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To cite this article: Shalva Amiranashvili and Elena Tobisch 2019 New J. Phys. 21 033029

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Extended criterion for the modulation instability

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Keywords: modilational instability, Lighthill criterion, four-wave mixing, generalized nonlinear Schrödinger equation

Abstract

PAPER

Modulation instability, following the classical Lighthill criterion, appears if nonlinearity and dispersion make opposite contributions to the wave frequency, e.g. in the framework of the one-dimensional nonlinear Schrödinger equation (NLSE). Several studies of the wave instabilities in optical fibers revealed four wave mixing instabilities that are not covered by the Lighthill criterion and require use of the generalized NLSE. We derive an extended criterion, which applies to all four wave interactions, covers arbitrary dispersion, and depends neither on the propagation equation nor on the slowly varying envelope approximation.

1. Introduction

Monochromatic waves are important special solutions of linear and weakly nonlinear dispersive systems. They may be destroyed by the growing self-modulations, appearance of such modulation instability (MI) is covered by the classical Lighthill criterion [1, 2]. MI discovery (for history see [3]) is related to the discovery of the nonlinear Schrödinger equation (NLSE) [4-10], which describes self-modulations making use of the slowly varying envelope approximation (SVEA).

Consider a modulated wave $\psi(z, t)e^{i(\beta_0 z - \omega_0 t)}$ in a one-dimensional setting, e.g. in an optical fiber. Its complex envelope $\psi(z, t)$ is described by the NLSE

$$\mathbf{i}(\partial_z \psi + \beta_1 \partial_t \psi) - \frac{\beta_2}{2} \partial_t^2 \psi + \gamma |\psi|^2 \psi = 0, \tag{1}$$

which maps $\psi|_{z=0}$ to $\psi|_{z=0}$. The dispersion coefficients β_i refer to the derivatives of the linear dispersion relation $k = \beta(\omega)$ at the carrier wave frequency

$$\beta_j = \beta^{(j)}(\omega_0), \qquad j = 0, 1, \dots, j_{\max}.$$
 (2)

 β_1 is the inverse group velocity, β_2 and γ quantify the group velocity dispersion (GVD) and the Kerr effect respectively [11].

A seed pump wave is given by $\psi = \sqrt{P_0} e^{i\gamma P_0 z}$, where P_0 is proportional to the pump power. The pump is destroyed by MI if

$$\beta_2 \gamma < 0, \tag{3}$$

which is the Lighthill criterion for the NLSE (1). MI manifests itself in the appearance of Stokes and anti-Stokes sideband waves first observed in water channels [6]. The sideband waves grow at the expense of the pump, generate a cascade, and are in turn destroyed by modulations [12–15].

MI in optical fibers was first observed in [16]. Single-mode fibers [17], which are in the focus of this work, offer important advantages for studies of nonlinear wave interactions. To derive the NLSE (1) for fibers, one rigorously eliminates two radial space coordinates [11]. Linear losses are small in the fiber transparency window such that $\beta(\omega)$ is actually real-valued. One can generate millions of pulses per second and collect their statistics to study MI and MI induced supercontinuum [18], wave turbulence [19], and rare extreme events [20, 21].

Dispersion of optical fibers, as opposed by water waves, is easy to manipulate [22]. Microstructured fibers may have several zero-dispersion frequencies (ZDFs) at which $\beta''(\omega) = 0$, which makes the criterion (3)

RECEIVED

OPEN ACCESS

29 June 2018 REVISED

7 January 2019

ACCEPTED FOR PUBLICATION 23 January 2019

PUBLISHED 28 March 2019

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degenerate. The spectral regions with minimum chromatic dispersion are of interest for optical communication. This triggered studies of MI for small or zero GVD by virtue of the generalized NLSE (GNLSE)

$$\mathrm{i}\partial_z \psi + \sum_{j=2}^{j_{\max}} \frac{\beta_j}{j!} (\mathrm{i}\partial_\tau)^j \psi + \gamma |\psi|^2 \psi = 0, \tag{4}$$

see equation (1), in which $\psi(z, t)$ is promoted to $\psi(z, \tau)$ with $\tau = t - \beta_1 z$. GNLSE (4) involves, e.g. 10 dispersion parameters [18] and goes beyond the SVEA [23, 24].

Without loss of generality we consider a focusing $\gamma > 0$ fiber. Parameter β_3 affects neither MI domain nor increment [25, 26]. MI at ZDF [27–29] where $\beta_2 = 0$ occurs if $\beta_4 < 0$. The competition of a small $\beta_2 \neq 0$ and β_4 may result in a new instability [30], the so-called four wave mixing (FWM) instability. It was observed in experiments [31–36] and further discussed in [37–41]. To summarize, a pump may be subject to

- (i) classical MI if (Lighthill) $\beta_2 < 0$,
- (ii) degenerate MI if $\beta_2 = 0$ and $\beta_4 < 0$,
- (iii) FWM instability if $\beta_2 \gtrsim 0$ and $\beta_4 < 0$.

The FWM instability is not covered by the classical criterion (3). In what follows we formulate an extended criterion. The new criterion avoids expansion of $\beta(\omega)$ and covers all three regimes.

2. Criterion

Both MI and FWM instability result from the resonant interaction of four waves. Two 'input' and two 'output' waves are involved in such interaction if [2, 42]

$$\omega_1 + \omega_2 = \omega_3 + \omega_4,\tag{5}$$

$$\beta(\omega_1) + \beta(\omega_2) = \beta(\omega_3) + \beta(\omega_4).$$
(6)

We set $\omega_{1,2} = \omega_0$ for the pump, $\omega_{3,4} = \omega_0 \pm \Omega$ for the anti-Stokes and Stokes daughter waves, and introduce [11] the wave vector mismatch \mathcal{M} versus offset Ω

$$\mathcal{M}(\Omega) = \frac{\beta(\omega_0 + \Omega) - 2\beta(\omega_0) + \beta(\omega_0 - \Omega)}{2}.$$
(7)

The SVEA $\Omega \ll \omega_0$ is not imposed. Equations (5)–(6) reduce to the phase matching condition

$$\mathcal{M}(\Omega) = 0, \tag{8}$$

which is independent on SVEA and NLSE. The phase matching condition was used to study MI at ZDF [27, 28] and FWM instability [36–38].

Equation (8) is insufficient for the instability, e.g. the solution $\Omega = 0$ may, but does not have to, yield MI. The classical MI condition (i) implies a local maximum of $\mathcal{M}(\Omega)$ at $\Omega = 0$. The mismatch is then locally negative. This is a key addition to equation (8), unstable sidebands result from small negative mismatches. The extended Lighthill criterion (ELC), which is the main result of this work, claims:

ELC The pump is unstable if for some frequency offset Ω_s both the mismatch vanishes, $\mathcal{M}(\Omega_s) = 0$, and $\mathcal{M}(\Omega) < 0$ for an interval of frequencies close to Ω_s .

The classical MI occurs if $\Omega_s = 0$, the FWM instability occurs if $\Omega_s \neq 0$. The ELC requires neither $\Omega_s \ll \omega_0$ nor the polynomial expansion of $\beta(\omega)$. It is consistent with all results reported in [27–40].

For instance, the mismatch shown in figure 1(a) should not lead to MI, yet considering larger offsets (figure 1(b)) we see that the pump is unstable. The mismatch in figure 1(c) leads to the classical MI, yet considering larger offsets (figure 1(d)) we see that two additional unstable bands appear. If both instabilities are present, MI dominates over the FWM instability. The latter can however appear without MI. Recall, that we consider a focusing fiber, for a defocusing one equation (8) should be combined with the locally positive mismatch.

3. Derivation

In this section we derive the ELC. To enjoy direct access to the wave vector mismatch, we use the GNLSE (4) in the expansion-independent form



Figure 1. Different shapes of $\mathcal{M}(\Omega)$ and frequency bands of the unstable offsets (red). (a), (c) Classical applications of the Lighthill criterion. (b), (d) New unstable Ω -bands appear for a more involved mismatch function.

$$i\partial_z \psi + \hat{\mathcal{D}}\psi + \gamma |\psi|^2 \psi = 0, \tag{9}$$

where the dispersion operator \hat{D} is defined in the frequency domain via $\hat{D}e^{-i\Omega\tau} = D(\Omega)e^{-i\Omega\tau}$ with

$$\mathcal{D}(\Omega) = \beta(\omega_0 + \Omega) - \beta_0 - \beta_1 \Omega$$

Linear losses are neglected, $\beta(\omega)$ is then real. If the SVEA applies, $\mathcal{D}(\Omega)$ is approximated by a truncated polynomial but this is not obligatory. Non-polynomial approximations were discussed in [43–46].

The pump solution of equation (9) will be used in the form

ſ

$$\psi = \sqrt{P_0} e^{ik_{nl}z}, \qquad k_{nl} = \gamma P_0, \tag{10}$$

where k_{nl} is a nonlinear correction to the pump wave vector. A perturbation, being imposed by two sideband waves with the frequencies $\omega_0 \pm \Omega$, reads

$$\psi(z,\tau) = \left[\sqrt{P_0} + u(z)e^{-i\Omega\tau} + \nu^*(z)e^{i\Omega\tau}\right]e^{ik_{nl}z},\tag{11}$$

behavior of u(z) and v(z) for $z \to \infty$ is of interest. With both u(z) and v(z) proportional to $e^{i\kappa z}$, a standard calculation leads to the characteristic equation

$$[\kappa - \mathcal{N}(\Omega)]^2 = \mathcal{M}(\Omega)[\mathcal{M}(\Omega) + 2k_{\rm nl}], \qquad (12)$$

the pump is unstable if a real Ω yields a complex κ . Here we split $\mathcal{D}(\Omega)$ into even and odd components [47]

$$\mathcal{M}(\Omega) = \frac{\mathcal{D}(\Omega) + \mathcal{D}(-\Omega)}{2}, \qquad \mathcal{N}(\Omega) = \frac{\mathcal{D}(\Omega) - \mathcal{D}(-\Omega)}{2}$$

The key observation is that the even component is identical to the mismatch (7).

A typical mismatch \mathcal{M} is much larger than k_{nl} . A generic Ω in equation (12) provides then $\kappa \in \mathbb{R}$. Frequency bands of the unstable perturbations are linked to the resonant offsets Ω_s with $\mathcal{M}(\Omega_s) = 0$. The sidebands are determined by the inequality

$$-2k_{\rm nl} < \mathcal{M}(\Omega) < 0. \tag{13}$$

They appear if the mismatch approaches zero from the negative side, which explains the ELC. This is illustrated in figure 2 for an exemplary Sellmeier dispersion law.

4. GNLSE with nonlinear dispersion

The GNLSE (9) utilizes a complicated operator \hat{D} to account for the linear dispersion, but uses one dispersionfree nonlinearity for all involved modulations. What happens, if nonlinearity depends on frequency? Note, that the sideband frequencies may considerably differ from ω_0 for the FWM instability. Therefore we generalize equation (9) to the form

$$i\partial_z \psi + \hat{\mathcal{D}}\psi + \gamma [f(|\psi|^2)\psi + |\psi|^2 \hat{\mathcal{A}}\psi + \psi^2 \hat{\mathcal{B}}\psi^*] = 0,$$
(14)





where the linear operators \hat{A} and \hat{B} quantify dispersion of the dominant nonlinearity. Higher-order nonlinear effects are included in $f(|\psi|^2)$, their dispersion is neglected. The leading order approximation of $f(|\psi|^2)$ is $|\psi|^2$.

The application-dependent operators $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$, being similar to $\hat{\mathcal{D}}$, will be represented by $\mathcal{A}(\Omega)$ and $\mathcal{B}(\Omega)$ in the frequency domain. $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ can, but do not have to, be approximated by differential operators. It is safe to set $\mathcal{A}(0) = \mathcal{B}(0) = 0$. In the SVEA limit we have $\hat{\mathcal{A}} \approx i\mathcal{A}'(0)\partial_{\tau}$ and $\hat{\mathcal{B}} \approx i\mathcal{B}'(0)\partial_{\tau}$.

The pump solution to equation (14), see equation (10), is

$$\psi = \sqrt{P_0} e^{ik_{\rm nl}z}, \qquad k_{\rm nl} = \gamma f(P_0), \tag{15}$$

its modulation is given by equation (11) with the new k_{nl} . The GNLSE (14) ignores terms like $(\partial_{\tau}\psi)^2\psi^*$ or $|\partial_{\tau}\psi|^2\psi$, as they appear in, e.g. the Lakshmanan–Porsezian–Daniel equation [48]. Such terms are quadratic in u(z) and v(z), they have no effect on pump modulations.

Equation (14) comprises the following special cases.

(A) The self-steepening [11] is reproduced by

$$f(|\psi|^2) = |\psi|^2, \quad \mathcal{A}(\Omega) = 2\Omega/\omega_0, \quad \mathcal{B}(\Omega) = \Omega/\omega_0,$$
$$i\partial_z \psi + \hat{\mathcal{D}}\psi + \gamma(1 + i\omega_0^{-1}\partial_\tau)|\psi|^2\psi = 0.$$

In a similar way one can obtain the derivative NLSE [49].

(B) For the Hirota equation [50] one should use

$$f(|\psi|^2) = |\psi|^2, \quad \mathcal{A}(\Omega) = T_H\Omega, \quad \mathcal{B}(\Omega) = 0,$$

and set $T_H = \beta_3 / \beta_2$ with $\beta_{j \ge 4} = 0$ to derive

$$\mathrm{i}\partial_z\psi + \sum_{j=2,3}\frac{\beta_j}{j!}(\mathrm{i}\partial_\tau)^j\psi + \gamma|\psi|^2\left(\psi + \mathrm{i}\frac{\beta_3}{\beta_2}\partial_\tau\psi\right) = 0.$$

One can also obtain a Sasa-Satsuma equation [51].

(C) Saturation of $f(|\psi|^2)$ was considered in [52]. Quadratic $\mathcal{A}(\Omega)$ and $\mathcal{B}(\Omega)$ were discussed in [53, 54]. Complex-valued $\mathcal{A}(\Omega)$, $\mathcal{B}(\Omega)$ yield non-Hermitian $\hat{\mathcal{A}}$, $\hat{\mathcal{B}}$, an evidence of the nonlinear losses (or gain).

(D) The interpulse Raman scattering [11] results in nonlinear losses, it is approximated by

$$f(|\psi|^2) = |\psi|^2, \quad \mathcal{A}(\Omega) = \mathcal{B}(\Omega) = iT_R\Omega,$$

$$i\partial_z \psi + \hat{\mathcal{D}}\psi + \gamma(|\psi|^2 - T_R\partial_\tau |\psi|^2)\psi = 0.$$

The value of T_R depends on material.

(E) To give a more involved example, consider GNLSE that fully accounts for the Raman scattering [47]

$$\mathrm{i}\partial_z\psi+\hat{\mathcal{D}}\psi+\gamma\bigg[f_K|\psi|^2+f_R\int_0^\infty h(s)|\psi_s|^2\mathrm{d}s\bigg]\psi=0,$$

where $\psi = \psi(z, \tau)$ and $\psi_s = \psi(z, \tau - s)$. Here f_K and f_R rank Kerr and Raman contributions, $f_K + f_R = 1$, and $\int_0^\infty h(s) ds = 1$. An equivalent representation is

$$i\partial_z \psi + \hat{\mathcal{D}}\psi + \gamma [|\psi|^2 + \hat{\mathcal{R}}|\psi|^2]\psi = 0,$$

with

$$\mathcal{R}(\Omega) = f_R \left[\int_0^\infty h(\tau) e^{i\Omega\tau} d\tau - 1 \right], \quad \mathcal{R}(0) = 0, \quad \mathcal{R}(\Omega) = \mathcal{R}^*(-\Omega).$$

We then deal with a non-polynomial complex-valued $\mathcal{R}(\Omega)$. The SVEA limit of $\hat{\mathcal{R}}$ is $-T_R \partial_{\tau}$, in general

$$\hat{\mathcal{R}}|\psi|^2 = \psi^* \hat{\mathcal{R}} \psi + \psi^* \hat{\mathcal{R}} \psi + ext{h.o.t.}$$

where, as explained after equation (15), the higher-order terms have no influence on the small pump modulations. One can then set $\hat{A} = \hat{B} = \hat{R}$ and employ the GNLSE (14).

5. MI for nonlinear dispersion

To summarize the previous section, equation (14) covers many if not all relevant modifications of equation (9). To address stability of the pump (15), we now insert the modulated pump (11) into equation (14), ignore all terms that are quadratic in u(z) and v(z), and obtain

$$\begin{bmatrix} i\partial_z + \Delta(\Omega) & \delta(\Omega) \\ \delta^*(-\Omega) & -i\partial_z + \Delta^*(-\Omega) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0,$$
(16)

with

$$\Delta(\Omega) = \gamma P_0[f'_0 + \mathcal{A}(\Omega)] + \mathcal{D}(\Omega), \qquad \delta(\Omega) = \gamma P_0[f'_0 + \mathcal{B}(\Omega)]$$

where $f'_0 = f'(P_0)$. Finally we insert u, $v \propto e^{i\kappa z}$ into equation (16) and derive the characteristic equation for κ

$$(\kappa - N - \gamma P_0 \mathcal{A}_a)^2 = \mathcal{M}[\mathcal{M} + 2\gamma P_0 (f'_0 + \mathcal{A}_s)] + \gamma^2 P_0^2 [(f'_0 + \mathcal{A}_s)^2 - (f'_0 + \mathcal{B}_s)^2 + \mathcal{B}_a^2],$$
(17)

where we introduced a symmetric $\mathcal{A}_{s}(\Omega)$ and asymmetric $\mathcal{A}_{a}(\Omega)$ components of $\mathcal{A}(\Omega)$, namely

$$\mathcal{A}_{s,a}(\Omega) = \frac{\mathcal{A}(\Omega) \pm \mathcal{A}^{*}(-\Omega)}{2}, \qquad \mathcal{A}_{s,a}(-\Omega) = \pm \mathcal{A}_{s,a}^{*}(\Omega),$$

and the same for $\mathcal{B}(\Omega)$.

Equation (17), extending equation (12), yields the unstable frequency bands for a given linear dispersion $\mathcal{D}(\Omega)$, nonlinearity $f(|\psi|^2)$, pump power P_0 , and nonlinear dispersion $\mathcal{A}(\Omega)$, $\mathcal{B}(\Omega)$. Let us now discuss how the nonlinear dispersion affects the ELC.

For real-valued $\mathcal{A}(\Omega)$ and $\mathcal{B}(\Omega)$, the unstable bunds predicted by equation (17) are still located close to the resonant frequencies at which $\mathcal{M}(\Omega)$ vanishes. The bands are slightly squeezed and shifted, as compared to those in figure 1, but never disappear completely. The ELC is perfectly applicable.

For complex-valued $\mathcal{A}(\Omega)$ and $\mathcal{B}(\Omega)$, the pump is formally unstable for all possible modulations. Consider, e.g. the GNLSE in the example (E), where $f'_0 = 1$ and $\mathcal{A}(\Omega) = \mathcal{B}(\Omega) = \mathcal{R}(\Omega)$. Equation (17) yields [39, 40]

$$(\kappa - \mathcal{N})^2 = \mathcal{M}[\mathcal{M} + 2\gamma P_0(1 + \mathcal{R})], \tag{18}$$

and $\kappa(\Omega)$ with Im $\kappa(\Omega) < 0$ is yielded by any Ω if only Im $\mathcal{R}(\Omega) \neq 0$. This is known as Raman gain [47]. The pump is destroyed because the nonlinear dissipation drains its energy, the new instability peaks appear where the drain is most efficient. These peaks, being still described by equation (17), are independent on wave mixing and ELC. They impose over the 'true' MI and FWM bands (figure 3) and vanish, if one ignores Im \mathcal{R} .

To summarize, ELC covers the instabilities that result from nonlinear interactions of four waves, be it linear or nonlinear dispersion. However, it does not cover the effect of the nonlinear gain or dissipation.

6. Discussion

MI is traditionally described by the NLSE, which depends on the GVD parameter. The classical Lighthill criterion predicts MI for a negative GVD in a focusing fiber. Fibers with minimum chromatic dispersion are not covered: a pump wave may go unstable for a positive GVD. Moreover, the growing daughter waves may be separated from the pump in the frequency domain (figure 2), which makes use of the NLSE questionable.

It is well known that both the spectrally-wide wave packets and all FWM instabilities in fibers are properly described by the GNLE. A natural question is whether it is possible to generalize the classical Lighthill criterion. We have found that it is convenient to promote the GVD to the mismatch of the FWM





$$\beta''(\omega_0) \quad \mapsto \quad \frac{\beta(\omega_0 + \Omega) - 2\beta(\omega_0) + \beta(\omega_0 - \Omega)}{2}$$

where the modulation frequency Ω may, but does not have to, be much smaller than the carrier frequency ω_0 . Pump stability is linked to a simple geometric property of the mismatch (ELC, figure 1), which generalizes the classical Lighthill criterion.

Last but no least, if $\Omega \leq \omega_0$, one has to address the nonlinear dispersion. We phenomenologically introduced equation (14), and demonstrated that it properly accounts for the nonlinear dispersion in all cases relevant for optical fibers, ranging from the integrable Schrödinger-type equations to dissipative models accounting for the Raman scattering. The resulting characteristic equation (17) provides the most general MI description in one spatial dimension. For instance, equation (17) is symmetric under the replacement $(\Omega, \kappa) \mapsto (-\Omega, -\kappa^*)$, which preserves Im κ . Therefore Stokes and anti-Stokes waves always grow with the same rate. The reported results, being primarily applicable to optical fibers, are of interest for many other nonlinear wave systems, e.g. for gravity water waves [55–57].

Acknowledgments

The authors thank A Vladimirov for discussions, S Pickartz for careful reading of the manuscript, and anonymous referees for useful remarks. Support of the German Research Foundation (DFG project 389251150) and the Austrian Science Foundation (FWF projects P31163 and P30887) is gratefully acknowledged.

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