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Abstract

The double dimer model is defined as the superposition of two independent uniformly distributed dimer covers of a graph. Its configurations can be viewed as disjoint collections of self-avoiding loops. Our first result is that in $\mathbb{Z}^d$, $d > 2$, the loops in the double dimer model are macroscopic. These are shown to behave qualitatively differently than in two dimensions. In particular, we show that, given two distant points of a large box, with uniformly positive probability there exists a loop visiting both points. Our second result involves the monomer double-dimer model, namely the double-dimer model in the presence of a density of monomers. These are vertices which are not allowed to be touched by any loop. This model depends on a parameter, the monomer activity, which controls the density of monomers. It is known from [2, 18] that a finite critical threshold of the monomer activity exists, below which a self-avoiding walk forced through the system is macroscopic. Our paper shows that, when $d > 2$, such a critical threshold is strictly positive. In other words, the self-avoiding walk is macroscopic even in the presence of a positive density of monomers.

1 Introduction

Dimer covers are perfect matchings of a graph, namely spanning sub-graphs such that every vertex has degree one. These mathematical objects attract interest from a wide range of perspectives, which include combinatorics, probability, statistical mechanics, and algorithm complexity studies. Our paper considers a random walk loop soups consisting in the superposition of two dimer covers. More precisely, we consider the double dimer model, which consists in the superposition of two independent uniformly distributed dimer covers, and the monomer double-dimer model, which corresponds to the double dimer model in the presence of a density of monomers (a monomer is a vertex which is not allowed to be touched by any loop). These models are related to the loop O(N) model and to other random walk loop soups which received increasing interest in the last few years, see for example [1, 3, 5, 13, 15, 16] for some references.

Our first result involves the double dimer model. The planar case was studied in [4, 10], in which conformal invariance properties of the scaling limit were proved. Relying on Kasteleyn's theorem [8, 19], the methods of these papers do not apply to higher dimension. Our first main theorem shows that the loops of the double dimer model in $\mathbb{Z}^d$, $d > 2$, are macroscopic. More precisely, we consider the double dimer model on a torus of $\mathbb{Z}^d$ of L^d sites and show that **(1):** the expected length of each loop is of order $O(L^d)$ **(2):** given two vertices on the Cartesian axis having distance of order $O(L)$, with uniformly positive probability a loop connects both. Contrary to our high dimensional case, in the planar case this probability converges to zero as $L \to \infty$ [10].

Secondly, we introduce the monomer double-dimer model. The model depends on a parameter, the *monomer activity*, which controls the density of monomers, and reduces to the double dimer model when the monomer activity is zero. We consider a version of this model where one of the loops is

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forced to be 'open' and then corresponds to a self-avoiding walk starting from the origin, which is allowed to end at an arbitrary vertex of the box. It is known from [2] that the length of the self-avoiding walk admits uniformly bounded exponential moments as the monomer activity is very large. It is known from [18] that the self-avoiding walk is 'macroscopic' if the monomer activity is zero and $d > 2$. Our second main result, Theorem 2.2, is that the self-avoiding walk keeps being macroscopic even if the monomer activity is strictly positive. In other words, the phase transition in this model is non-trivial and occurs at a strictly positive and finite threshold of the monomer activity. Our results also hold for an extension of this model in which the measure depends on a parameter N, which rewards the total number of loops which are present in the system, and N is a not too large integer.

Our results are an extension of the method developed in [18], in which the reflection positivity technique has been reformulated in the framework of random walk loop soups.

2 Definitions and results

Consider a finite undirected graph $G = (V, E)$. A dimer configuration is a spanning sub-graph of G such that every vertex has degree one. We let $\mathcal{D}_G$ be the set of all dimer configurations in G. Given a set $A \subset V$, we let G_A be the subgraph of G with vertex set $V \setminus A$ and with edge-set consisting of all the edges in E which do not touch any vertex in A. We let $\mathcal{D}_G(A)$ be the set of dimer configurations in G_A. We let $\mathbb{T}_L = (V_L, E_L)$ be a graph corresponding to the torus $\mathbb{Z}^d \setminus L\mathbb{Z}^d$, with edges connecting neighbour vertices. We let $o \in V_L$ be the origin.

The double-dimer model. The double-dimer model is sampled by superimposing two independent dimer covers with uniform distribution on the set of possible dimer covers. Each realisation of the model can then be viewed as a collection of disjoint self-avoiding loops, where some of these loops may consist of two dimers on the same edge.

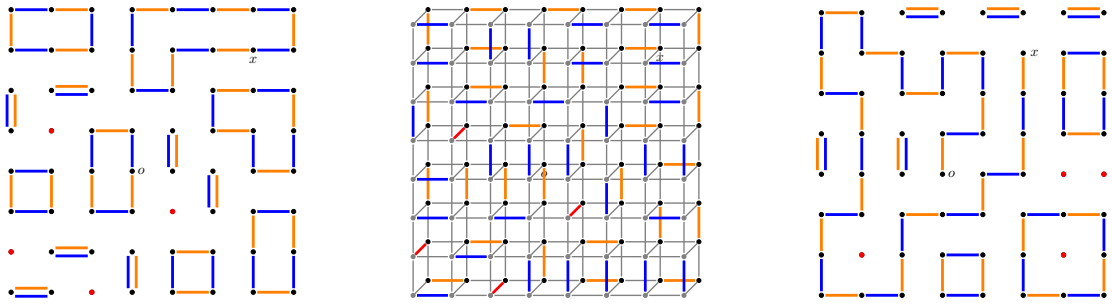
We let P_L be the counting measure on the set $\mathcal{D}_{\mathbb{T}_L} \times \mathcal{D}_{\mathbb{T}_L}$ normalized by the total number of double dimer coverings and we denote by E_L the corresponding expectation. We note that each pair $(d_1, d_2) \in \mathcal{D}_{\mathbb{T}_L} \times \mathcal{D}_{\mathbb{T}_L}$ can be viewed as a disjoint collection of self-avoiding loops. Some of these loops may correspond to the superposition of two dimers on the same edge. We denote by $\{o \leftrightarrow x\}$ the set of double dimer covers such that both $o, x \in V_L$ belong to the same loop. Further, we denote by $L_o = L_o(d_1, d_2)$ the loop that contains the origin and by $|L_o|$ we denote its length, namely,

$$|L_o| := \sum_{x \in V_L} \mathbb{1}_{\{x \in L_o\}}.$$

Our first theorem states that the expected length of the loop that contains the origin is of the same order of magnitude as the volume of the box and that the probability that a loop connects two vertices whose distance is of order $O(L)$ is uniformly positive. Moreover, our theorem also provides an upper bound on the expected loop length. To state the theorem, let $N_+ = \sum_{n>0} \mathbb{1}_{\{S_n = 0\}}$ be the number of returns to the origin of an independent simple random walk S_n in $\mathbb{Z}^d$ starting at the origin. Denoting its expectation by E^d, we set $r_d := E^d[N_+]$.

Theorem 2.1. *Suppose that $d > 2$. Then*

$$\left(\frac{1}{2d}\Big(1 - \frac{r_d}{2}\Big)\right)^2 \le \liminf_{\substack{L\to\infty \\ L \text{ even}}} \frac{1}{|V_L|} E_L\big[|L_o|\big] \le \frac{1}{2d}\Big(2 - \frac{1}{2d}\Big). \tag{2.1}$$



(a) (b) (c)

Figure 1: (a) A configuration of the monomer double-dimer model. (b) A dimer configuration on the duplicated torus. The red, orange and blue dimers represent the vertical dimers and the dimers of the lower and upper torus, respectively. Projecting the upper and lower torus onto the same torus leads to the configuration on the left. (c) A realisation $\omega = (M, d_1, d_2) \in \Omega(o, x)$. The set of monomers M is represented by red circles. The dimers of d_1 and d_2 are coloured blue and orange, respectively.

Moreover, for any $\varphi \in (0, \frac{1}{2d}(1 - \frac{r_d}{2}))$*, there exists* $\varepsilon = \varepsilon(\rho, d) \in (0, \frac{1}{2})$ *such that for any* $L \in 2\mathbb{N}$ *large enough and any odd integer* $n \in (0, \varepsilon L)$*,*

$$P_L(o \leftrightarrow n\, \boldsymbol{e}_1) \geq \varphi^2. \tag{2.2}$$

An exact computation made by Watson [20] shows that $0.51 < r_3 < 0.52$ and the Rayleigh monotonicity principle [14] implies that r_d is non-increasing in d. This implies that the right-hand side of (2.1) is uniformly positive for any $d > 2$. We refer to [7] for numerical estimates of r_d for $d \geq 3$.

Our Theorem 2.1 extends [16, Theorem 1.1], which states the occurrence of macroscopic loops for a very general random walk loop soup. That theorem, however, does not cover the double dimer model, since it holds only for random walk loop soups in which the vertices of the graph are allowed to be visited sufficiently many times by the loops.

The monomer double-dimer model. The monomer double-dimer model generalizes the double dimer model. The generalisation consists in allowing the presence of a density of monomers. These are controlled through an external parameter, the monomer activity. When the monomer activity is zero, the model reduces to the double-dimer model.

The configuration space of the monomer double-dimer model is denoted by Ω and it corresponds to the set of triplets $\omega = (M, d_1, d_2)$ such that $M \subset V_L$ and $(d_1, d_2) \in \mathcal{D}_{\mathbb{T}_L}(M) \times \mathcal{D}_{\mathbb{T}_L}(M)$. We refer to the first element of the triplet ω as set of monomers. We let $\mathcal{M} : \Omega \mapsto V_L$ be the random variable corresponding to the set of monomers, i.e, $\mathcal{M}(\omega) := M$ for each $\omega = (M, d_1, d_2) \in \Omega$. As one can see on the left of Figure 1, any such configuration can be viewed as a configuration of disjoint self-avoiding loops. As for the double dimer model, some of such loops may correspond to the superposition of two dimers on the same edge.

Alternatively, the configurations can be viewed as dimer configurations on a graph corresponding to two copies of a torus, with an edge connecting each vertex of one of the two tori to the corresponding vertex of the other torus. See also Figure 1. We refer to such a graph as *duplicated torus* and to the edges connecting the two tori as *vertical edges.* The representation is such that each monomer in the first representation corresponds to a dimer on a vertical edge in the second representation. In this representation, the monomer activity then rewards the number of dimers on vertical edges. A further alternative natural description of this model is to consider its configurations as permutations of the

vertices of the graph such that each vertex is mapped either to itself or to a nearest neighbour, see [18] for more details.

To state our second main result we need to introduce the *two point function* of the monomer double dimer model, which is defined by imposing that the dimers of the first dimer cover cannot touch two given vertices. More precisely, given any two pair of vertices $x, y \in V_L$, we define $\Omega(x, y)$ as the set of triplets $\omega = (M, d_1, d_2)$ such that $M \subset V_L \setminus \{x, y\}$, $d_1 \in \mathcal{D}_{\mathbb{T}_L}(M \cup \{x, y\})$, and $d_2 \in \mathcal{D}_{\mathbb{T}_L}(M)$. As one can see on the right of Figure 1, each element $\omega \in \Omega(x, y)$ can be viewed as a collection of disjoint self-avoiding loops with a self-avoiding walk starting at x and ending at y. For any $\omega \in \Omega(x, y)$, we let $\mathcal{L}(\omega)$ be the number of loops in ω (we consider a loop also the object corresponding to the superposition of two dimers on the same edge). We let $N \geq 0$, define the partition functions,

$$\mathbb{Z}_{L,N,\rho}(x, y) := \sum_{\omega \in \Omega(x,y)} \rho^{|\mathcal{M}(\omega)|} \left(\frac{N}{2}\right)^{\mathcal{L}(\omega)},$$

$$\mathbb{Z}_{L,N,\rho} := \sum_{\omega \in \Omega} \rho^{|\mathcal{M}(\omega)|} \left(\frac{N}{2}\right)^{\mathcal{L}(\omega)},$$

and introduce the *two-point function*,

$$\mathbb{G}_{L,N,\rho}(x, y) := \frac{\mathbb{Z}_{L,N,\rho}(x, y)}{\mathbb{Z}_{L,N,\rho}}.$$

Recalling the description of the monomer-double dimer model as a dimer model on the duplicated torus, the two-point function can then be viewed as the ratio between the weight of all dimer configurations on the duplicated torus with two monomers displaced at two vertices of such a graph and the weight of all configurations with no such monomers. When $\rho = 0$ and $N = 2$, our two point function reduces to the monomer-monomer correlation of the *dimer model* on the torus,

$$\mathbb{G}_{L,2,0}(x, y) = \frac{|\mathcal{D}_{\mathbb{T}_L}(\{x, y\})|}{|\mathcal{D}_{\mathbb{T}_L}|},$$

which is known to decay polynomially with the distance between the two monomers on $\mathbb{Z}^2$ [4] and to be uniformly positive on $\mathbb{Z}^d$ with $d > 2$ [18]. Our second main theorem states that the two-point function stays uniformly positive even for strictly positive values of the monomer activity (and even for integer values of N different from two, which are not too large). Since it is known from [2] that the two-point function decays exponentially fast as the monomer activity is large enough, our result implies the occurrence of a phase transition at a strictly positive thershold for the monomer activity when $d > 2$.

An alternative formulation of our result involves the length of a self-avoiding walk forced through the system of loops. Define the set $\Omega^w := \cup_{x \in V_L} \Omega(o, x)$, whose elements can be viewed as systems of mutually-disjoint self-avoiding loops with a self-avoiding walk starting at the origin and ending at an arbitrary vertex of the box. Define a probability measure on this set,

$$\forall \omega \in \Omega^w \quad \mathbb{P}_{L,N,\rho}(\omega) := \frac{\rho^{|\mathcal{M}(\omega)|} \left(\frac{N}{2}\right)^{\mathcal{L}(\omega)}}{\sum_{\omega \in \Omega^w} \rho^{|\mathcal{M}(\omega)|} \left(\frac{N}{2}\right)^{\mathcal{L}(\omega)}}.$$

For any $\omega \in \Omega^w$, let $X = X(\omega)$ be the random end-point of the self-avoiding walk which is not the origin (if $\omega \in \Omega(o, o)$, then the self-avoiding walk is degenerate and consists of a single monomer at o, in this case we set $X = o$). Our second main theorem, Theorem 2.2 below, states that, if the

monomer activity is small enough (possibly positive) and N is a not too large integer, then the distance between the two end-points of the self-avoiding walk is of the same order of magnitude as the diameter of the box. Contrary to this, it is known from [2, 17] that the length of the walk is of order $O(1)$ in the limit of large L if the monomer activity is large enough.

Theorem 2.2. *Suppose that $d > 2$, that N is an integer in $\big(0, \frac{2\,(3\pi-4)}{\pi\, r_d}\big)$ and that $\rho \in \big[0, 1-\frac{\pi\, r_d\, N}{2\,(3\pi-4)}\big)$. Then,*

$$\liminf_{\substack{L\to\infty \\ L \text{ even}}} \frac{1}{|V_L|} \sum_{x\in V_L} \mathbb{G}_{L,N,\rho}(o,x) \;\geq\; \frac{1}{4d}\Big(\frac{3\pi-4}{\pi N}\,(1-\rho) - \frac{r_d}{2}\Big) =: C. \tag{2.3}$$

Moreover, for any $\varphi \in (0, \frac{C}{2})$, there exists $\varepsilon > 0$ such that for any $L \in 2\mathbb{N}$, any integer $n \in (-\varepsilon\, L, \varepsilon\, L)$ and any $i \in \{1, \dots, d\}$,

$$\mathbb{G}_{L,N,\rho}(o, n\, \boldsymbol{e}_i) \geq \varphi. \tag{2.4}$$

Finally, under the same assumptions, there exists $c > 0$ such that for any $L \in 2\mathbb{N}$,

$$\mathbb{P}_{L,N,\rho}\big(\, |X|_1 \,>\, c\, L\,\big) > c. \tag{2.5}$$

For $d = 3$, the right-hand side of (2.3) is strictly positive for any $N \in \{1, \dots, 6\}$. Our theorem extends Theorems 2.2 and 2.3 in [18], since our lower bounds (2.3), (2.4), and (2.5) hold even for strictly positive values of the monomer activity.

3 Proof of the theorems

Our Theorem 2.1 follows from [18, Theorem 1] and from an argument from C. Kenyon *et al.* in [9]. Theorem 1 in [18] shows that the monomer-monomer correlation of the the dimer model is uniformly positive. Combining this result with an argument from [9], we deduce that the expected loop length in the double dimer model is uniformly positive. By using a general monotonicity property for two-point functions which was proved in [12], we also deduce that the probability of existence of a loop connecting two given distant vertices on the Cartesian axis is also uniformly positive.

The starting point of the proof of our Theorem 2.2 is the lower bound for the Cesàro sum of the two-point function, equation (3.7) below, which was derived in [18] using the reflection positivity method. This inequality can be viewed as a version of the so-called *Infrared Bound*, which is used in the framework of spin systems [6]. For spin systems, the uniform positivity of the Cesàro sum follows immediately from the Infrared bound, since the term $o - o$ of the two-point function stays uniformly positive in the external parameter (the so-called inverse temperature) and the bound gets better and better as the inverse temperature gets larger. This is not the case for monomer double dimer model, in which the term $o - o$ of the two-point function vanishes as the monomer activity goes to zero. Hence, we deal with an alternative version of the Infrared bound, represented by equation (3.7) below. To obtain non-trivial informations from such an inequality one needs to show that the right-hand side of the inequality is uniformly positive. For that, one needs to provide a uniform lower bound to the third term in the right-hand side of (3.7), which involves a sum of two-point functions at the even sites of the torus multiplied by positive and negative coefficients, and compare it with the term $\mathbb{G}_{L,N,\rho}(o, \boldsymbol{e}_1)$, which is shown to be uniformly positive and non-decreasing with the monomer activity. To control such a sum, the assumption $\rho = 0$ was made in [18]. Under this assumption the even two-point function is just zero, hence such a sum vanishes. Here we use simple analytic methods to provide a general lower bound to this term which holds for positive values of the monomer activity and is finite (but negative)

uniformly in the size of the torus. Combined with the uniform positivity of the term $\mathbb{G}_{L,N,\rho}(o, \boldsymbol{e}_1)$, this allows us to deduce that the Cesàro sum in the left-hand side is uniformly positive even for positive values of the monomer activity.

Our simple method can then be viewed as a general strategy to deduce long range order from the Infrared bound for models in which the term $o - o$ of the two-point function is small and it is only the odd two point function which is large in the regime in which one expects long range order.

3.1 Proof of Theorem 2.1

We introduce the subsets of even and odd vertices of V_L, namely,

$$V_L^e := \{x \in V_L : \, d(o,x) \in 2\mathbb{N}_0\}, \qquad V_L^o := \{x \in V_L : \, d(o,x) \in 2\mathbb{N}_0 + 1\},$$

where $d(o,x)$ is the graph distance in $\mathbb{T}_L$. For lighter notation, we will omit sub-scripts where appropriate.

The starting point of our proof of Theorem 2.1 is the following uniform positivity result for the monomer-monomer correlation of the dimer model.

Theorem 3.1 ([18, Theorem 2.1])**.** *Suppose that $d > 2$. Then*

$$\liminf_{\substack{L\to\infty \\ L \text{ even}}} \frac{1}{|V_L^o|} \sum_{x\in V_L^o} \frac{|D(\{o,x\})|}{|D(\emptyset)|} \geq \frac{1}{2d}\Big(1 - \frac{r_d}{2}\Big). \tag{3.1}$$

Moreover, for any $\varphi \in (0, \frac{1}{2d}(1 - \frac{r_d}{2}))$, there exists a constant $c = c(\varphi, d) \in (0, \frac{1}{2})$ such that for any $L \in 2\mathbb{N}$ large enough and any odd integer $n \in (0, c\,L)$,

$$\frac{|D(o, n\,\boldsymbol{e}_1)|}{|D(\emptyset)|} \geq \varphi. \tag{3.2}$$

In our proof we will first bound the number of double dimer coverings containing a loop connecting o and x from below by $|D(\{o,x\})|^2$ and then apply Theorem 3.1 to deduce uniform positivity of the expected loop length in the double dimer model.

Proof of Theorem 2.1. To begin, we show that for any $x \in V_L^o$, it holds that

$$|D(\{o,x\})|^2 \leq |D(\emptyset)^2 : \, o \leftrightarrow x|. \tag{3.3}$$

The upper bound (3.3) follows from the proof of [9, Theorem 2]. For comprehensiveness, we repeat the main argument given there. Note that if x is adjacent to the origin, then (3.3) trivially holds true since for any pair of configurations $(d_1, d_2) \in D(\{o,x\})^2$ we can add precisely one dimer on the edge $\{o,x\}$ in each of the two configurations d_1 and d_2, thus obtaining a configuration in $D(\emptyset)^2$ which contains a loop consisting of two dimers on $\{o,x\}$. Suppose now that x is not adjacent to the origin. We introduce the map $\phi : D(\{o,x\}) \times D(\{\boldsymbol{e}_1, x + \boldsymbol{e}_1\}) \to \{D(\emptyset)^2 : \, o \leftrightarrow x\}$, which for each configuration $(d_1, d_2) \in D(\{o,x\}) \times D(\{\boldsymbol{e}_1, x + \boldsymbol{e}_1\})$ acts by

(i) colouring the dimers in d_1 orange and the dimers in d_2 blue,

(ii) superimposing both (coloured) configurations,

(iii) adding an orange dimer on $\{o, \boldsymbol{e}_1\}$ and on $\{x, x+\boldsymbol{e}_1\}$, by then switching the colours of the dimers along the path from $x+\boldsymbol{e}_1$ to $\boldsymbol{e}_1$ which does not touch x from blue to orange and orange to blue, respectively,

(iv) defining a new configuration $(d_1', d_2') \in D(\emptyset)^2$, where d_1' consists of all orange and d_2' consists of all blue dimers.

Note that the configuration (d_1', d_2') contains a loop which visits amongst others the set $\{o, \boldsymbol{e}_1, x, x+\boldsymbol{e}_1\}$. For an illustration see also [9, Figure 2]. The map ϕ is well-defined and an injection as shown in [9]. Since $|D(\{\boldsymbol{e}_1, x+\boldsymbol{e}_1\})| = |D(\{o, x\})|$, this proves (3.3).

Using the Cauchy-Schwarz inequality, we then have that

$$\frac{1}{|V_L^o|}\sum_{x\in V_L^o}\frac{|D(\{o,x\})|}{|D(\emptyset)|} \le \left(\frac{1}{|V_L^o|}\sum_{x\in V_L^o}\frac{|D(\{o,x\})|^2}{|D(\emptyset)|^2}\right)^{\frac{1}{2}} \le \left(\frac{1}{|V_L^o|}\sum_{x\in V_L^o} P_L(o\leftrightarrow x)\right)^{\frac{1}{2}}, \quad (3.4)$$

where we used (3.3) in the last step. Since the number of even and odd vertices in each loop coincide we further have that

$$\frac{1}{|V_L|}E_L\big[|L_o|\big] = \frac{1}{|V_L^o|}\sum_{x\in V_L^o} E_L\big[\mathbb{1}_{\{x\in L_o\}}\big] = \frac{1}{|V_L^o|}\sum_{x\in V_L^o} P_L(o\leftrightarrow x). \quad (3.5)$$

From (3.1), (3.4) and (3.5), we deduce the lower bound in (2.1). The upper bound follows from (3.5) and from the site-monotonicity property,

$$P_L(o\leftrightarrow x) \le P_L(o\leftrightarrow \boldsymbol{e}_1) = \frac{1}{2d}\left(2-\frac{1}{2d}\right) \quad (3.6)$$

for all $x \in V_L$. In (3.6) we applied [12, Theorem 2.1] noting that the quantity $P_L(o \leftrightarrow x)$ can be expressed in the language of the *random path model* for specific choices of the parameters (see e.g. [11, 12, 16]). The rightmost term in (3.6) corresponds to the probability that at least one of the two dimer configurations has a dimer on $\{o, \boldsymbol{e}_1\}$. From (3.3) and (3.2) we further deduce (2.2). This concludes the proof of the theorem. □

3.2 Proof of Theorem 2.2

In this section we prove Theorem 2.2. We will use the notation

$$\mathbb{G}_{L,N,\rho}(x) := \mathbb{G}_{L,N,\rho}(o,x)$$

for any $x \in V_L$. The proof of Theorem 2.2 relies on the following theorem.

Theorem 3.2 ([18, Theorem 5.1]). *For any $d, N \in \mathbb{N}$, $L \in 2\mathbb{N}$, $\rho \in [0,\infty)$, we have that,*

$$\sum_{x\in V_L}\frac{\mathbb{G}_{L,N,\rho}(x)}{|V_L|} \ge \frac{1}{2}\left(\mathbb{G}_{L,N,\rho}(\,\boldsymbol{e}_1\,) - \mathcal{I}_L(d) + \frac{2}{|V_L|}\sum_{\substack{x\in V_L^e:\\ x_2=\ldots=x_d=0}} \Upsilon_L(x)\mathbb{G}_{L,N,\rho}(x)\right), \quad (3.7)$$

where $\big(\mathcal{I}_L(d)\big)_{L\in\mathbb{N}}$ is a sequence of real numbers whose limit $L\to\infty$ exists and satisfies

$$\lim_{L\to\infty}\mathcal{I}_L(d) = \frac{r_d}{4d}, \quad (3.8)$$

and $(\Upsilon_L)_{L\in\mathbb{N}}$ *is a sequence of real-valued functions, defined by*

$$\forall x \in \mathbb{Z}^d \qquad \Upsilon_L(x) := \mathit{Re}\Big(\sum_{\substack{k\in V_L^*:\\ k_1\in(-\frac{\pi}{2},\frac{\pi}{2}]}} e^{-ik\cdot(x-\boldsymbol{e}_1)} \Big).$$

Here, $V_L^* := \{\frac{2\pi}{L}x : x \in V_L\}$ *denotes the vertex set of the Fourier dual torus of* $\mathbb{T}_L$*.*

We will obtain (2.3) by deriving a lower bound for the third term appearing on the right-hand side of (3.7).

Proof of Theorem 2.2. We will show that

$$\liminf_{\substack{L\to\infty\\ L \text{ even}}} \frac{2}{|V_L|} \sum_{\substack{x\in V_L^e:\\ x_2=\ldots=x_d=0}} \Upsilon_L(x)\mathbb{G}_{L,N,\rho}(x) \geq \frac{\pi-4}{2\pi} \liminf_{\substack{L\to\infty\\ L \text{ even}}} \mathbb{G}_{L,N,\rho}(\,\boldsymbol{e}_1\,) + \frac{1}{\pi} \liminf_{\substack{L\to\infty\\ L \text{ even}}} \mathbb{G}_{L,N,\rho}(o). \tag{3.9}$$

Suppose that $L \in 2\mathbb{N}$. Fix $x \in V_L$ such that $x_1 \in 2\mathbb{N}$ and $x_2 = \cdots = x_d = 0$. Some basic calculations, which are given in the appendix, show that

$$\Upsilon_L(x) = \begin{cases} -L^{d-1} \cos\left(\frac{\pi\, x_1}{2}\right) \cot\left(\frac{\pi}{L}\,(x_1-1)\right) & \text{if } L \in 4\mathbb{N}, \\ -L^{d-1} \cos\left(\frac{\pi\, x_1}{2}\right) \csc\left(\frac{\pi}{L}\,(x_1-1)\right) & \text{if } L \in 2\mathbb{N}\setminus 4\mathbb{N}. \end{cases} \tag{3.10}$$

We will derive (3.9) for $L \in 4\mathbb{N}$, namely $L = 4m$ for some $m \in \mathbb{N}$. The case $L \in 2\mathbb{N}\setminus 4\mathbb{N}$ is then similar. By (3.10) and using that $\cot(-x) = -\cot(x)$ for any $x \in \mathbb{R}$ and $\mathbb{G}_{L,N,\rho}(x) = \mathbb{G}_{L,N,\rho}(-x)$ for any $x \in V_L$, we have that,

$$\begin{aligned}
&\frac{1}{|V_L|} \sum_{\substack{x\in V_L^e:\\ x_2=\ldots=x_d=0}} \Upsilon_L(x)\mathbb{G}_{L,N,\rho}(x) \\
&= -\frac{1}{4m} \sum_{n=-m+1}^{m} \cos(\pi\, n)\, \cot\big(\frac{\pi}{4m}\,(2n-1)\big)\, \mathbb{G}_{L,N,\rho}\big((2n,0,\ldots,0)\big) \\
&= \frac{1}{4m} \sum_{\substack{n=1\\ \text{n odd}}}^{m-1} \mathbb{G}_{L,N,\rho}\big((2n,0,\ldots,0)\big) \Big(\cot\big(\frac{\pi}{4m}\,(2n-1)\big) - \cot\big(\frac{\pi}{4m}\,(2n+1)\big)\Big) \\
&\quad - \frac{1}{4m} \sum_{\substack{n=1\\ \text{n even}}}^{m-1} \mathbb{G}_{L,N,\rho}\big((2n,0,\ldots,0)\big) \Big(\cot\big(\frac{\pi}{4m}\,(2n-1)\big) - \cot\big(\frac{\pi}{4m}\,(2n+1)\big)\Big) \\
&\quad + \frac{1}{4m}\mathbb{G}_{L,N,\rho}\big((0,0,\ldots,0)\big)\, \cot(\frac{\pi}{4m}) - \frac{1}{4m}\cos(m\,\pi)\cot(\frac{\pi}{2}-\frac{\pi}{4m})\, \mathbb{G}_{L,N,\rho}\big((2m,0,\ldots,0)\big).
\end{aligned} \tag{3.11}$$

We now derive a lower bound for (3.11) in the limit $m \to \infty$. By [18, Theorem 2.3] it holds that $\mathbb{G}_{L,N,\rho}(x) \leq \frac{1}{dN}$ for any $x \in V_L$. The last term of (3.11) thus vanishes to zero since $\cot(\frac{\pi}{2}) = 0$. The first term is non-negative since $\cot(x)$ is monotonically decreasing on the interval $(0,\pi)$. From

these considerations and using that $\lim_{m\to\infty} \frac{1}{4m}\cot(\frac{\pi}{4m}) = \frac{1}{\pi}$, we obtain from (3.11) that

$$\begin{aligned}
&\liminf_{\substack{L\to\infty\\ L\in 4\mathbb{N}}} \frac{1}{|V_L|} \sum_{\substack{x\in V_L^e:\\ x_2=\ldots=x_d=0}} \Upsilon_L(x)\,\mathbb{G}_{L,N,\rho}(x) \\
&\geq -\liminf_{\substack{L\to\infty\\ L\in 4\mathbb{N}}} \mathbb{G}_{L,N,\rho}(\boldsymbol{e}_1)\ \liminf_{m\to\infty} \frac{1}{4m}\sum_{n=1}^{\frac{m-1}{2}} \Big(\cot\big(\frac{\pi}{4m}(4n-1)\big) - \cot\big(\frac{\pi}{4m}(4n+1)\big)\Big) \\
&\qquad + \frac{1}{\pi}\liminf_{\substack{L\to\infty\\ L\in 4\mathbb{N}}} \mathbb{G}_{L,N,\rho}(o).
\end{aligned} \tag{3.12}$$

To conclude the proof of (3.9) it remains to derive a lower bound for the first term of (3.12). Using expansion into Laurent series and the Leibniz formula for π we obtain that

$$\begin{aligned}
&\frac{1}{4m}\sum_{n=1}^{\frac{m-1}{2}} \Big(\cot\big(\frac{\pi}{4m}(4n-1)\big) - \cot\big(\frac{\pi}{4m}(4n+1)\big)\Big) \\
&= \frac{1}{\pi}\sum_{n=1}^{\frac{m-1}{2}} \Big(\frac{1}{4n-1} - \frac{1}{4n+1}\Big) + \frac{1}{4m}\sum_{n=1}^{\frac{m-1}{2}}\sum_{l=0}^{\infty} c_{2l+1}\Big(\frac{\pi}{4m}\Big)^{2l+1}\Big((4n-1)^{2l+1} - (4n+1)^{2l+1}\Big) \\
&= \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1} + \frac{1}{4m}\sum_{n=1}^{\frac{m-1}{2}}\sum_{l=0}^{\infty} c_{2l+1}\Big(\frac{\pi}{4m}\Big)^{2l+1}\Big((4n-1)^{2l+1} - (4n+1)^{2l+1}\Big) \\
&= \frac{\pi-4}{4\pi} + \frac{1}{4m}\sum_{n=1}^{\frac{m-1}{2}}\sum_{l=0}^{\infty} c_{2l+1}\Big(\frac{\pi}{4m}\Big)^{2l+1}\Big((4n-1)^{2l+1} - (4n+1)^{2l+1}\Big),
\end{aligned} \tag{3.13}$$

for a sequence of real-valued coefficients $(c_{2l+1})_{l\in\mathbb{N}_0}$, which does not depend on m. We now show that we can neglect the second term on the right-hand side of (3.13) in the limit $m\to\infty$. For this, fix $l\in\mathbb{N}_0$ and set $k := 2l+1$. Applying the binomial theorem, we deduce that

$$\Big|\sum_{n=1}^{\frac{m-1}{2}}\Big((4n-1)^k - (4n+1)^k\Big)\Big| = 2\sum_{n=1}^{\frac{m-1}{2}}\sum_{\substack{j=0\\ j\text{ odd}}}^{k}\binom{k}{j}(4n)^{k-j} \leq k^{k+1}\,(m-1)^k\,2^{k-1},$$

and thus

$$\Big(\frac{\pi}{4m}\Big)^k \sum_{n=1}^{\frac{m-1}{2}}\Big((4n-1)^k - (4n+1)^k\Big) = o(m). \tag{3.14}$$

Using the reverse Fatou lemma and (3.14), we thus have that

$$\begin{aligned}
0 &\leq \liminf_{m\to\infty}\frac{1}{4m}\sum_{n=1}^{\frac{m-1}{2}}\sum_{l=0}^{\infty} c_{2l+1}\Big(\frac{\pi}{4m}\Big)^{2l+1}\Big((4n-1)^{2l+1} - (4n+1)^{2l+1}\Big) \\
&\leq \sum_{l=0}^{\infty} c_{2l+1}\,\limsup_{m\to\infty}\frac{1}{4m}\sum_{n=1}^{\frac{m-1}{2}}\Big(\frac{\pi}{4m}\Big)^{2l+1}\Big((4n-1)^{2l+1} - (4n+1)^{2l+1}\Big) = 0.
\end{aligned} \tag{3.15}$$

From (3.12), (3.13) and (3.15) we thus deduce (3.9). It remains to provide a lower bound on $\mathbb{G}_{L,N,\rho}(\boldsymbol{e}_1)$. Let Ω^ℓ denote the set of triples $\pi = (M, d_1, d_2)$ such that $M\subset V_L$ and $d_1, d_2 \in D_{\mathbb{T}_L}(M)$. Each

triplet can be seen as a system of monomers and disjoint self-avoiding loops. For any $\pi \in \Omega^\ell$, we let $\tilde{\mathcal{L}}(\pi)$ be the number of loops in π. We define a probability measure on Ω^ℓ,

$$\forall \pi \in \Omega^\ell \qquad \tilde{\mathbb{P}}_{L,N,\rho}(\pi) := \frac{\rho^{|M(\pi)|}\big(\frac{N}{2}\big)^{\tilde{\mathcal{L}}(\pi)}}{\tilde{Z}_{L,N,\rho}},$$

where $\tilde{Z}_{L,N,\rho}$ is a normalization constant. We have the equality (see e.g. [18]),

$$\mathbb{G}_{L,N,\rho}(\boldsymbol{e}_1) = \frac{1}{d\,N}\,\big(1 - \tilde{\mathbb{P}}_{L,N,\rho}(o \text{ is a monomer})\big). \tag{3.16}$$

The probability appearing on the right-hand side of (3.16) can be reformulated as probability in the *random path model* for special choices of the parameters, see e.g. [11, 18]. This reformulation allows the application of the Chessboard Estimate, see e.g. [18, Proposition 4.5], namely

$$\tilde{\mathbb{P}}_{L,N,\rho}(o \text{ is a monomer}) \le \tilde{\mathbb{P}}_{L,N,\rho}(\forall x \in V_L,\, x \text{ is a monomer})^{\frac{1}{|V_L|}} \le \rho,$$

which together with (3.16) implies that

$$\mathbb{G}_{L,N,\rho}(\boldsymbol{e}_1) \ge \frac{1}{dN}(1-\rho). \tag{3.17}$$

Combining (3.7), (3.8), (3.9) and (3.17) concludes the proof of (2.3). From (2.3) and [12, Theorem 2.2] we further deduce (2.4). For the last statement, equation (2.5), note that for any $\alpha \in (0,1)$,

$$\mathbb{P}_{L,N,\rho}\big(\,|X|_1 \,\le\, \alpha\, L\,\big) = \frac{\sum\limits_{x \in V_{\lfloor \alpha\, L \rfloor}} \mathbb{G}_{L,N,\rho}(o,x)}{\sum\limits_{x \in V_L} \mathbb{G}_{L,N,\rho}(o,x)} \le \frac{\alpha^d}{C},$$

where we used (2.3) and that $\mathbb{G}_{L,N,\rho}(x) \le 1$ for all $x \in V_L$ (see [18, Theorem 2.3]) in the last step. Choosing $\alpha > 0$ small enough and setting $c = \min(\alpha, 1 - \frac{\alpha^d}{C})$ proves (2.5) and concludes the proof of the theorem. □

A Appendix

Proof of (3.10) *in the proof of Theorem 2.2.* Let $L \in 2\mathbb{N}$ and $x \in V_L$ such that $x_1 \in 2\mathbb{N}$ and $x_2 = \cdots = x_d = 0$. We have that

$$\Upsilon_L(x) = \begin{cases} L^{d-1}\,\mathsf{Re}\Big(\sum\limits_{k=-\frac{L}{4}+1}^{\frac{L}{4}} e^{-i\,\frac{2\pi}{L}\,k\,(x_1-1)}\Big) & \text{if } L \in 4\mathbb{N}, \\ L^{d-1}\,\mathsf{Re}\Big(\sum\limits_{k=-\frac{L-2}{4}+1}^{\frac{L-2}{4}} e^{-i\,\frac{2\pi}{L}\,k\,(x_1-1)}\Big) & \text{if } L \in 2\mathbb{N}\setminus 4\mathbb{N}. \end{cases} \tag{A.1}$$

We now derive (3.10) for $L \in 4\mathbb{N}$. The case $L \in 2\mathbb{N}\setminus 4\mathbb{N}$ is then similar. Consider $L = 4n$ for some $n \in \mathbb{N}$ and take $x \in \mathbb{Z}$ odd. Using that $\cos(\frac{\pi}{2}x) = 0$, we have that

$$\mathsf{Re}\Big(\sum_{k=-n+1}^{n} e^{-i\,\frac{\pi}{2n}\,k\,x}\Big) = \mathsf{Re}\Big(e^{i\,\frac{\pi}{2}\,x}\,e^{-i\,\frac{\pi}{2n}\,x}\sum_{k=0}^{2n-1}\Big(e^{-i\,\frac{\pi}{2n}\,x}\Big)^k\Big) \;= 2\,\sin(\frac{\pi}{2}\,x)\,\mathsf{Im}\Big(e^{-i\,\frac{\pi}{2n}\,x}\,\frac{1}{e^{-i\,\frac{\pi}{2n}\,x}-1}\Big), \tag{A.2}$$

where in the last step we used that $\cos(\pi\, x) = -1$ and $\sin(\pi\, x) = 0$. We have that

$$\mathrm{Im}\left(e^{-i\,\frac{\pi}{2n}\,x}\,\frac{1}{e^{-i\,\frac{\pi}{2n}\,x}-1}\right) = \cos(\frac{\pi}{2n}\,x)\,\mathrm{Im}\left(\frac{1}{e^{-i\,\frac{\pi}{2n}\,x}-1}\right) - \sin(\frac{\pi}{2n}\,x)\,\mathrm{Re}\left(\frac{1}{e^{-i\,\frac{\pi}{2n}\,x}-1}\right). \tag{A.3}$$

Solving $\frac{1}{e^{-i\,\frac{\pi}{2n}\,x}-1} = a + ib$ for a and b gives $a = -\frac{1}{2}$ and $b = \frac{\sin(\frac{\pi}{2n}\,x)}{2-2\cos(\frac{\pi}{2n}x)}$. From (A.3), we thus obtain that

$$\mathrm{Im}\left(e^{-i\,\frac{\pi}{2n}\,x}\,\frac{1}{e^{-i\,\frac{\pi}{2n}\,x}-1}\right) = \frac{1}{2}\,\frac{\sin(\frac{\pi}{2n}\,x)}{1-\cos(\frac{\pi}{2n}\,x)} = \frac{1}{2}\,\cot(\frac{\pi}{4n}\,x). \tag{A.4}$$

From (A.1), (A.2) and (A.4) we deduce (3.10). □

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