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Representations of Finite Groups

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Abstract. The workshop *Representations of Finite Groups* was organised by Olivier Dudas (Marseille), Meinolf Geck (Stuttgart), Radha Kessar (Manchester), and Gabriel Navarro (Valencia). It covered a wide variety of aspects of the representation theory of finite groups and related topics, and showcased several recent breakthrough results.

Mathematics Subject Classification (2020): 20Cxx.

Introduction by the Organizers

The workshop Representations of Finite Groups was organised by Olivier Dudas (Marseille), Meinolf Geck (Stuttgart), Radha Kessar (Manchester), and Gabriel Navarro (Valencia). It was attended by 48 participants (and 3 remote) with broad geographic representation. It covered a wide variety of aspects of the representation theory of finite groups and related topics, notably algebraic groups.

There were 26 lectures, either 30 or 50 minutes long, in which recent progress, connections to related areas and emerging new directions of research were presented. Besides the lectures, there was plenty of time for informal discussions among participants; furthermore, two evening sessions provided an informal forum for presenting open research problems.

Among other highlights, this meeting will be remembered as the occasion on which the solution to Brauer's Height Zero Conjecture was presented by G. Malle. Brauer's conjecture has been with us since 1955, and several of our meetings have reported important progress in this area. Now, thanks to a new approach, the

conjecture has become a theorem (by Malle, Navarro, Schaeffer-Fry, and Tiep), completing the work of decades of many mathematicians.

Another driving force in our field is the McKay conjecture and its generalizations, which need advancing our understanding of groups of Lie type and Deligne–Lusztig theory. Important progress in this direction was presented by G. Lusztig himself, describing a new relation between his work on strata in reductive groups and his theory of character sheaves. He was preceded by a talk from J. Hetz, a very recent PhD, who removed the last ambiguity in the (generalised) Springer correspondence. C. Bonnafé stated far reaching conjectures on the unipotent characters and character sheaves based on actions of braid groups and J. Michel gave new insight on Lusztig's Fourier transform.

The meeting was opened by A. Schaeffer-Fry, who in joint work with L. Ruhstorfer, also a recent PhD, announced the solution of the Galois–McKay conjecture for the prime 2. One day later, Ruhstorfer himself announced that groups of Lie types B and C satisfy the Alperin–McKay conjecture, bringing us one step closer to a solution of this fundamental conjecture.

Regarding the Galois version of the Alperin-McKay conjecture, C. Vallejo established the fields of values of the 2-height zero characters, providing a surprising new consequence of and support for the conjecture. Finally, regarding the so-called "counting conjectures," J. Semeraro, in joint work with Kessar and Malle, presented strong evidence that the celebrated Alperin Weight Conjecture can hold more generally, even when finite groups are not present! This re-launches the Spetses program started 30 years ago.

In the spirit of M. Broué's pioneering approach of finding structural explanations for all these counting conjectures, S. Bouc showed us how to study p-permutation equivalences in families, that is as objects in a well-behaved category. R. Boltje closed our meeting by explaining how such equivalences encode some of the numerical coincidences we have observed in the past years.

Several outstanding problems in our field were also discussed. L. Margolis provided the state of the art of the celebrated counterexample for the prime $p = 2$ on the Modular Isomorphism Problem and the hopes for finding a p odd one. On the other hand, P. Tiep announced advances on a long-open conjecture by J. Thompson on simple groups and their generation as the square of a conjugacy class. C. Eaton reported on the latest developments on the Donovan conjecture, bringing together years of work of several participants of the workshop, including S. Koshitani whose talk focused on the prime $p = 2$. A breakthrough result is the reduction theorem for the conjecture in the abelian defect group case, obtained by Eaton and Livesey.

On cohomology, M. Linckelmann presented very recent results and conjectures obtained for the symmetric groups. Modules with finitely generated cohomology where at the heart of D. Benson's talk, who gave new perspectives on how to characterize them. In the case of groups of Lie type, J. Grodal explained how

string topology can be used to relate their cohomology with that of the corresponding complex loop groups, leading to new evidence for the genericity of their representation theory.

Representation theory also governs the structure of finite groups. H. P. Tong-Viet gave interesting variations of an old theorem of Baer–Suzuki. G. Robinson provided bounds on the index of abelian normal subgroups of finite subgroups of $\mathrm{GL}(n, \mathbb{C})$ with restricted composition factors. This last result illustrates the impact of our community in other branches of mathematics, as it was motivated by problems encountered by Coulembier–Ostrik–Etingof to adapt Deligne's work on symmetric tensor categories to positive characteristic. In the same spirit, G. Hiss explained how one could answer a problem on fixed point theory on manifolds thanks to our knowledge on finite groups of Lie type.

N. Mazza reported on the state of the art of the ambitious programme, joint with Carlson–Grodal–Nakanao, of describing the structure of the group of endotrivial modules of finite groups of Lie type. C. Lassueur gave an overview of a series of papers with G. Hiss on the classification of trivial source modules in cyclic blocks. J. Murray and B. Sambale's talks presented new results on duality in character theory. In a similar, though more combinatorial vein, M. Fayers proposed interesting new approaches to the Mullineux involution on the linear and projective modular characters of symmetric groups.

For this edition, we organized a late session of open problems which aroused great interest among the participants, so much so that we had to organize a second one. Participants were able to freely state nascent questions that we hope will spark research in the area for many years.

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Workshop: Representations of Finite Groups

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Abstracts

Navarro's Galois-McKay Conjecture for the Prime 2

Mandi A. Schaeffer Fry
(joint work with Lucas Ruhstorfer)

Let p be a prime, G a finite group, and P a Sylow p-subgroup of G. The long-standing McKay conjecture posits that the size of the set of irreducible ordinary characters of G with degree relatively prime to p, $\mathrm{Irr}_{p'}(G)$, should be the same as the size of the corresponding set, $\mathrm{Irr}_{p'}(N_G(P))$, for the normalizer of P. The McKay–Navarro conjecture (sometimes also called the Galois-McKay conjecture), a refinement due to G. Navarro in [10], says that there should further be a bijection between $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}_{p'}(N_G(P))$ that commutes with the action of a certain subgroup $\mathcal{H}$ of the absolute Galois group $\mathcal{G} = \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$. Namely, $\mathcal{H}$ is comprised of the Galois automorphisms that map all p'-roots of unity to a given p-power of themselves. (Equivalently, this is saying that the bijection should commute with the action of the absolute Galois group over the p-adic field $\mathbb{Q}_p$.)

The ordinary McKay conjecture was reduced in [2] by Isaacs–Malle–Navarro to proving certain "inductive McKay conditions" for every finite nonabelian simple group. In particular, the conditions require that for every quasisimple group G, there exists some $\mathrm{Aut}(G)_P$-stable $N_G(P) \leq M < G$ and a bijection

$$\Omega\colon \mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(M)$$

that is $\mathrm{Aut(G)}_P$-equivariant and satisfies several other strong properties.

Here at the MFO in 2014, G. Malle and B. Späth announced the proof of the ordinary McKay conjecture for the prime $p = 2$ (see [8]) as a consequence of proving these inductive conditions. In particular, their work completes the desired $\mathrm{Aut(G)}_P$-equivariant bijections in the case of groups of Lie type when $p = 2$. In [11], G. Navarro, B. Späth, and C. Vallejo proved a corresponding reduction theorem for the McKay–Navarro conjecture, stating that the conjecture holds for all finite groups if certain "inductive McKay–Navarro conditions" hold for every finite nonabelian simple group. These conditions build on the inductive McKay conditions, requiring that the bijection Ω from before is further $(\mathrm{Aut}(G)_P \times \mathcal{H})$-equivariant and satisfies a strong condition about $\mathcal{H}$-compatible extensions of $\chi \in \mathrm{Irr}_{p'}(G)$ and $\Omega(\chi)$ to $G \rtimes \mathrm{Aut}(G)_{P,\chi}$, respectively $M \rtimes \mathrm{Aut}(G)_{P,\chi}$.

In this talk, I discuss joint work with L. Ruhstorfer, in which we have completed the proof of the inductive McKay–Navarro conditions for $p = 2$, and hence the proof of:

Main Theorem. *The McKay–Navarro conjecture holds for $p = 2$.*

The case of the alternating groups for $p = 2$ follows quickly by the work of R. Nath [9]. Many groups of Lie type with exceptional Schur multipliers were checked by B. Johansson [3, 4], and we use GAP to verify the desired conditions for those remaining and the sporadic groups (the latter were also checked by C. Vallejo

in unpublished work). The case of groups of Lie type in defining characteristic was settled by my coauthor, L. Ruhstorfer [13], with some outlying cases completed in [3]. Hence, we are left to consider the groups of Lie type in non-defining characteristic.

Our work builds on the bijections Ω from [5, 6, 21, 1, 8] for proving the inductive McKay conditions for groups of Lie type for $p = 2$. In [17], I describe the action of $\mathcal{G}$ on the Howlett-Lehrer parameters for characters of groups with a BN pair in order to complete the proof began in [16, 19] of another conjecture of Navarro from [10], which now also follows as a consequence of the Main Theorem above. Here at the MFO in 2019, I discussed how I extended the techniques from [17] in [18], also building on joint work with J. Taylor [20], to show that for many groups of Lie type defined in odd characteristic, the map Ω can be chosen to further be $\mathcal{H}$-equivariant. For many of these groups (perhaps most notably the case $G = \mathrm{Sp}_{2n}(q)$, which required distinct treatment), L. Ruhstorfer and I were able to complete the extension portion of the conditions, and hence the inductive McKay–Navarro conditions for $p = 2$, in [14]. Several additional outlying cases were completed by B. Johansson in [4].

In the final installment [15] leading to the Main Theorem, Ruhstorfer and I completed the final pieces, in particular verifying both the $\mathcal{H}$-equivariance and extension parts of the inductive McKay–Navarro conditions for $p = 2$ and the groups whose underlying algebraic groups are of type A and D. These results use work from [8, 7, 19, 12], describing the specific Lusztig and Harish-Chandra series in which the odd-degree characters lie in these cases. We also use techniques developed in [14] and supplemented in [4] for working with extensions to field automorphisms, as well as rationality properties of characters in $\mathrm{Irr}_{2'}(G)$ in these cases in order to study the desired extensions to the rest of the automorphism group.

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Generating functions for the Hochschild cohomology of symmetric groups

Markus Linckelmann
(joint work with Dave Benson, Radha Kessar)

This talk is about the implications of homological stablility for the Hochschild cohomology of symmetric group algebras. The Hochschild cohomology of a finite-dimensional algebra A over a field is the Ext-algebra

$$HH^*(A) = \mathsf{Ext}^*_{A\otimes_k A^{\mathrm{op}}}(A, A).$$

We have canonical isomorphisms $HH^0(A) \cong Z(A)$ and $HH^1(A) \cong \mathsf{Der}(A)/\mathsf{IDer}(A)$, where $\mathsf{Der}(A)$ is the subspace of $\mathsf{End}_k(A)$ of derivations on A and $\mathsf{IDer}(A)$ the subspace of inner derivations. The space $\mathsf{Der}(A)$ is a Lie subalgebra of $\mathsf{End}_k(A)$ with respect to the Lie bracket $[f, g] = f \circ g - g \circ f$, where f, $g \in \mathsf{End}_k(A)$, and $\mathsf{IDer}(A)$ is a Lie ideal in $\mathsf{Der}(A)$, implying that $HH^1(A)$ inherits a Lie algebra structure. By results of Gerstenhaber, $HH^*(A)$ is a graded-commutative algebra, and $HH^*(A)$ has a graded Lie algebra structure of degree -1 which extends the Lie algebra structure on $HH^1(A)$.

Throughout this paper p is a prime. For n a non-negative integer, we denote by $p(n)$ the number of partitions of n, with the convention $p(0) = 1$. The partition function is the associated generating function $\mathcal{P}(t) = \sum_{n=0}^{\infty} p(n)\, t^n$. Since $p(n)$ is the number of conjugacy classes of $\mathfrak{S}_n$, we have $p(n) = \dim_{\mathbb{F}_p}(HH^0(\mathbb{F}_p\mathfrak{S}_n))$. In degree 1, we have the following combinatorial description. We write as usual $\lambda \vdash n$ if λ is a partition of n. We write λ_k for the number of parts of λ of length k, and

we use the notation $\lambda = (n^{\lambda_n} \dots 2^{\lambda_2} 1^{\lambda_1})$. For integers n, k such that $n \geqslant 0$ and $k \geqslant 1$ we set $F_k(n) = \sum_{\lambda \vdash n} \lambda_k$. Thus $F_k(n)$ is the total number of parts of length k in all partitions of n.

Theorem 1 ([2, Corollary 4.7])**.** *For any positive integer n we have*

$$\dim_{\mathbb{F}_p}(HH^1(\mathbb{F}_p\mathfrak{S}_n)) = \begin{cases} 2F_2(n) & p = 2, \\ F_p(n) & p \geqslant 3. \end{cases}$$

We similarly have combinatorial expressions for the dimensions of $HH^2(\mathbb{F}_p\mathfrak{S}_n)$. In terms of generating functions, we have the following.

Theorem 2 ([2, Theorem 1.2])**.** *The generating functions for the dimensions of the Hochschild cohomology of the group algebra of the symmetric group $\mathfrak{S}_n$ on n letters in low degrees are given by*

(i) $$\sum_{n=0}^{\infty} \dim_{\mathbb{F}_p}(HH^0(\mathbb{F}_p\mathfrak{S}_n))\, t^n = \mathcal{P}(t)$$

(ii) $$\sum_{n=0}^{\infty} \dim_{\mathbb{F}_p}(HH^1(\mathbb{F}_p\mathfrak{S}_n))\, t^n = \begin{cases} \dfrac{2t^2}{1-t^2}\,\mathcal{P}(t) & p = 2, \\ \dfrac{t^p}{1-t^p}\,\mathcal{P}(t) & p \geqslant 3. \end{cases}$$

(iii) $$\sum_{n=0}^{\infty} \dim_{\mathbb{F}_p}(HH^2(\mathbb{F}_p\mathfrak{S}_n))\, t^n = \begin{cases} \dfrac{2t^2+3t^4-t^6}{(1-t^2)(1-t^4)}\,\mathcal{P}(t) & p = 2, \\ \dfrac{t^p}{(1-t^p)(1-t^{2p})}\,\mathcal{P}(t) & p \geqslant 3. \end{cases}$$

The formula for degree one has independently been obtained by Briggs and Rubio y Degrassi [5]. The pattern of the above formulae carries over to arbitrary degrees.

Theorem 3 ([2, Theorem 1.3])**.** *For any integer $r \geqslant 0$ there exists a rational function $R(t)$ with integer coefficients, depending on r and on p, such that*

$$\sum_{n=0}^{\infty} \dim_{\mathbb{F}_p}(HH^r(\mathbb{F}_p\mathfrak{S}_n))\, t^n = R(t)\mathcal{P}(t).$$

One of the main tools in the proofs of these results is the centraliser decomposition for the Hochschild cohomology of finite group algebras

$$HH^*(kG) \cong \bigoplus_{g \in G/\sim} H^*(C_G(g); k),$$

where g runs over a set of representatives of the conjugacy classes in G (see e. g. [1, Theorem 2.11.2]), and where k is a commutative ring. This isomorphism is graded k-linear, unique up to unique isomorphism, but this is not an isomorphism

as graded k-algebras. In particular, in degree 1 we have a decomposition

$$HH^1(kG) \cong \bigoplus_{g \in G/\sim} \mathsf{Hom}(C_G(g); k),$$

where $\mathsf{Hom}(C_G(g), k)$ is the k-space of group homomorphisms from $C_G(g)$ to the additive group $(k, +)$. If k is a field of characteristic zero, then this space is zero since $C_G(g)$ is a finite group. If k is a field of characteristic p, then $(k, +)$ has exponent p, and hence the non-vanishing of $HH^1(kG)$ is in that case equivalent to the existence of an element $g \in G$ such that $O^p(C_G(g)) < C_G(g)$, or equivalently, such that $C_G(g)$ has a non-trivial p-group as homomorphic image. As a consequence of results by Fleischmann, Janiszczak and Lempken [7], using the classification of finite simple groups, this is always the case if p divides the order of G. In order to apply the centraliser decomposition to symmetric groups, we use the well-known description of centralisers in symmetric groups as direct products of certain wreath products, in conjunction with the Künneth formula. A second key tool at this point is the work on homological stablilty for symmetric groups due to Nakaoka [10]; more precisely, we are making use of a generalisation of one of Nakaoka's results by Hatcher and Wahl in [8].

In order to extend the above results to the Hochschild cohomology of blocks of symmetric groups, we denote by B_n the principal block of $\mathbb{F}_p\mathfrak{S}_n$, for all non-negative integers n (with the convention $B_0 = \mathbb{F}_p$). Nakayama's conjecture, proved by Brauer [4] and G. de B. Robinson [11], states that the blocks of $\mathbb{F}_p\mathfrak{S}_n$ are parametrised in terms of p-core partitions. By a result of Chuang and Rouquier in [6], if B is a block of $\mathbb{F}_p\mathfrak{S}_n$, then B is derived equivalent to the principal block B_{pw} of $\mathbb{F}_p\mathfrak{S}_{pw}$, for some integer w such that $0 \leqslant w \leqslant \frac{n}{p}$, called the *weight of* B. Since Hochschild cohomology is invariant under derived equivalences, this can be used to prove the following results.

Theorem 4 ([3, Theorem 1.2])**.** *Let B be a weight w-block of a symmetric group algebra over $\mathbb{F}_p$. If $p = 2$, then*

$$\dim_{\mathbb{F}_p} HH^1(B) = 2 \sum_{j=0}^{w-1} \dim_{\mathbb{F}_p}(Z(B_{pj})).$$

If $p \geqslant 3$, then

$$\dim_k HH^1(B) = \sum_{j=0}^{w-1} \dim_{\mathbb{F}_p}(Z(B_{pj})).$$

This shows in particular, that for any block B of $\mathbb{F}_p\mathfrak{S}_n$ of positive weight we have $HH^1(B) \neq 0$. It remains an open question (cf. [9, Question 7.4]), whether for every block B of a finite group with non-trivial defect groups we have $HH^1(B) \neq 0$. In terms of the generating functions

$$\mathcal{P}(t) = \sum_{n=0}^{\infty} p(n)\, t^n,$$

$$\mathcal{Z}(t) = \sum_{n=0}^{\infty} \dim_{\mathbb{F}_p}(Z(B_{pn}))\, t^n$$

we have the following.

Theorem 5 ([3, Theorem 1.3])**.** *For any positive integer r, there exists a rational function $\phi(t)$ (depending on p and r) with $\phi(0)$ non-zero, such that*

$$\sum_{n=0}^{\infty} \dim_{\mathbb{F}_p}(HH^r(B_{pn}))\, t^n = t\phi(t)\mathcal{Z}(t)$$

and

$$\sum_{n=0}^{\infty} \dim_{\mathbb{F}_p}(HH^r(k\mathfrak{S}_n))\, t^n = t^p\phi(t^p)\mathcal{P}(t).$$

This shows that $t^p\phi(t^p) = R(t)$ where $R(t)$ is as in Theorem 3, giving an affirmative answer to a question by Ken Ono. If one drops the condition $\phi(0) \neq 0$, then this formula holds also for $r = 0$ with $\phi(t) = \frac{1}{t}$. Since $\mathbb{F}_p$ is a splitting field for $\mathbb{F}_p\mathfrak{S}_n$, the dimensions of Hochschild cohomology of blocks of symmetric group algebras remain unchanged upon replacing $\mathbb{F}_p$ by any field of characteristic p. The authors conjecture that $HH^1(\mathbb{F}_p\mathfrak{S}_n)$ should be a solvable Lie algebra, for any positive integer n.

In addition to the work of Hatcher and Wahl already mentioned, homological stability has received a considerable amount of attention in recent papers by R. Boyd, R. Hepworth, P. Patzt, and others, where this phenomenon is extended to numerous classes of algebras. This raises the question whether the above results on generating functions for Hochschild cohomology can be extended to (blocks of) alternating groups, other finite Coxeter groups, Hecke algebras, and diagram algebras.

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Brauer's height zero conjecture

Gunter Malle

(joint work with Gabriel Navarro, Mandi Schaeffer Fry, Pham Huu Tiep)

Let G be a finite group, p a prime, b a Brauer p-block of G with defect group D and

$$\mathrm{Irr}_0(b) := \{\chi \in \mathrm{Irr}(b) \mid \chi(1)_p = |G : D|_p\}$$

the complex irreducible characters in b of *height zero*. Based upon rather scarce evidence, Richard Brauer in 1955 proposed the following:

Conjecture 1. $\mathrm{Irr}_0(b) = \mathrm{Irr}(b) \iff D$ *abelian.*

This became known as the Brauer Height Zero conjecture (BHZ). It has had a crucial influence on research in representation theory of finite groups since then, and is a prominent example of the unsolved local-global conjectures which have since been formulated. The BHZ had been verified for the class of p-solvable groups as well as for various classes of nearly simple groups by various authors. In 2013, R. Kessar and the author completed the proof of the "$\Leftarrow$" implication of BHZ, building on fundamental work of Lusztig as well as of Fong–Srinivasan, Cabanes–Enguehard and Bonnafé–Rouquier on representations of finite reductive groups, and later verified the other direction for all p-blocks of all quasi-simple groups. In 2022, L. Ruhstorfer was able to settle the missing direction for the prime $p = 2$, as a consequence (shown by Navarro–Späth) of his work on the Alperin–McKay conjecture. We presented and discussed the proof of our result:

Theorem 2. *The "$\Rightarrow$" implication of BHZ holds for all primes $p > 2$.*

By the earlier results cited above, this means that almost 70 years after it was first conceived, BHZ is now known in full generality.

Our proof proceeds by a careful analysis of a minimal counter-example, using all of the above-mentioned results as well as further ones by Külshammer–Puig and Koshitani–Späth. Along the way it requires two statements about blocks b of quasi-simple groups S. The first asserts the existence of enough characters in b not conjugate under the full automorphism group of S, the second shows that for b with abelian defect, the pointwise stabiliser in $\mathrm{Out}(S)$ of all characters in b has cyclic Sylow p-subgroups. Our proof of these two claims relies on the classification of finite simple groups as well as on the above-mentioned deep work of Lusztig and others on characters and blocks of finite reductive groups.

Splendid Morita equivalences for principal 2-blocks with wreathed defect groups

Shigeo Koshitani
(joint work with Caroline Lassueur, Benjamin Sambale)

We classify principal 2-blocks of finite groups G with Sylow 2-subgroups isomorphic to a wreathed 2-group $C_{2^n} \wr C_2$ with $n \geq 2$ up to splendid Morita equivalence (= Puig equivalence). As a consequence, obtain that Puig's Finiteness Conjecture holds for such blocks. Furthermore, we obtain a classification of such groups modulo $O_{2'}(G)$, which is a purely group theoretical result and of independent interest. This is the **first** case where the case of **wild** representation type is treated.

Given an integer $t \geq 0$ and a positive prime power q, we let

$$\mathrm{SL}_2^t(q) := \{A \in \mathrm{GL}_2(q) \mid \det(A)^{2^t} = 1\} \text{ and } \mathrm{SU}_2^t(q) := \{A \in \mathrm{GU}_2(q) \mid \det(A)^{2^t} = 1\}.$$

Theorem 0.1. *Let k be an algebraically closed field of characteristic 2 and let G be a finite group with a Sylow 2-subgroup P isomorphic to a wreathed 2-group $C_{2^n} \wr C_2$ for a fixed integer $n \geq 2$. Then the following assertions hold.*

(a) *The principal 2-block $B_0(kG)$ of kG is splendidly Morita equivalent to the principal 2-block of precisely one of the following finite groups:*

(W1) $C_{2^n} \wr C_2$;
(W2) $(C_{2^n} \times C_{2^n}) \rtimes \mathfrak{S}_3$;
(W3) $\mathrm{SL}_2^n(q)$ *where q is a positive power of a prime such that $(q-1)_2 = 2^n$;*
(W4) $\mathrm{SU}_2^n(q)$ *where q is a positive power of a prime such that $(q+1)_2 = 2^n$;*
(W5) $\mathrm{PSL}_3(q)$ *where q is a positive power of a prime such that $(q-1)_2 = 2^n$;*
(W6) $\mathrm{PSU}_3(q)$ *where q is a positive power of a prime such that $(q+1)_2 = 2^n$.*

Moreover, in all cases, the splendid Morita equivalence is induced by the Scott module $\mathrm{Sc}(G \times G_2, \Delta P)$.

(b) *In* (a) *if G_1 and G_2 are two groups such that $|G_1|_2 = |G_2|_2$ and which are both of type* (W3)*, both of type* (W4)*, both of type* (W5)*, or both of type* (W6)*, then $B_0(kG_1)$ and $B_0(kG_2)$ are splendidly Morita equivalent.*

As a byproduct of our main work we obtain a classification of principal 2-blocks of all finite groups with the wreath product Sylow 2-subgroups just as it has been done by Erdmann around 1990 for the case of tame representation type, and therefore our result should be a new result.

Corollary 0.2. *For every integer $n \geq 2$ there are only finitely many splendid Morita equivalence classes of principal 2-blocks with defect groups isomorphic to a wreathed 2-group $C_{2^n} \wr C_2$. Namely, Puig's Finiteness Conjecture holds for such blocks.*

On the generalised Springer correspondence

Jonas Hetz

Let p be a prime, $\mathbf{G}$ a connected reductive group over $k = \overline{\mathbb{F}}_p$, $\mathbf{W}$ the Weyl group of $\mathbf{G}$ and $\mathscr{N}_{\mathbf{G}}$ the set of all pairs $(\mathscr{O}, \mathscr{E})$ where $\mathscr{O} \subseteq \mathbf{G}$ is a unipotent class and $\mathscr{E}$ is a $\mathbf{G}$-equivariant irreducible local system on $\mathscr{O}$ (taken up to isomorphism).

The Springer correspondence (originally defined by Springer [7] for p not too small; for arbitrary p see Lusztig [2]) is an injective map $S_{\mathbf{G}} : \mathrm{Irr}(\mathbf{W}) \hookrightarrow \mathscr{N}_{\mathbf{G}}$, which plays a crucial role, for example, in the determination of the Deligne–Lusztig Green functions. However, $S_{\mathbf{G}}$ is in general not surjective. In order to understand the missing pairs, Lusztig [3] developed a generalisation of Springer's correspondence, which is a substantial ingredient in the program of determining the complete character tables of finite groups of Lie type. For instance, this has been utilised recently to complete the computation of unipotent characters at unipotent elements for the groups $E_6(q)$ and $E_7(q)$ where q is a power of p.

With very few exceptions, the generalised Springer correspondence has been determined explicitly by Lusztig and Spaltenstein [3, 4, 5, 6]; the last open problems occur for groups of type E_8 and $p = 3$. These remaining indeterminacies have been resolved in [1], which thus concludes the determination of the generalised Springer correspondence. The proof is based on considering the Hecke algebra associated to the finite group $E_8(q)$ (for q a power of 3) and its natural (B, N)-pair exploiting a well-known formula relating characters of this Hecke algebra with the unipotent principal series characters of $E_8(q)$.

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Unipotent character sheaves and strata of a reductive group

George Lusztig

Let G be a connected reductive group over an algebraically closed field. We define a decomposition of G into finitely many strata each of which is a union of conjugacy classes of fixed dimension. The strata are indexed by a set independent of the characteristic. The strata can be described purely in terms of the Weyl group. We show that the set of unipotent character sheaves on G can be naturally mapped surjectively to the set of strata of G.

On the source-algebra equivalence class of a block with cyclic defect groups

Caroline Lassueur
(joint work with Gerhard Hiß)

This talk was a report on a series of three joint articles [HL21, HL23a, HL23b] with Gerhard Hiß (RWTH Aachen). Considering a finite group G and an algebraically closed field k of positive characteristic p, the starting point was the following question.

Question A. *Given a block* $\mathbf{B}$ *of the group algebra* kG *with a non-trivial cyclic defect defect group* D *of order* p^n *(*$n \in \mathbb{Z}_{\geq 1}$*), can we give a concrete classification of the indecomposable* $\mathbf{B}$*-modules with a trivial source, up to isomorphism?*

In [Jan69] Janusz gave a constructive classification of all indecomposable $\mathbf{B}$-modules, up to isomorphism, using directed connected subgraphs of the Brauer tree called *paths*, which encode composition factors, submodules and quotients. Much later, in 2012, Hiß and Naehrig [HN12] gave a classification of all liftable $\mathbf{B}$-modules in terms of Janusz' classification of the indecomposables. We observe that both these classification problems can be considered up to Morita equivalence (hence understood from the data given by the embedded Brauer tree of the block), whereas an answer to Question A depends on the *source-algebra equivalence class* of the block (also called *Puig equivalence class* or *splendid Morita equivalence class*). Now, by the work of Linckelmann [Lin88] the source-algebra equivalence class of $\mathbf{B}$ is parametrised by three parameters: the embedded Brauer tree $\sigma(\mathbf{B})$ of $\mathbf{B}$, a sign function $\tau(\mathbf{B})$ on the vertices of $\sigma(\mathbf{B})$, and an indecomposable capped endo-permutation kD-module $W(\mathbf{B})$, on which the unique subgroup of order p of D acts trivially. Thus, building on [HN12], the main result of [HL21] is a concrete classification of the indecomposable trivial source modules belonging $\mathbf{B}$ in terms of Janusz' classification of the indecomposable modules and in terms of the parameters $\sigma(\mathbf{B})$, $\tau(\mathbf{B})$, $W(\mathbf{B})$. Moreover, [HL21] describes the precise position of the modules in the Auslander-Reiten quiver of the block in terms of distance to the rim. The precise combinatorics of the result is too complex to be printed down here, we refer the reader directly to [HL21, Theorem 5.3 and Proposition B.1].

In order to make concrete calculations for specific groups it is necessary to have a good understanding of the three parameters $\sigma(\mathbf{B})$, $\tau(\mathbf{B})$, and $W(\mathbf{B})$. The sign function $\tau(\mathbf{B})$ is understood in terms of values of the ordinary characters of $\mathbf{B}$ evaluated at a generator of the unique cyclic subgroup of order p of D. Feit [Fei84] considered the problem of describing all possible Brauer trees, which he solves through a reduction to quasi-simple groups. Now, when p is odd there are 2^{n-1} (resp. 2^{n-2} when $p = 2$) isomorphism classes of kD-modules that can occur as $W(\mathbf{B})$ and it is known that they all occur for some p-soluble group (in which case $\sigma(\mathbf{B})$ is a star with exceptional vertex at its centre). Thus, this leads naturally to the following questions, to which we give partial answers in [HL23a, HL23b].

Question B.

(1) Can $W(\mathbf{B})$ be read off from the character table of G?
(2) Can we determine $W(\mathbf{B})$ for all (non-uniserial) cyclic blocks $\mathbf{B}$ of finite groups?
(3) Can we determine $W(\mathbf{B})$ for all cyclic blocks $\mathbf{B}$ of finite quasi-simple groups?
(4) Can (b) be reduced to (c)?

To begin with, assuming p is odd, we prove that Question B(a) has an affirmative answer.

Theorem C ([HL23a, Theorem 3.4])**.** *Let p be an odd prime. Let $\mathbf{B}$ be a block of kG with a cyclic defect group $D \cong C_{p^n}$ where $n \geq 1$. For each $1 \leq i \leq n$ let u_i be a generator of the subgroup D_i of D of order p^i. Let χ be a non-exceptional irreducible character lying in $\mathrm{Irr}_{\mathbb{C}}(\mathbf{B})$. Then, the kD-module $W(\mathbf{B})$ is determined up to isomorphism by the signs of the character values $\{\chi(u_i)\}_{1\leq i\leq n}$ (which are all integers).*

Next, we give some criteria which ensure that $W(\mathbf{B}) \cong k$, the trivial kD-module. In particular, $W(\mathbf{B}) \cong k$ provided $\mathbf{B}$ is the principal block or $C_G(D) = C_G(D_1)$. This allows us to deal with the following classes of finite quasi-simple groups, where we note that for the purpose of the proofs it is good to treat the groups of Lie type with an exceptional Schur multiplier separately.

Theorem D ([HL23a, Propositions 6.1–6.5])**.** *Let p be an odd prime. Let $\mathbf{B}$ be a block of kG with a non-trivial cyclic defect group, where G is a finite quasi-simple group. Then $W(\mathbf{B}) \cong k$ provided $S := G/Z(G)$ is:*

(i) *a sporadic simple group or the Tits simple group;*
(ii) *an alternating group $\mathfrak{A}_n$ with $n \geq 5$;*
(iii) *a group of Lie type in defining characteristic;*
(iv) *a classical group of Lie type in cross-characteristic such that "the prime number p is large with respect to the Lie rank of G"; or*
(v) *an exceptional group of Lie type (including the Suzuki and Ree groups) in cross-characteristic and $p > 3$, respectively $p > 5$ if S is of type E_8.*

More precisely, the groups of type (iv) are the following classical groups. First, the universal covering group $\hat{G}$ of S is one of $\mathrm{SL}_n(q)$ $(n \geq 2)$, $\mathrm{SU}_n(q)$ $(n \geq 3)$,

$\mathrm{Sp}_n(q)$ ($n \geq 4$ even), $\mathrm{Spin}_n(q)$ ($n \geq 7$ odd), or $\mathrm{Spin}_n^{\pm}(q)$ ($n \geq 8$ even). Moreover, if d denotes the order of q modulo p, then $pd > n$ provided $\hat{G}$ is one of $\mathrm{SL}_n(q)$, $\mathrm{Sp}_n(q)$ or $\mathrm{Spin}_n^-(q)$; $pd > n-1$ provided $\hat{G} = \mathrm{Spin}_n(q)$ with n odd; $pd > n-2$ provided $\hat{G} = \mathrm{Spin}_n^+(q)$ with n even; and finally $pd > 2n$ provided $\hat{G} = \mathrm{SU}_n(q)$.

In [HL23b], we examine the remaining cases and describe all non-trivial invariants $W(\mathbf{B})$ which can occur. For quasi-simple classical groups we reduce this task to the computation of the endo-permutation kD-modules $W(\mathbf{B})$ arising from $\mathrm{GL}_n(q)$ or covering groups of $\mathrm{PSL}_n(q)$ when $p \mid (q-1)$, or from $\mathrm{GU}_n(q)$ or covering groups of $\mathrm{PSU}_n(q)$ when $p \mid (q+1)$. Moreover, if G is an exceptional group of Lie type, non-trivial modules $W(\mathbf{B})$ in fact only exist for $p = 3$. This work reveals an interesting close connection between the structure of the module $W(\mathbf{B})$ and Deligne-Lusztig theory. Also, the results depend, somehow mysteriously, on the congruence class of p modulo 4, namely, if $p \equiv -1 \pmod 4$, the number of non-isomorphic kD-modules $W(\mathbf{B})$ arising is roughly cubic in n, whereas if $p \equiv 1 \pmod 4$ this number is $n-1$. These investigations show that, within the class of all cyclic blocks $\mathbf{B}$ of finite quasi-simple groups, non-trivial modules $W(\mathbf{B})$ can also occur when the Brauer tree is not a star, although not for any shape of tree occuring.

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Braid group action on the cohomology of Deligne-Lusztig varieties

Cédric Bonnafé

(joint work with M. Broué, O. Dudas, J. Michel, R. Rouquier)

To an element b of the braid monoid associated with a split finite reductive group G^F, one can attach a Deligne-Lusztig variety $X(b)$ which is acted on by G^F. We construct an action of the centralizer of b in the braid group on the cohomology of $X(b)$, which does not come from an action on the variety. The aim of this talk is to describe this construction and give some of its properties, the most important one being the compatibility with Deligne-Lusztig induction. If time permits, we

will discuss some questions related to the $\mathrm{SL}_2(\mathbb{Z})$-action on the space spanned by unipotent characters.

Character bounds for finite groups of Lie type and Thompson's conjecture

Pham Huu Tiep

In this talk, we discuss recent results, obtained in joint work of the speaker with various collaborators, on the following problem:

Problem 1. *Let G be a finite almost quasisimple group, $g \in G \setminus Z(G)$.*

(A) *Find an explicit, and as small as possible, $0 < \gamma = \gamma(g) < 1$, such that*

$$\frac{|\chi(g)|}{\chi(1)} \leq \gamma, \ \forall\ \chi \in \mathrm{Irr}(G) \text{ with } \chi(1) > 1.$$

(B) *Find an explicit, and as small as possible, $0 < \alpha = \alpha(g) < 1$, such that*

$$|\chi(g)| \leq \chi(1)^\alpha, \ \forall\ \chi \in \mathrm{Irr}(G).$$

Results on Problem 1 will be useful for a number of applications, which usually involve using *Frobenius character formula*, and include

(i) *Ore conjecture* [14] and *Waring-type problems* [11, 12] for finite simple groups
(ii) Random generation of simple groups, and representation varieties $\mathrm{Hom}(\Gamma, G)$, where $G = \mathsf{S}_n$, $\mathcal{G}(\mathcal{F})$ for a simple algebraic group $\mathcal{G}$, and Γ a Fuchsian group, see [17],
(iii) Random walks on Cayley graphs and McKay graphs [18, 19],
(iv) *Thompson conjecture* [1]: *If G is a finite non-abelian simple group, then $G = C^2$ for some conjugacy class C of G.*

The most general result on Problem 1(A), which combines results of Gluck [Gl], Gluck and Magaard [5], and Guralnick and the author [9], says that one can take $\gamma = 79/80$, unless $E(G) = \mathsf{A}_n$.

Our main focus will be on finite classical groups $G = \mathrm{Cl}(V)$, where $V = \mathbb{F}_q^n$. For any $g \in \mathrm{Cl}(V)$, the *support* $\mathsf{supp}(g)$ is defined to be.

$$\mathsf{supp}(g) = \inf_{\lambda \in \overline{\mathbb{F}_q}} \mathrm{codimKer}(g - \lambda \cdot 1_{\tilde{V}}),$$

where $\tilde{V} = V \otimes \overline{\mathbb{F}_q}$. A result of Larsen, Shalev, and the author [11] states that, for any classical group $G = \mathrm{Cl}(V)$, any irreducible character $1_G \neq \chi \in \mathrm{Irr}(G)$, and any $g \in G$

$$\frac{|\chi(g)|}{\chi(1)} \leq \frac{1}{q^{\sqrt{\mathsf{supp}(g)}/481}}. \tag{1}$$

The first significant result on Problem 1(B) for symmetric groups was obtained by Fomin and Lulov [4]. It has been vastly generalized by Larsen and Shalev [10].

For finite groups of Lie type $G = \mathcal{G}^F$, where $\mathcal{G}$ is a connected reductive group of semisimple rank r in good characteristic p, strong exponential bounds of the form

$$|\chi(g)| \leq f(r)\chi(1)^{\alpha(\mathcal{L}^F}$$

have been obtained in [2], [23], for the elements $g \in G$ such that $C_G(g) \leq \mathcal{L}^F$ for some proper F-stable Levi subgroup $\mathcal{L}$. Ignoring the (explicit) function $f(r)$, the (explicit) exponent $\alpha(\mathcal{L}^F)$ is sharp in several cases. However, these bounds leave out the elements $g \in G$ whose centralizer is not contained in any proper Levi subgroup, for instance, if g is unipotent.

An alternate approach towards Problem 1(B) has been developed in [6], [7], which applies particularly well in the situation where either $\chi(1)$ or $|C_G(g)|$ is not too large, compared to $|G|$ logarithmically. The task of bounding character values can then be accomplished using the concept of *character level*. Relying on *Howe's duality* and *Deligne-Lusztig theory*, we relate the level to the degree and the Lusztig's label for any irreducible character of $G = \mathrm{Cl}_n(q)$.

Using level theory, we prove:

Theorem 2. *For any $0 < \varepsilon < 1$, there is $\delta = \delta(\varepsilon) > 0$ such that the following statement holds. For any finite quasisimple group G of Lie type, and for any $g \in G$ with $|C_G(g)| \leq |G|^\delta$,*

$$|\chi(g)| \leq \chi(1)^\varepsilon$$

for all $\chi \in \mathrm{Irr}(G)$.

For instance, if $G = \mathrm{SL}_n(q)$ or $\mathrm{SU}_n(q)$, and $\varepsilon = 8/9$, one can take $\delta = 1/12$.

Building on [6, 7], we have recently proved the following uniform exponential character bound, which works for all elements in all finite quasisimple groups of Lie type:

Theorem 3. [13] *There exists an explicit constant $c > 0$ such that for all finite quasisimple groups G of Lie type, all $\chi \in \mathrm{Irr}(G)$, and all $g \in G$, we have*

$$|\chi(g)|/\chi(1) \leq \chi(1)^{-c\frac{\log|g^G|}{\log|G|}}.$$

We remark that, up to the factor c, the exponent in Theorem 3 is optimal.

Here are some consequences of Theorem 3.

- A linear upgrade of the LST-bound (1):

  $$|\chi(g)/\chi(1)| \leq q^{-\sigma\cdot\mathsf{supp}(g)}$$

  for a uniform constant $\sigma > 0$.
- "Swapping" ε and δ in Theorem 2 :
  For any $0 < \delta < 1$, there is $0 < \varepsilon = \varepsilon(\delta) > 0$ such that the following statement holds. For any finite quasisimple group G of Lie type, and for any $g \in G$ with $|C_G(g)| \leq |G|^\delta$,

  $$|\chi(g)| \leq \chi(1)^\varepsilon$$

  for all $\chi \in \mathrm{Irr}(G)$.

- *Lubotzky's conjecture* (also of Shalev) [20, 22]: *If G is simple of Lie type, then the mixing time of the random walk on the Cayley graph $\Gamma(G, g^G)$ is of the same magnitude as its diameter; and it is $\Omega(n/\mathsf{supp}(g))$ if $G = \mathrm{Cl}_n(q)$.*
  The diameter part was established by [16].
- *Liebeck-Shalev-Tiep's conjecture* [17]: *If G is simple and α a faithful character of G, then the diameter of the McKay graph $\mathcal{M}(G, \alpha)$ is*
$$\Omega(\log |G| / \log \alpha^*(1)),$$
where α^* is the sum of distinct irreducible constituents of α.
  The alternating case was proved by [18], relying on a recent result of Sellke [21] for symmetric groups.

Strategy of proof of Theorem 3. We use *probability theory* to solve a representation-theoretic problem. The starting idea is that the expected value of $\chi(g^{x_1}g^{x_2}...g^{x_m})$, with $x_1, ..., x_m \in G$ chosen uniformly at random and $\chi \in \mathrm{Irr}(G)$, is $\chi(g)^m/\chi(1)^{m-1}$. Then we prove *mixing theorems*, which show that the probability that a product of m random conjugates of $g \in G$ lands in a small conjugacy class is small. Hence with high probability one can apply Theorem 2. In fact, one needs to do it in two bootstrappings. First, bootstrap a mid-size class to apply the GLT-bound in Theorem 2. Then bootstrap any class to land "almost surely" in a mid-size class. □

Theorem 3 allows us to make significant progress on Thompson's conjecture:

Theorem 4. [13] *If $G = \mathrm{Cl}_n(q)$ is a simple classical group of large enough rank, and, furthermore,*

$(*)$: $(q+1)|n$ *if* $G = \mathrm{PSU}_n(q)$, $\epsilon = (-1)^{n(q-1)/4}$ *and* $2|n$ *if* $G = P\Omega_n^\epsilon(q)$ *with* $2 \nmid q$

then Thompson's conjecture holds for G, i.e. $G = C^2$ for some $C = g^G$.

Strategy of proof. By [3], we may assume $q \leq 7$. Choose y of big support. Theorem 3 allows us to prove that $(y^G)^2$ contains all elements $g \in G$ except for the ones of small support. To cover the latter, say $g = \mathrm{diag}(h, I_{n-k})$ with small k, choose $x = \mathrm{diag}(y, y')$ with y' real. The above condition $(*)$ is needed to treat the case where the primary eigenvalue of g is not 1. Our proof also makes use of the fact that every element in the Schur cover of G is a commutator [15]. □

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The inductive Alperin–McKay condition for groups of Lie type B and C

Lucas Ruhstorfer
(joint work with Julian Brough)

In the representation theory of finite groups some of the most important conjectures relate the representation theory of a finite group G to the ones of its ℓ-local subgroups for a prime ℓ dividing the order of G. One of these conjectures is the Alperin–McKay conjecture, which forms a blockwise generalization of the McKay conjecture. For an ℓ-block b of G we denote by $\mathrm{Irr}_0(b)$ the subset of height zero characters of the characters of the block b.

Conjecture (Alperin–McKay)**.** *Let b be an ℓ-block of G with defect group D and B its Brauer correspondent in $\mathrm{N}_G(D)$. Then*

$$|\mathrm{Irr}_0(b)| = |\mathrm{Irr}_0(B)|.$$

In [Spä13], the Alperin–McKay conjecture was reduced to the verification of the so-called *inductive Alperin–McKay condition* (iAM) for all finite simple groups and primes ℓ.

For groups of Lie type B or C, it suffices to validate the iAM condition for all isolated ℓ-blocks, see [Ruh22]. In this talk we explain how an iAM-bijection can be obtained in this case. This is joint work in progress with Julian Brough.

In what follows, let $\mathbf{G}$ be a simple algebraic group of simply connected type B or C defined over an algebraically closed field of characteristic $p > 0$. We assume that $F : \mathbf{G} \to \mathbf{G}$ is a Frobenius endomorphism defining an $\mathbb{F}_q$-structure on $\mathbf{G}$ and we let $\ell \geq 5$ be a prime with $\ell \neq p$. Then every isolated ℓ-block b of $\mathbf{G}^F$ is labeled by a d-cuspidal pair $(\mathbf{L}, \lambda)$, i.e. a pair consisting of a d-split Levi subgroup $\mathbf{L}$ of $(\mathbf{G}, F)$ and a d-cuspidal character $\lambda \in \mathrm{Irr}(\mathbf{L}^F)$. In our situation $\mathrm{Z}(\mathbf{L})_\ell^F$ is a characteristic subgroup of a defect group D of b and there exists a unique block B of $\mathrm{N}_G(\mathbf{L})$ covering $b_{\mathbf{L}^F}(\lambda)$. One can show that in order to prove the iAM-conditions for the block b it suffices to construct an $\mathrm{Aut}(\mathbf{G}^F)_{b,D}$-equivariant bijection $\mathrm{Irr}_0(b) \to \mathrm{Irr}_0(B)$. We construct such a bijection by giving a parametrization of both the characters of $\mathrm{Irr}_0(B)$ and of $\mathrm{Irr}_0(b)$.

One of the key results for parametrizing the characters in the block B is the following proposition about extending characters from $\mathbf{L}^F$ to their inertia group in $\mathrm{N}_{\mathbf{G}^F}(\mathbf{L})$:

Proposition. *For every character $\theta \in \mathrm{Irr}(b_{\mathbf{L}^F}(\lambda))$ there exists an extension $\Lambda(\theta) \in \mathrm{Irr}(\mathrm{N}_{\mathbf{G}^F}(\mathbf{L})_\theta)$. In particular, we have a bijection*

$$\mathrm{Irr}(W_G(\mathbf{L}, \theta)) \to \mathrm{Irr}(\mathrm{N}_G(\mathbf{L}) \mid \theta), \eta \mapsto \mathrm{Ind}_{\mathrm{N}_G(\mathbf{L})_\lambda}^{\mathrm{N}_G(\mathbf{L})}(\Lambda(\theta)\eta).$$

In the "McKay-situation" such a bijection was already known by the work of Cabanes–Späth [CS19]. Using properties of the constructed extensions $\Lambda(\theta)$ it is then possible to describe the stabilizer of the elements of $\mathrm{Irr}_0(B)$ in $\mathrm{Aut}(\mathbf{G}^F)_{b,D}$.

For parameterizing the global characters height zero characters, we use the properties of d-Harish-Chandra theory. For a d-cuspidal character $\theta \in \mathrm{Irr}(\mathbf{L}^F)$ denote by $\mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \theta))$ the set of constituents of $R_{\mathbf{L}}^{\mathbf{G}}(\theta)$. Then we have the following:

Proposition. *For every $\theta \in \mathrm{Irr}(b_{\mathbf{L}^F}(\lambda))$ there exists a bijection*

$$\mathrm{Irr}(W_G(\mathbf{L}, \theta)) \to \mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \theta))$$

Moreover, every height zero character in $\mathrm{Irr}_0(b)$ appears in the d-Harish-Chandra series $\mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \theta))$ for some $\theta \in \mathrm{Irr}(b_{\mathbf{L}^F}(\lambda))$.

Note that for characters in $\mathrm{Irr}(\mathbf{G}^F)$ the stabilizer in $\mathrm{Aut}(\mathbf{G}^F)$ is understood by work of Cabanes–Späth [CS19]. Using this, one can show that the parametrizations from above yield an $\mathrm{Aut}(\mathbf{G}^F)_{b,D}$-equivariant bijection $\mathrm{Irr}_0(b) \to \mathrm{Irr}_0(B)$. As a consequence we obtain:

Theorem. *Let G be a quasi-simple group of Lie type B_n or C_n ($n \geq 2$) defined over the finite field $\mathbb{F}_q$ for q a prime power of an odd prime and let $\ell \geq 5$ not dividing q. Then every ℓ-block of G satisfies the iAM-condition.*

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The Mullineux map

Matt Fayers

Let sgn denote the 1-dimensional sign module for the symmetric group $\mathfrak{S}_n$. If M is an irreducible $\mathfrak{S}_n$-module, then $M \otimes \mathrm{sgn}$ is again irreducible, and it is natural to ask which irreducible it is. In characteristic zero, the answer is well-known: the irreducible modules are the *Specht modules* S^λ labelled by partitions of n, and it is a classical result that $\mathrm{S}^\lambda \otimes \mathrm{sgn} \cong \mathrm{S}^{\lambda'}$, where λ' is the conjugate partition. In characteristic p, the situation is more difficult: now the irreducible modules are the *James modules* D^λ labelled by p-regular partitions of n, so there is a function m_p (the *Mullineux map*) from p-regular partitions to p-regular partitions defined by $\mathrm{D}^\lambda \otimes \mathrm{sgn} \cong \mathrm{D}^{m_p(\lambda)}$. The *Mullineux problem* is to describe m_p combinatorially. This problem was studied in the 1970s by Mullineux, who conjectured a solution but was unable to prove his conjecture. Then in 1994 Kleshchev gave a very different solution based on his modular branching rules. Ford and Kleshchev then showed that the combinatorial functions defined by Mullineux and Kleshchev coincide, thereby proving the Mullineux conjecture. Since then, several other solutions to the Mullineux problem have been found, although the function m_p remains somewhat mysterious. In particular, there is an intriguing interplay between the Mullineux map and James's regularisation map, which has consequences for the

decomposition number problem for the symmetric groups. I describe a new algorithm for the Mullineux map based on generalisations (found by my student Diego Millan Berdasco) of the regularisation map. This allows the combinatorial proof of results of Bessenrodt–Olsson–Xu and the author relating the Mullineux map with regularisation. I then briefly describe some work in progress joint with Nicolas Jacon aimed at describing m_p in terms of m_{2p} using bipartitions.

Local determination of Frobenius–Schur indicators

Benjamin Sambale

It is a difficult problem to decide whether a real-valued character χ of a finite group G can be afforded by a real representation. Although this information is encoded by the *Frobenius–Schur indicator* (F-S indicator)

$$\epsilon(\chi) := \frac{1}{|G|}\sum_{g\in G}\chi(g^2)$$

(being 1 if and only if χ comes from a real representation), it cannot, for instance, be read off from the character table of G (as D_8 and Q_8 witness). On the other hand, the F-S indicators of characters in a given block B are, as always, influenced by a defect group of B. Motivated by John Murray's results on cyclic defect groups, I obtained the number of real characters in nilpotent blocks (this number is not invariant under Morita equivalence).

Theorem 1. *Let B be a real, nilpotent p-block of a finite group G with defect group D. Let b_D be a Brauer correspondent of B in $D\mathrm{C}_G(D)$. Then the number of real characters in* $\mathrm{Irr}(B)$ *of height h coincides with the number of characters $\lambda \in \mathrm{Irr}(D)$ of degree p^h such that $\lambda^t = \overline{\lambda}$ where*

$$\mathrm{N}_G(D, b_D)^*/D\mathrm{C}_G(D) = \langle tD\mathrm{C}_G(D)\rangle.$$

If $p > 2$, then all real characters in $\mathrm{Irr}(B)$ *have the same F-S indicator.*

Here $\mathrm{N}_G(D, b_D)^* := \{g \in \mathrm{N}_G(D) : b_D^g \in \{b_D, \overline{b_D}\}\}$ is the *extended* stabilizer of b_D. For $p = 2$, Rod Gow and John Murray have introduced the *extended* defect group E of B such that $|E : D| = 2$ unless B is the principal block. It seems that the pair (D, E) fully determines the F-S indicators in nilpotent blocks as follows:

Conjecture 2. *Let B be a real, nilpotent, non-principal 2-block of a finite group G with defect pair (D, E). Then there exists a height preserving bijection Γ :* $\mathrm{Irr}(D) \to \mathrm{Irr}(B)$ *such that*

$$\boxed{\epsilon(\Gamma(\lambda)) = \frac{1}{|D|}\sum_{e\in E\setminus D}\lambda(e^2)}$$

for all $\lambda \in \mathrm{Irr}(D)$.

Theorem 3. *Conjecture 2 holds in each of the following cases:*

(i) D is abelian or a dihedral group.
(ii) G is solvable or quasisimple.

In general, Conjecture 2 is implied by the following local conjecture:

Conjecture 4. *Let B be a real, non-principal 2-block with defect pair (D, E). Let (x, b) be a B-subsection with defect pair $(\mathrm{C}_D(x), \mathrm{C}_E(x))$ such that b has a unique projective indecomposable character Φ. Then for every $y \in G$ with $y^2 = x$, we have*

$$[\Phi_{\mathrm{C}_G(y)}, 1_{\mathrm{C}_G(y)}] = |y^{\mathrm{C}_G(x)} \cap E \setminus D|,$$

where $y^{\mathrm{C}_G(x)}$ denotes the conjugacy class of y in $\mathrm{C}_G(x)$.

A consequence of Conjecture 4 is an interesting formula involving the generalized decomposition numbers $d^x_{\chi\varphi}$ where $\mathrm{IBr}(b) = \{\varphi\}$:

$$\sum_{\chi \in \mathrm{Irr}(B)} \epsilon(\chi) d^x_{\chi\varphi} = |\{y \in E \setminus D : y^2 = x\}|.$$

Murray has also investigated blocks with dihedral defect groups. Using similar idea as above, I excluded two families appearing in his classification.

Theorem 5. *Let B be real block of a finite group G with dihedral defect group D of order $2^d \geq 8$ and extended defect group E. Let $\epsilon_1, \ldots, \epsilon_4$ be the F-S indicators of the four irreducible characters of height 0 in B. There is a unique family of 2-conjugate characters of height 1 in $\mathrm{Irr}(B)$ of size 2^{d-3}. Let μ be the common F-S indicator of those characters. The possible values for $\epsilon_1, \ldots, \epsilon_4, \mu$ are given as follows, while the remaining $2^{d-3}-1$ characters (of height 1) all have F-S indicator 1:*

Morita equivalence class	$l(B)$	E	$\epsilon_1, \ldots, \epsilon_4; \mu$
D (*nilpotent*)	1	D, $D \times C_2$	$1,1,1,1;1$
		$D * C_4$	$1,1,1,1;-1$
		$D_{2^{d+1}}$	$0,0,1,1;1$
		$SD_{2^{d+1}}$	$0,0,1,1;-1$
		$C_{2^{d-1}} \rtimes C_2^2$, $d \geq 4$	$1,1,1,1;0$
$\mathrm{PGL}(2,q)$, $\lvert q-1\rvert_2 = 2^{d-1}$	2	D, $D \times C_2$	$1,1,1,1;1$
		$C_{2^{d-1}} \rtimes C_2^2$, $d \geq 4$	$1,1,1,1;0$
$\mathrm{PGL}(2,q)$, $\lvert q+1\rvert_2 = 2^{d-1}$	2	D, $D \times C_2$	$1,1,1,1;1$
$\mathrm{PSL}(2,q)$, $\lvert q-1\rvert_2 = 2^d$	3	D, $D \times C_2$	$1,1,1,1;1$
		$D_{2^{d+1}}$	$0,0,1,1;1$
		$SD_{2^{d+1}}$	$0,0,1,1;-1$
		$C_{2^{d-1}} \rtimes C_2^2$, $d \geq 4$	$1,1,1,1;0$
$\mathrm{PSL}(2,q)$, $\lvert q+1\rvert_2 = 2^d$	3	D, $D \times C_2$	$0,0,1,1;1$
		$D_{2^{d+1}}$	$1,1,1,1;1$
A_7, $d = 3$	3	D, $D \times C_2$	$1,1,1,1;1$

All cases occur for all d as indicated.

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Modules with finitely generated cohomology

Dave Benson

Let G be a finite group and k a field of characteristic p. A theorem of Lenny Evens [4] shows that if M is a finitely generated kG-module then $H^*(G, M)$ is a finitely generated module over the graded commutative Noetherian k-algebra $H^*(G, k)$. What about the converse? If G is a finite p-group then you only need $H^0(G, M)$ to be finite dimensional in order for M to be finitely generated. Furthermore, if M has no projective summands and there exists $i > 0$ such that $H^i(G, M)$ is finite dimensional then M is finitely generated. However, for more general finite groups there can be infinite dimensional modules M with no projective summands but $H^*(G, M) = 0$. For example for $G = \mathbb{Z}/3 \times \Sigma_3$ in characteristic three, there is a three dimensional module with this property.

Let us work in the stable module category $\mathsf{StMod}(kG)$, where we have quotiented out those maps which factor through a projective module. This is a triangulated category whose compact objects are the finitely generated modules. The only difference this makes to cohomology is that we have to replace $H^*(G, M)$ by $\hat{H}^{\geqslant 0}(G, M)$, which is identical except in degree zero. So by a no cohomology module, we mean an object M in $\mathsf{StMod}(kG)$ with $\hat{H}^{\geqslant 0}(G, M) = 0$, and when we speak of modules with finitely generated cohomology we mean that $\hat{H}^{\geqslant 0}(G, M)$ is finitely generated over $H^*(G, k)$, and when we speak of no cohomology modules we mean that $\hat{H}^{\geqslant 0}(G, M) = 0$.

Conjecture 1. The thick subcategory of $\mathsf{StMod}(kG)$ consisting of modules with finitely generated cohomology is generated by the thick subcategories of finitely generated modules and of no cohomology modules.

Now let BG be the classifying space of G. Write C^*BG for the cochains on BG with coefficients in k, and $\mathsf{D}(C^*BG)$ for its derived category. There are several equivalent ways to construct this. If we regard C^*BG as a differential graded algebra then we take differential graded modules and homotopy classes of morphisms, and we invert the quasi-isomorphisms. Alternatively, if we regard C^*BG as H^*BG with an A_∞-algebra structure, then we take A_∞-modules and homotopy classes of maps. The quasi-isomorphisms in this picture are automatically invertible. Finally, we might want to regard C^*BG as a commutative S-algebra in the sense of Elmendorf, Kříž, Mandell and May [3], in which case we take the category of modules and invert the weak equivalences.

We write $\mathsf{D}^{\mathsf{b}}(C^*BG)$ for the thick subcategory consisting of those objects X in $\mathsf{D}(C^*BG)$ such that $H^*(BG;X) = \mathsf{Hom}_{\mathsf{D}(C^*BG)}(C^*BG, X)$ is finitely generated as a module over the ring H^*BG. This definition is introduced and justified in Greenlees and Stevenson [5].

Conjecture 2 (Greenlees)**.** The subcategory $\mathsf{D}^{\mathsf{b}}(C^*BG)$ is equal to the thick subcategory of $\mathsf{D}(C^*BG)$ generated by the module C^*BS, where S is a Sylow p-subgroup of G.

Greenlees and I have recently proved that given a finite group G and a field k, Conjectures 1 and 2 are equivalent. The connection between the two statements goes via the intermediary of the category $\mathsf{KInj}(kG)$. This is the homotopy category of complexes of injective (= projective) kG-modules. Inside this category, the acyclic complexes give a copy of $\mathsf{StMod}(kG)$ via Tate resolutions, while the localising subcategory generated by ik, an injective resolution of the trivial module, gives a copy of $\mathsf{D}(C^*BG)$.

Next, we introduce the notion of the nucleus [2]. This has a group theoretic definition and a representation theoretic definition, and the statement that the two coincide is proved in [1]. Recall that Quillen [6] gave a description of $V_G = \mathsf{Spec}\, H^*(G,k)$ in terms of the category of elementary abelian p-subgroups of G and their conjugations and inclusions. In terms of this, the group theoretic nucleus is the subset defined as the union of the images of $V_E \to V_G$ as E ranges over the elementary abelian p-subgroups whose centraliser is not p-nilpotent. The representation theoretic nucleus is the union of the supports of the no cohomology modules in the principal block of kG. We shall write Θ_G for the nucleus defined through either of these routes. The relevance of the nucleus is partly explained by the following theorem from [1].

Theorem 3. *The stable category of the principal block of kG is generated by the trivial module k together with the modules whose support is contained in Θ_G.*

We are able to prove Conjectures 1 and 2 in the case where Θ_G is $\varnothing$ or $\{0\}$; namely groups in which the centraliser of every element of order p is p-nilpotent. In these cases, the support of the singularity category

$$\mathsf{D}_{\mathsf{sg}}(C^*BG) = \mathsf{D}^{\mathsf{b}}(C^*BG)/\mathsf{Thick}(C^*BG)$$

is exactly equal to the nucleus. We conjecture that this is the case in general. If Conjecture 2 is true then using Theorem 3, one can show that the support is contained in the nucleus, but the other direction seems harder.

Finally, Jon Carlson and I have managed to prove Conjecture 1 in two cases with non-trivial nucleus, namely $G = \mathbb{Z}/3 \times \Sigma_3$ in characteristic three, and $G = \mathbb{Z}/2 \times A_4$ in characteristic two. The proofs are not easy, and do not readily generalise.

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Cohomology of finite groups of Lie type and loop groups

Jesper Grodal
(joint work with Anssi Lahtinen)

My talk was about a new relationship between $H^*(BG(\mathbb{F}_q);\mathbb{F}_\ell)$, the cohomology of finite groups of Lie type $G(\mathbb{F}_q)$, and $H^*(BLG(\mathbb{C});\mathbb{F}_\ell)$, the cohomology of the classifying space of the corresponding loop group $LG(\mathbb{C})$, and was based on a recent preprint with Lahtinen [GL]. Here G is a connected split reductive group defined over $\mathbb{Z}$, and we can for our purposes model the loop group $LG(\mathbb{C})$ by the space of continuous maps from the circle S^1 to $G(\mathbb{C})$ with the analytic topology, a group under pointwise multiplication. (Taking smooth or polynomial maps does not change the cohomology in this case.) The space $BLG(\mathbb{C})$ also identifies with $LBG(\mathbb{C}) = \mathrm{map}(S^1, BG(\mathbb{C}))$.

Quillen [Qui72] calculated in 1972 that, when $\ell \mid q-1$,

$$H^*(B\mathrm{GL}_n(\mathbb{F}_q);\mathbb{F}_\ell) \cong \mathbb{F}_\ell[x_1,\dots,x_n] \otimes \wedge_{\mathbb{F}_\ell}(y_1,\dots,y_n),$$

where the polynomial generators x_i are in degree $2i$ and the exterior generators y_i are in degree $2i-1$ (only additively when $\ell = 2$ unless $4 \mid q-1$). This answer agrees with the cohomology of $H^*(LB\mathrm{GL}_n(\mathbb{C});\mathbb{F}_\ell)$. Note that on the level of spaces $LB\mathrm{GL}_1(\mathbb{C}) \simeq S^1 \times \mathbb{C}P^\infty$ whereas $B\mathrm{GL}_1(\mathbb{F}_q) \simeq B\mathbb{Z}/(q-1)$, so the underlying spaces are certainly not homotopy equivalent, even after completing at the prime ℓ, ruling out the existence of an underlying direct map of spaces. Despite this, subsequent calculations have revealed agreements between $H^*(BG(\mathbb{F}_q);\mathbb{F}_\ell)$ and $H^*(LBG(\mathbb{C});\mathbb{F}_\ell)$ for other connected split reductive algebraic groups G, as long as $\ell \mid q-1$. This led Tezuka to ask in 1998 [Tez98] if it could always be the case for any connected split reductive algebraic group? Evidence for this has been mounting, yet without displaying any direct link between the two objects.

In recent joint work with Anssi Lahtinen [GL] we combine string topology à la Chas-Sullivan with the theory of ℓ-compact groups, to show that $H^*(G(\mathbb{F}_q);\mathbb{F}_\ell)$ admits a module structure over $H^*(LBG(\mathbb{C});\mathbb{F}_\ell)$, equipped with a Chas-Sullivan type string product. We furthermore show, with a few potential exceptions, that this module structure is free of rank one over $H^*(LBG(\mathbb{C});\mathbb{F}_\ell)$ whenever $\ell \mid q-1$. We also show how to get around the congruence condition, by passing to a ℓ-compact group:

Theorem 1 (G-Lahtinen [GL]). *Suppose G is a simple simply connected split linear algebraic group over $\mathbb{Z}$, and ℓ a prime. Suppose (G,ℓ) is not one of the following 8 pairs: $(E_8,5)$, $(E_i,3)$, $(F_4,3)$, and $(E_i,2)$, $i=6,7,8$.*

If $\ell \mid q-1$, then $H^*(BG(\mathbb{F}_q);\mathbb{F}_\ell)$ is free of rank 1 over $H^*(LBG(\mathbb{C});\mathbb{F}_\ell)$ equipped with a Chas-Sullivan string product.

Without the congruence condition there still exists an ℓ–compact group BH, depending only on G and the multiplicative order of q modulo ℓ, so that $H^*(BG(\mathbb{F}_q);\mathbb{F}_\ell)$ is free of rank 1 over $H^*(LBH;\mathbb{F}_\ell)$.

We establish the result by first constructing the module structure and then proving that it is free of rank one if and only if a certain natural map $\mathbb{F}_\ell \cong H_d(G(\mathbb{C});\mathbb{F}_\ell) \to H_d(G(\mathbb{F}_q);\mathbb{F}_\ell)$, induced from a 'Lang map' of ℓ-complete spaces $G(\mathbb{C})\hat{}_\ell \to BG(\mathbb{F}_q)\hat{}_\ell$, is non-trivial, where d is the dimension of the maximal compact subgroup of $G(\mathbb{C})$. We then verify that it is non-trivial e..g, under the assumptions stated in Theorem 1. We also investigate what other structure the isomorphism between $H^*(BG(\mathbb{F}_q);\mathbb{F}_\ell)$ and $H^*(LBH;\mathbb{F}_\ell)$ preserves. We refer to [GL] for more detail.

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The fields of values of the 2-height zero characters

Carolina Vallejo Rodríguez

(joint work with Gabriel Navarro, Lucas Ruhstorfer, Pham Tiep)

It is not difficult to show that every finite abelian extension of $\mathbb{Q}$ can be realized as the field of values of some irreducible character of a finite (solvable) group. We recall the if χ is a character of G, then the field of values of χ is

$$\mathbb{Q}(\chi) = \mathbb{Q}(\chi(g) \mid g \in G).$$

We can more generally define $\mathbb{F}(\chi)$, the field of values of χ over any number field $\mathbb{F}$, as the composite field $\langle \mathbb{F}, \mathbb{Q}(\chi)\rangle$. In this report, we write $\mathbb{Q}_n = \mathbb{Q}(e^{2\pi i/n})$ for every positive integer n. With this convention, note that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_{|G|}$ for every character of the finite group G.

In 2019, Isaacs, Liebeck, Navarro and Tiep began the study of the field of values of the irreducible characters of odd degree in [1]. Their goal was to understand which quadratic extensions of $\mathbb{Q}$ could appear as those, as computations suggested that, quite surprisingly, fields as $\mathbb{Q}(\sqrt{\pm 2})$ and $\mathbb{Q}(\sqrt{3})$ were never realized as the field of values of an odd-degree irreducible character of a finite group. They succeeded in this goal [1, Theorem A] and the key was to show that an odd-degree

irreducible character χ of a finite group G is either 2-rational, that is, $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_m$ for some odd integer m, or $i \in \mathbb{Q}(\chi)$ [1, Theorem C]. Later, in 2021, Navarro and Tiep generalized these results by completely characterizing the fields of values of the irreducible odd-degree characters in [3, Theorems A1 and A2].

Odd-degree irreducible characters are precisely the height zero characters lying in 2-blocks of maximal defect. Our present research aims at understanding the fields of values of height zero characters in arbitrary 2-blocks. We can start by noticing that $\mathbb{Q}(\sqrt{3})$ does not appear as the field of values of any odd-degree irreducible character [1, Theorem A], but there are many 2-blocks that have a height zero character χ with $\mathbb{Q}(\chi) = \mathbb{Q}(\sqrt{3})$. For instance, if we look in a double cover of S_5. We remark that such a χ is not 2-rational and $i \notin \mathbb{Q}(\chi)$.

In order to state our main results it is convenient to introduce one last piece of notation. If $\mathbb{F}$ is an abelian number field, the conductor of $\mathbb{F}$ is the minimum k such that $\mathbb{F} \subseteq \mathbb{Q}_k$. We define the conductor $c(\chi)$ of a character χ as the conductor of $\mathbb{Q}(\chi)$ its field of values. Note that if χ is an irreducible character with $\mathbb{Q}(\chi) = \mathbb{Q}(\sqrt{3})$, then $c(\chi) = 12$ and $i \in \mathbb{Q}_3(\chi) = \mathbb{Q}_{12}$.

Theorem A. Let χ be an irreducible character of a finite group G having height zero in its 2-block. Write $c(\chi) = 2^a m$, where m is an odd prime, and $\mathbb{F} = \mathbb{Q}_m$. Then $\mathbb{Q}_{2^a} \subseteq \mathbb{F}(\chi)$ (so $\mathbb{F}(\chi) = \mathbb{Q}_n$).

While Theorem A does not strictly generalize [1, Theorem C], it has three important consequences: firstly, Theorem A allows us to completely determine the abelian number fields that appear as the field of values of irreducible characters of 2-height zero in finite groups (those are precisely the abelian number fields $\mathbb{F}$ that satisfy $\langle \mathbb{F}, \mathbb{Q}_m \rangle = \mathbb{Q}_n$ where $n = 2^a m$ is the conductor of $\mathbb{F}$ and m is odd); in particular, we can characterize the quadratic extensions of $\mathbb{Q}$ that appear as the field of values of 2-height zero irreducible characters; and lastly, we can give a new characterization of 2-rational irreducible 2-height zero characters.

Moreover the statement of Theorem A follows from the so-called Alperin-McKay-Navarro conjecture [2, Conjecture B]. This contrast the situation in [1] and [3], where the statements of the main results do not follow from the McKay-Navarro conjecture [2, Conjecture A].

As the McKay conjecture and its refinements deal with irreducible characters of degree not divisible by a fixed but arbitrary prime p; it is natural to wonder what are the fields of values the irreducible characters of degree not divisible by p for odd primes p, and more generally, of the p-height zero characters. In [3], it was conjectured that the set constituted by the fields of values of the irreducible characters of degree coprime to p of finite groups is exactly the set conformed by the abelian number fields $\mathbb{F}$ with $|\mathbb{Q}_{p^a} : \mathbb{Q}_{p^a} \cap \mathbb{F}|$ not divisible by p, where the conductor of $\mathbb{F}$ is $p^a m$ and m is coprime to p. As the authors notice in [3], the above conjecture does not seem to follow from the McKay-Navarro conjecture. We propose the following.

Conjecture B. Let p be a prime. Let $\mathbb{F}$ be an abelian number field. Write n the conductor of $\mathbb{F}$ as $n = p^a m$ where m is coprime to p. There is some irreducible

character χ of a finite group G having height zero in its p-block with $\mathbb{Q}(\chi) = \mathbb{F}$ if, and only if, $|\mathbb{Q}_n : \langle \mathbb{F}, \mathbb{Q}_m \rangle|$ is coprime to p.

We show that Conjecture B is implied by the statement Alperin-McKay-Navarro conjecture. Moreover, we can prove "if" implication in Conjecture B and show that the "only if" implication holds if it holds for the height zero characters lying in p-blocks of quasi-simple groups.

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Bounding the order of finite complex linear groups with restricted composition factors

Geoffrey Robinson

We consider a family $\mathcal{F}$ of finite groups which is closed under taking normal subgroups and homomorphic images, and we wish to determine under which conditions there is a constant $\alpha(\mathcal{F})$ such that whenever $G \in \mathcal{F}$ and also $G \subseteq \mathrm{GL}(n, \mathbb{C})$, we have

$$[G : F(G)] \leq \alpha(\mathcal{F})^{n-1}$$

and related questions of a similar nature. This question was motivated by a question from P. Etingof related to work of Coulembier, Etingof and Ostrik on asymptotic growth properties in symmetric tensor categories, but we obtain other applications as well. Since we do not require the family $\mathcal{F}$ to be closed under extensions, this restricts the use of some standard Clifford-theoretic reductions.

It turns out (using the classification of finite simple groups) that $\alpha(\mathcal{F})$ exists if and only if $\mathcal{F}$ contains only finitely many alternating groups. This suffices to deal with covers the questions of Etingof, which concerned finite complex linear groups of order coprime to a chosen prime p, and finite complex linear groups with an Abelian Sylow p-subgroups. For we note that when $\mathcal{F}$ is the family of all finite groups of order prime to p (for p an odd prime), then $\mathcal{F}$ contains no alternating group Alt_k with $k \geq p$. Similarly, when $\mathcal{F}$ is the family of all finite groups with Abelian Sylow p-subgroups (for p an odd prime), then $\mathcal{F}$ contains no alternating group Alt_k with $k \geq p^2$.

An examples of the results we prove is:

Theorem A: *Let p be an odd prime greater than 11, and let G be a finite subgroup of $\mathrm{GL}(n, \mathbb{C})$ with Abelian Sylow p-subgroups. Then G has an Abelian normal subgroup A with $[G : A] \leq \left((p^2-1)!\right)^{\frac{n-1}{p^2-3}}$.*

Our methods also allow us to produce a relatively short proof of the following variant of theorems of B. Weisfeiler and M.J. Collins:

Let G be a finite subgroup of $\mathrm{GL}(n,\mathbb{C})$, and suppose that Alt_m is the largest alternating group which occurs as a composition factor of G. Then G has an Abelian normal subgroup A with

$$[G:A] \leq \left(\max\{60, m!^{\frac{1}{m-2}}\}\right)^{n-1}.$$

Furthermore, if $[G:A] > 60^{n-1}$, then $m > 151$ and

$$[G:A] \leq (m!)^{\frac{n-1}{m-2}} \leq (n+1)!.$$

On endotrivial modules for finite groups

Nadia Mazza
(joint work with Jon Carlson, Jesper Grodal, Dan Nakano)

Let G be a finite group and let k be a field of positive characteristic p dividing $|G|$. We consider finitely generated kG-modules. A kG-module M is *endotrivial* if $\mathrm{End}_k M$ is isomorphic (as kG-module) to the direct sum of the trivial module k and some projective module, or equivalently, $M^* \otimes M \cong k$ in the stable module category $\underline{\mathrm{mod}}(kG)$ where $M^* = \mathrm{Hom}(M,k)$ and $\otimes = \otimes_k$. The stable isomorphism classes of endotrivial modules form an abelian group, denoted $T(G)$ and called the *group of endotrivial modules* for G.

The stable isomorphism classes of endotrivial modules are the invertible elements of Green ring $\underline{\mathrm{mod}}(kG)$, and they also are the "building bricks" of the source modules of simple modules for finite p-soluble groups. Our ultimate objective is to classify endotrivial modules for *all* finite groups, or at least determine $T(G)$ as an abelian group. Numerous results have already been obtained in that direction. In particular, $T(G)$ is finitely generated, which allows us to split the question into two: finding the torsion subgroup $TT(G)$ of $T(G)$, and a torsion free complement $TF(G)$ of $TT(G)$ in $T(G)$.

The main, and often unique, example of endotrivial module of dimension greater than one is the syzygy $\Omega(k)$ of the trivial module, i.e. $\Omega(k) = \ker(P \twoheadrightarrow k)$ of a projective cover of k.

Henceforth, let $S \in \mathrm{Syl}_p(G)$, and set $N = N_G(S)$. We record the following results on endotrivial modules (cf. [4] for details and background):

- $T(N) \cong T(S)^N \oplus \mathrm{Hom}(N, k^\times)$, the direct sum of the N-stable elements of $T(S)$ and the one-dimensional kN-modules. Moreover, $T(S)$ is known by generators and relations.
- The rank n of $TF(G)$ as abelian group is equal to the number of components of the orbit space $\mathcal{A}_p^{\geq 2}(G)/G$ where $\mathcal{A}_p^{\geq 2}(G)$ is the poset of noncyclic elementary abelian p-subgroups of G. In particular, $n = 0$ if G has p-rank less than 2 and $n = 1$ if G has p-rank greater than p if p is odd, or 4 if $p = 2$. In any case, $n \leq p+1$ if p is odd, and $n \leq 5$ if $p = 2$. Moreover, if

$n \geq 1$, then $[\Omega(k)]$ generates an infinite cyclic direct summand in $TF(G)$. A result of Grodal uses homotopy methods, and in the cases of interest for the present work, reduces, in some sense, the computation of $K(G)$ to a p-local analysis.

In our joint work, we aim to determine $T(G)$ for all finite groups of Lie type G. By a *finite group of Lie type* we mean the group of fixed points $G = \mathbf{G}^F$ of a connected reductive algebraic group $\mathbf{G}$ defined over an algebraically closed field of positive characteristic under a Steinberg endomorphism (e.g. $\mathrm{SL}_n(q)$). We refer to [3] for the theoretical background.

Results for TF($\mathbf{G}$). Let G be a finite group of Lie type (of order divisible by p). The following hold.

- $TF(G) = 0$ whenever G has p-rank one.
- $TF(G) = \langle [\Omega(k)] \rangle \cong \mathbb{Z}$ unless G is isomorphic to one of the following:
  - $\mathrm{PGL}_p(q)$ with $p \mid q-1$, or $\mathrm{PGU}_p(q)$ with $p \mid q+1$, or ${}^3D_4(q)$ with $p = 3$ and $\gcd(3, q) = 1$.
  - $G/Z(G)$ is isomorphic to $\mathrm{PSL}_3(p), \mathrm{PSU}_3(p), \mathrm{PGL}_3(p), \mathrm{PSpin}_5(p)$, $\mathrm{SO}_5(p)$ or $G_2(p)$, with some further restrictions on the values of p.

  Detailed results are provided in [1, Theorem A].

Results for TT($\mathbf{G}$). Let G be a finite group of Lie type (of order divisible by p). The following hold.

- If G has p-rank one, then $\mathrm{Res}^G_{N_G(Z)} : T(G) \to T(N_G(Z))$ is an isomorphism, where Z is the unique subgroup of S of order p.
- $TT(G) = \mathrm{Hom}(G, k^\times) \cong G/[G, G]S$ unless G is isomorphic to a product $H \times K$ with K a p'-group and H is isomorphic to one of the following:
  - $\mathrm{SU}_3(2), \mathrm{Sp}_4(2), G_2(2)$ or $\mathrm{SL}_4(2)$ with $p = 3$.
  - $\mathrm{Sp}_8(2), F_4(2), E_7(2)$ or $E_7(3)_{sc}$ with $p = 5$, where the subscript sc denotes the fixed points $\mathbf{E_7}^{F_7}$ under the Frobenius endomorphism F_7 of the simply connected reductive group $\mathbf{E_7}$.

  Detailed results are provided in [2, Theorem A], together with the results for the very twisted groups.

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Spetses from the p-local perspective

Jason Semeraro
(joint work with Radha Kessar, Gunter Malle)

Let ℓ be a prime and k be an algebraically closed field of characteristic ℓ. It is often possible to describe the local side of a local-global counting conjecture in terms of fusion systems. To a finite group G one can associate a *saturated fusion system* $\mathcal{F}_S(G)$ on a Sylow ℓ-subgroup S of G. In general a saturated fusion system $\mathcal{F}$ on an ℓ-group S is a category on the subgroups of an ℓ-group whose morphisms are group homomorphisms satisfying particular axioms (all satisfied when $\mathcal{F} = \mathcal{F}_S(G)$). Such a category is called *exotic* if it is not realised by any finite group G in this way. It is a theorem that any ℓ-block B of kG determines a saturated fusion system $\mathcal{F}_D(B)$ on its defect group D and an open problem to show that $\mathcal{F}_D(B)$ is never exotic. The author was motivated by the following invariant for fusion systems. Let $d \geq 0$ and $\mathcal{F}$ be a saturated fusion system on S. A subgroup $Q \leq S$ is said to be $\mathcal{F}$-*centric* if $C_S(P) \leq P$ for all $P \in Q^{\mathcal{F}}$ (that is for all subgroups $\mathcal{F}$-conjugate to Q). For each $\mathcal{F}$-centric subgroup Q, define:

$$\mathcal{N}_Q := \{\sigma = (1 = X_0 \trianglelefteq X_1 \trianglelefteq X_2 \trianglelefteq \cdots \trianglelefteq X_m) \mid X_m \leq_\ell \mathrm{Out}_{\mathcal{F}}(Q), m \geq 0\}$$

to be the set of all normal chains of ℓ-subgroups of $\mathrm{Out}_{\mathcal{F}}(Q) = \mathrm{Aut}_{\mathcal{F}}(Q)/\mathrm{Inn}(Q)$ of length $|\sigma| = m$ and for $\sigma \in \mathcal{N}_Q$, write $I(\sigma) = \mathrm{stab}_{\mathrm{Out}_{\mathcal{F}}(Q)}(\sigma)$ for its stabiliser. Now, following [7], set

$$w_Q(\mathcal{F}, d) := \sum_{\sigma \in \mathcal{N}_Q/\,\mathrm{Out}_{\mathcal{F}}(Q)} (-1)^{|\sigma|} \sum_{\mu \in \mathrm{Irr}^d(Q)/I(\sigma)} z(kI(\sigma, \mu)),$$

where $\mathrm{Irr}^d(Q) = \{\mu \in \mathrm{Irr}(Q) \mid v_\ell(|Q|/\mu(1)) = d\}$ is the number of characters of ℓ-defect d and $z(-)$ denotes the number of ℓ-defect zero blocks. Finally, set

$$\mathbf{m}(\mathcal{F}, d) = \sum_Q w_Q(\mathcal{F}, d),$$

where the sum runs over all $\mathcal{F}$-centric subgroups Q up to $\mathcal{F}$-conjugacy. Robinson's Ordinary Weight Conjecture (OWC) asserts that for the principal ℓ-block B_0 of any finite group G there is an equality

$$|\,\mathrm{Irr}^d(B_0)| = \mathbf{m}(\mathcal{F}_S(G), d)$$

where $\mathrm{Irr}^d(B_0)$ denotes the set of ordinary characters of G in B_0 of ℓ-defect d. Since the RHS of this equality makes perfect sense for arbitrary saturated fusion systems one is naturally led to ask whether the integer $\mathbf{m}(\mathcal{F}, d)$ is of any importance/significance for exotic fusion systems. Indeed in 2010, Markus Linckelmann asked whether particular properties of this integer might be even used to *detect* the exoticity of a fusion system. This question motivated the author to calculate $\mathbf{m}(\mathcal{F}, d)$ for large collections of exotic fusion systems on a computer. Somewhat surprisingly, the results did not produce significant observable differences with the group case. This 'good' behaviour, in turn, led to several conjectures about

the invariant in joint work with Justin Lynd, Radha Kessar and Markus Linckelmann (see [7, Section 2]). For example, we conjectured that $\mathbf{m}(\mathcal{F}, d) \geq 0$ for any saturated fusion system and non-negative integer d.

A particularly interesting family of fusion systems on which to test these conjectures is the (Benson–Solomon) family of exotic simple 2-fusion systems with q an odd prime power. Calculation of the integers $\mathbf{m}(\mathcal{F}, d)$ in [10] yielded a further surprise: not only were they always positive, they were also generic polynomials in the 2-part of $q^2 - 1$! In an attempt to explain this, and realising that for certain groups of Lie type $\mathrm{Irr}^d(B_0)$ could be calculated in terms of the underlying fusion system and lists of unipotent degrees, the author employed the theory of spetses (see below) to define an 'ad-hoc' analogue of $\mathrm{Irr}^d(B_0)$ for $\mathrm{BSol}(q)$. For this set, he then conjectured and proved a version of (OWC).

These observations seemed to point to an unexplored connection between ℓ-compact groups and spetses which was the starting point for a joint project [8] with Radha Kessar and Gunter Malle. An *ℓ-compact group BX* is a connected pointed ℓ-complete topological space whose loop space has finite mod ℓ-cohomology. Such an object may be regarded as a certain homotopical version of a compact connected Lie group $\mathbf{G}$. In particular $(B\mathbf{G})^\wedge_\ell$ is an ℓ-compact group, but there are also exotic examples which are not of this form. Like a compact Lie group, any ℓ-compact group BX has a *maximal torus* BT and *Weyl group* W such that $(W, \pi_2(T))$ is an ℓ-adic reflection group. Moreover, $(W, \pi_2(T))$ controls the structure of BX see [1, 2]. Recall that an *ℓ-adic reflection group* is a finite subgroup W of $GL(L)$, (L a $\mathbb{Z}_\ell$-lattice) generated by *reflections*, i.e. elements of W which pointwise fix a sublattice of L of corank 1. Note that, via extension of scalars, ℓ-adic relection groups form a sub-class of complex reflection groups.

Now if BX is a simply connected ℓ-compact group with Weyl group W and q is a prime power with $(\ell, q) = 1$ then a result of Broto–Møller [3] shows that to the pair (W, q) one can associate a saturated fusion system $\mathcal{F}$ on a finite ℓ-group. More explicitly, it is shown that the classifying space of $\mathcal{F}$ is the space of homotopy fixed points of BX under an unstable Adams operation ψ^q. For example if $\mathbf{G}$ a compact connected Lie group and $BX = (B\mathbf{G})^\wedge_\ell$, $\mathcal{F}$ is the fusion system of (the principal ℓ-block of) the corresponding finite group of Lie type, [6, 9]. When BX is exotic, we also obtain exotic fusion systems corresponding to complex reflection groups $W = G(m, r, n)$, $r > 2, n \geq \ell$ and $\ell \mid (q - 1)$ (ℓ odd) and the Benson–Solomon fusion systems when $W = G_{24}$ and $\ell = 2$).

In a completely different direction, if $W \leq \mathrm{GL}(V)$ is a *spetsial* complex reflection group and $\varphi \in N_{\mathrm{GL}(V)}(W)$, the Broué–Malle–Michel theory of spetses [4] explains how to associate to the pair $\mathbb{G} := (W, \varphi)$:

- a set $\mathrm{Uch}(\mathbb{G})$ of "unipotent" characters, and for each $\gamma \in \mathrm{Uch}(\mathbb{G})$, a "degree" $\gamma(1) \in \mathbb{Q}_W[x]$, where $\mathbb{Q}_W$ is the character field of W;
- for a root of unity ζ, a partition of $\mathrm{Uch}(\mathbb{G})$ into ζ-Harish–Chandra series;
- a cyclotomic Hecke Algebra: $\mathcal{H}(\mathbb{G}, \mathbf{u})$ (see also, [5]).

If $W = W(\mathbf{G})$ is the Weyl group of a connected reductive group $\mathbf{G}$, $q = p^f$ then:

- Uch($\mathbb{G}$) corresponds to the set Uch($\mathbf{G}(q)$) of unipotent characters of $\mathbf{G}(q)$ and for each $\gamma \in \mathrm{Uch}(\mathbb{G})$ $\gamma(1)(q)$ is the correct degree;
- the ζ-series partition of Uch($\mathbb{G}$) corresponds to the ℓ-block partition of $\mathbf{G}(q)$, for $q \equiv \zeta \pmod{\ell}$;
- $\mathcal{H}(W, \mathbf{u})$ specialises to Iwahori–Hecke algebra of $\mathbf{G}(q)$.

It turns out that one can combine these two sets of data to formulate a good notion of the characters in the principal ℓ-block $\mathbb{B}_0$ of an ℓ-adic spets $\mathbb{G} := (W, \mathrm{Id})$ as follows. Let BX be the ℓ-compact group attached to W and $q \equiv \zeta \pmod{\ell}$ be a prime power coprime with ℓ. Let $(S, \mathcal{F})$ be the fusion system corresponding to (BX, q) and assume that BX is simply connected. Now to the pair $(\mathbb{G}, q)$ we associate a set

$$\mathrm{Irr}(\mathbb{B}_0) := \bigcup_{s \in S/\mathcal{F}} \mathcal{E}(\mathbb{G}, s)_\zeta, \text{ where } \mathcal{E}(\mathbb{G}, s) \leftrightarrow \mathrm{Uch}(C_{\mathbb{G}}(s))_\zeta$$

of "irreducible characters" in $\mathbb{B}_0$. This has the property that if $(\mathbb{G}, q)$ corresponds to $\mathbf{G}(q)$ then there exists a degree-preserving bijection between $\mathrm{Irr}(\mathbb{B}_0)$ and $\mathrm{Irr}(B_0)$, where B_0 is the principal ℓ-block of $\mathbf{G}(q)$. As a consequence we can formulate a version of (OWC) for $\mathbb{B}$ which simply asserts that $|\mathrm{Irr}^d(\mathbb{B}_0)| = \mathbf{m}(\mathcal{F}, d)$ for all $d \geq 0$. This was shown to hold in [8] in the case $W = G(e, r, \ell)$ when $q \equiv 1 \pmod{\ell}$ and, after slightly modifying $\mathbb{B}_0$, for $\ell = 2$ in the case $W = G_{24}$ when $q \equiv 1 \pmod{4}$ by applying the main result of [10].

This talk will be survey of the above, together with a discussion of recent progress in generalising unipotent blocks, their dimensions and character values on semisimple ℓ-elements. Our results and conjectures provide significant evidence of a substantial ℓ-local enrichment of the existing Broué-Malle-Michel theory of spetses and offer a valuable new perspective on the local-global conjectures of modular representation theory.

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The $E1$-property of finite groups

Gerhard Hiss
(joint work with Rafał Lutowski)

Motivated by fixed point theory on manifolds, Dekimpe, de Rock and Penninckx formulated a slightly weaker form of the following conjecture.

Conjecture (following [1, Conjecture 4.8]). Let G be a finite group and V a non-trivial, irreducible, odd-dimensional $\mathbb{R}G$-module. Write $\rho : G \to \mathrm{GL}(V)$ for the representation of G afforded by V. Then, for every $n \in N_{\mathrm{GL}(V)}(\rho(G))$ of finite order, there is $g \in G$ such that $\rho(g)n$ has eigenvalue 1. □

The aim of the joint project with Rafał Lutowski is to prove the above conjecture, which would imply the truth of [1, Conjecture 4.8]. In order to present some of the intermediate results we have achieved on this way, let me set up some terminology.

Let G be a finite group and let V be a finite-dimensional $\mathbb{R}G$-module. Denote by ρ the corresponding representation of G. For the purpose of this talk assume that ρ is faithful, and consider G as a subgroup of $\mathrm{GL}(V)$ via the embedding ρ. For $n \in \mathrm{GL}(V)$ of finite order normalizing G, we say that (V, n) *has the* $E1$*-property*, if there is $g \in G$ such that gn has eigenvalue 1. We say that V *has the* $E1$*-property*, if this condition holds for all such n. Finally G *has the* $E1$*-property*, if V has the $E1$-property for all non-trivial, irreducible, odd-dimensional $\mathbb{R}G$-modules V. Thus the conjecture predicts that every finite group has the $E1$-property.

The trivial $\mathbb{R}G$-module (for the trivial group G) does not have the $E1$-property. It is easy to see that every elementary abelian 2-group does have the $E1$-property. In [1, Example 4.9] the authors show that the absolute irreducible 4-dimensional real representation of the extraspecial group 2_+^{1+4} does not have the $E1$-property.

The following result gives a sufficient criterion for the $E1$-property.

Proposition 1. Assume that V is absolutely irreducible and let $n \in \mathrm{GL}(V)$ be of finite order normalizing G. Suppose that there exist $g \in G$ such that $x := gn$ satisfies

$$\dim(V) > (|x| - 1)|C_G(y)|^{1/2} \qquad \text{for all } 1 \neq y \in \langle x \rangle.$$

Then (V, n) has the $E1$-property. □

From now on assume that V is non-trivial and odd-dimensional. Then V is absolutely irreducible. We have reduced the conjecture to finite simple groups.

Theorem 1. A minimal counterexample to the conjecture is a non-abelian finite simple group. □

As a corollary, a finite solvable group has the $E1$-property. From now on let G be a non-abelian simple group. It is not difficult to see that G has the $E1$-property

if $\mathrm{Aut}(G) \cong G$ or if $\mathrm{Aut}(G)$ is a split extension of $\mathrm{Inn}(G)$ with a group of order 2. Hence a sporadic simple group or a simple alternating group (except, possibly A_6) is not a minimal counterexample to the conjecture.

It thus remains to consider the finite simple group of Lie type. Let G be such a group, let p be its characteristic, and let $B = UT$ denote a Borel subgroup of G, where T is a maximally split torus and U a Sylow p-subgroup of G.

Assume first that p is odd. Then the restriction of V to B contains a homogeneous component V_1 of odd dimension, such that the irreducible submodules of V_1 have dimension 1 and their character λ satisfies $\lambda^2 = 1_B$. This allows to prove, in most cases, that V satisfies the $E1$-property by restricting V to suitable parabolic subgroups of V. At the time of this writing, the $E1$-property for some instances (V, n) still has to be checked.

Suppose now $p = 2$. Here, our approach further subdivides into two cases. In the first of these, we can give a complete answer.

Theorem 2. Suppose that $G = \underline{G}^F$ for some simple algebraic group $\underline{G}$ of adjoint type of characteristic 2, and some suitable Steinberg morphism F of $\underline{G}$. Assume also that $G \ncong P\Omega_8^+(q)$. Then G has the $E1$-property. □

Let G be a group as in Theorem 2, and let (V, n) be a pair arising from G as in the conjecture. Modifying n by an element of G, we may assume that n normalizes U and T. Now restrict V to U, the Sylow 2-subgroup of G. Under our hypotheses, this restriction contains, after a further modification of n by an element of T, an n-invariant homogeneous component of odd dimension, consisting of a direct sum of non-trivial 1-dimensional simple modules. As n has eigenvalue 1 or -1 on V_1, this shows that (V, n) has the $E1$-property.

This leaves to consider the remaining groups of Lie type of characteristic 2. These are the simple groups $\mathrm{PSL}_d(q)$ and $\mathrm{PSU}_d(q)$ with $\gcd(d, q-1) > 1$ respectively $\gcd(d, q+1) > 1$, the simple groups $E_6(q)$ and ${}^2E_6(q)$ with $3 \mid q-1$ respectively $3 \mid q+1$, and the groups $P\Omega_8^+(q)$, where, in all cases, q is even. Most instances of V are ruled out with Proposition 1. This requires a detailed knowledge of the fixed point subgroups of non-inner automorphisms of G of prime order. When these methods do not apply, i.e. when $\dim(V)$ is too small, we show that V is Harish-Chandra induced from a suitable odd-dimensional real module of a suitable Levi subgroup of G. This step relies on the classification of the characters of the possible V via Lusztig's generalized Jordan decomposition.

Although some steps are still missing, I am confident that the methods we have acquired so far will lead to a proof of the conjecture.

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Variations of Baer-Suzuki theorem and applications

Hung P. Tong-Viet
(joint work with R. Guralnick and G. Tracey)

The Baer-Suzuki theorem, a classical result in finite group theory, states that if C is a conjugacy class of a finite group G and if every two elements in C generate a nilpotent subgroup, then C generates a nilpotent normal subgroup of G. This gives a nice characterization of the Fitting subgroup of G, that is, the largest nilpotent normal subgroup of G. This theorem was originally proved by Baer in 1957 and later by M. Suzuki in 1965. A more direct and elementary proof was obtained by Alperin and Lyons in [1]. This theorem was used by M. Suzuki in the classification of finite simple groups. Especially, it was used to show that every finite nonabelian simple group possesses a nontrivial real element of odd order. Many generalizations of this theorem have been proposed and studied over the years.

In [5, Theorem A], Guralnick and Robinson show that for a finite group G and a prime p, if $x \in G$ is an element of order p, and $[x, g]$ is a p-element for all elements $g \in G$, then $x \in O_p(G)$. Guralnick and G. Malle extend this to all p-elements in [3, Theorem 1.4]. These results have recently been used in [2] to show that if $x \in G$ is a p-element and xy is a p-element for all p-elements $y \in G$, then $x \in O_p(G)$.

In [6], we extend this result as follows.

Theorem 1. *Let G be a finite group and let p be a prime. If $x \in G$ is a p-element and xy is either 1 or p-singular for every p-element $y \in G$, then $x \in O_p(G)$.*

Recall that an element is p-singular if its order is divisible by p. Following Guralnick and Moretó [4], an irreducible complex character χ of a finite group G is said to be multiplicative if $\chi(xy) = \chi(x)\chi(y)$ for all nonidentity elements $x, y \in G$ with $\gcd(o(x), o(y)) = 1$. Clearly every linear character is multiplicative.

As an application of Theorem 1, we prove in [6] the following which partially answers a question raised in [4].

Theorem 2. *Let G be a finite group and let χ be a nonlinear irreducible complex character of G. Then χ is multiplicative if and only if there is a prime p such that χ vanishes off $O_p(G)$.*

In [6], we also generalize several other results in [5] and [2].

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Modular Isomorphism Problem – progress, solution and open challenges

Leo Margolis
(joint work with D. García-Lucas, Á. del Río, T. Sakurai, M. Stanojkovski)

Say we are given only the R-algebra structure of a group ring RG of a finite group G over a commutative ring R. Can we then find the isomorphism type of G as a group? This so-called Isomorphism Problem has obvious negative answers, considering e.g. abelian groups over the complex numbers, but more specific formulations have led to many deep results and beautiful mathematics. The last classical open formulation was the so-called Modular Isomorphism Problem: Does the isomorphism type of kG as a ring determine the isomorphism type of G as a group, if G is a p-group and k a field of characteristic p? This question goes back at least to Brauer's influential survey [Bra63].

We start with a short overview of some known results before presenting a rather recent general solution to the problem: there is a series of non-isomorphic 2-groups which have isomorphic group algebras over any field of characteristic 2 [GLMdR22]. These groups are 2-generated with cyclic derived subgroup and of nilpotency class 3. Several natural questions remain open though, including:

- Does the Modular Isomorphism Problem have a positive answer for groups of odd order?
- Does it have a positive answer for groups of nilpotency class 2?
- What is the role of the field in the problem: might there in particular exist p-groups G and H and a field k of characteristic p such that $\mathbb{F}_pG \ncong \mathbb{F}_pH$, but $kG \cong kH$?
- As the isomorphism type of kG can not be recovered from kG, which are actually the properties of G which can be read from the structure of kG?

I will present some new ideas concerning these questions partly covered by [MSS21, GLdRS22, GL22].

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Tower equivalence and Lusztig's truncated Fourier transform

Jean Michel

Let $W \in \mathrm{GL}(V)$ where $V = \mathbb{C}^n$ be an irreducible well-generated complex reflection group. We call a tower T a maximal chain $1 = W_0 \subset W_1 \subset \ldots \subset W_n = W$ of parabolic subgroups. If $\mathrm{Ref}(W)$ are the reflections, we associate to a tower a set of "Jucys-Murphy" elements

$$J_T^i = \sum_{r \in \mathrm{Ref}(W_i) - \mathrm{Ref}(W_{i-1})} r$$

and a commutative subalgebra $J_T = \mathbb{C}[J_T^1, \ldots, J_T^n] \subset \mathbb{C}(W)$. Two complex characters $\chi, \chi' \in \mathrm{Irr}(W)$ are said T-equivalent if they have the same restriction to J_T. The characters χ and χ' are said tower-equivalent, written $\chi \equiv \chi'$, if they are tower-equivalent for all towers.

A coxeter element is an element c of W which has $e^{2i\pi/h}$ as an eigenvalue on V, where h is the highest reflection degree of W. The main result of [1] is the tower equivalence

$$\sum_{\chi \in \mathrm{Irr}(W)} \chi(c^{-1})\chi \equiv \sum_{i=0}^{n} (-1)^i \chi_{\Lambda^i(V)},$$

where on the RHS the sum is over the exterior powers of the reflection representation. They show this case-by-case.

I have a more conceptual proof for spetsial reflection groups using Lusztig's truncated Fourier transform. I give the definition for Weyl groups, it is extended verbatim to spetsial groups.

Let $\mathbf{G}^F$ be a split finite reductive group with Weyl group W, and let $R_{\mathbf{T}_w}^{\mathbf{G}}\mathrm{Id}$ be the Deligne-Lusztig induced from a torus of type $w \in W$, and let U_χ for $\chi \in \mathrm{Irr}(W)$ be the principal series unipotent character indexed by χ.

We define the truncated Lusztig Fourier transform as the application which maps $\chi \in \mathrm{Irr}(W)$ to the class function $f(\chi)$ on W defined by

$$\text{for } w \in W,\ f(\chi)(w) = \langle R_{\mathbf{T}_w}^{\mathbf{G}}\mathrm{Id}, U_\chi \rangle_{\mathbf{G}^F}.$$

The definition is extended by linearity to all class functions.

The properties I prove from which I deduce the result of [1] are

- For any class function on W, we have $\chi \equiv f(\chi)$.
- For $\chi \in \mathrm{Irr}(W)$ we have $f(\chi)(c) = \begin{cases} (-1)^i & \text{if } \chi = \chi_{\Lambda^i(V)}, \\ 0 & \text{otherwise.} \end{cases}$.
- $f(\sum_{\chi \in \mathrm{Irr}(W)} \chi(w)\chi) = \sum_{\chi \in \mathrm{Irr}(W)} f(\chi)(w)\chi$.
- For any $\chi \in \mathrm{Irr}(W)$ we have $\chi \equiv \overline{\chi}$.

It follows from the first item that the image of $\mathrm{Id} - f$ is a vector space of class functions tower equivalent to 0. I call kernel of the tower equivalence the space of all class functions tower equivalent to 0. Many computations support the conjecture:

Conjecture 1. *For any irreducible spetsial group except G_{32}, the image of* $\mathrm{Id} - f$ *is equal to the kernel of the tower equivalence.*

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Functorial equivalence of blocks

Serge Bouc
(joint work with Deniz Yılmaz)

Let k be an algebraically closed field of characteristic $p > 0$, let R be a commutative ring, and let $\mathbb{F}$ be an algebraically closed field of characteristic 0. We consider ([BY20], [BY22]) the following category Rpp_k^Δ:

- The objects of Rpp_k^Δ are the finite groups.
- For finite groups G and H, the set of morphisms $\mathrm{Hom}_{Rpp_k^\Delta}(G,H)$ from G to H in Rpp_k^Δ is equal to $RT^\Delta(H,G) = R \otimes_{\mathbb{Z}} T^\Delta(H,G)$, where $T^\Delta(H,G)$ is the Grothendieck group of the category of *diagonal p-permutation (kH, kG)-bimodules*. These are p-permutation bimodules which are projective when considered as left or right modules (or equivalently, p-permutation bimodules which admit only indecomposable direct summands with twisted diagonal vertices).
- The composition in Rpp_k^Δ is induced by R-linearity from the usual tensor product of bimodules: if G, H, and K are finite groups, if M is a diagonal p-permutation (kH, kG)-bimodule and N is a diagonal p-permutation (kK, kH)-bimodule, then $N \otimes_{kH} M$ is a diagonal p-permutation (kK, kG)-bimodule. The composition of (the isomorphism class of) N and (the isomorphism class of) M is by definition (the isomorphism class of) $N \otimes_{kH} M$.
- The identity morphism of the group G is the (isomorphism class of the) (kG, kG)-bimodule kG.

The category Rpp_k^Δ is an R-linear category. The R-linear functors from Rpp_k^Δ to the category ${}_R\mathsf{Mod}$ of R-modules are called *diagonal p-permutation functors* over R. These functors, together with natural transformations between them, form an R-linear abelian category $\mathcal{F}^\Delta_{Rpp_k}$.

Our first main result is the following:

Theorem 1. *The category $\mathcal{F}^\Delta_{\mathbb{F}pp_k}$ of diagonal p-permutation functors over $\mathbb{F}$ is semisimple.*

In order to describe the simple objects of $\mathcal{F}^{\Delta}_{\mathbb{F}pp_k}$, we consider *pairs* (L,u), where L is a finite p-group and u is a generator of a p'-group acting on L. We write $L\langle u\rangle := L \rtimes \langle u\rangle$ for the corresponding semi-direct product. We say that two pairs (L,u) and (M,v) are *isomorphic* if there is a group isomorphism $f : L\langle u\rangle \to M\langle v\rangle$ that sends u to a conjugate of v. We denote by $\mathrm{Aut}(L,u)$ the group of automorphisms of $L\langle u\rangle$ sending u to a conjugate of u, and by $\mathrm{Out}(L,u)$ the quotient of $\mathrm{Aut}(L,u)$ by the subgroup of inner automorphisms of $L\langle u\rangle$. We say that a pair (L,u) is a D^{Δ}*-pair* if $C_{\langle u\rangle}(L) = 1$.

Theorem 2. *The simple diagonal p-permutation functors over $\mathbb{F}$, up to isomorphism, are parametrized by isomorphism classes of triples (L,u,V), where (L,u) is a D^{Δ}-pair and V is a simple $\mathbb{F}\mathrm{Out}(L,u)$-module.*

Let (G,b) be a pair of a finite group G and a block idempotent b of kG. Then kGb is a diagonal p-permutation (kG,kG)-bimodule, and its class in $RT^{\Delta}(G,G)$ is an idempotent endomorphism of G in Rpp^{Δ}_k. We denote by $RT^{\Delta}_{G,b}$ or $RT^{\Delta}(-,G)b$ the diagonal p-permutation functor over R defined by $RT^{\Delta}_{G,b} = RT^{\Delta}(-,G)\circ kGb$. Then $RT^{\Delta}_{G,b}$ is a direct summand of the representable functor $RT^{\Delta}(-,G)$, so it is a projective object of $\mathcal{F}^{\Delta}_{Rpp_k}$, by the Yoneda lemma.

Definition 3. *Let G and H be finite groups, let b be a block idempotent of kG, and c be a block idempotent of kH. We say that the pairs (G,b) and (H,c) are* functorially equivalent over R *if the functors $RT^{\Delta}_{G,b}$ and $RT^{\Delta}_{H,c}$ are isomorphic in $\mathcal{F}^{\Delta}_{Rpp_k}$.*

By the Yoneda lemma, the pairs (G,b) and (H,c) are functorially equivalent if and only if there are elements $\sigma \in cRT^{\Delta}(H,G)b$ and $\tau \in bRT^{\Delta}(G,H)c$ such that $\tau\circ\sigma = kGb$ in $bRT^{\Delta}(G,G)b$ and $\sigma\circ\tau = kHc$ in $cRT^{\Delta}(H,H)c$. In particular, functorial equivalence over $\mathbb{Z}$ implies functorial equivalence over any R. It is slightly weaker than p-permutation equivalence of blocks ([BX08], [BP20]), which requires in addition that τ be equal to the opposite of σ.

By Theorem 1, the functor $\mathbb{F}T^{\Delta}_{(G,b)}$ splits as a direct sum of simple functors in $\mathcal{F}^{\Delta}_{\mathbb{F}pp_k}$. We obtain three equivalent formulas for the multiplicity of a given simple functor $S_{L,u,V}$ as a direct summand of $\mathbb{F}T^{\Delta}_{G,b}$:

- One in terms of fixed points of some subgroups of $\mathrm{Out}(L,u)$ on V.
- One in terms of "u-invariant" (G,b)-Brauer pairs (P,e).
- One in terms of the "u-invariant" local pointed groups P_γ on kGb.

Using these formulas, we show that functorial equivalence preserves some important invariants of blocks:

Theorem 4. *Let b be a block idempotent of kG and c a block idempotent of kH. If (G,b) and (H,c) are functorially equivalent over $\mathbb{F}$, then:*

(i) $\mathrm{l}(kGb) = \mathrm{l}(kHc)$.
(ii) $\mathrm{k}(kGb) = \mathrm{k}(kHc)$.
(iii) b and c have isomorphic defect groups.

We also get the following characterization of nilpotent blocks:

Theorem 5. *Let b a block idempotent of kG with defect group D. The following are equivalent:*

(i) The block b is nilpotent.
(ii) If $S_{L,u,V}$ is a simple summand of $\mathbb{F}T^{\Delta}_{G,b}$, then $u = 1$.
(iii) If $S_{L,u,\mathbb{F}}$ is a simple summand of $\mathbb{F}T^{\Delta}_{G,b}$, then $u = 1$.
(iv) (G, b) is functorially equivalent to $(D, 1)$ over $\mathbb{F}$.

Finally, we prove the following finiteness result:

Theorem 6. *Let D be a finite p-group. Then there is only a finite number of pairs (G, b), where G is a finite group, and b is a block idempotent of kG with defect groups isomorphic to D, up to functorial equivalence over $\mathbb{F}$.*

We refer to [BY23] (Stable functorial equivalence of blocks) and [Y22] (Isotypic blocks are functorially equivalent) for recent additional results related to functorial equivalence of blocks.

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A real version of Gallagher's theorem

John Murray

The subject of this talk is Clifford Theory. This relates the irreducible characters Irr(G) of a finite group G to those of a normal subgroup N. In [2] P. X. Gallagher showed that for $\theta \in \mathrm{Irr(N)}$, the number of irreducible characters of G over θ equals the number of θ-good conjugacy classes of G_θ/N. Our real version of Gallagher's theorem counts the number of real irreducible characters of G over θ as a difference of certain θ-good classes of G_θ/N.

In a second theorem, we compute the Frobenius-Schur indicator of the induced character $\theta{\uparrow}^G$, as a sum over the involutions t in the quotient of the extended centraliser G^*_θ/N, with t weighted by Gow-Frobenius-Schur indicators of θ with respect to t. The latter indicator, which takes one of the values $0, \pm 1$, was introduced by R. Gow in [1]. The recent preprint [3] interprets this indicator in terms of the Brauer-Wall group of $\mathbb{R}$.

One corollary of our second theorem is that if θ is real and of 2-defect zero, the Frobenius-Schur indicator of $\theta{\uparrow}^G$ equals the number of involutions in G_θ/N.

We outline an ingenious alternative argument, which uses ideas of R. Gow and G. Robinson, to shows that under the hypothesis, θ has a canonical extension to G_θ. The conclusion then follows from standard Clifford theory.

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Progress on Donovan's conjecture, and challenges

Charles Eaton

I give an overview of some progress on Donovan's conjecture, which states that for a fixed finite p-group P, there are only finitely many Morita equivalence classes of blocks B of finite groups G with defect group $D \cong P$. This conjecture may be stated over $\mathcal{O}$ (the $\mathcal{O}$-Donovan conjecture) or k (the k-Donovan conjecture), where $\mathcal{O}$ is a suitable complete discrete valuation ring with residue field $k = \mathcal{O}/J(\mathcal{O})$ of characteristic p. For a given P, this conjecture is equivalent to both of the following conjectures holding for P:

Conjecture 1 (Weak Donovan conjecture, based on a question of Brauer)**.** *There is a bound on the entries of the Cartan matrix of blocks with defect group isomorphic to P.*

Conjecture 2 (Morita-Frobenius number conjecture)**.** *There is a bound on the Morita-Frobenius numbers of blocks with defect group isomorphic to P.*

The Morita-Frobenius number $\mathrm{mf}_k(B)$ of a block B is the smallest $m \geq 1$ such that $\sigma^m(B)$ is Morita equivalent to B, where σ is a ring automorphism of the group algebra induced by raising coefficients of group elements in kG to the power of p. A similar number $\mathrm{mf}_{\mathcal{O}}(B)$ is defined for $\mathcal{O}$-blocks.

This equivalence was proved for k in [6] and for $\mathcal{O}$ in [1]. It is more convenient to work with the related *strong $\mathcal{O}$-Frobenius number* $\mathrm{sf}_{\mathcal{O}}(B)$ that we do not define here, but which differs from $\mathrm{mf}_{\mathcal{O}}(B)$ by a factor depending on the defect group. In addition to being a Morita invariant, a key fact about $\mathrm{sf}_{\mathcal{O}}(B)$ is as follows. Let $N \lhd G$ and let B be a block of $\mathcal{O}G$ covering a block b of $\mathcal{O}N$. Suppose that B has defect group D and b has defect group $N \cap D$ such that $G = C_D(D \cap N)N$ (for example as happens if D is abelian, b is G-stable and G/N is a p-group). Then $\mathrm{sf}_{\mathcal{O}}(B) \leq \mathrm{sf}_{\mathcal{O}}(b)$. This was proved for D abelian in [3] and in general in [4].

Absolute bounds on the Morita-Frobenius numbers of blocks of quasisimple groups were obtained by Farrell and Kessar in [5]. As a result, in [1] it was proved that Donovan's conjecture for abelian defect groups is equivalent to the Weak Donovan conjecture for such blocks of quasisimple groups. Since the Weak Donovan conjecture is known to hold for 2-blocks with abelian defect groups by [2],

Donovan's conjecture follows in this case. Finally, Donovan's conjecture is proved for blocks with defect group $Q_8 \times Q_8$ and $Q_8 \times C_{2^n}$ in [4].

Progress on Donovan's conjecture, as well as on related conjectures and classifications of Morita equivalence classes of blocks with some given defect groups, is recorded at the following wiki site:

https://wiki.manchester.ac.uk/blocks/index.php/Main_Page.

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The Broué invariant of a p-permutation equivalence

Robert Boltje

Let G and H be finite groups, let $(K, \mathcal{O}, F)$ be a p-modular system (large enough for G and H), and let $A \in \mathrm{Bl}(\mathcal{O}G)$ and $B \in \mathrm{Bl}(\mathcal{O}H)$ be block algebras.

A result of Broué states that if $I\colon R_K(A) \xrightarrow{\sim} R_K(B)$ is a perfect isometry and $\chi \in \mathrm{Irr}(A)$ then

$$\frac{|G|/\chi(1)}{|H|/I(\chi)(1)} \in \mathbb{Q}$$

has p-value zero and its residue class in $\mathbb{F}_p^\times$ is independent of $\chi \in \mathrm{Irr}(A)$. We denote this residue class by $\beta(I)$ and call it the *Broué invariant* of I.

If $C_\bullet$ is a splendid Rickard complex for A and B then $\sum_{n\in\mathbb{Z}}(-1)^n[C_n]$ is a p-permutation equivalence in the representation group $T^\Delta(A, B)$, and if γ is a p-permutation equivalence between A and B then its character μ induces a perfect isometry $I\colon R_K(A) \xrightarrow{\sim} R_K(B)$.

Theorem A *If a perfect isometry I between A and B comes from a p-permutation equivalence (or even a splendid Rickard complex) then, up to a sign, $\beta(I) \in \mathbb{F}_p^\times$ is independent of I and equal to $\beta(A)/\beta(B)$, for certain invariants $\beta(A), \beta(B) \in \mathbb{F}_p^\times$. If additionally A and B correspond via Brauer's first main theorem then $\beta(I) \in \{\pm 1\}$.*

Suppose that A and B correspondent via Brauer's first main thoerem. Recall that the Alperin-McKay conjecture predicts a bijection

$$\Omega\colon\ \mathrm{Irr}_K^0(A) \xrightarrow{\sim} \mathrm{Irr}_K^0(B)$$

between the sets of irreducible characters of height zero.

Theorem B *Suppose that A and B are Brauer correspondents and let $\tilde{A} \in \mathrm{Bl}(\mathbb{Z}_pG)$ and $\tilde{B} \in \mathrm{Bl}(\mathbb{Z}_pH)$ be the unique block algebras with $A\tilde{A} \neq \{0\}$ and $B\tilde{B} \neq \{0\}$.*

(a) If there exists a p-permutation equivalence between A and B then there exists a bijection Ω preserving p-residues up to sign.

(b) If there exists a p-permutation equivalence between $\tilde{A}$ and $\tilde{B}$ then there exists a bijection Ω preserving p-residues up to sign and respecting the Galois action over $\mathbb{Q}_p$.

(c) If there exists a splendid Rickard equivalence between $\tilde{A}$ and $\tilde{B}$ then there exists a bijection Ω preserving p-residues up to sign, respecting the Galois action over $\mathbb{Q}_p$, and preserving Schur indices and endomorphism algebras over $\mathbb{Q}_p$.

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