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On Test Sets for Nonlinear Integer Maximization

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Abstract

A finite test set for an integer maximization problem enables us to verify whether a feasible point attains the global maximum. We establish in this paper several general results that apply to integer maximization problems with nonlinear objective functions.

Key words: Integer programming, test sets, certificates Hilbert basis, Gordan Lemma, superadditive

1 Introduction and related work

Given a feasible point x^* of an optimization problem P, one important concern is to establish a set of points $T = T(x^*, P)$ with which one can verify whether x^* is optimal for P. We refer to such a set T as a *test set*. Usually, for a test set T, we compare feasible elements of $\{x^* + t \ : \ t \in T\}$, via objective value, against x^*. Our goal is to establish, in various settings, the existence of a finite test set T. An initial feasible solution together with a description of a finite test set for any feasible point allows us to design an algorithm that iteratively builds up a sequence of better and better points.

It is preferred that a test set T does not depend on x^*, but this is not always possible. In general, it is interesting to establish a finite test set T and to understand the dependence of T on x^* and on the parameters that define P.

Before proceeding, we briefly set some notation. $\mathbb{Z}$ (resp., $\mathbb{R}$) denotes the set of integers (resp., real numbers). $\mathbb{Z}_+$ (resp., $\mathbb{Z}_-$) denotes the set of non-negative (resp., non-positive) integers, and we use $\mathbb{Z}_{-\infty} := \mathbb{Z} \cup \{-\infty\}$,

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$\mathbb{Z}_{+\infty} := \mathbb{Z} \cup \{+\infty\}$ and $\mathbb{Z}_{\pm\infty} := \mathbb{Z} \cup \{\pm\infty\}$. Analogously, we use such notation for $\mathbb{R}$. For $S \subset \mathbb{Z}^n$, and simple variable bounds $l \in \mathbb{Z}^n_{-\infty}$, $u \in \mathbb{Z}^n_{+\infty}$, with $l \leq u$, let

$$F(S, l, u) := \{x \in S \; : \; l \leq x \leq u\}.$$

For a function $f : \mathbb{R}^n \to \mathbb{R}$, we consider the optimization problem

$$P(f(x), S, l, u) : \quad \max\{f(x) \; : \; x \in F(S, l, u)\}.$$

Often, we focus on the case in which S is defined by linear equations. For $A \in \mathbb{Z}^{m\times n}$ and right-hand side $b \in \mathbb{Z}^m$ (l, u as above), we sometimes consider S of the form $S := \{x \in \mathbb{Z}^n : Ax = b\}$, and we write

$$F(A, b, l, u) := \{x \in \mathbb{Z}^n \; : \; Ax = b, \; l \leq x \leq u\}$$

and

$$P(f(x), A, b, l, u) : \quad \max\{f(x) \; : \; x \in F(A, b, l, u)\}.$$

Also, we write $L(A) := \{x \in \mathbb{R}^n \; : \; Ax = 0\}$. An *augmentation* for $x^* \in F(A, b, l, u)$ is an $h \in \mathbb{Z}^n$ such that $x^* + h \in F(A, b, l, u)$. Necessarily, an augmentation h is in $L(A)$. The augmentation h is *improving* if $f(x^*) < f(x^* + h)$.

We let O^n_j denote the j-th orthant of $\mathbb{R}^n$, for integers j satisfying $0 \leq j < 2^n$. Specifically, the j-th orthant of $\mathbb{R}^n$ is defined, for $0 \leq j < 2^n$, by having $x_k \geq 0$ (resp., $x_k \leq 0$) if bit k is 0 (resp., 1) in the binary representation of j. Hence $O^n_0 = \mathbb{R}^n_+$.

While it is no surprise that for a linear *continuous* optimization problem a finite test set (appropriately defined) can be given, the situation becomes more difficult when we consider linear *integer* optimization. The following general result establishes a finite test set for linear integer optimization.

Theorem 1.1 (Graver [3]) *For all $A \in \mathbb{Z}^{m\times n}$, there exists a finite set $T(A) \subset \mathbb{Z}^n$ such that for every $c \in \mathbb{R}^n$, $b \in \mathbb{Z}^m$, and $l \in \mathbb{Z}^n_{-\infty}$, $u \in \mathbb{Z}^n_{+\infty}$, with $l \leq u$, the point $x^* \in F(A, b, l, u)$ is optimal for $P(c^T x, A, b, l, u)$ if and only if $c^T(x^* + t) \leq c^T x^*$ for all $t \in T(A)$ such that $x^* + t \in F(A, b, l, u)$.*

One such set $T(A)$ is the so-called *Graver basis* $G(A)$, which will be defined and used in Section 2. Interestingly, Graver's result can be refined and generalized in order to provide us with an optimality criterion for $P(f(x), A, b, l, u)$ when $f(x)$ has a certain property related to superadditivity. We refer to Section 3 for the precise definition of this class of functions. This class of functions contains all functions that one can represent as a sum of univariate integer concave functions composed with affine functions, thus we generalize the following.

Let $\mathcal{F}$ be the set of functions $f : \mathbb{R}^n \to \mathbb{R}$ of the form $f(x) = \sum_{i=1}^r \phi_i(c_i^T x)$, where $c_i \in \mathbb{Z}^n$ and $\phi_i : \mathbb{R} \to \mathbb{R}$ is concave (univariate), for $i = 1, \ldots, r$.

Theorem 1.2 (Murota, Saito and Weismantel [5]; Hemmecke[4]) *For all $f \in \mathcal{F}$, $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$, and $l \in \mathbb{Z}^n_{-\infty}$, $u \in \mathbb{Z}^n_{+\infty}$, with $l \leq u$, there exists a finite set $T(f(x), A) \subset \mathbb{Z}^n \cap L(A)$ such that $x^* \in F(A, b, l, u)$ is optimal for $P(f(x), A, b, l, u)$ if and only if $f(x^* + t) \leq f(x^*)$ for all $t \in T(f(x), A)$ such that $x^* + t \in F(A, b, l, u)$.*

It is our intention with this note to demonstrate the existence of finite test sets for quite general integer optimization problems $P(f(x), S, l, u)$. More precisely, in Section 2, we study the general problem $P(f(x), S, l, u)$ with varying l, u. In this case it is possible to derive, for every feasible point x^*, a finite set for verifying its optimality. In Section 3 we introduce the notion of oriented subadditive and superadditive functions and exploit their structure. In Section 4, we construct finite test sets for integer optimization problems where the objective function is oriented subadditive or superadditive. In this setting, the sets that we construct are always universal in that they do not depend on the feasible point x^* that we test for optimality.

2 Finite test sets for feasible points

Define a partial order $\sqsubseteq$ on $\mathbb{Z}^n$ that extends the coordinate-wise partial order $\leq$ on $\mathbb{Z}^n_+$ as follows: For a pair of vectors $u, v \in \mathbb{Z}^n$, we write $u \sqsubseteq v$ and say that u is *conforms* to v if $|u_i| \leq |v_i|$ and $u_i v_i \geq 0$ for $i = 1, \ldots, n$, that is, u and v lie in the same orthant of $\mathbb{Z}^n$, and each component of u is bounded by the corresponding component of v in absolute value. Points with some zero components are in multiple orthants, but it is easy to see that $\sqsubseteq$ is well defined.

Here and throughout the paper, we make heavy use of the following natural extension to $\sqsubseteq$ and $\mathbb{Z}^n$ of the well-known Gordan Lemma [2] for $\leq$ and $\mathbb{Z}^n_+$.

Lemma 2.1 (Extended Gordan Lemma) *For every set $S \subset \mathbb{Z}^n$, the set $T(S) \subset S$ of $\sqsubseteq$-minimal elements of S is finite.*

We can now define the Graver basis $G(A)$ of an $m \times n$ integer matrix A (mentioned in the discussion following Theorem 1.1): it is defined to be the set $G(A) := T(S)$ of $\sqsubseteq$-minimal elements in $S := \{x \in \mathbb{Z}^n \cap L(A) \ : \ x \neq 0\}$.

Theorem 2.2 *For every set $S \subset \mathbb{Z}^n$, function $f : \mathbb{Z}^n \to \mathbb{R}$, and point $x^* \in S$, there is a finite set $T(x^*, f(x), S) \subset \mathbb{Z}^n$ such that, for every $l \in \mathbb{Z}^n_{-\infty}$, $u \in \mathbb{Z}^n_{+\infty}$ with $l \leq x^* \leq u$, the point x^* is optimal for $P(f(x), S, l, u)$ if and only if there is no $t \in T(x^*, f(x), S)$ with $l \leq x^* + t \leq u$.*

PROOF. Let

$$H(x^*, f(x), S) := \{h \in \mathbb{Z}^n \ : \\ x^* + h \in S, \ f(x^*) < f(x^* + h)\}.$$

We claim that the set $T(x^*, f(x), S) \subset H(x^*, f(x), S)$ of $\sqsubseteq$-minimal elements, guaranteed to be finite by the Extended Gordan Lemma 2.1, is the desired set. Consider any $l \in \mathbb{Z}^n_{-\infty}$, $u \in \mathbb{Z}^n_{+\infty}$ with $l \leq x^* \leq u$. If x^* is not optimal for $P(f(x), S, l, u)$, then there is an $\bar{x} \in S$ with $l \leq \bar{x} \leq u$ and $f(x^*) < f(\bar{x})$, and hence $h := \bar{x} - x^* \in H(x^*, f(x), S)$. Therefore there is a $t \in T(x^*, f(x), S)$ with $t \sqsubseteq h$. Now $t \sqsubseteq h$ and $l \leq x^*, x^* + h = \bar{x} \leq u$ imply that $l \leq x^* + t \leq u$.

Conversely, if there is a $t \in T(x^*, f(x), S)$ with $l \leq x^* + t \leq u$, then $T(x^*, f(x), S) \subset H(x^*, f(x), S)$ implies that $x^* + t \in F(S, l, u)$ and $f(x^*) < f(x^* + t)$, and therefore x^* is not optimal for $P(f(x), S, l, u)$. This completes the proof. □

We next present a refined result for the case in which the set S is defined using linear equations. In this case the test set does not depend on the set S but only on the matrix A defining the system of equations, and it applies to the family of integer problems with varying right hand sides. We make use of the Graver basis of the defining matrix.

Theorem 2.3 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function, and let $A \in \mathbb{Z}^{m \times n}$. For every $x^* \in \mathbb{Z}^n$, there exists a finite set $T(x^*, f(x), A) \subset \mathbb{Z}^n \cap L(A)$ such that for all $l \in \mathbb{Z}^n_{-\infty}$, $u \in \mathbb{Z}^n_{+\infty}$ with $l \leq x^* \leq u$, letting $b^* := Ax^*$, $x^* \in F(A, b^*, l, u)$ is optimal for $P(f(x), A, b^*, l, u)$ if and only if there is no $t \in T(x^*, f(x), A)$ with $l \leq x^* + t \leq u$.*

PROOF. Let $G(A) = \{g_1, \ldots, g_k\}$ be the Graver basis of A and let G be the $n \times k$ matrix with columns $g_1, \ldots, g_k$. Let

$$\mathcal{A}(x^*) := \{\alpha \in \mathbb{Z}_+^k : f(x^*) < f(x^* + G\alpha)\} .$$

Let $\mathcal{B}(x^*) \subset \mathcal{A}(x^*)$ be the subset of $\mathcal{A}(x^*)$ of $\leq$-minimal elements, which is finite by the (standard) Gordan Lemma. We claim that the desired test set is provided by

$$T(x^*, f(x), A) := \{G\beta : \beta \in \mathcal{B}(x^*)\} .$$

First, note that $T(x^*, f(x), A) \subset L(A)$ and therefore, for all $t \in T$, we have $A(x^* + t) = Ax^* = b^*$. Also, for all $t \in T(x^*, f(x), A)$, we have $f(x^*) < f(x^* + t)$ by the construction of $T(x^*, f(x), A)$. So if there is a $t \in T(x^*, f(x), A)$ with $l \leq x^* + t \leq u$ then $x^* + t$ is a better feasible point than x^*, so x^* is not optimal. Conversely, suppose x^* is not optimal and let x' be a better feasible point. Put $h := x' - x^*$. Then $h \in L(A)$ and therefore there is an $\alpha \in \mathbb{Z}_+^k$ providing a *conformal decomposition* of h into Graver bases elements, that is, $h = G\alpha$ and $g_j \sqsubseteq h$ whenever $\alpha_j > 0$. Now $f(x^*) < f(x') = f(x^* + G\alpha)$ implies that $\alpha \in \mathcal{A}(x^*)$ and hence there is a $\beta \in \mathcal{B}(x^*)$ satisfying $\beta \leq \alpha$. Consider the element $t := G\beta$ in $T(x^*, f(x), A)$. Then $h = G\alpha$ being a conformal decomposition of h and $\beta \leq \alpha$ imply $t \sqsubseteq h$. Now $l \leq x^*, x^* + h = x' \leq u$ imply $l \leq x^* + t \leq u$. $\square$

Note that, in an actual construction of the test set in the proof of Theorem 2.3, it might be useful to represent the Graver basis as the union of its intersections $\mathcal{H}_j := G(A) \cap O_j^n$ with the orthants of $\mathbb{R}^n$, for $0 \leq j < 2^n$. Then each $\mathcal{H}_j$ is the so-called *Hilbert basis* of the rational cone $O_j^n \cap L(A)$, and, using the so-called *integer Caratheodory property* (see [6]), one can restrict attention in the definition of the set $\mathcal{A}(x^*)$ in the proof of Theorem 2.3 to those $\alpha \in \mathbb{Z}_+^k$ with at most $2n - 2$ nonzero components corresponding to elements of some $\mathcal{H}_j$.

3 Oriented sub/superadditive functions

In this section we introduce the notion of oriented subadditive (and superadditive) functions and show how to manipulate these functions. The next definition makes precise what we mean by this.

Definition 3.1 Let $X, D_1, D_2 \subset \mathbb{R}^n$ be given. A function $f : \mathbb{Z}^n \to \mathbb{R}$ is (X, D_1, D_2)-*oriented superadditive* if for all integral $x \in X,\ y \in D_1, z \in D_2$ such that $x + y,\ x + z,\ x + y + z \in X$, we have

$$f(x + y + z) + f(x) \geq f(x + y) + f(x + z) .$$

Definition 3.2 The function f is *oriented subadditive* if $-f$ is oriented superadditive.

Note that the definition does not depend on the order of D_1 versus D_2 . That is, f is (X, D_1, D_2)-oriented superadditive if and only if f is (X, D_2, D_1)-oriented superadditive. In the special case when $D = D_1 = D_2$, then a function is (X, D, D)-oriented superadditive (subadditive) if and only if the family of functions $g_x : D \cap \mathbb{Z}^n \to \mathbb{R}$ defined by

$$g_x(y) = f(x + y) - f(x)$$

is superadditive (subadditive), for all $x \in X$.

Various functions are readily seen to be (X, D_1, D_2)-oriented superadditive. Trivially, all affine functions are $(\mathbb{R}^n, \mathbb{R}^n, \mathbb{R}^n)$-oriented superadditive. Any univariate convex function is (X, D_1, D_2)-oriented superadditive for suitably chosen (X, D_1, D_2) . More precisely, in [5] it was shown that a univariate convex function is $(\mathbb{R}_+, \mathbb{R}_+, \mathbb{R}_+)$-oriented superadditive as well as $(\mathbb{R}_+, \mathbb{R}_- \mathbb{R}_-)$-oriented superadditive. Other superadditive functions can be defined based on rounding. For example, for $c \in \mathbb{R}^n$, the function

$$f : \mathbb{Z}^n \to \mathbb{Z},\ f(x) = \left\lfloor c^T x \right\rfloor$$

is $(\mathbb{R}^n_+, \mathbb{R}^n_+, \mathbb{R}^n_+)$-oriented superadditive, but not $(\mathbb{R}^n, \mathbb{R}^n, \mathbb{R}^n)$-oriented superadditive (even for $n = 1$) as, for example, using $c = \frac{1}{2}$ and evaluating at $(x = 1, y = -5, z = 3)$, we have that $\lfloor(0.5 - 2.5 + 1.5)\rfloor + \lfloor 0.5 \rfloor \not\geq \lfloor(0.5-2.5)\rfloor + \lfloor(0.5+1.5)\rfloor$. In order to exploit the property of oriented sub/superadditivity, we need the notion of an orthant refinement of a linear space.

Definition 3.3 For a d-dimensional subspace $L \subset \mathbb{R}^n$, an *orthant refinement* of L is a finite set $\mathcal{C}$ of d-dimensional (convex) polyhedral cones such that:

(1) $L = \cup_{C \in \mathcal{C}} C$;
(2) $\text{int}(C) \cap \text{int}(D) = \emptyset$, for $C \neq D$, $C, D \in \mathcal{C}$;
(3) For all $C \in \mathcal{C}$: $\text{int}(C) \subset O^n_j$, for some $0 \leq j < 2^n$.

Of course we trivially have that the set of orthants is an orthant refinement of $\mathbb{R}^n$.

We can perform operations on oriented sub/superadditive functions. In particular, we obtain

Theorem 3.4 *Let $\mathcal{C}$ be an orthant refinement of $\mathbb{R}^n$, and let $f : \mathbb{Z}^n \to \mathbb{R}$ be $(\mathbb{R}^n, C, C)$-oriented superadditive (subadditive) for all $C \in \mathcal{C}$. Let $W \in \mathbb{Z}^{n \times m}$, and define the linear function $w : \mathbb{Z}^m \to \mathbb{Z}^n$ by $w(x) := Wx$, for all $x \in \mathbb{R}^m$. Then there exists an orthant refinement $\tilde{\mathcal{C}}$ of $\mathbb{R}^m$, such that the composed function*

$$f \circ w : \mathbb{Z}^m \to \mathbb{R}$$

is $(\mathbb{R}^m, \tilde{C}, \tilde{C})$-oriented superadditive (subadditive), for all $\tilde{C} \in \tilde{\mathcal{C}}$.

PROOF. Without loss of generality, we consider the case where the function f is $(\mathbb{R}^n, C, C)$-oriented superadditive. For $0 \leq j < 2^m$ and $C \in \mathcal{C}$, we define

$$\tilde{C}_j := \{x \in O^m_j \ : \ Wx \in C\} ,$$

and we let

$$\tilde{\mathcal{C}} := \{\tilde{C}_j \ : \ 0 \leq j < 2^m \text{ and } C \in \mathcal{C}\} .$$

Since the family of cones $\mathcal{C}$ is an orthant refinement of $\mathbb{R}^n$, the family of cones $\tilde{\mathcal{C}}$ is an orthant refinement of $\mathbb{R}^m$. Moreover, for all $x \in \mathbb{Z}^m$ and $y, z \in \tilde{C}_j \cap \mathbb{Z}^m$, we have that $Wy, Wz \in C$, and hence $f(Wx+Wy+Wz) + f(Wx) \geq f(Wx+Wy) + f(Wx+Wz)$. □

Theorem 3.4 illustrates that, indeed, the family of oriented sub/superadditive functions is not pathological. In combination with an orthant refinement of $\mathbb{R}^n$, the structure of a oriented sub/superadditive objective functions f allows us to establish finite universal test sets for the family of optimization problems $P(f(x), A, b, l, u)$ with varying integral data l, u . This is the topic of the next section.

4 Oriented sub/superadditive integer maximization

Our first result applies to the family of optimization problems $P(f(x), A, b, l, u)$ with varying integral data l, u when f is oriented subadditive in correspondence with an orthant refinement of $\mathbb{R}^n$. The proof follows directly from Theorem 6 of [5].

Theorem 4.1 *Let $A \in \mathbb{Z}^{m \times n}$, and let $f : \mathbb{Z}^n \to \mathbb{R}$ be $(\mathbb{R}^n, C, C)$-oriented subadditive, for all $C \in \mathcal{C}$, with respect to some orthant refinement $\mathcal{C}$ of $L(A)$. Then there is a finite set $T(f(x), A) \subset \mathbb{Z}^n \cap L(A)$, such that $x^* \in$*

$F(A,b,l,u)$ is optimal for $P(f(x),A,b,l,u)$ if and only if $f(x^*+t) \leq f(x^*)$ for all $t \in T(f(x),A)$ with $l \leq x^*+t \leq u$.

As a next step, we consider the family of optimization problems $P(f(x),A,b,l,u)$ when f is oriented superadditive. It turns out that this situation is more complex than the oriented subadditive case. Henceforth, we proceed in two stages. As a first step we consider the family of problems $P(f(x),A,b,0,\infty)$, where the explicit bounds describe $\mathbb{R}^n_+$ (i.e., $l_i = 0$, $u_i = \infty$, for all $i = 1,\ldots,n$). Then, we can deduce that the set of all non-optimal solutions for this infinite family of problems with varying right-hand side b has a nice combinatorial structure, namely it is *closed up*. More precisely, we have

Theorem 4.2 *Let $A \in \mathbb{Z}^{m\times n}$, and let $f : \mathbb{Z}^n \to \mathbb{R}$ be $(\mathbb{R}^n_+,\mathbb{R}^n_+,\mathbb{R}^n)$-oriented superadditive. Then there is a finite set $T(f(x),A) \subset \mathbb{Z}^n \cap L(A)$, such that for any $x^* \in \mathbb{Z}^n_+$, letting $b^* := Ax^*$, the point $x^* \in F(A,b^*,0,\infty)$ is optimal for $P(f(x),A,b^*,0,\infty)$ if and only if $f(x^*+t) \leq f(x^*)$ for all $t \in T(f(x),A)$ with $0 \leq x^*+t$.*

PROOF. Note that $x^* \in F(A,b^*,0,\infty)$ is not optimal for $P(f(x),A,b^*,0,\infty)$ if and only if it belongs to

$$X(f(x),A) := \{x' \in \mathbb{Z}^n_+ \ : \ \exists\, t \in \mathbb{Z}^n \cap L(A) \ , \\ x'+t \geq 0, \ f(x') < f(x'+t)\} \ .$$

We claim that the set $X(f(x),A)$ is *closed up*; that is, $x' \in X(f(x),A)$ implies that $x'+h \in X(f(x),A)$, for all $h \in \mathbb{Z}^n_+$.

To see this, suppose that $x' \in X(f(x),A)$. Therefore, there is a $t \in \mathbb{Z}^n \cap L(A)$ with $x'+t \geq 0$ and $f(x') < f(x'+t)$. We wish to show that $x'+h \in X(f(x),A)$, for all $h \in \mathbb{Z}^n_+$. To do this, we just need to show that there is a $\hat{t} \in \mathbb{Z}^n \cap L(A)$ such that $(x'+h)+\hat{t} \geq 0$ and $f(x'+h) < f((x'+h)+\hat{t})$.

We simply choose $\hat{t} := t$. Then we check

$$f((x'+h)+t) - f(x'+h) \geq f(x'+t) - f(x') > 0,$$

using $(\mathbb{R}^n_+,\mathbb{R}^n_+,\mathbb{R}^n)$-oriented superadditivity of f .

We have actually shown that every improving augmentation for a feasible point x' is also an improving augmentation for every feasible point that dominates x'.

By the Gordan Lemma, there exists a finite set $\tilde{X}(f(x),A) \subset X(f(x),A)$ such that for all $x^* \in X(f(x),A)$ there exists $\tilde{x}^* \in \tilde{X}(f(x),A)$ with $\tilde{x}^* \leq x^*$. By what we have already shown, it follows that $x^* \in X(f(x),A)$ if and only if there is a $\tilde{x}^* \in \tilde{X}(f(x),A)$ with $\tilde{x}^* \leq x^*$.

Now we just take $T(f(x),A)$ to consist of one improving augmentation for each point in $\tilde{X}(f(x),A)$, and the proof is complete. □

It remains to establish existence of a finite test set for verifying optimality in the presence of lower and upper bounds. We obtain

Theorem 4.3 *Let $A \in \mathbb{Z}^{m\times n}$, and let $f : \mathbb{Z}^n \to \mathbb{R}$ be $(O^n_j,O^n_j,\mathbb{R}^n)$-oriented superadditive for all $0 \leq j < 2^n$. Then there is a finite set $T(f(x),A) \subset \mathbb{Z}^n \cap L(A)$ such that, for every $x^* \in \mathbb{Z}^n$, $l \in \mathbb{Z}^n_{-\infty}$, $u \in \mathbb{Z}^n_{+\infty}$ with $l \leq x^* \leq u$, letting $b^* := Ax^*$, the point $x^* \in F(A,b^*,l,u)$ is optimal for $P(f(x),A,b^*,l,u)$ if and only if $f(x^*+t) \leq f(x^*)$ for all $t \in T(f(x),A)$ with $l \leq x^*+t \leq u$.*

PROOF. Our point of departure is the proof of Theorem 2.3. More precisely, in this proof we defined a test set $T(x^*,f(x),A)$ that depends on the point x^*. Next, we establish that, using the hypothesized oriented superadditivity condition, the set

$$T(f(x), A) := \bigcup_{x^* \in \mathbb{Z}^n} T(x^*, f(x), A)$$

is finite. This will imply the result.

In order to verify that $T(f(x), A)$ is finite we resort to the notation introduced in the proof of Theorem 2.3. For every $0 \leq j < 2^n$ and $h \in G(A)$, let

$$a(h,j) := \sup \Big\{ \alpha_h \ : \ \alpha \in \mathcal{B}(x^*) \\ \text{for some } x^* \in O_j^n \cap \mathbb{Z}^n \Big\} .$$

We want to argue that $a(h,j)$ is finite , so that the sup operator can indeed be replaced by the max operator. For the purpose of deriving a contradiction, let us assume that $a(h,j)$ is infinite. Then there exists an infinite sequence of points $\bar{x}^1, \bar{x}^2, \bar{x}^3, \ldots \in O_j^n \cap \mathbb{Z}^n$, and an infinite sequence $\bar{\alpha}^1, \bar{\alpha}^2, \ldots \in \mathbb{Z}_+^r$, such that

$$\bar{\alpha}^i \in \mathcal{B}(\bar{x}^i) \ , \ \text{and } \bar{\alpha}_h^1 < \bar{\alpha}_h^2 < \cdots$$

Next, we define the set

$$\bar{E} := \Big\{ (\bar{x}^1, \bar{\alpha}^1), (\bar{x}^2, \bar{\alpha}^2), \ldots \Big\} \ .$$

By the Gordan Lemma, there exists a finite $\widetilde{E} \subset \bar{E}$, such that for every $(\bar{x}^i, \bar{\alpha}^i) \in \bar{E}$, there exists $(\tilde{x}^i, \tilde{\alpha}^i) \in \widetilde{E}$, satisfying $|\tilde{x}^i| \leq |\bar{x}^i|$ and $\tilde{\alpha}^i \leq \bar{\alpha}^i$ (where $|\cdot|$ is component wise). Therefore, there exists a pair of indices $k < i$ for which $\bar{\alpha}^k \leq \bar{\alpha}^i$ and $\bar{\alpha}^k \neq \bar{\alpha}^i$. Then, using the fact that $\mathcal{B}(x)$ is the subset of $\mathcal{A}(x)$ of $\leq$-minimal elements (refering to the proof of Theorem 2.3), we obtain the following relations:

$$(*) \qquad \begin{aligned} & f(\bar{x}^k + G\alpha^k) - f(\bar{x}^k) > 0 \ , \\ & f(\bar{x}^i + G\alpha^k) - f(\bar{x}^i) \leq 0 \ . \end{aligned}$$

On the other hand, $\bar{x}^i = \bar{x}^k + v$ with $v \in O_j^n \cap \mathbb{Z}^n$. Using the hypothesized superadditivity condition, we obtain $f(\bar{x}^k + v + G\bar{\alpha}^k) + f(\bar{x}^k) \geq f(\bar{x}^k + v) + f(\bar{x}^k + G\bar{\alpha}^k)$, which is equivalent to $f(\bar{x}^i + G\bar{\alpha}^k) + f(\bar{x}^k) \geq f(\bar{x}^i) + f(\bar{x}^k + G\bar{\alpha}^k)$. Rearranging terms, we obtain

$$\begin{aligned} & f(\bar{x}^i + G\bar{\alpha}^k) - f(\bar{x}^i) \\ & \geq f(\bar{x}^k + G\bar{\alpha}^k) - f(\bar{x}^k) \ , \end{aligned}$$

which is in contradiction with $(*)$. In conclusion, $a(h,j)$ is a finite number, for all $0 \leq j < 2^n$, $h \in G$. This completes the proof. $\square$

Finally, we demonstrate that if the function f satisfies a property that is very closely related to that of being oriented $(O_j^n, O_j^n, \mathbb{R}^n)$-oriented superadditive, then again a universal test set exists.

Theorem 4.4 *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a function such that*

$$f(x+y+z) + f(x) \geq f(x+y) + f(x+z)$$

for all $x, y, z \in \mathbb{Z}^n$ *such that* x, y *lie in the same orthant, and let* $A \in \mathbb{Z}^{m \times n}$. *Then there exists a finite set* $T(f(x), A) \subset \mathbb{Z}^n \cap L(A)$ *such that for for every* $x^* \in \mathbb{Z}^n$, $l \in \mathbb{Z}_{-\infty}^n$, $u \in \mathbb{Z}_{+\infty}^n$ *with* $l \leq x^* \leq u$, *letting* $b^* := Ax^*$, $x^* \in F(A, b^*, l, u)$ *is optimal for* $P(f(x), A, b^*, l, u)$ *if and only if* $f(x^* + t) \leq f(x^*)$ *for all* $t \in T(f(x), A)$ *with* $l \leq x^* + t \leq u$.

PROOF. Let $G(A) = \{g_1, \ldots, g_k\}$ be the Graver basis of A and let G be the $n \times k$ matrix with columns $g_1, \ldots, g_k$. Let

$$\begin{aligned} \mathcal{A} := \{ & (x, \alpha) : x \in \mathbb{Z}^n, \alpha \in \mathbb{Z}_+^k \ , \\ & f(x) < f(x + G\alpha) \}. \end{aligned}$$

Let $\mathcal{B} \subset \mathcal{A}$ be the subset of $\mathcal{A}$ of $\sqsubseteq$-minimal elements, which is finite by the Extended Gordan Lemma. We claim that the desired test set is provided by

$$T(f(x), A) := \{G\beta : \exists x \in \mathbb{Z}^n \ (x, \beta) \in \mathcal{B}\}.$$

First, note that $T(f(x), A) \subset L(A)$ and therefore, for all $t \in T(f(x), A)$, we have $A(x^* + t) = Ax^* = b^*$. So if there is a $t \in T(f(x), A)$ with $l \leq x^* + t \leq u$ and $f(x^*) < f(x^* + t)$ then x^* is not optimal.

Conversely, suppose x^* is not optimal and let x' be a better feasible point. Put $h := x' - x^*$. Then $h \in L(A)$ and therefore there is an $\alpha \in \mathbb{Z}_+^k$ providing a conformal decomposition $h = G\alpha$ of h into Graver basis elements. Now $f(x^*) < f(x') = f(x^* + G\alpha)$ implies $(x^*, \alpha) \in \mathcal{A}$ and hence there is a $(y^*, \beta) \in \mathcal{B}$ satisfying $y^* \sqsubseteq x^*$ $\beta \leq \alpha$.

Consider the element $t := G\beta$ in $T(f(x), A)$. Then $h = G\alpha$ being a conformal decomposition of h and $\beta \leq \alpha$ imply $t \sqsubseteq h$. So now $l \leq x^*, x^* + h = x' \leq u$ imply $l \leq x^* + t \leq u$. So $x^* + t$ is feasible. We claim that it is also better than x^*. Let $v := x^* - y^*$. Then $y^* \sqsubseteq x^*$ implies that y^* and v lie in the same orthant. Also, $(y^*, \beta) \in \mathcal{B} \subset \mathcal{A}$ implies $f(y^*) < f(y^* + G\beta) = f(y^* + t)$. Now, by the property of f, we find that, as claimed,

$$\begin{aligned} f(x^* + t) - f(x^*) &= f(y^* + v + t) - f(y^* + v) \\ &\geq f(y^* + t) - f(y^*) \ > \ 0 \ . \end{aligned}$$

$\square$

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