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Absence of percolation in graphs based on stationary point processes with degrees bounded by two

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Abstract

We consider undirected graphs that arise as deterministic functions of stationary point processes such that each point has degree bounded by two. For a large class of point processes and edge-drawing rules, we show that the arising graph has no infinite connected component, almost surely. In particular, this extends our previous result for signal-to-interference ratio graphs based on stabilizing Cox point processes and verifies the conjecture of Balister and Bollobás that the bidirectional k-nearest neighbor graph of a two-dimensional homogeneous Poisson point process does not percolate for $k = 2$.

KEYWORDS

bidirectional k-nearest neighbor graph, continuum percolation, degree bounds, deletion-tolerance, stationary point processes

1 | INTRODUCTION

Continuum percolation was introduced by Gilbert [10] in order to model connectivity in large telecommunication networks. In his graph model, the vertices form a homogeneous Poisson point process (PPP) in $\mathbb{R}^2$, and two points are connected whenever their distance is less than a fixed connection radius $r > 0$. He showed that this model undergoes a phase transition: if the spatial intensity $\lambda > 0$ of the PPP is sufficiently small, then the graph consists of finite components only, almost surely, whereas for large enough λ, the graph *percolates*, that is, it has an unbounded connected component, also almost surely.
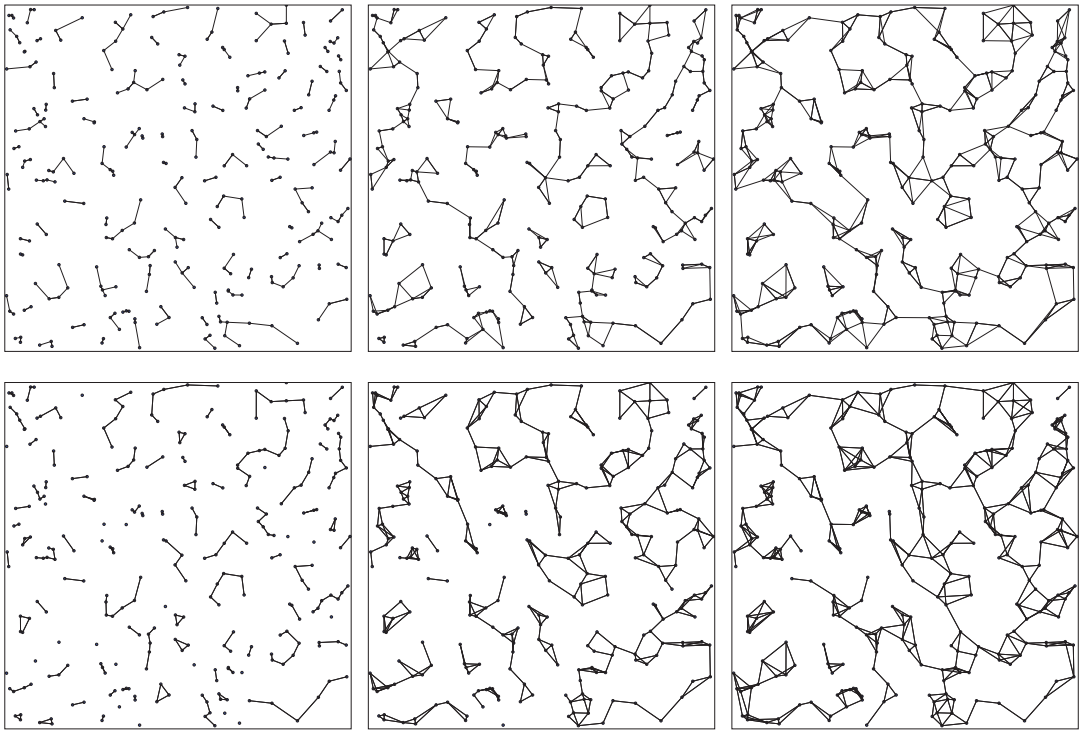
FIGURE 1 Top: Realizations of **U**-kNN graphs based on a PPP with $k = 1$ (left), $k = 2$ (center) and $k = 3$ (right). Bottom: Realizations of **B**-kNN graphs based on a PPP with $k = 2$ (left), $k = 4$ (center) and $k = 5$ (right)

This model has been widely extended, for instance to the case of random connection radii and for various point processes, see [2, 4, 8, 12, 16, 17, 21, 22]. A drawback of Gilbert's model is that it allows for an arbitrarily large degree of the vertices, whereas for many applications, it is a reasonable assumption that the vertices should have bounded degree. Incorporating this property, Häggström and Meester [13] studied percolation in the so-called *undirected k-nearest neighbor* (**U**-kNN) *graph*, based on a stationary PPP in $\mathbb{R}^d$, $d \geq 1$, see top line of Figure 1. Here, all points of the point process are connected to their k-nearest neighbors, for some fixed $k \in \mathbb{N}$. This results in a graph that is the undirected variant of a directed graph with out-degrees bounded by k, which itself also has degrees larger than k. Let us write $k_{\mathbf{U},d}$ for the minimum of all $k \in \mathbb{N}$ such that the **U**-kNN-graph of the stationary PPP in $\mathbb{R}^d$ percolates with positive probability. It was shown in [13] that $k_{\mathbf{U},d} > 1$ for all $d \in \mathbb{N}$, however, $k_{\mathbf{U},d} = 2$ for all sufficiently large d. This was complemented in [26] by the assertion that $k_{\mathbf{U},d} < \infty$ for all $d \geq 2$.

Balister and Bollobás [1] studied the case $d = 2$. They also introduced another undirected graph, which is contained in the **U**-kNN graph, called the *bidirectional k-nearest neighbor* (**B**-kNN) *graph*, see bottom line of Figure 1. Here, one connects two points of the point process if and only if they are mutually among the k-nearest neighbors of each other. This graph has in fact degrees bounded by k, which immediately implies that there is no percolation for $k = 1$, whatever the vertex set is (note that in the PPP case this also follows from the results of [13]). Define the critical out-degree $k_{\mathbf{B},d}$ analogously to $k_{\mathbf{U},d}$ but with **U** replaced by **B**. It was shown in [1] that $k_{\mathbf{U},2} \leq 11$ and $k_{\mathbf{B},2} \leq 15$. Further, "high-confidence results" of [1] indicate $k_{\mathbf{U},2} = 3$ and $k_{\mathbf{B},2} = 5$. By "high-confidence results," the authors of that paper meant that these assertions follow once one shows that a certain

deterministic integral exceeds a certain deterministic value; however, the integrals were only evaluated via Monte–Carlo methods so far. Hence, from a mathematical point of view, these are still open conjectures, but there is very strong numerical evidence that they are true.

Another line of research on percolation of bounded-degree spatial graphs with unbounded-range dependencies, which is also close to applications in wireless networks, is *signal-to-interference plus noise ratio (SINR)* percolation, introduced in [6, 7]. Here, a transmission in the network is considered successful if and only if, measured at the receiver, the incoming signal power of the transmitter is larger than a given threshold times the interference (sum of signal powers) coming from all other users plus some external noise. Then, the SINR graph is constructed by drawing an edge between two vertices whenever the transmission between them is successful in both directions, see Section 4.2 for more details. This graph has bounded degrees (see [6, theorem 1]), where the smallest degree bound k depends on the model parameters. If the transmitted signal powers are all equal, then the SINR graph is contained in the corresponding **B**-kNN graph (see [24, lemma 4.1.13]) and hence also of the **U**-kNN graph.

In general, if in an undirected graph all degrees are bounded by $k = 2$, all infinite connected components must be path graphs (no cycles, no branchings), infinite in one or two directions, which makes the graph similar to a one-dimensional continuum percolation model, indicating that under rather general conditions, there should be no infinite connected component. Certainly, there are deterministic point processes where percolation is possible, but a little bit of randomness can be expected to suffice for nonpercolation. In our recent paper [15], we showed that in SINR graphs based on general stationary Cox point processes (CPPs) in any dimension, under rather general choices of the parameters resulting in degrees bounded by 2, there is no percolation.

In the present paper, moving away from the particular setting of SINR graphs, we present analogous results in a general framework, extending the methods of the proof of [15, theorem 2.2]. We consider a generalization of the **B**-kNN graph, called the f-kNN graph. Here, points of the underlying marked point process are connected by an edge whenever they are mutually among the k nearest neighbors of each other with respect to an ordering that may also depend on some marks of the points, apart from the (not necessarily Euclidean) distance of the points. The ordering is expressed in terms of a function f, hence the name f-kNN graph. We show that under suitable conditions on the underlying stationary marked point process, the f-kNN graph does not percolate for $k = 2$. This in particular implies nonpercolation of subgraphs of the f-kNN graph depending on additional randomness. In fact, our results extend to all stationary point processes that are *deletion-tolerant* in the sense of [14]. This includes all CPPs satisfying a basic nondegeneracy condition, and a large class of Gibbs point processes.

As a special case, our results imply that for general stationary CPPs, the **B**-2NN graph does not percolate. This in particular implies that $k_{\mathbf{B},2} \geq 3$, which provides a partial verification of the high-confidence results of [1]. Note that this result does not follow from [15, theorem 2.2] because in general, if the SINR graph is contained in a **B**-2NN graph, it is a proper subgraph of it with substantially less edges. After stating and proving our main results, we also present examples of graphs with degrees bounded by 2 that are not contained in an f-2NN graph but where our proof techniques are also applicable, and also ones where they are not applicable.

Our setting is also related to the line of research on *outdegree-one* graphs, which were introduced in [3]. However, our results do not follow from the results of that paper, and also not the other way around. We will comment on the similarities and differences of the two models in Section 4.4.

The rest of the paper is organized as follows. In Section 2 we present our setting and main result. In Section 3 we provide the proofs for our main result. Section 4 is devoted to examples, extensions of our methods, and discussions.

2 | MODEL DEFINITION AND MAIN RESULT

In this section we present our model definition and main results. Our setting is as follows. Let $d \in \mathbb{N}$, and let $\| \cdot \|$ be an arbitrary norm on $\mathbb{R}^d$. Further, let $\mathcal{B}(\mathbb{R}^d)$ denote the Borel-σ-algebra of $\mathbb{R}^d$ (clearly, $\|\cdot\|$ generates the standard topology of $\mathbb{R}^d$, which generates $\mathcal{B}(\mathbb{R}^d)$). Moreover, consider the measurable space $(M, \mathcal{M})$, which serves as a mark space.

Next, let $\mathbf{X} = \{(X_i, P_i)\}_{i \in I}$ be a marked point process in $\mathbb{R}^d \times M$, so that $X = \{X_i\}_{i \in I}$ is a stationary point process in $\mathbb{R}^d$ with finite intensity $\lambda = \mathbb{E}[X([0,1]^d)]$, that is *nonequidistant*. This means that for all $i, j, k, l \in I$, $\|X_i - X_j\| = \|X_k - X_l\| > 0$ implies $\{i, j\} = \{k, l\}$ and $\|X_i\| = \|X_j\|$ implies $i = j$, almost surely. Clearly, this property implies that the point process X is *simple*, that is, $\mathbb{P}(X_i \neq X_j, \forall i, j \in I$ with $i \neq j) = 1$. For illustration, note that the randomly shifted lattice $\mathbb{Z}^d + U$, where U is a uniform random variable in $[0,1]^d$, is a simple stationary but not nonequidistant point process on $\mathbb{R}^d$.

Next, we introduce a total ordering of the points. For this, let $f : [0, \infty) \times M \to [0, \infty)$ be a measurable function such that $v \mapsto f(v, q)$ is monotone decreasing for all $q \in M$. We call such a function an *ordering function*.

Definition 2.1. Let f be an ordering function and $(v_1, q_1), (v_2, q_2), (v_3, q_3) \in \mathbb{R}^d \times M$. We say that v_2 *is f-closer to v_1 than to v_3* if one of the following conditions is satisfied:

1 $f(\|v_1 - v_2\|, q_1) < f(\|v_3 - v_2\|, q_3)$, or
2 $f(\|v_1 - v_2\|, q_1) = f(\|v_3 - v_2\|, q_3)$ and $\|v_1 - v_2\| < \|v_3 - v_2\|$.

Then, it is elementary to verify the following lemma.

Lemma 2.2. *Let f be an ordering function. For $\mathbf{X}$ defined as above and X nonequidistant, almost surely, the following holds. For all $i \in I$, the relation "X_i is f-closer to X_j than to X_l" is a total ordering (i.e., irreflexive, antisymmetric and transitive, with any two elements being comparable) on the set $\{(j, l) \in I \times I : j \neq i$ and $l \neq i\}$, which we call the f-ordering.*

Thus, if $\boldsymbol{x} = \{(x_i, p_i)\}_{i \in I}$ is a deterministic, locally finite, infinite, and nonequidistant set of points in $\mathbb{R}^d \times M$ (for some countable index set I) and $v_o \in x := \{x_i\}_{i \in I}$, we can represent x as $x = \{v_n^f(v_o, \boldsymbol{x})\}_{n \in \mathbb{N}_0}$, where $v_0^f(v_o, \boldsymbol{x}) = v_o$, and $v_n^f(v_o, \boldsymbol{x})$ is the nth nearest neighbor of v_o in x with respect to the f-ordering for any $n \in \mathbb{N}_0$. Next, we build a graph based on the f-ordering.

Definition 2.3. Let f be an ordering function, $k \in \mathbb{N}$ and $\mathbf{X}$ defined as above with X nonequidistant, almost surely. The *f-k-nearest neighbor (f-k NN) graph* $g_{k,f}(\mathbf{X})$ is the random graph having vertex set X and for all $i \in I$ and $n \in \{1, \dots, k\}$ an edge between X_i and $v_n^f(X_i, \mathbf{X})$ whenever $X_i \in \{v_1^f(v_n^f(X_i, \mathbf{X}), \mathbf{X}), \dots, v_k^f(v_n^f(X_i, \mathbf{X}), \mathbf{X})\}$.

As the next example shows, the **B**-kNN graph is an f-kNN graph for a point process with trivial marks. Let us write $\{\star\}$ for the one-point measurable space (with $\mathcal{M} = \{\emptyset, \{\star\}\}$).

Example 2.4. Consider a nonequidistant point process X in $\mathbb{R}^d$, $d \geq 1$, and equip X with trivial marks in $M = \{\star\}$. Then, $f(v, q) = f(v) = v$ yields the **B**-k NN graph based on X.

We will explain the relations between f-kNN graphs and SINR graphs in Section 4.2.

Apart from the basic requirement of being nonequidistant, the property of *deletion-tolerance* introduced in [14] is the most important requirement on the marked point process. An **X**-*point* is an

$(\mathbb{R}^d \times M)$-valued random variable $\mathbf{z}$, defined on the same probability space as $\mathbf{X}$, such that $\mathbf{z} \in \mathbf{X}$ a.s., and one says that $\mathbf{X}$ is *deletion-tolerant* if for any $\mathbf{X}$-point $\mathbf{z}$, the distribution of $\mathbf{X} \backslash \{\mathbf{z}\}$ is absolutely continuous with respect to the one of $\mathbf{X}$. See [14, theorem 1.1] for the equivalent formulations of this property for nonmarked point processes. We will present examples of marked point processes that are deletion-tolerant below. Equipped with the above definitions, we are now able to state our main result.

Theorem 2.5. *Let f be an ordering function and let the deletion-tolerant marked point process $\mathbf{X}$ be such that the underlying point process X is stationary, nonequidistant and has a finite intensity. Then, we have*

$$\mathbb{P}(g_{2,f}(\mathbf{X}) \text{ percolates}) = 0.$$

The proof of this theorem is carried out in Section 3. In Section 4.2 we discuss the relation between the proof of Theorem 2.5 and the one of [15, theorem 2.2]. In Section 4.3 we will explain how it extends to other graphs that are defined similarly, have degrees bounded by 2, but are not subgraphs of f-kNN graphs. A key ingredient in the proof of Theorem 2.5 and its aforementioned extensions is the so-called *edge-preserving property* of the underlying graph, which we will introduce in Definition 3.2 below.

Note that deletion-tolerance is satisfied by many point processes, as is shown by the following proposition, see also [14] for more examples without marks.

Proposition 2.6. *Any i.i.d.-marked CPP $\mathbf{X}$ on $\mathbb{R}^d \times M$ is deletion-tolerant. Further, any infinite-volume Gibbs point process $\mathbf{X}$ on $\mathbb{R}^d \times M$ based on an Hamiltonian H is deletion-tolerant, whenever M is a Polish space equipped with the associated Borel σ-algebra, and for all bounded measurable $\Lambda \subset \mathbb{R}^d$ and almost-all boundary conditions $\mathbf{y}$*

$$Z_\Lambda(\mathbf{y}) = \int \mathcal{P}_\Lambda(\mathrm{d}\mathbf{x}_\Lambda) \exp(-H_\Lambda(\mathbf{x}_\Lambda \cup \mathbf{y}_{\Lambda^c})) < \infty,$$

where $\mathcal{P}_\Lambda$ is an i.i.d.-marked Poisson point process in Λ and $\mathbf{y}_{\Lambda^c} = \{(v, q) \in \mathbf{y} : v \in \mathbb{R}^d \backslash \Lambda\}$.

The proof of this proposition is presented in Section 4.1. Note that the class of stationarity and nonequidistant CPPs includes the homogeneous PPPs. The class of Gibbs point processes satisfying the above condition is very rich and in particular includes the classical examples of superstable Hamiltonians, see [5] and references therein. Moreover, there are also well-known point processes that are not deletion-tolerant. A very straightforward class of such processes is the following. As introduced in [9, 18], we say that the point process X is *number rigid* if for any $\mathcal{K} \subset \mathbb{R}^d$ compact, there exists a deterministic measurable function $h_\mathcal{K}$ such that,

$$\#(X \cap \mathcal{K}) = h_\mathcal{K}(X \backslash \mathcal{K}),$$

almost surely, that is, X outside $\mathcal{K}$ determines the number of points of X in $\mathcal{K}$. The following proposition states that number rigid point processes fail to be deletion-tolerant.

Proposition 2.7. *If the nonmarked version X of the marked point process $\mathbf{X}$ is stationary and number rigid with positive intensity, then $\mathbf{X}$ is not deletion-tolerant.*

This proposition follows immediately from results of [14], see Section 4.1 for a proof.

According to [9], the Ginibre ensemble and the Gaussian zero process are number rigid point processes in $\mathbb{R}^2$, which are also stationary, nonequidistant, and of positive intensity. Hence, the proof of Theorem 2.5 is not applicable for them. We nevertheless believe that they satisfy the assertion of the theorem, but the proof would require additional arguments.

3 | PROOF OF THEOREM 2.5

The proof of the Theorem 2.5 proceeds along the following line of arguments. We first show that with probability 1, clusters, that is, maximally connected components, are either finite or they consist only of points of degree 2, see Lemma 3.1 below. Next, we assume for a contradiction that there exists an infinite cluster with positive probability. Then, we introduce a procedure that removes a finite subprocess from the infinite cluster that is closest to the origin in a certain sense associated with the f-ordering. (Below we will provide a formal definition of a finite subprocess of **X**). In the resulting configuration, the infinite cluster still remains infinite, but it contains a vertex of degree 1. Hence, the probability that the process takes values in the set of the resulting configurations is zero. Then it remains to show that also the probability that the process takes place in the set of original configurations is zero, which leads to the desired contradiction. This last step relies on the deletion-tolerance of **X** and uses a certain result from [14] that we will recall below.

Note that for the proof of Theorem 2.5, we can assume that the intensity λ of the underlying stationary point process is positive, since otherwise Theorem 2.5 is trivially true. We start with the following, previously proven lemma, which excludes existence of infinite clusters that have a degree-one point in the case of general random graphs based on stationary marked point processes.

Lemma 3.1. [15, *Lemma 5.4*] *Let $g(\mathbf{X})$ be a random graph based on a marked point process* **X** *such that the degree of all $X_i \in X$, $\deg(X_i)$, is bounded by 2, almost surely. Let X be stationary and nonequidistant with a finite intensity, and consider the point process of degree-one points in infinite clusters*

$$\mathcal{X}_0 = \sum_{i \in I} \delta_{X_i} \mathbb{1}\{\deg(X_i) = 1, \ X_i \ \text{is part of an infinite cluster in} \ g(\mathbf{X})\}.$$

Then, $\mathbb{P}(\mathcal{X}_0(\mathbb{R}^d) = 0) = 1$.

The proof is based on a certain variant of the *mass-transport principle* (see [3, section 4.2] for instance). Informally speaking, the proof goes as follows: If there was an infinite cluster having a point of degree one, then by stationarity, the point process of degree-1 points of infinite clusters $\mathcal{X}_0$ would have to have a positive density. This, however, leads to a contradiction because any infinite cluster can only contain at most one degree-1 point and must contain infinitely many degree-2 points, which implies that the aforementioned density must be equal to zero. We refer the reader to [15, section 5.2] for further details.

In will be convenient to assume that **X** takes values in the set **N** of marked point configurations **x** in $\mathbb{R}^d \times M$ such that $x = \{x_i : (x_i, p_i) \in \mathbf{x}\}$ is an infinite, locally finite, nonequidistant point configuration on $\mathbb{R}^d$. We will denote by N the set of such point configurations x and equip **N** and N with the corresponding evaluation σ-fields.

One essential property that we will use in the proof of Theorem 2.5 is the so-called *edge-preserving property* of the underlying f-kNN graph. Informally speaking, this means that if we remove points from

a configuration, then edges between remaining points are preserved. Now we provide the definition of this property corresponding to our configuration spaces.

Definition 3.2. Let $g : \mathbf{N} \to \mathbf{N} \times (\mathbf{N} \times \mathbf{N})$, $\mathbf{x} \mapsto g(\mathbf{x}) = (x, E_g(\mathbf{x}))$ be a function that maps a marked point configuration $\mathbf{x}$ to a graph with vertex set x. We say that g is edge-preserving if for all $\mathbf{y}, \mathbf{x} \in \mathbf{N}$ with $\mathbf{y} \subseteq \mathbf{x}$, for all $(v_1, q_1), (v_2, q_2) \in \mathbf{y}$ such that $(v_1, v_2) \in E_g(\mathbf{x})$, one has $(v_1, v_2) \in E_g(\mathbf{y})$.

Let us, for the remainder of this section, fix an ordering function f. It is then easy to see that the f-kNN graph $g_{k,f} : \mathbf{x} \mapsto g_{k,f}(\mathbf{x})$ is edge-preserving for all $k \in \mathbb{N}$. See Sections 4.2 and 4.3 for further examples of edge-preserving graphs.

Now, for $\mathbf{x} \in \mathbf{N}$ and $v_o \in x$, we consider the sequence $(\mathbf{v}_n(v_o, \mathbf{x}))_{n \in \mathbb{N}_0}$ of the marked points of $\mathbf{x}$ ordered in increasing f-order of $\mathbf{x}$, measured from v_o. Then, nonboldface $v_i(v_o, \mathbf{x})$, as defined in Section 2, is the first component of $\mathbf{v}_i(v_o, \mathbf{x})$, which we call the *$i$th nearest f-neighbor of v_o*. In particular, recall that $v_0(v_o, \mathbf{x}) = v_o$.

Next, if v_o has degree two in $g_{2,f}(\mathbf{x})$, then v_o is connected by an edge to both $v_1(v_o, \mathbf{x})$ and $v_2(v_o, \mathbf{x})$. Moreover, both $v_1(v_o, \mathbf{x})$ and $v_2(v_o, \mathbf{x})$ also have v_o as one of their first two nearest f-neighbors, that is,

$$v_o \in \{v_1(v_i(v_o, \mathbf{x}), \mathbf{x}), v_2(v_i(v_o, \mathbf{x}), \mathbf{x})\},$$

for all $i \in \{1, 2\}$. These k-nearest f-neighbor relations hold almost surely and in particular for every nonequidistant configuration $\mathbf{x}$. Thanks to the choice of our configuration space $\mathbf{N}$, we can entirely exclude configurations that offend the degree bound or the k-nearest f-neighbor relations or are not nonequidistant. In particular, the f-2NN graph $g_{2,f}(\mathbf{x})$ is well-defined for all $\mathbf{x} \in \mathbf{N}$.

Given this, for $\mathbf{x} \in \mathbf{N}$, we define $\ell(\mathbf{x}) \in [0, \infty]$ as the number of infinite clusters in $g_{2,f}(\mathbf{x})$, and in case $\ell(\mathbf{x}) > 0$, we let $\mathbf{z}(\mathbf{x}) = (z(\mathbf{x}), r(\mathbf{x}))$ denote the closest point of x to the origin such that $z(\mathbf{x})$ has degree two and is contained in an infinite cluster in $g_{2,f}(\mathbf{x})$, and we write $C_1(\mathbf{x})$ for the cluster containing $z(\mathbf{x})$.

Now, in order to show that the probability that $g_{2,f}(\mathbf{X})$ percolates is zero, it suffices to verify that

$$\mathbb{P}(\ell(\mathbf{X}) \geq 1) = 0. \tag{3.1}$$

Next, for $\mathbf{x} \in \mathbf{N}$ with $\ell(\mathbf{x}) \geq 1$, we define the third-nearest f-neighbor of $z(\mathbf{x})$ within the infinite cluster $C_1(\mathbf{x})$ by

$$\tau(\mathbf{x}) = \inf\{i \geq 3 : v_i(z(\mathbf{x}), \mathbf{x}) \in C_1(\mathbf{x})\}.$$

Indeed, among the points $v_j(z(\mathbf{x}), \mathbf{x})$, $j < \tau(\mathbf{x})$, precisely two (namely, $v_1(z(\mathbf{x}), \mathbf{x})$ and $v_2(z(\mathbf{x}), \mathbf{x})$) are contained in $C_1(\mathbf{x})$. In case $\ell(\mathbf{x}) = 0$, we put $\tau(\mathbf{x}) = \infty$. Then, we have the following assertion.

Proposition 3.3. *Under the assumptions of Theorem 2.5, for any natural number $i \geq 3$, we have*

$$\mathbb{P}(\{\ell(\mathbf{X}) \geq 1\} \cap \{\tau(\mathbf{X}) = i\}) = 0. \tag{3.2}$$

Before proving Proposition 3.3, let us show why it implies Theorem 2.5.

Proof of Theorem 2.5. Noting that $\{\ell(\mathbf{X}) \geq 1\} \subset \{\tau(\mathbf{X}) < \infty\}$ and using a union bound, Proposition 3.3 implies $\mathbb{P}(\ell(\mathbf{X}) \geq 1) = 0$, which is (3.1), and thus finishes the proof of Theorem 2.5. ∎

In order to verify Proposition 3.3, let us now present a preliminary result. First, we say that a point process $\mathbf{Y}$ is a *finite subprocess* of $\mathbf{X}$ if $\mathbf{Y}$ is defined on the same probability space as $\mathbf{X}$, satisfies $\#\mathbf{Y} < \infty$ and $\mathbf{Y} \subset \mathbf{X}$ almost surely. Then we have the following claim: If $\mathbf{X}$ is deletion-tolerant, then for any subprocess $\mathbf{Y}$ of $\mathbf{X}$, the law of $\mathbf{X}\backslash\mathbf{Y}$ is absolutely continuous with respect to the one of $\mathbf{X}$. Indeed, for $M = \mathbb{R}^l$, $l \geq 0$, the claim follows from [14, theorem 1.1]. Now we verify it in the case of general M analogously to the proof presented for the case $M = \mathbb{R}^l$ in [14, section 3]. Let us assume that for some measurable subset $\mathcal{A}$ of $\mathbf{N}$ we have $\mathbb{P}(\mathbf{X}\backslash\mathbf{Y} \in \mathcal{A}) > 0$. Then, there must exist some $n \in \mathbb{N}$ such that $\mathbb{P}(\{\mathbf{X}\backslash\mathbf{Y} \in \mathcal{A}\} \cap \{\#\mathbf{Y} = n\}) > 0$. Set $\mathbf{Y}_n = \mathbf{Y}$ if $\#\mathbf{Y} = n$ and set $\mathbf{Y}_n = \emptyset$ otherwise. Then, it follows that $\mathbb{P}(\mathbf{X}\backslash\mathbf{Y}_n \in \mathcal{A}) > 0$. Since $\mathbf{X}$ is deletion-tolerant, it follows that $\mathbb{P}(\mathbf{X} \in \mathcal{A}) > 0$, as claimed.

Next, we observe the following. For $\mathbf{x} \in \mathbf{N}$ with $\ell(\mathbf{x}) \geq 1$, by definition, we have that $z(\mathbf{x})$ is connected by an edge both to $v_1(z(\mathbf{x}), \mathbf{x})$ and $v_2(z(\mathbf{x}), \mathbf{x})$ in $g_{2,f}(\mathbf{x})$. Further, since the degrees are bounded by two, for such $\mathbf{x}$, $v_1(z(\mathbf{x}), \mathbf{x})$ and $v_2(z(\mathbf{x}), \mathbf{x})$ have no further joint neighbor in $g_{2,f}(\mathbf{x})$ since otherwise $C_1(\mathbf{x})$ has a loop and cannot be infinite. Hence, for any $i \geq 3$, there exists $l \in \{1, 2\}$ such that $v_i(z(\mathbf{x}), \mathbf{x})$ and $v_l(z(\mathbf{x}), \mathbf{x})$ are not connected by an edge in $g_{2,f}(\mathbf{x})$. Let us denote the corresponding $v_l(z(\mathbf{x}), \mathbf{x})$ by $m_i(\mathbf{x})$, and define $m_i(\mathbf{x}) = v_1(z(\mathbf{x}), \mathbf{x})$ if neither $v_1(z(\mathbf{x}), \mathbf{x})$ nor $v_2(z(\mathbf{x}), \mathbf{x})$ is connected to $v_i(z(\mathbf{x}), \mathbf{x})$ by an edge. The element of $\{v_1(z(\mathbf{x}), \mathbf{x}), v_2(z(\mathbf{x}), \mathbf{x})\}$ not being equal to $m_i(\mathbf{x})$ is denoted by $n_i(\mathbf{x})$. We will write $q_i(\mathbf{x})$ for the mark of $m_i(\mathbf{x})$.

Let us define the set of configurations in which the infinite cluster closest to the origin is one-armed

$$B = \{\mathbf{x} \in \mathbf{N} : \ell(\mathbf{x}) \geq 1 \text{ and } C_1(\mathbf{x}) \text{ contains a point of degree one}\},$$

and note that $B \subset \{\mathbf{x} \in \mathbf{N} : \ell(\mathbf{x}) \geq 1\}$. With this, for $\mathbf{x} \in \mathbf{N}$, let us define the finite point configuration

$$\mathbf{y}(\mathbf{x}) = \begin{cases} \{v_3(z(\mathbf{x}), \mathbf{x}), \ldots, v_{\tau(\mathbf{x})-1}(z(\mathbf{x}), \mathbf{x}), (m_{\tau(\mathbf{x})}(\mathbf{x}), q_{\tau(\mathbf{x})}(\mathbf{x}))\} & \text{if } \ell(\mathbf{x}) \geq 1 \text{ and } \mathbf{x} \notin B, \\ \emptyset, & \text{if } \ell(\mathbf{x}) = 0 \text{ or } \mathbf{x} \in B. \end{cases}$$

Let us fix $i \geq 3$. Let $\mathbf{x} \in \mathbf{N}$ be such that $\ell(\mathbf{x}) \geq 1$ and $\tau(\mathbf{x}) = i$. Then, we define a thinned configuration

$$\mathbf{x}^i = \mathbf{x}\backslash\mathbf{y}(\mathbf{x}),$$

and put $x^i = \{v : (v, q) \in \mathbf{x}^i\}$ for the corresponding nonmarked configuration.

We claim that $\{\ell(\mathbf{x}) \geq 1\} \cap \{\tau(\mathbf{x}) = i\}$ is a subset of $\{\ell(\mathbf{x}^i) \geq 1\}$. For this, first note that the removal of finitely many points of the point process can change the edge structure of the remaining points. Nevertheless, it cannot remove edges that were already present before the removal of points since $g_{2,f}$ is edge-preserving. Hence, all edges between two points of x^i in $g_{2,f}(\mathbf{x})$ also exist in $g_{2,f}(\mathbf{x}^i)$. In particular, if $g_{2,f}(\mathbf{x})$ contains an infinite cluster, then $g_{2,f}(\mathbf{x}^i)$ contains all but at most finitely many edges of this cluster. This implies that $\ell(\mathbf{x}^i) \geq 1$, hence the claim.

Then, the next claim is that for $\mathbf{x} \in \mathbf{N}$ with $\ell(\mathbf{x}) \geq 1$ and $\tau(\mathbf{x}) = i$, we have that $\mathbf{x}^i$ is contained in B. The proof of this claim is clear if $\mathbf{x} \in B$ since then $\mathbf{x}^i = \mathbf{x}$. Thus, we now consider the case $\mathbf{x} \notin B$. This case is illustrated in Figure 2. Recall that for $\mathbf{x} \in \mathbf{N}$, $z(\mathbf{x})$ cannot have degree higher than two in $g_{2,f}(\mathbf{x}^i)$, whereas it has degree at least one and its cluster $C_1(\mathbf{x}^i)$ is infinite in $g_{2,f}(\mathbf{x}^i)$. Note also that the edge between $z(\mathbf{x})$ and $n_i(\mathbf{x})$ still exists in $g_{2,f}(\mathbf{x}^i)$. Further, if $z(\mathbf{x})$ has degree two in $g_{2,f}(\mathbf{x}^i)$, then it is connected to the second-nearest f-neighbor toward $z(\mathbf{x})$ in $\mathbf{x}^i$, which is $v_2(z(\mathbf{x}), \mathbf{x}^i) = v_i(z(\mathbf{x}), \mathbf{x})$, whereas $v_1(z(\mathbf{x}), \mathbf{x}^i) = n_i(\mathbf{x})$. Now, since $\mathbf{x} \notin B$, $\ell(\mathbf{x}) \geq 1$ and $v_i(z(\mathbf{x}), \mathbf{x}) \in C_1(\mathbf{x})$, it follows that $v_i(z(\mathbf{x}), \mathbf{x})$ has degree equal to two in $g_{2,f}(\mathbf{x})$. Further, it is neither connected to $m_i(\mathbf{x})$ by an edge nor to $z(\mathbf{x})$ in this graph. Hence, both edges adjacent to $v_i(z(\mathbf{x}), \mathbf{x})$ also exist in $g_{2,f}(\mathbf{x}^i)$. But since $v_i(z(\mathbf{x}), \mathbf{x})$
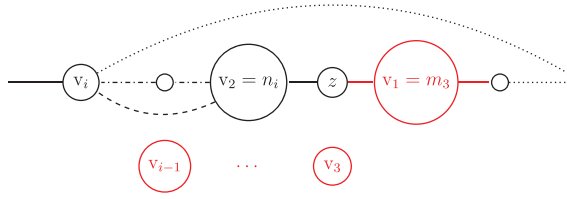
FIGURE 2 An illustration of the removal of the finite subconfiguration $\mathbf{y}(\mathbf{x}) = \{(m_i, q_i), \mathbf{v}_3(z(\mathbf{x})), \dots, \mathbf{v}_{n-1}(z(\mathbf{x}))\}$ from the configuration $\mathbf{x}$ satisfying $\ell(\mathbf{x}) \geq 1$, $\tau(\mathbf{x}) = i \geq 3$ and $\mathbf{x} \notin B$. The point $\mathbf{v}_i = \mathbf{v}_i(z(\mathbf{x}), \mathbf{x})$ is contained in the infinite cluster $C_1 = C_1(\mathbf{x})$ including $z = z(\mathbf{x})$, and it is not a neighbor of $m_i = m_i(\mathbf{x})$, which equals $\mathbf{v}_1 = \mathbf{v}_1(z(\mathbf{x}), \mathbf{x})$ here, whereas $\mathbf{v}_2 = \mathbf{v}_2(z(\mathbf{x}), \mathbf{x}) = n_i = n_i(\mathbf{x})$. (In the figure we use the short-hand notations $\mathbf{v}_j = \mathbf{v}_j(z(\mathbf{x}), \mathbf{x})$ for all values of j.) Hence, if $\mathbf{v}_i$ has degree two in C_1, then there are various possibilities respecting the degree bound of two to connect $\mathbf{v}_i$ to C_1 so that it is not connected to m_i by an edge. $\mathbf{v}_i$ can either be a direct neighbor of $\mathbf{v}_2$ (see dashed line) or a further point of the path from z to infinity starting with the edge from z to $\mathbf{v}_2$ (dash-dotted lines) or a nondirect neighbor of $\mathbf{v}_1$ on the path from z to infinity starting with the edge from z to $\mathbf{v}_1$ (dotted lines). Now, removing the finite subconfiguration $\mathbf{y}(\mathbf{x})$ from the realization (the nonmarked points corresponding to $\mathbf{y}$ are colored red in the figure), in the resulting f-2NN graph of $\mathbf{x}^i = \mathbf{x} \backslash \mathbf{y}(\mathbf{x})$, both edges adjacent to $\mathbf{v}_i$ are preserved. Also all edges from z to infinity starting with the edge from z to $\mathbf{v}_2$ are preserved, hence z is still contained in an infinite cluster, but the edge from z to $\mathbf{v}_1$ is removed. In the resulting new configuration, the second-nearest f-neighbor toward z is $\mathbf{v}_i$, and hence this is the only point of the configuration that could be connected to z by an edge. But $\mathbf{v}_i$ still cannot have degree 3 or more, hence it cannot be connected to z. Thus, in the new configuration, z is in an infinite cluster and has degree 1

has degree at most two in $g_{2,f}(\mathbf{x}^i)$, it follows that $z(\mathbf{x})$ and $\mathbf{v}_i(z(\mathbf{x}), \mathbf{x})$ are not connected by an edge in this graph. Hence, $\mathbf{x}^i \in B$, which implies the claim.

Note that by Lemma 3.1, the set B is a $\mathbb{P}$-null set, that is,

$$\mathbb{P}(\mathbf{X} \in \{\mathbf{x}^i : \mathbf{x} \in \mathbf{N}, \ell(\mathbf{x}) \geq 1 \text{ and } \tau(\mathbf{x}) = i\}) = 0. \tag{3.3}$$

This implies (3.2). Next, we have the following lemma.

Lemma 3.4. *Under the assumptions of Theorem 2.5, for any $i \geq 3$, $\mathbb{P}(\{\ell(\mathbf{X}) \geq 1\} \cap \{\tau(\mathbf{X}) = i\}) > 0$ implies $\mathbb{P}(\mathbf{X} \in \{\mathbf{x}^i : \mathbf{x} \in \mathbf{N}, \ell(\mathbf{x}) \text{ and } \tau(\mathbf{x}) = i\}) > 0$.*

Before proving the lemma, let us now explain why it implies Proposition 3.3.

Proof of Proposition 3.3.   By Lemma 3.4, where we show that if the collection of thinned configurations is contained in a $\mathbb{P}$-null set, also the nonthinned configurations form a $\mathbb{P}$-null set, we see that (3.3) implies (3.2), which concludes the proof of Proposition 3.3. ∎

Finally, it remains to verify Lemma 3.4. Its proof strongly relies on deletion-tolerance.

Proof of Lemma 3.4.   Let us fix $i \geq 3$ and assume that $\mathbb{P}(\{\ell(\mathbf{X}) \geq 1\} \cap \{\tau(\mathbf{X}) = i\}) > 0$. For the finite subprocess $\mathbf{Y} = \mathbf{y}(\mathbf{X})$ of $\mathbf{X}$ this means that

$$0 < \mathbb{P}(\{\ell(\mathbf{X}) \geq 1\} \cap \{\tau(\mathbf{X}) = i\}) = \mathbb{P}(\mathbf{X} \in \{\mathbf{x} \in \mathbf{N} : \ell(\mathbf{x}) \geq 1, \tau(\mathbf{x}) = i\})$$
$$\leq \mathbb{P}\left(\mathbf{X} \backslash \mathbf{Y} \in \{\mathbf{x}^i : \ell(\mathbf{x}) \geq 1, \tau(\mathbf{x}) = i\}\right).$$

Next, since $\mathbf{X}$ is assumed to be deletion-tolerant and $\mathbf{Y}$ is a finite subprocess of $\mathbf{X}$, by the equivalent characterization of deletion-tolerance as presented below Proposition 3.3, it follows that the law of $\mathbf{X} \backslash \mathbf{Y}$ is absolutely continuous with respect to the one of $\mathbf{X}$. This yields

$$0 < \mathbb{P}\left(\mathbf{X} \in \{\mathbf{x}^i : \ell(\mathbf{x}) \geq 1, \tau(\mathbf{x}) = i\}\right).$$

This implies the lemma. ∎

4 | EXAMPLES, DISCUSSION, AND EXTENSIONS

4.1 | Examples of deletion-tolerance

In this section, we verify Propositions 2.6 and 2.7. We first carry our the proof of Proposition 2.6.

Proof of Proposition 2.6. First, let $\mathbf{X}$ be an i.i.d. marked CPP, where X is stationary and non-equidistant. Then, due to the i.i.d. markings, $\mathbf{X}$ can be seen as a PPP with random directing measure $\Gamma(\mathrm{d}v) \otimes Q(\mathrm{d}q)$, where Q is the distribution of the marks and Γ is the (random) intensity of the unmarked CPP, see for example, [20, section 13]. Consider an event $\mathcal{A}$ such that $\mathbb{P}(\mathbf{X} \in \mathcal{A}) = 0$, then, by the definition of deletion tolerance, it suffices to show that for any bounded measurable set $B \subset \mathbb{R}^d$, we have

$$\mathbb{E}\left[\sum_{X_i \in X \cap B} \mathbb{1}\{\mathbf{X} \backslash \mathbf{X}_i \in \mathcal{A}\}\right] = 0.$$

But, using the Mecke formula for PPPs in general measurable spaces, see [20, theorem 13.8], we have that

$$\mathbb{E}\left[\sum_{X_i \in X \cap B} \mathbb{1}\{\mathbf{X} \backslash \mathbf{X}_i \in \mathcal{A}\}\right] = \mathbb{E}[\Gamma(B)\mathbb{P}(\mathbf{X} \in \mathcal{A}|\Gamma)] = 0,$$

as desired.

Let now $\mathbf{X}$ be a stationary Gibbs point processes. As in the proof of Proposition 2.7 below, we want to employ an equivalent characterization of deletion tolerance via almost-sure positivity of conditional void probabilities, see [14, theorem 1.1] for the case of unmarked point processes. In order to lift this to the level of marked point processes we extend the arguments as exhibited in [14]. However, first note that by our assumptions, for all bounded measurable $\Lambda \subset \mathbb{R}^d$, and almost all realizations $\mathbf{X}_{\Lambda^c}$ of $\mathbf{X}$ in Λ^c, we have

$$\mathbb{P}(\mathbf{X}_\Lambda = \emptyset|\mathbf{X}_{\Lambda^c}) \geq e^{-\lambda|\Lambda|} Z_\Lambda^{-1}(\mathbf{X}_{\Lambda^c}) > 0,$$

where $\lambda > 0$ denotes the intensity of the underlying unmarked Poisson point process.

Using this, on a general level, if for some bounded $\Lambda \subset \mathbb{R}^d$ and measurable $\mathcal{A} \subset \mathbb{R}^d \times M$, we have $\mathbb{P}(\mathbf{X}_{\Lambda^c} \in \mathcal{A}) > 0$, then $\mathbb{P}(\mathbf{X} \in \mathcal{A}) \geq \mathbb{P}(\mathbf{X}_{\Lambda^c} \in \mathcal{A}, \mathbf{X}_\Lambda = \emptyset) = \mathbb{E}[\mathbb{1}\{\mathbf{X}_{\Lambda^c} \in \mathcal{A}\}\mathbb{P}(\mathbf{X}_\Lambda = \emptyset|\mathbf{X}_{\Lambda^c})] > 0$ and hence, the law of $\mathbf{X}_{\Lambda^c}$ is absolutely continuous with respect to the law of $\mathbf{X}$. In order to deduce deletion tolerance from this absolute continuity, we consider open balls $B(\mathbf{x}, r)$ in $\mathbb{R}^d \times M$, where $\mathbf{x} \in \mathbb{Q}^d \times D$, with D a countable-dense subset of the Polish space M, and $r \geq 0$ is rational. We let C denote the set of unions of finitely many such balls. Then, by the local finiteness of $\mathbf{X}$, for all $\mathbf{X}$-points $\mathbf{Z}$, there exists a C-valued random variable $\boldsymbol{\Gamma}$ such that $\mathbb{P}(\mathbf{X}_{\boldsymbol{\Gamma}} = \delta_{\mathbf{Z}}) = 1$. But then, if for some measurable $\mathcal{A} \subset \mathbb{R}^d \times M$ we have $\mathbb{P}(\mathbf{X} - \delta_{\mathbf{Z}} \in \mathcal{A}) > 0$, it follows that

$$0 < \mathbb{P}(\mathbf{X} - \delta_{\mathbf{Z}} \in \mathcal{A}, \mathbf{X}_{\boldsymbol{\Gamma}} = \delta_{\mathbf{Z}}) = \mathbb{P}(\mathbf{X}_{\boldsymbol{\Gamma}^c} \in \mathcal{A}) = \sum_{\Gamma \in C} \mathbb{P}(\mathbf{X}_{\Gamma^c} \in \mathcal{A}, \boldsymbol{\Gamma} = \Gamma),$$

and hence $\mathbb{P}(\mathbf{X}_{\Gamma^c} \in \mathcal{A}) > 0$ for some Γ. But this implies $\mathbb{P}(\mathbf{X} \in \mathcal{A}) > 0$ by the absolute continuity asserted above. ∎

Finally, we prove Proposition 2.7. The proof is based on the following lemma.

Lemma 4.1. *Let the marked point process* **X** *be deletion-tolerant. Then, the underlying point process* X *is also deletion-tolerant.*

In order to keep the proof short, we again assume that **X** takes values in the configuration space **N** and X in the configuration space N, which were defined in Section 3.

Proof. Let A be any measurable subset of **N**. Further, let Y be any X-point. Let P be the mark of Y (defined realizationwise). Then (Y, P) is an **X**-point and we have

$$\mathbb{P}(X \in A) = \mathbb{P}\left(X \in A, (P_i)_{i \in \mathbb{N}} \in M^{\mathbb{N}}\right) = \mathbb{P}(\mathbf{X} \in A \times M^{\mathbb{N}}).$$

Since $A \times M^{\mathbb{N}}$ is a measurable subset of **N** and **X** is deletion-tolerant, we have

$$0 < \mathbb{P}(\mathbf{X} \setminus \{(Y, P)\} \in A \times M^{\mathbb{N}}) = \mathbb{P}(X \setminus \{Y\} \in A).$$

Hence, the distribution of $X \setminus \{Y\}$ is absolutely continuous with respect to the one of X, as wanted. ∎

Proof of Proposition 2.7. Thanks to Lemma 4.1, it suffices to show that X is not deletion-tolerant. Further, as already mentioned in the proof of Proposition 2.6, the following assertion is known from [14, theorem 1.1]: If X is deletion-tolerant, then for any bounded $S \in \mathcal{B}(\mathbb{R}^d)$, we have that almost surely

$$\mathbb{P}(X \cap S = \emptyset | X \cap S^c) > 0. \tag{4.1}$$

Assume now that the conditions of the proposition are satisfied. Let $\mathcal{K} \in \mathcal{B}(\mathbb{R}^d)$ be compact. Since X is stationary with positive intensity, there exists some $k \in \mathbb{N}$ such that

$$\mathbb{P}(\#(X \cap \mathcal{K}) = k) > 0.$$

Let us now consider the measurable set $F = h_{\mathcal{K}}^{-1}(k)$. We have

$$\mathbb{P}((X \cap \mathcal{K}^c) \in F) > 0.$$

Hence, the conditional probability

$$\mathbb{P}(X \cap \mathcal{K} = \emptyset | X \cap \mathcal{K}^c \in F), \tag{4.2}$$

is well-defined. However, since $\{X \cap \mathcal{K}^c \in F\} \subseteq \{\#(X \cap \mathcal{K}) = k\}$, (4.2) equals zero. This shows that for our choice of X and for $S = \mathcal{K}$, the assertion (4.1) does not hold, and hence X is not deletion-tolerant. ∎

4.2 | SINR graphs as subgraphs of f-kNN graphs

Let us briefly summarize the relation between Theorem 2.5 and [15, theorem 2.2]. Indeed, the two proofs are similar, in particular, two steps of the proof, Lemma 3.1 and the part of Proposition 2.6 regarding the Cox case already appeared in [15]. However, in [15] we focused on the particular case of SINR graphs based on stationary and nonequidistant CPPs, having concrete applications in

telecommunications in mind, and we did not aim at checking whether our proof works also for a wider class of point processes or graphs. Thus, the main novelty in this paper is not that we exploit new proof techniques (although the proofs of Proposition 2.7 and the part of Proposition 2.6 regarding Gibbs point processes have no analogues in [15]). Instead, we highlight that apart from the general combinatorial condition of working with undirected and stationary random graphs with degrees bounded by two, two properties are crucial for a straightforward generalization of the proof in [15]: (1) the deletion tolerance of the underlying point process and (2) the edge-preserving property of the graph. The latter observation allows for the extensions of Theorem 2.5 presented in Section 4.3.

This puts the result into a general framework and allows for generalizations of the result both with respect to the type of graph and with respect to the kind of point process. Here, let us note that the SINR graph is not a special case of an f-kNN graph, but a proper subgraph of an f-2NN graph under particular choices of the parameters, which is itself edge-preserving. Nonpercolation in f-2NN graphs was not even known before the present paper in the simplest case represented by the **B**-2NN graph.

In order to make the relation between f-kNN graphs and SINR graphs explicit we recall the definition and interpretation of the latter graphs. Let $M = N = [0, \infty)$, $\|\cdot\| = \|\cdot\|_2$, P_o be a nonnegative random variable, and $\mathbf{X} = \{(X_i, P_i)\}_{i \in I}$ an i.i.d. marked point process in $\mathbb{R}^d \times [0, \infty)$ such that all P_i are distributed as P_o. Let $\ell : (0, \infty) \to [0, \infty)$, the so-called *path-loss function*, be a monotone decreasing function. Typical examples of path-loss functions correspond to *Hertzian propagation* (see [6, 7]), for example, for $\alpha > d$, the unbounded function $\ell(r) = r^{-\alpha}$, its truncated variant $\ell(r) = \min\{1, r^{-\alpha}\}$, and its "shifted" variant $\ell(r) = (1 + r)^{-\alpha}$. Now, define $f(x, p) = 1/(p\ell(\|x\|))$. In a telecommunication context, for $(X_i, P_i) \in \mathbf{X}$ and $x \in \mathbb{R}^d$, P_i expresses the signal power transmitted by a device at spatial position X_i, and ℓ describes propagation of signal strength over distance. Note that ℓ need not be strictly decreasing, which gives relevance to Part (2) of Definition 2.1 in order to make the f-ordering well-defined. We observe that in case P_o is almost surely equal to a fixed positive constant, then the arising f-kNN graph is the **B**-kNN graph.

In this setting, the *SINR graph* [6] is usually introduced in the following way. Let N_o be another nonnegative random variable independent of $\mathbf{X}$. Choose two further parameters $\gamma, \tau > 0$, the so-called *interference-cancellation factor* and the *SINR threshold*, respectively, and for $i, j \in I$, $i \neq j$, connect X_i and X_j by an edge whenever the *SINR constraint* is satisfied in both directions, that is,

$$P_i \ell(|X_i - X_j|) > \tau \left(N_o + \gamma \sum_{k \in I \setminus \{i,j\}} P_k \ell(|X_k - X_j|) \right), \tag{4.3}$$

and the same holds with the roles of i and j interchanged. Then, it is known from [6, theorem 1] that if $\mathbf{X}$ is a simple point process (even if not stationarity or not nonequidistant), all degrees in the SINR graph are less than $k = 1 + 1/(\tau\gamma)$. Using the elementary arguments of [24, lemma 4.1.13], one can easily verify that if $X = \{X_i\}_{i \in I}$ is also nonequidistant, then the SINR graph is a subgraph of the f-kNN graph of the present example. Further, if $N_o > 0$ is deterministic, then the SINR graph has bounded edge lengths and hence is a subgraph of the Gilbert graph introduced in [10]. The same assertion holds also for $N_o = 0$ in case ℓ has bounded support. For positive assertions about percolation in SINR graphs based on various kinds of point processes, we refer the reader, for example to [2, 6, 7, 15, 19, 24, 25].

Hence, for point processes satisfying the conditions of Theorem 2.5, there is no percolation in the SINR graph if its degrees are bounded by 2, which is always the case if $\gamma \geq 1/(2\tau)$. Thanks to Proposition 2.6, in particular, Gibbs point processes are covered by this result. To the best of our knowledge, there have been no results about SINR percolation for Gibbs point processes before (apart from the degree bounds themselves). Regarding nonpercolation in case degrees are bounded by two, the case of CPPs was handled in [15, theorem 2.2]. Here, based on the observation [15, section 5.2,

proof of proposition 5.8] that SINR graphs are edge-preserving on their own right in the sense of Definition 3.2, we carried out a certain variant of the proof of Theorem 2.5 for the SINR graph directly, with no direct reference to f-kNN (or even **B**-kNN) graphs.

The aforementioned positive results on SINR percolation guarantee that for various kinds of point processes, the SINR graph percolates with positive probability for some positive γ given that λ is sufficiently large, while all the other parameters (depending on the type of point process) are fixed. In other words, we know that $k^* < \infty$ holds for the infimum k^* of all degree bounds k such that there exists an SINR graph with largest degree equal to k that percolates. There are multiple interesting open questions related to this. First, what is the smallest value of such k^*, and how does it depend on the type of the point process? The main results of the present paper imply that for stationary and nonequidistant point processes that are deletion-tolerant, we have $k^* \geq 3$. Further, according to the high-confidence results of [1], $k^* \geq 5$ for the two-dimensional PPP. Second, is the smallest such degree bound the same for SINR graphs as for the underlying **B**-kNN graph? While the relationship between Gilbert graphs and SINR graphs is clear (viz., Gilbert graphs are increasing limits of SINR graphs as $\gamma \downarrow 0$), we are not aware of results stating that the **B**-kNN graph is an increasing limit of certain SINR graphs with degree bound k, and such a result may not be true in general. Namely, it may be the case that an SINR constraint of the form (4.3) with degree bound $k \in \mathbb{N}$ poses stronger restrictions on the edges of the graph than a **B**-kNN constraint for the same k. We defer the investigation of such questions to future work, noting that numerical evidence indicates that the two critical degree bounds are not the same in general, see for example, Figure 3.

4.3 | Extensions and limitations of the proof of Theorem 2.5

We now present examples of graphs with degrees bounded by two that are not contained in an f-2NN graph but have rather similar properties to it, to the extent that the proof techniques of Theorem 2.5 are applicable to it.

Example 4.2 (Locally furthest neighbors)**.** The edge-preserving property of f-k NN graphs (see Definition 3.2) also holds if we replace the "k-nearest neighbors with respect to the f-ordering" in their definition by "k-furthest neighbors in a bounded (possibly random) set shifted to the point, w.r.t. f-ordering". For the sake of simplicity of notation, let us explain how this works in the case of **B**-k NN graph. The case of general f-k NN graphs can be handled similarly, taking into account also the marks and using the f-ordering instead of the ordering of Euclidean norms. We assume throughout this discussion that the point process X is stationary, nonequidistant, and deletion-tolerant.

Let us fix a deterministic measurable set $A \subseteq \mathbb{R}^d$ of finite Lebesgue measure and define a random graph with vertex set **X** via connecting two different points X_i, X_j of the point process X by an edge whenever X_j is one of the $k \in \mathbb{N}$ furthest neighbors in $(A + X_i) \setminus \{X_i\}$ and the same holds with the roles of i and j interchanged. It is easy to see that this graph is well-defined and edge-preserving. Clearly, for $k = 2$ it has degrees bounded by two. Hence, non-percolation of the graph can be verified along the lines of the proof of Theorem 2.5 also in the case of this graph. If A is bounded, then the graph has bounded edge lengths (unlike the **B**-k NN graph).

This approach can be extended to the case when the deterministic set $A + X_i$ is replaced by a random set A_{X_i} in such a way that $\{A_{X_i}\}_{i \in I}$ are stationary, since the edge-preserving property and the degree bound of two are still preserved. E.g., the proof techniques of Theorem 2.5 are still applicable if $\{A_{X_i}\}_{i \in I}$ is a Boolean model with random radii based on $X = \{X_i\}_{i \in I}$ [21]. Instead of connecting X_i to the two furthest neighbors in $X \cap A_{X_i}$ by an edge, one can also connect it to the two nearest neighbors in $X \cap A_{X_i}$ and obtain the same result. We refrain from presenting further details here.
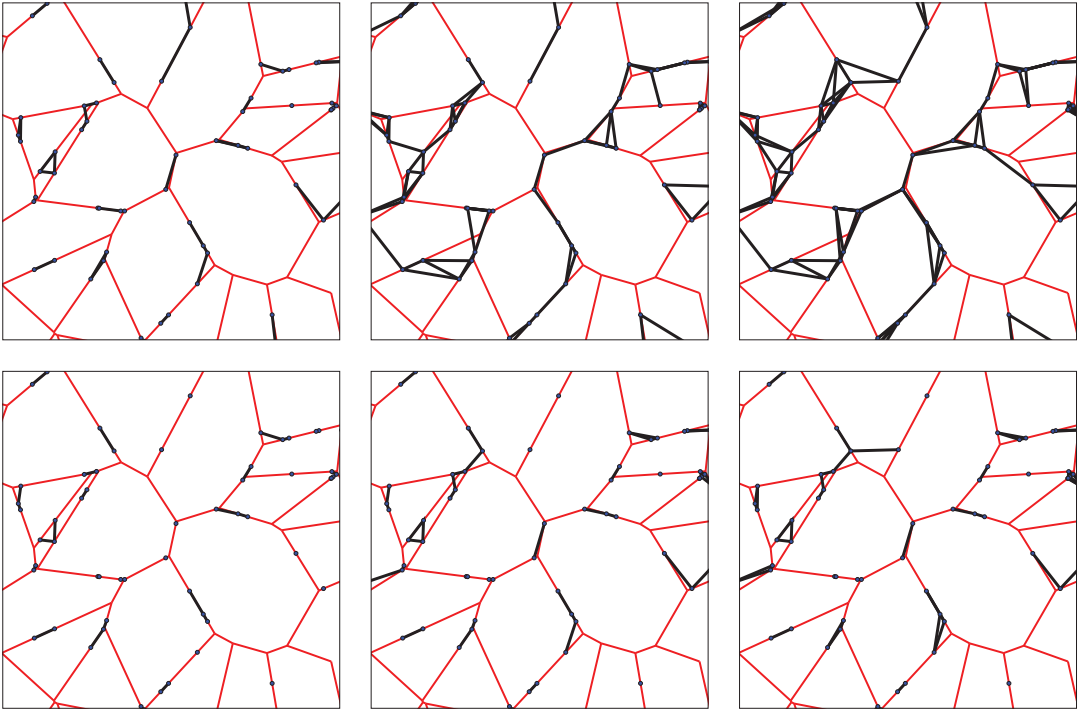
FIGURE 3 **B**-kNN graphs (in the first line) and signal-to-interference plus noise ratio (SINR) graphs with degree bound k (in corresponding order in the second line) for $k = 2, 4, 5$, for X being a stationary Cox point process. The random intensity measure Λ is given as the edge-length measure (i.e., the one-dimensional Hausdorff measure) of a two-dimensional Poisson–Voronoi tessellation. The simulation leads to the conjecture that the smallest k such that the **B**-kNN graph percolates is $k = 5$, which would mean that it equals the one of the two-dimensional PPP (which is 5 according to the high-confidence results of [1]). Further, it is known from [25] that in this case, for large enough λ and accordingly chosen small $\gamma > 0$, there is also percolation in the SINR graph. However, it does not seem to be the case that this already happens when the degree bound equals 5, as the simulation suggests. Here, γ is just slightly bigger than $1/(5\tau)$, that is, a small further increase of γ would increase the degree bound to 6, but the SINR graph is still much less connected than the corresponding **B**-5NN graph

The next example shows that there are graphs defined very similarly to the f-2NN graph for which our methods are not applicable.

Example 4.3 (k_1th and k_2th nearest neighbors). Let $k_1, k_2 \in \mathbb{N}$ such that $k_1 < k_2$. Similarly to Definition 2.3, the f-k_1th or k_2th-nearest neighbor (f-(k_1, k_2)NN) *graph* $g_{(k_1,k_2),f}(\mathbf{X})$ is defined as the random graph having vertex set X and for all $i \in I$ and $n \in \{1, 2\}$ an edge between X_i and $\mathrm{v}_n^f(X_i, \mathbf{X})$ whenever $X_i \in \{\mathrm{v}_{k_1}^f(\mathrm{v}_n^f(X_i, \mathbf{X}), \mathbf{X}), \ldots, \mathrm{v}_{k_2}^f(\mathrm{v}_n^f(X_i, \mathbf{X}), \mathbf{X})\}$. In the case $k_1 = 1$ and $k_2 = 2$, $g_{(1,2),f}(\mathbf{X})$ is equal to the f-2NN graph $g_{2,f}(\mathbf{X})$. However, it is easy to see that if $(k_1, k_2) \neq (1, 2)$, then the f-(k_1, k_2)NN graph is in general not edge-preserving in the sense of Definition 2.3. This is a major obstacle for generalizing the proof of Theorem 2.5 to the case $(k_1, k_2) \neq (1, 2)$, despite the fact that many of the proof ingredients of the theorem are still available in this case.

4.4 | Relation of our model to outdegree-one graphs

In the setting of outdegree-one graphs [3], one considers directed percolation in a directed graph arising as a deterministic and stationary function of a PPP in $\mathbb{R}^d$, where each vertex has precisely one out-degree. It was shown in [3] that under certain stabilization and looping conditions of the

edge-drawing mechanism, this model does not percolate, in the sense that the out-component (or the in-component) of any vertex is almost-surely finite, see also [11]. In [23], it was shown that this result is applicable for the example of the *kth nearest neighbor graph*, where the outgoing edge of a vertex points to the *k*th nearest neighbor of the vertex in the point process. This setting looks rather similar to the one that we are considering but is still different from it, for at least two reasons. First, although it is tempting to think that the **B**-*k*NN graph can be obtained as a deterministic transformation of a (stationary) outdegree-one graph satisfying the conditions of [3], we were not able to find such an outdegree-one graph. Second, the *k*th nearest neighbor graph is only contained in the **U**-*k*NN graph, not in the **B**-*k*NN one; in particular, the results of [23] cannot be derived from our ones.

ACKNOWLEDGMENTS

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