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# **SINR percolation for Cox point processes with random powers**

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## Abstract

Signal-to-interference plus noise ratio (SINR) percolation is an infinite-range dependent variant of continuum percolation modeling connections in a telecommunication network. Unlike in earlier works, in the present paper the transmitted signal powers of the devices of the network are assumed random, i.i.d. and possibly unbounded. Additionally, we assume that the devices form a stationary Cox point process, i.e., a Poisson point process with stationary random intensity measure, in two or higher dimensions. We present the following main results. First, under suitable moment conditions on the signal powers and the intensity measure, there is percolation in the SINR graph given that the device density is high and interferences are sufficiently reduced, but not vanishing. Second, if the interference cancellation factor  $\gamma$  and the SINR threshold  $\tau$  satisfy  $\gamma \geq 1/(2\tau)$ , then there is no percolation for any intensity parameter. Third, in the case of a Poisson point process with constant powers, for any intensity parameter that is supercritical for the underlying Gilbert graph, the SINR graph also percolates with some small but positive interference cancellation factor.

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## 1 Introduction and main results

Let  $\mathbb{X}^\lambda = \{(X_i, P_i)\}_{i \in I}$  be an i.i.d. marked *Cox point process* (CPP) in  $\mathbb{R}^d \times [0, \infty)$  for  $d \geq 2$ , with directing measure  $\lambda \Lambda \otimes \zeta$  where  $\Lambda$  is stationary with  $\mathbb{E}[\Lambda(Q_1)] = 1$  and  $Q_n = [-n/2, n/2]^d$  for  $n > 0$ . We consider the *SINR graph* with vertex set given by the first component of  $\mathbb{X}^\lambda$ , which we denote by  $X^\lambda$ . Here, every pair  $X_i \neq X_j \in X^\lambda$  of vertices is connected by an edge if and only if

$$\begin{aligned} P_i \ell(|X_i - X_j|) &> \tau (\mathcal{N} + \gamma \sum_{k \in I \setminus \{i, j\}} P_k \ell(|X_k - X_j|)) \quad \text{and} \\ P_j \ell(|X_i - X_j|) &> \tau (\mathcal{N} + \gamma \sum_{k \in I \setminus \{i, j\}} P_k \ell(|X_k - X_i|)). \end{aligned} \quad (1.1)$$

In (1.1),  $\tau > 0$  is fixed and called the *SINR threshold*,  $\mathcal{N} \geq 0$  represents *noise*,  $r \mapsto \ell(r) \in [0, \infty)$  is referred to as the *path-loss function* and  $\gamma \geq 0$  is called the *interference-cancellation factor*. The random variables  $\{P_i\}_{i \in I}$  are often called *random powers* and the term

$$I(X_i, X_j, \mathbb{X}^\lambda) = \sum_{k \in I \setminus \{i, j\}} P_k \ell(|X_k - X_j|)$$

is referred to as *interference*. We will use the notation  $g_{(\gamma, \zeta)}(\mathbb{X}^\lambda)$  to indicate the SINR graph, suppressing the dependencies on  $\tau, \mathcal{N}$  and  $\ell$ , but highlighting the dependence on  $\gamma$  and the distribution of the powers  $\zeta$ . We refer to [DBT05, Section 1] for further interpretation of the modeling parameters.

The SINR graph has a nice interpretation in the study of device-to-device telecommunication systems where the devices  $X^\lambda$  can communicate directly with each other if their mutual distance, represented by the path-loss function, and their individual powers, are sufficiently strong to overcome thermal noise plus all the interference coming from the other devices. If this is the case, then the possibility to communicate is represented by an undirected edge. The SINR graph has been the subject of, by now, a large body of works, which we will further elaborate on in Section 2.

Our main interest lies in percolation properties of the SINR graph, as has been first studied in [DBT05, DFM<sup>+</sup>06, FM07]. We say that  $g_{(\gamma, \zeta)}(\mathbb{X}^\lambda)$  *percolates* if  $g_{(\gamma, \zeta)}(\mathbb{X}^\lambda)$  contains an unbounded connected component. Here we focus on the following key quantities. First, the *critical interference-cancellation factor* is defined as

$$\gamma_\zeta(\lambda) = \sup \{ \gamma > 0 : \mathbb{P}(g_{(\gamma, \zeta)}(\mathbb{X}^\lambda) \text{ percolates}) > 0 \}. \quad (1.2)$$

In words, it represents the maximal amount of interference that can be added to the system and still maintain percolation. Second, the *critical intensity* is defined as

$$\lambda_\zeta = \inf \{ \lambda > 0 : \gamma_\zeta(\lambda') > 0, \forall \lambda' > \lambda \}, \quad (1.3)$$

which describes the smallest intensity such that for all larger intensities the addition of a small amount of interference does not destroy percolation.

For the statement of our first main result, we assume certain decorrelation and connectivity properties for the directing measure  $\Lambda$  of the underlying CPP. The precise definitions for  $\Lambda$  to be *stabilizing*, *b-dependent* or *asymptotically essentially connected* will be presented in Definitions 2.1 and 2.2 in Section 2, where we will also mention a number of relevant examples of random measures satisfying these definitions. We denote by  $P_o$  a generic power random variable distributed according to  $\zeta$ . We put  $P_{\sup} = \text{ess sup } \zeta$ . Our first result establishes existence of a supercritical regime of percolation for the SINR graph based on CPPs with random powers.

**Theorem 1.1.** Let  $d \geq 2$ ,  $\mathcal{N}, \tau > 0$ ,  $P_{\sup} = \infty$ , and let  $\Lambda$  be stabilizing. Further, let  $\ell$  satisfy the following assumptions:

- (i)  $\ell$  is continuous, constant on  $[0, d_o]$  for some  $d_o \geq 0$ , and on  $[d_o, \infty) \cap \text{supp}(\ell)$  it is strictly decreasing,
- (ii)  $\int_0^\infty \ell(r) dr < \infty$ , and
- (iii)  $1 \geq \ell(0)$ .

Then  $\lambda_\zeta < \infty$  holds if at least one of the following conditions is satisfied:

- 1  $\ell$  has unbounded support,  $\Lambda$  is  $b$ -dependent, and  $\mathbb{E}[\exp(\alpha \Lambda(Q_1))] < \infty$  as well as  $\mathbb{E}[\exp(\alpha P_o)] < \infty$  holds for some  $\alpha > 0$ , or
- 2  $\ell$  has bounded support,  $\mathbb{E}[P_o] < \infty$ , and  $\Lambda$  is asymptotically essentially connected, or
- 3  $\ell$  has bounded support,  $\mathbb{E}[P_o] < \infty$ , and  $\text{supp } \text{supp}(\ell)$  is sufficiently large depending on  $\Lambda$ .

The proof of Theorem 1.1 uses some arguments of the proof of a previous result, Proposition 2.3, which covers the case of bounded powers but does not tell anything about the case  $P_{\sup} = \infty$ . We will discuss the relation to these results in detail in Section 2.

Our second main result establishes a uniform upper bound on the critical interference-cancellation factor.

**Theorem 1.2.** Let  $d \geq 1$ ,  $\mathcal{N} \geq 0$  and  $\tau > 0$ , then  $\gamma_\zeta(\lambda) \leq 1/(2\tau)$ .

Note that we do not require any stabilization or connectedness. The proof of Theorem 1.2 rests on showing absence of percolation in the SINR graph with a maximal degree given by 2.

Finally, our third main result states that the critical intensity parameter for the SINR graph can be represented as the critical threshold for percolation of an associated Gilbert graph in any dimension. For this we assume a simpler setting in which  $\Lambda(dx)$  equals the Lebesgue measure  $dx$ , i.e., the CPP is in fact a Poisson point process (PPP), and the powers are non-random and given by  $P > 0$ . The associated SINR

graph is denoted by  $g_{(\gamma,P)}(X^\lambda)$  and correspondingly we write  $\lambda_P$  for the critical intensity. Then, note that for  $\gamma = 0$ , the SINR graph is in fact a *Poisson–Gilbert graph* (see [Gil61]) with connectivity threshold given by

$$r_B = \ell^{-1}(\tau\mathcal{N}/P). \quad (1.4)$$

We denote this Gilbert graph by  $g_{r_B}(X^\lambda)$ . It is a standard result in continuum percolation that for the Poisson–Gilbert graph with connectivity threshold  $0 < r < \infty$ , there exists a unique critical intensity  $0 < \lambda_c(r) < \infty$  that separates a supercritical regime, where  $\lambda > \lambda_c(r)$ , in which the Gilbert graph percolates with probability one and a subcritical regime, where  $\lambda < \lambda_c(r)$ , in which the Gilbert graph does not percolate almost surely, see for example [MR96, Section 3].

**Theorem 1.3.** Let  $d \geq 2$ ,  $\mathcal{N}, \tau, P > 0$  and  $\Lambda(dx) = dx$ . Further, if  $\ell$  satisfies the assumption stated in Theorem 1.1, then  $\lambda_P = \lambda_c(r_B)$ .

Theorem 1.3 extends the result [DFM<sup>+</sup>06, Theorem 1] to dimensions  $d \geq 3$  using new techniques, see Section 2 for details.

In the following section, we lay out the strategies for the proofs of our main results, make references to preceding work and comment on limitations and further extensions of the statements presented.

## 2 Strategy of proofs

The study of percolation properties of random graphs traces back many decades and results are available in textbooks, see for example [MR96, Gri99]. The first results for percolation in the continuum were presented in the landmark paper by Gilbert [Gil61], where non-trivial percolation was established for the *Poisson–Gilbert graph*  $g_r(X^\lambda)$ , consisting of vertices given by a homogeneous PPP  $X^\lambda$  in  $\mathbb{R}^2$  with intensity  $\lambda > 0$  and edges connecting any pair of vertices with distance less than  $r > 0$ . The context of telecommunications was already mentioned there.

Recently, in [HJC19], existence of a unique non-trivial critical intensity threshold was established for Gilbert graphs where the underlying point process is a stationary CPP with directing measure  $\lambda\Lambda$  under some conditions on  $\Lambda$  that we state here for subsequent reference. Let  $Q_n(x) = Q_n + x$  denote the box with side length  $n$ , centered at  $x \in \mathbb{R}^d$ , and  $\text{dist}(x, A) = \inf\{|x - y| : y \in A\}$ .

**Definition 2.1** (Stabilization). The random measure  $\Lambda$  is called *stabilizing* if there exists a random field of *stabilization radii*  $R = \{R_x\}_{x \in \mathbb{R}^d}$  defined on the same probability space as  $\Lambda$  such that, writing

$$R(Q_n(x)) = \sup_{y \in Q_n(x) \cap \mathbb{Q}^d} R_y, \quad n \geq 1, x \in \mathbb{R}^d,$$

the following hold.

- 1  $(\Lambda, R)$  is jointly stationary,

2  $\lim_{n \uparrow \infty} \mathbb{P}(R(Q_n) < n) = 1$ , and

3 for all  $n \geq 1$ , non-negative bounded measurable functions  $f$ , and finite  $\varphi \subset \mathbb{R}^d$  with  $\text{dist}(x, \varphi \setminus \{x\}) > 3n$  for all  $x \in \varphi$ , the following random variables are independent:

$$f(\Lambda_{Q_n(x)}) \mathbb{1}\{R(Q_n(x)) < n\}, \quad x \in \varphi.$$

A strong form of stabilization is when  $\Lambda$  is *b-dependent* for some  $b > 0$ , that is, the restrictions  $\Lambda_A$  and  $\Lambda_B$  of  $\Lambda$  to the measurable sets  $A, B \subset \mathbb{R}^d$  are independent whenever  $\text{dist}(A, B) > b$ . For *b*-dependence of subsets of  $\mathbb{Z}^d$  we will use the analogous definition but with  $\text{dist}$  replaced by the  $\ell^\infty$ -distance.

**Definition 2.2** (Asymptotic essential connectedness). The stabilizing random measure  $\Lambda$  with stabilization radii  $R$  is *asymptotically essentially connected* if for all  $n \geq 1$ , whenever  $R(Q_{2n}) < n/2$ , we have that

- 1  $\text{supp}(\Lambda_{Q_n})$  contains a connected component of diameter at least  $n/3$ ,
- 2 any two connected components of  $\text{supp}(\Lambda_{Q_n})$  of diameter at least  $n/9$  are contained in the same connected component of  $\text{supp}(\Lambda_{Q_{2n}})$ .

The class of stabilizing random measures includes a number of interesting and relevant examples, for instance directing measures given via random tessellations based on PPPs. As already mentioned in [HJC19], for example the edge-length measures of *Poisson–Voronoi* and *Poisson–Delaunay tessellations* are asymptotically essentially connected but not *b*-dependent. However, the edge-length measures of Poisson line tessellations in  $\mathbb{R}^2$  are not even stabilizing. Stabilizing random measures that are absolutely continuous with respect to the Lebesgue measure are, e.g., the directing measure of some *modulated PPPs* or *shot-noise fields* with compactly supported kernel. In particular, a modulated PPP [CSK<sup>+</sup>13, Section 5.2.2] can be defined with directing  $\Lambda(dx) = \lambda_1 \mathbb{1}\{x \in \Xi\}dx + \lambda_2 \mathbb{1}\{x \notin \Xi\}dx$ , for some Poisson–Boolean model  $\Xi$ , see Section 3.3 for a proper introduction, and  $\lambda_1, \lambda_2 \geq 0$ , in which case it is even *b*-dependent. Here we see that if  $\lambda_1$  and  $\lambda_2$  are positive, then  $\Lambda$  is asymptotically essentially connected and there exist examples, both for  $\lambda_1 > 0$  and  $\lambda_2 = 0$  as well as  $\lambda_1 = 0$  and  $\lambda_2 > 0$ , such that asymptotic essential connectedness fails. However, if  $\Xi$  is in the supercritical regime for percolation and  $\lambda_1 > 0$ , then it can be seen that  $\Lambda$  is asymptotically essentially connected. Also shot-noise fields are not asymptotically essentially connected in general, see [HJC19], but in some relevant cases they are, see [Tób19a, Section 2.5.1]. Moreover, they are always *b*-dependent.

For a stabilizing directing measure and fixed connectivity threshold  $r \geq 0$ , in [HJC19] it was proved that for sufficiently small intensity  $\lambda$  the process is subcritical. On the other hand, for any  $r > 0$  and asymptotically essentially connected directing measures, [HJC19] establishes existence of a supercritical percolation phase. It will be important for our proofs that in [Tób19a] it was additionally verified that for all sufficiently large  $r > 0$  the requirement of asymptotically essentially connectedness can be replaced by stabilization and still existence of a supercritical percolation regime is guaranteed for sufficiently large  $\lambda$ .

In the context of telecommunications, the extension of Poisson–Gilbert graphs towards Gilbert graphs based on CPPs allows to study long-range communication properties in device-to-device networks where

devices are placed according to a PPP in random environment that is represented by the directing measure  $\Lambda$ . Standard examples of asymptotically-essentially-connected environments with applications in telecommunications are Poisson-Voronoi, or Poisson-Delaunay tessellations, see for example [HJC19, CGH<sup>+</sup>18]. However, the edge-drawing mechanism in these *Cox–Gilbert graphs* remains as in the classical case.

Another line of research aimed towards a different kind of extension of the Poisson–Gilbert graph with respect to the edges. Starting with the papers [DBT05, DFM<sup>+</sup>06], still based on a homogeneous PPP in  $\mathbb{R}^2$ , the edge-drawing mechanism was replaced by the one described in (1.1) with constant powers, giving rise to the SINR graph on PPPs, or the *Poisson–SINR graph*. This introduces long-range dependencies for the construction of edges into the system. However, using comparison techniques with the Poisson–Gilbert graph, again non-trivial percolation properties could be established. Let us mention that the SINR graph has very different monotonicity properties compared to the Poisson–Gilbert graph. To see this, note that in the presence of interference, an increase of the intensity  $\lambda$  also leads to an increase of the interference and thus to the potential loss of edges. On the other hand, for the Poisson–Gilbert graph, the connectivity increases with the intensity.

In [Tób19a], the two extensions described above were for the first time considered jointly, giving rise to the SINR graph based on CPPs, the *Cox–SINR graph*. There it was established, in the case for non-random powers  $P > 0$ , that for sufficiently large  $\lambda$  and asymptotically essentially connected directing measures  $\Lambda$ , the graph  $g_{(\gamma, P)}(\mathbb{X}^\lambda)$  percolates almost surely at least for some  $\gamma > 0$ , and thus in particular  $\lambda_P < \infty$  in all dimensions  $d \geq 2$  in case  $\Lambda(dx) = dx$ .

So far, none of the presented graphs used an additional randomness for the construction of edges, other than the vertex positions. A canonical way to introduce such a randomness is by marking every vertex  $X_i$  with an i.i.d. random variable  $P_i$ , its power. This power value defines the connection radius  $\ell^{-1}(\tau N_0/P_i)$  corresponding to  $X_i$  (cf. (1.4)). In the context of the Poisson–Gilbert graph, two vertices may be connected if and only if their distance is smaller than the sum of their connection radii. The percolation properties of the associated *Poisson–Gilbert graph with random radii* is well-understood, see [MR96]. Corresponding general results for the *Cox–Gilbert graph with random radii* are not available in the literature yet. However, using couplings with Cox–Gilbert graphs with constant radii, it is easy to derive existence of a supercritical phase under stabilization assumptions on  $\Lambda$  and lower bounds on the essential infimum of  $\zeta$ , cf. [Tób19b, Section 4.2.3.4], but the main questions around existence of a subcritical phase for unbounded radii are completely open.

For the case of the SINR graph with random powers based on PPPs, or *Poisson–SINR graph with random powers*, the paper [KY07] presents first results similar to the assertions presented in [DBT05, DFM<sup>+</sup>06] under very strong boundedness assumptions on the powers. In [Tób19b, Section 4.2.3.4] a short explanation is provided on how to lift those results to the Cox setting. Let us note that the definition of an SINR graph with random powers already occurs in [DBT05], but the only proven result of this paper for the setting with random radii is about degree bounds (cf. Section 2.2). The first steps towards understanding the case of unbounded powers were made recently in [Löf19]. In this master thesis, supervised by the authors, it was shown that in the case for PPP and  $d \geq 2$ , finiteness of  $\lambda_\zeta < \infty$  holds under stronger assumptions than presented in Theorem 1.1. [Löf19] also provides sufficient conditions for the absence of



percolation for small intensities  $\lambda$ .

After having introduced some important definition, examples and general context, we now give further details about our three main results.

## 2.1 Strategy for the proof of Theorem 1.1

The statement of Theorem 1.1 is an extension of the results of [Löf19] to the case of stabilizing CPPs. For the proof, we combine the approach used for [Löf19, Theorem 4.5] for handling random radii and the approach used for [Tób19a, Theorem 2.4] for dealing with the spatial correlations of the directing measure  $\Lambda$  of the CPP. To begin with, by an easy coupling argument, [Tób19b, Section 4.2.3.4] implies that as long as the powers are bounded, all positive results of [Tób19a] about percolation in the Cox-SINR graph for asymptotically essentially connected  $\Lambda$  are applicable. More precisely, we have the following proposition for the Cox-SINR graph with random bounded powers.

**Proposition 2.3.** *[Tób19a] Let  $d \geq 2$ ,  $\mathcal{N}, \tau > 0$ ,  $\mathbb{P}(P_o > 0) > 0$ ,  $\Lambda$  be stabilizing and  $\ell$  satisfy the assumption stated in Theorem 1.1. If  $P_{\sup} < \infty$  and  $\ell(0) > \tau \mathcal{N} / P_{\sup}$ , then  $\lambda_\zeta < \infty$  holds if at least one of the following conditions is satisfied:*

- 1  *$\ell$  has unbounded support,  $\Lambda$  is  $b$ -dependent and  $\mathbb{E}[\exp(\alpha \Lambda(Q_1))] < \infty$  holds for some  $\alpha > 0$ , and at least one of the following conditions hold:  $\Lambda$  is asymptotically essentially connected, or  $P_{\sup}$  is sufficiently large, or*
- 2  *$\ell$  has bounded support, and  $\Lambda$  is asymptotically essentially connected, or*
- 3  *$\ell$  has bounded support, and  $\sup \text{supp}(\ell)$  and  $P_{\sup}$  are both sufficiently large.*

Note that we have formulated the condition (1) in Proposition 2.3 more generally than what was stated in [Tób19a]. However, the proof from [Tób19a] can also be adapted to this more general case. Given Proposition 2.3, in the present paper it suffices to consider the case when  $P_{\sup} = \infty$ , hence the formulation of Theorem 1.1.

Let us comment on some aspects of Theorem 1.1. First, as for condition (2) in Theorem 1.1, an extension to the general stabilizing case is not possible in general. Indeed, even if  $P_o$  has very heavy tails, as soon as  $\text{supp}(\ell)$  is bounded, the radii of the associated Cox–Gilbert graph with random radii are bounded. Then, it is not hard to exhibit examples of stabilizing directing measures  $\Lambda$ , such that  $\lambda_c(r) = \infty$ , see the examples in [Tób19a, Section 2.5.1].

Second, if  $\Lambda$  is such that  $\Lambda(Q_1)$  is almost surely bounded, then the exponential-moment condition

$$\mathbb{E}[\exp(\alpha \Lambda(Q_1))] < \infty \tag{2.1}$$

of condition (1) in Theorem 1.1 clearly holds for all  $\alpha > 0$ . E.g., this is the case for the modulated PPP with  $\lambda_1, \lambda_2 \geq 0$ . Further, (2.1) holds for shot-noise fields for all  $\alpha > 0$ , see e.g. [Tób19a, Section 2.5.1].

For Poisson–Voronoi and Poisson–Delaunay tessellations, the  $b$ -dependence assumption in (2.1) fails, and hence percolation in the SINR graph can only be concluded for compactly supported  $\ell$ . On the other hand, it was verified in [JT19] that for these two kinds of tessellations in two dimensions,  $\mathbb{E}[\exp(\alpha\Lambda(Q_1))] < \infty$  holds for all  $\alpha > 0$ ; it is not known whether the same holds in higher dimensions.

Third, the moment conditions on  $P_o$  may look surprising at first. Indeed, why do we need to upper bound moments of  $P_o$  in order to guarantee percolation in an SINR graph? This is indeed counterintuitive in view of the Gilbert graph since there larger radii would lead to better connectivity. However, in the SINR graph, as mentioned above, larger powers also increase interference and thus also might decrease connectivity. The classical approach used in [DFM<sup>+</sup>06, BY13, Tób19a] to establish percolation in SINR graphs is to show that the underlying Gilbert graph satisfies some strong connectivity properties and at the same time the interferences can be uniformly bounded on large connected areas. We follow this approach as well, however, the random powers dictate several workarounds.

Finally, the condition (1) in Theorem 1.1 is not necessarily optimal. However, we believe that if percolation with unbounded  $\text{supp}(\ell)$  and without exponential moments of  $P_o$  is possible, then the proof for this statement must be rather different from ours. An interference-control argument may not be possible at all, instead one should be able to show that the SINR values are sufficiently large for many transitions yielding satisfactory connectivity of the network for percolation. Let us mention a similar problem. It was conjectured in [DBT05] that in the case with constant powers, in order to have percolation in the SINR graph for large  $\lambda$ ,  $\ell$  has only to have integrable tails but it may explode at zero. However, the setting where  $\lim_{r \downarrow 0} \ell(r) = \infty$  is such that the classical interference-control argument, as exhibited in [DFM<sup>+</sup>06] and adapted to the case of random powers in Section 3.1.1, certainly cannot work. Indeed, the interferences are almost-surely finite but they have infinite expectation, see [Dal71], hence there is no hope to apply a version of the exponential Markov inequality. Let us also note that the results of [Dal71] also imply that, if the tails of  $\ell$  are not integrable, then SINR graphs with  $\gamma > 0$  have no edges. We prove Theorem 1.1 in Section 3.1.

## 2.2 Strategy for the proof of Theorem 1.2

As already pointed out in [DBT05, Theorem 1], for  $\gamma > 0$ , all degrees in  $g_{(\gamma, \zeta)}(\mathbb{X}^\lambda)$ , where  $X^\lambda$  is a PPP, are less than  $1 + 1/(\tau\gamma)$  for any choice of  $\lambda, \tau > 0$  and  $\mathcal{N} \geq 0$ . In other words, all vertices in  $g_{(\gamma, \zeta)}(\mathbb{X}^\lambda)$  have at most  $1 + 1/(\tau\gamma)$  neighbors. It is not hard to see that this property remains true if the PPP is replaced by a CPP, or even any simple point process, see [Tób19b, Section A.3]. Thanks to the degree bounds, any such Cox-SINR graph with random powers for which  $\gamma \geq 1/\tau$  has no infinite cluster since it has degrees bounded by 1. For  $\gamma \in [1/(2\tau), 1/\tau)$ , we have an *a priori* degree bound of 2, which implies that all maximal connected components of SINR graphs are finite cycles or paths that are infinite in zero, one or two directions. This reminds of a one-dimensional percolation model, and thus the conjecture is that it contains no infinite clusters under general assumptions on the directing measure of the CPP, see Figure 1 for an illustration. The following proposition shows that this is indeed true for the Cox-SINR graph with random powers.

**Proposition 2.4.** *Let  $d \geq 1, \mathcal{N} \geq 0, \tau > 0$  and  $\gamma \geq 1/(2\tau)$ , then*

$$\mathbb{P}(g_{(\gamma, \zeta)}(\mathbb{X}^\lambda) \text{ percolates}) = 0.$$

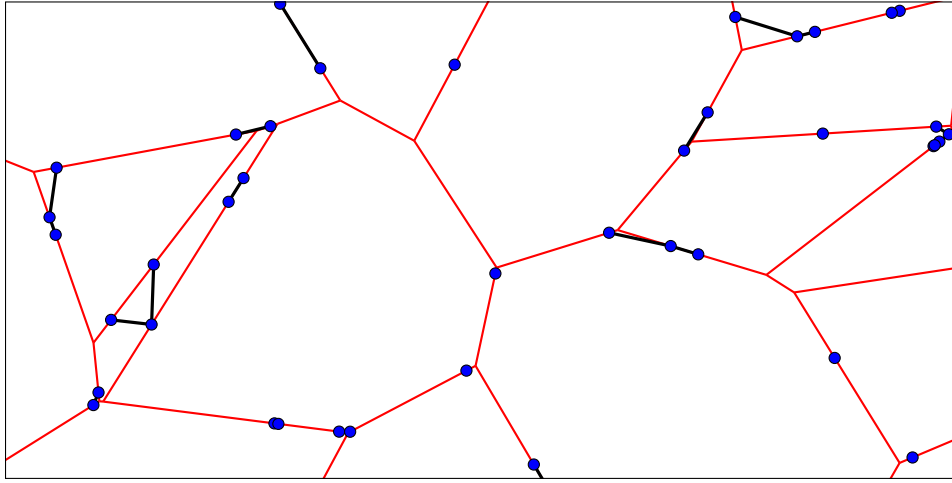


Figure 1: A typical realization of a Cox-SINR graph (with blue vertices and black edges) with directing measure given by the edge-length measure of a two-dimensional Poisson–Voronoi tessellation (in red) in a box, with  $\mathcal{N} = P_o = \tau = 1$  and a suitable path-loss function  $\ell$ . The interference-cancellation factor is set to  $\gamma = 1/(2\tau)$ . We see only a few vertices having degree two, the largest connected component is of size three, and there are no cycles in the graph. As indicated by Proposition 2.4 the graph is highly disconnected.

The statement of Theorem 1.2 is an immediate consequence of Proposition 2.4, the proof of which can be found in Section 3.3. The proof of non-percolation employs a fine configuration-wise analysis of the SINR graph, which seems to be new in the literature. Moreover, we expect the proof to hold for SINR graphs based on general simple nonequidistant stationary point processes, where nonequidistance is defined in Section 3.2.

### 2.3 Strategy for the proof of Theorem 1.3

As mentioned previously, we have  $g_{(0,P)}(X^\lambda) = g_{r_B}(X^\lambda)$  for all  $\lambda > 0$  in the Poisson-SINR graph with fixed powers, where  $r_B$  is defined in (1.4). Moreover, note that the increase of the interference-cancellation factor  $\gamma$  can only lead to edges being removed from the graph and hence there is a monotonicity of  $g_{(\gamma,P)}(X^\lambda)$  with respect to  $\gamma$ . Additionally, there is a monotonicity of  $g_{r_B}(X^\lambda)$  with respect to  $\lambda$ , which together implies that  $\lambda_P \geq \lambda_c(r_B)$ . We have the following equivalence result from [DFM<sup>+</sup>06] for the two-dimensional Poisson-SINR graphs.

**Theorem 2.5.** [DFM<sup>+</sup>06] Let  $d = 2$ ,  $\mathcal{N}, \tau, P > 0$  and  $\Lambda(dx) = dx$ . Further, let  $\ell$  satisfy the assumption stated in Theorem 1.1. Then  $\lambda_P = \lambda_c(r_B)$ .

In words, this result states that for any  $\lambda > 0$  such that the Poisson–Gilbert graph  $g_{r_B}(X^\lambda)$  is supercritical, there exists  $\gamma > 0$  such that also the Poisson-SINR graph  $g_{(\gamma,P)}(X^\lambda)$  percolates. In an extended context of SINR graphs, it was shown that this percolation is preserved if the transmitters forming a PPP experience additional interference coming from a weakly  $\alpha$ -sub-PPP, see [BY13].

The proof of Theorem 2.5 employs Russo–Seymour–Welsh type arguments about the Poisson–Gilbert graph in two dimensions, see [MR96, Section 4] and [DFM<sup>+</sup>06, Section 3]. These arguments do not have a known analogue in the Poisson case for  $d \geq 3$ , or in the general Cox case even for  $d = 2$ . Note that the results of [Tób19a] only imply that  $\lambda_P < \infty$  for  $d \geq 3$  and  $\Lambda(dx) = dx$ . However, [HJC19] includes some further observations about Gilbert graphs in  $d \geq 3$  dimensions, originating from results of [PP96], that allow us to conclude the analogue of Theorem 2.5 for the higher-dimensional Poisson case. The proof of Theorem 1.3 will be carried out in Section 3.3.

### 3 Proofs

For the proofs it will be convenient to define the SINR of  $X_i \neq X_j \in \mathbb{X}^\lambda$  via

$$\text{SINR}(X_i, X_j, \mathbb{X}^\lambda) = \frac{P_i \ell(|X_i - X_j|)}{\mathcal{N} + \gamma \sum_{k \in I \setminus \{i, j\}} P_k \ell(|X_k - X_j|)}. \quad (3.1)$$

#### 3.1 Proof of Theorem 1.1

Let us first carry out the proof under Condition (1) in Section 3.1.1. The proof under Condition (2) is presented in Section 3.1.2.

##### 3.1.1 Proof of Theorem 1.1 part (1)

For fixed  $\lambda$  and  $\gamma$ , in order to show that  $g_{(\gamma, \zeta)}(\mathbb{X}^\lambda)$  percolates, it suffices to verify that a subgraph of it contains an infinite cluster. Our proof consists of four steps. First, for  $\gamma, \lambda > 0$ , we define a subgraph that is included in a Cox–Gilbert graph with constant radii. Second, we map this subgraph to a lattice percolation model and show that this discrete model percolates for large  $\lambda$  for a suitable choice of auxiliary parameters. In particular, since  $\Lambda$  is only assumed stabilizing, the connection radius of the Gilbert graph must be large enough so that the graph percolates for large  $\lambda$ . In this step, we are able to employ multiple arguments of [DFM<sup>+</sup>06, HJC19, Tób19a]. Our interference-control assertion, Proposition 3.2, is presented here. Third, using the subgraph, we make a choice of  $\gamma > 0$  such that percolation in the discrete model implies percolation in the SINR graph  $g_{(\gamma, \zeta)}(\mathbb{X}^\lambda)$ , which is done analogously to [DFM<sup>+</sup>06]. Fourth, we carry out the proof of Proposition 3.2, combining arguments of [DFM<sup>+</sup>06, Tób19a] for SINR graphs with constant powers and arguments used in [Löff19] for Poisson–SINR graphs with random powers.

**STEP 1.** *A subgraph of the SINR graph.*

We first present a general construction of a subgraph of  $g_{(\gamma, \zeta)}(\mathbb{X}^\lambda)$  for  $\gamma, \lambda > 0$ . Let  $r_o > d_o$ . Since both  $P_o$  and  $\text{supp}(\ell)$  are unbounded, we have

$$p(r_o) = \mathbb{P}(\ell^{-1}(\tau \mathcal{N}/P_o) \geq r_o) > 0.$$

Let us define the independent thinning

$$X^{\lambda,-} = \{X_i \in X^\lambda : \ell^{-1}(\tau\mathcal{N}/P_i) \geq r_o\}$$

of  $X^\lambda$  with survival probability  $p(r_o)$ . Now, let us define a subgraph  $g_{(\gamma,\zeta)}^-(\mathbb{X}^\lambda)$  of  $g_{(\gamma,\zeta)}(\mathbb{X}^\lambda)$  as follows. The vertex set is  $X^{\lambda,-}$ , and two vertices  $X_i, X_j \in X^{\lambda,-}$ ,  $i \neq j$ , are connected by an edge if and only if

$$\text{SINR}^-(X_i, X_j, \mathbb{X}^\lambda) = \frac{(\tau\mathcal{N}/\ell(r_o))\ell(|X_i - X_j|)}{\mathcal{N} + \gamma \sum_{k \in I \setminus \{i,j\}} P_k \ell(|X_k - X_j|)} > \tau. \quad (3.2)$$

Note that for  $X_i, X_j \in X^{\lambda,-}$ , in the numerator of  $\text{SINR}(X_i, X_j, \mathbb{X}^\lambda)$ , for the power of  $X_i$  we have  $P_i \geq \tau\mathcal{N}/\ell(r_o)$ , whereas the denominators of (3.1) and (3.2) are equal. Hence,  $g_{(\gamma,\zeta)}^-(\mathbb{X}^\lambda)$  is indeed a subgraph of  $g_{(\gamma,\zeta)}(\mathbb{X}^\lambda)$  for any  $\gamma \geq 0$ . As for  $\gamma = 0$ ,  $g_{(0,\zeta)}^-(\mathbb{X}^\lambda)$  equals the Cox–Gilbert graph  $g_{r_o}(X^{\lambda,-})$  with constant radius  $r_o$  and vertex set  $X^{\lambda,-}$ . In words, in order to obtain  $g_{(\gamma,\zeta)}^-(\mathbb{X}^\lambda)$  from  $g_{(\gamma,\zeta)}(\mathbb{X}^\lambda)$ , one first thins out vertices with small powers, in order to get rid of vertices with small values of the connection radius  $r_B^i$ , where

$$r_B^i = \ell^{-1}(\tau\mathcal{N}/P_i). \quad (3.3)$$

Then, one bounds the powers of the remaining vertices by  $\tau\mathcal{N}/\ell(r_o)$  from below.

**STEP 2.** *Mapping the subgraph to a lattice-percolation problem and percolation on the lattice.*

Now we are in a position to adapt to the setting of [Tób19a, Section 3.2.2] and use strong connectivity of  $g_{r_o}(X^{\lambda,-})$  in case  $r_o$  is sufficiently large and  $\lambda$  is chosen according to  $r_o$ . Together with an interference-control argument presented below, this will allow us to verify Theorem 1.1 part (1).

For  $\varrho > 0$ , let  $Y^\varrho$  be a PPP with intensity measure (directing measure)  $\varrho\text{Leb}$ . Let  $\varrho_c(1)$  be such that the Poisson–Gilbert graph  $g_1(Y^{\varrho_c(1)})$  is critical. Then, due to the scale invariance of Poisson–Gilbert graphs [MR96, Section 2.2], for  $\varrho > \varrho_c(1)$ , we can choose a smaller intensity  $\varrho' < \varrho$  such that  $g_1(Y^{\varrho'})$  is still supercritical. Now, for  $r > d_o$ , we define  $\lambda(r) = \varrho' r^{-d}$ ,  $r_o(r) = r\varrho/\varrho'$  and  $P(r) = \tau\mathcal{N}/\ell(r_o(r))$ . Then  $r^{-1}g_r(X^{\lambda(r),-})$  converges to the supercritical graph  $g_1(Y^{\varrho'})$  on compact sets, as  $r$  tends to infinity, see [HJC19, Section 7.1]. Further, recalling that  $R$  denotes the stabilization radii of  $\Lambda$ , we put  $R(Q) = \sup_{x \in Q \cap \mathbb{Q}^d} R_x$  for any measurable set  $Q \subseteq \mathbb{R}^d$ .

Using these notions, we construct a renormalized percolation process on  $\mathbb{Z}^d$  as follows. For  $n \geq 1$  and  $r > d_o$ , the site  $z \in \mathbb{Z}^d$  is  $(r, n)$ -good if

- 1  $R(Q_{6rn}(rnz)) < rn/2$ ,
- 2  $X^{\lambda(r),-} \cap Q_{rn}(rnz) \neq \emptyset$ , and
- 3 every  $X_i, X_j \in X^{\lambda(r),-} \cap Q_{3rn}(rnz)$  are connected by a path in  $g_r(X^{\lambda(r),-}) \cap Q_{6rn}(rnz)$ .

The site  $z \in \mathbb{Z}^d$  is  $(r, n)$ -bad if it is not  $(r, n)$ -good. Note that the process of  $(r, n)$ -good sites is 7-dependent thanks to the definition of stabilization. The following lemma has been verified in [Tób19a, Section 3.2.2] based on arguments of [HJC19, Section 5.2].

**Lemma 3.1.** *[Tób19a] Assume that the conditions of Theorem 1.1 part (1) hold. Then, for all sufficiently large  $\lambda > 0$  and for all  $n \geq 1$  and  $r > d_o$  with  $rn$  sufficiently large, there exists  $q_A = q_A(\lambda, rn) < 1$  such that for any  $N \in \mathbb{N}$  and pairwise distinct  $z_1, \dots, z_N \in \mathbb{Z}^d$ ,*

$$\mathbb{P}(z_1, \dots, z_N \text{ are all } (r, n)\text{-bad}) \leq q_A^N.$$

Further, for any  $\varepsilon > 0$ , one can choose  $\lambda$  and  $rn$  sufficiently large such that  $q_A < \varepsilon$ .

We further proceed similarly to [DFM<sup>+</sup>06, Tób19a] by defining ‘shifted’ versions of the path-loss function  $\ell$ . For  $a \geq 0$ , define

$$\ell_a(r) = \ell(0)\mathbb{1}\{r < a\sqrt{d}/2\} + \ell(r - a\sqrt{d}/2)\mathbb{1}\{r \geq a\sqrt{d}/2\}. \quad (3.4)$$

Note that  $\ell_0 = \ell$ . Now, we define the shot-noise processes

$$I_a(x) = \sum_{X_i \in X^\lambda} P_i \ell_a(|x - X_i|), \quad I(x) = \sum_{X_i \in X^\lambda} P_i \ell(|x - X_i|), \quad x \in \mathbb{R}^d,$$

and note that  $I_0(x) = I(x)$ . By the triangle inequality, for  $a \geq 0$ ,  $I(x) \leq I_a(z)$  holds for any  $z \in \mathbb{R}^d$  and  $x \in Q_a(z)$ . Now, the interference-control argument consists in verifying the following proposition. For  $z \in \mathbb{Z}^d$ , let us write  $B_{r,n,M}(z) = \{I_{6rn}(rnz) \leq M\}$ .

**Proposition 3.2.** *Assume that the conditions of Theorem 1.1 part (1) hold. Then, for all  $\lambda > 0$ , for all  $n \geq 1$  and  $r > d_o$  with  $rn$  sufficiently large and for all  $M > 0$  sufficiently large, there exists  $q_B = q_B(\lambda, rn, N) < 1$  such that for all  $N \in \mathbb{N}$  and for all pairwise distinct  $z_1, \dots, z_N \in \mathbb{Z}^d$  we have*

$$\mathbb{P}(B_{r,n,M}(z_1)^c \cap \dots \cap B_{r,n,M}(z_N)^c) \leq q_B^N. \quad (3.5)$$

Further, for any  $\varepsilon > 0$  and  $\lambda > 0$ , one can choose  $rn$  and  $M$  sufficiently large such that  $q_B < \varepsilon$ .

The proof of this proposition is postponed until Step 4. Once we have shown Proposition 3.2, one can derive the following corollary using a standard argument (see e.g. the proof of [DFM<sup>+</sup>06, Proposition 3] or the one of [Tób19a, Proposition 3.1]). For  $z \in \mathbb{Z}^d$  let us define  $C_{r,n,M}(z) = \{z \text{ is } (r, n)\text{-good}\} \cap \{I_{6rn}(rnz) \leq M\}$ .

**Corollary 3.3.** *Assume that the conditions of Theorem 1.1 part (1) hold. Then, for all sufficiently large  $\lambda > 0$ , for all  $r > d_o$  and  $n \geq 1$  with  $rn$  sufficiently large and for all  $M > 0$  sufficiently large, there exists  $q_C = q_C(\lambda, rn, M) < 1$  such that for all  $N \in \mathbb{N}$  and for all pairwise distinct  $z_1, \dots, z_N \in \mathbb{Z}^d$  we have*

$$\mathbb{P}(C_{r,n,M}(z_1)^c \cap \dots \cap C_{r,n,M}(z_N)^c) \leq q_C^N.$$

Further, for any  $\varepsilon > 0$ , one can choose  $\lambda, rn, M$  sufficiently large such that  $q_C < \varepsilon$ .

**STEP 3.** *Percolation in the subgraph of the SINR graph.*

Having Corollary 3.3 and employing a Peierls argument (cf. [Gri99, Section 1.4]), we conclude that for  $\lambda, rn, M$  sufficiently large, the process of  $(r, n)$ -good sites  $z \in \mathbb{Z}^d$  such that  $I_{6rn}(rnz) \leq M$  percolates. Using arguments of [HJC19, Section 5.2], this implies percolation of the Cox–Gilbert graph  $g_{(0,\zeta)}^-(\mathbb{X}^{\lambda(r)}) = g_{r_o(r)}(X^{\lambda(r),-})$ . From this point of the proof it is classical to derive that  $g_{(\gamma,\zeta)}^-(\mathbb{X}^{\lambda(r)})$  percolates for small  $\gamma > 0$ , see [DFM<sup>+</sup>06, Section 3.3]. For the convenience of the reader, let us give the details here. We define

$$\gamma' = \frac{\mathcal{N}}{P(r)M} \left( \frac{\ell(r)}{\ell(r_o(r))} - 1 \right) = \frac{\ell(r_o(r))}{\tau M} \left( \frac{\ell(r)}{\ell(r_o(r))} - 1 \right) > 0,$$

where the strict inequality holds because  $r_o(r) > r > d_o$  and  $\ell$  has unbounded support. Then we have

$$\frac{P(r)\ell(r)}{\mathcal{N} + \gamma'P(r)M} = \tau.$$

Now, let  $X_i, X_j \in X^{\lambda(r),-}$  be situated in  $Q_{rn}(rnz)$  respectively  $Q_{rn}(rnz')$  for some sites  $z, z' \in \mathbb{Z}^d$  included in the same infinite cluster of the process of  $(r, n)$ -good sites  $z \in \mathbb{Z}^d$  satisfying  $I_{6rn}(rnz) \leq M$  such that  $|X_i - X_j| < r$ . Then, for  $\gamma < \gamma'$ , we have

$$\text{SINR}(X_i, X_j, \mathbb{X}^\lambda) \geq \text{SINR}^-(X_i, X_j, \mathbb{X}^\lambda) > \frac{P(r)\ell(r)}{\mathcal{N} + \gamma'P(r)M} = \tau.$$

Thus,  $X_i$  and  $X_j$  are connected by an edge in  $g_{(\gamma,\zeta)}^-(\mathbb{X}^\lambda)$ . Hence,  $g_{(\gamma,\zeta)}(\mathbb{X}^\lambda)$  also percolates. Thus, we can conclude Theorem 1.1 as soon as we have verified Proposition 3.2.

**STEP 4.** *Proof of Proposition 3.2: the interference-control argument.*

Similarly to [Tób19a, Section 3.1.1], we split the interference into two parts. For  $x \in \mathbb{R}^d$ ,  $n \geq 1$  and  $r > 0$ , we put

$$I_{6rn}^{\text{in}}(x) = \sum_{X_i \in X^\lambda \cap Q_{12rn\sqrt{d}}(x)} \ell_{6rn}(|X_i - x|), \quad I_{6rn}^{\text{out}}(x) = \sum_{X_i \in X^\lambda \setminus Q_{12rn\sqrt{d}}(x)} \ell_{6rn}(|X_i - x|).$$

Then, for  $M > 0$ , if  $I_{6rn}(x) > M$ , then  $I_{6rn}^{\text{in}}(x) > M/2$  or  $I_{6rn}^{\text{out}}(x) > M/2$ . Using a union bound and the fact that in Proposition 3.2,  $M$  can be chosen arbitrarily large, it suffices to conclude the proposition both with  $B_{r,n,M}(z_i)$  replaced by  $B_{r,n,M}^{\text{in}}(z_i)$  and with  $B_{r,n,M}(z_i)$  replaced by  $B_{r,n,M}^{\text{out}}(z_i)$  everywhere in (3.5) for all  $i \in \{1, \dots, N\}$ , where for  $z \in \mathbb{Z}^d$  we write  $B_{r,n,M}^{\text{in}}(z) = \{I_{6rn}^{\text{in}}(rnz) \leq M\}$  and  $B_{r,n,M}^{\text{out}}(z) = \{I_{6rn}^{\text{out}}(rnz) \leq M\}$ . Indeed, having these assertions, we can combine them similarly to Corollary 3.3.

We now verify Proposition 3.2 with  $B_{r,n,M}(\cdot)$  replaced by  $B_{r,n,M}^{\text{in}}(\cdot)$  everywhere. For this assertion, instead of the assumption that  $P_o$  and  $\Lambda(Q_1)$  have some exponential moments, it suffices if they have a first moment (for  $\Lambda(Q_1)$  this is automatic since  $\mathbb{E}[\Lambda(Q_1)] = 1$  by assumption). To be more precise, we prove the following lemma.

**Lemma 3.4.** *Assume that for  $\ell$  the conditions of Theorem 1.1 part (1) hold. Further, let  $\Lambda$  be stabilizing and  $\mathbb{E}[P_o] < \infty$ . Then, for all  $\lambda > 0$ , for all  $n \geq 1$  and  $r > d_o$  with  $rn$  sufficiently large and for all  $M > 0$  sufficiently large, there exists  $q_B = q_B(\lambda, rn, N) < 1$  such that for all  $N \in \mathbb{N}$  and for all pairwise distinct  $z_1, \dots, z_N \in \mathbb{Z}^d$  we have*

$$\mathbb{P}(B_{r,n,M}^{\text{in}}(z_1)^c \cap \dots \cap B_{r,n,M}^{\text{in}}(z_N)^c) \leq q_B^N. \quad (3.6)$$

Further, for any  $\varepsilon > 0$  and  $\lambda > 0$ , one can choose  $rn$  and  $M$  sufficiently large such that  $q_B < \varepsilon$ .

*Proof.* We use the following auxiliary discrete percolation process. A site  $z \in \mathbb{Z}^d$  is  $(r, n)$ -tame if

- 1  $R(Q_{12rn\sqrt{d}}(rnz)) < rn/2$ , and
- 2  $I_{6rn}^{\text{in}}(rnz) \leq M$ .

A site  $z \in \mathbb{Z}^d$  is  $(r, n)$ -wild if it is not  $(r, n)$ -tame. The process of  $(r, n)$ -tame sites is  $\lceil 12\sqrt{d} + 1 \rceil$ -dependent according to the definition of stabilization. Thus, it follows from dependent-percolation theory [LSS97, Theorem 0.0] that, in order to verify Lemma 3.4, it suffices to show that for all  $\lambda > 0$ ,  $\mathbb{P}(o \text{ is } (r, n)\text{-wild})$  can be made arbitrarily close to zero by choosing first  $rn$  sufficiently large and then  $M$  large enough accordingly. We have

$$\mathbb{P}(o \text{ is } (r, n)\text{-wild}) \leq \mathbb{P}(R(Q_{12rn\sqrt{d}}(rnz)) \geq rn/2) + \mathbb{P}(I_{6rn}^{\text{in}}(rnz) > M).$$

The first term can be made arbitrarily small by choosing  $rn$  large enough, thanks to the definition of stabilization. Moreover, by the definition of  $\ell_a$ , see (3.4),

$$I_{6rn}^{\text{in}}(o) = \sum_{X_i \in X^\lambda \cap Q_{12rn\sqrt{d}}(o)} P_i \ell_{6rn}(|X_i|) \leq \ell(0) \sum_{X_i \in X^\lambda \cap Q_{12rn\sqrt{d}}(o)} P_i.$$

In particular, using that the point process  $\mathbb{X}^\lambda$  is independently marked with  $P_i$  having marginal distribution  $\zeta$ , and that  $\Lambda$  is stationary with  $\mathbb{E}[\Lambda(Q_1)] = 1$ , it follows that

$$\mathbb{E}[I_{6rn}^{\text{in}}(o)] \leq \ell(0) \lambda \mathbb{E}[P_o] \mathbb{E}[\Lambda(Q_{12rn\sqrt{d}})] = (12rn\sqrt{d})^d \ell(0) \lambda \mathbb{E}[P_o].$$

Thus, for any  $n \geq 1$  and  $r > 0$ ,  $\mathbb{P}(I_{6rn}^{\text{in}}(o) > M)$  can be made arbitrarily small by choosing  $M$  large enough, given that  $\mathbb{E}[P_o] < \infty$ . Thus, the lemma follows.  $\square$

It remains to verify Proposition 3.2 with  $B_{r,n,M}(\cdot)$  replaced by  $B_{r,n,M}^{\text{out}}(\cdot)$  everywhere. More precisely, thanks to the exponential-moment and  $b$ -dependence assumption on  $\Lambda$ , the proof can be completed analogously to the proof of [Tób19a, Proposition 3.3] starting from [Tób19a, Equation (3.15)], as soon as we have verified the following lemma.



**Lemma 3.5.** *Under the assumptions of Theorem 1.1 part (1), there exists a constant  $c_o = c_o(\zeta, \ell) > 0$  such that for all sufficiently small  $s > 0$ , for all  $\lambda > 0$ ,  $n \geq 1$  and  $r > d_o$  with  $rn > 0$  sufficiently large and for all large enough  $M > 0$ , for all  $N \in \mathbb{N}$  and pairwise distinct  $z_1, \dots, z_N \in \mathbb{Z}^d$  we have*

$$\begin{aligned} & \mathbb{P}(B_{r,n,M}^{\text{out}}(z_1)^c \cap \dots \cap B_{r,n,M}^{\text{out}}(z_N)^c) \\ & \leq \mathbb{E} \left[ \exp \left( c_o \lambda s \sum_{i=1}^N \int_{\mathbb{R}^d \setminus Q_{12rn\sqrt{d}}(rnz_i)} \ell_{6rn}(|rnz_i - x|) \Lambda(dx) \right) \right]. \end{aligned} \quad (3.7)$$

*Proof.* We start with an estimate originating from [DFM<sup>+</sup>06, Section 3.2]. By Markov's inequality, for any  $s > 0$ ,

$$\begin{aligned} & \mathbb{P}(B_{r,n,M}^{\text{out}}(z_1)^c \cap \dots \cap B_{r,n,M}^{\text{out}}(z_N)^c) = \mathbb{P}(I_{6rn}^{\text{out}}(rnz_1) > M, \dots, I_{6rn}^{\text{out}}(rnz_N) > M) \\ & \leq \mathbb{P} \left( \sum_{i=1}^N I_{6rn}^{\text{out}}(rnz_i) > NM \right) \\ & \leq e^{-sNM} \mathbb{E} \left[ \exp \left( s \sum_{i=1}^N \sum_{X_k \in X^\lambda \setminus Q_{12rn\sqrt{d}}(rnz_i)} P_k \ell_{6rn}(|rnz_i - X_k|) \right) \right]. \end{aligned} \quad (3.8)$$

The randomness of the power values  $P_k$  prevents us from continuing the proof analogously to [DFM<sup>+</sup>06, Tób19a]. On the other hand, similarly to [Löf19, Section 4.3] in the Poisson case, we can argue as follows. According to the Marking Theorem [Kin93, Section 5.2], the independently marked CPP  $\mathbb{X}^\lambda = (X_i, P_i)_{i \in I}$  is a CPP in  $\mathbb{R}^d \times [0, \infty)$  with directing measure  $\Lambda \otimes \zeta$ , where we recall that  $\zeta = \mathbb{P} \circ P_o^{-1}$  is the distribution of  $P_o$ . Hence, applying the Laplace functional of a CPP (cf. [Kin93, Sections 3.2, 6]) to the function  $f: \mathbb{R}^d \times [0, \infty) \rightarrow [0, \infty)$ ,

$$f(x, p) = s \sum_{i=1}^N p \ell_{6rn}(|x - rnz_i|) \mathbb{1}\{x \in \mathbb{R}^d \setminus Q_{12rn\sqrt{d}}(rnz_i)\},$$

we obtain

$$\begin{aligned} & \mathbb{E} \left[ \exp \left( s \sum_{i=1}^N \sum_{X_k \in X^\lambda \setminus Q_{12rn\sqrt{d}}(rnz_i)} P_k \ell_{6rn}(|rnz_i - X_k|) \right) \right] \\ & = \mathbb{E} \left[ \exp \left( \lambda \int_{\mathbb{R}^d \setminus Q_{12rn\sqrt{d}}(rnz_i)} \int_0^\infty \left( \exp \left( sp \sum_{i=1}^N \ell_{6rn}(|rnz_i - x|) \right) - 1 \right) \zeta(dp) \Lambda(dx) \right) \right]. \end{aligned} \quad (3.9)$$

Thanks to the exponential-moment assumption on  $P_o$  from (1), the moment-generating function

$$\alpha \mapsto \mathbb{E}[\exp(\alpha P_o)] = \int_0^\infty e^{\alpha p} \zeta(dp)$$

is infinitely differentiable at  $\alpha = 0$  with first derivative  $\int_0^\infty p\zeta(dp) = \mathbb{E}[P_o] < \infty$ . Note that  $\sum_{i=1}^N \ell_{6rn}(|rnz_i - x|)$  is uniformly bounded in  $x \in \mathbb{R}^d$ ,  $rn$ ,  $N$  and pairwise distinct  $z_1, \dots, z_N$ , see [Tób19a, Lemma 3.6]. Consequently, for any  $C > 1$ , the following holds for all sufficiently small  $s > 0$  (depending on  $C$ ),

$$\int_0^\infty \left( \exp \left( sp \sum_{i=1}^N \ell_{6rn}(|rnz_i - x|) \right) - 1 \right) \zeta(dp) \leq Cs \mathbb{E}[P_o] \sum_{i=1}^N \ell_{6rn}(|rnz_i - x|). \quad (3.10)$$

For such  $s$ , plugging (3.10) back into (3.9), starting from (3.8) we obtain

$$\begin{aligned} & \mathbb{P}(B_{r,n,M}^{\text{out}}(z_1)^c \cap \dots \cap B_{r,n,M}^{\text{out}}(z_N)^c) \\ & \leq \mathbb{E} \left[ \exp \left( C \mathbb{E}[P_o] \lambda s \sum_{i=1}^N \int_{\mathbb{R}^d \setminus Q_{12rn\sqrt{d}}(rnz_i)} \ell_{6rn}(|rnz_i - x|) \Lambda(dx) \right) \right], \end{aligned} \quad (3.11)$$

which is (3.7) with  $c_o = C \mathbb{E}[P_o]$ . With this we conclude the lemma.  $\square$

### 3.1.2 Proof of Theorem 1.1 part (2).

Since  $\Lambda$  is asymptotically essentially connected,  $\lambda_c(r) < \infty$  holds for any  $r > 0$  according to [HJC19, Theorem 2.4]. Note further that the connection radii  $(r_B^i)_{i \in I}$ , defined in (3.3), are bounded by  $d_{\max} = \sup\{x \geq 0 : x \in \text{supp}(\ell)\}$ . The proof of Theorem 1.1 part (2) can be obtained as an adaptation of the proof of part (1) of the same theorem as follows.

First, one defines the subgraph of the SINR graph analogously to Step 1 of the proof of Theorem 1.1 part (1). Next, one takes the Step 2 but for  $r \in (d_o, d_{\max})$  arbitrary and fixed instead of letting  $r \uparrow \infty$ , and for  $r_o(r) > r$  such that  $r_o(r)$  still lies in the interval  $(d_o, d_{\max})$  on which  $\ell$  is strictly decreasing. This way, choosing  $rn$  sufficiently large will be equivalent to choosing  $n$  large enough (for fixed  $r$ ). Further, one alters the choice of  $\lambda(r)$ : now,  $\lambda(r)$  has to be chosen so large that the process of  $(r, n)$ -good sites percolates for some  $n \geq 1$ , which is possible for any fixed  $r \in (d_o, d_{\max})$  since  $\Lambda$  is asymptotically essentially connected, cf. [HJC19, Section 5.2]. Next, Step 3 is also applicable for all choices of the parameters where the underlying discrete model percolates. Finally, let us explain how to complete the proof of Proposition 3.2 under the mere assumption that  $\mathbb{E}[P_o] < \infty$ . Since  $\text{supp}(\ell)$  is bounded, for all sufficiently large  $n \geq 1$  the following holds for all  $z \in \mathbb{Z}^d$

$$\begin{aligned} I_{6rn}(rnz) &= \sum_{X_i \in X^\lambda \cap Q_{6rn+2d_{\max}}(rnz)} P_i \ell_{6rn}(|X_i - rnz|) \\ &\leq \sum_{X_i \in X^\lambda \cap Q_{12rn\sqrt{d}}(rnz)} P_i \ell_{6rn}(|X_i - rnz|) = I_{6rn}^{\text{in}}(rnz). \end{aligned} \quad (3.12)$$

Hence, it remains to control the inner part of the interference, which can be done analogously to Lemma 3.4 once  $\mathbb{E}[P_o] < \infty$ , given that  $\Lambda$  is stabilizing. Hence, we conclude Theorem 1.1 part (2).

### 3.1.3 Proof of Theorem 1.1 part (3).

In the case when  $\Lambda$  is only stabilizing and  $P_{\sup} = \infty$ , we observe that the proof of Theorem 1.1 part (1) stays valid if  $\text{supp}(\ell)$  is bounded but the following assumption holds:  $\sup \text{supp}(\ell)$  is sufficiently large such that

$$\sup \text{supp}(\ell) > \inf\{r > 0: \text{there exists } n \geq 1 \text{ and } \lambda > 0 \text{ such that } (r, n)\text{-good sites percolate}\},$$

where the infimum is finite because  $\Lambda$  is stabilizing. Indeed, in this situation, Lemma 3.1, as in [Tób19a, Section 3.2.2], holds as well. Further, (3.12) holds for all sufficiently large  $n$  for all  $z \in \mathbb{Z}^d$ , and therefore one can complete the proof under the assumptions of Lemma 3.4, i.e., for  $\Lambda$  stabilizing and  $P_o$  such that  $\mathbb{E}[P_o] < \infty$ , without requiring  $b$ -dependence of  $\Lambda$  or existence of exponential moments of  $\Lambda(Q_1)$  or  $P_o$ .

## 3.2 Proof of Proposition 2.4

We can assume that  $\mathbb{P}(P_o > 0) > 0$  it what follows, since otherwise the statement is trivially true. We start the proof with the following lemma, which excludes infinite paths that have an endpoint in case the degrees are bounded by two, in a substantially more general setting.

**Lemma 3.6.** *Let  $g(\mathbb{X})$  be a random graph based on a stationary marked point process  $\mathbb{X} = \{(X_i, M_i)\}_{i \in I}$ , with vertex set  $X = \{X_i\}_{i \in I}$  such that the degree of all  $X_i \in X$ ,  $\deg(X_i)$ , is bounded by 2, almost surely. Let  $X$  have a finite intensity and consider the point process of degree-one points in infinite clusters*

$$\mathcal{X}_0 = \sum_{i \in I} \delta_{X_i} \mathbb{1}\{\deg(X_i) = 1, X_i \text{ is part of an infinite cluster in } g(\mathbb{X})\}.$$

Then,  $\mathbb{P}(\mathcal{X}_0(\mathbb{R}^d) = 0) = 1$ .

*Proof.* First, using the union bound and stationarity, it is enough to show that  $\mathbb{E}[\mathcal{X}_0(Q_1)] = 0$ . Let us define the point process of points in infinite clusters in  $Q_1$  that are at distance equal to  $k \in \mathbb{N}_o$  from a point in  $\mathcal{X}_0$ ,

$$\mathcal{X}_k = \sum_{i \in I} \delta_{X_i} \mathbb{1}\{X_i \text{ is part of an infinite cluster and has graph distance } k \text{ from } \mathcal{X}_0\}.$$

Thanks to the degree bound, every infinite cluster has at most one point in  $\mathcal{X}_0$  and  $\mathbb{E}[\mathcal{X}_k(Q_1)] = \mathbb{E}[\mathcal{X}_0(Q_1)]$ , for all  $k \in \mathbb{N}_o$ , by stationarity. However,  $\sum_{k \geq 0} \mathbb{E}[\mathcal{X}_k(Q_1)] \leq \mathbb{E}[X(Q_1)] < \infty$  and thus  $\mathbb{E}[\mathcal{X}_0(Q_1)] = 0$ .  $\square$

Let  $X^{\lambda,*}$  denote the Palm version [HJC19, Section 2.2] of  $X^\lambda$ . Note that  $X^{\lambda,*}$  can be interpreted as a CPP conditioned to have a point at the origin, in particular it is a simple point process. Further, it is almost-surely nonequidistant, i.e., for any  $X_i, X_j, X_k, X_l \in X^{\lambda,*}$ ,  $|X_i - X_j| = |X_k - X_l| > 0$  implies  $\{i, j\} = \{k, l\}$ . Let  $\mathbb{X}^{\lambda,*} = \{(X_i, P_i)\}_{i \in J}$  be an independently-marked point process with  $\{X_i\}_{i \in J} = X^{\lambda,*}$  and conditional on  $X^{\lambda,*}$ ,  $\{P_i\}_{i \in J}$  are i.i.d.  $\zeta$ -distributed random variables. Then  $\mathbb{X}^{\lambda,*}$  is the Palm

version of  $\mathbb{X}^\lambda$  with respect to the  $X_i$ -coordinate, which has a point of the form  $(o, P_*)$  where  $P_*$  is  $\zeta$ -distributed and independent of  $X^{\lambda,*}$  and all other power values. In particular, thanks to the simpleness of  $X^{\lambda,*}$ ,  $g_{(\gamma,\zeta)}(\mathbb{X}^{\lambda,*})$  has degrees bounded by two under the assumptions of Proposition 2.4. Now, Lemma 3.6 implies the following.

**Corollary 3.7.** *Under the assumptions of Proposition 2.4, almost surely, the cluster containing  $o$  in the SINR graph  $g_{(\gamma,\zeta)}(\mathbb{X}^{\lambda,*})$  is finite or it consists only of points of degree two.*

*Proof.* Assume otherwise and let  $\mathcal{C}$  denote the cluster containing  $o$  in  $g_{(\gamma,\zeta)}(\mathbb{X}^{\lambda,*})$ . We have

$$\mathbb{P}(\#\mathcal{C} = \infty \text{ and } \mathcal{C} \text{ contains a point of degree 1}) > 0.$$

But then, according to the definition of the Palm version, it follows that

$$\mathbb{E}[\#\{X_i \in X^\lambda \cap Q_1 : \text{the cluster of } X_i \text{ in } g_{(\gamma,\zeta)}(\mathbb{X}^\lambda) \text{ is infinite and contains a point of degree 1}\}] > 0$$

holds, which contradicts with Lemma 3.6. This implies the corollary.  $\square$

*Proof of Proposition 2.4.* Using Palm calculus, it suffices to show that

$$\mathbb{P}(\#\mathcal{C} = \infty) = 0. \quad (3.13)$$

We view  $\mathbb{X}^{\lambda,*}$  as the canonical process  $\mathbb{X}^{\lambda,*}(\omega) = \omega$  on the set  $\mathbf{N}^*$  of marked point configurations  $\omega$  in  $\mathbb{R}^d \times \text{supp } \zeta \subseteq \mathbb{R}^d \times [0, \infty)$  such that  $\omega = \{x_i : (x_i, p_i) \in \omega\}$  is an infinite locally-finite simple and nonequidistant point configuration on  $\mathbb{R}^d$  such that  $o \in \omega$ . The set of such point configurations  $\omega$  will be denoted by  $\mathbf{N}^*$ . We equip  $\mathbf{N}^*$  and  $\mathbf{N}^*$  with the corresponding evaluation  $\sigma$ -fields. We can then assume that  $\mathbb{P}$  is the distribution of  $\omega$ . Note that if  $\omega, \omega' \in \mathbf{N}^*$  are such that  $\omega \subseteq \omega'$ , then for any  $x, y \in \omega$  such that  $\text{SINR}((x, p), (y, q), \omega') > \tau$ , we also have  $\text{SINR}((x, p), (y, q), \omega) > \tau$ . Hence,  $g_{(\gamma,\zeta)}(\omega)$  contains all edges of  $g_{(\gamma,\zeta)}(\omega')$  that connect two points of  $\omega$ .

For a given configuration  $\omega = \{(x_i, p_i)\}_{i \in J} \in \mathbf{N}^*$  and a point  $x_o \in \omega$ , we can uniquely order the points according to the transmitted signal strength received at  $x_o$ . More precisely, let us write  $\mathbf{V}(x_o, \omega) = ((x_o, p_o), (x_1, p_1), (x_2, p_2), \dots)$  for the vector of marked points such that the following conditions are satisfied:

- 1  $i \mapsto p_i \ell(|x_i - x_o|)$  is decreasing on  $\mathbb{N}$ , and
- 2 for all  $i, j \in \mathbb{N}$  with  $i < j$  and  $p_i \ell(|x_i - x_o|) = p_j \ell(|x_j - x_o|)$ , we have  $|x_i - x_o| < |x_j - x_o|$ .

Let us write  $V(x_o, \omega)$  for the vector of the first components of  $\mathbf{V}(x_o, \omega)$  and  $V_i(x_o, \omega)$  for the  $i$ -th entry of  $V(x_o, \omega)$ , which we call the  $i$ -th strongest transmitter towards  $x_o$ . In particular,  $V_0(x_o, \omega) = x_o$ . We will use the notation  $V_i(\omega) = V_i(o, \omega)$  and also write  $\mathbf{V}_i(x_o, \omega)$  for the  $i$ -th entry of  $\mathbf{V}(x_o, \omega)$ .

Note also that despite the nonequidistance condition, ties of the form  $p_i \ell(|x_i - x_o|) = p_j \ell(|x_j - x_o|)$  may occur with probability one for example if  $\ell$  has bounded support. Also, in case of a constant signal power  $P_o$ ,  $V_i(x_o, \omega)$  is simply the  $i$ -th nearest neighbor of  $x_o$  in  $\omega$  with respect to Euclidean distance.

It was noted in [Tób19b, Section A.3] that the degree bound of two holds for *every* realization of  $g_{(\gamma, \zeta)}(\mathbb{X}^{\lambda, *})$  under the assumption that  $\gamma \geq 1/(2\tau)$ . Thus, the following can be derived analogously to [Tób19a, Section 2.4.1], where the case of constant powers was considered. For such  $\gamma$ , if  $o$  has degree two in  $g_{(\gamma, \zeta)}(\mathbb{X}^{\lambda, *})$ , then  $o$  must be connected by an edge to both  $V_1 = V_1(\mathbb{X}^{\lambda, *})$  and  $V_2$  since the degree bound applies already for the edges towards  $o$ . Moreover, both  $V_1$  and  $V_2$  must also have  $o$  as one of their first two strongest transmitters towards them, that is,

$$o \in \{V_1(V_i(\mathbb{X}^{\lambda, *}), \mathbb{X}^{\lambda, *}), V_2(V_i(\mathbb{X}^{\lambda, *}), \mathbb{X}^{\lambda, *})\},$$

for all  $i \in \{1, 2\}$ . These strongest-transmitter relations hold almost surely, in particular for every simple and nonequidistant configuration of  $\mathbb{X}^{\lambda, *}$ . The goal of using the configuration space  $\mathbf{N}^*$  is to entirely exclude configurations that offend the degree bound or the strongest-transmitter relations.

Hence, Proposition 2.4 immediately follows once we have verified the following proposition. For this recall that  $\mathcal{C}$  denotes the cluster attached to  $o$  in  $g_{(\gamma, \zeta)}(\mathbb{X}^{\lambda, *})$ .

**Proposition 3.8.** *Let us define the random variable*

$$I = \inf\{i \geq 3 : V_i \in \mathcal{C}\}$$

*and the set  $A = \{\#\mathcal{C} = \infty\}$ . Then, for any  $i \geq 3$ , we have*

$$\mathbb{P}(A \cap \{I = i\}) = 0. \quad (3.14)$$

Indeed, using a union bound and noting that  $A \subset \{I < \infty\}$ , Proposition 3.8 implies  $\mathbb{P}(A) = 0$ , which is (3.13) and thus finishes the proof of Proposition 2.4.  $\square$

*Proof of Proposition 3.8.* Thanks to Corollary 3.7, in the event  $A$ ,  $o$  is connected by an edge both to  $V_1$  and  $V_2$  in this SINR graph. Further, thanks to the degree bound of 2, in the event  $A$ ,  $V_1$  and  $V_2$  have no further joint neighbor in the SINR graph since otherwise  $\mathcal{C}$  has a loop and can not be infinite by the degree bound. This way, for any  $i \geq 3$ , there exists  $j \in \{1, 2\}$  such that  $V_i$  and  $V_j$  are not connected by an edge in  $g_{(\gamma, \zeta)}(\mathbb{X}^{\lambda, *})$ . Let us denote the corresponding  $V_j$  by  $M_i$ , and define  $M_i = V_1$  if neither  $V_1$  nor  $V_2$  is connected to  $V_i$  by an edge. The element of  $\{V_1, V_2\}$  not being equal to  $M_i$  is denoted by  $N_i$ . We will write  $Q$  for the power value associated to  $M_i$ .

Let us fix  $i \geq 3$ . Let  $\omega \in A$  be such that  $I(\omega) = i$ . Let us define a thinned configuration

$$\omega^i = \omega \setminus \{(M_i(\omega), Q), V_3(\omega), \dots, V_{i-1}(\omega)\}.$$

We claim that  $\omega^i \in \mathbf{N}^*$ . Indeed, for any  $\omega \in \mathbf{N}^*$ ,  $\omega$  minus a finite set of points in  $\mathbb{R}^d \setminus \{o\}$  is an element of  $\mathbf{N}^*$ , which holds in particular for  $\omega^i$ .

Next, we claim for  $\mathbb{P}$ -almost all  $\omega \in A$ ,  $\omega^i \in A$ . Indeed, thanks to Corollary 3.7, for  $\mathbb{P}$ -almost all  $\omega \in A$  with  $I(\omega) = i$ , the following two conditions are both satisfied.

- (i) There are precisely two edge-disjoint infinite paths in  $g_{(\gamma, \zeta)}(\omega)$  starting from  $o$ . Hence, in particular, at least one of these paths does not pass through  $M_i(\omega)$ ,
- (ii)  $V_3(\omega), \dots, V_{i-1}(\omega) \notin \mathcal{C}(\omega)$  by the definition of  $I$  and the fact that  $I(\omega) = i$ .

Now, for  $\omega$  satisfying (i) and (ii), since all edges between two points of  $\omega^i$  in  $g_{(\gamma, \zeta)}(\omega)$  also exist in  $g_{(\gamma, \zeta)}(\omega^i)$ , we conclude that  $\#\mathcal{C}(\omega^i) = \infty$  in  $g_{(\gamma, \zeta)}(\omega^i)$ . This implies the claim.

Our next claim is that for  $\omega$  satisfying (i) and (ii),  $\omega$  or  $\omega^i$  is contained in

$$B = \{\eta: \#\mathcal{C}(\eta) = \infty \text{ and } \mathcal{C}(\eta) \text{ contains a point of degree 1}\} \subset A,$$

which is a  $\mathbb{P}$ -nullset. The proof of this claim in the simplest case  $i = 3$  is illustrated in Figure 2. Indeed,

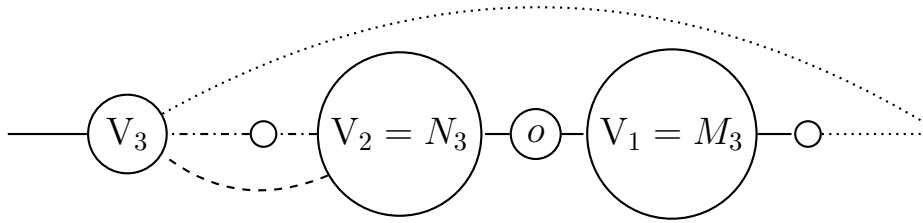


Figure 2: An illustration of the case  $I = i = 3$ .  $V_3$  is contained in the infinite cluster  $\mathcal{C}$  including  $o$ , and it is not a neighbor of  $M_3$ , which in this example equals  $V_1$ , whereas  $V_2 = N_3$ . Hence, if  $V_3$  has degree two in  $\mathcal{C}$ , then there are various possibilities respecting the degree bound of 2 to connect  $V_3$  to  $\mathcal{C}$  so that it is not connected to  $M_3$  by an edge.  $V_3$  can either be a direct neighbor of  $V_2$  (see dashed line) or a later point of the path from  $o$  to infinity starting with the edge from  $o$  to  $V_2$  (dash-dotted lines) or a non-direct neighbor of  $V_1$  on the path from  $o$  to infinity starting with the edge from  $o$  to  $V_1$  (dotted lines). Now, removing  $M_3$  from the realization, both edges adjacent to  $V_3$  are preserved. Also all edges from  $o$  to infinity starting with the edge from  $o$  to  $V_2$  are preserved, hence  $o$  is still contained in an infinite cluster, but the edge from  $o$  to  $V_1$  is removed. In the obtained new configuration, the second-strongest transmitter towards  $o$  is  $V_3$ , and hence this is the only point of the configuration that could be connected to  $o$  by an edge. But  $V_3$  still cannot have degree 3 or more, hence it cannot be connected to  $o$ , which implies that the new configuration is contained in the nullset where  $o$  is in an infinite cluster containing a point of degree 1.

recall that  $o$  cannot have degree higher than two in  $g_{(\gamma, \zeta)}(\omega^i)$ , whereas it has degree at least one and its cluster  $\mathcal{C}(\omega^i)$  is infinite. Note also that the edge between  $o$  and  $N_i(\omega)$  still exists in  $g_{(\gamma, \zeta)}(\omega^i)$ . Further, if  $o$  has degree two in  $g_{(\gamma, \zeta)}(\omega^i)$ , then it is connected to the second-nearest transmitter towards  $o$  in  $\omega^i$ , which is  $V_2(\omega^i) = V_i(\omega)$ , whereas  $V_1(\omega^i) = N_i(\omega)$ . Now, there are two possibilities. If  $\omega \in B$ , then there is nothing to show. Else, since  $\omega \notin B$ ,  $\omega \in A$  and  $V_i(\omega) \in \mathcal{C}(\omega)$ , it follows that  $V_i(\omega)$  has degree equal to two in  $g_{(\gamma, \zeta)}(\omega)$ . Further, it is neither connected to  $M_i(\omega)$  by an edge nor to  $o$  in this graph. Hence, both edges adjacent to  $V_i(\omega)$  also exist in  $g_{(\gamma, \zeta)}(\omega^i)$ . But since  $V_i(\omega)$  has degree at most two in  $g_{(\gamma, \zeta)}(\omega^i)$ , it follows that  $o$  and  $V_i(\omega)$  are not connected by an edge in this graph. Hence,  $\omega^i \in B$ , which implies the claim.

Summarizing, in the event  $\{I = i\} \cap A$ ,  $(\mathbb{X}^{\lambda,*})^i$  is contained in the  $\mathbb{P}$ -nullset  $B$ . In other words, since  $\mathbb{X}^{\lambda,*}$  is the canonical process on  $\mathbb{N}^*$  with distribution  $\mathbb{P}$ ,

$$\mathbb{P}(\{\omega^i: \omega \in A \cap \{I = i\}\}) = 0. \quad (3.15)$$

This implies (3.14) and concludes the proof of Proposition 3.8 as soon as the following lemma is verified.

**Lemma 3.9.** *For any  $i \geq 3$ ,  $\mathbb{P}(A \cap \{I = i\}) > 0$  implies  $\mathbb{P}(\{\omega^i: \omega \in A \cap \{I = i\}\}) > 0$ .*

By Lemma 3.9, where we show that if the collection of thinned configurations is contained in a  $\mathbb{P}$ -nullset, also the non-thinned configurations form a  $\mathbb{P}$ -nullset, we see that (3.15) implies (3.14), which concludes the proof of Proposition 3.8.  $\square$

*Proof of Lemma 3.9.* Let us fix  $i \geq 3$  and assume that  $\mathbb{P}(A \cap \{I = i\}) > 0$ . Then, by continuity of measures, there exists  $K > 0$  such that

$$\mathbb{P}(\{\omega \in A: I(\omega) = i, \forall j \in \{1, \dots, i\}, V_j(\omega) \in B_K(o)\}) > 0,$$

where  $B_K(o)$  denotes the open Euclidean ball of radius  $K$  in  $\mathbb{R}^d$ . Hence, there exists  $n \geq i$  such that  $\mathbb{P}(C_{i,K,n}) > 0$ , where

$$C_{i,K,n} = \{\omega \in A: I(\omega) = i, \#(\omega \cap B_K(o)) = n + 1 \text{ and } V_j(\omega) \in B_K(o), \\ \forall j \in \{1, \dots, i\}\}.$$

Conditional on the event  $C_{i,K,n}$ , the marked CPP  $(\mathbb{X}^{\lambda,*} \setminus \{o, P_*\}) \cap B_K(o)$  has precisely  $n$  points  $\mathbf{X}_1, \dots, \mathbf{X}_n$ . Let us define a random thinning function  $F: C_{i,K,n} \rightarrow \mathbb{N}^*$  such that it discards each point of  $(\mathbb{X}^{\lambda,*} \setminus \{o, P_*\}) \cap B_K(o)$  independently with probability  $1 - p$  and keeps the rest of the points. To be more precise, we choose a set  $\{\mathcal{I}_1, \dots, \mathcal{I}_n\}$  of i.i.d. Bernoulli random variables with parameter  $p \in (0, 1)$  (with a slight abuse of notation we let  $\mathbb{P}$  also govern this i.i.d. sequence), and we define

$$F(\omega) = \{(o, P_*)\} \cup (\omega \setminus B_K(o)) \cup \{\mathbf{X}_i(\omega): \mathcal{I}_i = 1\}, \quad \text{for all } \omega \in C_{i,K,n}.$$

Now we have that

$$\mathbb{P}(\{F(\omega): \omega \in C_{i,K,n}\}) \geq \mathbb{P}(C_{i,K,n} \cap \{F(\omega) = \omega\}) = \mathbb{P}(C_{i,K,n})p^n > 0,$$

and thus we can use elementary conditioning to conclude

$$\begin{aligned} \mathbb{P}(\{\omega^i: \omega \in A \cap \{I = i\}\}) &\geq \mathbb{P}(\{\omega^i: \omega \in C_{i,K,n}\}) \\ &\geq \mathbb{P}(\{F(\omega): \omega \in C_{i,K,n}, F(\omega) = \omega^i\}) \\ &= \mathbb{P}(\{F(\omega): \omega \in C_{i,K,n}, F(\omega) = \omega^i\} | \{F(\omega): \omega \in C_{i,K,n}\}) \\ &\geq \mathbb{P}(C_{i,K,n})p^n p^{n-i+2}(1-p)^{i-2} > 0. \end{aligned} \quad (3.16)$$

This implies the lemma.  $\square$

### 3.3 Proof of Theorem 1.3

This proof is similar to the one of Theorem 1.1 part (1) but simpler. The new proof ingredient that we use here is the strong connectivity of *any* supercritical Poisson–Boolean model [PP96] in case  $d \geq 2$ , which allows us to improve the result that  $\lambda_P < \infty$  to  $\lambda_P = \lambda_c(r_B)$ . First we introduce an adequate discrete percolation model and then we control the interferences.

Throughout the proof  $X^\lambda = \{X_i\}_{i \in I}$  denotes a homogeneous PPP with intensity  $\lambda$  in  $\mathbb{R}^d$ . Let us introduce the notion and elementary properties of Boolean models with constant radius  $r > 0$ . The *Poisson–Boolean model*  $B(X^\lambda, r)$  is defined as

$$B(X^\lambda, r) = \bigcup_{i \in I} B_r(X_i) = X^\lambda \oplus B_r(o).$$

Connecting any two different points  $X_i, X_j \in X^\lambda$  by an edge whenever

$$|X_i - X_j| < 2r, \tag{3.17}$$

we obtain the Poisson–Gilbert graph  $g_{2r}(X^\lambda)$  with connection radius  $2r$ . Percolation in this Gilbert graph is equivalent to the existence of an unbounded connected component in  $B(X^\lambda, r)$ , which we also refer to as percolation. This way, one can speak about subcritical, critical and supercritical Poisson–Boolean models.

Recall the definition of the radius  $r_B$  from (1.4) and let us fix  $\lambda > \lambda_c(r_B)$  for the remainder of this section. Thanks to scale invariance of Poisson–Boolean models [MR96, Section 2.2] and our assumptions on  $\ell$ , we can fix  $r \in (d_o, r_B)$  such that the Poisson–Boolean model  $B(X^\lambda, r/2)$  associated to  $g_r(X^\lambda)$  is still supercritical. The next lemma is an immediate consequence of the results in [PP96, Section 1].

**Lemma 3.10** ([PP96]). *Let  $B(X^\lambda, r/2)$  be a supercritical Poisson–Boolean model and let  $x \in \mathbb{R}^d$ . With probability tending to one as  $n \uparrow \infty$ , we have that*

- (1)  $B(X^\lambda, r/2) \cap Q_n(x)$  contains a connected component of diameter at least  $n/3$ ,
- (2) any two connected components of  $B(X^\lambda, r/2) \cap Q_n(x)$  of diameter at least  $n/9$  each are contained in the same connected component of  $B(X^\lambda, r/2) \cap Q_{2n}(x)$ .

Using Lemma 3.10, we construct a renormalized percolation process on  $\mathbb{Z}^d$ . For  $z \in \mathbb{Z}^d$ , let  $\Xi_n(z)$  denote the union of all connected components of  $B(X^\lambda, r/2) \cap Q_n(z)$  that are of diameter at least  $n/3$ . For  $n \geq 1$ , we say that the site  $z \in \mathbb{Z}^d$  is *n-good* if

- (1)  $\Xi_n(nz) \neq \emptyset$ , and
- (2) for any  $z' \in \mathbb{Z}^d$  with  $|z - z'|_\infty \leq 1$ , it holds that all pairs of connected components  $C$  of  $\Xi_n(nz)$  and  $C'$  of  $\Xi_n(nz')$  are contained in the same connected component of  $B(X^\lambda, r/2) \cap Q_{6n}(nz)$ .

The site  $z \in \mathbb{Z}^d$  is *n-bad* if  $z$  is not *n-good*. We have the following lemma.



**Lemma 3.11.** *For all  $n \geq 1$  sufficiently large, there exists  $q_A = q_A(\lambda, n) \in (0, 1)$  such that for any  $N \in \mathbb{N}$  and pairwise distinct  $z_1, \dots, z_N \in \mathbb{Z}^d$  we have*

$$\mathbb{P}(z_1, \dots, z_N \text{ are all } n\text{-bad}) \leq q_A^N.$$

*Further, for any  $\varepsilon > 0$ , for all large enough  $n$  one can choose  $q_A$  such that  $q_A < \varepsilon$ .*

*Proof.* For  $z \in \mathbb{Z}^d$ ,  $\mathbb{1}\{z \text{ is } n\text{-good}\}$  is measurable with respect to  $X^\lambda \cap (Q_{6n}(nz) \oplus B_{r/2}(o))$ , which is contained in  $X^\lambda \cap Q_{7n}(nz)$  for all  $n$  large enough, hence for all sufficiently large  $n$  the process of  $n$ -good sites is 7-dependent thanks to the independence property of the PPP  $X^\lambda$ . Hence, using arguments of [DFM<sup>+</sup>06, Section 3.2.], it suffices to verify that

$$\limsup_{n \uparrow \infty} \mathbb{P}(o \text{ is } n\text{-bad}) = 0. \quad (3.18)$$

The limit (3.18) can be verified along the lines of the proof of [HJC19, Theorem 2.6] using an adequate interpretation of the Poisson–Boolean model. More precisely, in view of Definition 2.2, the assertion of Lemma 3.10 is equivalent to the statement [HJC19, Section 2.1] that the  $b$ -dependent directing random measure  $\Lambda$  given as  $\Lambda(dx) = \lambda_1 \mathbb{1}\{x \in B(X^\lambda, r/2)\}dx$  is asymptotically essentially connected, where  $\lambda_1 > 0$  is such that  $\mathbb{E}[\Lambda(Q_1)] = 1$ .  $\square$

The other essential proof ingredient is the interference control. We recall the “shifted” path-loss functions  $\ell_a$  (3.4) and the shot-noise processes  $I_a(x), I(x)$  from Section 3.1, and also that by the triangle inequality, for  $a \geq 0$ ,  $I(x) \leq I_a(z)$  holds for any  $z \in \mathbb{R}^d$  and  $x \in Q_a(z)$ .

For  $n \geq 1$  and  $M > 0$ , we say that  $z \in \mathbb{Z}^d$  is  $(n, M)$ -tame if  $I_{7n}(nz) \leq M$  and  $(n, M)$ -wild otherwise. Then we have the following assertion, which holds for all  $\lambda$  such that  $B(X^\lambda, r/2)$  is supercritical.

**Lemma 3.12.** *[Tób19a] For fixed  $n \geq 1$ , for all sufficiently large  $M > 0$ , there exists  $q_B = q_B(\lambda, n, M) \in (0, 1)$  such that for any  $N \in \mathbb{N}$  and pairwise distinct  $z_1, \dots, z_N \in \mathbb{Z}^d$  we have*

$$\mathbb{P}(z_1, \dots, z_N \text{ are all } (n, M)\text{-wild}) \leq q_B^N.$$

*Further, for  $\varepsilon > 0$ , for any  $n \geq 1$ , for all sufficiently large  $M$  one can choose  $q_B$  such that  $q_B < \varepsilon$ .*

Equipped with these results, we can now prove our main theorem.

*Proof of Theorem 1.3.* For  $n \geq 1$  and  $M > 0$ , we say that the site  $z \in \mathbb{Z}^d$  is  $(n, M)$ -nice if it is both  $n$ -good and  $(n, M)$ -tame. We claim that for all sufficiently large  $n$  and accordingly chosen large enough  $M$ , the process of  $(n, M)$ -nice sites percolates. Indeed, this follows by combining the estimates of Lemmas 3.10 and 3.12 similarly to Corollary 3.3 and carrying out a Peierls argument.

We claim that this assertion implies percolation in  $g_{(\gamma, P)}(X^\lambda)$  for small  $\gamma > 0$ . Indeed, let  $n, M$  be so large that the process of  $(n, M)$ -nice sites percolates, and such that  $Q_{6n}(o) \oplus B_{r/2}(o) \subseteq Q_{7n}(o)$ . Using a standard argument [DFM<sup>+</sup>06], one can choose  $\gamma > 0$  sufficiently small such that for any  $(n, M)$ -tame site  $z$ , all connections in  $g_r(X^\lambda) \cap Q_{7n}(nz)$  also exist in  $g_{(\gamma, P)}(X^\lambda) \cap Q_{7n}(nz)$ .

Now, analogously to [HJC19, Section 5.2], we can argue as follows. Let  $\mathcal{C}$  be an infinite connected component of the process of sites that are  $(n, M)$ -nice. Let  $z, z' \in \mathcal{C}$  and  $\{z_0 = z, z_1, \dots, z_{k-1}, z_k = z'\}$  a path in  $\mathcal{C}$  connecting  $z$  and  $z'$ . Then, thanks to  $n$ -goodness, for any  $j = 0, \dots, k$  and for any  $X_j \in X^\lambda$  such that  $B_{r/2}(X_j) \cap Q_n(nz_j) \subseteq \Xi_n(nz_j)$  we have that  $X_j$  and  $X_{j+1}$  are in the same connected component of  $B(X^\lambda, r/2) \cap Q_{6n}(nz_j)$ . In other words  $X_j$  and  $X_{j+1}$  are connected in the Poisson–Gilbert graph  $g_r(X^\lambda)$  via a path in  $Q_{7n}(nz_j)$ , where the additional unit of  $n$  comes from the fact that centers of balls in the Boolean model might lie in a neighboring box. Hence, using  $(n, M)$ -tameness, we conclude that all edges of this path in  $g_r(X^\lambda)$  also exist in  $g_{(\gamma, P)}(X^\lambda)$ . Thus,  $g_{(\gamma, P)}(X^\lambda)$  also percolates. Since  $\lambda > \lambda_c(r_B)$  was arbitrary, the theorem follows.  $\square$

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