Mathematisches Forschungsinstitut Oberwolfach

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Miniworkshop: Geometry of Operators

June 23rd – June 29th, 2002

The purpose of this mini–workshop was to bring geometers, analysts, topologists, and mathematical physicists together in an interdisciplinary workshop. It was in fact a follow up meeting of two previous meetings of the same type organized in the year 1998 at the M.F.O. and in the year 2000 at the C.I.R.M. (Luminy, France). There were two types of talks: 75 min. talks of survey type delivered by senior researchers and shorter talks of 45 min. duration.

The participants appreciated the informal atmosphere of this small scale meeting as well as the perfect infrastructure of the institute.

The talks focused on the following topics:

1. Methods of noncommutative geometry and index problems on singular structures, e.g. foliations (Benameur, Pflaum, Wahl),
2. String theory, gerbes and higher categories (Bunke, Carey, Mickelsson, Stevenson),
3. Geometry of infinite–dimensional manifolds (Lesch, Magnot, Paycha, Wurzbacher),
4. Renormalization (Rosenberg),
5. Spectral Geometry (Burghelea, Cardona, Park, Wojciechowski).

The participants agreed that a follow up meeting in about two years would be desirable.

Below the reader will find the abstracts of the talks given.

Abstracts

A higher Lefschetz formula for foliations

Moulay-Tahar Benameur

Let (M, F) be a smooth compact foliated manifold and let H be a topologically cyclic compact Lie group which acts on M by $F-$preserving diffeomorphisms. Denote by $h \in H$ a generator of H and let (E, d) be an elliptic pseudo-differential complex along the leaves of (M, F) which is $H-$invariant.

In a first introductory part of the talk, we shall recall the definition of the Lefschetz class $L(h; E, d)$ of h with respect to (E, d) as the localization of the $H-$equivariant Connes-Skandalis index of (E, d) with respect to the prime ideal associated with h in the representation ring $R(H)$ of H. Therefore this class lives in the localization of the $H-$equivariant $K-$theory of the Connes C^*-algebra of the foliation. We shall also explain how one can exploit the index theory for foliations to prove a Lefschetz fixed point formula relating $L(h; E, d)$ to topological data over the fixed point submanifold M^H of H. This formula takes place at the level of $K-$theory.

The main part of the talk concerns higher Lefschetz formulae. We shall then use classical Noncommutative Geometry techniques to prove scalar Lefschetz fixed point formulae. A first example is furnished by any holonomy invariant transverse measure, in which case we show that we can deduce the Heitsch-Lazarov theorem. More interesting is the fact that higher cyclic cohomology can be used to state and prove new higher fixed point theorems. One can take advantage of the Haefliger homology of the foliation and we shall show that any Haefliger current gives rise to a Hochschild cohomology class and that this class comes from cyclic cohomology if the Haefliger current is closed. Moreover, the main theorem that we shall explain is the existence of a pairing between Haefliger's homology and $H-$equivariant $K-$theory whenever H preserves the leaves (not only the bundle F). As a corollary we deduce the announced higher Lefschetz formula by proving a commutation lemma with Morita extensions.

One consequence will be detailed. It concerns higher rigidity theorems which generalize some classical well-known ones.

Families of Dirac operators, Deligne cohomology, and field theory

Ulrich Bunke

In two talks we discussed line bundles, gerbes and higher generalizations from a categorical point of view. In this picture the symmetric monoidal category of line bundles gives rise to a two category with one object, with line bundles as morphisms, tensor product as composition, and isomorphisms of line bundles as two-morphisms. The sheafification of this two-category gives the two-category of gerbes. Higher generalizations should be defined iteratively.

Using Chech theory we indicated a way to iterate this construction without going into higher category theory. The idea is to represent bundles, gerbes and the higher objects by Chech cocycles. Isomorphism classes are then described by Deligne cohomology. We explained how families of Dirac operators give rise to the determinant bundle, the index gerbe, and the higher objects in this picture.

Laplace transform: from topology to spectral geometry

DAN BURGHELEA

In this talk I will explain how one can recover the counting functions of instantons and closed trajectories of a vector field X (which is gradient like for a pair (ω, g) on a closed manifold M of dimension n) from the spectral theory of the one parameter family of elliptic operators $\Delta^q + t(L_X + L_X^*) + t^2||\omega||^2 Id$, $0 \leq q \leq n$, $t \in R_+$. Here ω is a closed one form with all critical points non-degenerate, g a Riemannian metric Δ^q the Laplacian on $q-forms$ with respect to g, L_X the Lie derivative and L_X^* the formal adjoint of L_X. This is done via Laplace transform of Dirichlet series.

Trace anomalies and phase anomalies in Chern-Simons theories

ALEXANDER CARDONA

Unlike ordinary traces, ζ-regularized traces on the algebra of classical PDO's are neither cyclic nor do they commute with exterior differentiation, thus giving rise to *tracial anomalies*. In this talk we show how tracial anomalies can lead to anomalous phenomena in quantum field theory, in particular in the case of Chern-Simons theory. Since tracial anomalies can be expressed in terms of Wodzicki residues, they have some local feature which is in turn reflected on the locality of anomalies in quantum field theory. For the particular case of Chern-Simons theory, we describe variations of η-invariants as integrated tracial anomalies, thus giving an interpretation of the local term arising in the Atiyah-Patodi-Singer theorem for families as an integrated tracial anomaly and, in th case of families of signature operators in dimension 3, giving rise to a phase anomaly interpreted here as an integrated tracial anomaly. It leads, via the APS theorem, to the well-known Chern-Simons term in topological quantum field theory.

Gerbes and Strings I

ALAN CAREY

I surveyed the theory of bundle gerbes following the original articles of Murray and Stevenson. As the applications to string theory are formulated in terms of Hitchen's notion of local gerbes I phrased the discussion in that way. Local gerbes are a useful way to think about bundle gerbes by virtue of the notion of stable equivalence.

One starts from M a manifold, $\{U_\alpha\}_{\alpha \in A}$ a good cover and $\{B_\alpha\}_{\alpha \in A}$ a family of closed two forms with B_α defined on U_α. The relations $dB_\alpha = dB_\beta$ hold on overlaps $U_\alpha \cap U_\beta$ so there is a global 3-form H which is equal to dB_α on U_α. There are additional conditions which are equivalent to the definition of the gerbe multiplication and these were explained later in Stevenson's talk. The case of interest is when the Dixmier-Douady class of the local gerbe is non-trivial. For the non-torsion case this means that the class $[H]$ of H in integral de Rham cohomology is non-trivial.

Gerbes and Strings II

Alan Carey

I introduced $D-$branes as submanifolds Q of the spacetime M. In forming the world sheet action one is interested in defining a suitable function (the world sheet action) on the space $\Sigma_Q(M)$ of smooth maps from a Riemann surface Σ into M which takes the boundary $\partial\Sigma$ into Q. We make the simplifying assumption that $\partial\Sigma = S^1$ so that there is a map $\partial : \Sigma_Q(M) \to L(Q)$ to the loop space of Q given by restricting an element of $\Sigma_Q(M)$ to $\partial\Sigma$. Bundle gerbes provide a geometric way to understand certain constructions of Gawedzki which in turn facilitate the definition of the world sheet action.

Starting from a 3-cocycle ξ on M a line bundle on $L_\xi \to L(M)$ is constructed by using the evaluation map $ev : S^1 \times L(M) \to M$ to pull back ξ to $S^1 \times L(M)$ then integrating over S^1 to give an integral 2-class and hence a line bundle on $L(M)$. Using ∂ we can also form the pullback line bundle $\partial^{-1}(L_\xi) \to \Sigma_Q(M)$.

There is a contribution to the world sheet action from the fermionic action on $\phi(\Sigma)$. This contribution is really a section of the Pfaffian bundle. Freed and Witten show that the obstruction to triviality of this Pfaffian bundle is a certain Stiefel-Whitney class $[\omega] \in H^3(Q, \mathbb{Z})$ (Čech cohomology). Freed and Witten argued that if the Čech class in $H^3(Q, \mathbb{Z})$ defined by the $B-$field and denoted $[H_Q]$ is such that $\partial^{-1}L_\omega \otimes \partial^{-1}L^*_{H_Q}$ is a trivial line bundle then one can define the action as a function by tensoring a section of the Pfaffian bundle $\partial^{-1}L_\omega$ with a section of $\partial^{-1}L^*_{H_Q}$ and dividing by a trivialisation of the tensor product.

I gave an overview of how bundle gerbes, bundle gerbe modules and the holonomy associated with connections on these provide a way to extend the Freed-Witten approach to anomaly cancellation in the world sheet action to general $B-$fields.

Unbounded Fredholm operators and Spectral Flow

Matthias Lesch

This talk was based on recent joint work with B. Booss–Bavnbek and J. Phillips.

I explained the two natural topologies on the space of (not necessarily bounded) self–adjoint Fredholm operators $\mathcal{F}$ in a separable Hilbert space: the Riesz topology and the gap (= "projection norm" = "graph norm") topology. I showed that the gap topology is characterized as the topology such that the Cayley transform is a homeomorphism onto a subspace of the space of unitary operators. A unitary U is in the image of the Caley transform iff $U + I$ is Fredholm and $U - I$ is injective. The last condition is unstable and therefore not easy to deal with.

I showed that $\mathcal{F}$ with the gap topology is connected. This is in contrast to the space of bounded Fredholm operators which is not connected.

Moreover, I presented a rigorous definition of spectral flow of a path in $\mathcal{F}$ and indicated a proof of the homotopy invariance.

Bundles of formal frames over manifolds of maps

Jean-Pierre Magnot

In this talk, we study principal bundles over manifolds of maps with structure group the invertible formals symbols of classical pseudo-differential operators of order 0. These bundles are derived from bundles of frames over manifolds of maps built in a classical way.

On these principal bundles, we consider Chern-type forms $\mathrm{res}(\Omega^k)$, where res is the Wodzicki residue and Ω is the curvature of a connection 1-form. All these forms are exact.

Then, we study some possible generalization of the finite dimensional Ambrose-Singer theorem to this setting, using the framework of ILH manifolds.

Twisted K–theory and QFT

Jouko Mickelsson

The aim of this talk is to explain how symmetry breaking in a quantum field theory problem leads to a study of projective bundles, Dixmier-Douady classes, and associated gerbes. A gerbe manifests itself in different equivalent ways. Besides the cohomological description as a DD class, it can be defined in terms of a family of local line bundles or as a prolongation problem for an (infinite-dimensional) principal bundle, with the fibre consisting of (a subgroup of) projective unitaries in a Hilbert space. The prolongation aspect is directly related to the appearance of central extensions of (broken) symmetry groups. We also discuss the construction of twisted K-theory classes by families of supercharges for the supersymmetric Wess-Zumino-Witten model.

Eta invariants for odd dimensional hyperbolic manifolds with cusps

Jinsung Park

In this talk, I introduce the recent results about the eta invariants of Dirac operators over the hyperbolic manifolds with cusps. We follow Werner Müller and use the relative traces to define eta invariants. We show regularity of the eta functions at $s = 0$. The Selberg trace formula and detailed analysis of the unipotent terms gives the relation between the eta invariant and Selberg zeta function of odd type. This allows us to show the vanishing of the unipotent contribution. We show that the eta invariant and Selberg zeta function of odd type satisfy certain functional equation . These results generalize earlier work of John Millson to the hyperbolic manifolds with cusps.

About Chern-Weil type classes on infinite dimensional vector bundles

Sylvie Paycha

Our purpose is to extend Chern-Weil calculus on finite rank vector bundles to a class of infinite rank vector bundles using ζ-regularized traces. In order to circumvent tracial anomalies that arise from the non cyclicity of these traces and from the fact that they do not commute with exterior differentiation, we suggest to work “modulo tracial anomalies”. We investigate in how far the Chern-Weil type forms we obtain this way relate to exotic Chern-Weil forms one can build from the Wodzicki residue. In the light of this infinite dimensional Chern Weil type calculus, we reinterpret results concerning Chern classes in infinite dimensions previously established by other authors in the context of loop groups (investigated by D.Freed) on one hand and in the context of the Bismut-Freed family index theorem on the other hand.

Homology theories of algebras of Whitney functions

Markus Pflaum

(joint work with J.-P. Brasselet)

In this talk we study several homology theories of the algebra of Whitney functions over a subanalytic set with a view towards noncommutative geometry. Using a localization method going back to Teleman we prove a Hochschild–Kostant–Rosenberg type theorem for Whitney functions under certain mild assumptions on the underlying space.

This includes the case of a subanalytic set. We also compute the Hochschild cohomology of the algebra of Whitney functions for a regular set with regularly situated diagonals and derive the cyclic and periodic cyclic theories. It is shown that the periodic cyclic homology coincides with the de Rham cohomology, thus generalizing a result of Feigin–Tsygan. Motivated by the algebraic de Rham theory of Grothendieck we finally prove that for subanalytic sets the de Rham cohomology of the algebra of Whitney functions coincides with the singular cohomology.

For the proof of this result we introduce the notion of a bimeromorphic subanalytic triangulation and show that every bounded subanalytic set admits such a triangulation.

Flat Connections in Mathematical Physics

Steven Rosenberg

Recent work of Connes and Kreimer has uncovered a Hopf algebra structure for controlling divergences in the loop expansions to all orders in perturbative quantum field theory. By dimensional regularization for e.g. ϕ^4 theory, this work produces a family of singular flat holomorphic connections over $\mathbb{P}^1$ from a Birkhoff decompostion depending on the choice of external momenta, coupling constant and scaling parameter. The well-known parameter independence of the pole term in dim reg translates into the isomonodromy of the family of connections. As a result, this family can be studied by Hitchin's development of Malgrange's theory of universal flat connections. It remains to be seen if this family forms a sub-integrable system of the universal family, along the lines of Hitchin's analysis of Frobenius manifolds.

Bundle Gerbes and String Structures

Daniel Stevenson

Given a principal bundle P with structure group $L(G)$, the free loops into a compact Lie group G, one can ask when it is possible to lift the structure group of P to the Kac-Moody group. The obstruction preventing this is a degree three integer cohomology class on M, known as the string class. In this talk I will explain how the string class has a natural interpretation as the Dixmier-Douady class of a certain bundle gerbe associated to P. Using the differential geometry of gerbes we are able to write down an explicit 3-form representing the image of the string class in real cohomology. If time permits I will show how this allows us to give a quick proof of a result of Killingback: in the case where the bundle P is obtained by taking loops on a principal G bundle Q on X, the string class of P is the integration over the circle of the pullback of the first Pontrjagin class of Q by the evaluation map.

A noncommutative index theorem for a manifold with corners

Charlotte Wahl

We transfer the result of Bunke and Koch obtained in "'The η-form and a generalized Maslov index"' to a noncommutative setting:

Let A be a C^*-algebra A and A_∞ a dense subalgebra fulfilling further technical conditions. Let I be an automorphism on A^2 inducing a symplectic structure with respect to the standard scalar product. Let $P_0, P_1 \in M_2(A_\infty)$ be projections onto mutually transverse Lagrangian subspaces. We associate a superconnection to the operator $I\frac{d}{dx}$ acting on $C^\infty([0,1], A^2)$ with boundary conditions $P_0 f(0) = 0$ and $P_1 f(1) = 0$ and use heat kernel methods in order to construct an η-form $\eta(P_0, P_1)$ taking values in a quotient of the universal graded differential algebra $\Omega_*(A_\infty)$. For three mutually transverse projections P_0, P_1, P_2 the sum $\eta(P_0, P_1) + \eta(P_1, P_2) + \eta(P_2, P_0)$ is – up to exact forms – the Chern character of an index of a Dirac operator on a two dimensional manifold with corners. It can be interpreted as a generalisation of the Maslov index associated to P_0, P_1, P_2.

Surgery of ζ–determinants

Krzystof Wojciechowski

In my talk I described my recent result (joint work with J. Park) on the adiabatic decomposition of the ζ-determinant of the Dirac Laplacian. The problem here is the non-locality of the ζ-determinant. This is why we have to involve an adiabatic process, which allows us to split determinant onto the contributions coming from different parts of the manifold. In some (important) cases the adiabatic procedure mentioned above causes the appearance of the small eigenvalues of the Dirac operator and corresponding boundary problems. The main achievement of my work is the analysis of the contribution determined by the "small" eigenvalues or in other words "large" time contribution. This contribution can be described in terms of the scattering matrices determined by the Dirac operator.

The homotopy type of the restricted linear group of a polarized Hilbert space

Tilmann Wurzbacher

Let $K = K_+ \oplus K_-$ be a polarized Hilbert space, G_{res} the associated restricted Grassmannian, and let GL_{res} resp. U_{res} be the associated restricted general linear resp. restricted unitary group. (See, e.g., [PS] or [W] for the precise definitions.) Both groups acting smoothly and transitively on G_{res} with contractible isotropies, the homotopy types of G_{res}, GL_{res} and U_{res} are the same.

The calculation of this homotopy type given in [PS] is not only very geometric, but yields also a direct identification of GL_{res} with $BU(\infty)$, the classifying space of the infinite unitary group $U(\infty) = \cup_{n\geq 1} U(n)$. Unfortunately there seems to be a gap in the crucial proof of their Proposition 6.2.4. Proving first a "Boardman-Vogt" type lemma (on deforming injective linear bounded maps) we give an elementary and direct proof of this proposition.

We also discuss applications of the knowledge of the homotopy and cohomology groups of GL_{res} to second quantization, string structures on loop spaces and characteristic classes.

References.

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