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Computability Theory

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Abstract. Computability and computable enumerability are two of the fundamental notions of mathematics. Interest in effectiveness is already apparent in the famous Hilbert problems, in particular the second and tenth, and in early 20th century work of Dehn, initiating the study of word problems in group theory. The last decade has seen both completely new subareas develop as well as remarkable growth in two-way interactions between classical computability theory and areas of applications. There is also a great deal of work on algorithmic randomness, reverse mathematics, computable analysis, and in computable structure theory/computable model theory.

The goal of this workshop is to bring together researchers representing different aspects of computability theory to discuss recent advances, and to stimulate future work.

Mathematics Subject Classification (2010): 03Dxx; Secondary: 03C57, 68Qxx.

Introduction by the Organisers

Computability theory is one of the main branches of mathematical logic. It explores the computational limitations of mathematics. At the center of this area are notions of relative computation and various induced hierarchies. Classical concepts include the degrees of unsolvability, the arithmetical and analytic hierarchies, and many other methods of calibrating relative complexity. Principal applications have been to algorithmic randomness (e.g., Kolmogorov complexity), mathematical logic, algebra, analysis, and proof theory (such as reverse mathematics).

A number of deep tools have been developed in the area. These include priority methods, effective forcing methods, and sophisticated coding techniques. While there have been some memorable recent results clarifying the pure theory, much of current research is devoted to using these techniques to distill the effective content of and give insight into applications. This activity has given rise to several new international conferences: *Computability and Complexity in Analysis (CCA)* since 1995, *Computability, Complexity and Randomness (CCR)* since 2004, and *Computability in Europe (CiE)* since 2005, which all emphasize particular applications; the CiE meetings also include physicists, biologists, and linguists.

Moreover, there have been several recent major advances, such as the solution of a 50 year-old question on the definability of the total degrees (M. Cai, Ganchev, Lempp, J. Miller, M. Soskova), solutions to longstanding questions on tilings and entropy (Hochman, Meyerovich, 2010), and proofs that torsion-free abelian groups cannot have useful invariants (Downey, Montalbán, 2008).

Here are some of the strands we seek to draw together at this meeting:

Fundamentals. The basic notion of the area is that of a "reduction": $A \leq B$ means that A "is computable" from B. The traditional notion is Turing reducibility, introduced by Turing in 1939. Turing reducibility yields the Turing degrees, used to measure the complexity of unsolvable problems. There has been a great deal of work on the structure of the Turing degrees and its restriction to the computably enumerable degrees. Post had asked whether all noncomputable c.e sets have the same Turing degree. The negative solution by Friedberg and Muchnik in the late 1950's introduced the "priority method", a signature method in the subject. A great deal is known about the Turing degrees and the c.e. degrees, but some fundamental problems remain open, in particular whether there is a nontrivial automorphism, and the related "Bi-Interpretability Conjecture".

There are other degree structures, for example, the *enumeration degrees.* The past few years have seen a great deal of progress in understanding the enumeration degrees and their connection with the Turing degrees. Much of the progress is due to M. Soskova and her collaborators. This work culminated in a result stating that the Turing degrees sit inside the enumeration degrees as a definable subset (M. Cai, Ganchev, Lempp, J. Miller, M. Soskova, 2016).

From these fundamentals we derive various hierarchies which align themselves with *logical definability.* For example, Σ^0_1 means that a problem is essentially some kind of countable computable unbounded search, like the famous Halting Problem; and being Σ^0_1-hard means being as complicated as any other Σ^0_1-problem and corresponds to classical problems like Hilbert's 10th problem. Similarly, Σ^1_1 correlates to a search through all $2^{\aleph_0}$ many functions from $\mathbb{N}$ to $\mathbb{N}$, so being Σ^1_1-hard means that this search cannot be simplified. Isomorphism between countable structures is naturally Σ^1_1, and hence, when it is shown that an isomorphism problem for a class of structures is Σ^1_1-hard, then there cannot be any simplifying invariants, like dimension. Downey and Montalbán used this method to prove that torsion-free abelian groups cannot have any useful classifying invariants.

Inspired by related work in algorithmic randomness, significant portions of recent work have focused on computable approximations of noncomputable objects via "tracing" and limit approximations. For example, the sets that can only compute functions f where we can give a computable set of possibilities as the value for $f(n)$ turn out to be precisely the sets that cannot derandomize a certain class of random reals. A major recent unifying program for approximations was set out by Cholak, Downey and Greenberg.

Algorithmic Randomness. A natural arena for computability theory is the area of algorithmic randomness. This area tries to give meaning to randomness for individual sequences and strings. Typical questions are: When is a real more random than another, what is the computational power of a random real, or sets of random strings, how can we understand "almost everywhere" behavior in mathematics? The hierarchies associated with algorithmic randomness and those of computability theory interrelate. A remarkable example of this stems from work on "K-trivial" sets. This analysis has led to new results on the structure of the c.e. sets, "natural" solutions to Post's problem, new randomness notions (Bienvenu, Greenberg, Kučera, Nies, Turetsky, 2016) and entirely new algorithmic methods.

Analysis and randomness. The early interactions between computability and randomness have developed into widespread applications in computable analysis, ergodic theory, subshifts of finite type, tiling problems and even number theory. Randomness and genericity align themselves to differentiability of effective functions (Brattka, J. Miller, Nies).

Randomness is tied to effective dimension ergodic theory starting with V'yugin's proof (1997) that Martin-Löf randomness suffices for the effective Birkhoff theorem. Work in symbolic dynamics shows close relationships between entropy, effective and classical Hausdorff dimension, and Kolmogorov complexity (Simpson, 2015). A standard notion in randomness is the halting probability, and this has been found to be quite natural and to turn up in places apparently removed from such considerations, e.g., in the classification of dimensions of subshifts of finite type (Hochman, Meyerovich, 2010), and the complexity of Julia Sets (Braverman, Yampolsky, 2006). Classical computable analysis remains a very active field of research, and these new interactions with randomness are invigorating both areas.

Computable model theory. In computable model theory, we consider structures with effective presentations. Typically, we look at the interplay of definability and algorithmic behavior. For example, a sufficiently decidable structure is computably categorical iff it can be "named" by an infinitary computable formula. Many of the original limitations were established by unnatural "pathological" examples, and much recent work seeks to answer what "tame", or "natural", behavior is. The *degree spectrum* of a relation R on a structure $\mathcal{A}$ is the set of Turing degrees of images of R in computable copies of $\mathcal{A}$. There are examples of computable structures with a relation whose degree spectrum is strange. Harrison-Trainor has investigated spectra of relations "on a cone" and showed a number of dichotomies on spectra. Csima and Harrison-Trainor considered degrees of categoricity on a cone. Again, there are examples with strange degrees of categoricity, but on a cone, there is tame behavior, the degree of categoricity is Δ^0_α for some α.

A recurrent idea is that nice model-theoretic properties of the theory should make it easier to understand the complexity of the models. There has been major progress recently, bounding the complexity of countable models of strongly minimal theories (Andrews, Knight), and bounding the possibilities on which countable models of strongly minimal theories are computable (Andrews, Lempp).

Of course, this work is related to computable algebra, where we deal with concrete algebraic structures like groups, rings and fields. Hirschfeldt, Khoussainov, Slinko, and Shore in 2002 gave general conditions on a class of structures that permit effective coding and decoding of graphs. Among the original examples of classes that satisfy the conditions are partial orderings, lattices, rings, integral domains, commutative semigroups and 2-step nilpotent groups. Now, fields have been added fields to this list (R. Miller, Schoutens, 2012; R. Miller, Poonen, Schoutens, Shlapentokh). For the fine structure of *particular* algebraic objects, new methods and complex computability techniques seem to be needed. Recent results include an analysis of the complexity of radicals in rings by Conidis, and the isomorphism problem for completely decomposable groups by Downey and Melnikov. A new line of research is to work in uncountable structures.

Reverse mathematics and proof theory. Reverse mathematics seeks to classify mathematical results according to the proof-theoretical resources needed for them. The techniques of this area and computability theory are close, and there is much cross-fertilization. Initially, the program of reverse mathematics found many important mathematical theorems equivalent to just one of five systems, linearly ordered. More recent work, especially on principles related to combinatorics, has produced a large number of new systems. A new result in this area was announced at the 2012 Oberwolfach meeting by Chong, Slaman and Y. Yang, who gave an amazing proof separating certain variations of Ramsey's Theorem using non-standard models. This technique is certain to yield further results.

Related to this are pre-orderings suggested by Kolmogorov on "problems"; Medvedev and Muchnik, respectively, made this intuition precise as strong and weak reducibilities on "mass problems", i.e., subsets of Baire space or Cantor space. Among the mass problems, we may consider those asking for a copy of a given structure, or certain collections of random sets. In recent years a number of new reducibilities, e.g. Weihrauch reducibility and computable reducibility, emerged that generalize the Medvedev and Muchnik reducibilities in the sense that parameterized "problems" are considered. These reducibilities allow a more resource sensitive and uniform version of reverse mathematics that can be directly approached with computability-theoretic techniques. So far, problems from computable analysis (Brattka, Gherardi, 2011) and combinatorial problems (Dorais, Dzhafarov, Hirst, Mileti, Shafer, 2016) have been classified in this approach.

The workshop is organized by Vasco Brattka, Rodney G. Downey, Julia F. Knight, and Steffen Lempp, and includes 28 talks and two open problem sessions. Some of the open problems discussed are listed in the last section of this report. The slides for the talks can be found on Steffen Lempp's website at `http://www.math.wisc.edu/~lempp/conf/OW18/OW18slides.htm`.

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Workshop: Computability Theory

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Abstracts

Pigeons do not jump high

Ludovic Patey
(joint work with Benoit Monin)

The infinite pigeonhole principle asserts that every set of integers admits an infinite subset in it or its complement. This seemingly trivial principle surprisingly admits highly non-trivial computability-theoretic features, when considering non-computable instances. In particular, one may wonder whether there is an instance of the pigeonhole principle whose solutions are all high. The answer to this question will be given during this talk, although the careful reader might find a subtle clue in the title. This is a joint work with Benoit Monin.

Uniformly multiply permitting c.e. sets and lattice embeddings into the c.e. degrees

Klaus Ambos-Spies
(joint work with Nadine Losert)

We introduce a new permitting property of computably enumerable (c.e.) sets called uniform multiple permitting. This notion is wtt-invariant and the c.e. wtt-degrees which do not contain any sets with this property form an ideal. The Turing degrees of the sets with the uniform multiple property are the not totally ω-c.a. c.e. degrees. So this new property gives an alternative characterization of the permitting power of those degrees. We apply our permitting notion in order to obtain some new results on the not totally ω-c.a. degrees. For instance we show that there are some finite lattices such that a c.e. degree bounds an embedding of such a lattice in the c.e. degrees if and only if the degree is not totally ω-c.a. An example of such a lattice is the 7-element lattice S_7 which is meet-semidistributive but not join-semidistributive.

Computability and incomputability of projection functions in Euclidean space

Alberto Marcone
(joint work with Guido Gherardi)

Projecting a point over a non-empty closed subset of the Euclidean space is deeply grounded in our geometrical intuition of the spatial continuum and has many important applications in several areas of mathematics. We study the complexity of this projection operator (and of a natural approximate version of it) in the Weihrauch lattice. The answer does depend on the way the closed set is given and, in some cases, on the dimension of the Euclidean space.

Let $n \geq 1$ and consider the computable metric space $\mathbb{R}^n$. The *(exact) negative, positive and total projection operators on* $\mathbb{R}^n$ are the partial multi-valued functions $\mathsf{Proj}^-_{\mathbb{R}^n}$, $\mathsf{Proj}^+_{\mathbb{R}^n}$ and $\mathsf{Proj}_{\mathbb{R}^n}$ which associate to every $x \in \mathbb{R}^n$ (with Cauchy representation) and every nonempty closed $A \subseteq \mathbb{R}^n$ (with negative, positive and total representation, respectively) the set

$$\{y \in A \mid d(x, y) = d(x, A)\}.$$

Fix $\varepsilon > 0$. The *ε-approximate negative, positive and total projection operators on* $\mathbb{R}^n$ are the partial multi-valued functions $\varepsilon\text{-}\mathsf{Proj}^-_{\mathbb{R}^n}$, $\varepsilon\text{-}\mathsf{Proj}^+_{\mathbb{R}^n}$ and $\varepsilon\text{-}\mathsf{Proj}_{\mathbb{R}^n}$ which associate to every $x \in \mathbb{R}^n$ (with Cauchy representation) and every nonempty closed $A \subseteq \mathbb{R}^n$ (with negative, positive and total representation, respectively) the set

$$\{y \in A \mid d(x, y) \leq (1 + \varepsilon) d(x, A)\}.$$

Our main results are summarized in the following table:

projection	**representation**	**dimension**	**Weihrauch degree**
Exact	negative	$n = 1$	$>_W \mathsf{lim}$, $>_W \mathsf{BWT}_2$, $<_W \mathsf{BWT}_{\mathbb{R}}$
		$n \geq 2$	$>_W \mathsf{lim}$, $>_W \mathsf{BWT}_2$, $\leq_W \mathsf{BWT}_{\mathbb{R}}$
	positive	$n = 1$	$>_W \mathsf{lim}$, $>_W \mathsf{BWT}_2$, $<_W \mathsf{BWT}_{\mathbb{R}}$
		$n \geq 2$	$\equiv_W \mathsf{BWT}_{\mathbb{R}}$
	total	$n = 1$	$\equiv_W \mathsf{C}_2$
		$n \geq 2$	$\equiv_W \mathsf{C}_{2^{\mathbb{N}}}$
Approximate	negative	$n \geq 1$	$\equiv_W \mathsf{C}_{\mathbb{R}}$
	positive	$n \geq 1$	$\equiv_W \mathsf{Sort}$
	total	$n \geq 1$	computable

Here $\mathsf{Sort} : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ maps p to $0^n 1^\omega$ if $|\{i \mid p(i) = 0\}| = n$ and to 0^ω if p has infinitely many 0's.

Computable paths intersect in a computable point

Klaus Weihrauch

Consider two paths $f, g : [0; 1] \to [0; 1]^2$ on the unit square such that $f(0) = (0, 0)$, $f(1) = (1, 1)$, $g(0) = (0, 1)$, and $g(1) = (1, 0)$. By continuity of f and g there is a point of intersection. We prove that there is a computable point of intersection if the paths are computable.

Almost-total enumeration degrees, graph-cototal enumeration degrees and a theorem by Sierpiński

Arno Pauly

(joint work with Steffen Lempp, Takayuki Kihara, and Keng Meng Ng)

Generalizing an earlier definition by Miller [4], Kihara and Pauly extended the notion of Turing reducibility to points in represented spaces. The collection of degrees realized in a space is called its spectrum, and can be seen as a topological invariant. Many classes of degrees studied in computability theory are precisely the spectrum of certain topological spaces. In particular, there is a tight connection between the study of substructures of the enumeration degrees and second-countable topological spaces. Here, we illustrate this by a particular open question.

Definition. *An enumeration degree A is called* almost total*, if for all total $e \not\leq_e A$ we find that $e \oplus A$ is total.*

Definition. *An enumeration degree A is called* graph-cototal*, if it has a representative of the form* $\mathrm{Graph}(f)^C$ *for a total function $f : \mathbb{N} \to \mathbb{N}$.*

Question 1. *Is every almost-total degree graph-cototal?*

Theorem (AIMS [1])**.** *The almost total degrees are the spectrum of the Hilbert cube* $[0,1]^\omega$*.*

Theorem. *Let* $\mathbb{N}_{\mathrm{cof}}$ *denote the integers with the cofinite topology. Then the graph-cototal degrees are the spectrum of* $(\mathbb{N}_{\mathrm{cof}})^\omega$*.*

Now Question 1 is equivalent to asking whether there exists a countable decomposition of $[0,1]^\omega$ such that each piece embeds into $(\mathbb{N}_{\mathrm{cof}})^\omega$. A classical theorem by Sierpiński implies that no non-trivial compact connected Polish space embeds into $(\mathbb{N}_{\mathrm{cof}})^\omega$. Metric spaces not admitting any compact connected subspaces are called *punctiform*, and we thus arrive at the question whether $[0,1]^\omega$ admits a countable decomposition into punctiform spaces. While $[0,1]^\omega$ does not admit a countable decomposition into zero-dimensional spaces, there actually are punctiform infinite dimensional spaces.

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Characterizing the continuous degrees

Joseph S. Miller

(joint work with Uri Andrews, Greg Igusa, and Mariya Soskova.)

The continuous degrees measure the computability-theoretic content of elements of computable metric spaces. They properly extend the Turing degrees and naturally embed into the enumeration degrees [4]. Although nontotal (i.e., non-Turing) continuous degrees exist, they are all very close to total: joining a continuous degree with a total degree that is not below it always results in a total degree. We call this curious property *almost totality*.

We prove that the almost total degrees coincide with the continuous degrees. Since the total degrees are definable in the partial order of enumeration degrees [1], we see that the continuous degrees are also definable. Applying earlier work on the continuous degrees [4], this shows that the relation "PA above" on the total degrees is definable in the enumeration degrees.

In order to prove that every almost total degree is continuous, we pass through another characterization of the continuous degrees that slightly simplifies one of Kihara and Pauly [3]. Like them, we identify our almost total degree as the degree of a point in a computably regular space with a computable dense sequence, and then we apply the effective version of Urysohn's metrization theorem [5, 2] to reveal our space as a computable metric space.

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Randomness and uniform distribution modulo one

Verónica Becher

(joint work with Serge Grigorieff and Theodore Slaman)

How is algorithmic randomness related to the classical theory of uniform distribution of number theory? In this talk we consider the definition of Martin-Löf randomness for real numbers in terms of uniform distribution of sequences. First, we present a necessary condition for a real number to be Martin-Löf random, in terms of classical uniform distribution. Then, we introduce a notion of uniform distribution relative to the computable enumerable open subsets of $[0, 1]$. Based on this notion we give a sufficient condition for a real number to be Martin-Löf random.

Intuitively, a real number is random if it has the propeties of almost all real numbers, that is, if it belongs to no set of probability zero. The intuition can not be taken literally because every real number belongs to the singleton set containing it, which has probability zero. To prevent the property being void one can restrict to computably defined sets, as done by Martin-Löf in [7]: A Martin-Löf test is a computable sequence $(V_n)_{n\geq 1}$ of computably enumerable open subsets V_n of real numbers with Lebesgue measure less than 2^{-n}. A real x passes the test $(V_n)_{n\geq 1}$ if x is not in the measure zero set given by $\bigcap_{n\geq 1} V_n$. A real x is Martin-Löf random if it passes all Martin-Löf tests. Since there are only countably many of these tests, there are also only countably of these measure zero sets, which implies that almost all real numbers, with respect to Lebesgue measure, are Martin-Löf random. The definition entails that the fractional expansions of Martin-Löf real numbers obey all the usual probability laws. It follows that all the computable real numbers, such as the irrational algebraic numbers and the usual mathematical constants including π and e, are not Martin-Löf random. For a presentation of the theory of randomness see [3].

An infinite sequence $(x_n)_{n\geq 1}$ of real numbers is uniformly distributed modulo 1, abbreviated u.d. mod 1, if the sequence formed by the fractional parts of each term is equidistributed in the unit interval. This means that for each subinterval of the unit interval, asymptotically, the proportion of terms falling within that subinterval is equal to its length. For a real number x, we write $\lfloor x \rfloor$ to denote its integer part and $\{x\} = x - \lfloor x \rfloor$ for its fractional expansion. Thus, a sequence $(x_n)_{n\geq 1}$ of real numbers is u.d. mod 1 if for every half open interval $[a, b)$ in $[0, 1]$,

$$\lim_{N\to\infty} \frac{1}{N} \big\{ n : 1 \leq n \leq N \text{ and } \{x_n\} \in [a, b) \big\} = b - a.$$

A presentation of the theory of uniform distribution can be read from [6, 4, 2].

In the years 1909 and 1910 Bohl, Sierpiński and Weyl independently established that a number x is irrational exactly when the sequence $(n\,x)_{n\geq 1}$ is u.d. mod 1. The subset of the irrational numbers that Borel called absolutely normal [1] also has a characterization in terms of uniform distribution: A real number is absolutely normal exactly when, for every integer b greater than 1, the sequence $(b^n x)_{n\geq 1}$ is u.d. mod 1, see [6, 2]. The Martin-Löf random real numbers are a proper subset of the irrational numbers and also of the absolutely normal numbers, so one can expect to characterize the Martin-Löf random real numbers with a class of sequences that includes $(n\,x)_{n\geq 1}$ and $(b^n x)_{n\geq 1}$, for every integer b greater than 1. We found that the class to be considered is that in Koksma's General Metric theorem [5] but restricted to a suitable computability condition. We call the effective Koksma class $\mathcal{K}$ to the class of computable sequences $(u_n)_{n\geq 1}$ of computable functions from an interval $[a, b]$ on the real numbers satisfying the conditions determined in Koksma's General Metric Theorem. In Theorem 1 we give a necessary condition for Martin-Löf randomness: we show that if a real number x is Martin-Löf random then for every $(u_n)_{n\geq 1}$ in $\mathcal{K}$ the sequence $(u_n(x))_{n\geq 1}$ is u.d. mod 1.

To capture when uniform distribution entails Martin-Löf randomness we need to strengthen the classic definition of uniform distribution. It is possible to replace intervals by Jordan-measurable subsets of the unit interval (see [6, Notes in Chapter 1, Section 1]) but one cannot replace intervals by all Lebesgue measurable subsets. To see why consider any given sequence and consider the set determined by the range of the sequence itself: since the range is a countable set, it has Lebesgue measure zero, but for the first N elements sequence the proportion of elements in the set is equal to 1, for every N. We can prevent the notion to be void if we consider Lebesgue measurable subsets with a computability constraint. With this aim we define a notion of uniform distribution relative to the computably enumerable open subsets of the unit interval. We call the notion Σ^0_1-u.d. Theorem 2 is the main result of this talk and it gives a sufficient condition in terms of Σ^0_1-u.d. for a real number to be Martin-Löf random.

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Rogers Semilattices of Generalized computable numberings

S. S. Goncharov

A comprehensive and extensive study of generalized computable numberings was initiated at the end of the past century, through a unifying approach towards a notion of a computable numbering for a family of sets of constructive objects, suggested in the paper of S.Goncharov and A.Sorbi [1]. A program of such a study was outlined in the authors' paper [2]. The study of ershov hierarchy was started by S.Goncharov and S.Lempp [3] and for analytical hierarchy by Owing [4]. One of interesting question in analytical hierarchy is connected with Friedberg computable numbering. Owing [4] proved that the family of all Σ^1_1 - sets has not Σ^1_1-computable Friedberg numbering but M. Dorzhieva (in print) proved the Owing result without metarecursion and proved that the family of all Σ^1_2 - sets has not Σ^1_2-computable Friedberg numbering. Nevertheless the questions for $n > 2$ are open.

The question for Rogers semilatticas R^0_n of arithmetical Σ^0_n-computable sets and Σ^0_n-computable numberings was solved in [5] and [6], but it is open about non-isomorphism for R^0_n R^0_{n+1} if $n \geq 3$.

It is interesting questions about computable numberings relative to classes of Ershov Hierarchy.

Together with A.Sorbi and S. Badaev was proved the next result.

Theorem There exist the infinite family S of $\Sigma^{-1}_n, n \geq 2$ sets with Rogers semilattes $R^{-1}_n(S)$ of all Σ^{-1}_n(-numberings of the family S with exactly two different elements α and β such that the Σ^{-1}_n-computable numbering α is Friedberg numbering and for any Σ^{-1}_n-computable numbering γ of this femily if α is not equivalent to γ then $\beta \leq \gamma$.

The question is open about finite family with these properties. Some another open question is about the existence of family.

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Who asked *us*? How the theory of computing answers questions that weren't about computing

Jack H. Lutz

It is rare for the theory of computing to be used to answer open mathematical questions whose statements do not involve computation or related aspects of logic. This talk discusses recent developments that do exactly this. After a brief review of algorithmic information and dimension, we describe the point-to-set principle (with N. Lutz [1]) and its application to two new results in geometric measure theory. These are N. Lutz and D. Stull's strengthened lower bounds on the Hausdorff dimensions of generalized Furstenberg sets [2] and N. Lutz's extension of the fractal intersection formulas for Hausdorff dimensions in Euclidean spaces from Borel sets to arbitrary sets [3].

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On building models of Solovay theories

Uri Andrews

If M is a recursive structure, then the $\exists_n$-fragment of the theory of M is uniformly Σ^0_n. In general, we call a theory with this property a Solovay theory. Knight (with some shared credit to Solovay) gave a characterization of precisely the degrees that compute some model of every Solovay theory. It is the same as the degrees that compute a nonstandard model of true arithmetic. One can ask how hard it is to compute every model of a Solovay theory. Of course, no degree does this for every Solovay theory, as some Solovay theories have continuum many models (true arithmetic for example). When we restrict to a nice class $\mathfrak{C}$ of theories which do not have continuum many countable models, this question becomes meaningful and important again. I think of this as asking what understanding we need to have of a theory in $\mathfrak{C}$, beyond the obvious, to know how to build its models. I will discuss results giving the answer for the class of countably categorical theories (due to Knight '94 improving a result of Lerman and Schmerl '79) and recent results about the class of strongly minimal theories.

Topologizing the degree theory

Takayuki Kihara

(joint work with Steffen Lempp, Keng Meng Ng, and Arno Pauly)

I will survey my recent works with collaborators on the topological generalization of the degree theory. My first motivation for topologizing the Turing degree theory came from my attempt to solve the generalized Jayne-Rogers conjecture, one of the most attractive open problems in descriptive set theory. After a few years of work, the topologized Turing degree theory has turned out to have many other applications. This theory has clarified the relationship between the Turing degree theory and infinite dimensional topology (joint with Arno Pauly). This theory has enabled us to utilize nonmetrizable topology to explore the structure of the enumeration degrees (joint with Steffen Lempp, Keng Meng Ng, and Arno Pauly).

In the latter part of this talk, I will focus on a generalization of the truth-table degrees in computable metric spaces, which is recently introduced by McNicholl-Rute. We apply the Jayne-Rogers theorem to characterize the notion of the generalized *tt*-degree in the context of the first-level Borel isomorphism. This characterization is the guidepost which indicates the right way to go. First-level Borel isomorphisms have appeared in several literatures in topological dimension theory. Thus the theory of effective topological dimension naturally emerges. For instance, by the topological method called "condensation of singularities," we show that, for any positive integer n, every weakly 1-generic Turing degree contains a tt-degree which is $(n+1)$-dimensional, but not n-dimensional.

Strong jump traceability and superlow preservation

Keng Meng Ng

Strong jump traceability is a combinatorial notion introduced to understand the relationship between algorithmic randomness and computability. It has shown to be an extremely robust class with many different equivalent definitions from randomness. We show that it can also be characterised from the point of view of classical computability. The related degree theoretic notions come from the ideals generated by considering all Δ^0_2 degrees $\boldsymbol{x}$ such that $\boldsymbol{x} \cup \boldsymbol{z} \in \mathcal{C}$ for every $\boldsymbol{z} \in \mathcal{C}$, where $\mathcal{C}$ is a class of Turing degrees closed downwards under the Turing reducibility.

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When is a property expressed in infinitary logic also pseudo-elementary?

Matthew Harrison-Trainor
(joint work with Barbara Csima and Nancy Day)

It is well-known that many properties, such as the well-foundedness of a linear order or the connectedness of a graph, cannot be expressed in elementary first-order logic. Two natural ways of extending first-order logic are to allow second-order quantifiers and to allow conjunctions and disjunctions over infinite sets. More particularly, a property is pseudo-elementary if it can be expressed using an existential second-order quantifier, and definable in the infinitary logic $\mathcal{L}_{\omega_1\omega}$ if it can be expressed using conjunctions and disjunctions over countable sets of formulas. We ask: what is the intersection of these two extensions of first-order logic? This is joint work with Barbara Csima and Nancy Day.

Some non-elementary properties, such as reachability in a graph, are expressible both in infinitary logic and in a pseudo-elementary (or co-pseudo-elementary) way. On the other hand, some properties, such as well-foundedness, are expressible in only one of these ways. Our main result is that the properties which are expressible both in a pseudo-elementary way and using infinitary logic are exactly the properties expressible in infinitary logic using countable conjunctions but only finite disjunctions (after putting the formula in normal form). The proof in one direction passes through a version of Craig's Interpolation Theorem for infinitary logic, and in the other direction by expanding a structure by adding a (possible non-standard) model of arithmetic which can talk about computable infinitary sentences.

Cardinal characteristics, highness classes, Weihrauch reducibility, and forcing

Noam Greenberg
(joint work with Rutger Kuyper and Dan Turetsky)

Vojtaš showed that many cardinal characteristics of the continuum, including all those appearing in the Cichon diagram, arise from binary relations considered as problems, with the cardinal being the smallest size of a set which includes solutions for all instances. ZFC-provable cardinal inequalities arise from morphisms between these binary relations.

In his thesis, Rupprecht introduced highness classes in computability which correspond to these cardinals by considering the very same binary relations (see also Brendle, Brooke-Taylor, Nies and Ng). He showed that computable morphisms prove implications between these highness classes. The morphisms are (not necessarily uniform) strong Weihrauch reductions.

Some implications in both set theory and computability require not only morphisms but manipulations of problems, operations which have been studies in both the Weihrauch lattice and in set theory. We discuss this correspondence, how it relates to several results in algorithmic randomness, and how to calibrate computational strength by working over ideals.

Effective Classification and Measure for Countable Structures

Russell Miller
(joint work with Johanna Franklin)

Following work in model theory and descriptive set theory, we consider the class of all atomic diagrams of structures with domain ω belonging to a given class $\mathcal{C}$ of structures. The primary example is the class Alg^* of atomic diagrams of algebraic fields of characteristic 0 (equivalently, subfields of the algebraic closure $\overline{\mathbb{Q}}$). As a subset of Cantor space, this class has the subspace topology. We can then mod out by the relation of isomorphism, yielding the quotient topological space $Alg^*/\cong$. This space, and similar spaces built the same way from other classes of countable structures, are the focus of this talk.

The asterisk in Alg^* denotes that we have expanded the language of fields before executing the above steps. If we had used the usual signature $(+, \cdot, 0, 1)$ for fields, the resulting space $Alg/\cong$ is distinctly unfamiliar to most mathematicians: it is not Hausdorff, and in fact it has one single element which lies in every nonempty open set, and another which lies in no open set except the entire space. In contrast, we add d-ary *root predicates* R_d to the signature, for all $d > 1$, defined by:

$$R_x(a_0, \ldots, a_{d-1}) \iff (\exists x) x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0 = 0.$$

In this signature, the space $Alg^*/\cong$ turns out to be computably homeomorphic to Cantor space 2^ω, with the bits of an element of 2^ω corresponding to the choices made by the field about which polynomials have roots in the field. The signature with these root predicates is the "sweet spot" for this classification: if one expands the signature further, the topology changes further, and the space of indices for algebraic fields becomes a refinement of Cantor space, once again not nearly as recognizable or familiar as 2^ω itself.

The computable homeomorphism allows us to transfer the usual Lebesgue measure from Cantor space onto the space $Alg^*/\cong$. However, the resulting measure on $Alg^*/\cong$ does depend on the choice of the computable sequence of polynomials used to describe the homeomorphism. An alternative measure, *Haar-compatible measure*, has a similar definition, but refined in such a way that, for each Galois extension K of $\mathbb{Q}$ of (finite) degree d, the class of those fields containing K has Haar-compatible measure $\frac{1}{d}$. (For finite non-normal fields K, the measure of this class remains dependent on the choice of the sequence of polynomials.) Since $\frac{1}{d}$ is exactly the Haar measure of the pointwise stabilizer (within $\mathrm{Aut}(\mathbb{Q})$) of a Galois extension K of degree d, we view this measure as being compatible with the Haar measure on the compact group $\mathrm{Aut}(\mathbb{Q})$, justifying its name.

These new measures allow us to pose questions about the prevalence of specific properties of algebraic fields. It is readily seen that the property of being normal has measure 0, as does the property of having relatively intrinsically computable root set (in the signature without the root predicates). A more involved proof shows that the property of uniform computable categoricity has measure 1. When the root predicates are included in the language, this is not so surprising, and indeed in this situation a single Turing functional Φ shows measure-1-many of these fields to be uniformly computably categorical. It is still true, but more surprising, that even in the original language of fields, the property of uniform computable categoricity still has full measure. The result is not quite as strong here, though, as no single functional suffices for that many fields. One proves the theorem by fixing a rational $\epsilon > 0$ in advance and giving a functional under which all but ϵ-many fields are uniformly computably categorical.

In joint work with Franklin, we have investigated notions of randomness for algebraic fields, using the computable homeomorphism given above and considering a field to be random just if the corresponding real is random. Using this notion, the proof for computable categoricity shows that all Schnorr-random fields are uniformly computably categorical, but we can produce a Kurtz-random field which is not uniformly computably categorical. The notion of randomness defined here appears to be compatible with Khoussainov's concept of a random structure, as in [2], and merits further investigation along these lines.

Finally, we have investigated other classes of structures along these same lines. Of course, non-smooth classes such as graphs will not yield to our methods, as the isomorphism problem for graphs is Σ_1^1-complete; the same holds for other classes of structures complete in this sense, as found in [1, 4]. Therefore, we have restricted ourselves to classes for which the isomorphism problem is relatively simple. For finite-branching (infinite) trees, it is natural to adjoin unary *branching predicates* stating the number of immediate successors of a given node in the tree. In this signature, the class is computably homeomorphic to Baire space ω^ω. (If we restricted binary-branching trees, then in this signature it would be homeomorphic to 2^ω instead.) This offers several possibilities for measures to be placed on the class, and Franklin and Miller have investigated some of these possibilities. The question of computable categoricity for trees is more involved, but Franklin and Miller conjecture that (in the original signature, without branching predicates) measure-0-many of these trees are uniformly computably categorical; however, if one uses instead the class of finite-branching trees with no terminal nodes, then measure-1-many of the trees are uniformly computably categorical.

The class of all subrings of $\mathbb{Q}$ is a natural and simple example of this process: it becomes homeomorphic to 2^ω in the signature including a unary predicate for invertibility. Some further work is under way on torsion-free abelian groups of finite rank, and on archimedean real closed fields; parts of this work are joint with Fokina, Friedman, Rossegger, and San Mauro. In some cases, the classification may be not by Cantor space or Baire space themselves, but by a quotient of one of those spaces, using standard Borel equivalence relations. Questions of measure and randomness are then likely to turn on whether the individual equivalence classes all have measure 0, and on whether these equivalence relations respect the notion of randomness involved.

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Exponents of irrationality and transcendence and effective Hausdorff dimension

Theodore A. Slaman
(joint work with Verónica Becher, Jan Reimann)

We discuss the similarities between measuring the describability of a real number in terms of Diophantine Approximation or in terms of Kolmogorov Complexity.

For a real number ξ, the *irrationality exponent of* ξ is the least upper bound of the set of real numbers z such that

$$0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{q^z}$$

is satisfied by an infinite number of integer pairs (p, q) with $q > 0$. It provides a quantitative measure of how well ξ can be approximated by rational numbers.

To motivate our discussion, we may define the *incomputability exponent* of ξ is the least upper bound of the set of real numbers z such that

$$0 < |\xi - R_e| < \frac{1}{e^z}$$

is satisfied by an infinite number of integers e, where R_e is the real number computed by the eth program (in a universal computable enumeration of such.) We show that for a real number ξ, the effective Hausdorff dimension of ξ is equal to the reciprocal of its incomputability exponent, where effective Hausdorff dimension is meant in the sense introduced by Lutz in the context of algorithmic information theory and effective randomness.

The connection between the two notions is as follows. For every $a \geq 2$ and every b in $[0, 2/a]$, there is a real number ξ such that ξ has irrationality exponent a and effective Hausdorff dimension b.

We explore further connections between transcendental number theory and computability theory, such as investigating the Fourier dimension of sets that arise in the two fields and investigating the properties of a random real in a set of positive Fourier dimension.

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Randomness relative to an enumeration oracle

Mariya Soskova
(joint work with Joe Miller)

We initiate the study of algorithmic randomness relative to an enumeration oracle. The enumeration degrees can be viewed as an extension of the Turing degrees: the substructure of the total enumeration degrees is an isomorphic copy of the Turing degrees within the wider context of the enumeration degrees. In this sense, the notion of randomness for total enumeration oracles can be inferred from the corresponding notion for Turing oracles. There are two distinct approaches to extending this notion to non-total enumeration degrees. The first one is to define randomness relative to a non-total enumeration oracle in terms of the total enumeration oracles that are comparable with it.

On the other hand, most of the notions in algorithmic randomness, and all of those most closely related to Martin-Löf randomness, can be expressed in terms of c.e. sets. So when we relativize these notions to an oracle X, we are usually interested in the X-c.e. sets. The second approach to generalizing from Turing degrees to enumeration degrees is straightforward: to relativize a randomness notion to the enumeration degree of a set A we simply replace X-c.e. with c.e. in every enumeration of A (i.e., enumeration reducible to A). The resulting definitions are easily seen to be unchanged for the total degrees, demonstrating that this notion of randomness does lead to an extension of the original one. We show that there are oracles for which this notion does not coincide with the notions obtained via our first approach. Thus moving to the context of the enumeration degrees gives rise to a notion of relative randomness that does not reduce to one expressible in the Turing world by relativizing to sets of oracles. For non-total degrees, we find that some familiar theorems are preserved, perhaps with more subtle proofs, while other theorems may fail. For example, there need not be a universal Martin-Löf test relative to the enumeration degree of A, and there are continuum many enumeration degrees that are low for randomness. This is joint work with Joe Miller.

On low for speed sets

Laurent Bienvenu
(joint work with Rod Downey)

Relativizing computations of Turing machines to an oracle is a central concept in the theory of computation, both in complexity theory and in computability theory(!). Inspired by lowness notions from computability theory, Allender introduced the concept of "low for speed" oracles. An oracle A is low for speed if relativizing to A has essentially no effect on computational complexity, meaning that if a decidable language can be decided in time $f(n)$ with access to oracle A, then it can be decided in time $poly(f(n))$ without any oracle. The existence of non-computable such A's was later proven by Bayer and Slaman, who even constructed a computably enumerable one, and exhibited a number of properties of these oracles as well as interesting connections with computability theory. In this paper, we pursue this line of research, answering the questions left by Bayer and Slaman and give further evidence that the structure of the class of low for speed oracles is a very rich one.

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Partial results on the complexity of roots of polynomials over Hahn fields and Puiseux fields

Reed Solomon
(joint work with Julia Knight and Karen Lange)

Let K be an algebraically closed field of characteristic zero and G be a divisible ordered abelian group. The associated Hahn field $K((G))$ and the field of Puiseux series $K\{\{t\}\}$ are both algebraically closed. We consider the complexity of finding roots in these fields when K and G are countable. In [1], Knight and Lange provided bounds on the length of a root for a polynomial over $K((G))$ in terms of the length of the coefficients of the polynomial. For the field of Puiseux series, we give preliminary results on the complexity of finding roots in computability theoretic terms. For the Hahn field, we give an upper bound for the complexity of computing initial segments of a root and show that in some low level cases, this bound is sharp. We end with some open questions including whether these bounds are sharp for all initial segments.

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A point-to-set principle for separable metric spaces

Elvira Mayordomo

J. Lutz and N. Lutz (2017) have recently proven a point-to-set principle for Euclidean and Cantor spaces. This result is a characterization of classical Hausdorff dimension in terms of relativized effective dimension. This implies that geometric measure results regarding Hausdorff dimension can be shown using only effective methods. Several interesting classical results have already been proven using this principle (N. Lutz and Stull 2017, N. Lutz 2017).

We present here a point-to-set principle for any separable metric space. We will first introduce an effectivization of dimension in terms of Kolmogorov complexity that is valid for any separable space, and then show that the classical Hausdorff dimension of any set is the minimum on all oracles of the relativized effective dimension. We expect that better Hausdorff dimension bounds will be proven as consequences of this theorem.

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Algorithmic learning of probability distributions from random data in the limit

George Barmpalias
(joint work with Frank Stephan and Nan Fang)

We study the problem of identifying a probability distribution for some given randomly sampled data in the limit, in the context of algorithmic learning theory as proposed recently by Vinanyi and Chater [5]. We show that there exists a computable partial learner for the computable probability measures, while by Bienvenu, Monin and Shen [4] it is known that there is no computable learner for the computable probability measures. We characterize of the oracles that compute explanatory learners for the computable (continuous) probability measures as the high oracles. This provides an analogue of a well-known result of Adleman and Blum [1] in the context of learning computable probability distributions. We prove that for certain families of probability measures that are nicely parametrized by reals, learnability of a subclass of probability measures is equivalent to learnability of the class of the corresponding real parameters. This equivalence allows to transfer results from classical algorithmic theory to learning theory of probability measures. We present a number of such applications, providing new results regarding EX and BC learnability of classes of measures.

This is joint work with Frank Stephan [2] and Nan Fang [3].

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Finite final segments of the d.c.e. degrees

Steffen Lempp
(joint work with Yiqun Liu, Yong Liu, Selwyn Ng, Cheng Peng, Guohua Wu, and Yue Yang)

Ever since the existence of a maximal incomplete d.c.e. degree (i.e., a Turing degree containing the difference of two c.e. sets) was established [1, 2], the question of exactly which finite final segments exist in the d.c.e. degrees has been an interesting open question.

In this talk, I will report on progress toward establishing that all finite distributive lattices are final segments. some partial results have been established, but challenges lie ahead for the full result, and there is a larger class, the so-called finite interval dismantlable lattices, which may be realized using the same techniques. More specifically, we know that the finite Boolean algebras and the 3-element chain can be realized, and we have ideas how to generalize our techniques toward all finite distributive lattices.

This is joint work with Yiqun Liu, Yong Liu, Selwyn Ng, Cheng Peng, Guohua Wu and Yue Yang, all from Singapore.

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Martingales and Restricted Ratio Betting

Satyadev Nandakumar
(joint work with Sumedh Masulkar and Keng-Meng Ng)

Let A and B be two finite sets of computable real numbers which denote the allowable *wagers* that martingales can make. Following the terminology in Chalcraft et. al., an A-martingale is a martingale whose wagers are limited to elements in A, and a B-martingale has wagers limited to elements in B. In Chalcraft et. al., the authors establish necessary and sufficient conditions for some A-martingale to succeed betting on sequences that B-martingales can succeed betting on.

In this paper, we investigate the analogous question of comparative betting power of martingales when the *ratios* of bets are restricted to a finite set which excludes 1. This contrasts with the setting of simple martingales and almost simple martingales as investigated by Ambos-Spies and Mayordomo. Without loss of generality, we will restrict the ratios to rational numbers, and the general case of finite sets of computable real ratios is similar. We derive necessary and sufficient conditions for deciding when a set of ratios allows greater power in betting as compared to another.

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Determined Borel codes in Reverse Math

Linda Brown Westrick

(joint work with Eric Astor, Damir Dzhafarov, Antonio Montalbán and Reed Solomon)

The standard definition of a Borel code in reverse math [3] doesn't require the model to believe that each real is either in the coded set or in its complement. In fact, the statement "for every Borel coded set, either it or its complement is non-empty" already implies ATR_0 [1]. We define a *determined Borel code* to be a Borel code with the property that every real is contained either in the coded set or in its complement. Then we consider the following statement.

Definition. *Let* DPB *be the statement "Every determined Borel set has the property of Baire."*

How to formalize "having the property of Baire" can be found in [1]. Both DPB and the statement "every Borel set has the property of Baire" follow from ATR_0 over RCA_0 [1]. The latter is equivalent to ATR_0 due to the technicality mentioned above [1], but here we show that DPB is a strictly weaker theorem. First we show that any ω-model of DPB must be closed under hyperarithmetic reduction. This leads to the question of whether DPB could reverse to ATR_0, or perhaps be a statement of hyperarithmetic analysis (see [2]). Neither is true:

Theorem. DPB *does not hold in* HYP.

Theorem. *There is an* ω*-model of* DPB *that does not satisfy* ATR_0.

The ω-model constructed above is made by adding many hyperarithmetic generics to HYP. We show it is necessary to include such elements in any ω-model.

Theorem. DPB *implies the existence of hyperarithmetic generics in* ω*-models.*

The theorem can be stated more formally as follows: If $\mathcal{I}$ is a HYP ideal which satisfies DPB, then for every $X \in \mathcal{I}$, there is a $G \in \mathcal{I}$ such that G is Δ^1_1-generic relative to X.

Both Theorem 1 and the stronger Theorem 3 are proved using overflow methods to construct ill-founded Borel codes which seem well-founded and determined in a given model.

Many questions remain concerning the exact strength of DPB. One could also consider determined versions of other theorems about Borel sets.

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All things Diophantine: stability, definability and undecidability

Alexandra Shlapentokh

We discuss the current state of affairs with respect to using elliptic curves (and abelian varieties in general) to show first-order and Diophantine definability of $\mathbb{Z}$ over subrings of algebraic extensions of $\mathbb{Q}$, proving that thew first-order or existential theory of these rings is undecidable.

Algorithmic Randomness For Amenable Groups

Adam R. Day

We develop the theory of algorithmic randomness for the space A^G where A is a finite alphabet and G is a computable amenable group. We give an effective version of the Shannon-McMillan-Breiman theorem in this setting. We also extend a result of Simpson equating topological entropy and Hausdorff dimension. This proof makes use of work of Ornstein and Weiss which we also present.

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On finitely presented expansions of semigroups, groups, and algebras

Bakhadyr Khoussainov

Finitely presented structures, such as groups and semigroups, are of foundational interest in algebra and computation. Finitely presented structures necessarily have c.e. word equality and these systems are finitely generated. Not all c.e. and finitely generated structures are finitely presented. This work is concerned with finding finitely presented expansions of finitely generated structures. Expansions of structures, such as turning groups into rings or distinguishing subsets in the underlying structures, is an important method used in algebra, model theory, and various areas of computer science. Bergstra and Tucker [1] [2] proved that all c.e. algebraic systems with decidable word problem have finitely presented expansions. Then they [1] [2] and, independently, [3] asked if every finitely generated c.e. algebraic system has a finitely presented expansion. We build examples of finitely generated c.e. semigroups, algebras (rings that are vectors spaces over fields), and groups that fail to possess finitely presented expansions. We also construct examples of a residually finite and immune group, answering the question of Miasnikov and Osin from [4]. Our proofs are based on the interplay between key constructions, concepts, and results from computability theory (simple set constructions), universal algebra (residually finite structures), and classical algebra (Golod- Shafaverevich theorem [7]). The work is joint with D. Hirschfeldt [5], and independently with A. Miyasnikov [6].

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Existence of fixed points for monotone operators

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(joint work with Sean Walsh)

We study the logical strength of the existence of fixed points for monotone operators. It is easy to see that repeatedly applying such an operator to the empty set must result in a fixed point, and in fact, this will be the least fixed point under inclusion. The strength of the existence of least fixed points was analyzed already by Cenzer and Remmel [1], but the situation for arbitrary fixed points, as opposed to least, is quite different. For example, Cenzer and Remmel show that the existence of least fixed points for computable monotone operators is equivalent to ACA_0, while we show that the existence of fixed points in general is equivalent to WKL_0. We study the existence of fixed points for various other classes of operators. Part of our motivation is understanding the complexity of Kripke's construction of a language (in three-valued logic) interpreting its own truth predicate [3], which is obtained as a fixed point for a particular monotone operator. In particular, we formalize Kripke's proof in Π^1_1-CA_0, and also establish preliminary lower bounds on its strength. Finally, we turn to formal theories of truth, and the conservation results of Feferman [2] for the theory KF. The original proofs of these results proceed via ordinal analysis, but we find comparatively simpler proofs using model-extension methods familiar from the study of subsystems of second-order arithmetic.

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Open Problems

Veronica Becher, Theodore Slaman, Carl Jockusch, Julia Knight, Steffen Lempp, Andre Nies, Arno Pauly, Liang Yu

In this section, we list some of the problems discussed during the open problem sessions.

(1) (Veronica Becher and Theodore Slaman) First we recall the classical definition of discrepancy of a sequence of real numbers. If x is a real number, we write $\{x\}$ to denote $x - \lfloor x \rfloor$.

Definition. *For a sequence $(x_n)_{n\geq 1}$ of real numbers the discrepancy of its first N elements is*

$$D_N((x_n)_{n\geq 1}) = \sup_{0\leq a<b<1} \left| \frac{1}{N} \# \Big\{ n : 1 \leq n \leq N \text{ and } \{x_n\} \in [ab) \Big\} - (b-a) \right| .$$

Question 1. *Is there a Martin-Löf random real x such that for every N large enough $D_N((2^n x)_{n\geq 1})$ is $O((\log N)/N)$?*

An answer to this question will also answer the longstanding open question about the existence of a normal number with this low discrepancy, asked first by Korobov in 1956 [3].

(2) (Carl Jockusch) Let $\mathrm{HT}_k^{=n}$ be Hindman's Theorem for k colors, restricted to sums of length exactly n. Thus, $\mathrm{HT}_k^{=n}$ asserts that whenever the natural numbers are colored with k colors there is an infinite set H such that all sums of n distinct elements of H have the same color. It follows immediately from my 1972 paper "Ramsey's Theorem and Recursion Theory" in the JSL that each computable instance of $\mathrm{HT}_k^{=n}$ has a Π_n^0 solution. This is known to be best possible for $n = 2$, in the sense that there is a computable instance of $\mathrm{HT}_2^{=2}$ with no Σ_2^0 solution, by work of Csima, Dzhafarov, Hirschfeldt, Solomon, Westrick, and Jockusch (in preparation). The open question is to determine for what values of n there is a computable instance of $\mathrm{HT}_2^{=n}$ with no Σ_n^0 solution. This is open even for $n = 3$. If a computable instance of $\mathrm{HT}_2^{=n}$ with no Σ_n^0 solution exists for infinitely many n, it follows on general grounds that Hindman's Theorem for two colors is not provable in ACA_0. This would resolve a long-standing open question of Blass, Hirst, and Simpson in their 1987 paper "Logical analysis of some theorems in combinatorics and topological dynamics" in the book *Logic and Combinatorics*, published by the AMS. For further information, see the paper "Effectiveness of Hindman's Theorem for bounded sums" by Dzhafarov, Solomon, Westrick, and Jockusch, pages 134–142 in the volume "Computability and Complexity", LNCS 10010 (the Rod Downey Festschrift).

(3) (Julia Knight) Is there a computable finitely presented group with no d-Σ_2 Scott sentence?

The complexity of an optimal Scott sentence is a measure of the internal complexity of a countable structure. For a finitely generated group, there is always a Scott sentence that is Σ_3, and if the group is computable, then the Scott sentence can be taken to be computable Σ_3. Often there is a simpler Scott sentence. It had been conjectured that every finitely generated group has a Scott sentence that is d-Σ_2, and that every computable finitely generated group has a Scott sentence that is computable d-Σ_2. Harrison-Trainor and Ho disproved both conjectures, constructing a computable finitely presented group with no d-Σ_2 Scott sentence. Their group is not finitely presented.

(4) (Julia Knight) Gromov introduced an informal notion of "random" group. Gromov's idea been made precise in several different ways. Using the approach of Kapovich and Schupp, we make an off-hand comment of Benjamin Fine, that in the limiting density sense, all groups look free, into a conjecture. Let $G(n, s)$ be the set of group presentations with n generators and a single relator of length at most s. For a sentence φ in the language of groups, let $P(n, s, \varphi)$ be the proportion of presentations in $G(n, s)$ such that the resulting group satisfies φ.

Conjecture. Let $n \geq 2$. For each sentence φ, the limiting density $lim_{s\to\infty} P(n, s, \varphi)$ exists, always with value 0 or 1. Moreover, the set T of sentences for which the limiting density is 1 is the theory of the non-Abelian free groups.

(5) (Steffen Lempp) In the 1970's, Ershov asked for which finite families P and Q of c.e. sets, the Rogers semilattices of P and Q are isomorphic. (Recall that the Rogers semilattice of a family of c.e. sets is simply the collection of uniformly c.e. enumerations of the family, factored by interreducibility.)

There are essentially only two known theorems about this question: In 1978, Denisov showed that for any $n > 0$, the families $\{\emptyset, \{0\}\}$ and $\{\emptyset, \{0\}, \ldots, \{n\}\}$ are isomorphic. And in 2003, Ershov showed that if the Rogers semilattices of P and Q are isomorphic, then P and Q are isomorphic partial orders under set inclusion once the maximal elements in each are removed. The conjecture is that the converse of Ershov's theorem holds, but this appears to be a very hard question.

(6) (Andre Nies) One says that a K-trivial set A is *smart* if every ML-random that computes A computes all the K-trivials. Show that being smart is an arithmetical property. Reference: Greenberg, Miller, Nies and Turetsky, Martin-Loef reducibility and cost functions, arxiv.org/abs/1707.00258.

(7) (Andre Nies) For an order function h, let $IOE(h)$ denote the mass problem of functions that equal infinitely often each computable function bounded by h. It is not hard to show that $IOE(h(n))$ is Muchnik equivalent to $IOE(n \to h(2n))$. Give an example of an h where this cannot be improved to Medvedev equivalence. Reference: Monin and Nies, Muchnik degrees and cardinal characteristics, arXiv:1712.00864.

(8) (Arno Pauly) This is a question about Weihrauch degrees, see [1] for context. Let $\mathrm{UC}_{\mathbb{N}^\mathbb{N}}$ denote the task *Given a countably branching tree with exactly one infinite path, find that path.* Let $\mathrm{List}_{2^\mathbb{N}}$ denote the task *Given a binary tree with countably many infinite path, produce a list comprising exactly those.* We know that $\mathrm{List}_{2^\mathbb{N}} \leq_{\mathrm{W}} \mathrm{UC}_{\mathbb{N}^\mathbb{N}}$.

Question 2. *Is the reduction* $\mathrm{List}_{2^\mathbb{N}} \leq_{\mathrm{W}} \mathrm{UC}_{\mathbb{N}^\mathbb{N}}$ *strict?*

The question was answered in the affirmative during the workshop by Takayuki Kihara.

(9) (Arno Pauly) It is a classic result in topology that compact Hausdorff spaces are regular. However, the classic proof does not effectivize directly. We thus ask:

Question 3. *Are computably compact computably Hausdorff spaces also computably regular?*

For countably-based spaces, an easy direct argument establishes that the answer is yes. For context, see [4] and [2].

(10) (Liang Yu) Wang, Wu, and Yu proved that there exists a model of ZF in which there exists a cofinal chain of Turing degrees of order type ω_1 but there is no a well ordering of reals. Also $ZFC + CH$ implies that there exists a cofinal maximal chain of Turing degrees of order type ω_1. My question is:

Does there exist a model of ZF in which there exists a cofinal maximal chain of Turing degrees of order type ω_1 but there is no a well ordering of reals?

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