for  $j = 0$ , we obtain as first equation

$$dK_0 + \frac{1}{n+2(p-1)} L \cdot \delta K_0 = 0 .$$

Thus  $K_0$  is a trace-free conformal Killing tensor with respect to  $g$  and hence parallel because of Proposition 6.6 from [3].

**Lemma 3.1.** *Let  $\xi$  be a linear combination with constant coefficients of the  $g$ -parallel vector fields  $\xi_1, \dots, \xi_{n-1}$ . We denote by  $'$  the derivative in direction of  $\xi_n$ . Then the following holds*

- (i)  $\nabla_\xi K = 0$  and  $\nabla_\xi K_j = 0$  for all  $j$
- (ii)  $df = f' \xi_n$ ,  $\nabla_\xi df = 0$
- (iii)  $dK = \xi_n \cdot \nabla_{\xi_n} K$ ,  $\delta K = -\xi_n \lrcorner \nabla_{\xi_n} K$
- (iv)  $K(\xi, \dots, \xi, \xi_n)_x = 0$  at all points  $x \in T^n$  where  $f'(x) \neq 0$ .

*Proof.* (i) Since  $\nabla K_0 = 0$ ,  $\nabla \xi = 0$  and

$$(8) \quad \mathcal{L}_\xi = \nabla_\xi - (\nabla \xi)_* = \nabla_\xi,$$

it follows  $\mathcal{L}_\xi K_0 = 0$ . The Lie derivative  $\mathcal{L}_\xi$  preserves the space of trace-free tensors as well as that of Killing tensors and of course  $\mathcal{L}_\xi L = 0$ . Hence, taking the Lie derivative in the standard decomposition  $K = \sum_{j \geq 0} L^j K_j$ , we obtain that the standard decomposition of the Killing tensor  $\mathcal{L}_\xi K$  is

$$(9) \quad \mathcal{L}_\xi K = L \cdot (\mathcal{L}_\xi K_1 + L \cdot \mathcal{L}_\xi K_2 + \dots) .$$

The operator  $L \cdot$  is injective and commutes with differential  $d$ . Thus a symmetric tensor  $Q$  is Killing if and only if  $L \cdot Q$  is Killing. From (9), we conclude that  $\mathcal{L}_\xi K_1 + L \cdot \mathcal{L}_\xi K_2 + \dots$  is Killing with trace-free part  $\mathcal{L}_\xi K_1$ . Repeating the argument above we get  $\mathcal{L}_\xi^2 K_1 = 0$ . Taking the scalar product with  $K_1$  and integrating over  $(T^n, g)$  yields:

$$0 = \int_{T^n} g(\mathcal{L}_\xi^2 K_1, K_1) d\mu = \int_{T^n} \mathcal{L}_\xi(g(\mathcal{L}_\xi K_1, K_1)) - g(\mathcal{L}_\xi K_1, \mathcal{L}_\xi K_1) d\mu = - \int_{T^n} |\mathcal{L}_\xi K_1|^2 d\mu$$

since,  $\xi$  being parallel, the integral over  $M$  of  $\xi(\psi)$  vanishes for every function  $\psi$  by the Stokes formula. This shows that  $\mathcal{L}_\xi K_1 = 0$ . By immediate induction using (9) we obtain  $\mathcal{L}_\xi K_j = 0$  for all  $j$  and finally also  $\mathcal{L}_\xi K = 0$ . We conclude using (8) again.

(ii) The function  $f$  only depends on the last coordinate  $x_n$ , i.e.  $df = f'(x_n) \xi_n$ . Thus  $\nabla_\xi df = f'(x_n) \nabla_\xi \xi_n = 0$ .

(iii) is a direct consequence of (i) and (ii).

(iv) Using (iii) and (6) we get  $(\xi_n \cdot \nabla_{\xi_n} K)(\xi, \dots, \xi) = -(L \cdot f' \xi_n \lrcorner K)(\xi, \dots, \xi)$ . Since  $\xi_n$  is orthogonal to  $\xi$ , we obtain by (2) that  $0 = -p(p+1)f'K(\xi, \dots, \xi, \xi_n)$  and the statement follows.  $\square$

We now introduce the functions

$$\alpha_j := K_j(\xi, \dots, \xi, \xi_n) = g(K_j, \xi^{p-2j-1} \cdot \xi_n),$$

for  $0 \leq j \leq \lfloor \frac{p-1}{2} \rfloor$ , where  $\xi$  is as above a linear combination with constant coefficients of the  $g$ -parallel vector fields  $\xi_1, \dots, \xi_{n-1}$ . For convenience we assume  $\xi$  to have constant length one. Note that  $\alpha_0$  is constant since  $K_0$  is parallel. We want to show that all  $\alpha_j$  have to vanish. First we find that

$$(\delta K_j)(\xi, \dots, \xi) = -(\xi_n \lrcorner \nabla_{\xi_n} K_j)(\xi, \dots, \xi) = -\alpha'_j.$$

Since  $\xi$  is orthogonal to  $\xi_n$  we have

$$(dK_j)(\xi, \dots, \xi) = (\xi_n \cdot \nabla_{\xi_n} K_j)(\xi, \dots, \xi) = 0.$$

Taking the scalar product with  $\xi^{p-2j+1}$  in (7) and using (2) we obtain for  $0 \leq j \leq \lfloor \frac{p-1}{2} \rfloor$

$$(10) \quad \frac{(p-2j+1)(p-2j)}{n+2(p-2j-1)} \alpha'_j + \frac{2j(p-2j+1)(p-2j)}{n+2(p-2j-1)} f' \alpha_j = \frac{1}{n+2(p-2j+1)} \alpha'_{j-1} + \frac{n+2p-2j}{n+2(p-2j+1)} f' \alpha_{j-1}.$$

From this system of ODE's it will follow that the functions  $\alpha_j$  are polynomials in  $e^{-2f}$  of degree  $j$ . Indeed we have

**Lemma 3.2.** *Let  $\{b_j, c_j \mid 0 \leq j \leq l\}$  be real constants and let  $\{f, \alpha_j \mid 0 \leq j \leq l\}$  be a set of smooth real valued functions on an open interval  $I \subset \mathbb{R}$  satisfying following system of differential equations*

$$\alpha'_j + 2j f' \alpha_j = b_j \alpha'_{j-1} + c_j f' \alpha_{j-1}.$$

*Assume moreover that  $\alpha_0$  is a constant. Then every  $\alpha_j$  is either identically zero or a polynomial in  $e^{-2f}$  of degree  $j$ .*

*Proof.* The statement is proved by induction. It is true for  $j = 0$  since  $\alpha_0$  is constant. We set  $\varphi = e^{2f}$ . Assume the statement to be true for all  $k$  with  $0 \leq k \leq j-1 < l$ . Multiplying the equation for  $\alpha_j$  with  $\varphi^j$  and using  $2f'\varphi = \varphi'$  we get

$$(11) \quad (\varphi^j \alpha_j)' = b_j \varphi^j \alpha'_{j-1} + c_j f' \varphi^j \alpha_{j-1}.$$

By assumption, either  $\alpha_{j-1} = 0$ , or there is a polynomial  $P$  of degree  $(j-1)$  with  $\varphi^{j-1} \alpha_{j-1} = P(\varphi)$ . In the first case, (11) implies that  $\varphi^j \alpha_j$  is constant and thus the induction hypothesis follows. Otherwise we have

$$\varphi^{j-1} \alpha'_{j-1} = 2f' \varphi P'(\varphi) - 2(j-1)f' P(\varphi) = f' Q(\varphi),$$

where  $Q$  is a polynomial of degree  $(j-1)$ . Substituting this into (11) we arrive at

$$(\varphi^j \alpha_j)' = \frac{b_j}{2} Q(\varphi) \varphi' + \frac{c_j}{2} P(\varphi) \varphi'.$$

Finally, integrating this equation shows that  $\varphi^j \alpha_j$  is polynomial of degree  $j$  in  $\varphi$ , proving the induction hypothesis for every  $j \leq l$ .  $\square$

An easy calculation shows that  $\Lambda(\xi^k \cdot \xi_n) = k(k-1)\xi^{k-2} \cdot \xi_n$  for  $k \geq 2$  and therefore

$$g(L^j \cdot K_j, \xi^{p-1} \cdot \xi_n) = d_j g(K_j, \xi^{p-2j-1} \cdot \xi_n)$$

for  $0 \leq j \leq \lfloor \frac{p-1}{2} \rfloor$ , where  $d_j := \frac{(p-1)!}{(p-2j-1)!}$ . Moreover,  $g(L^j \cdot K_j, \xi^{p-1} \cdot \xi_n) = 0$  if  $p$  is even and  $j = \frac{p}{2}$ . Taking the scalar product with  $\xi^{p-1} \cdot \xi_n$  in the standard decomposition  $K = \sum L^j K_j$  thus yields

$$K(\xi, \dots, \xi, \xi_n) = \sum_{j=0}^{\lfloor \frac{p-1}{2} \rfloor} d_j \alpha_j .$$

By Lemma 3.1 (iv), the left hand side vanishes on the open set of points with  $f' \neq 0$ . Since by (10) and Lemma 3.2, each  $\alpha_j$  is either zero or a polynomial in  $e^{-2f}$  of degree  $j$ , this equation can only hold if all  $\alpha_j$  vanish identically. This shows that

$$(12) \quad K_j(\xi, \dots, \xi, \xi_n) = 0, \quad \forall j \leq \left\lfloor \frac{p-1}{2} \right\rfloor$$

on the open set where  $f' \neq 0$ .

Next we expand  $K_0$  in powers of  $\xi_n$ , i.e.  $K_0 = \sum_{j \geq 0} \xi_n^j \cdot P_j$ , where every  $P_j$  is a polynomial of degree  $p-j$  in the vector fields  $\xi_1, \dots, \xi_{n-1}$ . Moreover  $P_1 = 0$  by (12). Applying the contraction  $\Lambda$  to the expansion of  $K_0$  leads to the following relation of polynomials

$$0 = \Lambda K_0 = \sum_{j \geq 0} \Lambda(\xi_n^j \cdot P_j) = \Lambda P_0 + \sum_{j \geq 2} j(j-1)\xi_n^{j-2} \cdot P_j + \xi_n^j \cdot \Lambda P_j .$$

Comparing the coefficients of  $\xi_n^j$ , this implies by immediate induction that  $P_j = 0$  for every odd  $j$  with  $1 \leq j \leq p$ . Hence  $K_0$  is a polynomial in  $\xi_1, \dots, \xi_{n-1}$  and  $\xi_n^2$ . The same argument applies for all tensors  $K_j$  from the standard decomposition  $K = \sum_{j \geq 0} L^j \cdot K_j$ . Hence we can write  $K_j = \sum_{k \geq 0} P_{jk} \xi_n^{2k}$ , where  $P_{jk}$  are polynomials in  $\xi_1, \dots, \xi_{n-1}$  of degree  $p-2j-2k$ . By Lemma 3.1 (i), the coefficients of  $P_{jk}$  are constant in the variables  $x_1, \dots, x_{n-1}$ . We now rewrite

$$(13) \quad K = \sum_{j,k \geq 0} L^j \cdot P_{jk} \cdot \xi_n^{2k} = \sum_{j,k \geq 0} L^j \cdot P_{jk} \cdot (L - \sum_{a=1}^{n-1} \xi_a^2)^k = \sum_{k \geq 0} \tilde{L}^k \cdot Q_k ,$$

where  $\tilde{L} = e^{-2f}L$  and  $Q_k$  is a polynomial in  $\xi_1, \dots, \xi_{n-1}$  whose coefficients are constant in  $x_1, \dots, x_{n-1}$  but may depend on  $x_n$ . Since  $\nabla_{\xi_n} \tilde{L} = -2f' \tilde{L}$ , we compute from (13):

$$(14) \quad \nabla_{\xi_n} K = \sum_{k \geq 0} (-2k f' \tilde{L}^k \cdot Q_k + \tilde{L}^k \cdot \nabla_{\xi_n} Q_k) = \sum_{k \geq 0} \tilde{L}^k \cdot (\nabla_{\xi_n} Q_k - 2k f' Q_k) .$$

On the other side, since  $\xi_n \lrcorner \tilde{L} = 2e^{-2f} \xi_n$ , we have

$$(15) \quad d f \lrcorner K = f' \xi_n \lrcorner K = \sum_{k \geq 0} 2k f' e^{-2f} \xi_n \cdot \tilde{L}^{k-1} \cdot Q_k .$$

Now, the original equation (6) together with Lemma 3.1 yields

$$\xi_n \cdot \nabla_{\xi_n} K = dK = -e^{2f} \tilde{L} \cdot df \lrcorner K$$

and taking (13)–(15) into account, this leads to

$$\xi_n \cdot \sum_{k \geq 0} \tilde{L}^k \cdot (\nabla_{\xi_n} Q_k - 2k f' Q_k) = - \sum_{k \geq 0} 2k f' \xi_n \cdot \tilde{L}^k \cdot Q_k .$$

It follows that  $\xi_n \cdot \sum_{k \geq 0} \tilde{L}^k \cdot \nabla_{\xi_n} Q_k = 0$  and thus  $\sum_{k \geq 0} \tilde{L}^k \cdot \nabla_{\xi_n} Q_k = 0$ . Since  $Q_k$  are polynomials in  $\xi_1, \dots, \xi_{n-1}$ , this leads to  $\nabla_{\xi_n} Q_k = 0$  for all  $k$ , i.e.  $Q_k$  are polynomials in  $\xi_1, \dots, \xi_{n-1}$  with constant coefficients. It follows that  $K$  itself is a polynomial in  $\xi_1, \dots, \xi_{n-1}$  and  $\tilde{L}$  with constant coefficients on any connected component of the open set where  $f' \neq 0$ . But since the Killing equation is of finite type, two Killing tensors which coincide on some non-empty open set must coincide everywhere. This proves the statement of Theorem 1.1.  $\square$

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