Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

A local projection stabilization/continuous Galerkin–Petrov method for incompressible flow problems

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submitted: December 7, 2016

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No. 2347
Berlin 2016

2010 *Mathematics Subject Classification.* 65M12, 65M15, 65M80.

Key words and phrases. Evolutionary Oseen problem, inf-sup stable pairs of finite element spaces, local projection stabilization (LPS) methods, continuous Galerkin–Petrov (cGP) methods.

Julia Novo's research was financed by Spanish MICINN under grant MTM.

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Abstract. The local projection stabilization (LPS) method in space is considered to approximate the evolutionary Oseen equations. Optimal error bounds independent of the viscosity parameter are obtained in the continuous-in-time case for the approximations of both velocity and pressure. In addition, the fully discrete case in combination with higher order continuous Galerkin–Petrov (cGP) methods is studied. Error estimates of order $k+1$ are proved, where k denotes the polynomial degree in time, assuming that the convective term is time-independent. Numerical results show that the predicted order is also achieved in the general case of time-dependent convective terms.

1. Introduction

The behavior of incompressible flows is modeled by the incompressible Navier–Stokes equations. Analyzing numerical schemes for these equations faces several difficulties. First, the unresolved problem of the uniqueness of the weak solution of the Navier–Stokes equations in three dimensions requires to assume uniqueness, which is usually done by assuming sufficient regularity of the weak solution. Moreover, the estimate of the nonlinear term often uses the Gronwall lemma, such that an exponential factor occurs in the error bounds, depending on some norm of the velocity, e.g., on $\|\nabla \boldsymbol{u}\|_\infty$ as in [20]. As result, the obtained estimates are by far too pessimistic in practice. For these reasons, this paper will deal, with respect to the numerical analysis, with a related but simpler problem, namely the evolutionary or transient Oseen equations. They read in dimensionless form as follows:

Find $\boldsymbol{u}(t,\boldsymbol{x}) : (0,T] \times \Omega \to \mathbb{R}^d$, $d \in \{2,3\}$, and $p(t,\boldsymbol{x}) : (0,T] \times \Omega \to \mathbb{R}$ such that

$$
\begin{aligned}
\partial_t \boldsymbol{u} - \nu \Delta \boldsymbol{u} + (\boldsymbol{b} \cdot \nabla)\boldsymbol{u} + \sigma \boldsymbol{u} + \nabla p &= \boldsymbol{f} && \text{in } (0,T] \times \Omega, \\
\operatorname{div} \boldsymbol{u} &= 0 && \text{in } (0,T] \times \Omega, \\
\boldsymbol{u} &= \boldsymbol{0} && \text{on } (0,T] \times \partial\Omega, \\
\boldsymbol{u}(0,\cdot) &= \boldsymbol{u}_0 && \text{in } \Omega,
\end{aligned}
\tag{1}
$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $\nu = \mathrm{Re}^{-1} > 0$ (viscosity) and $\sigma > 0$ are positive constants, $\boldsymbol{b}(t, \boldsymbol{x})$ is a given velocity field with $\operatorname{div} \boldsymbol{b} = 0$, $\boldsymbol{u}_0$ is the initial velocity field, and T is a given final time. Without loss of generality, one can assume $\sigma > 0$, since if it is not the case then a simple change of variable transforms the problem into (1) with $\sigma > 0$, see [21, Sect. 1].

The numerical solution of (1) requires discretizations in time and space. Concerning the temporal discretization, continuous Galerkin–Petrov methods of order $k \geq 1$, cGP(k), will be considered. With respect to space, finite element methods will be studied. Since the paper will study the convection-dominated regime, where ν is smaller than an appropriate norm of $\boldsymbol{b}$ by several orders of magnitude, a stabilization of the standard finite element discretization becomes necessary.

Considering the situation that the viscosity is much smaller than the convection in the practical relevant case of the Navier–Stokes equations, the flow becomes turbulent. The simulation of turbulent flows requires the use of a turbulence model. There are many models proposed in the literature, like, e.g., the Smagorinsky model, variational multiscale (VMS) methods, or deconvolution models. In particular, the residual-based VMS method from [12] is an extension of the well known streamline upwind Petrov–Galerkin (SUPG) method from [17,23] by higher order (with respect to the residual) terms. Often, the SUPG stabilization is used in combination with the pressure-stabilization Petrov–Galerkin (PSPG) method, which stabilizes the violation of the discrete inf-sup condition [30]. However, the SUPG/PSPG method possesses some drawbacks. As explained in [16], the SUPG/PSPG approach introduces a velocity-pressure coupling for which no physical explanation is known and also the non-symmetry of the stabilization might be of disadvantage. In the time-dependent case, the consistent application of the method leads to a number of additional terms which have to be assembled, including an approximation of the temporal derivative, see [29, 32]. Because of the drawbacks of the SUPG/PSPG method, we think that it is worth to study different approaches in detail, in particular such approaches that are symmetric and that do not introduce an additional velocity-pressure coupling. Local projection stabilization (LPS) methods belong to this class of methods and will be the topic of this paper.

A different approach was studied recently in [21], where a grad-div stabilized method is used to discretize the evolutionary Oseen equations. Optimal bounds for the divergence of the velocity and the $L^2(\Omega)$ norm of the pressure are proved for this method.

The LPS method was originally proposed for the Stokes problem in [13] and it was successfully extended to transport problems in [14]. Numerical analysis for the LPS method applied to the stationary Oseen equations can be found in [15,35] and to convection-diffusion-reaction problems in [5, 7, 11, 36]. The stabilization term of the LPS method is based on a projection defined on the finite element space that approximates the solution into a discontinuous space. Compared with the standard Galerkin approach, the LPS method gives additional control over (parts of) the fluctuation of the gradient. The method is weakly consistent but the consistency error can be bounded to achieve an optimal rate of convergence. Originally, the LPS method was proposed as a two-level approach, where the projection spaces are defined on coarser grids. This approach introduces additional couplings between neighboring mesh cell and hence, the sparsity of the matrix decreases. This drawback does not appear in the one-level approach, where both spaces are defined on the same grid. In this approach, the approximation spaces have to be enriched compared with the standard finite element spaces. The additional degrees of freedoms which are introduced due to the enrichment can be eliminated using static condensation. Altogether, the one-level approach is, in our opinion, more appealing from the point of view of implementation and this variant of the LPS method will be considered in this paper.

Recently, in [19] the time-dependent Oseen problem was considered using LPS methods with stabilization of the streamline derivative together with grad-div stabilization. In the case of using methods of order k without compatibility condition, error bounds are obtained under a restriction on the mesh size: a certain measure for the mesh size should be of order of the square root of the viscosity. In order to avoid the restriction on the mesh size for small viscosity, the authors of [19] considered pairs satisfying a certain element-wise compatibility condition between the discrete velocities on the fine mesh and in the projection space. Even in that case, optimal error bounds for the pressure were not obtained in [19]. In [8], a LPS method for the time-dependent Navier–Stokes equations was analyzed. As in [19], the LPS approach is applied to the streamline derivative and to a grad-div stabilization term, which is a different LPS method than considered here. Error estimates for the velocity in the continuous-in-time situation were derived in [8]. An analysis of the fully discretized so-called high-order term-by-term LPS method can be found in [2].

As mentioned above, cGP(k) methods will be considered as temporal discretization. For incompressible flow problems, usually θ-schemes are used. These schemes are simple to implement, however, they are at most of second order, like the Crank–Nicolson scheme or the fractional-step θ-scheme. In addition, they do not allow an efficient adaptive time step control. There are only few studies, like [25, 28, 31] which consider higher order schemes, like diagonally implicit Runge–Kutta (DIRK) methods, Rosenbrock–Wanner (ROW) methods, or just cGP(2). To the best of our knowledge, there is no numerical analysis available for the first two classes of schemes applied to incompressible flow problems or even to convection-diffusion equations. The situation is different for cGP(k) that treats the temporal derivative in a finite element way. The cGP(k) methods are a class of finite element methods using discrete solution spaces in time that consist of continuous piecewise polynomials of degree less than or equal to k and test spaces which are built by discontinuous polynomials of degree up to order $k-1$. This choice enables the performance of a standard time marching algorithm and it avoids the solution of a global system in space and time as in space-time finite element methods.

The cGP method in time for the heat equation has been investigated in [10]. Optimal error estimates and super-convergence results are derived at the end point of the discrete time intervals. The methods cGP(k) have been studied in [38] even in an abstract Hilbert space setting and for nonlinear systems of ordinary differential equations in d space dimensions. A-stability and optimal error estimates were proved. Moreover, it was shown that cGP(k) methods have an energy decreasing property for the gradient flow equation of an energy functional. Recently, in [5], transient convection-diffusion-reaction equations were considered using cGP(k) in time combined with LPS in space. Optimal a-priori error estimates were derived for the fully discrete scheme. It has been shown numerically that cGP(k) is super-convergent of order $(k+2)$ in the integrated norm and of order $2k$ at discrete time points. Moreover, the obtained results were compared with discontinuous Galerkin (dG) time stepping schemes. Numerical studies for the time-dependent Stokes equations in [24], the transient Oseen equations in [4], and transient convection-diffusion-reaction equations in [5] showed the expected orders of convergence for cGP(k), $k \in \{1, 2\}$. The dG(k) method was analyzed for the transient Stokes equations in [1]. In addition, the higher order convergence of cGP(2) compared with the discontinuous Galerkin discretization dG(1), both methods possessing the same complexity, was demonstrated. An efficient adaptive time step control is also possible with cGP(k) methods, e.g., as applied in [3] to transient convection-diffusion-reaction equations. The adaptive time step control is based on a post-processed discrete solution. It has been shown that the adaptive time step control leads to lengths of the time steps that properly reflect the dynamics of the solution.

However, there is also a certain drawback of cGP(k) methods for $k \geq 2$: a coupled system of k equations has to be solved at each discrete time. By a clever construction proposed in [38], the coupling is not strong, but it cannot be removed completely. Efficient solvers for this coupled problem in case of the Navier–Stokes equations have been studied in [25], where a coupled multigrid method with Vanka-type smoothers was utilized.

Altogether, cGP(k) is in our opinion an attractive alternative to θ-schemes since a higher order in time can be achieved and an efficient time step control is possible at affordable computational costs.

The goal of this paper consists in studying the combination of the LPS method in space with the cGP(k) method in time. The numerical analysis will be performed for the transient Oseen equations (1). Thus, this paper presents the first numerical analysis of a higher order time stepping scheme for an incompressible flow problem with convection. In the continuous-in-time case, optimal error bounds for velocity and pressure with constants that do not depend on the viscosity parameter ν are obtained with the assumption that the solution is sufficiently smooth. In addition, error estimates for the fully discrete problem of order $k+1$ are proved, assuming, as in other recently published papers, that the convective term is time-independent. Numerical results show that the predicted order can be also observed in the case of time-dependent convective terms.

The remainder of the paper is organized as follows: Section 2 introduces the basic notation, it presents some preliminaries, and the semi-discretization (continuous-in-time) of the LPS method will be described. In Section 3, the error bounds for the semi-discrete problem are derived. Section 4 presents the error analysis of the fully discrete problem using a temporal discretization with a cGP(k) method. Numerical studies can be found in Section 5.

2. Preliminaries

Throughout this paper, standard notation and conventions will be used. For a measurable set $G \subset \mathbb{R}^d$, the inner product in $L^2(G)$, $L^2(G)^d$, and $L^2(G)^{d\times d}$ will be denoted by $(\cdot,\cdot)_G$. The norm and the semi-norm in $W^{m,p}(G)$ are given by $\|\cdot\|_{m,p,G}$ and $|\cdot|_{m,p,G}$, respectively. In the case $p = 2$, $H^m(G)$, $\|\cdot\|_{m,G}$, and $|\cdot|_{m,G}$ are written instead of $W^{m,2}(G)$, $\|\cdot\|_{m,2,G}$, and $|\cdot|_{m,2,G}$. If $G = \Omega$, the index G in inner products, norms, and semi-norms will be omitted. The dual pairing between a space Z and its dual Z' will be denoted by $\langle\cdot,\cdot\rangle$. The temporal derivative of a function f is denoted by $\partial_t f$ and the i-th temporal derivative by $\partial_t^i f$. The subspace of functions from $H^1(\Omega)$ having zero boundary trace is denoted by $H_0^1(\Omega)$. Its dual space is denote by $H^{-1}(\Omega)$ with the associated norm $\|v\|_{-1} = \sup_{\varphi \in H_0^1(\Omega)\setminus\{0\}} \frac{\langle v,\varphi\rangle}{\|\nabla\varphi\|_0}$. Let Z be a Banach space with norm $\|\cdot\|_Z$, then the following spaces are defined

$$
\begin{aligned}
L^2(0,t;Z) &:= \left\{ v \ : \ (0,t) \to Z \ : \ \int_0^t \|v(s)\|_Z^2 \, ds < \infty \right\}, \\
H^1(0,t;Z) &:= \left\{ v \in L^2(0,t;Z) \ : \ \partial_t v \in L^2(0,t;Z) \right\}, \\
C(0,t;Z) &:= \{ v \ : \ (0,t) \to Z \ : \ v \text{ is continuous with respect to time} \},
\end{aligned}
$$

where $\partial_t v$ is the time derivative of v in the sense of distributions. If $t = T$, then the abbreviations $L^2(Z)$, $H^1(Z)$, and $C(Z)$ are used and it will not be indicated whether it is a scalar-valued or vector-valued space.

In order to derive a variational form of (1), the spaces

$$
V := H_0^1(\Omega)^d, \quad Q := L_0^2(\Omega), \quad X := \left\{ \boldsymbol{v} \in L^2(V),\ \partial_t \boldsymbol{v} \in L^2(V') \right\}
$$

and the bilinear form

$$
a\big((\boldsymbol{u},p);(\boldsymbol{v},q)\big) := \nu(\nabla\boldsymbol{u},\nabla\boldsymbol{v}) + ((\boldsymbol{b}\cdot\nabla)\boldsymbol{u},\boldsymbol{v}) + (\sigma\boldsymbol{u},\boldsymbol{v}) - (\operatorname{div}\boldsymbol{v},p) + (\operatorname{div}\boldsymbol{u},q)
$$

are introduced. Then, a variational form of (1) reads as follows:

Find $\boldsymbol{u} \in X$ and $p \in L^2(Q)$ such that

(2)

$$
\langle \partial_t \boldsymbol{u}(t), \boldsymbol{v}(t)\rangle + a\big((\boldsymbol{u}(t),p(t));(\boldsymbol{v}(t),q(t))\big) = (\boldsymbol{f}(t),\boldsymbol{v}(t)) \quad \forall\ \boldsymbol{v} \in L^2(V),\ q \in L^2(Q)
$$

for almost all $t \in (0,T]$ and $\boldsymbol{u}(0,\cdot) = \boldsymbol{u}_0$. Note that this initial condition is well defined since functions belonging to X are continuous in time.

If the initial condition $\boldsymbol{u}_0$ is different from $\boldsymbol{0}$, the velocity $\boldsymbol{u}$ can be decomposed in the form

$$
\boldsymbol{u}(t) = \boldsymbol{u}_0 + \boldsymbol{\psi}(t), \quad \boldsymbol{\psi} \in X_0 := \{ \boldsymbol{v} \in X : \ \boldsymbol{v}(0,\cdot) = \boldsymbol{0} \}.
$$

Then for the given initial velocity field $\boldsymbol{u}_0$, one has to find $\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{\psi}(t)$, with $\boldsymbol{\psi}(t) \in X_0$, and $p \in L^2(Q)$, where $(\boldsymbol{\psi},p)$ is the solution of the problem

$$
(\partial_t\boldsymbol{\psi}(t),\boldsymbol{v}(t)) + a\big((\boldsymbol{\psi}(t),p(t));(\boldsymbol{v}(t),q(t))\big) = (\boldsymbol{g}(t),\boldsymbol{v}(t))
$$

with

$$
(\boldsymbol{g},\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v}) - \nu(\nabla\boldsymbol{u}_0,\nabla\boldsymbol{v}) - ((\boldsymbol{b}\cdot\nabla)\boldsymbol{u}_0,\boldsymbol{v}) - (\sigma\boldsymbol{u}_0,\boldsymbol{v}).
$$

For this reason, one can assume $\boldsymbol{u}_0 = \boldsymbol{0}$, which will be done in the sequel. Note that this choice of the initial condition will result in errors bounds that do not contain contributions depending on $\boldsymbol{u}_0$.

Let $\Pi \ : \ L^2(\Omega)^d \to H^{\text{div}}$ be the Leray projector that maps each function in $L^2(\Omega)^d$ onto its divergence-free part, where the Hilbert space H^{div} is defined by $H^{\text{div}} = \{\boldsymbol{v} \in L^2(\Omega)^d \ : \ \nabla\cdot\boldsymbol{v} = 0,\ \boldsymbol{v}\cdot\boldsymbol{n}|_{\partial\Omega} = 0\}$. The Stokes operator in Ω is given by

$$
A \ : \ \mathcal{D}(A) \subset H^{\text{div}} \to H^{\text{div}}, \quad A = -\Pi\Delta, \quad \mathcal{D}(A) = H^2(\Omega)^d \cap V^{\text{div}},
$$

where the space $V^{\text{div}} = \left\{\boldsymbol{v} \in H_0^1(\Omega)^d : \nabla\cdot\boldsymbol{v} = 0\right\}$ is equipped with the inner product of $H_0^1(\Omega)^d$.

Let $\{\mathcal{T}_h\}$ be a family of shape-regular triangulations of Ω into compact d-simplices, quadrilaterals, or hexahedra such that $\overline{\Omega} = \cup_{K\in\mathcal{T}_h} K$. The diameter of $K \in \mathcal{T}_h$ will be denoted by h_K and the mesh size h is defined by $h := \max\limits_{K\in\mathcal{T}_h} h_K$. Let $Y_h \subset H_0^1(\Omega)$

be a finite element space of scalar, continuous, piecewise mapped polynomial functions over $\mathcal{T}_h$. The finite element space V_h for approximating the velocity field is given by $V_h := Y_h^d \cap V$. The pressure is discretized using a finite element space $Q_h \subset Q$ of continuous or discontinuous functions with respect to $\mathcal{T}_h$. In this paper, inf-sup stable pairs (V_h, Q_h) will be considered, i.e., there is a positive constant β_0, independent of the triangulation, such that

$$\inf_{q_h \in Q_h \setminus \{0\}} \sup_{\boldsymbol{v}_h \in V_h \setminus \{\boldsymbol{0}\}} \frac{(\operatorname{div} \boldsymbol{v}_h, q_h)}{|\boldsymbol{v}_h|_1 \|q_h\|_0} \geq \beta_0 > 0. \tag{3}$$

Since it will be assumed that the family of meshes is regular, the following inverse inequality holds

$$\|\boldsymbol{v}_h\|_{m,K} \leq C_{\text{inv}} h_K^{l-m} \|\boldsymbol{v}_h\|_{l,K} \tag{4}$$

for each $\boldsymbol{v}_h \in V_h$ and $0 \leq l \leq m \leq 1$, see, e.g., [18, Thm. 3.2.6].

The space of discretely divergence-free functions is denoted by

$$V_h^{\text{div}} = \{\boldsymbol{v}_h \in V_h \ : \ (\nabla \cdot \boldsymbol{v}_h, q_h) = 0 \quad \forall\, q_h \in Q_h\}.$$

The linear operator $A_h \ : \ V_h^{\text{div}} \to V_h^{\text{div}}$ is defined by

$$(A_h \boldsymbol{v}_h, \boldsymbol{w}_h) = (\nabla \boldsymbol{v}_h, \nabla \boldsymbol{w}_h) \quad \forall\, \boldsymbol{w}_h \in V_h^{\text{div}}. \tag{5}$$

Note that from this definition, it follows that

$$\|A_h^{1/2} \boldsymbol{v}_h\|_0 = \|\nabla \boldsymbol{v}_h\|_0, \quad \|\nabla A_h^{-1/2} \boldsymbol{v}_h\|_0 = \|\boldsymbol{v}_h\|_0 \quad \forall\, \boldsymbol{v}_h \in V_h^{\text{div}}. \tag{6}$$

The so-called discrete Leray projection $\Pi_h^{\text{div}} \ : \ L^2(\Omega)^d \to V_h^{\text{div}}$ is introduced, being the L^2-orthogonal projection of $L^2(\Omega)^d$ onto V_h^{div}

$$(\Pi_h^{\text{div}} \boldsymbol{v}, \boldsymbol{w}_h) = (\boldsymbol{v}, \boldsymbol{w}_h) \quad \forall\, \boldsymbol{w}_h \in V_h^{\text{div}}. \tag{7}$$

By definition, it follows that the projection is stable in the L^2 norm: $\|\Pi_h^{\text{div}} \boldsymbol{v}\|_0 \leq \|\boldsymbol{v}\|_0$ for all $\boldsymbol{v} \in L^2(\Omega)^d$.

The continuous-in-time standard Galerkin finite element method applied to (2) consists in finding $\boldsymbol{u}_h \in H^1(V_h)$ with $\boldsymbol{u}_h(0) = \boldsymbol{0}$ and $p_h \in L^2(Q_h)$ such that

$$(\partial_t \boldsymbol{u}_h(t), \boldsymbol{v}_h) + a\big((\boldsymbol{u}_h(t), p_h(t)); (\boldsymbol{v}_h, q_h)\big) = (\boldsymbol{f}(t), \boldsymbol{v}_h) \quad \forall\, \boldsymbol{v}_h \in V_h,\ q_h \in Q_h.$$

In the convection-dominated case, it is well-known that this method is unstable, unless h is sufficiently small. The use of a stabilized discretization becomes necessary.

This paper concentrates on the one-level variant of the LPS method in which approximation and projection spaces are defined on the same mesh. Let $D(K)$, $K \in \mathcal{T}_h$, be local finite-dimensional spaces and $\pi_K \ : \ L^2(K) \to D(K)$ the local L^2 projection into $D(K)$. The local fluctuation operator $\kappa_K \ : \ L^2(K) \to L^2(K)$ is given by $\kappa_K v := v - \pi_K v$. It is applied component-wise to vector-valued and tensor-valued arguments. The stabilization term S_h is defined by

$$S_h(\boldsymbol{u}_h, \boldsymbol{v}_h) := \sum_{K \in \mathcal{T}_h} \mu_K \big(\kappa_K \nabla \boldsymbol{u}_h, \kappa_K \nabla \boldsymbol{v}_h\big)_K,$$

where μ_K, $K \in \mathcal{T}_h$, are non-negative constants. This kind of LPS method gives additional control on the fluctuation of the gradient. Also other variants of this method are possible, e.g., by replacing in both arguments of $S_h(\cdot,\cdot)$ the gradient $\nabla \boldsymbol{w}_h$ by the derivative in the streamline direction $(\boldsymbol{b} \cdot \nabla)\boldsymbol{w}_h$ or, even better [33,34], by $(\boldsymbol{b}_K \cdot \nabla)\boldsymbol{w}_h$, where $\boldsymbol{b}_K$ is a piecewise constant approximation of $\boldsymbol{b}$. But in this method, one has to add the so-called grad-div term $(\operatorname{div} \boldsymbol{u}_h, \operatorname{div} \boldsymbol{v}_h)$ to S_h, see [37].

For the numerical analysis, the linear operator $C_h : V_h^{\text{div}} \to V_h^{\text{div}}$ with

$$(C_h \boldsymbol{v}_h, \boldsymbol{w}_h) = \sum_{K \in \mathcal{T}_h} \mu_K (\kappa_K \nabla \boldsymbol{v}_h, \kappa_K \nabla \boldsymbol{w}_h)_K \quad \forall\, \boldsymbol{v}_h, \boldsymbol{w}_h \in V_h^{\text{div}}, \tag{8}$$

the linear operator $D_h : L^2(\Omega) \to V_h^{\text{div}}$ with

$$(D_h q, \boldsymbol{w}_h) = (\operatorname{div} \boldsymbol{w}_h, q) \quad \forall\, \boldsymbol{w}_h \in V_h^{\text{div}}, \tag{9}$$

the stabilized bilinear form

$$a_h\big((\boldsymbol{u}, p), (\boldsymbol{v}, q)\big) = a\big((\boldsymbol{u}, p); (\boldsymbol{v}, q)\big) + S_h(\boldsymbol{u}, \boldsymbol{v})$$

on the product space (V_h, Q_h), and the mesh-dependent norm

$$|||\boldsymbol{v}||| := \left\{ \nu |\boldsymbol{v}|_1^2 + \sigma \|\boldsymbol{v}\|_0^2 + \sum_{K \in \mathcal{T}_h} \mu_K \|\kappa_K \nabla \boldsymbol{v}\|_{0,K}^2 \right\}^{1/2}$$

are defined.

It will be assumed that $\boldsymbol{b} \in L^\infty(L^\infty(\Omega) \cap H^{\text{div}}(\Omega))$ and $\nabla \cdot \boldsymbol{b}(t) = 0$ for almost all $t \in [0, T]$. Then, a straightforward calculation shows that

$$a_h\left((\boldsymbol{v}_h, q_h), (\boldsymbol{v}_h, q_h)\right) = |||\boldsymbol{v}_h|||^2 \quad \forall\, \boldsymbol{v}_h \in V_h,\ q_h \in Q_h. \tag{10}$$

The stabilized semi-discrete problem reads:

Find $\boldsymbol{u}_h \in H^1(V_h)$ with $\boldsymbol{u}_h(0) = \boldsymbol{0}$ and $p_h \in L^2(Q_h)$ such that

$$\big(\partial_t \boldsymbol{u}_h, \boldsymbol{v}_h\big) + a_h\left(\big(\boldsymbol{u}_h, p_h\big); (\boldsymbol{v}_h, q_h)\right) = \big(\boldsymbol{f}, \boldsymbol{v}_h\big) \quad \forall\, \boldsymbol{v}_h \in V_h,\ q_h \in Q_h \tag{11}$$

for almost every $t \in (0, T]$.

For performing the analysis of LPS schemes, certain compatibility conditions between the approximation space and local projection space have to be satisfied, see [35].

Assumption A1. There are interpolation operators $j_h : H^2(\Omega)^d \to V_h$ and $i_h : H^2(\Omega) \to Q_h$ with the approximation properties

$$\|\boldsymbol{w} - j_h \boldsymbol{w}\|_{0,K} + h_K |\boldsymbol{w} - j_h \boldsymbol{w}|_{1,K} \le C h_K^l \|\boldsymbol{w}\|_{l,K} \quad \forall\, \boldsymbol{w} \in H^l(K)^d,\ 2 \le l \le r+1, \tag{12}$$

$$\|q - i_h q\|_{0,K} + h_K |q - i_h q|_{1,K} \le C h_K^l \|q\|_{l,K} \quad \forall\, q \in H^l(K),\ 2 \le l \le r, \tag{13}$$

for all $K \in \mathcal{T}_h$. The pressure interpolation operator i_h satisfies the orthogonality condition

$$(q - i_h q, r_h)_K = 0 \qquad \forall\, q \in Q \cap H^2(\Omega),\ r_h \in D(K). \tag{14}$$

The pairs $V_h/Q_h = \mathbb{Q}_r/\mathbb{P}_{r-1}^{\text{disc}}$ together with $D(K) = \mathbb{P}_{r-1}(K)$ fulfill for $r \ge 2$ assumption A1 if j_h is the usual Lagrangian interpolation operator and i_h the L^2 projection. Further examples of inf-sup stable pairs V_h/Q_h, associated interpolation operators j_h and i_h, and projection spaces fulfilling assumption A1 can be found in [37].

Assumption A2. The fluctuation operator satisfies the following approximation property

$$\left\|\kappa_K q\right\|_{0,K} \le C h_K^l \left|q\right|_{l,K} \qquad \forall\ K \in \mathcal{T}_h,\ \forall\ q \in H^l(K),\ 0 \le l \le r. \tag{15}$$

For performing the numerical analysis, the steady-state Stokes problem

$$\begin{aligned} -\nu\Delta\boldsymbol{u} + \nabla p &= \boldsymbol{g} && \text{in } \Omega,\\ \boldsymbol{u} &= \boldsymbol{0} && \text{on } \partial\Omega,\\ \nabla\cdot\boldsymbol{u} &= 0 && \text{in } \Omega, \end{aligned} \tag{16}$$

will be considered. The standard Galerkin approximation $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$ is the solution of the mixed finite element approximation to (16), given by

$$\begin{aligned} \nu(\nabla\boldsymbol{u}_h, \nabla\boldsymbol{v}_h) - (\operatorname{div}\boldsymbol{v}_h, p_h) &= (\boldsymbol{g}, \boldsymbol{v}_h) && \forall\ \boldsymbol{v}_h \in V_h,\\ (\nabla\cdot\boldsymbol{u}_h, q_h) &= 0 && \forall\ q_h \in Q_h. \end{aligned} \tag{17}$$

Following [22, 26] one gets the estimates

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_1 \le C\left(\inf_{\boldsymbol{v}_h\in V_h}\|\boldsymbol{u} - \boldsymbol{v}_h\|_1 + \nu^{-1}\inf_{q_h\in Q_h}\|p - q_h\|_0\right), \tag{18}$$

$$\|p - p_h\|_0 \le C\left(\nu\inf_{\boldsymbol{v}_h\in V_h}\|\boldsymbol{u} - \boldsymbol{v}_h\|_1 + \inf_{q_h\in Q_h}\|p - q_h\|_0\right), \tag{19}$$

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_0 \le Ch\left(\inf_{\boldsymbol{v}_h\in V_h}\|\boldsymbol{u} - \boldsymbol{v}_h\|_1 + \nu^{-1}\inf_{q_h\in Q_h}\|p - q_h\|_0\right). \tag{20}$$

It can be observed that the error bounds for the velocity depend on negative powers of ν.

As suggested in [21], a projection of $(\boldsymbol{u}, p)$ into $V_h \times Q_h$ is used, where the bounds for the velocity are uniform in ν. For the Oseen problem, let $(\boldsymbol{u}, p)$ be the solution of (1) with $\boldsymbol{u} \in H^1\left(V \cap H^{l+1}(\Omega)^d\right)$, $p \in L^2\left(Q \cap H^l(\Omega)\right)$, $l \ge 1$, and define the right-hand side of the Stokes problem (16) by

$$\boldsymbol{g} = \boldsymbol{f} - \partial_t\boldsymbol{u} - (\boldsymbol{b}\cdot\nabla)\boldsymbol{u} - \sigma\boldsymbol{u} - \nabla p. \tag{21}$$

Then $(\boldsymbol{u}, 0)$ is the solution of (16). Denoting the corresponding Galerkin approximation in $V_h \times Q_h$ by $(\boldsymbol{s}_h, l_h)$, one obtains from (18)–(20)

$$\|\boldsymbol{u} - \boldsymbol{s}_h\|_0 + h\|\boldsymbol{u} - \boldsymbol{s}_h\|_1 \le Ch^{l+1}\|\boldsymbol{u}\|_{l+1}, \tag{22}$$

$$\|l_h\|_0 \le C\nu h^l\|\boldsymbol{u}\|_{l+1}, \tag{23}$$

where the constant C does not depend on ν.

Remark 1. *Assuming the necessary smoothness in time and considering* (16) *with*

$$\boldsymbol{g} = \boldsymbol{g}^i = \partial_t^i\left(\boldsymbol{f} - \partial_t\boldsymbol{u} - (\boldsymbol{b}\cdot\nabla)\boldsymbol{u} - \sigma\boldsymbol{u} - \nabla p\right), \quad i \ge 1,$$

one can derive error bounds of form (22) *and* (23) *also for* $\partial_t^i\boldsymbol{u} - \boldsymbol{s}_h(\boldsymbol{g}^i)$ *and* $l_h(\boldsymbol{g}^i)$, *where* $\left(\boldsymbol{s}_h(\boldsymbol{g}^i), l_h(\boldsymbol{g}^i)\right)$ *denotes the solution of* (17) *with right-hand side* $\boldsymbol{g} = \boldsymbol{g}^i$. *Hence, the estimates*

$$\begin{aligned} \|\partial_t^i\boldsymbol{u} - \boldsymbol{s}_h(\boldsymbol{g}^i)\|_0 + h\|\partial_t^i\boldsymbol{u} - \boldsymbol{s}_h(\boldsymbol{g}^i)\|_1 &\le Ch^{l+1}\|\partial_t^i\boldsymbol{u}\|_{l+1},\\ \|l_h(\boldsymbol{g}^i)\|_0 &\le C\nu h^l\|\partial_t^i\boldsymbol{u}\|_{l+1}, \end{aligned}$$

can be obtained.

3. Error analysis for the continuous-in-time case

In this section, error bounds for velocity and pressure will be derived with constants independent of ν for a sufficiently smooth solution. The analysis follows the lines of [21].

Theorem 2. *Let $(\boldsymbol{u}, p)$ be the solution of (2) and let $(\boldsymbol{u}_h, p_h)$ be the solution of (11). Assume $\boldsymbol{b} \in L^\infty(L^\infty)$ and the regularities*

$$(\boldsymbol{u}, p) \in L^2(H^{r+1}) \times L^2(H^r), \quad \partial_t \boldsymbol{u} \in L^2(H^r). \tag{24}$$

Choosing the stabilization parameters of the LPS method such that $\mu_K \sim 1$ with respect to the mesh width, then the following error estimate holds for all $t \in (0,T]$

$$\begin{aligned} \|(\boldsymbol{u}-\boldsymbol{u}_h)(t)\|_0^2 &+ \nu\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|^2_{L^2(0,t;L^2)} + \sigma\|\boldsymbol{u}-\boldsymbol{u}_h\|^2_{L^2(0,t;L^2)} \\ &+ \sum_{K\in\mathcal{T}_h} \|\kappa_K \nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|^2_{L^2(0,t;L^2(K))} \\ &\le Ch^{2r}\left(\|\boldsymbol{u}\|^2_{L^2(0,t;H^{r+1})} + \|\partial_t \boldsymbol{u}\|^2_{L^2(0,t;H^r)} + \|p\|^2_{L^2(0,t;H^r)}\right), \end{aligned} \tag{25}$$

where $C = C\left(\sigma, \|\boldsymbol{b}\|_{L^\infty(0,t;L^\infty)}\right)$ is independent of ν and h.

Proof. The proof of the error estimate is based on the comparison of the Galerkin approximation $(\boldsymbol{u}_h, p_h)$ in (11) with the approximation $(\boldsymbol{s}_h, l_h)$ of the Stokes equations with right-hand side (21). Let $\boldsymbol{e}_h = \boldsymbol{u}_h - \boldsymbol{s}_h$, then a straightforward calculation yields

$$\begin{aligned} (\partial_t \boldsymbol{e}_h, \boldsymbol{v}_h) &+ a_h((\boldsymbol{e}_h, p_h - l_h), (\boldsymbol{v}_h, q_h)) \\ &= (\partial_t(\boldsymbol{u}-\boldsymbol{s}_h), \boldsymbol{v}_h) + ((\boldsymbol{b}\cdot\nabla)(\boldsymbol{u}-\boldsymbol{s}_h) + \sigma(\boldsymbol{u}-\boldsymbol{s}_h), \boldsymbol{v}_h) \\ &\quad + (\nabla p, \boldsymbol{v}_h) - S_h(\boldsymbol{s}_h, \boldsymbol{v}_h) \quad \forall\ \boldsymbol{v}_h \in V_h,\ q_h \in Q_h. \end{aligned} \tag{26}$$

Taking $(\boldsymbol{v}_h, q_h) = (\boldsymbol{e}_h, p_h - l_h)$ in (26), one gets with integrating by parts, using that $\boldsymbol{e}_h$ has discrete divergence equal to zero, and (14)

$$(\nabla p, \boldsymbol{e}_h) = -(p, \nabla\cdot\boldsymbol{e}_h) = -(p - i_h p, \nabla\cdot\boldsymbol{e}_h) = (i_h p - p, \kappa_K \nabla\cdot\boldsymbol{e}_h).$$

With the Cauchy–Schwarz inequality and Hölder's inequality, it follows that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\boldsymbol{e}_h\|_0^2 + \nu\|\nabla\boldsymbol{e}_h\|_0^2 + \sigma\|\boldsymbol{e}_h\|_0^2 + \sum_{K\in\mathcal{T}_h} \mu_K\|\kappa_K\nabla\boldsymbol{e}_h\|^2_{0,K} \\ &\le \|\partial_t(\boldsymbol{u}-\boldsymbol{s}_h)\|_0\|\boldsymbol{e}_h\|_0 + \|\boldsymbol{b}\|_\infty\|\nabla(\boldsymbol{u}-\boldsymbol{s}_h)\|_0\|\boldsymbol{e}_h\|_0 + \sigma\|\boldsymbol{u}-\boldsymbol{s}_h\|_0\|\boldsymbol{e}_h\|_0 \\ &\quad + \left(\sum_{K\in\mathcal{T}_h} \mu_K^{-1}\|p - i_h p\|^2_{0,K}\right)^{1/2}\left(\sum_{K\in\mathcal{T}_h} \mu_K\|\kappa_K\nabla\boldsymbol{e}_h\|^2_{0,K}\right)^{1/2} + |S_h(\boldsymbol{s}_h, \boldsymbol{e}_h)|. \end{aligned}$$

Now, the term with the stabilization has to be bounded. The Cauchy–Schwarz inequality gives

$$\begin{aligned} S_h(\boldsymbol{s}_h, \boldsymbol{e}_h) &= S_h(\boldsymbol{s}_h - \boldsymbol{u}, \boldsymbol{e}_h) + S_h(\boldsymbol{u}, \boldsymbol{e}_h) \\ &\le S_h^{1/2}(\boldsymbol{s}_h - \boldsymbol{u}, \boldsymbol{s}_h - \boldsymbol{u})S_h^{1/2}(\boldsymbol{e}_h, \boldsymbol{e}_h) + S_h^{1/2}(\boldsymbol{u}, \boldsymbol{u})S_h^{1/2}(\boldsymbol{e}_h, \boldsymbol{e}_h). \end{aligned} \tag{27}$$

Applying the stability of the fluctuation operator κ_K and the choice $\mu_K \sim 1$ of the stabilization parameters yields

$$S_h(\boldsymbol{s}_h, \boldsymbol{e}_h) \le C\left(\|\boldsymbol{s}_h - \boldsymbol{u}\|_1 + \|\kappa_K \nabla \boldsymbol{u}\|_0\right)\left(\sum_{K\in\mathcal{T}_h} \|\kappa_K \nabla \boldsymbol{e}_h\|_{0,K}^2\right)^{1/2}, \tag{28}$$

such that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{e}_h\|_0^2 + \nu\|\nabla \boldsymbol{e}_h\|_0^2 + \sigma\|\boldsymbol{e}_h\|_0^2 + \sum_{K\in\mathcal{T}_h} \mu_K \|\kappa_K \nabla \boldsymbol{e}_h\|_{0,K}^2 \\
&\quad \le \|\partial_t(\boldsymbol{u} - \boldsymbol{s}_h)\|_0 \|\boldsymbol{e}_h\|_0 + \|\boldsymbol{b}\|_\infty \|\nabla(\boldsymbol{u} - \boldsymbol{s}_h)\|_0 \|\boldsymbol{e}_h\|_0 + \sigma \|\boldsymbol{u} - \boldsymbol{s}_h\|_0 \|\boldsymbol{e}_h\|_0 \\
&\qquad + C\big(\|p - i_h p\|_0 + \|\boldsymbol{s}_h - \boldsymbol{u}\|_1 + \|\kappa_K \nabla \boldsymbol{u}\|_0\big)\left(\sum_{K\in\mathcal{T}_h} \|\kappa_K \nabla \boldsymbol{e}_h\|_{0,K}^2\right)^{1/2}.
\end{aligned}$$

With Young's inequality and hiding terms on the left-hand side, one obtains

$$\begin{aligned}
&\frac{d}{dt}\|\boldsymbol{e}_h\|_0^2 + 2\nu\|\nabla \boldsymbol{e}_h\|_0^2 + \sigma\|\boldsymbol{e}_h\|_0^2 + \sum_{K\in\mathcal{T}_h} \mu_K \|\kappa_K \nabla \boldsymbol{e}_h\|_{0,K}^2 \\
&\quad \le C\left(\|\partial_t(\boldsymbol{u} - \boldsymbol{s}_h)\|_0^2 + \|\boldsymbol{b}\|_\infty^2 \|\nabla(\boldsymbol{u} - \boldsymbol{s}_h)\|_0^2 + \sigma^2\|\boldsymbol{u} - \boldsymbol{s}_h\|_0^2\right) \\
&\qquad + C\left(\|p - i_h p\|_0^2 + \|\boldsymbol{s}_h - \boldsymbol{u}\|_1^2 + \|\kappa_K \nabla \boldsymbol{u}\|_0^2\right).
\end{aligned} \tag{29}$$

Assuming now for $t \le T$ the regularities (24), integrating (29) on $(0, t)$, taking into account that $\boldsymbol{e}_h(0) = \boldsymbol{0}$, since $\boldsymbol{u}_0 = \boldsymbol{0}$, and applying estimates (22), (13), and (15), one gets

$$\begin{aligned}
&\|\boldsymbol{e}_h(t)\|_0^2 + 2\nu\|\nabla \boldsymbol{e}_h\|_{L^2(0,t;L^2)}^2 + \sigma\|\boldsymbol{e}_h\|_{L^2(0,t;L^2)}^2 + \sum_{K\in\mathcal{T}_h} \|\kappa_K \nabla \boldsymbol{e}_h\|_{L^2(0,t;L^2(K))}^2 \\
&\quad \le C h^{2r}\left(\|\boldsymbol{u}\|_{L^2(0,t;H^{r+1})}^2 + \|\partial_t \boldsymbol{u}\|_{L^2(0,t;H^r)}^2 + \|p\|_{L^2(0,t;H^r)}^2\right),
\end{aligned} \tag{30}$$

where $C = C\left(\sigma, \|\boldsymbol{b}\|_{L^\infty(0,t;L^\infty)}\right)$ is independent of ν and h.

The final result is obtained by applying the triangle inequality to the left-hand side of (25) and using (30) and (22). $\square$

The next step in the error analysis consists in obtaining a bound for the pressure error.

Theorem 3. *Let the assumptions of Theorem 2 hold and let $\nu \le 1$ then*

$$\|p - p_h\|_{L^2(0,t;L^2)} \le C h^r \quad \forall\ t \in (0, T], \tag{31}$$

where $C = C\left(\beta_0^{-1}, \|\boldsymbol{u}\|_{L^2(0,t;H^{r+1})}, \|\partial_t \boldsymbol{u}\|_{L^2(0,t;H^r)}, \|p\|_{L^2(0,t;H^r)}, \sigma, \|\boldsymbol{b}\|_{L^\infty(0,t;L^\infty)}\right)$ is independent of ν and h.

Proof. This bound is derived as usual on the basis of the discrete inf-sup condition (3). In particular, a bound for $\|\partial_t \boldsymbol{e}_h\|_{-1}$ is needed. By definition, it is

$$\|\partial_t \boldsymbol{e}_h\|_{-1} = \sup_{\boldsymbol{\varphi}\in H_0^1(\Omega)^d\setminus\{\boldsymbol{0}\}} \frac{|\langle \partial_t \boldsymbol{e}_h, \boldsymbol{\varphi}\rangle|}{\|\nabla \boldsymbol{\varphi}\|_0}.$$

The first step consists in reducing the bound of $\|\partial_t \boldsymbol{e}_h\|_{-1}$ to a bound of $\|A_h^{-1/2}\partial_t \boldsymbol{e}_h\|_0$. From [9, Lemma 3.11], it is known that

$$\|\partial_t \boldsymbol{e}_h\|_{-1} \le C h \|\partial_t \boldsymbol{e}_h\|_0 + C\|A^{-1/2}\Pi \partial_t \boldsymbol{e}_h\|_0, \tag{32}$$

where Π is the Leray projector introduced in Section 2. Applying [9, (2.15)], one obtains

$$\|A^{-1/2}\Pi\partial_t \boldsymbol{e}_h\|_0 \le Ch\|\partial_t \boldsymbol{e}_h\|_0 + \|A_h^{-1/2}\partial_t \boldsymbol{e}_h\|_0, \tag{33}$$

with A_h defined in (5). From (32), (33), the symmetry of A_h, (6), and the inverse inequality (4), it follows that

$$\begin{aligned}\|\partial_t \boldsymbol{e}_h\|_{-1} &\le Ch\|\partial_t \boldsymbol{e}_h\|_0 + C\|A_h^{-1/2}\partial_t \boldsymbol{e}_h\|_0\\ &= Ch\|A_h^{1/2}A_h^{-1/2}\partial_t \boldsymbol{e}_h\|_0 + C\|A_h^{-1/2}\partial_t \boldsymbol{e}_h\|_0\\ &= Ch\|\nabla(A_h^{-1/2}\partial_t \boldsymbol{e}_h)\|_0 + C\|A_h^{-1/2}\partial_t \boldsymbol{e}_h\|_0\\ &\le C\|A_h^{-1/2}\partial_t \boldsymbol{e}_h\|_0. \end{aligned}\tag{34}$$

Next, a bound for $\|A_h^{-1/2}\partial_t \boldsymbol{e}_h\|_0$ will be derived. Projecting the error equation (26) onto the discretely divergence-free space V_h^{div} and using integration by parts, one gets

$$\begin{aligned}(\partial_t \boldsymbol{e}_h, \boldsymbol{v}_h) + \nu(\nabla \boldsymbol{e}_h, \nabla \boldsymbol{v}_h) + ((\boldsymbol{b}\cdot\nabla)\boldsymbol{e}_h + \sigma \boldsymbol{e}_h, \boldsymbol{v}_h) + S_h(\boldsymbol{e}_h, \boldsymbol{v}_h)&\\ = (\partial_t(\boldsymbol{u}-\boldsymbol{s}_h), \boldsymbol{v}_h) + ((\boldsymbol{b}\cdot\nabla)(\boldsymbol{u}-\boldsymbol{s}_h) + \sigma(\boldsymbol{u}-\boldsymbol{s}_h), \boldsymbol{v}_h)&\\ - S_h(\boldsymbol{s}_h, \boldsymbol{v}_h) - (p - i_h p, \nabla\cdot \boldsymbol{v}_h).&\end{aligned}$$

Recalling definition (9), one has $(p - i_h p, \nabla\cdot\boldsymbol{v}_h) = (D_h(p-i_hp), \boldsymbol{v}_h)$, such that

$$\begin{aligned}\partial_t \boldsymbol{e}_h = -\nu A_h \boldsymbol{e}_h - \Pi_h^{\mathrm{div}}\big((\boldsymbol{b}\cdot\nabla)\boldsymbol{e}_h + \sigma\boldsymbol{e}_h\big) - C_h\boldsymbol{e}_h + \Pi_h^{\mathrm{div}}\big(\partial_t(\boldsymbol{u}-\boldsymbol{s}_h)\big)&\\ + \Pi_h^{\mathrm{div}}\big((\boldsymbol{b}\cdot\nabla)(\boldsymbol{u}-\boldsymbol{s}_h) + \sigma(\boldsymbol{u}-\boldsymbol{s}_h)\big) - C_h(\boldsymbol{s}_h)&\\ - D_h(p - i_h p).&\end{aligned}\tag{35}$$

With (8), the Cauchy–Schwarz inequality, (6), the L^2 stability of the fluctuation operator κ_K, and $\mu_K \sim 1$, one obtains for all $\boldsymbol{v}_h \in V_h^{\mathrm{div}}$

$$\begin{aligned}\|A_h^{-1/2}C_h\boldsymbol{v}_h\|_0 &= \sup_{\boldsymbol{w}_h\in V_h^{\mathrm{div}}\setminus\{\boldsymbol{0}\}} \frac{\big|\langle C_h\boldsymbol{v}_h, A_h^{-1/2}\boldsymbol{w}_h\rangle\big|}{\|\boldsymbol{w}_h\|_0}\\ &= \sup_{\boldsymbol{w}_h\in V_h^{\mathrm{div}}\setminus\{\boldsymbol{0}\}} \frac{\big|\sum_{K\in\mathcal{T}_h}(\kappa_K\nabla\boldsymbol{v}_h, \kappa_K\nabla(A_h^{-1/2}\boldsymbol{w}_h))_{0,K}\big|}{\|\boldsymbol{w}_h\|_0}\\ &\le \sup_{\boldsymbol{w}_h\in V_h^{\mathrm{div}}\setminus\{\boldsymbol{0}\}} \frac{\big(\sum_{K\in\mathcal{T}_h}\|\kappa_K\nabla\boldsymbol{v}_h\|_{0,K}^2\big)^{1/2} C\|\nabla(A_h^{-1/2}\boldsymbol{w}_h)\|_0}{\|\boldsymbol{w}_h\|_0}\\ &\le C \sup_{\boldsymbol{w}_h\in V_h^{\mathrm{div}}\setminus\{\boldsymbol{0}\}} \frac{\big(\sum_{K\in\mathcal{T}_h}\|\kappa_K\nabla\boldsymbol{v}_h\|_{0,K}^2\big)^{1/2}\|\boldsymbol{w}_h\|_0}{\|\boldsymbol{w}_h\|_0}\\ &= C\left(\sum_{K\in\mathcal{T}_h}\|\kappa_K\nabla\boldsymbol{v}_h\|_{0,K}^2\right)^{1/2}.\end{aligned}\tag{36}$$

The above argument applied to $\|A_h^{-1/2}D_h(p-i_hp)\|_0$ yields

$$\|A_h^{-1/2}D_h(p-i_hp)\|_0 \le C\|p - i_h p\|_0. \tag{37}$$

Definition (7) and the symmetry of A_h gives for any $\boldsymbol{g} \in L^2(\Omega)^d$ the equality $(A_h^{-1/2}\Pi_h^{\mathrm{div}}\boldsymbol{g}, \boldsymbol{v}_h) = (\boldsymbol{g}, A_h^{-1/2}\boldsymbol{v}_h)$ for all $\boldsymbol{v}_h \in V_h^{\mathrm{div}}$. It follows with $\boldsymbol{v}_h = A_h^{-1/2}\Pi_h^{\mathrm{div}}\boldsymbol{g} \in V_h^{\mathrm{div}}$ and (6) that

$$\|A_h^{-1/2}\Pi_h^{\mathrm{div}}\boldsymbol{g}\|_0^2 \le \|\boldsymbol{g}\|_{-1}\|\nabla(A_h^{-1/2}A_h^{-1/2}\Pi_h^{\mathrm{div}}\boldsymbol{g})\|_0 = \|\boldsymbol{g}\|_{-1}\|A_h^{-1/2}\Pi_h^{\mathrm{div}}\boldsymbol{g}\|_0$$

and hence

$$\|A_h^{-1/2}\Pi_h^{\mathrm{div}}\boldsymbol{g}\|_0 \le \|\boldsymbol{g}\|_{-1} \quad \forall\, \boldsymbol{g} \in L^2(\Omega)^d. \tag{38}$$

Next, $A_h^{-1/2}$ is applied to (35). Using (36), (37), and (38), one gets

$$\begin{aligned} &\|A_h^{-1/2}\partial_t \boldsymbol{e}_h\|_0 \\ &\quad\le \nu\|A_h^{1/2}\boldsymbol{e}_h\|_0 + \|(\boldsymbol{b}\cdot\nabla)\boldsymbol{e}_h + \sigma\boldsymbol{e}_h\|_{-1} + \left(\sum_{K\in\mathcal{T}_h} \mu_K\|\kappa_K\nabla\boldsymbol{e}_h\|_{0,K}^2\right)^{1/2} \\ &\qquad + \|\partial_t(\boldsymbol{u}-\boldsymbol{s}_h)\|_{-1} + \|(\boldsymbol{b}\cdot\nabla)(\boldsymbol{u}-\boldsymbol{s}_h) + \sigma(\boldsymbol{u}-\boldsymbol{s}_h)\|_{-1} \\ &\qquad + \left(\sum_{K\in\mathcal{T}_h} \mu_K\|\kappa_K\nabla\boldsymbol{s}_h\|_{0,K}^2\right)^{1/2} + \|p - i_h p\|_0. \end{aligned} \tag{39}$$

Taking the square of (39) and integrating on $(0,t)$ yields

$$\begin{aligned} &\int_0^t \|A_h^{-1/2}\partial_t\boldsymbol{e}_h(s)\|_0^2\,ds \\ &\le C\Bigg(\int_0^t \nu^2\|A_h^{1/2}\boldsymbol{e}_h(s)\|_0^2\,ds + \int_0^t \|((\boldsymbol{b}\cdot\nabla)\boldsymbol{e}_h + \sigma\boldsymbol{e}_h)(s)\|_{-1}^2\,ds \\ &\quad + \int_0^t \sum_{K\in\mathcal{T}_h}\mu_K\|\kappa_K\nabla\boldsymbol{e}_h(s)\|_{0,K}^2\,ds + \int_0^t \|\partial_t(\boldsymbol{u}-\boldsymbol{s}_h)(s)\|_{-1}^2\,ds \\ &\quad + \int_0^t \|(p-i_hp)(s)\|_0^2\,ds + \int_0^t \|((\boldsymbol{b}\cdot\nabla)(\boldsymbol{u}-\boldsymbol{s}_h) + \sigma(\boldsymbol{u}-\boldsymbol{s}_h))(s)\|_{-1}^2\,ds \\ &\quad + \int_0^t \sum_{K\in\mathcal{T}_h}\mu_K\|\kappa_K\nabla\boldsymbol{s}_h(s)\|_{0,K}^2\,ds\Bigg). \end{aligned} \tag{40}$$

It will be proved that all the terms on the right-hand-side of (40) are $\mathcal{O}(h^{2r})$. The desired asymptotic behavior is obtained for the first and third term directly from (30). For the second term in (40), the definition of the $H^{-1}(\Omega)^d$ norm, integrating by parts, and Poincaré's inequality lead to

$$\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{e}_h + \sigma\boldsymbol{e}_h\|_{-1} \le C\left(\|\boldsymbol{b}\|_\infty + \sigma\right)\|\boldsymbol{e}_h\|_0.$$

Hence, one obtains

$$\int_0^t \|((\boldsymbol{b}\cdot\nabla)\boldsymbol{e}_h + \sigma\boldsymbol{e}_h)(s)\|_{-1}^2\,ds \le C\int_0^t \|\boldsymbol{e}_h(s)\|_0^2\,ds,$$

such that the desired order of convergence can be again deduced from (30). Concerning $\|\partial_t(\boldsymbol{u}-\boldsymbol{s}_h)\|_{-1}$, the definition of the $H^{-1}(\Omega)^d$ norm and Poincaré's inequality are applied to bound this term by $C\|\partial_t(\boldsymbol{u}-\boldsymbol{s}_h)\|_0$. Now, (22) is applied (see Remark 1) and with the regularity assumptions (24), the estimate for $\|\partial_t(\boldsymbol{u}-\boldsymbol{s}_h)\|_{-1}$

is $\mathcal{O}(h^r)$. Once this term is bounded, it is clear that the integral of its square is also bounded

$$\int_0^t \|\partial_t(\boldsymbol{u}-\boldsymbol{s}_h)(s)\|_{-1}^2 \ ds \leq Ch^{2r}\|\partial_t \boldsymbol{u}\|_{L^2(0,t;H^r)}^2.$$

The term involving the pressure is estimated with (13). One can argue as in (27)–(28) to obtain the bound

$$\begin{aligned}\int_0^t \sum_{K\in\mathcal{T}_h} \mu_K \|\kappa_K \nabla \boldsymbol{s}_h(s)\|_{0,K}^2 \ ds &\leq C\int_0^t \left(\|(\boldsymbol{s}_h-\boldsymbol{u})(s)\|_1^2 + \|\kappa_K \nabla \boldsymbol{u}(s)\|_0^2\right) \ ds \\ &\leq Ch^{2r}\|\boldsymbol{u}\|_{L^2(0,t;H^{r+1})}^2,\end{aligned}$$

where (22) and (15) were applied in the last inequality. Finally, arguing as for the second term, one obtains

$$\|(\boldsymbol{b}\cdot\nabla)(\boldsymbol{u}-\boldsymbol{s}_h)+\sigma(\boldsymbol{u}-\boldsymbol{s}_h)\|_{-1} \leq C\left(\|\boldsymbol{b}\|_\infty+\sigma\right)\|\boldsymbol{u}-\boldsymbol{s}_h\|_0,$$

from what follows that

$$\int_0^t \|((\boldsymbol{b}\cdot\nabla)(\boldsymbol{u}-\boldsymbol{s}_h)+\sigma(\boldsymbol{u}-\boldsymbol{s}_h))(s)\|_{-1}^2 \ ds \leq C\int_0^t \|(\boldsymbol{u}-\boldsymbol{s}_h)(s)\|_0^2 \ ds.$$

The bound for this term is concluded by applying (22). Combining the estimates for (40) with (34), it is shown that

$$\int_0^t \|\partial_t(\boldsymbol{e}_h)(s)\|_{-1}^2 \ ds = \mathcal{O}(h^{2r}). \tag{41}$$

Using now the discrete inf-sup condition (3) and (26), one obtains

$$\begin{aligned}&\beta_0\|p_h - i_h p\|_0 \\ &\leq \nu\|\nabla \boldsymbol{e}_h\|_0 + \|(\boldsymbol{b}\cdot\nabla)\boldsymbol{e}_h + \sigma\boldsymbol{e}_h\|_{-1} + \|\partial_t \boldsymbol{e}_h\|_{-1} + C\left(\sum_{K\in\mathcal{T}_h}\mu_K\|\kappa_K\nabla\boldsymbol{e}_h\|_{0,K}^2\right)^{1/2} \\ &\quad + \|\partial_t(\boldsymbol{u}-\boldsymbol{s}_h)\|_{-1} + \|(\boldsymbol{b}\cdot\nabla)(\boldsymbol{u}-\boldsymbol{s}_h)+\sigma(\boldsymbol{u}-\boldsymbol{s}_h)\|_{-1} \\ &\quad + C\left(\sum_{K\in\mathcal{T}_h}\mu_K\|\kappa_K\nabla\boldsymbol{s}_h\|_{0,K}^2\right)^{1/2} + \|p-i_h p\|_0 + \|l_h\|_0.\end{aligned}$$

Taking the square and integrating on $(0,t)$ leads to

$$\begin{aligned}&\beta_0^2\int_0^t \|(p_h-i_h p)(s)\|_0^2 \ ds \\ &\leq C\Bigg(\int_0^t \nu^2\|\nabla\boldsymbol{e}_h(s)\|_0^2 \ ds + \int_0^t \|((\boldsymbol{b}\cdot\nabla)\boldsymbol{e}_h+\sigma\boldsymbol{e}_h)(s)\|_{-1}^2 \ ds \\ &\quad + \int_0^t \|\partial_t(\boldsymbol{e}_h)(s)\|_{-1}^2 \ ds + \int_0^t \sum_{K\in\mathcal{T}_h}\mu_K\|\kappa_K\nabla\boldsymbol{e}_h(s)\|_{0,K}^2 \ ds \\ &\quad + \int_0^t \|\partial_t(\boldsymbol{u}-\boldsymbol{s}_h)(s)\|_{-1}^2 \ ds + \int_0^t \|((\boldsymbol{b}\cdot\nabla)(\boldsymbol{u}-\boldsymbol{s}_h)(s)+\sigma(\boldsymbol{u}-\boldsymbol{s}_h))(s)\|_{-1}^2 \ ds \\ &\quad + \int_0^t \sum_{K\in\mathcal{T}_h}\mu_K\|\kappa_K\nabla\boldsymbol{s}_h(s)\|_{0,K}^2 \ ds + \int_0^t \|(p-i_h p)(s)\|_0^2 \ ds + \int_0^t \|l_h(s)\|_0^2 \ ds\Bigg).\end{aligned}$$

Arguing exactly as for the estimates on the right-hand side of (40), using (41) for $\int_0^t \|\partial_t(\boldsymbol{e}_h)(s)\|_{-1}^2\, ds$, (23) to bound the last term, and finally the triangle inequality, proves (31). □

4. Error analysis for the fully discrete method with cGP(k)

The continuous Galerkin–Petrov method is applied as temporal discretization. To this end, consider a partition $0 = t_0 < t_1 < \ldots < t_N = T$ of the time interval $I := [0,T]$ and set $I_n = (t_{n-1}, t_n]$, $\tau_n = t_n - t_{n-1}$, $n = 1, \ldots N$, and $\tau := \max_{1\le n\le N} \tau_n$. For a given non-negative integer k, define the time-continuous and time-discontinuous velocity spaces as follows

$$X_k^{\mathrm{c}} := \{\boldsymbol{u} \in C(V_h) : \boldsymbol{u}|_{I_n} \in \mathbb{P}_k(I_n, V_h)\}\,,\ X_k^{\mathrm{dc}} := \left\{\boldsymbol{u} \in L^2(V_h) : \boldsymbol{u}|_{I_n} \in \mathbb{P}_k(I_n, V_h)\right\},$$

and time-continuous and time-discontinuous pressure spaces by

$$Y_k^{\mathrm{c}} := \{q \in C(Q_h) : q|_{I_n} \in \mathbb{P}_k(I_n, Q_h)\}\,,\ Y_k^{\mathrm{dc}} := \left\{q \in L^2(Q_h) : q|_{I_n} \in \mathbb{P}_k(I_n, Q_h)\right\},$$

for $n = 1, \ldots, N$. Here

$$\mathbb{P}_k(I_n, W_h) := \left\{u : I_n \to W_h : u(t) = \sum_{i=0}^{k} U_i t^i,\ \forall\ t \in I_n, U_i \in W_h, \forall\ i\right\} \tag{42}$$

denotes the space of W_h-valued polynomials of order k in time. The functions in the spaces X_k^{dc} and Y_k^{dc} are allowed to be discontinuous at the nodes t_n. In the following, the combination of the LPS method as spatial discretization and the cGP(k) time stepping scheme is denoted by LPS/cGP.

Denote by $X_{k,0}^{\mathrm{c}} := X_k^{\mathrm{c}} \cap X_0$ the subspace of X_k^{c} with zero initial condition and introduce a bilinear form b_h given by

$$b_h\left((\boldsymbol{u}, p); (\boldsymbol{v}, q)\right) := \int_0^T \left[\big(\partial_t \boldsymbol{u}, \boldsymbol{v}\big) + a_h\big((\boldsymbol{u}, p); (\boldsymbol{v}, q)\big)\right]\, dt.$$

The LPS/cGP method reads as follows:

Find $\boldsymbol{u}_{h,\tau} \in X_{k,0}^{\mathrm{c}}$ and $p_{h,\tau} \in Y_k^{\mathrm{c}}$ such that

$$b_h\big((\boldsymbol{u}_{h,\tau}, p_{h,\tau}); (\boldsymbol{v}_{h,\tau}, q_{h,\tau})\big) = \int_0^T \big(\boldsymbol{f}, \boldsymbol{v}_{h,\tau}\big)\, dt \quad \forall\ \boldsymbol{v}_{h,\tau} \in X_{k-1}^{\mathrm{dc}},\ q_{h,\tau} \in Y_{k-1}^{\mathrm{dc}}, \tag{43}$$

where the index h, τ refers to the discretization in space and time. The associated continuous problem is defined as follows:

Find $\boldsymbol{u} \in X_0$ and $p \in L^2(Q)$ such that

$$\int_0^T \left[\big(\partial_t \boldsymbol{u}(t), \boldsymbol{v}(t)\big) + a\big((\boldsymbol{u}(t), p(t)); (\boldsymbol{v}(t), q(t))\big)\right]\, dt = \int_0^T \big(\boldsymbol{f}(t), \boldsymbol{v}(t)\big)\, dt \tag{44}$$

for all $\boldsymbol{v} \in L^2(V)$, $q \in L^2(Q)$.

For a function w which is smooth on each time interval I_n, the operator π_{k-1} is defined by

$$(\pi_{k-1} w)|_{I_n}(t) = \sum_{k=1}^{k} w(\tilde{t}_{n,i}) \tilde{L}_{n,i}(t), \tag{45}$$

where $\tilde{t}_{n,i}$ denote the Gauss quadrature points on I_n and $\tilde{L}_{n,i} \in \mathbb{P}_{k-1}(I_n)$ are the associated Lagrange basis functions. Definition (45) gives $\pi_{k-1}\boldsymbol{w}_{h,\tau} \in X_{k-1}^{\mathrm{dc}}$ for all

$\boldsymbol{w}_{h,\tau} \in X_k^{\mathrm{c}}$ and $\pi_{k-1} q_{h,\tau} \in Y_{k-1}^{\mathrm{dc}}$ for all $q_{h,\tau} \in Y_k^{\mathrm{c}}$. Furthermore, one has for all $\boldsymbol{w}_{h,\tau} \in X_k^{\mathrm{c}}$ that

$$\int_{I_n} (\boldsymbol{w}_{h,\tau}(t) - \pi_{k-1}\boldsymbol{w}_{h,\tau}(t)) t^j \, dt = \mathbf{0}, \quad j = 0, \ldots, k-1, \ n = 1, \ldots, N, \tag{46}$$

where $\mathbf{0}$ denotes the zero element in V_h.

The analysis considers the mesh-dependent norm

$$\|\boldsymbol{v}\|_{\mathrm{cGP}} := \left(\int_0^T |||\pi_{k-1}\boldsymbol{v}|||^2 dt + \frac{1}{2} \|\boldsymbol{v}(T)\|_0^2 \right)^{1/2}.$$

Note that, as observed in [5], $||\cdot||_{\mathrm{cGP}}$ is on $X_k^{\mathrm{c}} \subset X_k^{\mathrm{dc}}$ not only a semi-norm but a norm. Indeed, the first term inside the definition of $||\boldsymbol{v}||_{\mathrm{cGP}}$ guarantees that $||\boldsymbol{v}||_{\mathrm{cGP}} = 0$ results in a function $\boldsymbol{v}$ which is on each time interval I_n given by $L_k^{(n)}(t)\boldsymbol{\varphi}_h(x)$, where $L_k^{(n)}$ is the transformed k-th Legendre polynomial on I_n and $\boldsymbol{\varphi}_h \in V_h$. Due to $\boldsymbol{v}(T) = 0$ and $L_k^{(N)}(T) = 1$ the function $\boldsymbol{v}$ vanishes on the last time interval I_N. The continuity of $\boldsymbol{v}$ on I gives then $\boldsymbol{v}(t_{N-1}) = \mathbf{0}$. By recursion, one obtains $\boldsymbol{v} = \mathbf{0}$ on I and hence $||\cdot||_{\mathrm{cGP}}$ is a norm.

The following lemma will show a property of the bilinear form b_h that will be used to get the error bounds for the approximation to the velocity.

Lemma 4. *Assume that $\boldsymbol{b}$ and σ are constant with respect to time. Then, there exists a constant $C > 0$ independent of ν, h, and τ such that*

$$b_h\left((\boldsymbol{v}_{h,\tau}, q_{h,\tau}); (\pi_{k-1}\boldsymbol{v}_{h,\tau}, \pi_{k-1}q_{h,\tau})\right) = \|\boldsymbol{v}_{h,\tau}\|_{\mathrm{cGP}}^2 \quad \forall\ (\boldsymbol{v}_{h,\tau}, q_{h,\tau}) \in X_k^{\mathrm{dc}} \times Y_k^{\mathrm{dc}}$$

holds true.

Proof. It is

$$\begin{aligned} & b_h\left((\boldsymbol{v}_{h,\tau}, q_{h,\tau}); (\pi_{k-1}\boldsymbol{v}_{h,\tau}, \pi_{k-1}q_{h,\tau})\right) \\ & \qquad = \int_0^T \left[\left(\partial_t \boldsymbol{v}_{h,\tau}, \pi_{k-1}\boldsymbol{v}_{h,\tau}\right) + a_h\left((\boldsymbol{v}_{h,\tau}, q_{h,\tau}); (\pi_{k-1}\boldsymbol{v}_{h,\tau}, \pi_{k-1}q_{h,\tau})\right) \right] dt. \end{aligned}$$

Using the fact that the convection and reaction are time-independent functions and taking into account that

$$\begin{aligned} & \int_0^T \left[-\left(q_{h,\tau}, \operatorname{div} \pi_{k-1}\boldsymbol{v}_{h,\tau}\right) + \left(\pi_{k-1}q_{h,\tau}, \operatorname{div} \boldsymbol{v}_{h,\tau}\right) \right] dt \\ & \qquad = \int_0^T \left[-\left(\pi_{k-1}q_{h,\tau}, \operatorname{div} \pi_{k-1}\boldsymbol{v}_{h,\tau}\right) + \left(\pi_{k-1}q_{h,\tau}, \operatorname{div} \pi_{k-1}\boldsymbol{v}_{h,\tau}\right) \right] dt = 0 \end{aligned}$$

and (10), one obtains

$$\begin{aligned} & \int_0^T a_h\left((\boldsymbol{v}_{h,\tau}, q_{h,\tau}); (\pi_{k-1}\boldsymbol{v}_{h,\tau}, \pi_{k-1}q_{h,\tau})\right) dt \\ & = \int_0^T a_h\left((\pi_{k-1}\boldsymbol{v}_{h,\tau}, \pi_{k-1}q_{h,\tau}); (\pi_{k-1}\boldsymbol{v}_{h,\tau}, \pi_{k-1}q_{h,\tau})\right) dt = \int_0^T |||\pi_{k-1}\boldsymbol{v}_{h,\tau}|||^2 \, dt. \end{aligned}$$

Concerning the first term, it is noted that $\partial_t \boldsymbol{v}_{h,\tau}$ is a discontinuous function in time of degree $k-1$. Using $\boldsymbol{v}_{h,\tau}(0) = \boldsymbol{0}$ yields

$$\int_0^T \left(\partial_t \boldsymbol{v}_{h,\tau}, \pi_{k-1}\boldsymbol{v}_{h,\tau}\right)\, dt = \int_0^T \left(\partial_t \boldsymbol{v}_{h,\tau}, \boldsymbol{v}_{h,\tau}\right)\, dt = \frac{1}{2}\int_0^T \frac{d}{dt}\left\|\boldsymbol{v}_{h,\tau}\right\|_0^2\, dt = \frac{1}{2}\left\|\boldsymbol{v}_{h,\tau}(T)\right\|_0^2.$$

□

The derivation of error bounds makes use of a time interpolation of a sufficiently smooth function w: $\widetilde{w} \in C(H)$, where H can be either a velocity space V or a pressure space Q, and $\widetilde{w}|_{I_n} \in \mathbb{P}_k(I_n, H)$, defined by

$$\widetilde{w}(t_{n-1}) = w(t_{n-1}), \quad \widetilde{w}(t_n) = w(t_n), \quad \int_{I_n} \left(w(t) - \widetilde{w}(t), z(t)\right)\, dt = 0, \tag{47}$$

for all $z \in \mathbb{P}_{k-2}(I_n, H)$. The standard interpolation error estimate

$$\left(\int_{I_n} \|w - \widetilde{w}\|_m^2\, dt\right)^{1/2} \le C\tau_n^{k+1}\left(\int_{I_n} \|w^{(k+1)}\|_m^2\, dt\right)^{1/2} \tag{48}$$

holds true for $m \in \{0,1\}$ and all time intervals I_n, $n = 1, \ldots, N$.

Theorem 5. *Assume that the spaces V_h, Q_h satisfy Assumptions A1 and A2, $\mu_K \sim 1$ for all $K \in \mathcal{T}_h$, and $\nu \le 1$. Let $(\boldsymbol{u}, p)$ be the solution of* (44) *and $(\boldsymbol{u}_{h,\tau}, p_{h,\tau})$ the solution of* (43)*. Further, assume that the solution $(\boldsymbol{u}, p)$ is smooth enough such that all the norms on the right-hand side of* (49) *are bounded. Then, there exists a positive constant C independent of ν, h, and τ such that the error estimate*

$$\begin{aligned}\left\|\boldsymbol{u} - \boldsymbol{u}_{h,\tau}\right\|_{\mathrm{cGP}} \le{}& Ch^r\left(\|\boldsymbol{u}\|_{L^2(H^{r+1})} + \|\boldsymbol{u}\|_{H^1(H^r)} + \|p\|_{L^2(H^r)} + h\|\boldsymbol{u}(T)\|_{r+1}\right)\\ &+ C\tau^{k+1}\|\boldsymbol{u}\|_{H^{k+1}(H^1)}\end{aligned} \tag{49}$$

holds true.

Proof. The error analysis starts by decomposing the errors $\boldsymbol{e}_{h,\tau} = \boldsymbol{u}_{h,\tau} - \boldsymbol{u}$ into $\boldsymbol{\theta}_h := \tilde{\boldsymbol{s}}_h - \boldsymbol{u}$ and $\boldsymbol{\xi}_{h,\tau} := \boldsymbol{u}_{h,\tau} - \tilde{\boldsymbol{s}}_h$ with the velocity solution $\tilde{\boldsymbol{s}}_h$ of (17) where the right-hand side $\boldsymbol{g}$ in (21) is defined using $\tilde{\boldsymbol{u}}$ instead of $\boldsymbol{u}$. Then

$$\boldsymbol{u}_{h,\tau} - \boldsymbol{u} = \boldsymbol{e}_{h,\tau} = \boldsymbol{\theta}_h + \boldsymbol{\xi}_{h,\tau}.$$

For the discrete error $\boldsymbol{\xi}_{h,\tau}$ Lemma 4 provides

$$\|\boldsymbol{\xi}_{h,\tau}\|_{\mathrm{cGP}}^2 = b_h\big((\boldsymbol{\xi}_{h,\tau}, p_{h,\tau}); (\pi_{k-1}\boldsymbol{\xi}_{h,\tau}, \pi_{k-1}p_{h,\tau})\big). \tag{50}$$

A straightforward calculation gives

$$\begin{aligned}b_h\big((\boldsymbol{\xi}_{h,\tau}, &p_{h,\tau}); (\pi_{k-1}\boldsymbol{\xi}_{h,\tau}, \pi_{k-1}p_{h,\tau})\big)\\ &= \int_0^T \left(\partial_t \boldsymbol{u} - \partial_t \tilde{\boldsymbol{s}}_h, \pi_{k-1}\boldsymbol{\xi}_{h,\tau}\right)\, dt + \int_0^T \nu\big(\nabla(\boldsymbol{u} - \tilde{\boldsymbol{s}}_h), \nabla(\pi_{k-1}\boldsymbol{\xi}_{h,\tau})\big)\, dt\\ &\quad + \int_0^T \left((\boldsymbol{b}\cdot\nabla)(\boldsymbol{u} - \tilde{\boldsymbol{s}}_h), \pi_{k-1}\boldsymbol{\xi}_{h,\tau}\right)\, dt + \int_0^T \left(\sigma(\boldsymbol{u} - \tilde{\boldsymbol{s}}_h), \pi_{k-1}\boldsymbol{\xi}_{h,\tau}\right)\, dt\\ &\quad + \int_0^T \left(\nabla p, \pi_{k-1}\boldsymbol{\xi}_{h,\tau}\right)\, dt - \int_0^T S_h(\tilde{\boldsymbol{s}}_h, \pi_{k-1}\boldsymbol{\xi}_{h,\tau})\, dt.\end{aligned} \tag{51}$$

The six terms on the right-hand side will be bounded.

For the first one, the error is split in two terms

$$\begin{aligned}\int_0^T \left(\partial_t \boldsymbol{u} - \partial_t \tilde{\boldsymbol{s}}_h, \pi_{k-1}\boldsymbol{\xi}_{h,\tau}\right)\ dt = &\int_0^T \left(\partial_t(\boldsymbol{u} - \tilde{\boldsymbol{u}}), \pi_{k-1}\boldsymbol{\xi}_{h,\tau}\right)\ dt \\ &+ \int_0^T \left(\partial_t(\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{s}}_h), \pi_{k-1}\boldsymbol{\xi}_{h,\tau}\right)\ dt. \end{aligned} \tag{52}$$

Integration by parts and using (47) yield for the first term on the right-hand side of (52)

$$\begin{aligned}&\int_0^T \left(\partial_t(\boldsymbol{u} - \tilde{\boldsymbol{u}}), \pi_{k-1}\boldsymbol{\xi}_{h,\tau}\right)\ dt \\ &\quad = \sum_{n=1}^N \left(- \int_{I_n} (\boldsymbol{u} - \tilde{\boldsymbol{u}}, \partial_t(\pi_{k-1}\boldsymbol{\xi}))\ dt + (\boldsymbol{u} - \tilde{\boldsymbol{u}}, \pi_{k-1}\boldsymbol{\xi}_{h\tau})\Big|_{t_{n-1}}^{t_n} \right) = 0. \end{aligned} \tag{53}$$

For the second term on the right-hand side of (52), the application of the Cauchy–Schwarz inequality and (22) gives

$$\begin{aligned}&\int_0^T \left(\partial_t(\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{s}}_h), \pi_{k-1}\boldsymbol{\xi}_{h,\tau}\right)\ dt \\ &\quad \le \sum_{n=1}^N \int_{I_n} \|\partial_t \tilde{\boldsymbol{u}} - \partial_t \tilde{\boldsymbol{s}}_h\|_0 \|\pi_{k-1}\boldsymbol{\xi}_{h,\tau}\|_0\ dt \\ &\quad \le \left(\sum_{n=1}^N \int_{I_n} \|\partial_t \tilde{\boldsymbol{u}} - \partial_t \tilde{\boldsymbol{s}}_h\|_0^2\ dt \right)^{1/2} \left(\sum_{n=1}^N \|\pi_{k-1}\boldsymbol{\xi}_{h,\tau}\|_0^2\ dt \right)^{1/2} \\ &\quad \le Ch^r \left(\sum_{n=1}^N \int_{I_n} \|\partial_t \tilde{\boldsymbol{u}}\|_r^2\ dt \right)^{1/2} \left(\sum_{n=1}^N \int_{I_n} \sigma \|\pi_{k-1}\boldsymbol{\xi}_{h,\tau}\|_0^2\ dt \right)^{1/2} \\ &\quad \le Ch^r \|\boldsymbol{u}\|_{H^1(H^r)} \|\boldsymbol{\xi}_{h,\tau}\|_{\mathrm{cGP}}, \end{aligned} \tag{54}$$

where in the last estimate the inequality $\|\tilde{\boldsymbol{u}}\|_{H^1(H^r)} \le C\|\boldsymbol{u}\|_{H^1(H^r)}$ was applied. Thus, from (52), (53), and (54) one derives the bound for the first term on the right-hand side of (51)

$$\int_0^T \left(\partial_t \boldsymbol{u} - \partial_t \tilde{\boldsymbol{s}}_h, \pi_{k-1}\boldsymbol{\xi}_{h,\tau}\right)\ dt \le Ch^r \|\boldsymbol{u}\|_{H^1(H^r)} \|\boldsymbol{\xi}\|_{\mathrm{cGP}}. \tag{55}$$

To bound the third term on the right-hand side of (51), the error splitting

$$\begin{aligned}\int_0^T \left((\boldsymbol{b}\cdot\nabla)(\boldsymbol{u} - \tilde{\boldsymbol{s}}_h), \pi_{k-1}\boldsymbol{\xi}_{h,\tau}\right)\ dt = &\int_0^T \left((\boldsymbol{b}\cdot\nabla)(\boldsymbol{u} - \tilde{\boldsymbol{u}}), \pi_{k-1}\boldsymbol{\xi}_{h,\tau}\right)\ dt \\ &+ \int_0^T \left((\boldsymbol{b}\cdot\nabla)(\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{s}}_h), \pi_{k-1}\boldsymbol{\xi}_{h,\tau}\right)\ dt \end{aligned}$$

is used. Then, applying (48) and (22) yields

$$
\begin{aligned}
&\int_0^T \left((\boldsymbol{b}\cdot\nabla)(\boldsymbol{u}-\tilde{\boldsymbol{s}}_h), \pi_{k-1}\boldsymbol{\xi}_{h,\tau}\right)\ dt \\
&\quad\le \int_0^T \|\boldsymbol{b}\|_\infty \left(\|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|_1 + \|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}_h\|_1\right)\|\pi_{k-1}\boldsymbol{\xi}_{h,\tau}\|_0\ dt \\
&\quad\le C\left[\left(\sum_{n=1}^N \tau_n^{2k+2}\int_{I_n}\|\boldsymbol{u}^{(k+1)}\|_1^2\ dt\right)^{1/2} + \left(h^{2r}\sum_{n=1}^N\int_{I_n}\|\tilde{\boldsymbol{u}}\|_{r+1}^2\ dt\right)^{1/2}\right] \\
&\qquad\quad \times\left(\sum_{n=1}^N\int_{I_n}|||\pi_{k-1}\boldsymbol{\xi}|||_0^2\ dt\right)^{1/2} \\
&\quad\le \left(C\tau^{k+1}\|\boldsymbol{u}\|_{H^{k+1}(H^1)} + Ch^r\|\boldsymbol{u}\|_{L^2(H^{r+1})}\right)\|\boldsymbol{\xi}_{h,\tau}\|_{\mathrm{cGP}}. \qquad (56)
\end{aligned}
$$

Arguing exactly as before gives for the second and the fourth term on the right-hand side of (51)

$$
\begin{aligned}
&\int_0^T \nu\left(\nabla(\boldsymbol{u}-\tilde{\boldsymbol{s}}_h), \nabla(\pi_{k-1}\boldsymbol{\xi}_{h,\tau})\right)\ dt + \int_0^T \left(\sigma(\boldsymbol{u}-\tilde{\boldsymbol{s}}_h), \pi_{k-1}\boldsymbol{\xi}_{h,\tau}\right)\ dt \qquad (57)\\
&\le \left(C(\nu^{1/2}+\sigma^{1/2}h)h^r\|\boldsymbol{u}\|_{L^2(H^{r+1})} + C(\nu^{1/2}+\sigma^{1/2})\tau^{k+1}\|\boldsymbol{u}\|_{H^{k+1}(H^1)}\right)\|\boldsymbol{\xi}_{h,\tau}\|_{\mathrm{cGP}}.
\end{aligned}
$$

To bound the fifth term on the right-hand side of (51) observe that

$$
\int_0^T (\nabla p, \pi_{k-1}\boldsymbol{\xi}_{h,\tau})\ dt = \int_0^T -(p, \nabla\cdot\pi_{k-1}\boldsymbol{\xi}_{h,\tau})\ dt = \int_0^T -(p, \pi_{k-1}\nabla\cdot\boldsymbol{\xi}_{h,\tau})\ dt,
$$

since the time projection π_{k-1} and the divergence commute. In addition, it is

$$
\begin{aligned}
\int_{I_n}(i_h p, \pi_{k-1}\nabla\cdot\boldsymbol{\xi}_{h,\tau})\ dt &= \int_{I_n}(\pi_{k-1}(i_h p), \pi_{k-1}\nabla\cdot\boldsymbol{\xi}_{h,\tau})\ dt \\
&= \int_{I_n}(\pi_{k-1}(i_h p), \nabla\cdot\boldsymbol{\xi}_{h,\tau})\ dt = 0, \qquad (58)
\end{aligned}
$$

since $\tilde{\boldsymbol{s}}_h$ has discrete divergence equal to zero and the relation $\int_{I_n}(\nabla\cdot\boldsymbol{u}_{h,\tau}, q_{h,\tau})\ dt = 0$ holds by definition for all $q_{h,\tau}\in Y_{k-1}^{\mathrm{dc}}$. Thus, for the fifth term on the right-hand side of (51), integration by parts with respect to space, applying the orthogonality condition (14), using (58), $\mu_K\sim 1$, and (13) lead to

$$
\begin{aligned}
&\int_0^T (\nabla p, \pi_{k-1}\boldsymbol{\xi}_{h,\tau})\ dt \\
&\quad = \int_0^T (i_h p - p, \pi_{k-1}\nabla\cdot\boldsymbol{\xi}_{h,\tau})\ dt = \int_0^T \sum_{k\in\mathcal{T}_h}(i_h p - p, \kappa_K\pi_{k-1}\nabla\cdot\boldsymbol{\xi}_{h,\tau})_K\ dt \\
&\quad\le \int_0^T \left(\sum_{K\in\mathcal{T}_h}\mu_K^{-1}\|i_h p - p\|_{0,K}^2\right)^{1/2}\left(\sum_{K\in\mathcal{T}_h}\mu_K\|\kappa_K\nabla\pi_{k-1}\boldsymbol{\xi}_{h,\tau}\|_{0,K}^2\right)^{1/2}\ dt \\
&\quad\le C\left(\int_0^T\|i_h p - p\|_0^2\ dt\right)^{1/2}\|\boldsymbol{\xi}_{h,\tau}\|_{\mathrm{cGP}} \le Ch^r\|p\|_{L^2(H^r)}\|\boldsymbol{\xi}_{h,\tau}\|_{\mathrm{cGP}}. \qquad (59)
\end{aligned}
$$

Finally, to bound the last term on the right-hand side of (51), the following decomposition is considered

$$\int_0^T S_h(\tilde{\boldsymbol{s}}_h, \pi_{k-1}\boldsymbol{\xi}_{h,\tau})\ dt = \int_0^T S_h(\tilde{\boldsymbol{s}}_h - \tilde{\boldsymbol{u}}, \pi_{k-1}\boldsymbol{\xi}_{h,\tau})\ dt$$
$$+ \int_0^T S_h(\tilde{\boldsymbol{u}} - \boldsymbol{u}, \pi_{k-1}\boldsymbol{\xi}_{h,\tau})\ dt + \int_0^T S_h(\boldsymbol{u}, \pi_{k-1}\boldsymbol{\xi}_{h,\tau})\ dt. \tag{60}$$

For the first term on the right-hand side of (60), the L^2 stability of the fluctuation operator κ_K, $\mu_K \sim 1$, and (22) are applied to obtain

$$\begin{aligned}
&\int_0^T S_h(\tilde{\boldsymbol{s}}_h - \tilde{\boldsymbol{u}}, \pi_{k-1}\boldsymbol{\xi}_{h,\tau})\ dt \\
&\le \int_0^T \left(\sum_{K\in\mathcal{T}_h} \mu_K \|\kappa_K \nabla(\tilde{\boldsymbol{s}}_h - \tilde{\boldsymbol{u}})\|_{0,K}^2\right)^{1/2} \left(\sum_{K\in\mathcal{T}_h} \mu_K \|\kappa_K \pi_{k-1}\nabla\boldsymbol{\xi}_{h,\tau}\|_{0,K}^2\right)^{1/2} dt \\
&\le \left(\int_0^T \sum_{K\in\mathcal{T}_h} \mu_K \|\kappa_K \nabla(\tilde{\boldsymbol{s}}_h - \tilde{\boldsymbol{u}})\|_{0,K}^2\ dt\right)^{1/2} \left(\int_0^T |||\pi_{k-1}\boldsymbol{\xi}_{h,\tau}|||^2\ dt\right)^{1/2} \\
&\le C h^r \|\boldsymbol{u}\|_{L^2(H^{r+1})} \|\boldsymbol{\xi}_{h,\tau}\|_{\mathrm{cGP}}.
\end{aligned} \tag{61}$$

Applying the stability of the fluctuation operator κ_K, $\mu_K \sim 1$, and (48) gives for the second term on the right-hand side of (60)

$$\begin{aligned}
&\int_0^T S_h(\tilde{\boldsymbol{u}} - \boldsymbol{u}, \pi_{k-1}\boldsymbol{\xi}_{h,\tau})\ dt \\
&\le \int_0^T \left(\sum_{K\in\mathcal{T}_h} \mu_K \|\kappa_K \nabla(\tilde{\boldsymbol{u}} - \boldsymbol{u})\|_{0,K}^2\right)^{1/2} \left(\sum_{K\in\mathcal{T}_h} \mu_K \|\kappa_K \pi_{k-1}\nabla\boldsymbol{\xi}_{h,\tau}\|_{0,K}^2\right)^{1/2} dt \\
&\le \left(\int_0^T \sum_{K\in\mathcal{T}_h} \mu_K \|\kappa_K \nabla(\tilde{\boldsymbol{u}} - \boldsymbol{u})\|_{0,K}^2\ dt\right)^{1/2} \left(\int_0^T |||\pi_{k-1}\boldsymbol{\xi}_{h,\tau}|||^2\ dt\right)^{1/2} \\
&\le C \tau^{k+1} \|\boldsymbol{u}\|_{H^{k+1}(H^1)} \|\boldsymbol{\xi}_{h,\tau}\|_{\mathrm{cGP}}.
\end{aligned} \tag{62}$$

To finish the estimate of the last term on the right-hand side of (51), the Cauchy–Schwarz inequality, the approximation properties (15) of the fluctuation operator κ_K, and $\mu_K \sim 1$ are used to get

$$\begin{aligned}
&\int_0^T S_h(\boldsymbol{u}, \pi_{k-1}\boldsymbol{\xi}_{h,\tau})\ dt \\
&\le \int_0^T \left(\sum_{K\in\mathcal{T}_h} \mu_K \|\kappa_K \nabla\boldsymbol{u}\|_{0,K}^2\right)^{1/2} \left(\sum_{K\in\mathcal{T}_h} \mu_K \|\kappa_K \pi_{k-1}\nabla\boldsymbol{\xi}_{h,\tau}\|_{0,K}^2\right)^{1/2} dt \\
&\le \left(\int_0^T \sum_{K\in\mathcal{T}_h} \mu_K \|\kappa_K (\nabla\boldsymbol{u})\|_{0,K}^2\ dt\right)^{1/2} \left(\int_0^T |||\pi_{k-1}\boldsymbol{\xi}_{h,\tau}|||^2\ dt\right)^{1/2} \\
&\le C h^r \|\boldsymbol{u}\|_{L^2(H^{r+1})} |||\boldsymbol{\xi}|||_{\mathrm{cGP}}.
\end{aligned} \tag{63}$$

Inserting (61), (62), and (63) in (60) gives

$$\int_0^T S_h(\tilde{\boldsymbol{s}}_h, \pi_{k-1}\boldsymbol{\xi}_{h,\tau})\, dt \le \left(Ch^r \|\boldsymbol{u}\|_{L^2(H^{r+1})} + C\tau^{k+1}\|\boldsymbol{u}\|_{H^{k+1}(H^1)}\right) |||\boldsymbol{\xi}|||_{\mathrm{cGP}}. \tag{64}$$

Inserting (51) in (50) and utilizing (55), (56), (57), (59), and (64) lead to

$$|||\boldsymbol{\xi}_{h,\tau}|||_{\mathrm{cGP}} \le Ch^r \left[\|\boldsymbol{u}\|_{L^2(H^{r+1})} + \|\boldsymbol{u}\|_{H^1(H^r)} + \|p\|_{L^2(H^r)}\right] + C\tau^{k+1}\|\boldsymbol{u}\|_{H^{k+1}(H^1)}. \tag{65}$$

Applying the triangle inequality, the bound (22), and the interpolation error estimates in time gives the statement of the theorem. □

Arguing similarly as in [5, Thm. 3.4], one can prove the following theorem.

Theorem 6. *Under the assumptions of Theorem 5, the following error estimate is valid*

$$\begin{aligned}\left(\int_0^T \|\boldsymbol{u}(t) - \boldsymbol{u}_{h,\tau}(t)\|_0^2\, dt\right)^{1/2} &\\ \le C(1+T^{1/2})h^r &\left[\|\boldsymbol{u}\|_{L^2(H^{r+1})} + \|\boldsymbol{u}\|_{H^1(H^r)} + \|p\|_{L^2(H^r)}\right] \\ &+ C(1+T^{1/2})\tau^{k+1}\|\boldsymbol{u}\|_{H^{k+1}(H^1)},\end{aligned} \tag{66}$$

with C independent of ν, h, and τ.

Proof. Denoting as before $\boldsymbol{\xi}_{h,\tau} = \boldsymbol{u}_{h,\tau} - \tilde{\boldsymbol{s}}_h$ and applying the ideas leading to (65) not only on $[0, T]$ but also on $[0, t_n]$, $n = 1, \ldots, N$, result in the estimate

$$\begin{aligned}\int_0^{t_n} |||\pi_{k-1}\boldsymbol{\xi}_{h,\tau}(t)|||^2\, dt + \frac{1}{2}\|\boldsymbol{\xi}_{h,\tau}(t_n)\|_0^2 &\\ \le Ch^{2r}\left[\|\boldsymbol{u}\|^2_{L^2(H^{r+1})} + \|\boldsymbol{u}\|^2_{H^1(H^r)} + \|p\|^2_{L^2(H^r)}\right] &+ C\tau^{2k+2}\|\boldsymbol{u}\|^2_{H^{k+1}(H^1)},\end{aligned}$$

where the integrals on the right-hand side were extended from $[0, t_n]$ to $[0, T]$ by monotonicity. After neglecting the non-negative integral on the left-hand side and multiplying by τ_n, a summation over $n = 1, \ldots, N$ provides

$$\begin{aligned}\sum_{n=1}^N \tau_n \|\boldsymbol{\xi}_{h,\tau}(t_n)\|_0^2 \le \left(\sum_{n=1}^N \tau_n\right) Ch^{2r}&\left[\|\boldsymbol{u}\|^2_{L^2(H^{r+1})} + \|\boldsymbol{u}\|^2_{H^1(H^r)} + \|p\|^2_{L^2(H^r)}\right] \\ + \left(\sum_{n=1}^N \tau_n\right) &C\tau^{2k+2}\|\boldsymbol{u}\|^2_{H^{k+1}(H^1)}.\end{aligned} \tag{67}$$

Since $\boldsymbol{\xi}_{h,\tau}$ is a piecewise polynomial of degree less than or equal to k in time, a norm equivalence on finite-dimensional spaces gives

$$\int_{t_{n-1}}^{t_n} \|\boldsymbol{\xi}_{h,\tau}\|_0^2\, dt \le C_k \left(\int_{t_{n-1}}^{t_n} \|\pi_{k-1}\boldsymbol{\xi}_{h,\tau}(t)\|_0^2\, dt + \tau_n \|\boldsymbol{\xi}_{h,\tau}(t_n)\|_0^2\right),$$

where C_k depends on the polynomial degree k but it is independent of τ_n and h. Hence, applying (67) and (65) yields

$$
\begin{aligned}
\int_0^T \|\boldsymbol{u}_{h,\tau}(t)-\tilde{\boldsymbol{s}}_h(t)\|_0^2\,dt &\le C_k \sum_{n=1}^{N}\left(\int_{t_{n-1}}^{t_n} \|\pi_{k-1}\boldsymbol{\xi}_{h,\tau}(t)\|_0^2\,dt + \tau_n\|\boldsymbol{\xi}_{h,\tau}(t_n)\|_0^2\right)\\
&\le C_k\left(\int_0^T |||\pi_{k-1}\boldsymbol{\xi}_{h,\tau}(t)|||_0^2\,dt + \sum_{n=1}^{N}\tau_n\|\boldsymbol{\xi}_{h,\tau}(t_n)\|_0^2\right)\\
&\le C(1+T)h^{2r}\Big[\|\boldsymbol{u}\|^2_{L^2(H^{r+1})}+\|\boldsymbol{u}\|^2_{H^1(H^r)}+\|p\|^2_{L^2(H^r)}\Big]\\
&\quad + C(1+T)\tau^{2k+2}\|\boldsymbol{u}\|^2_{H^{k+1}(H^1)}. \qquad (68)
\end{aligned}
$$

Now, the statement of the theorem follows by applying the triangle inequality and the time interpolation error estimates (48) together with (22). □

Theorem 7. *Let the assumptions of Theorem 5 hold and let in addition $(\boldsymbol{u},p)$ be smooth enough such that the norms on the right-hand side of (69) are bounded. Then, there exists a positive constant C independent of ν, h, and τ such that the error estimate*

$$
\begin{aligned}
&\left(\int_0^T \|\pi_{k-1}\left(p_{h,\tau}(t)-p(t)\right)\|_0^2\,dt\right)^{1/2}\\
&\qquad \le C(1+T)h^r\left[\|\boldsymbol{u}\|_{H^1(H^{r+1})}+\|\boldsymbol{u}\|_{H^2(H^r)}+\|p\|_{H^1(H^r)}\right]\\
&\qquad\quad + C(1+T)\tau^k(1+\tau)\|\boldsymbol{u}\|_{H^{k+2}(H^1)} + C\tau^{k+1}\|p\|_{H^{k+1}(L^2)}\\
&\qquad\quad + Ch^r\left[\|\boldsymbol{u}\|_{L^2(H^{r+1})}+\|\boldsymbol{u}\|_{H^1(H^r)}+\|p\|_{H^1(H^r)}\right] \qquad (69)
\end{aligned}
$$

holds.

Proof. A straightforward calculation shows that for all $\boldsymbol{v}_{h,\tau}\in X_{k-1}^{\mathrm{dc}}$ and $q_{h,\tau}\in Y_{k-1}^{\mathrm{dc}}$ it holds

$$
\begin{aligned}
&b_h((\boldsymbol{u}_{h,\tau}-\tilde{\boldsymbol{s}}_h,p_{h,\tau});(\boldsymbol{v}_{h,\tau},q_{h,\tau}))\\
&=\int_0^T(\partial_t\boldsymbol{\xi}_{h,\tau},\boldsymbol{v}_{h,\tau})\,dt+\int_0^T\nu(\nabla\boldsymbol{\xi}_{h,\tau},\nabla\boldsymbol{v}_{h,\tau})\,dt+\int_0^T\big((\boldsymbol{b}\cdot\nabla)\boldsymbol{\xi}_{h,\tau},\boldsymbol{v}_{h,\tau}\big)\,dt\\
&\quad+\int_0^T\sigma(\boldsymbol{\xi}_{h,\tau},\boldsymbol{v}_{h,\tau})\,dt-\int_0^T(\nabla\cdot\boldsymbol{v}_{h,\tau},p_{h,\tau})\,dt+\int_0^T S_h(\boldsymbol{\xi}_{h,\tau},\boldsymbol{v}_{h,\tau})\,dt\\
&=\int_0^T(\partial_t(\boldsymbol{u}-\tilde{\boldsymbol{s}}_h),\boldsymbol{v}_{h,\tau})\,dt+\int_0^T\nu(\nabla(\boldsymbol{u}-\tilde{\boldsymbol{s}}_h),\nabla\boldsymbol{v}_{h,\tau})\,dt\\
&\quad+\int_0^T((\boldsymbol{b}\cdot\nabla)(\boldsymbol{u}-\tilde{\boldsymbol{s}}_h),\boldsymbol{v}_{h,\tau})\,dt+\int_0^T\sigma(\boldsymbol{u}-\tilde{\boldsymbol{s}}_h,\boldsymbol{v}_{h,\tau})\,dt\\
&\quad-\int_0^T S_h(\tilde{\boldsymbol{s}}_h,\boldsymbol{v}_{h,\tau})\,dt+\int_0^T(\nabla p,\boldsymbol{v}_{h,\tau})\,dt.
\end{aligned}
$$

From this equation, one obtains

$$
\begin{aligned}
&\int_0^T (p_{h,\tau} - i_h\tilde{p}, \nabla\cdot \boldsymbol{v}_{h,\tau})\ dt \\
&= \int_0^T (p - i_h\tilde{p}, \nabla\cdot \boldsymbol{v}_{h,\tau})\ dt + \int_0^T (\partial_t \boldsymbol{\xi}_{h,\tau}, \boldsymbol{v}_{h,\tau})\ dt + \int_0^T \nu(\nabla\boldsymbol{\xi}_{h,\tau}, \nabla\boldsymbol{v}_{h,\tau})\ dt \\
&\quad + \int_0^T ((\boldsymbol{b}\cdot\nabla)\boldsymbol{\xi}_{h,\tau}, \boldsymbol{v}_{h,\tau})\ dt + \int_0^T \sigma(\boldsymbol{\xi}_{h,\tau}, \boldsymbol{v}_{h,\tau})\ dt + \int_0^T S_h(\boldsymbol{\xi}_{h,\tau}, \boldsymbol{v}_{h,\tau})\ dt \\
&\quad + \int_0^T (\partial_t(\tilde{\boldsymbol{s}}_h - \boldsymbol{u}), \boldsymbol{v}_{h,\tau})\ dt + \int_0^T \nu(\nabla(\tilde{\boldsymbol{s}}_h - \boldsymbol{u}), \nabla\boldsymbol{v}_{h,\tau})\ dt \\
&\quad + \int_0^T ((\boldsymbol{b}\cdot\nabla)(\tilde{\boldsymbol{s}}_h - \boldsymbol{u}), \boldsymbol{v}_{h,\tau})\ dt + \int_0^T \sigma(\tilde{\boldsymbol{s}}_h - \boldsymbol{u}, \boldsymbol{v}_{h,\tau})\ dt + \int_0^T S_h(\tilde{\boldsymbol{s}}_h, \boldsymbol{v}_{h,\tau})\ dt.
\end{aligned}
\tag{70}
$$

To derive the error estimates, the Gauss quadrature rule with k points will be used for the numerical integration of the time integral. Hence, one has

$$
\int_0^T q_{2k-1}(t)\ dt = \sum_{n=1}^N \frac{\tau_n}{2} \sum_{i=1}^k \hat{\omega}_i q_{2k-1}(\tilde{t}_{n,i}) \tag{71}
$$

for all $q_{2k-1} \in \mathbb{P}_{2k-1}(I_n)$ where $\tilde{t}_{n,i}$ denote the corresponding quadrature points on I_n and $\hat{\omega}_i$ are the weights of the Gauss formula on $(-1,1)$ which fulfill $\hat{\omega}_i > 0$. Let $\tilde{t}_{n,0} = t_{n-1}$ be an additional point.

Using the discrete inf-sup condition (3), one can construct $\boldsymbol{w}_{h,\tau} \in \mathbb{P}_k(I_n, V_h)$ such that

$$
\beta_0 \|\pi_{k-1}\left(p_{h,\tau}(\tilde{t}_{n,i}) - i_h\tilde{p}(\tilde{t}_{n,i})\right)\|_0^2 \le \left(\pi_{k-1}\left(p_{h,\tau}(\tilde{t}_{n,i}) - i_h\tilde{p}(\tilde{t}_{n,i})\right), \nabla\cdot\boldsymbol{w}_{h,\tau}(\tilde{t}_{n,i})\right), \tag{72}
$$

$$
\|\boldsymbol{w}_{h,\tau}(\tilde{t}_{n,i})\|_1 = \|\pi_{k-1}\left(p_{h,\tau}(\tilde{t}_{n,i}) - i_h\tilde{p}(\tilde{t}_{n,i})\right)\|_0. \tag{73}
$$

Since $\boldsymbol{w}_{h,\tau} \in \mathbb{P}_k(I_n, V_h)$, it follows that $\pi_{k-1}\boldsymbol{w}_{h,\tau} \in \mathbb{P}_{k-1}(I_n, V_h)$. Setting $\boldsymbol{v}_{h,\tau} = \pi_{k-1}\boldsymbol{w}_{h,\tau}$ and using (45), (46), one obtains

$$
\begin{aligned}
\int_0^T (p_{h,\tau} - i_h\tilde{p}, \nabla\cdot\boldsymbol{v}_{h,\tau})\ dt &= \sum_{n=1}^N \int_{I_n} ((p_{h,\tau} - i_h\tilde{p}), \pi_{k-1}(\nabla\cdot\boldsymbol{w}_{h,\tau}))\ dt \\
&= \sum_{n=1}^N \int_{I_n} (\pi_{k-1}(p_{h,\tau} - i_h\tilde{p}), \nabla\cdot\boldsymbol{w}_{h,\tau})\ dt \\
&\ge \beta_0 \int_0^T \|\pi_{k-1}(p_{h,\tau} - i_h\tilde{p})\|_0^2\ dt,
\end{aligned}
\tag{74}
$$

where the exactness of the quadrature rule for polynomials of degree $(2k-1)$, the positivity of the quadrature weights, (71), and (72) were used.

Setting $\boldsymbol{v}_{h,\tau} = \pi_{k-1}\boldsymbol{w}_{h,\tau}$ in (70), using (74), the assumption that $\boldsymbol{b}$ and σ are constants with respect to time, and (45), it follows that

$$\begin{aligned}
&\beta_0 \int_0^T \|\pi_{k-1}(p_{h,\tau} - i_h\tilde{p})\|_0^2 \, dt \\
&\le \int_0^T (p_{h,\tau} - i_h\tilde{p}, \pi_{k-1}(\nabla\cdot\boldsymbol{w}_{h,\tau}))\, dt \\
&= \int_0^T (\pi_{k-1}(p - i_h\tilde{p}), \pi_{k-1}(\nabla\cdot\boldsymbol{w}_{h,\tau}))\ dt + \int_0^T (\partial_t\boldsymbol{\xi}_{h,\tau}, \pi_{k-1}\boldsymbol{w}_{h,\tau})\ dt \\
&\quad + \int_0^T \nu(\pi_{k-1}(\nabla\boldsymbol{\xi}_{h,\tau}), \pi_{k-1}(\nabla\boldsymbol{w}_{h,\tau}))\ dt + \int_0^T ((\boldsymbol{b}\cdot\nabla)\pi_{k-1}\boldsymbol{\xi}_{h,\tau}, \pi_{k-1}\boldsymbol{w}_{h,\tau})\ dt \\
&\quad + \int_0^T \sigma(\pi_{k-1}\boldsymbol{\xi}_{h,\tau}, \pi_{k-1}\boldsymbol{w}_{h,\tau})\ dt + \int_0^T S_h(\pi_{k-1}\boldsymbol{\xi}_{h,\tau}, \pi_{k-1}\boldsymbol{w}_{h,\tau})\ dt \\
&\quad + \int_0^T (\partial_t(\tilde{\boldsymbol{s}}_h - \boldsymbol{u}), \pi_{k-1}\boldsymbol{w}_{h,\tau})\ dt + \int_0^T \nu(\pi_{k-1}\nabla(\tilde{\boldsymbol{s}}_h - \boldsymbol{u}), \pi_{k-1}(\nabla\boldsymbol{w}_{h,\tau}))\ dt \\
&\quad + \int_0^T (\pi_{k-1}(\boldsymbol{b}\cdot\nabla)(\tilde{\boldsymbol{s}}_h - \boldsymbol{u})_{h,\tau}, \pi_{k-1}\boldsymbol{w}_{h,\tau})\ dt \\
&\quad + \int_0^T \sigma(\pi_{k-1}(\tilde{\boldsymbol{s}}_h - \boldsymbol{u}), \pi_{k-1}\boldsymbol{w}_{h,\tau})\ dt + \int_0^T S_h(\pi_{k-1}\tilde{\boldsymbol{s}}_h, \pi_{k-1}\boldsymbol{w}_{h,\tau})\ dt.
\end{aligned} \tag{75}$$

The seventh term on the right-hand side of (75) is decomposed in the form

$$\begin{aligned}
\int_0^T (\partial_t(\tilde{\boldsymbol{s}}_h - \boldsymbol{u}), \pi_{k-1}\boldsymbol{w}_{h,\tau})\ dt &= \int_0^T (\partial_t(\tilde{\boldsymbol{s}}_h - \tilde{\boldsymbol{u}}), \pi_{k-1}\boldsymbol{w}_{h,\tau})\ dt \\
&\quad + \int_0^T (\partial_t(\tilde{\boldsymbol{u}} - \boldsymbol{u}), \pi_{k-1}\boldsymbol{w}_{h,\tau})\ dt.
\end{aligned}$$

For the second term on the right-hand side, integrating by parts with respect to time and using (47) yield

$$\begin{aligned}
&\int_0^T (\partial_t(\tilde{\boldsymbol{u}} - \boldsymbol{u}), \pi_{k-1}\boldsymbol{w}_{h,\tau})\ dt \\
&\qquad = -\sum_{n=1}^{N}\left(\int_{I_n} (\tilde{\boldsymbol{u}} - \boldsymbol{u}, \partial_t(\pi_{k-1}\boldsymbol{w}_{h,\tau}))\ dt + (\boldsymbol{u} - \tilde{\boldsymbol{u}}, \pi_{k-1}\boldsymbol{w}_{h,\tau})\Big|_{t_{n-1}}^{t_n}\right) = 0.
\end{aligned}$$

It follows that

$$\begin{aligned}
\int_0^T (\partial_t(\tilde{\boldsymbol{s}}_h - \boldsymbol{u}), \pi_{k-1}\boldsymbol{w}_{h,\tau})\ dt &\le \int_0^T \|\partial_t(\tilde{\boldsymbol{s}}_h - \tilde{\boldsymbol{u}})\|_0 \|\pi_{k-1}\boldsymbol{w}_{h,\tau}\|_0\ dt \\
&\le C\int_0^T \|\partial_t(\tilde{\boldsymbol{s}}_h - \tilde{\boldsymbol{u}})\|_0 \|\nabla\pi_{k-1}\boldsymbol{w}_{h,\tau}\|_0\ dt,
\end{aligned}$$

where Poincaré's inequality was applied in the last line.

Using (71), (45), and (73) gives

$$\begin{aligned}\int_0^T \|\nabla\pi_{k-1}\boldsymbol{w}_{h,\tau}\|_0^2\ dt &= \sum_{n=1}^N \frac{\tau_n}{2}\sum_{i=1}^k \hat{\omega}_i\|\pi_{k-1}\nabla\boldsymbol{w}_h(\tilde{t}_{n,i})\|_0^2\\ &= \sum_{n=1}^N \frac{\tau_n}{2}\sum_{i=1}^k \hat{\omega}_i\|\nabla\boldsymbol{w}_h(\tilde{t}_{n,i})\|_0^2\\ &= \sum_{n=1}^N \frac{\tau_n}{2}\sum_{i=1}^k \hat{\omega}_i\|\pi_{k-1}\left(p_{h,\tau}(\tilde{t}_{n,i}) - i_h\tilde{p}(\tilde{t}_{n,i})\right)\|_0^2\\ &= \int_0^T \|\pi_{k-1}\left(p_{h,\tau} - i_h\tilde{p}\right)\|_0^2\ dt \end{aligned} \tag{76}$$

where $\tilde{t}_{n,i}$, $i = 1,\dots,k$, denote the node of Gaussian quadrature on I_n and $\hat{\omega}_i$, $i = 1,\dots,k$, are the corresponding weight on $[-1,1]$.

Applying (76) yields

$$\begin{aligned}&\int_0^T (\partial_t(\tilde{\boldsymbol{s}}_h - \boldsymbol{u}), \pi_{k-1}\boldsymbol{w}_{h,\tau})\ dt\\ &\qquad\le C\int_0^T \|\partial_t(\tilde{\boldsymbol{s}}_h - \tilde{\boldsymbol{u}})\|_0^2\ dt + \frac{\beta_0}{12}\int_0^T \|\pi_{k-1}\left(p_{h,\tau} - i_h\tilde{p}\right)\|_0^2\ dt.\end{aligned}$$

Arguing in the same way for the rest of the terms on the right-hand side of (75) leads to

$$\begin{aligned}&\int_0^T \|\pi_{k-1}(p_{h,\tau} - i_h\tilde{p})\|_0^2\ dt\\ &\le C\Bigg[\int_0^T \|\pi_{k-1}(p - i_h\tilde{p})\|_0^2\ dt + \int_0^T \|\partial_t\boldsymbol{\xi}_{h,\tau}\|_{-1}^2\ dt + \int_0^T |||\pi_{k-1}\boldsymbol{\xi}_{h,\tau}|||^2\ dt\\ &\qquad + \int_0^T \|\partial_t(\tilde{\boldsymbol{s}}_h - \tilde{\boldsymbol{u}})\|_0^2\ dt + \int_0^T \nu\|\pi_{k-1}\nabla(\tilde{\boldsymbol{s}}_h - \boldsymbol{u})\|_0^2\ dt\\ &\qquad + \int_0^T (\|\boldsymbol{b}\|_\infty + \sigma)\|\pi_{k-1}(\tilde{\boldsymbol{s}}_h - \boldsymbol{u})\|_0^2\ dt + \int_0^T \sum_{K\in\mathcal{T}_h}\mu_K\|\kappa_K\pi_{k-1}\nabla\tilde{\boldsymbol{s}}_h\|_{0,K}^2\ dt\Bigg].\end{aligned} \tag{77}$$

Now, the terms on the right-hand side of (77) need to be bounded. The estimates for the third term follows from Theorem 5. In the following, the L^2 stability of the projection π_{k-1} and the interpolation operator with respect to time, i.e.,

$$\int_{I_n} \|\pi_{k-1}\boldsymbol{v}\|_0\, dt \le C\int_{I_n}\|\boldsymbol{v}\|_0\, dt \quad \text{and} \quad \int_{I_n}\|\tilde{\boldsymbol{v}}\|_0\, dt \le C\int_{I_n}\|\boldsymbol{v}\|_0\, dt$$

will be often used. For the first term on the right-hand side of (77), applying (48) and (14) gives

$$\begin{aligned}\int_0^T \|\pi_{k-1}(p - i_h\tilde{p})\|_0^2 \, dt &\le C\left(\int_0^T \|p-\tilde{p}\|_0^2 \, dt + \int_0^T \|\tilde{p} - i_h\tilde{p}\|_0^2 \, dt\right)\\ &\le C\tau^{2k+2}\int_0^T \|p^{(k+1)}\|_0^2 \, dt + Ch^{2r}\int_0^T \|\tilde{p}\|_{2r}^2 \, dt\\ &\le C\left(\tau^{2k+2}\|p\|_{H^{k+1}(L^2)}^2 + h^{2r}\|p\|_{L^2(H^r)}^2\right).\end{aligned}$$

For bounding the second term on the right-hand side of (77), one first observes that $\int_0^T \|\partial_t \boldsymbol{\xi}_{h,\tau}\|_{-1} \, dt \le \int_0^T \|\partial_t \boldsymbol{\xi}_{h,\tau}\|_0 \, dt$. Now, since it is assumed that $\boldsymbol{b}$ and σ are independent of t, the error bounds for $\|\boldsymbol{\xi}_{h,\tau}\|_0$ can also be applied to its time derivative so that applying (68) to $\partial_t\boldsymbol{\xi}_{h,\tau}$ leads to

$$\begin{aligned}\int_0^T \|\partial_t\boldsymbol{\xi}_{h,\tau}\|_0^2 \, dt \le C(1+T)h^{2r}&\left[\|\partial_t\boldsymbol{u}\|_{L^2(H^{r+1})}^2 + \|\partial_t\boldsymbol{u}\|_{H^1(H^r)}^2 + \|\partial_t p\|_{L^2(H^r)}^2\right]\\ &+ C(1+T)\tau^{2k}\|\partial_t\boldsymbol{u}\|_{H^{k+1}(H^1)}^2.\end{aligned}$$

For the truncation errors involving $\tilde{\boldsymbol{s}}_h - \boldsymbol{u}$ (the last four terms), one argues as in Theorem 5 to get

$$\begin{aligned}\int_0^T \|\partial_t(\tilde{\boldsymbol{s}}_h - \tilde{\boldsymbol{u}})\|_0^2 \, dt &\le Ch^{2r}\|\boldsymbol{u}\|_{H^1(H^r)}^2,\\ \int_0^T \nu\|\pi_{k-1}\nabla(\tilde{\boldsymbol{s}}_h - \boldsymbol{u})\|_0^2 \, dt &\le C\nu\left(h^{2r}\|\boldsymbol{u}\|_{L^2(H^{r+1})}^2 + \tau^{2k+2}\|\boldsymbol{u}\|_{H^{k+1}(H^1)}^2\right),\\ \int_0^T (\|\boldsymbol{b}\|_\infty + \sigma)\|\tilde{\boldsymbol{s}}_h - \boldsymbol{u}\|_0^2 \, dt &\le C\left(h^{2r}\|\boldsymbol{u}\|_{L^2(H^r)}^2 + \tau^{2k+2}\|\boldsymbol{u}\|_{H^{k+1}(H^1)}^2\right).\end{aligned}$$

The bound for the last term (similarly as in the estimates (60)–(63)) uses the error splitting with respect to space and time, the L^2 stability of the fluctuation operator κ_K, $\mu_K \sim 1$, and the approximation properties of κ_K. One obtains

$$\begin{aligned}&\int_0^T \sum_{K\in\mathcal{T}_h} \mu_K\|\kappa_K\pi_{k-1}\nabla\tilde{\boldsymbol{s}}_h\|_{0,K}^2 \, dt\\ &\quad\le 3\int_0^T \sum_{K\in\mathcal{T}_h}\|\kappa_K\nabla(\tilde{\boldsymbol{s}}_h - \tilde{\boldsymbol{u}})\|_{0,K}^2 \, dt + 3\int_0^T \sum_{K\in\mathcal{T}_h}\|\kappa_K\nabla(\tilde{\boldsymbol{u}} - \boldsymbol{u})\|_{0,K}^2 \, dt\\ &\qquad + 3\int_0^T \sum_{K\in\mathcal{T}_h}\|\kappa_K\nabla\boldsymbol{u}\|_{0,K}^2 \, dt\\ &\quad\le C\left(h^{2r}\|\boldsymbol{u}\|_{L^2(H^{r+1})}^2 + \tau^{2k+2}\|\boldsymbol{u}\|_{H^{k+1}(H^{r+1})}^2\right).\end{aligned}$$

The statement of the theorem follows by collecting the bounds for all terms on the right-hand side of (77), and by applying the triangle inequality and the bounds (14) and (48) for the interpolation errors in space and time. □

Remark 8. *Instead of using* $\int_0^T \|\partial_t\boldsymbol{\xi}_{h,\tau}\|_{-1}^2 \, dt \le \int_0^T \|\partial_t\boldsymbol{\xi}_{h,\tau}\|_0^2 \, dt$ *one could use*

$$\int_0^T \|\partial_t\boldsymbol{\xi}_{h,\tau}\|_{-1}^2 \, dt \le C\int_0^T \|A_h^{-1/2}\partial_t\boldsymbol{\xi}_{h,\tau}\|_0^2 \, dt$$

and then argue as in the proof of Theorem 3. However, since it is assumed that $\boldsymbol{b}$ is time-independent, the proof presented above is shorter although it requires a higher regularity of the solution.

5. Numerical studies

This section presents numerical simulations that support the theoretical results obtained in the previous sections. Two examples will be presented. In the first example, an analytical solution is considered and very small time steps are applied to support the error analysis of Section 3. In the second example the solution is polynomial in the space such that the approximation will be exact in the spatial part and the discretization error in time dominates. This example will support the analytical results from Section 4.

All simulations were performed on uniform quadrilateral grids where the coarsest grid (level 1) is obtained by dividing the unit square into four squares. Mapped finite element spaces [18] were used, where the enriched spaces on the reference cell $\hat{K} = [-1,1]^2$ are given by

$$\mathbb{Q}_r^{\text{bubble}}(\hat{K}) := \mathbb{Q}_r(\hat{K}) + \operatorname{span}\left\{\hat{b}_\Box \hat{x}_i^{r-1},\ i = 1,2\right\}$$

with the biquadratic bubble function $\hat{b}_\Box = (1-\hat{x}_1^2)(1-\hat{x}_2^2)$. The combination $\mathbb{Q}_r^{\text{bubble}}(\hat{K})$ with $D(K) = \mathbb{P}_{r-1}(K)$ provides for $r \geq 2$ suitable spaces for LPS methods, see [37]. The simulations were performed with the code MooNMD [27].

Example 9. *An example with negligible temporal error.* Consider the Oseen problem (1) with $\Omega = (0,1)^2$, $\nu = 10^{-10}$, $\boldsymbol{b} = \boldsymbol{u}$, $\sigma = 1$, and $T = 1$. The right-hand side $\boldsymbol{f}$ and the initial condition $\boldsymbol{u}_0$ were chosen such that

$$\boldsymbol{u}(t,x,y) = \sin(t)\begin{pmatrix}\sin(\pi x)\sin(\pi y)\\ \cos(\pi x)\cos(\pi y)\end{pmatrix},$$
$$p(t,x,y) = \sin(t)\left(\sin(\pi x) + \cos(\pi y) - \frac{2}{\pi}\right)$$

is the solution of (1) equipped with non-homogeneous Dirichlet boundary conditions.

This example studies the convergence order with respect to space. To this end, the time discretization scheme cGP(2) with the small time step length $\tau = 1/1280$ was used. Numerical studies concerning the choice of stabilization parameters for convection-dominated problems suggest that a good choice is $\mu_K \in (0,1)$, e.g., see [6]. Based on these studies and our own experience, the stabilization parameters were set to be $\mu_K = 0.1$. The convergence plots for simulations with the finite element spaces $V_h/Q_h = \mathbb{Q}_3^{\text{bubble}}/\mathbb{P}_2^{\text{disc}}$ and the projection space $D(K) = \mathbb{P}_2(K)$ are presented in Figure 1. One can see fourth order convergence for the $L^2(L^2)$ norm and the L^2 norm at the final time. For all other norms on the left-hand side of (25) and the $L^2(L^2)$ norm of pressure, third order of convergence can be observed. It can be seen in Figure 1 that $\|\kappa_K \nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|_{L^2(L^2)}$ is the dominant term among the velocity errors on the left-hand side of (25). Altogether, the order of convergence is exactly as predicted in (25) and (31).

Example 10. *An example with dominant temporal error.* Let $\Omega = (0,1)^2$, $\nu = 10^{-10}$, $\boldsymbol{b} = \boldsymbol{u}$, $\sigma = 1$, $T = 1$ and consider the Oseen equations (1) with the

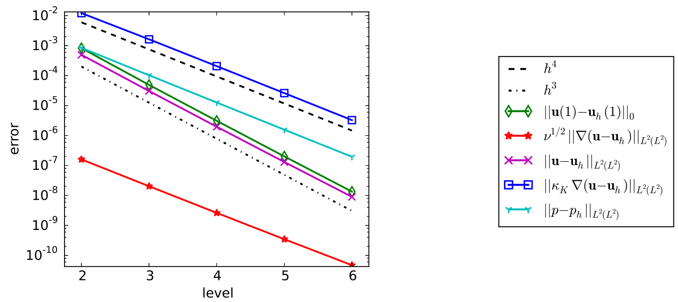

FIGURE 1. Example 9: Convergence of various errors with respect to the spatial mesh width.

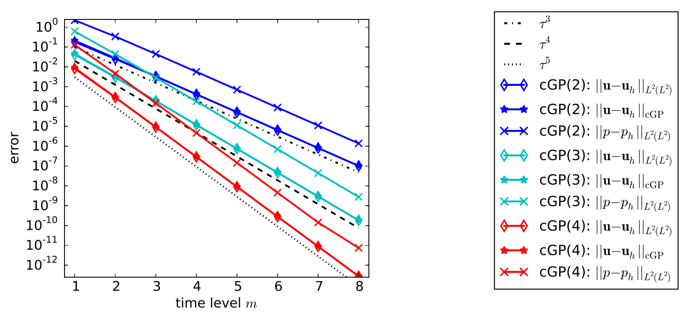

FIGURE 2. Example 10: Convergence of various errors with respect to the time step, where the time step is given by $\tau = 0.1 \cdot 2^{-m+1}$.

prescribed solution

$$\boldsymbol{u} = \begin{pmatrix} \sin(40t)y \\ \cos(t)x \end{pmatrix}, \quad p(t,x,y) = \cos(40t)(x-0.5) + \sin(40t)(2y-1).$$

In this example, the spaces $V_h/Q_h = \mathbb{Q}_2^{\text{bubble}}/\mathbb{P}_1^{\text{disc}}$ and the projection space $D(K) = \mathbb{P}_1(K)$ were considered. The mesh consisted of 16×16 squares. Note that for any time t the solution can be represented exactly by functions from the finite element spaces V_h and Q_h. Hence, all occurring errors will result from the temporal discretization.

Figure 2 reports the order of convergence for the methods cGP(k), $k \in \{2,3,4\}$, in combination with the LPS method. One can observe the predicted convergence order $k+1$ for the errors estimated in (49) and (66). Also for the pressure, order $k+1$ can be seen although estimate (69) predicts only order k.

6. SUMMARY

This paper analyzed a combination of higher order continuous Galerkin–Petrov schemes in time with the one-level variant of the LPS method in space applied to the transient Oseen equations. The continuous-in-time case and the fully discrete situation were considered. Optimal error bounds for velocity and pressure were obtained with constants that do not depend on the viscosity parameter ν. The theoretical results were confirmed by numerical simulations.

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