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Complete damage in linear elastic materials — modeling, weak formulation and existence results

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Abstract

We introduce a complete damage model with a time-depending domain for linear-elastically stressed solids under time-varying Dirichlet boundary conditions. The evolution of the system is described by a doubly nonlinear differential inclusion for the damage process and a quasi-static balance equation for the displacement field. For the introduced complete damage model, we propose a classical formulation and a corresponding suitable weak formulation in an SBV-framework. We show that the classical differential inclusion can be regained from the notion of weak solutions under certain regularity assumptions. The main aim of this work is to prove local-in-time existence and global-in-time existence in some weaker sense for the introduced model.

In contrast to incomplete damage theories, the material can be exposed to damage in the proposed model in such a way that the elastic behavior may break down on the damaged parts of the material, i.e. we loose coercivity properties of the free energy. This leads to several mathematical difficulties. For instance, it might occur that not completely damaged material regions are isolated from the Dirichlet boundary. In this case, the deformation field cannot be controlled in the transition from incomplete to complete damage. To tackle this problem, we consider the evolution process on a time-depending domain. In this context, two major challenges arise: Firstly, the time-dependent domain approach leads to jumps in the energy which have to be accounted for in the energy inequality of the notion of weak solutions. To handle this problem, several energy estimates are established by Γ-convergence techniques. Secondly, the time-depending domain might have bad smoothness properties such that Korn's inequality cannot be applied. To this end, a covering result for such sets with smooth compactly embedded domains has been shown.

1 Motivation

From a microscopic point of view, damage behavior originates from breaking atomic links in the material whereas a macroscopic theory may specify the damage by a scalar variable related to the quantity of damage. According to the latter perspective, phase-field models are quite common to model smooth transitions between damaged and undamaged material states.

In complete damage models, the elastic material is allowed to completely disintegrate. Mathematical works of such models covering global-in-time existence are rarely and are mainly focused on purely *rate-independent systems* [MR06, BMR09, MRZ10] by using Γ-convergence techniques to recover energetic properties in the limit. Existence results for *rate-dependent* complete damage systems in thermoviscoelastic materials are recently published in [RR12]. In contrast, much mathematical efforts have been made in understanding incomplete damage processes. Existence and uniqueness results for damage models of viscoelastic materials are proven in [BSS05] in the one dimensional case. Higher dimensional damage models are analytically investigated in [BS04, MT10, KRZ11] and, there, existence, uniqueness and regularity properties are shown. A coupled system describing incomplete damage, linear elasticity and phase separation appeared in [HK11, HK10]. All these works are based on the gradient-of-damage model proposed by Frémond and Nedjar [FN96] which describes the damage as a result from microscopic movements in the solid. The distinction between a balance law for the microscopic forces and constitutive relations of the material yield a satisfactory derivation of an evolution law for the damage propagation from the physical point of view. In particular, the gradient of the damage variable enters the resulting equations and serves as a regularization term for the mathematical analysis. When the evolution of the damage is assumed to be uni-directional, i.e. the damage is irreversible, the microforce balance law becomes a differential inclusion.

The reason why incomplete damage models are more feasible for mathematical investigations is that a coercivity assumption on the free energy prevents the material from a complete degeneration and dropping this assumption may lead to serious troubles. However, in the case of viscoelastic materials, the inertia terms circumvent this kind of problem in the sense that the deformation field still exists on the whole domain accompanied with a loss of spatial regularity (cf. [RR12]). Unfortunately, this result cannot be expected in the case of quasi-static mechanical equilibrium (see for instance [BMR09]).

The main aim of this work is to introduce and analyze a complete damage model for linear elastic materials which are assumed to be in quasi-static equilibrium. For the analytical discussion, we start with an incomplete damage model which is regularized in the equation of balance of forces as in [MR06, MRZ10, BMR09, RR12] such that already known existence results from the incomplete damage regime can be applied. The basis for a weak formulation of the regularized system is a notion introduced in [HK11]. This notion seems well adapted for the transition to complete damage (see also [RR12]). The advantage is that we can deal with low regularity solutions and we are able to use weak semi-continuity arguments for the passage. In our weak formulation, the evolution law for the damage variable which is classically described by a differential inclusion becomes some kind of variational inequality combined with a total energy inequality. Nevertheless, we are faced with several mathematical challenges since the system highly degenerates during the passage.

The major challenge is to establish a meaningful deformation field on regions where the damage is *not* complete in the limit system. For instance, it might happen that in the limit path-connected components of the not completely damaged material are isolated from the Dirichlet boundary. In this case, the deformation variable cannot be controlled in the transition to complete damage with Korn's inequality. We will overcome this particular issue by formulating the problem in terms of a *time-depending domain*. The domain contains all the not completely damaged path-connected components of the material which still possess a part of the Dirichlet boundary. Inside the domain, the damage evolution is still driven by a differential inclusion. The remaining area of the original domain consists of completely damaged material and of material parts which are not completely damaged and isolated from the Dirichlet boundary. Two further complicacies arise in this connection.

The first issue concerns the energy inequality. The time-depending domain approach leads to jumps in the energy which must be accounted for in the energy inequality of the notion of weak soultions as well. This issue is tackled with Γ-convergence techniques in order to keep track of the energy at jump points.

Secondly, the time-depending domain might have very bad smoothness properties which again might lead to a failure of Korn'n inequality. This problem is approached by proving some covering results for these sets with smooth domains where Korn's inequality can be applied. In this context, we introduce some special kind of local Sobolev spaces where we look for solutions in the limit system.

This paper is structured as follows. The next section provides an overview of the notation we are going to use while Section 3 develops our model first in a classical setting with enough smoothness properties and then in a rigorous mathematical setting by presenting a weak formulation with SBV-functions for the damage and local Sobolev functions for the deformation. It is shown in Theorem 3.7 that the weak notion reduces to the classical formulation when enough regularity is assumed. The main results, i.e. Theorem 4.1 and Theorem 4.2, are stated in Section 4 while the proofs are carried out in the subsequent Section 5. We first solve a simplified problem in Section 5.2. By Zorn's lemma, existence of solutions to the main problems will be proven in Section 5.3.

To the best of our knowledge, there are no global-in-time existence results of completely damage models of such degeneracy in the mathematical literature where the classical differential inclusion can be regained under some additional regularity assumptions.

2 Notation

Let $\Omega \subseteq \mathbb{R}^n$ denote a bounded Lipschitz domain, $D \subseteq \partial\Omega$ a part of the boundary with $\mathcal{H}^{n-1}(D) > 0$ and $T > 0$. The following table provides an overview of some elementary notation used in this paper:

$\mathcal{L}^n$, $\mathcal{H}^n$	*n-dimensional Lebesgue and Hausdorff measure*
$\mathcal{C}^l_x(A;\mathbb{R}^N)$	*l-times continuously differentiable function on open $A \subseteq \mathbb{R}^{n+1}$ with respect to the spatial variable x*
∂J	*subdifferential of a convex function $J : X \to \mathbb{R} \cup \{\infty\}$, X Banach space*
$B_\varepsilon(A)$	*ε-neighborhood of $A \subseteq \mathbb{R}^n$*
$\mathbb{1}_A$, I_A	*characteristic function and indicator function $X \to \mathbb{R} \cup \{\infty\}$ with respect to a subset $A \subseteq X$*
$\overline{A}, \mathrm{int}(A), \partial A$	*closure, interior and boundary of $A \subseteq \mathbb{R}^n$*
Ω_T, D_T	*$\Omega \times (0,T)$ and $D \times (0,T)$*
$\{v = 0\}$, $\{v > 0\}$	*level and super-level set given by $\{x \in \overline{\Omega} \,\vert\, v(x) = 0\}$ and $\{x \in \overline{\Omega} \,\vert\, v(x) > 0\}$ for functions $v \in W^{1,p}(\Omega)$, $p > n$, by employing the embedding $W^{1,p}(\Omega) \hookrightarrow \mathcal{C}(\overline{\Omega})$*
$\mathrm{supp}(v)$	*support of a function v*

Let $(X, \|\cdot\|)$ be a Banach space, $I \subseteq \mathbb{R}$ be an open interval and μ be a positive measure. The space $L^p(I,\mu;X)$, $1 \le p \le \infty$, denotes the p-Bochner μ-integrable functions with values in X (μ-essentially bounded for $p = \infty$, respectively). We write $L^p(I;X)$ for $L^p(I,\mathcal{L}^1;X)$. The subspace $H^q(I;X) \subseteq L^2(I;X)$, $q \in \mathbb{N}$, indicates L^2-functions which are q-times weakly differentiable with weak derivatives in L^2. Moreover, the subspace $BV(I;X) \subseteq L^1(I;X)$ consists of functions $f \in L^1(I;X)$ with

$$\operatorname{ess\,var}_I(f) := \inf\big\{\mathrm{var}_I(g) \,\big\vert\, g = f \ \mathcal{L}^1\text{-a.e. in } I\big\} < +\infty,$$

and

$$\mathrm{var}_I(f) := \sup\Big\{\sum_{i=1}^{k-1} \|f(t_{i+1}) - f(t_i)\| \,\Big\vert\, t_1 < t_2 < \ldots < t_k \text{ with } t_1, t_2, \ldots t_k \in I \text{ for } k \ge 2\Big\}.$$

To every $f \in BV(I;X)$, we can choose a representant (also denoted by f) with $\mathrm{var}_I(f) < +\infty$. Then the values $f(t^\pm) := \lim_{s\to t^\pm} f(s)$ exist for all $t \in \overline{I}$ (and are independent of the representant) by adapting the convention $f\big((\inf I)^-\big) := f\big((\inf I)^+\big)$ and $f\big((\sup I)^+\big) := f\big((\sup I)^-\big)$. The functions $f^+(t) := f(t^+)$ and $f^-(t) := f(t^-)$ are thus uniquely defined for every $t \in \overline{I}$ and do not coincide for at most countably many points, i.e. in the jump discontinuity set J_f. Furthermore, a regular measure $\mathrm{d}f$ with finite variation, i.e. $|\mathrm{d}f|(I) < \infty$, and with values in X (called *differential measure*) can be assigned such that $\mathrm{d}f((a, b]) = f^+(b) - f^+(a)$ for all $a, b \in \overline{I}$ with $a \leq b$, cf. [Din66]. If X is a finite dimensional vector space we refer to [AFP00] for a comprehensive introduction.

If X exhibits the Radon-Nikodym property (e.g. if X is reflexive) the differenial measure decomposes into $\mathrm{d}f = f'_\mu \mu$ for a (non uniquely) positive Radon measure μ and a function $f'_\mu \in L^1(I, \mu; X)$ [MV87]. The subspace $SBV(I; X) \subseteq BV(I; X)$ of special functions of bounded variation is defined as the space of functions $f \in BV(I; X)$ where the decomposition

$$\mathrm{d}f = f'\mathcal{L}^1 + (f^+ - f^-)\mathcal{H}^0\lfloor J_f$$

for an $f' \in L^1(I; X)$ exists. This function f' is called the absolutely continuous part of the differential measure and we also write $\partial_t^{\mathrm{a}} f$. If, additionally, $\partial_t^{\mathrm{a}} f \in L^p(I; X)$, $p \geq 1$, we write $f \in SBV^p(I; X)$.

For the analysis of the system given in the next chapter, it is convenient to introduce local Sobolev functions on shrinking sets. Let $G \subseteq \overline{\Omega_T}$ be a subset. The intersection of G at time $t \in [0, T]$, i.e. $G \cap (\overline{\Omega} \times \{t\})$, is denoted by $G(t) := \{x \in \overline{\Omega} \,|\, (x, t) \in G\}$. We call G *shrinking* if G is relatively open in $\overline{\Omega_T}$ and $G(s) \subseteq G(t)$ for arbitrary $0 \leq t \leq s \leq T$.

In the sequel, $G \subseteq \overline{\Omega_T}$ denotes a shrinking set. We define the following time-dependent local Sobolev space:

$$\begin{aligned} L_t^2 H_{x,\mathrm{loc}}^q(G; \mathbb{R}^N) := \Big\{ & v : G \to \mathbb{R}^N \text{ with } v|_{U \times (0,t)} \in L^2(0, t; H^q(U; \mathbb{R}^N)) \\ & \qquad \text{for all open sets } U \subset\subset G(t) \text{ and all } t \in (0, T] \Big\}. \end{aligned} \tag{1}$$

Here, $L_t^2 H_{x,\mathrm{loc}}^0(G; \mathbb{R}^N)$ coincides with $L_{\mathrm{loc}}^2(G; \mathbb{R}^N)$ (see Section 5.1.1). $L_{\mathrm{loc}}^2(G; \mathbb{R}^N)$ denotes the classical local L^2-Lebesgue space on G given by

$$L_{\mathrm{loc}}^2(G; \mathbb{R}^N) := \Big\{ v : G \to \mathbb{R}^N \text{ with } v|_V \in L^2(V; \mathbb{R}^N) \text{ for all open } V \subset\subset G \Big\}.$$

(Note that we do not demand that G should be open.) As usual, we set $L_t^2 H_{x,\mathrm{loc}}^q(G) := L_t^2 H_{x,\mathrm{loc}}^q(G; \mathbb{R})$. At fixed time points $t \in (0, T)$, we find $v(t) \in H_{\mathrm{loc}}^q(G(t); \mathbb{R}^N)$. If $q \geq 1$ we write ∇v for the weak derivative with respect to the spatial variable as well as $\epsilon(v) := \frac{1}{2}(\nabla v + (\nabla v)^{\mathrm{T}})$ for its symmetric part. The precise definition and characterization of ∇v can be found in Proposition 5.4.

Given $v \in L_t^2 H_{x,\mathrm{loc}}^q(G; \mathbb{R}^N)$ with $q \geq 1$, we say that $v = b$ on $D_T \cap G$ with $D \subseteq \partial\Omega$ and $\mathcal{H}^{n-1}(D) > 0$ if for every $t \in (0, T)$ and every open set $U \subset\subset G(t)$ with Lipschitz boundary

$$\widetilde{v}(s) = b(s) \text{ on } \partial U \cap D \text{ in the sense of traces for a.e. } s \in (0, t), \tag{2}$$

is fulfilled with $\widetilde{v} := v|_{U \times (0,t)} \in L^2(0, t; H^1(U; \mathbb{R}^N))$.

3 Modeling

3.1 Classical formulation

In the sequel, let the bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^n$ be the reference configuration of the regarded body which is also clamped at the Dirichlet boundary $D \subseteq \partial\Omega$ with $\mathcal{H}^{n-1}(D) > 0$. Even though more general forms are conceivable, we confine ourselves in this work to the following free energy density φ and dissipation potential density ϱ based on the gradient-of-damage theory:

$$\varphi(e,z,\nabla z) := \frac{1}{p}|\nabla z|^p + W(e,z), \quad \varrho(\dot{z}) := -\alpha\dot{z} + \beta|\dot{z}|^2 + I_{(-\infty,0]}(\dot{z}), \tag{3}$$

where e denotes the linearized strain tensor, z the damage phase-field variable, $\alpha \geq 0$ and $\beta > 0$. The function W indicates the elastic energy density. More general free energies incorporating appropriate convex potentials for the damage function can be employed with the obvious modifications but we confine the analysis to the case (3) in order to avoid overburden the presentation. The gradient exponent p satisfies $p > n$. The damage variable z specifies the degree of damage at each reference position $x \in \Omega$ in the material, i.e. $z(x) = 1$ stands for an undamaged and $z(x) = 0$ for a completely damage material point whereas intermediate values represent partial damage. Furthermore, the irreversibility of the damage process (the solid can not heal itself) is ensured by the indicator function in ϱ. We assume that no volume forces are acting. The microforce balance law yields the pointwise inclusion

$$0 \in -\mathrm{div}(\varphi_{,\nabla z}(e,z,\nabla z)) + \varphi_{,z}(e,z,\nabla z) + \partial I_{[0,\infty)}(z) + \partial_{\dot{z}}\varrho(\partial_t z). \tag{4}$$

Note that the subdifferential of the indicator function $I_{[0,\infty)}$ appears on the right hand side of the inclusion to account for the constraint $z \geq 0$.

This paper will cover elastic energy densities of the form

$$W(e,z) = \frac{1}{2}g(z)\mathbb{C}e : e \tag{5}$$

with a symmetric and positive definite stiffness tensor $\mathbb{C} \in \mathcal{L}(\mathbb{R}^{n\times n}_{\mathrm{sym}})$ and a function $g \in \mathcal{C}^1([0,1];\mathbb{R}^+)$ with the properties

$$\eta \leq g'(z), \qquad g(0) = 0 \tag{6}$$

for all $z \in [0,1]$ and some constant $\eta > 0$.

We use the small strain assumption, i.e. the strain calculates as

$$e = \epsilon(u) := \frac{1}{2}(\nabla u + (\nabla u)^{\mathrm{T}}), \tag{7}$$

where the right hand side denotes the symmetric gradient of the deformation field u.

By neglecting inertia effect, the momentum balance equation is a quasi-static mechanical equilibrium and reads as

$$0 = \mathrm{div}(W_{,e}(e,z)). \tag{8}$$

Note that complete damage is possible if and only if $g(0) = 0$. The case $g(0) > 0$ would describe incomplete damage which is already covered in the mathematical literature (see Section 1). As mentioned in the introduction, a regularization scheme is adapted, where a regularized elastic energy density W^ε, $\varepsilon > 0$, is used instead of W in the first instance. More precisely, W^ε is given by

$$W^\varepsilon(e,z) = \frac{1}{2}(g(z)+\varepsilon)\mathbb{C}e : e. \tag{9}$$

In the complete damage regime $\varepsilon = 0$, the deformation variable becomes meaningless on material fragment with maximal damage because the free energy density vanishes regardless of the values of u. Therefore, the balance laws (4) and (8) make obviously only sense pointwise in $\{z > 0\}$. Beyond that, as already mentioned in the introduction, a phenomenon (in the following called *material exclusion*) might cause severe troubles.

Suppose that at a specific time point t, a path-connected component P (relatively open in $\overline{\Omega}$) from $\{x \in \overline{\Omega}\,|\,z(x,t) > 0\}$ is isolated from the Dirichlet boundary, i.e. $\mathcal{H}^{n-1}(P \cap D) = 0$. In this case, Korn's inequality fails on P and, consequently, the deformation field u_ε for the regularized system cannot be controlled on P in the transition $\varepsilon \to 0^+$. To overcome this problem, path-connected components P of the not completely damaged area $\{z(t) > 0\}$ isolated from the Dirichlet boundary, i.e. $\mathcal{H}^{n-1}(P \cap D) = 0$, will be excluded from our considerations in our proposed model. On the one hand, this make our model accessible for a rigorous analysis and, on the other hand, for some applications, the detached parts might be of little interest anyway. This approach is illustrated in Figure 1 and motivates the definition of maximal admissible subsets.

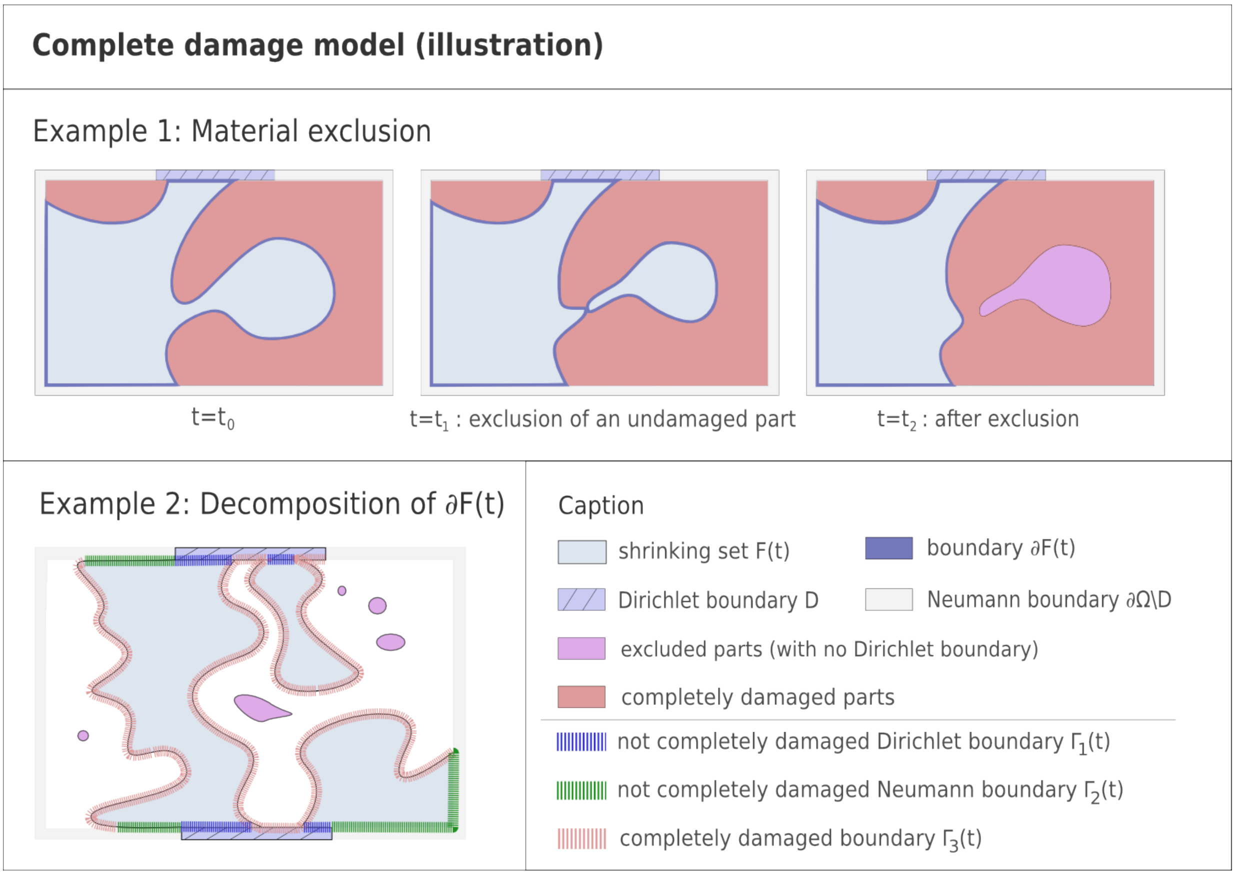

Figure 1: *The first example in the illustration above shows the exclusion of an undamaged material part during the evolutionary process in 2D. The dark blue curve encircles the maximal admissible subset $F(t) = \mathfrak{A}_D(\{z(t) > 0\})$ of the not completely damaged area $\{z(t) > 0\}$. The second example below pictures the different parts of the boundary of $F(t)$. There, the Dirichlet boundary D consists of two components.*

Definition 3.1 (Admissible subsets of $\overline{\Omega}$)

(i) *Let $F \subseteq \overline{\Omega}$ be a relatively open subset and*

$$P_F(x) := \big\{ y \in F \,|\, x \text{ and } y \text{ are connected by a path in } F \big\}$$

for $x \in F$. We say that F is admissible with respect to the Dirichlet boundary D if for every $x \in F$ the condition

$$\mathcal{H}^{n-1}(P_F(x) \cap D) > 0$$

is fulfilled. Furthermore, $\mathfrak{A}_D(F)$ denotes the maximal admissible subset of F with respect to D, i.e.

$$\mathfrak{A}_D(F) := \bigcup \{ G \subseteq F \,|\, G \text{ is admissible with respect to } D \} \, .$$

(ii) *For a relatively open subset $F \subseteq \overline{\Omega_T}$, the set $\mathfrak{A}_D(F)$ is given by $(\mathfrak{A}_D(F))(t) := \mathfrak{A}_D(F(t))$.*

In a nutshell, the evolutionary problem (4) and (8) is considered on a time-depending domain (a shrinking set) which is, for any time, admissible with respect to D. The whole evolution problem with its initial-boundary conditions can be summarized within a classical notion in the following way.

Definition 3.2 (Classical solution) *A pair of functions (u, z) defined on an admissible shrinking set $F \subseteq \overline{\Omega_T}$ with*

$$F(t) = \mathfrak{A}_D\Big(\bigcap_{s<t} F(s) \Big) \tag{10}$$

is called a classical solution to the initial-boundary data (z^0, b) if the following properties are satisfied:

(i) *Regularity:*

$$u \in \mathcal{C}^2_x(F; \mathbb{R}^n), \; z \in \mathcal{C}^2(\overline{F}; \mathbb{R})$$

(ii) *Evolution laws in F:*

$$\begin{aligned} &0 = \mathrm{div}(W_{,e}(\epsilon(u), z)), \\ &0 \in -\mathrm{div}(|\nabla z|^{p-2} \nabla z) + W_{,z}(\epsilon(u), z) - \alpha + \beta \partial_t z + \partial I_{(-\infty, 0]}(\partial_t z) \\ &0 < z \end{aligned}$$

(iii) *Initial-boundary conditions:*

$$\begin{aligned} &z(t = 0) = z^0, && \text{on } F(0), \\ &u(t) = b(t) && \text{on } \Gamma_1(t) := F(t) \cap D, \\ &W_{,e}(\epsilon(u(t)), z(t)) \cdot \nu = 0 && \text{on } \Gamma_2(t) := F(t) \cap (\partial\Omega \setminus D), \\ &z(t) = 0 && \text{on } \Gamma_3(t) := \partial F(t) \setminus F(t) \\ &\nabla z(t) \cdot \nu = 0 && \text{on } \Gamma_1(t) \cup \Gamma_2(t) \end{aligned}$$

Remark 3.3 *The time-depending boundary $\partial F(t)$ disjointly decomposes into $\Gamma_1(t) \cup \Gamma_2(t) \cup \Gamma_3(t)$, where $\Gamma_1(t)$ indicates the not completely damaged Dirichlet boundary, $\Gamma_2(t)$ the not completely damaged Neumann boundary and $\Gamma_3(t)$ the completely damaged boundary (see Figure 1). We have the following types of boundary conditions:*

$\Gamma_1(t)$	—	*Dirichlet boundary condition for u*
		Neumann boundary condition for z
$\Gamma_2(t)$	—	*Neumann boundary condition for u*
		Neumann boundary condition for z
$\Gamma_3(t)$	—	*degenerated boundary condition*

On the degenerated boundary, z vanishes (homogeneous Dirichlet boundary condition for z) and, therefore, if we assume that $\epsilon(u)$ can be continuously extended to Γ_3 the stress $W_{,e}(\epsilon(u), z)$ vanishes too.

The goal of the next section is to state a weak formulation such that existence can be proven. Due to the high degree of degeneracy and the non-smoothness of F, u can only be expected in some local Sobolev space on the shrinking set F introduced in (1).

3.2 Weak formulation and justification

For a weak formulation of the system presented in Section 3.1, we will take advantage of the free energy $\mathcal{E}$ whose density has already be given in (3). In contrast to [KRZ11] for incomplete damage models and related works, we will not use a purely energetic approach but rather a mixed variational/energetic formulation as presented in [HK11].

Definition 3.4 (Free energy) *Let $e \in L^2(\Omega; \mathbb{R}^{n\times n}_{\mathrm{sym}})$ and $z \in W^{1,p}(\Omega)$ be given. The associated free energy of the system in Definition 3.2 is given by*

$$\mathcal{E}(e, z) := \int_\Omega \frac{1}{p}|\nabla z|^p + W(e, z)\,\mathrm{d}x,$$

whereas its ε-regularization with $\varepsilon > 0$ (for later use) is defined as (see (9))

$$\mathcal{E}_\varepsilon(e, z) := \int_\Omega \frac{1}{p}|\nabla z|^p + W^\varepsilon(e, z)\,\mathrm{d}x.$$

If e is only defined on a measurable subset $H \subset \Omega$, i.e. $e \in L^2(H; \mathbb{R}^{n\times n}_{\mathrm{sym}})$, we use the convention $\mathcal{E}(e, z) := \mathcal{E}(\widetilde{e}, z)$, where $\widetilde{e} := e$ in H and $\widetilde{e} := 0$ in $\Omega \setminus H$.

We are now able to give a weak formulation of the system in an SBV-setting (with respect to the damage variable). In accordance to Definition 3.2, z is extended on whole $\overline{\Omega_T}$ and when viewed as an $SBV^2(0, T; L^2(\Omega))$-function has a jump at t if and only if a material exclusion occurs at t.

Definition 3.5 (Weak solution) *A pair (u, z) is called a weak solution of the system given in Definition 3.2 with the initial-boundary data (z^0, b) if* *(i) Regularity:*

$$z \in L^\infty(0,T;W^{1,p}(\Omega)) \cap SBV^2(0,T;L^2(\Omega)), \quad u \in L^2_t H^1_{x,\mathrm{loc}}(F;\mathbb{R}^n)$$

with $\epsilon(u) =: e \in L^2(F;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ *where* $F := \mathfrak{A}_D(\{z^- > 0\}) \subseteq \overline{\Omega_T}$ *is a shrinking set.*

(ii) Quasi-static mechanical equilibrium:

$$0 = \int_{F(t)} W_{,e}(e(t),z(t)) : \epsilon(\zeta)\,\mathrm{d}x \tag{11}$$

for a.e. $t \in (0,T)$ *and for all* $\zeta \in H^1_D(\Omega;\mathbb{R}^n)$. *Furthermore,* $u = b$ *on* $D_T \cap F$.

(iii) Damage variational inequality:

$$\int_{F(t)} |\nabla z(t)|^{p-2}\nabla z(t)\cdot\nabla\zeta + W_{,z}(e(t),z(t))\zeta\,\mathrm{d}x \geq \int_\Omega (\alpha - \beta\partial_t^{\mathrm{a}} z(t))\zeta\,\mathrm{d}x \tag{12}$$

$$0 \leq z(t) \text{ in } \Omega,$$
$$0 \geq \partial_t^{\mathrm{a}} z(t) \text{ in } \Omega$$

for a.e. $t \in (0,T)$ *and for all* $\zeta \in W^{1,p}(\Omega)$ *with* $\zeta \leq 0$. *The initial value is given by* $z^+(0) = z^0$ *with* $0 \leq z^0 \leq 1$ *in* Ω.

(iv) Damage jump condition:

$$z^+(t) = z^-(t)\mathbb{1}_{F(t)} \text{ in } \Omega \tag{13}$$

for all $t \in [0,T]$.

(v) Energy inequality:

$$\begin{aligned}\mathcal{E}(e(t),z(t)) + \int_0^t\int_{F(s)} \alpha|\partial_t^{\mathrm{a}} z| + \beta|\partial_t^{\mathrm{a}} z|^2\,\mathrm{d}(x,s) + \sum_{s\in J_z\cap(0,t]} \mathcal{J}_s \\ \leq \mathfrak{e}_0^+ + \int_0^t\int_{F(s)} W_{,e}(e,z) : \epsilon(\partial_t b)\,\mathrm{d}(x,s)\end{aligned} \tag{14}$$

for a.e. $t \in (0,T)$, *where the jump part* $\mathcal{J}_s$ *satisfies* $0 \leq \mathcal{J}_s$ *and is given by*

$$\mathcal{J}_s := \lim_{\tau\to s^-} \operatorname*{ess\,inf}_{\vartheta\in(\tau,s)} \mathcal{E}(e(\vartheta),z(\vartheta)) - \mathfrak{e}_s^+ \tag{15}$$

and the values $\mathfrak{e}_s^+ \in \mathbb{R}_+$ *satisfy the upper energy estimate*

$$\mathfrak{e}_s^+ \leq \mathcal{E}(\epsilon(b(s)+\zeta), z^+(s)) \tag{16}$$

for all $\zeta \in H^1_{\mathrm{loc}}(\{z^+(s) > 0\};\mathbb{R}^n)$ *with* $\zeta = 0$ *on* $D \cap \{z^+(s) > 0\}$.

Remark 3.6 *(i) Lemma C.1 ensures that for all time* t *we have* $z^-(t)\mathbb{1}_{F(t)} \in W^{1,p}(\Omega)$ *(see jump condition* (13)*).*

(ii) *Jump condition* (13) *and the definition of* F *imply* $\{z^+(t) > 0\} = F(t)$ *for all* $t \in [0, T]$. *By the convention introduced in Definition 3.4,*

$$\mathcal{E}(e(t), z(t)) = \int_{F(t)} \frac{1}{p}|\nabla z(t)|^p + W(e(t), z(t))\,\mathrm{d}x,$$

which equals $\int_{\{z(t)>0\}} \frac{1}{p}|\nabla z(t)|^p + W(e(t), z(t))\,\mathrm{d}x$ *for a.e.* $t \in (0, T)$.

(iii) *The jump term* $\mathcal{J}_s$ *equals the energy of the excluded material parts at time point* s, *i.e.* $\mathcal{J}_s = \mathcal{E}(s^-) - \mathcal{E}(s^+)$ *(for smooth solutions on* F*), where* $t \mapsto \mathcal{E}(e(t), z(t))$ *denotes the energy function along the trajectory. However, for less regular weak solutions as in Definition 3.5, the one-sided limits* $\mathcal{E}(s^-)$ *and* $\mathcal{E}(s^+)$ *possibly do not exist. But, in any case,* $\lim_{\tau \to s^-} \operatorname{ess\,inf}_{\vartheta \in (\tau, s)} \mathcal{E}(\vartheta)$ *clearly exists and coincides with* $\mathcal{E}(s^-)$ *for smooth solutions. The value* $\mathcal{E}(s^+)$, *on the other hand, can be avoided in a rather indirect way by using upper energy estimates. More precisely, it turns out that* $\mathcal{E}(s^+)$ *can be substituted by values (denoted by* $\mathfrak{e}_s^+$*) merely satisfying* (16). *Together with equations* (11)-(14), $\mathfrak{e}_s^+$ *is forced to coincide with* $\mathcal{E}(s^+)$ *for smooth solutions. This is particularly shown in the proof of the following theorem.*

Theorem 3.7 *Let* (u, z) *be a weak solution according to Definition 3.5. We assume the regularity properties* $u \in \mathcal{C}^2(\overline{\Omega_T}; \mathbb{R}^n)$ *with* $u = b$ *on* D_T *and* $z|_{\overline{F}} \in \mathcal{C}^2(\overline{F}; \mathbb{R})$. *Then,* $(u|_{\overline{F}}, z|_{\overline{F}})$ *is a classical solution according to Definition 3.2.*

Proof. The first equation and the last inequality in (ii) from Definition 3.2 as well as property (iii) follow immediately by classical integral calculus from the weak notion.

By the monotonicity of z with respect to t (coming from $0 \geq \partial_t^{\mathrm{a}} z$ and the jump condition (13)) and by Remark 3.6 (ii),

$$F(t) = \mathfrak{A}_D(\{z^-(t) > 0\}) = \mathfrak{A}_D\Big(\bigcap_{s<t}\{z^+(s) > 0\}\Big) = \mathfrak{A}_D\Big(\bigcap_{s<t} F(s)\Big).$$

Therefore, condition (10) from Definition 3.2 is shown.

Finally, we need to prove the differential inclusion in (ii) from Definition 3.2. The jump condition (13) and the regularity assumption yields for a.e. $(x, t) \in \Omega_T$

$$\partial_t^{\mathrm{a}} z(x,t) = \begin{cases} \partial_t z(x,t) & \text{if } (x,t) \in F, \\ 0 & \text{if } (x,t) \in \Omega_T \setminus F, \end{cases}$$

where $\partial_t z(x, t)$ is the classical time-derivative of z at (x, t). In the following, we will make use of this property. First, observe that by the regulartiy assumptions $q := (e, z) \in SBV(0, T; X)$ with $X := L^2(\Omega; \mathbb{R}^{n\times n}) \times W^{1,p}(\Omega)$.

Applying the chain rule (see Corollary B.2) for the continuously Fréchet-differentiable energy functional $\mathcal{E}$ and the X-valued SBV-function q shows that $\mathcal{E} \circ q$ is an SBV-function and

$$\begin{aligned} \mathcal{E}(q(t^+)) - \mathcal{E}(q(0^+)) &= \mathrm{d}(\mathcal{E} \circ q)\big((0, t]\big) \\ &= \int_0^t \langle \mathrm{d}_e \mathcal{E}(q(s)), \partial_t e(s)\rangle + \langle \mathrm{d}_z \mathcal{E}(q(s)), \partial_t^{\mathrm{a}} z(s)\rangle\,\mathrm{d}s \\ &\quad + \sum_{s \in J_z \cap (0,t]} \big(\mathcal{E}(q(s^+)) - \mathcal{E}(q(s^-))\big)\,. \end{aligned}$$

The two terms in the integral on the right hand side can be treated as follows.

- Taking into account $z = 0$ in $\Omega_T \setminus F$ and testing (11) with $\zeta = \partial_t u(s) - \partial_t b(s)$, yields

$$\begin{aligned}\langle \mathrm{d}_e \mathcal{E}(q(s)), \partial_t e(s)\rangle &= \int_\Omega W_{,e}(\epsilon(u(s)), z(s)) : \epsilon(\partial_t u(s))\,\mathrm{d}x \\ &= \int_{F(s)} W_{,e}(\epsilon(u(s)), z(s)) : \epsilon(\partial_t u(s))\,\mathrm{d}x \\ &= \int_{F(s)} W_{,e}(\epsilon(u(s)), z(s)) : \epsilon(\partial_t b(s))\,\mathrm{d}x.\end{aligned}$$

- Using the property $\partial_t^{\mathrm{a}} z = 0$ in $\Omega_T \setminus F$,

$$\begin{aligned}&\langle \mathrm{d}_z \mathcal{E}(q(s)), \partial_t^{\mathrm{a}} z(s)\rangle \\ &\quad= \int_\Omega |\nabla z(s)|^{p-2}\nabla z(s)\cdot\nabla\partial_t^{\mathrm{a}} z(s) + W_{,z}(\epsilon(u(s)), z(s))\partial_t^{\mathrm{a}} z(s)\,\mathrm{d}x \\ &\quad= \int_{F(s)} |\nabla z(s)|^{p-2}\nabla z(s)\cdot\nabla\partial_t z(s) + W_{,z}(\epsilon(u(s)), z(s))\partial_t z(s)\,\mathrm{d}x.\end{aligned}$$

Putting the pieces together, we end up with

$$\begin{aligned}&\mathcal{E}(q(t^+)) + \sum_{s\in J_z\cap(0,t]} \big(\mathcal{E}(q(s^-)) - \mathcal{E}(q(s^+))\big) \\ &\quad= \mathcal{E}(q(0^+)) + \int_0^t \int_{F(s)} W_{,e}(\epsilon(u), z) : \epsilon(\partial_t b)\,\mathrm{d}(x,s) \\ &\qquad+ \int_0^t \int_{F(s)} |\nabla z|^{p-2}\nabla z\cdot\nabla\partial_t z + W_{,z}(\epsilon(u), z)\partial_t z\,\mathrm{d}(x,s).\end{aligned} \tag{17}$$

Note that we have $\mathcal{E}(q(0^+)) = \mathfrak{e}_0^+$. Indeed, passing $t \to 0^+$ in (14) yields $\mathcal{E}(q(0^+)) \le \mathfrak{e}_0^+$. The "$\ge$"-inequality follows from (16) tested with $\zeta = u(0) - b(0)$.

Therefore, (14) particularly implies

$$\begin{aligned}&\mathcal{E}(q(t^+)) + \int_0^t \int_{F(s)} \alpha|\partial_t z| + \beta|\partial_t z|^2\,\mathrm{d}(x,s) + \sum_{s\in J_z\cap(0,t]} \mathcal{J}_s \\ &\quad\le \mathcal{E}(q(0^+)) + \int_0^t \int_{F(s)} W_{,e}(e, z) : \epsilon(\partial_t b)\,\mathrm{d}(x,s)\end{aligned} \tag{18}$$

Integrating (12) on $[0,t]$ with respect to time, testing it with $\zeta = \partial_t^{\mathrm{a}} z \le 0$, applying it to (17) and comparing the result with the energy inequality (18) shows

$$\begin{aligned}&\mathcal{E}(q(t^+)) + \sum_{s\in J_z\cap(0,t]} \big(\mathcal{E}(q(s^-)) - \mathcal{E}(q(s^+))\big) + \int_0^t \int_{F(s)} -\alpha\partial_t z + \beta|\partial_t z|^2\,\mathrm{d}(x,s) \\ &\quad\ge \mathcal{E}(q(0^+)) + \int_0^t \int_{F(s)} W_{,e}(\epsilon(u), z) : \epsilon(\partial_t b)\,\mathrm{d}(x,s) \\ &\quad\ge \mathcal{E}(q(t^+)) + \sum_{s\in J_z\cap(0,t]} \mathcal{J}_s + \int_0^t \int_{F(s)} -\alpha\partial_t z + \beta|\partial_t z|^2\,\mathrm{d}(x,s).\end{aligned} \tag{19}$$

Taking also (15) into account and using $\mathcal{E}(q(s^-)) = \lim_{\tau\to s^-} \operatorname{ess\,inf}_{\vartheta\in(\tau,s)} \mathcal{E}(q(\vartheta))$, estimate (19) yields

$$\sum_{s\in J_z} \mathcal{E}(q(s^+)) \leq \sum_{s\in J_z} \mathfrak{e}_s^+. \tag{20}$$

On the other hand, by (16), we find $\mathfrak{e}_s^+ \leq \mathcal{E}(q(s^+))$ for all $s \in J_z$. Combining this with (20) shows $\mathcal{E}(q(s^+)) = \mathfrak{e}_s^+$ for all $s \in J_z$.

Therefore, $\mathcal{J}_s = \mathcal{E}(q(s^-)) - \mathcal{E}(q(s^+))$ and (19) becomes an equality. Taking also (17) into account gives

$$0 = \int_0^t \int_{F(s)} |\nabla z|^{p-2}\nabla z \cdot \nabla\partial_t z + W_{,z}(\epsilon(u), z)\partial_t z - \alpha\partial_t z + \beta|\partial_t z|^2 \,\mathrm{d}(x,s).$$

Together with the variational inequality (12) and the regularity assumptions, we obtain

$$0 \leq \int_{F(s)} \Big(-\operatorname{div}(|\nabla z(s)|^{p-2}\nabla z(s)) + W_{,z}(\epsilon(u(s)), z(s)) - \alpha + \beta\partial_t z(s)\Big)(\zeta - \partial_t z(s)) \,\mathrm{d}x$$

for all $s \in (0,T)$ and for all $\zeta \in L^1(F(s))$ with $\zeta \leq 0$. This leads to

$$0 \leq \Big(-\operatorname{div}(|\nabla z|^{p-2}\nabla z) + W_{,z}(\epsilon(u), z) - \alpha + \beta\partial_t z\Big)(\zeta - \partial_t z)$$

for a.e. $(x,t) \in F$. By the regularity assumptions, this inequality holds pointwise in F. Therefore, the differential inclusion in Definition 3.2 (ii) is shown. $\square$

One main goal of this work is to prove existence of weak solutions according to Definition 3.5. Due to the application of Zorn's lemma used in the global existence proof, analytical problems arises when infinitely many exclusions of material parts occur in arbitrary short time intervals in the “future“, i.e. cluster points from the right of the jump set $J_{z^\star}$ (denoted by $C_{z^\star}$ in the following) where $z^\star \in SBV(0,T;L^2(\Omega))$ is given by $z^\star(t) := z(t)\mathbb{1}_{\mathfrak{A}_D(\{z^-(t)>0\})}$. In this case, we are only able to prove that the shrinking set F is approximately given by $\mathfrak{A}_D(\{z^- > 0\})$ whereas the strain e can still be represented as the symmetric gradient of u in $\mathfrak{A}_D(F)$.

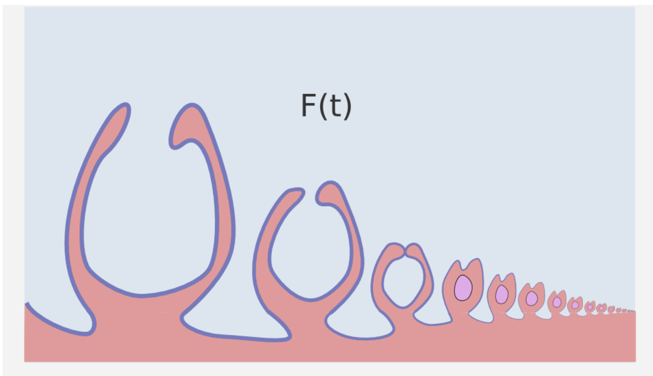

Figure 2: *An example of a shrinking set where infinitely many exclusions during an "infinitesimal" time-interval have occured.*

To be precise, we introduce the following notion.

Definition 3.8 (Approximate weak solution) *A triple (e, u, z) and a shrinking set $F \subseteq \overline{\Omega_T}$ is called an approximate weak solution with fineness $\eta > 0$ of the system according to Definition 3.2 to the initial-boundary data (z^0, b) if* *(i) Regularity:*

$$z \in L^\infty(0,T;W^{1,p}(\Omega)) \cap SBV^2(0,T;L^2(\Omega)), \quad u \in L^2_t H^1_{x,\mathrm{loc}}(\mathfrak{A}_D(F);\mathbb{R}^n),$$
$$e \in L^2(F;\mathbb{R}^{n\times n}_{\mathrm{sym}})$$

with $e = \epsilon(u)$ *in* $\mathfrak{A}_D(F)$.

(ii) Shrinking set properties:

$$F(t) \supseteq \mathfrak{A}_D(\{z^-(t) > 0\}) \text{ for all } t \in [0,T],$$
$$F(t) = \mathfrak{A}_D(\{z^-(t) > 0\}) \text{ for all } t \in [0,T] \setminus \bigcup_{t\in C_{z^\star}} [t, t+\eta),$$
$$\mathcal{L}^n\big(F(t) \setminus \mathfrak{A}_D(\{z^-(t) > 0\})\big) < \eta \text{ for all } t \in \bigcup_{t\in C_{z^\star}} [t, t+\eta).$$

(iii) Evolutionary equations:

Properties (ii)-(v) of Definition 3.5 are satisfied.

Remark 3.9 *If an approximate weak solution* (e,u,z) *on* F *according to Definition 3.8 satisfies* $C_{z^\star} = \emptyset$ *then* (u,z) *is a weak solution according to Definition 3.5.*

4 Main results

Theorem 4.1 (Global-in-time existence of approximate weak solutions)
Let $b \in W^{1,1}(0,T;W^{1,\infty}(\Omega;\mathbb{R}^n))$ *and* $z^0 \in W^{1,p}(\Omega)$ *with* $0 \le z^0 \le 1$ *in* Ω *and* $\{z^0 > 0\}$ *admissible with respect to* D *be initial-boundary data. Furthermore, let* $\eta > 0$ *and* W *be given by* (5) *satisfying* (6)*. Then there exists an approximate weak solution* (e,u,z) *with fineness* $\eta > 0$ *according to Definition 3.8.*

Theorem 4.2 (Local-in-time existence of weak solutions)
Let $b \in W^{1,1}(0,T;W^{1,\infty}(\Omega;\mathbb{R}^n))$ *and* $z^0 \in W^{1,p}(\Omega)$ *with* $0 < \kappa \le z^0 \le 1$ *in* Ω *be initial-boundary data. Furthermore, let* W *be given by* (5) *satisfying* (6)*. Then there exist a maximal value* $\widehat{T} > 0$ *with* $\widehat{T} \le T$ *and functions* u *and* z *defined on the time interval* $[0,\widehat{T}]$ *such that* (u,z) *is a weak solution according to Definition 3.5. Therefore, if* $\widehat{T} < T$, (u,z) *cannot be extended to a weak solution on* $[0,\widehat{T}+\varepsilon]$.

5 Proof of the main result

5.1 Preliminaries

5.1.1 Covering properties

The aim in this subsection is to prove covering results for shrinking sets.

Definition 5.1 (Fine representation) *Let $H \subseteq \overline{\Omega}$ be a relatively open subset. We call a countable family $\{U_k\}$ of open sets $U_k \subset\subset H$ a fine representation for H if for every $x \in H$ there exist an open set $U \subseteq \mathbb{R}^n$ with $x \in U$ and an $k \in \mathbb{N}$ such that $U \cap \Omega \subseteq U_k$.*

Remark 5.2 *Note that $H \cap \partial\Omega$ is not covered by $\{U_k\}$. See Figure 3 for an example.*

Lemma 5.3 *Let $G \subseteq \overline{\Omega_T}$ be a relatively open subset and the sequence $\{t_m\}$ containing T be dense in $[0,T]$. Furthermore, let $\{U_k^m\}_{k\in\mathbb{N}}$ be a fine representation for $G(t_m)$ for every $m \in \mathbb{N}$. Then, for every compact set $K \subseteq G$ there exist a finite set $I \subseteq \mathbb{N}$ and values $m_k \in \mathbb{N}$, $k \in I$, such that $K \cap \Omega_T \subseteq \bigcup_{k\in I} U_k^{m_k} \times (0, t_{m_k})$.*

Proof. To every element $p = (x,t) \in K$, we will construct a neighborhood $\Theta_p \subseteq \overline{\Omega_T}$ of p in the subspace topology of $\overline{\Omega_T}$ such that there exists $k, m \in \mathbb{N}$ with $\Theta_p \cap \Omega_T \subseteq U_k^m \times (0, t_m)$. Then the claim follows by the Heine-Borel theorem.

Indeed, to every $p = (x,t) \in K$ there exists an $\varepsilon > 0$ such that $B_\varepsilon(p) \cap \overline{\Omega_T} \subseteq G$ since $G \subseteq \overline{\Omega_T}$ is relatively open. Therefore, if $t < T$, $(x, t_m) \in G$ for all $m \in \mathbb{N}$ such that $t < t_m < t + \varepsilon$. This implies $(x, t_m) \in G \cap (\overline{\Omega} \times \{t_m\}) = G(t_m) \times \{t_m\}$. Then, we find $p \in G(t_m) \times J$ with $J = [0, t_m)$. In the case $t = T$, it holds $p \in G(T) \times J$ with $J = [0,T]$. Since $\{U_k^m\}_{k\in\mathbb{N}}$ is a fine representation of $G(t_m)$, let $\delta > 0$ such that $B_\delta(x) \cap \Omega \subseteq U_k^m$ for some $k \in \mathbb{N}$. Finally, $\Theta_p := (B_\delta(x) \cap \overline{\Omega}) \times J$ is the required neighborhood of p. $\square$

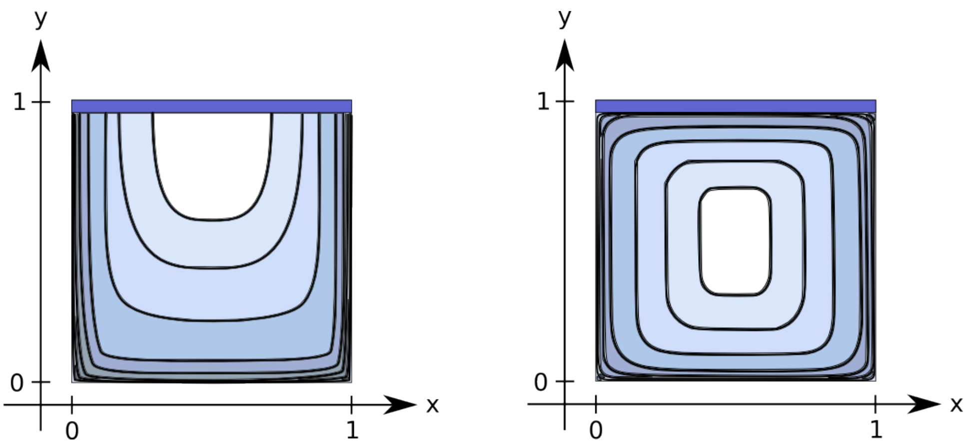

Figure 3: *The left illustration shows a fine representation for the relatively open subset $H = (0,1) \times (0,1]$ of $\overline{\Omega} = [0,1] \times [0,1]$ whereas the right picture shows not a fine representation for H.*

A simple consequence of Lemma 5.3 is $L_t^2 H_{x,\mathrm{loc}}^0(G; \mathbb{R}^N) = L_{\mathrm{loc}}^2(G; \mathbb{R}^N)$, provided that $G \subseteq \overline{\Omega_T}$ is shrinking. Moreover, we can characterize the function space $L_t^2 H_{x,\mathrm{loc}}^1(G; \mathbb{R}^N)$ as follows.

Proposition 5.4 *Let $G \subseteq \overline{\Omega_T}$ be a shrinking subset and let $\{t_m\}$ and $\{U_k^m\}$ be as in Lemma 5.3. Furthermore, let $v : G \to \mathbb{R}^N$ be a function.*

(a) The following statements are equivalent:

(i) $v \in L_t^2 H_{x,\mathrm{loc}}^1(G; \mathbb{R}^N)$

(ii) $v|_{U_k^m\times(0,t_m)} \in L^2(0,t_m;H^1(U_k^m;\mathbb{R}^N))$ *for all* $k,m\in\mathbb{N}$

(iii) $v\in L^2_{\mathrm{loc}}(G;\mathbb{R}^N)$ *and there exists a function* $g\in L^2_{\mathrm{loc}}(G;\mathbb{R}^{N\times n})$ *such that*

$$\int_G v\cdot \operatorname{div}(\zeta)\,\mathrm{d}(x,t) = -\int_G g:\zeta\,\mathrm{d}(x,t) \tag{21}$$

for all $\zeta\in\mathcal{C}_c^\infty(\mathrm{int}(G);\mathbb{R}^{N\times n})$

If one of these conditions is satisfied we write $\nabla v := g$ *and* $\epsilon(v) := \frac{1}{2}(\nabla v + \nabla v^{\mathrm{T}})$.

(b) Assume that each U_k^m *has a Lipschitz boundary. Then the following statements are equivalent:*

(i) $v=b$ *on the boundary* $D_T\cap G$

(ii) for every $k,m\in\mathbb{N}$, *condition* (2) *is satisfied for* $U=U_k^m$ *and* $t=t_m$

Proof.

(a) (i)$\Longrightarrow$(ii) and (iii)$\Longrightarrow$(i) are trivial.

(ii)$\Longrightarrow$(iii): Let the function $\widehat{g}:G\to\mathbb{R}^{N\times n}$ be $\mathcal{L}^{n+1}$-a.e. defined as follows. For each $k,m\in\mathbb{N}$, we set $\widehat{g}|_{U_k^m} := \widehat{g}_k^m$ where $\widehat{g}_k^m\in L^2(U_k^m\times(0,t_m);\mathbb{R}^{N\times n})$ is the weak derivative of $v|_{U_k^m\times(0,t_m)}$. The function $\widehat{g}$ is well-defined on $G\cap\Omega_T$ since

$$G\cap\Omega_T = \bigcup_{k,m\in\mathbb{N}} U_k^m\times(0,t_m)$$

and $\widehat{g}_{k_1}^{m_1} = \widehat{g}_{k_2}^{m_2}$ on $U_{k_1}^{m_1}\times(0,t_{m_1})\cap U_{k_2}^{m_2}\times(0,t_{m_2})$ for all $k_1,k_2,m_1,m_2\in\mathbb{N}$ in an $\mathcal{L}^{n+1}$-a.e. sense. Let $t\in(0,T]$ and $U\subset\subset G(t)$ be open. By Lemma 5.3, $U\times(0,t)$ can be covered by finitely many sets $U_k^m\times(0,t_m)$. In particular, $\widehat{g}|_{U\times(0,t)}\in L^2(0,t;L^2(U;\mathbb{R}^{N\times n}))$. Thus $\widehat{g}\in L^2_{\mathrm{loc}}(G;\mathbb{R}^{N\times n}))$.

Let $\zeta\in\mathcal{C}_c^\infty(\mathrm{int}(G);\mathbb{R}^{N\times n})$. Applying Lemma 5.3 again, there exists a finite set $I\subseteq\mathbb{N}$ such that $\operatorname{supp}(\zeta)\subseteq\bigcup_{k\in I}U_k^{m_k}\times(0,t_{m_k}) =: U$. By a partition of unity argument over U, (21) holds for $g=\widehat{g}$.

(b) (ii)$\Longrightarrow$(i): Let $t\in(0,T)$ and $U\subset\subset G(t)$ be an arbitrary open subset. By Lemma 5.3, we find a finite set $I\subseteq\mathbb{N}$ such that $U\subseteq\bigcup_{k\in I}U_k^{m_k}$ and $t_{m_k}\geq t$. The claim follows. ☐

If a relatively open set $H\subseteq\overline{\Omega}$ is admissible with respect to D we can construct a fine representation for H with Lipschitz domains in the following sense.

Lemma 5.5 (Lipschitz cover of admissible sets) *Let* $H\subseteq\overline{\Omega}$ *be relatively open and admissible with respect to* D. *Then there exists a fine representation* $\{U_m\}$ *for* H *such that*

(i) U_m *is a Lipschitz domain for all* $m\in N$,

(ii) $\mathcal{H}^{n-1}(\partial U_m\cap D)>0$ *for all* $m\in N$.

Proof. We will sketch a possible construction for reader's convenience.

We assume w.l.o.g. that H is path-connected because H can only have at most countably many path-connected components and for each component we can apply the construction below.

Let us choose a reference point $x_0 \in D \cap H$ with the property

$$\mathcal{H}^{n-1}\Big(\partial\big(B_\varepsilon(x_0) \cap \Omega\big) \cap D\Big) > 0 \text{ for all } \varepsilon > 0, \tag{22}$$

which is possible since $\mathcal{H}^{n-1}(D \cap H) > 0$. The relatively open subset $D_m \subseteq \overline{\Omega}$ for $m \in \mathbb{N}$ is defined as

$$D_m := H \setminus \overline{B_{1/m}(\overline{\Omega} \setminus H)}.$$

If m is large enough we have $x_0 \in D_m$ since $H \subseteq \overline{\Omega}$ is relatively open. We define

$$D_m' := \{x \in D_m \,|\, x \text{ is path-connected to } x_0 \text{ in } D_m\}.$$

Hence, we obtain an $\varepsilon > 0$ such that $B_\varepsilon(x_0) \cap \overline{\Omega} \subseteq D_m'$ since D_m' is relatively open in $\overline{\Omega}$. In combination with (22), this yields $\mathcal{H}^{n-1}(\partial D_m' \cap D) > 0$. Because of $D_m' \subset\subset H$, there exists a Lipschitz domain $U_m \subseteq \Omega$ with $D_m' \subseteq \overline{U_m} \subseteq H$ (e.g. the part of the boundary $\partial U_m \setminus \partial\Omega$ of U_m can be constructed by polygons such that ∂U_m fulfills the Lipschitz boundary condition). The family $\{U_m\}$ satisfies all the desired properties. □

Corollary 5.6 *Let $G \subseteq \overline{\Omega_T}$ be a shrinking set where $G(t)$ is admissible with respect to D for all $t \in [0,T]$. Furthermore, let $\{t_m\} \subseteq [0,T]$ be a dense sequence containing T.*

Then, there exists a countable family $\{U_k^m\}_{k\in\mathbb{N}}$ of Lipschitz domains $U_k^m \subset\subset G(t_m)$ for each $m \in \mathbb{N}$ such that

- *(i)* $\mathcal{H}^{n-1}(\partial U_k^m \cap D) > 0$ *for all* $m \in \mathbb{N}$,
- *(ii)* $\{U_k^m\}_{k\in\mathbb{N}}$ *is a fine representation for* $G(t_m)$ *for all* $m \in \mathbb{N}$,
- *(iii)* $G = \bigcup_{m=1}^\infty G(t_m) \times [0, t_m]$.

5.1.2 Γ-limit of the regularized energy

The construction of the values $\mathfrak{e}_s^+$ in (14) satisfying the lower energy bound (16) is based on Γ-convergence techniques which will be introduced below. We refer to [BMR09] for the utilization of Γ-convergence in the context with rate-independent complete damage models.

Proposition 5.7 (Γ-limit of the ε-regularized reduced energy)
Let $\mathfrak{E}_\varepsilon : H^1(\Omega;\mathbb{R}^n) \times W^{1,p}_{\mathrm{w}}(\Omega) \to \mathbb{R}_\infty$ be for $\varepsilon \geq 0$ the (regularized) reduced free energy defined by

$$\mathfrak{E}_\varepsilon(\xi, z) := \begin{cases} \inf_{\zeta \in H^1_D(\Omega;\mathbb{R}^n)} \mathcal{E}_\varepsilon(\epsilon(\xi+\zeta), z) & \text{if } 0 \leq z \leq 1, \\ \infty & \text{else.} \end{cases}$$

Then the Γ-limit of $\mathfrak{E}_\varepsilon$ as $\varepsilon \to 0^+$ with respect to the topology in $H^1(\Omega;\mathbb{R}^n) \times W^{1,p}_{\mathrm{w}}(\Omega)$ exists and is denoted by $\mathfrak{E}$ which is the lower semi-continuous envelope of $\mathfrak{E}_0$ (see [Bra02]). Here, $W^{1,p}_{\mathrm{w}}(\Omega)$ denotes the space $W^{1,p}(\Omega)$ with its weak topology.

Remark 5.8 *The existence of the* Γ*-limit above is ensured because* $\{\mathfrak{E}_\varepsilon\}$ *is non-negative and monotonically decreasing as* $\varepsilon \to 0^+$*. Furthermore,* $\mathfrak{E}$ *is lower semi-continuous in the* $H^1(\Omega;\mathbb{R}^n)\times W^{1,p}_{\mathrm{w}}(\Omega)$ *topology.*

To prove properties of the Γ-limit $\mathfrak{E}$ which are needed in Section 5.3, we will establish explicit recovery sequences. The proof relies on a substitution which is introduced in the following.

Assume that $u \in H^1(\Omega;\mathbb{R}^n)$ minimizes $\mathcal{F}_\varepsilon(\epsilon(\cdot), z)$ with Dirichlet data ξ on D. Then, by expressing the elastic energy density W^ε in terms of its derivative $W^\varepsilon_{,e}$, i.e. $W^\varepsilon = \frac{1}{2}W^\varepsilon_{,e} : e$, and by testing the Euler-Lagrange equation with $\zeta = u - \widetilde{u}$ for a function $\widetilde{u} \in H^1(\Omega;\mathbb{R}^n)$ with $\widetilde{u} = \xi$ on D, the elastic energy term in $\mathcal{E}_\varepsilon$ can be rewritten as

$$\int_\Omega W^\varepsilon(\epsilon(u), z)\,\mathrm{d}x = \int_\Omega \frac{1}{2}(g(z)+\varepsilon)\mathbb{C}\epsilon(u) : \epsilon(\widetilde{u})\,\mathrm{d}x. \tag{23}$$

Lemma 5.9 *For every* $\xi \in H^1(\Omega)$ *and* $z \in W^{1,p}(\Omega)$ *there exists a sequence* $\delta_\varepsilon \to 0^+$ *such that* $(\xi, (z-\delta_\varepsilon)^+) \to (\xi, z)$ *is a recovery sequence for* $\mathfrak{F}_\varepsilon \xrightarrow{\Gamma} \mathfrak{F}$ *as* $\varepsilon \to 0^+$ *where* $\mathfrak{F}$ *is the* Γ*-limit of* $\mathfrak{F}_\varepsilon : H^1(\Omega;\mathbb{R}^n) \times W^{1,p}_{\mathrm{w}}(\Omega) \to \mathbb{R}_\infty$ *given by*

$$\mathfrak{F}_\varepsilon(\xi, z) := \begin{cases} \min\limits_{\zeta \in H^1_D(\Omega;\mathbb{R}^n)} \mathcal{F}_\varepsilon(\epsilon(\xi+\zeta), z) & \text{if } 0 \le z \le 1, \\ \infty & \text{else.} \end{cases}$$

with

$$\mathcal{F}_\varepsilon(e, z) := \int_\Omega W^\varepsilon(e, z)\,\mathrm{d}x.$$

in the $H^1(\Omega;\mathbb{R}^n) \times W^{1,p}_{\mathrm{w}}(\Omega)$ *topology.*

Proof. The Γ-limit $\mathfrak{F}$ exists by the same argument as in Proposition 5.7. Let $(\xi_\varepsilon, z_\varepsilon) \to (\xi, z)$ be a recovery sequence. Since $z_\varepsilon \to z$ in $\mathcal{C}^{0,\alpha}(\overline{\Omega})$ due to the compact embedding $W^{1,p}(\Omega) \hookrightarrow \mathcal{C}^{0,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1 - \frac{n}{p}$, we can choose a sequence $\delta_\varepsilon \to 0^+$ such that $(z-\delta_\varepsilon)^+ \le z_\varepsilon$. Note that $(z-\delta_\varepsilon)^+ \in W^{1,p}(\Omega)$. Consider the arrangement

$$\mathfrak{F}_\varepsilon(\xi, (z-\delta_\varepsilon)^+) - \mathfrak{F}_\varepsilon(\xi_\varepsilon, z_\varepsilon) = \underbrace{\mathfrak{F}_\varepsilon(\xi, (z-\delta_\varepsilon)^+) - \mathfrak{F}_\varepsilon(\xi, z_\varepsilon)}_{A_\varepsilon} + \underbrace{\mathfrak{F}_\varepsilon(\xi, z_\varepsilon) - \mathfrak{F}_\varepsilon(\xi_\varepsilon, z_\varepsilon)}_{B_\varepsilon}.$$

We observe that $A_\varepsilon \le 0$ because of (note that $(z-\delta_\varepsilon)^+ \le z_\varepsilon$)

$$\mathcal{F}_\varepsilon(\epsilon(\xi+\zeta), (z-\delta_\varepsilon)^+) \le \mathcal{F}_\varepsilon(\epsilon(\xi+\zeta), z_\varepsilon)$$

for all $\zeta \in H^1_D(\Omega;\mathbb{R}^n)$. Let $u_\varepsilon, v_\varepsilon \in H^1_D(\Omega;\mathbb{R}^n)$ be given by

$$u_\varepsilon = \operatorname*{arg\,min}_{\zeta \in H^1_D(\Omega;\mathbb{R}^n)} \mathcal{F}_\varepsilon(\epsilon(\xi+\zeta), z_\varepsilon),$$
$$v_\varepsilon = \operatorname*{arg\,min}_{\zeta \in H^1_D(\Omega;\mathbb{R}^n)} \mathcal{F}_\varepsilon(\epsilon(\xi_\varepsilon+\zeta), z_\varepsilon).$$

Applying the substitution equation (23) for u_ε with $\widetilde{u} = v_\varepsilon$ and for v_ε with $\widetilde{u} = u_\varepsilon$, we obtain a calculation as follows:

$$B_\varepsilon = \mathcal{F}_\varepsilon(\epsilon(\xi+u_\varepsilon), z_\varepsilon) - \mathcal{F}_\varepsilon(\epsilon(\xi_\varepsilon+v_\varepsilon), z_\varepsilon)$$

$$
\begin{aligned}
&= \int_\Omega \frac{1}{2}(g(z_\varepsilon)+\varepsilon)\mathbb{C}\epsilon(\xi+u_\varepsilon):\epsilon(\xi+v_\varepsilon) - \frac{1}{2}(g(z_\varepsilon)+\varepsilon)\mathbb{C}\epsilon(\xi_\varepsilon+v_\varepsilon):\epsilon(\xi_\varepsilon+u_\varepsilon)\,\mathrm{d}x \\
&\leq \int_\Omega \frac{1}{2}(g(z_\varepsilon)+\varepsilon)\Big(\mathbb{C}\epsilon(\xi):\epsilon(\xi) - \mathbb{C}\epsilon(\xi_\varepsilon):\epsilon(\xi_\varepsilon)\Big)\,\mathrm{d}x \\
&\quad + \|\frac{1}{2}(g(z_\varepsilon)+\varepsilon)\mathbb{C}\epsilon(u_\varepsilon+v_\varepsilon)\|_{L^2(\Omega)}\|\epsilon(\xi-\xi_\varepsilon)\|_{L^2(\Omega)}
\end{aligned}
$$

Using $\xi_\varepsilon \to \xi$ in $H^1(\Omega)$, $z_\varepsilon \rightharpoonup z$ in $W^{1,p}(\Omega)$ and the boundednes of $\mathcal{F}_\varepsilon(\epsilon(\xi+u_\varepsilon), z_\varepsilon)$ and $\mathcal{F}_\varepsilon(\epsilon(\xi_\varepsilon + v_\varepsilon), z_\varepsilon)$ with respect to ε, we end up with $\limsup_{\varepsilon\to 0^+} B_\varepsilon \leq 0$. Consequently, taking also into account that $(\xi_\varepsilon, z_\varepsilon) \to (\xi, z)$ is a recovery sequence, we obtain

$$
\limsup_{\varepsilon\to 0^+} \mathfrak{F}_\varepsilon(\xi,(z-\delta_\varepsilon)^+) \leq \limsup_{\varepsilon\to 0^+} \mathfrak{F}_\varepsilon(\xi_\varepsilon, z_\varepsilon) + \limsup_{\varepsilon\to 0^+} A_\varepsilon + \limsup_{\varepsilon\to 0^+} B_\varepsilon \leq \mathfrak{F}(\xi, z).
$$

□

Corollary 5.10 *(i) For every $\xi \in H^1(\Omega;\mathbb{R}^n)$ and $z \in W^{1,p}(\Omega)$*

$$
\mathfrak{E}(\xi,z) = \int_\Omega \frac{1}{p}|\nabla z|^p\,\mathrm{d}x + \mathfrak{F}(\xi,z).
$$

(ii) The recovery sequence $(\xi,(z-\delta_\varepsilon)^+) \to (\xi,z)$ for $\mathfrak{F}_\varepsilon \xrightarrow{\Gamma} \mathfrak{F}$ constructed in Lemma 5.9 is a recovery sequence for $\mathfrak{E}_\varepsilon \xrightarrow{\Gamma} \mathfrak{E}$ as well.

(iii) Let $\xi \in H^1(\Omega;\mathbb{R}^n)$, $z \in W^{1,p}(\Omega)$ and $F \subseteq \Omega$ be open such that $\mathbb{1}_F z \in W^{1,p}(\Omega)$. Then $\mathfrak{E}(\xi, \mathbb{1}_F z) \leq \mathfrak{E}(\xi, z)$.

Proof.

(i) Let $(\xi_\varepsilon, z_\varepsilon) \to (\xi, z)$ be a recovery sequence for $\mathfrak{E}_\varepsilon \xrightarrow{\Gamma} \mathfrak{E}$. Hence, $\xi_\varepsilon \to \xi$ in $H^1(\Omega;\mathbb{R}^n)$ and $z_\varepsilon \rightharpoonup z$ in $W^{1,p}(\Omega)$. Applying "$\liminf_{\varepsilon\to 0^+}$" on each side of the identity

$$
\mathfrak{E}_\varepsilon(\xi_\varepsilon, z_\varepsilon) = \int_\Omega \frac{1}{p}|\nabla z_\varepsilon|^p\,\mathrm{d}x + \mathfrak{F}_\varepsilon(\xi_\varepsilon, z_\varepsilon) \tag{24}
$$

yields for a subsequence

$$
\mathfrak{E}(\xi,z) \geq \int_\Omega \frac{1}{p}|\nabla z|^p\,\mathrm{d}x + \mathfrak{F}(\xi,z).
$$

The "$\leq$" - part can be shown by considering a recovery sequence $(\xi,(z-\varepsilon)^+) \to (\xi,z)$ for $\mathfrak{F}_\varepsilon \xrightarrow{\Gamma} \mathfrak{F}$ according to Lemma 5.9 and applying "$\liminf_{\varepsilon\to 0^+}$" in (24) with $(\xi_\varepsilon, z_\varepsilon) = (\xi,(z-\varepsilon)^+)$ on both sides.

(ii) This follows from (i).

(iii) Without loss of generality, we assume $0 \leq z \leq 1$ on Ω. Let $(\xi,(z-\delta_\varepsilon)^+) \to (\xi,z)$ be a recovery sequence for $\mathfrak{E}_\varepsilon \xrightarrow{\Gamma} \mathfrak{E}$ as in (ii). By assumption, $\mathbb{1}_F(z-\delta_\varepsilon)^+ \in W^{1,p}(\Omega)$ and $\mathbb{1}_F(z-\delta_\varepsilon)^+ \to \mathbb{1}_F z$ in $W^{1,p}(\Omega)$ as $\varepsilon \to 0^+$.

Since $\mathcal{E}_\varepsilon(\epsilon(\xi+\zeta), \mathbb{1}_F(z-\delta_\varepsilon)^+) \leq \mathcal{E}_\varepsilon(\epsilon(\xi+\zeta),(z-\delta_\varepsilon)^+)$ for all $\zeta \in H^1_D(\Omega;\mathbb{R}^n)$, we obtain

$$\inf_{\zeta\in H^1_D(\Omega;\mathbb{R}^n)} \mathcal{E}_\varepsilon(\epsilon(\xi+\zeta), \mathbb{1}_F(z-\delta_\varepsilon)^+) \leq \inf_{\zeta\in H^1_D(\Omega;\mathbb{R}^n)} \mathcal{E}_\varepsilon(\epsilon(\xi+\zeta),(z-\delta_\varepsilon)^+).$$

Therefore,

$$\mathfrak{E}_\varepsilon(\xi, \mathbb{1}_F(z-\delta_\varepsilon)^+) \leq \mathfrak{E}_\varepsilon(\xi,(z-\delta_\varepsilon)^+).$$

Passing to $\varepsilon \to 0^+$ yields the claim. $\square$

Lemma 5.11 *Let $\xi \in H^1(\Omega;\mathbb{R}^n)$ and $z \in W^{1,p}(\Omega)$ with $0 \leq z \leq 1$. Furthermore, let $u \in H^1_{\mathrm{loc}}(\{z>0\};\mathbb{R}^n)$ and for every Lipschitz domain $U \subset\subset \{z>0\}$, $u=\xi$ on $D\cap\partial U$ in the sense of traces. Then*

$$\mathfrak{E}(\xi,z) \leq \mathcal{E}(\epsilon(u),z).$$

Proof. Consider an arbitrary $\varepsilon>0$ and define $z_\varepsilon := (z-\varepsilon)^+$. Since $z\in\mathcal{C}(\overline{\Omega})$, it holds the compact inclusion $\{z_\varepsilon>0\}\subset\subset\{z>0\}$. There exists an open set U with Lipschitz boundary such that $\{z_\varepsilon>0\}\subseteq\overline{U}\subseteq\{z>0\}$ (e.g. construction of $\partial U\setminus\partial\Omega$ by polygons such that ∂U fulfills the Lipschitz boundary condition).

Now, we have $u|_U\in H^1(U;\mathbb{R}^n)$ as well as $u=\xi$ on $\partial U\cap D$. There exists an extension $u_\varepsilon\in H^1(\Omega;\mathbb{R}^n)$ with $u_\varepsilon|_U=u|_U$ and $u_\varepsilon=\xi$ on D. The monotone decreasingness of $\{\mathfrak{E}_\varepsilon\}$ with respect to ε implies that $\mathfrak{E}$ is the lower semi-continuous envelope of $\widetilde{\mathfrak{E}}(\xi,z) := \inf_{\varepsilon>0}\mathfrak{E}_\varepsilon(\xi,z)$. in the $H^1(\Omega;\mathbb{R}^n)\times W^{1,p}_{\mathrm{w}}(\Omega)$-topology (cf. [Bra02]). By switching the infima, it holds

$$\widetilde{\mathfrak{E}}(\xi,z) = \begin{cases} \inf_{\zeta\in H^1_D(\Omega;\mathbb{R}^n)}\mathcal{E}(\epsilon(\xi+\zeta),z) & \text{if } 0\leq z\leq 1,\\ \infty & \text{else.}\end{cases}$$

Since $u=u_\varepsilon$ on $\{z_\varepsilon>0\}$, we get

$$\begin{aligned}\mathfrak{E}(\xi,z) &= \inf_{\xi_\varepsilon\to\xi \text{ in } H^1(\Omega;\mathbb{R}^n)}\ \inf_{\eta_\varepsilon\rightharpoonup z \text{ in } W^{1,p}(\Omega)}\ \liminf_{\varepsilon\to 0}\ \widetilde{\mathfrak{E}}(\xi_\varepsilon,\eta_\varepsilon)\\ &\leq \liminf_{\varepsilon\to 0}\ \widetilde{\mathfrak{E}}(\xi,z_\varepsilon) \leq \liminf_{\varepsilon\to 0}\ \mathcal{E}(\epsilon(u_\varepsilon),z_\varepsilon)\\ &\leq \liminf_{\varepsilon\to 0}\ \mathcal{E}(\epsilon(u),z_\varepsilon) = \mathcal{E}(\epsilon(u),z).\end{aligned}$$

$\square$

5.2 Simplified problem

In the first step of proving Theorem 4.1, an existence result of a simplified problem, where no exclusion of material parts are considered, will be shown. The statement we are going to prove in this subsection is given as follows.

Proposition 5.12 (Degenerate limit) *Let $b \in W^{1,1}(0,T;W^{1,\infty}(\Omega;\mathbb{R}^n))$ and $z^0 \in W^{1,p}(\Omega)$ with $0 \le z^0 \le 1$ and $\{z^0 > 0\}$ admissible with respect to D be initial-boundary data and let W be given by* (5) *satisfying* (6)*. Then there exist functions*

$$z \in L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega)), \quad u \in L^2_t H^1_{x,\mathrm{loc}}(\mathfrak{A}_D(\{z>0\});\mathbb{R}^n),$$
$$e \in L^2(\{z>0\};\mathbb{R}^{n\times n}_{\mathrm{sym}})$$

with $e = \epsilon(u)$ in $\mathfrak{A}_D(\{z>0\})$ such that the properties (ii)-(v) of Definition 3.5 are fulfilled for $F := \{z > 0\}$. Moreover, $\mathfrak{e}_0^+$ (see energy inequality (14)*) can be chosen to be $\mathfrak{E}(b^0, z^0)$ which satisfies* (16) *by Lemma 5.11.*

Remark 5.13 *Let us consider the functions e, u and z obtained above in the degenerate limit. We do not know that $F = \{z > 0\}$ equals $\mathfrak{A}_D(\{z>0\})$ and, if $F \setminus \mathfrak{A}_D(\{z>0\}) \neq \emptyset$, it is not clear whether u can be extended such that $e = \epsilon(u)$ also holds in F. On the other hand, we would like to stress that $(u, z^\star)$ with the truncated function $z^\star := z\mathbb{1}_{\mathfrak{A}_D(\{z>0\})}$ also do not necessarily form a weak solution in the sense of Definition 3.5. Because $z^\star$ viewed as an $SBV^2(0,T;L^2(\Omega))$-function may have jumps which needs to be accounted for in the energy inequality* (14)*. The construction of weak solutions will be performed in Section 5.3.*

Let $(b^0, z^0_\varepsilon) \to (b^0, z^0)$ with $z^0_\varepsilon := (z - \delta_\varepsilon)^+$ and $b^0 := b(0)$ be a recovery sequence of $\mathfrak{E}_\varepsilon \xrightarrow{\Gamma} \mathfrak{E}$ according to Lemma 5.10 (ii). A modification of the proof of Theorem 4.6 in [HK11] yields the following result.

Theorem 5.14 (ε-regularized problem - incomplete damage) *Let $\varepsilon > 0$. For the given initial-boundary data $z^0_\varepsilon \in W^{1,p}(\Omega)$ and $b \in W^{1,1}(0,T;W^{1,\infty}(\Omega;\mathbb{R}^n))$ there exists a pair $q_\varepsilon = (u_\varepsilon, z_\varepsilon)$ such that*

(i) Trajectory spaces:

$$z_\varepsilon \in L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega)), \quad u_\varepsilon \in L^\infty(0,T;H^1(\Omega;\mathbb{R}^n)).$$

(ii) Quasi-static mechanical equilibrium:

$$\int_\Omega W^\varepsilon_{,e}(\epsilon(u_\varepsilon(t)), z_\varepsilon(t)) : \epsilon(\zeta)\,\mathrm{d}x = 0 \tag{25}$$

for a.e. $t \in (0,T)$ and for all $\zeta \in H^1_D(\Omega;\mathbb{R}^n)$. Furthermore, $u_\varepsilon = b$ on the boundary D_T.

(iii) Damage variational inequality:

$$\int_\Omega |\nabla z_\varepsilon(t)|^{p-2}\nabla z_\varepsilon(t)\cdot\nabla\zeta + W^\varepsilon_{,z}(\epsilon(u_\varepsilon(t)), z_\varepsilon(t))\zeta\,\mathrm{d}x \ge \int_\Omega (\alpha - \beta\partial_t z_\varepsilon(t) - r_\varepsilon(t))\zeta\,\mathrm{d}x, \tag{26}$$
$$z_\varepsilon(t) \ge 0,$$
$$\partial_t z_\varepsilon(t) \le 0$$

for a.e. $t \in (0,T)$ and for all $\zeta \in W^{1,p}(\Omega)$ with $\zeta \le 0$ where $r_\varepsilon \in L^1(\Omega_T)$ satisfies

$$\int_\Omega r_\varepsilon(t)(\zeta - z_\varepsilon(t))\,\mathrm{d}x \le 0$$

for a.e. $t \in (0,T)$ and for all $\zeta \in W^{1,p}(\Omega)$ with $\zeta \ge 0$. The initial value is given by $z_\varepsilon(t = 0) = z^0_\varepsilon$ in $\overline{\Omega}$.

(iv) Energy inequality:

$$\begin{aligned}\mathcal{E}_\varepsilon(\epsilon(u_\varepsilon(t)),z_\varepsilon(t))&+\int_{\Omega_t}\alpha|\partial_t z_\varepsilon|+\beta|\partial_t z_\varepsilon|^2\,\mathrm{d}(x,s)\\&\leq\mathcal{E}_\varepsilon(\epsilon(u_\varepsilon^0),z_\varepsilon^0)+\int_{\Omega_t}W^\varepsilon_{,e}(\epsilon(u_\varepsilon),z_\varepsilon):\epsilon(\partial_t b)\,\mathrm{d}(x,s)\end{aligned}\tag{27}$$

holds for a.e. $t\in(0,T)$ *where* u_ε^0 *minimizes* $\mathcal{E}_\varepsilon(\epsilon(\cdot),z_\varepsilon^0)$ *in* $H^1(\Omega;\mathbb{R}^n)$ *with Dirichlet data* b^0 *on* D.

Moreover, r_ε *in (iv) can be chosen to be*

$$r_\varepsilon=-\chi_\varepsilon W_{,z}(\epsilon(u_\varepsilon),z_\varepsilon)\tag{28}$$

with $\chi_\varepsilon\in L^\infty(\Omega)$ *fulfulling* $\chi_\varepsilon=0$ *on* $\{z_\varepsilon>0\}$ *and* $0\leq\chi_\varepsilon\leq 1$ *on* $\{z_\varepsilon=0\}$.

We consider a sequence $\{\varepsilon_M\}_{M\in\mathbb{N}}\subseteq(0,1)$ with $\varepsilon_M\to 0^+$ as $M\to\infty$ and for every $M\in\mathbb{N}$ a weak solution $(u_{\varepsilon_M},z_{\varepsilon_M})$ of the incomplete damage problem according to Theorem 5.14. The index M is omitted in the following. We agree that $e_\varepsilon:=\epsilon(u_\varepsilon)$ denotes the strain of the regularized system. Our further analysis makes also use of the truncated strain $\widehat{e}_\varepsilon$ (the strain on the not completely damaged parts of Ω) given by

$$\widehat{e}_\varepsilon:=e_\varepsilon\mathbb{1}_{\{z_\varepsilon>0\}}.$$

We proceed by deriving suitable a-priori estimates for the incomplete damage problem with respect to ε.

Lemma 5.15 (A-priori estimates) *There exists a* $C>0$ *independent of* ε *such that*

(i) $\|\widehat{e}_\varepsilon\|_{L^2(\Omega_T;\mathbb{R}^{n\times n})}\leq C$,

(ii) $\sup_{t\in[0,T]}\|z_\varepsilon(t)\|_{W^{1,p}(\Omega)}\leq C$,

(iii) $\|\partial_t z_\varepsilon\|_{L^2(\Omega_T)}\leq C$,

(iv) $\|W^\varepsilon(e_\varepsilon,z_\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))}\leq C$.

Proof. Applying Gronwall's lemma to the energy estimate (27) and noticing the boundedness of $\mathcal{E}_\varepsilon(\epsilon(u_\varepsilon^0),z_\varepsilon^0)$ with respect to $\varepsilon\in(0,1)$ show (iii) and

$$\mathcal{E}_\varepsilon(e_\varepsilon(t),z_\varepsilon(t))\leq C\tag{29}$$

for a.e. $t\in(0,T)$ and all $\varepsilon\in(0,1)$ (cf. [HK11]) and in particular (iv). Taking the restriction $0\leq z_\varepsilon\leq 1$ into account, property (29) gives rise to $\|z_\varepsilon\|_{L^\infty(0,T;W^{1,p}(\Omega))}\leq C$. Together with the control of the time-derivative (iii), we obtain boundedness of $\|z_\varepsilon(t)\|_{W^{1,p}(\Omega)}\leq C$ for every $t\in[0,T]$ and $\varepsilon\in(0,1)$. Hence, (ii) is proven.

It remains to show (i). To proceed, we test inequality (26) with $\zeta\equiv -1$ and integrate from $t=0$ to $t=T$:

$$\int_{\Omega_T}W^\varepsilon_{,z}(e_\varepsilon,z_\varepsilon)+r_\varepsilon\,\mathrm{d}(x,t)\leq\int_{\Omega_T}\alpha-\beta\,\partial_t z_\varepsilon\,\mathrm{d}(x,t).\tag{30}$$

Applying (6), (28) and (30), yield

$$\int_{\Omega_T}\eta|\widehat{e}_\varepsilon|^2\,\mathrm{d}(x,t)=\int_{\{z_\varepsilon>0\}}\eta|e_\varepsilon|^2\,\mathrm{d}(x,t)\leq\int_{\Omega_T}W^\varepsilon_{,z}(e_\varepsilon,z_\varepsilon)\,\mathrm{d}(x,t)-\int_{\{z_\varepsilon=0\}}W^\varepsilon_{,z}(e_\varepsilon,z_\varepsilon)\,\mathrm{d}(x,t)$$

$$\leq \int_{\Omega_T} W^\varepsilon_{,z}(e_\varepsilon, z_\varepsilon) + r_\varepsilon(t)\,\mathrm{d}(x,t)$$
$$\leq \int_{\Omega_T} \alpha - \beta\,\partial_t z_\varepsilon\,\mathrm{d}(x,t).$$

This and the boundedness of $\int_{\Omega_T} \alpha - \beta\,\partial_t z_\varepsilon\,\mathrm{d}(x,t)$ with respect to ε shows (i). $\square$

Lemma 5.16 (Converging subsequences) *There exists functions*

$$\widehat{e} \in L^2(\Omega_T;\mathbb{R}^{n\times n}), \quad z \in L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega)),$$

where z is monotonically decreasing with respect to t, i.e. $\partial_t z \leq 0$, and a subsequence (we omit the index) such that for $\varepsilon \to 0^+$

(i) $z_\varepsilon \rightharpoonup z$ *in* $H^1(0,T;L^2(\Omega))$,
$z_\varepsilon \to z$ *in* $L^p(0,T;W^{1,p}(\Omega))$,
$z_\varepsilon(t) \rightharpoonup z(t)$ *in* $W^{1,p}(\Omega)$,
$z_\varepsilon \to z$ *in* $\overline{\Omega_T}$,

(ii) $\widehat{e}_\varepsilon \rightharpoonup \widehat{e}$ *in* $L^2(\Omega_T;\mathbb{R}^{n\times n})$,
$W^\varepsilon_{,e}(e_\varepsilon, z_\varepsilon) \rightharpoonup W_{,e}(\widehat{e}, z)$ *in* $L^2(\{z>0\};\mathbb{R}^{n\times n})$,
$W^\varepsilon_{,e}(e_\varepsilon, z_\varepsilon) \to 0$ *in* $L^2(\{z=0\};\mathbb{R}^{n\times n})$.

Proof. The a-priori estimates from Lemma 5.15 and classical compactness theorems as well as compactness theorems from J.-L. Lions and T. Aubin yield [Sim86]

$z_\varepsilon \overset{\star}{\rightharpoonup} z$ in $L^\infty(0,T;W^{1,p}(\Omega))$,
$z_\varepsilon \rightharpoonup z$ in $H^1(0,T;L^2(\Omega))$,
$z_\varepsilon \to z$ in $L^p(\Omega_T)$

$\widehat{e}_\varepsilon \rightharpoonup \widehat{e}$ in $L^2(\Omega_T;\mathbb{R}^{n\times n})$,
$W^\varepsilon_{,e}(e_\varepsilon, z_\varepsilon) \rightharpoonup w_e$ in $L^2(\Omega_T;\mathbb{R}^{n\times n})$,

as $\varepsilon \to 0^+$ for a subsequence and appropriate functions w_e, $\widehat{e}$ and z.

Proving the strong convergence of ∇z_ε in $L^p(\Omega_T;\mathbb{R}^n)$ differs not substantially from the proof presented in [HK11]. It is essentially based on the elementary inequality

$$C_{\mathrm{uc}}|x-y|^p \leq \left\langle \left(|x|^{p-2}x - |y|^{p-2}y\right), x-y \right\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the standard euclidean scalar product and on an approximation scheme $\{\zeta_\varepsilon\} \subseteq L^p(0,T;W^{1,p}(\Omega))$ with $\zeta_\varepsilon \geq 0$ and

$$\zeta_\varepsilon \to z \text{ in } L^p(0,T;W^{1,p}(\Omega)) \text{ as } \varepsilon \to 0^+, \tag{31a}$$
$$0 \leq \zeta_\varepsilon \leq z_\varepsilon \text{ a.e. in } \Omega_T \text{ for all } \varepsilon \in (0,1). \tag{31b}$$

Using the above properties, we obtain the estimate:

$$C_{\mathrm{uc}}\int_{\Omega_T} |\nabla z_\varepsilon - \nabla z|^p\,\mathrm{d}(x,t) \leq \int_{\Omega_T} (|\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon - |\nabla z|^{p-2}\nabla z)\cdot\nabla(z_\varepsilon - z)\,\mathrm{d}(x,t)$$
$$= \underbrace{\int_{\Omega_T} |\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon \cdot \nabla(z_\varepsilon - \zeta_\varepsilon)\,\mathrm{d}(x,t)}_{A_\varepsilon}$$

$$+\underbrace{\int_{\Omega_T} |\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon\cdot\nabla(\zeta_\varepsilon - z) - |\nabla z|^{p-2}\nabla z\cdot\nabla(z_\varepsilon - z)\,\mathrm{d}(x,t)}_{B_\varepsilon}.$$

The weak convergence property of $\{\nabla z_\varepsilon\}$ in $L^p(\Omega_T)$ and (31a) show $B_\varepsilon \to 0$ as $\varepsilon \to 0^+$. Property (26) tested with $\zeta(t) = \zeta_\varepsilon(t) - z_\varepsilon(t)$ and integration from $t = 0$ to $t = T$ yields

$$A_\varepsilon \le \underbrace{\int_{\Omega_T} W^\varepsilon_{,z}(\epsilon(u_\varepsilon), z_\varepsilon)(\zeta_\varepsilon - z_\varepsilon)\,\mathrm{d}(x,t)}_{\le 0 \text{ by (6) and (31b)}} + \underbrace{\int_{\Omega_T} (-\alpha + \beta(\partial_t z_\varepsilon(t)))(\zeta_\varepsilon - z_\varepsilon)\,\mathrm{d}(x,t)}_{\to 0 \text{ as } \varepsilon\to 0^+ \text{ by (31a)}}.$$

Here, we have used $r_\varepsilon\zeta = 0$ on Ω_T (see (28)). Therefore, (i) is also shown.

To prove (ii), we define N_ε to be $\{z_\varepsilon > 0\} \cap \{z > 0\}$. Consequently, we get

$$W^\varepsilon_{,e}(\widehat{e}_\varepsilon, z_\varepsilon)\mathbb{1}_{N_\varepsilon} = W^\varepsilon_{,e}(e_\varepsilon, z_\varepsilon)\mathbb{1}_{N_\varepsilon} \tag{32}$$

and the convergence

$$\mathbb{1}_{N_\varepsilon} \to \mathbb{1}_{\{z>0\}} \text{ in } \Omega_T \tag{33}$$

for $\varepsilon \to 0^+$ by using $z_\varepsilon \to z$ in $\overline{\Omega_T}$. Calculating the weak $L^1(\Omega_T;\mathbb{R}^{n\times n})$-limits in (32) for $\varepsilon \to 0^+$ on both sides by using the already proven convergence properties, we obtain $W_{,e}(\widehat{e}, z) = w_e$. The remaining convergence property in (ii) follow from Lemma 5.15 (iv). □

We now introduce the shrinking set $F \subseteq \overline{\Omega_T}$ by defining

$$F(t) := \{z(t) > 0\}$$

for all $t \in [0,T]$. This is a well-defined object since $F \subseteq \overline{\Omega_T}$ is relatively open by Theorem A.2 as well as $F(s) \subseteq F(t)$ for all $0 \le t \le s \le T$ by the monotone decrease of $z(x,\cdot)$.

Corollary 5.17 *Let $t \in [0,T]$ and $U \subset\subset F(t)$ be an open subset. Then $U \subseteq \{z_\varepsilon(s) > 0\}$ for all $s \in [0,t]$ provided that $\varepsilon > 0$ is sufficiently small. More precisely, there exist $0 < \varepsilon_0, \eta < 1$ such that*

$$z_\varepsilon(s) \ge \eta \text{ in } U$$

for all $s \in [0,t]$ and for all $0 < \varepsilon < \varepsilon_0$.

Proof. By assumption, we obtain the property $\mathrm{dist}(U, \{z(t) = 0\}) > 0$. Therefore, and by $z(t) \in \mathcal{C}(\overline{\Omega})$, we find an $\eta > 0$ such that $z(t) \ge 2\eta$ in U. By exploiting the convergence $z_\varepsilon(t) \to z(t)$ in $\mathcal{C}(\overline{\Omega})$ as $\varepsilon \to 0^+$ by Lemma 5.16 (b) and the compact embedding $W^{1,p}(\Omega) \hookrightarrow \mathcal{C}(\overline{\Omega})$, there exists an $\varepsilon_0 > 0$ such that $z_\varepsilon(t) \ge \eta$ on U for all $0 < \varepsilon < \varepsilon_0$. Finally, the claim follows from the fact that z_ε is monotonically decreasing with respect to t. □

Lemma 5.18 *There exists a function $u \in L^2_t H^1_{x,\mathrm{loc}}(\mathfrak{A}_D(F);\mathbb{R}^n)$ such that*

(i) $\epsilon(u) = \widehat{e}$ a.e. in $\mathfrak{A}_D(F)$,

(ii) $u = b$ *on the boundary* $D_T \cap \mathfrak{A}_D(F)$.

Proof. Let $\{U_k^m\}$ and $\{t_m\}$ be sequences satisfying the properties of Corollary 5.6 applied to $\mathfrak{A}_D(F)$. We get for each fixed $k, m \in \mathbb{N}$

$$U_k^m \times [0, t_m] \subseteq \{z_\varepsilon > 0\} \tag{34}$$

for all $0 < \varepsilon \ll 1$ due to Corollary 5.17. Inclusion (34) implies

$$\epsilon(u_\varepsilon) = \widehat{e}_\varepsilon \tag{35}$$

a.e. in $U_k^m \times (0, t_m)$. Korn's inequality applied on the Lipschitz domain U_k^m yields (note that $\mathcal{H}^{n-1}(\partial U_k^m \cap D) > 0$)

$$\begin{aligned}
\|u_\varepsilon\|^2_{L^2(0,t_m;H^1(U_k^m;\mathbb{R}^n))} &\le 2 \int_0^{t_m} \|u_\varepsilon(t) - b(t)\|^2_{H^1(U_k^m;\mathbb{R}^n)} + \|b(t)\|^2_{H^1(U_k^m;\mathbb{R}^n)} \,\mathrm{d}t \\
&\le C\left(1 + \int_0^{t_m} \|\epsilon(u_\varepsilon(t))\|^2_{L^2(U_k^m;\mathbb{R}^{n\times n})} \,\mathrm{d}t\right) \\
&\le C\left(1 + \int_0^{t_m} \|\widehat{e}_\varepsilon(t))\|^2_{L^2(\Omega;\mathbb{R}^{n\times n})} \,\mathrm{d}t\right).
\end{aligned}$$

with a constant $C = C(U_k^m, b) > 0$. Together with the boundedness of $\widehat{e}_\varepsilon$ in $L^2(\Omega_T; \mathbb{R}^{n\times n})$, we can find a subsequence $\varepsilon \to 0^+$ and a function $u^{(k,m)} \in L^2(0, t_m; H^1(U_k^m; \mathbb{R}^n))$ such that

$$u_\varepsilon \rightharpoonup u^{(k,m)} \text{ in } L^2(0, t_m; H^1(U_k^m; \mathbb{R}^n)). \tag{36}$$

Thus $\epsilon(u^{(k,m)}) = \widehat{e}$ in $U_k^m \times (0, t_m)$ because of (35) and the weak convergence property of $\widehat{e}_\varepsilon$. For each $k, m \in \mathbb{N}$, we can apply the argumention above. Therefore, by successively choosing subsequences and by applying a diagonalization argument, we obtain a subsequence $\varepsilon \to 0^+$ such that (36) holds for all $k, m \in \mathbb{N}$.

Since $u^{(k_1,m_1)} = u^{(k_2,m_2)}$ a.e. on $U_{k_1}^{m_1} \times (0, t_{m_1}) \cap U_{k_2}^{m_2} \times (0, t_{m_2})$ for all $k_1, k_2, m_1, m_2 \in \mathbb{N}$, we obtain an $u : \mathfrak{A}_D(F) \to \mathbb{R}^n$ such that $u|_{U_k^m \times (0,t_m)} \in L^2(0, t_m; H^1(U_k^m; \mathbb{R}^n))$ for all $m \in \mathbb{N}$. Proposition 5.4 (a) yields $u \in L_t^2 H^1_{x,\mathrm{loc}}(F; \mathbb{R}^n)$ and the symmetric gradient $\epsilon(u)$ coincides with $\widehat{e}$. Therefore, (i) is shown.

Furthermore, for every $k, m \in \mathbb{N}$, we have $u(t) = b(t)$ on $\partial U_k^m \cap D$ in the sense of traces for a.e. $t \in [0, t_m]$. By Proposition 5.4 (b), (ii) follows. □

We are now able to prove Proposition 5.12.

Proof of Proposition 5.12. Lemma 5.16 and Lemma 5.18 give the desired regularity properties of the functions (e, u, z) in Proposition 5.12. Here, we set $e := \widehat{e}|_F \in L^2(F; \mathbb{R}^{n\times n})$. The property $e = \epsilon(u)$ in $\mathfrak{A}_D(F)$ follows from Lemma 5.18.

In the following, we are going to prove that properties (ii)-(v) of Definition 3.5 are satisfied.

(ii) Lemma 5.16 (ii) allows us to pass to $\varepsilon \to 0^+$ in (25) integrated from $t = 0$ to $t = T$. Therefore, equation (11) holds for a.e. $t \in (0, T)$ and all $\zeta \in H^1_D(\Omega; \mathbb{R}^n)$. Moreover, the boundary condition $u = b$ on $D_T \cap \mathfrak{A}_D(F)$ is satisfied. Definition 3.1 immediately implies $D_T \cap F = D_T \cap \mathfrak{A}_D(F)$.

(iii) We first show (12). Let $\zeta \in L^\infty(0,T;W^{1,p}(\Omega))$ with $\zeta \le 0$. The variational inequality (26) and the representation for r_ε (28) imply

$$0 \le \int_{\Omega_T} |\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon \cdot \nabla\zeta + (-\alpha + \beta\partial_t z_\varepsilon)\zeta \,\mathrm{d}(x,t) + \int_{\{z_\varepsilon>0\}} W^\varepsilon_{,z}(e_\varepsilon, z_\varepsilon)\zeta \,\mathrm{d}(x,t). \tag{37}$$

In addition,

$$\begin{aligned}\int_{\{z_\varepsilon>0\}} W^\varepsilon_{,z}(e_\varepsilon, z_\varepsilon)\zeta \,\mathrm{d}(x,t) &\le \int_{F\cap\{z_\varepsilon>0\}} W^\varepsilon_{,z}(e_\varepsilon, z_\varepsilon)\zeta \,\mathrm{d}(x,t)\\ &= \int_F g'(z_\varepsilon)\mathbb{C}\widehat{e}_\varepsilon : \widehat{e}_\varepsilon\zeta \,\mathrm{d}(x,t)\end{aligned}$$

Lemma 5.16, a lower semi-continuity argument and $\mathbb{1}_{\{z_\varepsilon>0\}\cap\{z=0\}} \to \mathbb{1}_{\{z=0\}}$ a.e. in Ω_T (see proof of Lemma 5.16) yield

$$\limsup_{\varepsilon\to0^+} \int_{\{z_\varepsilon>0\}} W^\varepsilon_{,z}(e_\varepsilon, z_\varepsilon)\zeta \,\mathrm{d}(x,t) \le \int_F W^\varepsilon_{,z}(e, z)\zeta \,\mathrm{d}(x,t).$$

Therefore, applying "$\limsup_{\varepsilon\to0^+}$" on both sides of (37), using the above estimate and Lemma 5.16 yield

$$\int_F |\nabla z|^{p-2}\nabla z\cdot\nabla\zeta + W_{,z}(e,z)\zeta \,\mathrm{d}x \ge \int_\Omega (\alpha - \beta\partial_t z)\zeta \,\mathrm{d}x. \tag{38}$$

The properties $\partial_t z \le 0$ and $z \ge 0$ a.e. in Ω_T follow from Lemma 5.16 by taking $\partial_t z_\varepsilon \le 0$ and $z_\varepsilon \ge 0$ a.e. in Ω_T into account.

(iv) The jump condition (13) in (iv) of Definition 3.5 holds trivially since we have the regularity $z \in L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega))$.

(v) To complete the proof, we need to show the energy estimates (14). Since $\{b^0, z^0_\varepsilon\}$ is a recovery sequence, we get $\mathcal{E}_\varepsilon(\epsilon(u^0_\varepsilon), z^0_\varepsilon) \to \mathfrak{E}(b^0, z^0)$ as $\varepsilon \to 0^+$. Now, applying "$\limsup_{\varepsilon\to0^+}$" on both sides in (27) and using the convergence properties in Lemma 5.16 as well as lower semi-continuity arguments yield

$$\begin{aligned}&\mathfrak{E}(b^0, z^0) + \int_0^t\int_{F(s)} W_{,e}(e,z) : \epsilon(\partial_t b)\,\mathrm{d}(x,s)\\ &\ge \limsup_{\varepsilon\to0^+}\left(\mathcal{E}_\varepsilon(e_\varepsilon(t), z_\varepsilon(t)) + \int_{\Omega_t} \alpha|\partial_t z_\varepsilon| + \beta|\partial_t z_\varepsilon|^2 \,\mathrm{d}(x,s)\right)\\ &\ge \limsup_{\varepsilon\to0^+}\int_\Omega W^\varepsilon(e_\varepsilon(t), z_\varepsilon(t))\,\mathrm{d}x + \int_\Omega \frac{1}{p}|\nabla z(t)|^p\,\mathrm{d}x\\ &\quad + \int_{\Omega_t} \alpha|\partial_t z| + \beta|\partial_t z|^2\,\mathrm{d}(x,s).\end{aligned} \tag{39}$$

Indeed, for an arbitrary $t \in (0,T)$, we derive by Fatou's lemma and Lemma 5.16

$$\begin{aligned}\int_0^t \left(\limsup_{\varepsilon\to0^+}\int_\Omega W^\varepsilon(e_\varepsilon(s), z_\varepsilon(s))\,\mathrm{d}x\right)\mathrm{d}s &\ge \limsup_{\varepsilon\to0^+}\int_{\Omega_t} W^\varepsilon(e_\varepsilon, z_\varepsilon)\,\mathrm{d}(x,s)\\ &\ge \liminf_{\varepsilon\to0^+}\int_F (g(z_\varepsilon)+\varepsilon)\mathbb{C}\widehat{e}_\varepsilon : \widehat{e}_\varepsilon\,\mathrm{d}(x,s)\end{aligned}$$

$$\geq \int_F W(e,z)\,\mathrm{d}(x,s). \tag{40}$$

We have used the weak convergence property

$$\sqrt{g(z_\varepsilon)+\varepsilon}\,\widehat{e}_\varepsilon \rightharpoonup \sqrt{g(z)}\,\widehat{e} \text{ in } L^2(\Omega_T;\mathbb{R}^{n\times n})$$

as $\varepsilon \to 0^+$. To the end, (40) implies

$$\limsup_{\varepsilon\to 0^+} \int_\Omega W^\varepsilon(e_\varepsilon(t),z_\varepsilon(t))\,\mathrm{d}x \geq \int_{F(t)} W(e(t),z(t))\,\mathrm{d}x$$

for a.e. $t \in (0,T)$. Combining it with (39), estimate (14) is shown. □

5.3 Existence of weak solutions

By using the achievements in the previous section and Zorn's lemma, we will prove the main results, Theorem 4.1 and Theorem 4.2. To proceed, let $\eta > 0$ be fixed and $\mathcal{P}$ be the set

$$\begin{aligned}\mathcal{P} := \big\{(\widehat{T},e,u,z,F)\,|\, 0<\widehat{T}\leq T \text{ and } (e,u,z,F) \text{ is an approximate weak solution on}\\ [0,\widehat{T}] \text{ with fineness } \eta \text{ according to Definition 3.8 } \big\}.\end{aligned}$$

We introduce a partial ordering $\leq$ on $\mathcal{P}$ by

$$\begin{aligned}(\widehat{T}_1,e_1,u_1,z_1,F_1)\leq(\widehat{T}_2,e_2,u_2,z_2,F_2) \quad \Leftrightarrow \quad & \widehat{T}_1\leq\widehat{T}_2,\ e_2|_{[0,\widehat{T}_1]}=e_1,\ u_2|_{[0,\widehat{T}_1]}=u_1,\\ & z_2|_{[0,\widehat{T}_1]}=z_1,\ F_2|_{[0,\widehat{T}_1]}=F_1.\end{aligned}$$

The next two lemma prove the assumptions for Zorn's lemma.

Lemma 5.19 $\mathcal{P}\neq\emptyset$.

Proof. Let (e,u,z) be the tuple from Proposition 5.12 to the initial-boundary data (z^0,b). If there exists an $\varepsilon>0$ such that $J_{z^\star}\cap[0,\varepsilon]=\emptyset$ with $z^\star(t):=z(t)\mathbb{1}_{\mathfrak{A}_D(\{z^-(t)>0\})}$ then $(\varepsilon,e,u,z,F)\in\mathcal{P}$. Otherwise, we find $0\in C_{z^\star}$. We claim

$$\mathcal{L}^n\left(\{z^0>0\}\setminus\mathfrak{A}_D(\{z(t)>0\})\right)\to 0 \text{ as } t\to 0^+. \tag{41}$$

We consider the non-trivial case $z^0\not\equiv 0$. Let $x\in\{z^0>0\}\cap\Omega$. Since $\{z^0>0\}\subseteq\overline{\Omega_T}$ is relatively open and admissible with respect to D, there exists a Lipschitz domain $U\subset\subset\{z^0>0\}$ with $x\in U$ such that $\mathcal{H}^{n-1}(\partial U\cap D)>0$ by Lemma 5.5. Because of Theorem A.2, $z\in\mathcal{C}(\overline{\Omega_T})$ and, consequently, there exists a $t>0$ such that $U\subset\subset\{z(s)>0\}$ for all $0\leq s<t$. In particular, $x\in\mathfrak{A}_D(\{z(s)>0\})$ for all $0\leq s<t$. This proves (41). Finally, choose $\varepsilon>0$ so small such that $\varepsilon<\eta$ and (note the monotonicity of z with respect to t)

$$\mathcal{L}^n\left(\{z(t)>0\}\setminus\mathfrak{A}_D(\{z(t)>0\})\right)\leq\mathcal{L}^n\left(\{z^0>0\}\setminus\mathfrak{A}_D(\{z(t)>0\})\right)<\eta$$

for all $0\leq t<\varepsilon$. We have proved that (e,u,z) on $F:=\{z>0\}$ is an approximate weak solution with fineness η on the time interval $[0,\varepsilon]$, i.e. $(\varepsilon,e,u,z,F)\in\mathcal{P}$. □

Lemma 5.20 *Every totally ordered subset of $\mathcal{P}$ has an upper bound.*

Proof. Let $\mathcal{R} \subseteq \mathcal{P}$ be a totally ordered subset. We denote with $[0, T_R]$ the corresponding time interval of an element $R \in \mathcal{R}$. Let us select a sequence $\{T_\theta, e_\theta, u_\theta, z_\theta, F_\theta\}_{\theta \in (0,1)} \subseteq \mathcal{R}$, with $T_{\theta_1} \leq T_{\theta_2}$ for $\theta_2 \leq \theta_1$ and $\lim_{\theta \to 0^+} T_\theta = \sup_{Q \in \mathcal{R}} T_Q =: \widehat{T}$.

Let $t \in (0, \widehat{T})$. There exists a $\theta \in (0,1)$ with $T_\theta \geq t$ and we define

$$(e(t), u(t), z(t), F(t)) := (e_\theta(t), u_\theta(t), z_\theta(t), F_\theta(t)).$$

By construction, the functions (e, u, z) satisfy the properties (ii)-(v) of Definition 3.5 on $[0, \widehat{T}]$. It remains to show that $(e(t), u(t), z(t))$ are in the trajectory spaces as in Definition 3.8 (i) and that F satisfies Definition 3.8 (ii).

The energy estimate for $(e_\theta, u_\theta, z_\theta)$ implies

$$\begin{aligned} \mathcal{E}(e(t), z(t)) &+ \int_0^t \int_{F(s)} \alpha |\partial_t^{\mathrm{a}} z| + \beta |\partial_t^{\mathrm{a}} z|^2 \,\mathrm{d}(x,s) \\ &\leq \mathfrak{e}_0^+ + \int_0^t \int_{F(s)} W_{,e}(e, z) : \epsilon(\partial_t b) \,\mathrm{d}(x,s) \end{aligned} \tag{42}$$

for a.e. $t \in (0, \widehat{T})$. Gronwall's lemma yields boundedness of the left hand side of (42) with respect to a.e. $t \in (0, \widehat{T})$.

We immediately get

$$z \in L^\infty(0, \widehat{T}; W^{1,p}(\Omega)) \cap SBV^2(0, \widehat{T}; L^2(\Omega)), \tag{43}$$

Variational inequality (12) tested with $\zeta \equiv -1$ shows

$$\int_{F(t)} W_{,z}(e(t), z(t)) \,\mathrm{d}x \leq \int_\Omega \alpha \,\mathrm{d}x - \int_{F(t)} \beta \partial_t^{\mathrm{a}} z(t) \,\mathrm{d}x$$

for a.e. $t \in (0, \widehat{T})$. This implies

$$e \in L^2(F; \mathbb{R}^{n \times n}). \tag{44}$$

We know that $u|_{U \times (0,t)} \in L^2(0, t; H^1(U; \mathbb{R}^n))$ for all $t \in (0, \widehat{T})$ and all open subsets $U \subset\subset \mathfrak{A}_D(F(t))$. Let $\{U_k\}$ be a Lipschitz cover of the admissible set

$$F(\widehat{T}) := \mathfrak{A}_D(\{z^-(\widehat{T}) > 0\})$$

according to Lemma 5.5 (in particular, Definition 3.8 (ii) is fulfilled). For each $k \in \mathbb{N}$, we apply Korn's inequality and get for all $t \in (0, \widehat{T})$

$$\|u - b\|_{L^2(0,t; H^1(U_k; \mathbb{R}^n))} \leq C \|\epsilon(u)\|_{L^2(0,t; L^2(U_k; \mathbb{R}^n))},$$

where $C > 0$ depends on the domain U_k but not on the time t. Thus $u|_{U_k \times (0, \widehat{T})} \in L^2(0, T; H^1(U_k; \mathbb{R}^n))$. In conclusion,

$$u \in L_t^2 H_{x,\mathrm{loc}}^1(F; \mathbb{R}^n). \tag{45}$$

Therefore, property (i) of Definition 3.8 follows by (43)-(45). We end up with $\{\widehat{T}, e, u, z, F\} \in \mathcal{P}$ satisfying $\{T_\theta, e_\theta, u_\theta, z_\theta, F_\theta\} \le \{\widehat{T}, e, u, z, F\}$ for all $\theta \in (0,1)$. $\square$

Weak solutions exhibit the following concatenation property.

Lemma 5.21 *Let $t_1 < t_2 < t_3$ be real numbers. Suppose that*

$$\begin{aligned}
&\widetilde{q} := (\widetilde{e}, \widetilde{u}, \widetilde{z}, \widetilde{F}) \text{ is an approximate weak solution on } [t_1, t_2],\\
&\widehat{q} := (\widehat{e}, \widehat{u}, \widehat{z}, \widehat{F}) \text{ is an approximate weak solution on } [t_2, t_3]\\
&\qquad\qquad \text{with } \widehat{\mathfrak{e}}_{t_2}^+ = \mathfrak{E}(\widehat{b}(t_2), \widehat{z}^+(t_2)) \text{ (the value } \mathfrak{e}_{t_2}^+ \text{ for } \widehat{q} \text{ in Definition 3.5).}
\end{aligned}$$

Furthermore, suppose the compatibility condition $\widehat{z}^+(t_2) = \widetilde{z}^-(t_2)\mathbb{1}_{\mathfrak{A}_D(\{\widetilde{z}^-(t_2)>0\})}$ and the Dirichlet boundary data $b \in W^{1,1}(t_1, t_3; W^{1,\infty}(\Omega; \mathbb{R}^n))$. Then, we obtain that $q := (e, u, z, F)$ defined as $q|_{[t_1,t_2)} := \widetilde{q}$ and $q|_{[t_2,t_3]} := \widehat{q}$ is an approximate weak solution on $[t_1, t_3]$.

Proof. Applying "$\lim_{s\to t_2^-} \operatorname{ess\,inf}_{\tau\in(s,t_2)}$" on both sides of the energy estimate (14) for $(\widetilde{e}, \widetilde{u}, \widetilde{z}, \widetilde{F})$ yields

$$\begin{aligned}
\lim_{s\to t_2^-} \operatorname*{ess\,inf}_{\tau\in(s,t_2)} \mathcal{E}(e(\tau), z(\tau)) + \int_{t_1}^{t_2}\int_{F(s)} \alpha|\partial_t^{\mathrm{a}} z| + \beta|\partial_t^{\mathrm{a}} z|^2 \,\mathrm{d}(x,s) + \liminf_{s\to t_2^-} \sum_{\tau\in J_z\cap(t_1,s]} \mathcal{J}_\tau\\
\le \mathfrak{e}_{t_1}^+ + \int_{t_1}^{t_2}\int_{F(s)} W_{,e}(e,z) : \epsilon(\partial_t b) \,\mathrm{d}(x,s).
\end{aligned}$$

This estimate can be rewritten as

$$\begin{aligned}
&\mathfrak{E}(b(t_2), z^+(t_2)) + \int_{t_1}^{t_2}\int_{F(s)} \alpha|\partial_t^{\mathrm{a}} z| + \beta|\partial_t^{\mathrm{a}} z|^2 \,\mathrm{d}(x,s)\\
&+ \liminf_{s\to t_2^-} \sum_{\tau\in J_z\cap(t_2,s]} \mathcal{J}_\tau + \lim_{s\to t_2^-} \operatorname*{ess\,inf}_{\tau\in(s,t_2)} \mathcal{E}(e(\tau), z(\tau)) - \mathfrak{E}(b(t_2), z^+(t_2))\\
&\qquad \le \mathfrak{e}_{t_1}^+ + \int_{t_1}^{t_2}\int_{F(s)} W_{,e}(e,z) : \epsilon(\partial_t b) \,\mathrm{d}(x,s). \qquad (46)
\end{aligned}$$

In the following, we show that we may choose the value $\mathfrak{E}(b(t_2), z^+(t_2))$ for $\mathfrak{e}_{t_2}^+$. By the property (i) of Definition 3.5, we get $z^-(s) \rightharpoonup z^-(t_2)$ in $W^{1,p}(\Omega)$ and $b(s) \to b(t_2)$ in $W^{1,\infty}(\Omega; \mathbb{R}^n)$ as $s \to t_2^-$. In particular, by using Lemma C.1 and the monotone decrease of z^- with respect to t,

$$z^-(s)\mathbb{1}_{\mathfrak{A}_D(\{z^-(s)>0\})} \rightharpoonup z^-(t_2)\mathbb{1}_{\bigcap_{\tau\in(t_1,t_2)}\mathfrak{A}_D(\{z^-(\tau)>0\})} =: \chi$$

in $W^{1,p}(\Omega)$ as $s \to t_2^-$. By the definition of χ, the inclusion

$$\mathfrak{A}_D(\{z^-(t_2) > 0\}) \subseteq \bigcap_{\tau\in(t_1,t_2)} \mathfrak{A}_D(\{z^-(\tau) > 0\})$$

and the compatibility condition, we find $z^+(t_2) = \chi\mathbb{1}_{\mathfrak{A}_D(\{z^-(t_2)>0\})}$.

Thus, applying Lemma 5.11, lower semi-continuity of the Γ-limit $\mathfrak{E}$ and Corollary 5.10 (iii), we obtain

$$\lim_{s\to t_2^-} \operatorname*{ess\,inf}_{\tau\in(s,t_2)} \mathcal{E}(e(\tau), z(\tau)) = \lim_{s\to t_2^-} \operatorname*{ess\,inf}_{\tau\in(s,t_2)} \mathcal{E}(e(\tau), z^-(\tau))$$

$$
\begin{aligned}
&\geq \lim_{s\to t_2^-} \operatorname*{ess\,inf}_{\tau\in(s,t_2)} \mathcal{E}(\epsilon(u(\tau)), z^-(\tau)\mathbb{1}_{\mathfrak{A}_D(\{z^-(\tau)>0\})})\\
&\geq \lim_{s\to t_2^-} \operatorname*{ess\,inf}_{\tau\in(s,t_2)} \mathfrak{E}(b(\tau), z^-(\tau)\mathbb{1}_{\mathfrak{A}_D(\{z^-(\tau)>0\})})\\
&\geq \mathfrak{E}(b(t_2), \chi)\\
&\geq \mathfrak{E}(b(t_2), z^+(t_2)).
\end{aligned}
$$

This leads to

$$
\begin{aligned}
0 \leq \sum_{s\in J_z\cap(t_1,t_2]} \mathcal{J}_s \leq {}& \lim_{s\to t_2^-} \operatorname*{ess\,inf}_{\tau\in(s,t_2)} \mathcal{E}(e(\tau), z(\tau)) - \mathfrak{E}(b(t_2), z^+(t_2))\\
&+ \liminf_{s\to t_2^-} \sum_{\tau\in J_z\cap(t_1,s]} \mathcal{J}_\tau,
\end{aligned}
$$

where the second '$\leq$' becomes an '$=$' if $t_2 \in J_z$. Consequently, (46) becomes

$$
\begin{aligned}
\mathfrak{E}(b(t_2), z^+(t_2)) + \int_{t_1}^{t_2}\int_{F(s)} \alpha|\partial_t^{\mathrm{a}} z| + \beta|\partial_t^{\mathrm{a}} z|^2 \,\mathrm{d}(x,s) + \sum_{s\in J_z\cap(t_1,t_2]} \mathcal{J}_s\\
\leq \mathfrak{e}_{t_1}^+ + \int_{t_1}^{t_2}\int_{F(s)} W_{,e}(e,z) : \epsilon(\partial_t b)\,\mathrm{d}(x,s). \qquad (47)
\end{aligned}
$$

The energy inequality (14) for $(\widehat{e}, \widehat{u}, \widehat{z}, \widehat{F})$ (taking $\widehat{\mathfrak{e}}_{t_2}^+ = \mathfrak{E}(\widehat{b}(t_2), \widehat{z}^+(t_2))$ into account) can be expressed as

$$
\begin{aligned}
\mathcal{E}(e(t), z(t)) + \int_{t_2}^{t}\int_{F(s)} \alpha|\partial_t^{\mathrm{a}} z| + \beta|\partial_t^{\mathrm{a}} z|^2 \,\mathrm{d}(x,s) + \sum_{s\in J_z\cap(t_2,t]} \mathcal{J}_s\\
\leq \mathfrak{E}(b(t_2), z^+(t_2)) + \int_{t_2}^{t}\int_{F(s)} W_{,e}(e,z) : \epsilon(\partial_t b)\,\mathrm{d}(x,s) \qquad (48)
\end{aligned}
$$

for a.e. $t \in (t_2, t_3)$. Adding (47) and (48) shows that the energy estimate for (e, u, z, F) also holds for a.e. $t \in (t_2, t_3)$. It is now easy to verify that (e, u, z, F) is a approximate weak solution on the time interval $[t_1, t_3]$ according to Definition 3.5. ☐

Proof of Theorem 4.1. By Zorn's lemma, we deduce the existence of a maximal element $R = (\widetilde{T}, \widetilde{e}, \widetilde{u}, \widetilde{z}, \widetilde{F})$ in $\mathcal{P}$. In particular, a maximal element satisfies the properties in Theorem 4.1 on the interval $[0, \widetilde{T}]$. We deduce $T = \widetilde{T}$. Otherwise, we get another approximate weak solution $(\widehat{e}, \widehat{u}, \widehat{z}, \widehat{F})$ on $[\widetilde{T}, \widetilde{T} + \varepsilon]$ for an $\varepsilon > 0$ with initial datum $\widetilde{z}^-(\widetilde{T})\mathbb{1}_{\mathfrak{A}_D(\widetilde{z}^-(\widetilde{T})>0)}$ (which is an element of $W^{1,p}(\Omega)$ by Lemma C.1) as in the proof of Lemma 5.19 with $e_{\widetilde{T}}^+ = \mathfrak{E}(b(\widetilde{T}), z(\widetilde{T}))$ if $\widetilde{T} \in J_z$. By Lemma 5.21, $(\widetilde{e}, \widetilde{u}, \widetilde{z}, \widetilde{F})$ and $(\widehat{e}, \widehat{u}, \widehat{z}, \widehat{F})$ can be concatenate to an approximate weak solution on $[0, \widetilde{T} + \varepsilon]$ which is a contradiction. ☐

Proof of Theorem 4.2. Here, let us consider the set $\mathcal{P}$ given by

$$
\begin{aligned}
\mathcal{P} := \big\{(\widehat{T}, u, z) \,|\, 0 < \widehat{T} \leq T \text{ and } (u, z) \text{ is a weak solution on}\\
[0, \widehat{T}] \text{ according to Definition 3.5}\big\}
\end{aligned}
$$

with an ordering $\leq$ as above (except the conditions $e_2|_{[0,\widehat{T}_1]} = e_1$ and $F_2|_{[0,\widehat{T}_1]} = F_1$ which are not needed here). Proposition 5.12 shows $\mathcal{P} \neq \emptyset$ by noticing $z \in \mathcal{C}(\overline{\Omega_T})$ (see Theorem A.2) and $0 < \eta \leq z^0$. The property that every totally ordered subset of $\mathcal{P}$ has an upper bound can be shown as in Lemma 5.20. A maximal element satisfies the claim. $\square$

A Embedding Theorem

The embedding theorem A.2 in this appendix is a special version of a more general compactness result in [Sim86, Corollary 5]. However, we would like to present a different (short) proof which requires the following generalized version of Poincaré's inequality.

Theorem A.1 (Generalized Poincaré inequality [Alt99, Section 6.15]) *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain and $M \subseteq W^{1,p}(\Omega;\mathbb{R}^m)$ non-empty, convex and closed with $1 < p < \infty$. Furthermore, M satisfies the property*

$$u \in M,\ \alpha \geq 0 \implies \alpha u \in M.$$

Then the following statements are equivalent:

(i) There exists a $u_0 \in M$ and a constant $C_0 > 0$ such that for all $\xi \in \mathbb{R}^m$

$$u_0 + \xi \in M \implies |\xi| \leq C_0.$$

(ii) There exists a constant $C > 0$ such that for all $u \in M$

$$\|u\|_{L^p(\Omega;\mathbb{R}^m)} \leq C\|\nabla u\|_{L^p(\Omega;\mathbb{R}^{m\times n})}.$$

Theorem A.2 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain and $p > n$. Then*

$$L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega)) \subseteq \mathcal{C}(\overline{\Omega_T}).$$

Proof. Let $z \in L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega))$ and $(x_m,t_m) \in \overline{\Omega_T}$ be arbitrary with $(x_m,t_m) \to (x,t)$ in $\overline{\Omega_T}$ as $m \to \infty$. We have

$$|z(x,t) - z(x_m,t_m)| \leq \underbrace{|z(x,t) - z(x,t_m)|}_{A_m} + \underbrace{|z(x,t_m) - z(x_m,t_m)|}_{B_m}.$$

Assume that $A_m \not\to 0$ as $m \to \infty$. Then there exists a subsequence of $\{A_m\}$ (also denoted by $\{A_m\}$) such that $\lim_{m\to\infty} A_m > 0$. Using this subsequence, it holds $z(\cdot,t_m) \to z(\cdot,t)$ in $L^2(\Omega)$ due to $H^1(0,T;L^2(\Omega)) \subseteq \mathcal{C}([0,T];L^2(\Omega))$. We obtain again a subsequence (we omit the additional subscript) such that $z(y,t_m) \to z(y,t)$ as $m \to \infty$ for a.e. $y \in \Omega$. Therefore, we can choose $y_m \to x$ in $\overline{\Omega}$ such that $|z(y_m,t) - z(y_m,t_m)| \to 0$ as $m \to \infty$. It follows

$$|z(x,t) - z(x,t_m)| \leq \underbrace{|z(x,t) - z(y_m,t)|}_{A_m^1} + \underbrace{|z(y_m,t) - z(y_m,t_m)|}_{A_m^2} + \underbrace{|z(y_m,t_m) - z(x,t_m)|}_{A_m^3}$$

The continuity of $z(\cdot,t)$ due to $W^{1,p}(\Omega) \subseteq \mathcal{C}(\overline{\Omega})$ (note that $p > n$) implies $A_m^1 \to 0$ as $m \to \infty$. A_m^2 converges to 0 by the construction of $\{y_m\}$. To treat the term A_m^3, we apply the Poincaré inequality in Theorem A.1 to $M := \{u \in W^{1,p}(B_1(q_0)) \,|\, u(q_0) = 0\}$ and obtain

$$\|g\|_{L^p(B_1(q_0))} \leq C\|\nabla g\|_{L^p(B_1(q_0))} \tag{49}$$

for all $g \in W^{1,p}(B_1(q_0))$ with $g(q_0) = 0$, where $q_0 \in \mathbb{R}^n$ and $C > 0$ is independent of g and q_0. Note that, due to $g \in W^{1,p}(B_1(q_0)) \subseteq \mathcal{C}(\overline{B_1(q_0)})$, g is pointwise defined. By utilizing (49) and using a scaling argument, we gain a $C > 0$ such that for all $\varepsilon > 0$ and all $g \in W^{1,p}(B_\varepsilon(q_0))$ with $g(q_0) = 0$ follows

$$\|g\|_{\mathcal{C}(\overline{B_\varepsilon(q_0)})} = \|g(\varepsilon\cdot)\|_{\mathcal{C}(\overline{B_1(q_0)})} \leq C\|g(\varepsilon\cdot)\|_{W^{1,p}(B_1(q_0))} \leq C\|\varepsilon\nabla g(\varepsilon\cdot)\|_{L^p(B_1(q_0))} = C\varepsilon^{\frac{p-n}{p}}\|\nabla g\|_{L^p(B_\varepsilon(q_0))}.$$

By setting $g_m(\cdot) := z(y_m,t_m) - z(\cdot,t_m)$ and $\varepsilon_m := 2|y_m - x|$, we can estimate A_m^3 in the following way (note that $g_m(y_m) = 0$):

$$A_m^3 \leq \|g_m\|_{\mathcal{C}(\overline{B_{\varepsilon_m}(y_m)})} \leq C\varepsilon_m^{\frac{p-n}{p}}\|\nabla g_m\|_{L^p(B_{\varepsilon_m}(y_m))}.$$

Since $z \in L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega))$, $\|\nabla g_m\|_{L^p(B_{\varepsilon_m}(y_m))}$ is bounded with respect to m. In conclusion, $A_m^3 \to 0$ as $m \to \infty$. Hence, we end up with a contradiction. Therefore, $A_m \to 0$ as $m \to \infty$.

The convergence $B_m \to 0$ as $m \to \infty$ can be shown as for $A_m^3 \to 0$. □

B Chain-rule for vector-valued functions of bounded variation

Theorem B.1 (BV-chain rule [MV87]) *Let $I \subseteq \mathbb{R}$ be an interval, X be a real reflexive Banach space, $f \in BV_{\mathrm{loc}}(I;X)$ with $\mathrm{d}f = f'_\mu\mu$ for a non-negative Radon measure μ on I and $f'_\mu \in L^1_{\mathrm{loc}}(I,\mu;X)$. Moreover, let $E : X \to \mathbb{R}$ be continuously Fréchet-differentiable. Then $E \circ f \in BV_{\mathrm{loc}}(I;\mathbb{R})$ and $\mathrm{d}(E \circ f)$ admits as density relative to μ the function $t \mapsto \langle\theta(t), f'_\mu(t)\rangle$, where $\theta : I \to X^\star$ is defined as*

$$\theta(t) := \int_0^1 \mathrm{d}E((1-r)f(t^-) + rf(t^+))\,\mathrm{d}r.$$

Corollary B.2 *Suppose $f \in SBV(0,T;X)$ and $E : X \to \mathbb{R}$ is continuously Fréchet-differentiable. Then $E \circ f \in SBV(0,T)$ and for all $0 \leq a \leq b \leq T$:*

$$\mathrm{d}(E \circ f)((a,b]) = \int_a^b \langle \mathrm{d}E(f(s)), f'(s)\rangle\,\mathrm{d}s + \sum_{s \in J_f \cap (a,b]} \left(E(f(s^+)) - E(f(s^-))\right).$$

Proof. We apply Theorem B.1. By assumption, we obtain the decomposition $\mathrm{d}f = f'_\mu\mu$ with $\mu = \mathcal{L}^1 + \mathcal{H}^0\lfloor J_f$ and $f'_\mu(t) = f'(t) + f(t^+) - f(t^-)$ for all $t \in (0,T)$. Applying Theorem B.1 yields

$$\mathrm{d}(E \circ f)((a,b]) = \int_{(a,b]} \langle\theta(s), f'_\mu(s)\rangle\,\mathrm{d}\mu(s)$$

$$= \int_{(a,b]} \langle \theta(s), f'(s) \rangle \, \mathrm{d}\mathcal{L}^1(s) + \sum_{t \in J_f \cap (a,b]} \langle \theta(s), f(s^+) - f(s^-) \rangle$$

Since $f(s^+) = f(s^-) = f(s)$ for $\mathcal{L}^1-$a.e. $s \in (a,b]$, the first term on the right hand side becomes

$$\begin{aligned} \int_{(a,b]} \langle \theta(s), f'(s) \rangle \, \mathrm{d}\mathcal{L}^1(s) &= \int_{(a,b]} \Big\langle \int_0^1 \mathrm{d}E((1-r)f(s^-) + rf(s^+)) \, \mathrm{d}r, f'(s) \Big\rangle \, \mathrm{d}\mathcal{L}^1(s) \\ &= \int_{(a,b]} \langle \mathrm{d}E(f(s), f'(s) \rangle \, \mathrm{d}s, \end{aligned}$$

where $\mathrm{d}s := \mathrm{d}\mathcal{L}^1(s)$. Furthermore, by the classical chain rule,

$$\begin{aligned} \sum_{s \in J_f \cap (a,b]} \langle \theta(s), f(s^+) - f(s^-) \rangle &= \sum_{s \in J_f \cap (a,b]} \Big\langle \int_0^1 \mathrm{d}E((1-r)f(s^-) + rf(s^+)) \, \mathrm{d}r, f(s^+) - f(s^-) \Big\rangle \\ &= \sum_{s \in J_f \cap (a,b]} \int_0^1 \Big\langle \mathrm{d}E((1-r)f(s^-) + rf(s^+)), f(s^+) - f(s^-) \Big\rangle \, \mathrm{d}r \\ &= \sum_{s \in J_f \cap (a,b]} \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r} E((1-r)f(s^-) + rf(s^+)) \, \mathrm{d}r \\ &= \sum_{s \in J_f \cap (a,b]} \Big(E(f(s^+)) - E(f(s^-)) \Big). \end{aligned}$$

□

C Truncation property for Sobolev functions

Lemma C.1 *Let $D, \Omega \subseteq \mathbb{R}^n$ be open sets and $p > n$. Furthermore, assume that a function $f \in W^{1,p}(\Omega)$ fulfills $f = 0$ on $\partial D \backslash \partial \Omega$ (f is here considered as a continuous function due to the embedding $W^{1,p}(\Omega) \hookrightarrow \mathcal{C}(\overline{\Omega})$). Then $f \mathbb{1}_D \in W^{1,p}(\Omega)$.*

Proof. We can reduce the problem to one space dimension by using the following slicing result from [AFP00, Proposition 3.105] for functions $u \in L^p(\Omega)$:

$$\begin{aligned} u \in W^{1,p}(\Omega) \quad \iff \quad & \forall \nu \in \mathbb{S}^{n-1} : u_x^\nu \in W^{1,p}(\Omega_x^\nu) \text{ for } \mathcal{L}^{n-1}\text{-a.e. } x \in \Omega_\nu \\ & \text{and} \int_{\Omega_\nu} \int_{\Omega_x^\nu} |\nabla u_x^\nu|^p \, \mathrm{d}t \, \mathrm{d}y < \infty, \end{aligned} \tag{50}$$

where Ω_ν is the orthogonal projection of Ω to the hyperplane orthogonal to ν and $\Omega_x^\nu := \{t \in \mathbb{R} \,|\, x + t\nu \in \Omega\}$ as well as $u_x^\nu(t) := u(x + t\nu)$.

Applying this result to f, we obtain $f_x^\nu \in W^{1,p}(\Omega_x^\nu)$ for $\mathcal{L}^{n-1}$-a.e. $x \in \Omega_\nu$ and all $\nu \in \mathbb{S}^{n-1}$. Moreover, slices for the function $g := f \mathbb{1}_D$ are given by the equation

$$g_x^\nu = f_x^\nu \mathbb{1}_{D_x^\nu}.$$

The function f_x^ν is absolutely continuous. We claim that this is also the case for g_x^ν. To proceed, let $\varepsilon > 0$ be an arbitrary real. Then, we get some constant $\delta > 0$ such that

$$(a_k, b_k),\ k \in I, \text{ with } a_k \leq b_k \text{ are finitely many disjoint intervals of } \Omega_x^\nu \text{ with } \sum_{k\in I} |a_k - b_k| < \delta$$
$$\implies \quad \sum_{k\in I} |f_x^\nu(a_k) - f_x^\nu(b_k)| < \varepsilon. \tag{51}$$

The property (51) is also satisfied for g_x^ν. Indeed, let (a_k, b_k), $k \in I$, with $a_k \leq b_k$ be finitely many disjoint intervals of Ω_x^ν with $\sum_{k\in I} |a_k - b_k| < \delta$. We define the values $\widetilde{a}_k$ and $\widetilde{b}_k$ in the following way:

$$(\widetilde{a}_k, \widetilde{b}_k) := \begin{cases} (a_k, b_k) & \text{if } a_k, b_k \in D_x^\nu \text{ or } a_k, b_k \notin D_x^\nu, \\ (z, b_k) \text{ for an arbitrary fixed } z \in [a_k, b_k] \cap \partial D_x^\nu & \text{if } a_k \notin D_x^\nu \text{ and } b_k \in D_x^\nu, \\ (a_k, z) \text{ for an arbitrary fixed } z \in [a_k, b_k] \cap \partial D_x^\nu & \text{if } a_k \in D_x^\nu \text{ and } b_k \notin D_x^\nu. \end{cases}$$

We conclude $\sum_{k\in I} |\widetilde{a}_k - \widetilde{b}_k| \leq \sum_{k\in I} |a_k - b_k| \leq \delta$ and therefore $\sum_{k\in I} |f_x^\nu(\widetilde{a}_k) - f_x^\nu(\widetilde{b}_k)| < \varepsilon$ by (51). Taking

$$\sum_{k\in I} |g_x^\nu(a_k) - g_x^\nu(b_k)| = \sum_{k\in I} |g_x^\nu(\widetilde{a}_k) - g_x^\nu(\widetilde{b}_k)| \leq \sum_{k\in I} |f_x^\nu(\widetilde{a}_k) - f_x^\nu(\widetilde{b}_k)|$$

into account, shows that g_x^ν is absolutely continuous and we find $g_x^\nu \in W^{1,p}(\Omega_x^\nu)$.

Moreover, $\int_{\Omega_\nu} \int_{\Omega_x^\nu} |\nabla g_x^\nu|^p \,\mathrm{d}t\,\mathrm{d}y = \int_{D_\nu} \int_{D_x^\nu} |\nabla f_x^\nu|^p \,\mathrm{d}t\,\mathrm{d}y < \infty$. Applying (50) yields $g \in W^{1,p}(\Omega)$. ☐

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