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A local projection stabilization finite element method with nonlinear crosswind diffusion for convection-diffusion-reaction equations

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Abstract

An extension of the local projection stabilization (LPS) finite element method for convection-diffusion-reaction equations is presented and analyzed, both in the steady-state and the transient setting. In addition to the standard LPS method, a nonlinear crosswind diffusion term is introduced that accounts for the reduction of spurious oscillations. The existence of a solution can be proved and, depending on the choice of the stabilization parameter, also its uniqueness. Error estimates are derived which are supported by numerical studies. These studies demonstrate also the reduction of the spurious oscillations.

1. Introduction

The solution of convection-dominated convection-diffusion-reaction equations with finite element methods constitutes a very challenging (and open) problem. Over the last three decades, the amount of work devoted to this problem is impressive. The usual way of treating dominating convection, at least in the context of finite element methods, consists in adding extra terms to the standard Galerkin formulation, aimed at enhancing the stability of the discrete solution by means of introducing artificial diffusion. These new terms vary according to the method, and can be residual-based, as in the SUPG/GLS/SDFEM family (see [6, 16, 13, 14, 27]), or edge based, such as the CIP method (see [9, 7]). For an up-to-date and thorough review of these and other techniques, see [29]. It is striking to notice that, despite the impressive amount of work that has been devoted to this topic, up to now there is not a method that 'ticks all the boxes', i.e., a method that produces sharp layers while avoiding oscillations, see [1] for a recent review and a numerical assessment.

Among the various stabilized finite element methods, the local projection stabilization (LPS) method has received some attention over the last decade. Originally proposed for the Stokes problem in [2], and extended to the Oseen equations in [4] (see also [5, 28]), the LPS method has also been used recently to treat convection-diffusion equations (see

[24, 15, 22, 23]). The basic idea of this method consists in restricting the direct application of the stabilization to so-called fluctuations or resolved small scales, which are defined by local projections. It has several attractive features, such as adding symmetric terms to the formulation and avoiding the computation of second derivatives of the basis functions (thus using only information that is needed for the assembly of the matrices from the standard Galerkin method). Unfortunately, the solutions obtained with the LPS method possess the same deficiency like solutions computed, e.g., with the SUPG method: non-negligible spurious oscillations are often present in a vicinity of layers.

Motivated by the wish of recovering the monotonicity properties of the continuous problem, which might be crucial in applications, a number of so-called Spurious Oscillations at Layers Diminishing (SOLD) methods were proposed. SOLD methods add an extra term to the already stabilized formulation, which usually depends on the discrete solution in a nonlinear way, vanishes for small residuals (thus acting mostly at layers), and adds some extra, but different, diffusivity to the formulation. In particular, methods that add crosswind diffusion, like the one proposed in [11], have been proved to belong to the best SOLD methods in comprehensive studies [17, 18]. Although these methods diminish oscillations considerably, no single method succeeds to fully eliminate them [17, 18, 21]. Also, from a purely mathematical point of view, it is unknown if these methods lead to well-posed problems. In fact, existence of solutions is usually possible to prove, but, to our best knowledge, there is no nonlinear SOLD method that is known to produce a unique solution, see [25] and [7] for a discussion of this topic.

This paper proposes a LPS method with nonlinear crosswind diffusion for convection-diffusion-reaction equations. The crosswind diffusion term is chosen in such a way that, for a certain choice of the stabilization parameter, the existence and the uniqueness of the solution can be proved for the steady-state equation and for the time-dependent equation, which is discretized in time with an implicit one-step θ-scheme. To our best knowledge, this is the first nonlinear discretization for convection-diffusion-reaction equations for that both, existence and uniqueness of a solution can be shown. The form of the crosswind term is motivated by the Smagorinsky Large Eddy Simulation (LES) model which was analyzed in [26]. It involves fluctuations of a term mimicking a p-Laplacian. The crucial analytical property for proving the uniqueness of the solution is the strong monotonicity of the corresponding operator. In addition, a second variant of the stabilization parameter is studied, whose proposal is based on scaling arguments. For this parameter, the existence of a solution can be proved and the uniqueness for the time-dependent equation in the case of sufficiently small time steps.

The plan of the paper is as follows. In the remaining part of this introduction, the problems of interest are stated and some basic notations are given. Section 2 will summarize the main abstract hypothesis imposed on the different partitions of the domain and the finite element spaces considered. Section 3 presents the method for the steady-state case, whose well-posedness is analyzed in Section 3.1 and convergence and error estimates are presented in Section 3.2. In Section 4, the method for the time-dependent problem is presented. Well-posedness and stability are proved in Section 4.1 and error estimates in Section 4.2. Since the analysis is based on the abstract framework from Section 2, Section 5 presents some concrete examples that fit into this framework. Finally, numerical illustrations that support the analytical results and which demonstrate the reduction of spurious oscillations are presented in Section 6.

Throughout the paper, standard notations are used for Sobolev spaces and corresponding norms, see, e.g., [10]. In particular, given a measurable set $D \subset \mathbb{R}^d$, the inner product in $L^2(D)$ or $L^2(D)^d$ is denoted by $(\cdot,\cdot)_D$ and the notation $(\cdot,\cdot)$ is used instead of $(\cdot,\cdot)_\Omega$. The norm (seminorm) in $W^{m,p}(D)$ will be denoted by $\|\cdot\|_{m,p,D}$ ($|\cdot|_{m,p,D}$), with the convention $\|\cdot\|_{m,D} = \|\cdot\|_{m,2,D}$, and the same notation is used for scalar and vector-valued functions.

1.1. **The problems of interest.** Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded polygonal (polyhedral) domain with a Lipschitz-continuous boundary $\partial\Omega$ and let us consider the steady-state convection-diffusion-reaction equation

$$-\varepsilon\,\Delta u + \boldsymbol{b}\cdot\nabla u + c\,u = f \quad \text{in } \Omega\,, \qquad\qquad u = u_b \quad \text{on } \partial\Omega\,. \tag{1}$$

It is assumed that ε is a positive constant and $\boldsymbol{b} \in W^{1,\infty}(\Omega)^d$, $c \in L^\infty(\Omega)$, $f \in L^2(\Omega)$, and $u_b \in H^{1/2}(\partial\Omega)$ are given functions satisfying

$$\sigma := c - \frac{1}{2}\nabla\cdot\boldsymbol{b} \ge \sigma_0 > 0 \quad \text{in } \Omega\,, \tag{2}$$

where σ_0 is a constant. Then the boundary value problem (1) has a unique solution in $H^1(\Omega)$.

Besides the steady-state case, also the time-dependent convection-diffusion-reaction equation

$$\left\{ \begin{array}{rcll} u_t - \varepsilon\,\Delta u + \boldsymbol{b}\cdot\nabla u + c\,u &=& f & \text{in } (0,T]\times\Omega, \\ u &=& u_b & \text{in } [0,T]\times\partial\Omega, \\ u(0,\cdot) &=& u_0 & \text{in } \Omega, \end{array} \right. \tag{3}$$

will be considered. In (3), $[0,T]$ is a finite time interval, ε is assumed to be a positive constant, $\boldsymbol{b}(\cdot,t) \in W^{1,\infty}(\Omega)^d$, $c(\cdot,t) \in L^\infty(\Omega)$, $f(\cdot,t) \in L^2(\Omega)$, $u_b(\cdot,t) \in H^{1/2}(\partial\Omega)$

for all $t \in [0,T]$, and $u_0 \in H^1(\Omega)$ denotes the initial condition. Moreover, it is assumed that $\boldsymbol{b} \in L^\infty(0,T;W^{1,\infty}(\Omega)^d)$, $c \in L^\infty(0,T;L^\infty(\Omega))$, $f \in L^2(0,T;L^2(\Omega))$, and $u_b \in L^2(0,T;H^{1/2}(\partial\Omega))$. The function σ is defined analogously to (2) and the inequality (2) is assumed to hold for all $t \in [0,T]$.

2. Assumptions on approximation spaces and the set $\mathscr{M}_h$

From now on, C, $\tilde{C}$ or $\bar{C}$ denote generic constants which may take different values at different occurrences but are always independent of the data ε, $\boldsymbol{b}$, c, f, and u_b and the discretization parameters (h and δt in the following).

Given $h > 0$, let $W_h \subset W^{1,\infty}(\Omega)$ be a finite-dimensional space approximating the space $H^1(\Omega)$ and set $V_h = W_h \cap H^1_0(\Omega)$. Next, let $\mathscr{M}_h$ be a set consisting of a finite number of open subsets M of Ω such that $\overline{\Omega} = \cup_{M\in\mathscr{M}_h} \overline{M}$. It will be supposed that, for any $M \in \mathscr{M}_h$,

$$\operatorname{card}\{M' \in \mathscr{M}_h\,;\ M \cap M' \neq \emptyset\} \leq C\,, \tag{4}$$

$$h_M := \operatorname{diam}(M) \leq C\,h\,, \tag{5}$$

$$h_M \leq C\,h_{M'} \qquad \forall\ M' \in \mathscr{M}_h,\ M \cap M' \neq \emptyset\,, \tag{6}$$

$$h_M^d \leq C\operatorname{meas}_d(M)\,. \tag{7}$$

The space W_h is assumed to satisfy the local inverse inequality

$$|v_h|_{1,M} \leq C\,h_M^{-1}\,\|v_h\|_{0,M} \qquad \forall\ v_h \in W_h,\ M \in \mathscr{M}_h\,. \tag{8}$$

For any $M \in \mathscr{M}_h$, a finite-dimensional space $D_M \subset L^\infty(M)$ is introduced. It is assumed that there exists a positive constant β_{LP} independent of h such that

$$\sup_{v\in V_M} \frac{(v,q)_M}{\|v\|_{0,M}} \geq \beta_{LP}\,\|q\|_{0,M} \qquad \forall\ q \in D_M,\ \ M \in \mathscr{M}_h\,, \tag{9}$$

where $V_M = \{v_h \in V_h\,;\ v_h = 0 \text{ in } \Omega \setminus M\}$. This hypothesis will be needed in what follows for the construction of a special interpolation operator (see Lemma 6 below). Concrete examples of spaces W_h and D_M satisfying the assumptions formulated here will be presented in Section 5.

Furthermore, for any $M \in \mathscr{M}_h$, a finite-dimensional space $G_M \subset L^\infty(M)$ with $G_M \supset D_M$ is introduced such that

$$\left.\frac{\partial v_h}{\partial x_i}\right|_M \in G_M \qquad \forall\ v_h \in W_h,\ \ i = 1,\dots,d\,,$$

and it is assumed that, for any $p \in [1,\infty]$, there is a constant C such that

$$\|q\|_{0,p,M} \leq C\,h_M^{\frac{d}{p}-\frac{d}{2}}\,\|q\|_{0,M} \qquad \forall\ q \in G_M,\ M \in \mathscr{M}_h\,. \tag{10}$$

To characterize the approximation properties of the spaces W_h and D_M, it is assumed that there exist interpolation operators $i_h \in \mathscr{L}(H^2(\Omega), W_h) \cap \mathscr{L}(H^2(\Omega) \cap H^1_0(\Omega), V_h)$ and $j_M \in \mathscr{L}(H^1(M), D_M)$, $M \in \mathscr{M}_h$, such that, for some constants $l \in \mathbb{N}$ and $C > 0$ and for any set $M \in \mathscr{M}_h$, it holds

$$|v - i_h v|_{1,M} + h_M^{-1}\, \|v - i_h v\|_{0,M} \le C\, h_M^k\, |v|_{k+1,M} \qquad \forall\, v \in H^{k+1}(M),\ k = 1, \dots, l\,, \tag{11}$$

$$\|q - j_M q\|_{0,M} \le C\, h_M^k\, |q|_{k,M} \qquad \forall\, q \in H^k(M),\ k = 1, \dots, l\,. \tag{12}$$

In addition, it is assumed that, for any $p \in [1, 6]$,

$$|v - i_h v|_{1,p,M} \le C\, h_M^{k+\frac{d}{p}-\frac{d}{2}}\, |v|_{k+1,M} \qquad \forall\, v \in H^{k+1}(M),\ k = 1, \dots, l\,. \tag{13}$$

3. A local projection discretization of the steady-state problem

The weak form of problem (1) is: Find $u \in H^1(\Omega)$ such that $u = u_b$ on $\partial\Omega$ and

$$a(u, v) = (f, v) \qquad \forall\, v \in H^1_0(\Omega)\,, \tag{14}$$

where the bilinear form a is given by

$$a(u, v) := \varepsilon\, (\nabla u, \nabla v) + (\boldsymbol{b} \cdot \nabla u, v) + (c\, u, v)\,.$$

As it was mentioned in the introduction, the most often used approach to cure the instabilities of the Galerkin method consists in adding extra terms to the formulation. To build these additional terms for the method studied here, for any $M \in \mathscr{M}_h$, a continuous linear projection operator π_M is introduced which maps the space $L^2(M)$ onto the space D_M. It is assumed that

$$\|\pi_M\|_{\mathscr{L}(L^2(M), L^2(M))} \le C \qquad \forall\, M \in \mathscr{M}_h\,. \tag{15}$$

E.g., if π_M is the orthogonal L^2 projection, then $C = 1$. Using this operator, the fluctuation operator $\kappa_M := id - \pi_M$ is defined, where id is the identity operator on $L^2(M)$. Then, clearly

$$\|\kappa_M\|_{\mathscr{L}(L^2(M), L^2(M))} \le C \qquad \forall\, M \in \mathscr{M}_h\,. \tag{16}$$

Since κ_M vanishes on D_M, it follows from (16) and (12) that

$$\|\kappa_M\, q\|_{0,M} \le C\, h_M^k\, |q|_{k,M} \qquad \forall\, q \in H^k(M),\ M \in \mathscr{M}_h,\ k = 0, \dots, l\,. \tag{17}$$

An application of κ_M to a vector-valued function means that κ_M is applied component-wise.

For any $M \in \mathscr{M}_h$, a constant $\boldsymbol{b}_M \in \mathbb{R}^d$ is chosen such that

$$|\boldsymbol{b}_M| \le \|\boldsymbol{b}\|_{0,\infty,M}\,, \qquad \|\boldsymbol{b} - \boldsymbol{b}_M\|_{0,\infty,M} \le C\, h_M\, |\boldsymbol{b}|_{1,\infty,M}\,. \tag{18}$$

A typical choice for $\boldsymbol{b}_M$ is the value of $\boldsymbol{b}$ at one point of M, or the integral mean value of $\boldsymbol{b}$ over M. In addition, a function $\widetilde{u}_{bh} \in W_h$ is introduced such that its trace approximates the boundary condition u_b.

We are now ready to present the finite element method to be studied: Find $u_h \in W_h$ such that $u_h - \widetilde{u}_{bh} \in V_h$ and

$$a(u_h, v_h) + s_h(u_h, v_h) + d_h(u_h; u_h, v_h) = (f, v_h) \qquad \forall\ v_h \in V_h\,, \tag{19}$$

where

$$s_h(u, v) = \sum_{M \in \mathscr{M}_h} \tau_M\, (\kappa_M(\boldsymbol{b}_M \cdot \nabla u), \kappa_M(\boldsymbol{b}_M \cdot \nabla v))_M\,,$$
$$d_h(w; u, v) = \sum_{M \in \mathscr{M}_h} (\tau_M^{\mathrm{sold}}(w)\, \kappa_M(P_M \nabla u), \kappa_M(P_M \nabla v))_M\,,$$

and $P_M : \mathbb{R}^d \to \mathbb{R}^d$ is the projection onto the line (plane) orthogonal (crosswind) to the vector $\boldsymbol{b}_M$ defined by

$$P_M = \begin{cases} I - \dfrac{\boldsymbol{b}_M \otimes \boldsymbol{b}_M}{|\boldsymbol{b}_M|^2} & \text{if } \boldsymbol{b}_M \neq \boldsymbol{0}, \\ 0 & \text{if } \boldsymbol{b}_M = \boldsymbol{0}, \end{cases}$$

I being the identity tensor. The stabilization parameters are given by

$$\tau_M = \tau_0\ \min\left\{ \frac{h_M}{\|\boldsymbol{b}\|_{0,\infty,M}}, \frac{h_M^2}{\varepsilon} \right\}, \tag{20}$$
$$\tau_M^{\mathrm{sold}}(u_h) = \tilde{\tau}_M(u_h)\, |\kappa_M(P_M \nabla u_h)|\,,$$

where τ_0 is a positive constant and $\tilde{\tau}_M$ is a non-negative function of u_h and the data of (1). In particular, we shall investigate the properties of the discrete problem for

$$\tilde{\tau}_M = \beta\, h_M\, |\boldsymbol{b}_M|\,, \tag{21}$$

and for

$$\tilde{\tau}_M(u_h) = \begin{cases} \dfrac{\beta\, h_M^{1+d/2}\, |\boldsymbol{b}_M|}{|u_h|_{1,M}} & \text{if } |u_h|_{1,M} \neq 0, \\ 0 & \text{if } |u_h|_{1,M} = 0, \end{cases} \tag{22}$$

where β is a positive constant. The power of h_M in (22) assures a proper scaling of the parameter τ_M^{sold} with respect to the length scale of the problem. Note that the crosswind stabilization term is of p-Laplacian type with $p = 3$.

Remark.

- If $d = 2$ and $\boldsymbol{b}_M \neq \mathbf{0}$, one has $P_M = \boldsymbol{b}_M^\perp \otimes \boldsymbol{b}_M^\perp$ where $\boldsymbol{b}_M^\perp$ is a vector satisfying $\boldsymbol{b}_M^\perp \cdot \boldsymbol{b}_M = 0$ and $|\boldsymbol{b}_M^\perp| = 1$. Thus, in this case, the nonlinear stabilization term can be written in the form

$$d_h(w; u, v) = \sum_{M \in \mathscr{M}_h} (\tau_M^{\text{sold}}(w)\, \kappa_M(\boldsymbol{b}_M^\perp \cdot \nabla u), \kappa_M(\boldsymbol{b}_M^\perp \cdot \nabla v))_M\,.$$

- It is useful for the analysis of the discrete problem to note that $\kappa_M(\boldsymbol{b}_M \cdot \nabla u) = \boldsymbol{b}_M \cdot \kappa_M \nabla u$ and $\kappa_M(P_M \nabla u) = P_M \kappa_M \nabla u$. Note also that $\|P_M\|_2 = 1$.
- Finally, if $\tilde{\tau}_M$ is defined by (22), then, using (18), (16), and $\|P_M\|_2 = 1$, one obtains

$$\|\tau_M^{\text{sold}}(v)\|_{0,M} \le C\, h_M^{1+d/2}\, \|\boldsymbol{b}\|_{0,\infty,M} \qquad \forall\, v \in H^1(\Omega),\ M \in \mathscr{M}_h\,. \tag{23}$$

In the analysis, the error will be measured using the following mesh-dependent norm

$$\|v\|_{\text{LPS}} := \left(\varepsilon\, |v|_{1,\Omega}^2 + \|\sigma^{1/2}\, v\|_{0,\Omega}^2 + s_h(v, v)\right)^{1/2}\,,$$

and a term involving the crosswind derivative of the error. Note that integrating by parts gives

$$a(v, v) + s_h(v, v) = \|v\|_{\text{LPS}}^2 \qquad \forall\, v \in H_0^1(\Omega)\,. \tag{24}$$

3.1. **Well-posedness of the nonlinear discrete problem.** This section studies the existence and uniqueness of solutions for the nonlinear discrete problem (19). Let us define the nonlinear operator $T_h : V_h \to V_h$ by

$$(T_h z_h, v_h) = a(z_h + \widetilde{u}_{bh}, v_h) + s_h(z_h + \widetilde{u}_{bh}, v_h) + d_h(z_h + \widetilde{u}_{bh}; z_h + \widetilde{u}_{bh}, v_h) - (f, v_h) \tag{25}$$

for any $z_h, v_h \in V_h$. Then $u_h \in W_h$ is a solution of (19) if and only if $u_h|_{\partial\Omega} = \widetilde{u}_{bh}|_{\partial\Omega}$ and

$$T_h(u_h - \widetilde{u}_{bh}) = 0\,,$$

or, equivalently, $u_h = \widetilde{u}_h + \widetilde{u}_{bh} \in W_h$ is a solution of (19) if $\widetilde{u}_h \in V_h$ and $T_h(\widetilde{u}_h) = 0$. Thus, our aim is to prove that the operator T_h has a zero in V_h. To this end, the properties of the form d_h shall be investigated first. As these properties are different with respect to the definition of $\tilde{\tau}_M$, we start supposing that $\tilde{\tau}_M$ is given by (21).

Lemma 1. *Let $\tilde{\tau}_M$ be defined by (21). Consider any $u, v, z \in W^{1,3}(\Omega)$ and set $w := u - v$. Then*

$$d_h(u; u, w) - d_h(v; v, w) \geq \frac{1}{7} \sum_{M \in \mathscr{M}_h} \tilde{\tau}_M \, \|\kappa_M(P_M \nabla w)\|_{0,3,M}^3 = \frac{1}{7} \, d_h(w; w, w)\,, \tag{26}$$

$$\begin{aligned} |d_h(u; u, z) - d_h(v; v, z)| \leq \sum_{M \in \mathscr{M}_h} \tilde{\tau}_M \, (\|\kappa_M(P_M \nabla u)\|_{0,3,M} + \|\kappa_M(P_M \nabla v)\|_{0,3,M}) \times \\ \times \|\kappa_M(P_M \nabla w)\|_{0,3,M} \, \|\kappa_M(P_M \nabla z)\|_{0,3,M}\,. \end{aligned} \tag{27}$$

Proof. Let us denote

$$d_h(u; u, z) - d_h(v; v, z) = \sum_{M \in \mathscr{M}_h} N_M(u, v, z)\,, \tag{28}$$

where

$$N_M(u, v, z) := \big(\tau_M^{\mathrm{sold}}(u) \, \kappa_M(P_M \nabla u) - \tau_M^{\mathrm{sold}}(v) \, \kappa_M(P_M \nabla v), \kappa_M(P_M \nabla z)\big)_M\,.$$

For $t \in [0, 1]$, let us define $u^t := tu + (1 - t)v$ and set

$$g(t) := \tilde{\tau}_M \, |\kappa_M(P_M \nabla u^t)| \, \kappa_M(P_M \nabla u^t)\,, \qquad t \in [0, 1]\,.$$

Then

$$N_M(u, v, z) = \big(g(1) - g(0), \kappa_M(P_M \nabla z)\big)_M = \left(\int_0^1 g'(t) \, \mathrm{d}t, \kappa_M(P_M \nabla z) \right)_M\,.$$

Since

$$g'(t) = \tilde{\tau}_M \, \frac{\kappa_M(P_M \nabla u^t)}{|\kappa_M(P_M \nabla u^t)|} \, \kappa_M(P_M \nabla u^t) \cdot \kappa_M(P_M \nabla w) + \tilde{\tau}_M \, |\kappa_M(P_M \nabla u^t)| \, \kappa_M(P_M \nabla w)\,,$$

one has

$$\begin{aligned} |g'(t)| &\leq 2 \, \tilde{\tau}_M \, |\kappa_M(P_M \nabla u^t)| \, |\kappa_M(P_M \nabla w)| \\ &\leq 2 \, \tilde{\tau}_M \, (t \, |\kappa_M(P_M \nabla u)| + (1 - t) \, |\kappa_M(P_M \nabla v)|) \, |\kappa_M(P_M \nabla w)|\,, \end{aligned}$$

which implies (27). On the other hand,

$$N_M(u, v, w) \geq \left(\tilde{\tau}_M \int_0^1 |\kappa_M(P_M \nabla u^t)| \, \mathrm{d}t \, \kappa_M(P_M \nabla w), \kappa_M(P_M \nabla w) \right)_M\,. \tag{29}$$

Next, clearly

$$\int_0^1 |\kappa_M(P_M \nabla u^t)| \, \mathrm{d}t \geq \max_{i=1,\ldots,d} \int_0^1 |t \, \kappa_M(P_M \nabla u)_i + (1 - t) \, \kappa_M(P_M \nabla v)_i| \, \mathrm{d}t\,.$$

Denoting

$$I(a, b) = \int_0^1 |ta + (1 - t)b| \, \mathrm{d}t\,, \qquad a, b \in \mathbb{R}\,,$$

a direct computation gives

$$I(a,b) = \frac{|a|+|b|}{2} \quad \text{if } a\,b \geq 0\,, \qquad I(a,b) = \frac{1}{2}\,\frac{a^2+b^2}{|a|+|b|} \quad \text{if } a\,b < 0\,.$$

Thus, for any $a, b \in \mathbb{R}$, it follows

$$I(a,b) \geq \frac{|a|+|b|}{4} \geq \frac{|a-b|}{4}\,.$$

Consequently,

$$\int_0^1 |\kappa_M(P_M\nabla u^t)|\,\mathrm{d}t \geq \frac{1}{4}\max_{i=1,\dots,d}|\kappa_M(P_M\nabla w)_i| \geq \frac{1}{4\sqrt{d}}\,|\kappa_M(P_M\nabla w)| \geq \frac{1}{7}\,|\kappa_M(P_M\nabla w)|\,.$$

Combining this estimate with (29) and using (28) gives (26). □

Next, the properties of d_h are explored for the case that $\tilde{\tau}_M$ is defined by (22).

Lemma 2. *Let $\tilde{\tau}_M$ be defined by (22). Consider any $u, v, z \in W^{1,4}(\Omega)$. Then*

$$|d_h(u;v,z)| \leq C \sum_{M\in\mathscr{M}_h} h_M^{1+d/2}\,\|\boldsymbol{b}\|_{0,\infty,M}\,\|\kappa_M(P_M\nabla v)\|_{0,4,M}\,\|\kappa_M(P_M\nabla z)\|_{0,4,M}\,, \tag{30}$$

$$\begin{aligned}|d_h(u;u,z) - d_h(v;v,z)| \leq C \sum_{M\in\mathscr{M}_h} h_M^{1+d/2}\,\|\boldsymbol{b}\|_{0,\infty,M}\,\zeta_M(u,v)\times \\ \times\left(\|\kappa_M(P_M\nabla u)\|_{0,4,M} + \|\kappa_M(P_M\nabla v)\|_{0,4,M}\right)\|\kappa_M(P_M\nabla z)\|_{0,4,M}\,,\end{aligned} \tag{31}$$

where

$$\zeta_M(u,v) = \begin{cases} \dfrac{|u-v|_{1,M}}{|u|_{1,M}+|v|_{1,M}} & \text{if } |u|_{1,M} \neq 0 \text{ or } |v|_{1,M} \neq 0, \\ 0 & \text{if } |u|_{1,M} = |v|_{1,M} = 0. \end{cases}$$

Proof. Denoting

$$d_M(u;v,z) = \left(\tau_M^{\mathrm{sold}}(u)\,\kappa_M(P_M\nabla v), \kappa_M(P_M\nabla z)\right)_M\,,$$

it is easy to realize that

$$d_h(u;v,z) = \sum_{M\in\mathscr{M}_h} d_M(u;v,z)\,.$$

Applying Hölder's inequality yields

$$|d_M(u;v,z)| \leq \|\tau_M^{\mathrm{sold}}(u)\|_{0,M}\,\|\kappa_M(P_M\nabla v)\|_{0,4,M}\,\|\kappa_M(P_M\nabla z)\|_{0,4,M}\,,$$

which, using (23), gives

$$|d_M(u;v,z)| \leq C\,h_M^{1+d/2}\,\|\boldsymbol{b}\|_{0,\infty,M}\,\|\kappa_M(P_M\nabla v)\|_{0,4,M}\,\|\kappa_M(P_M\nabla z)\|_{0,4,M}\,, \tag{32}$$

thus proving (30). Now let us prove that

$$
\begin{aligned}
|d_M(u;u,z) - d_M(v;v,z)| \le C\, h_M^{1+d/2}\, \|\boldsymbol{b}\|_{0,\infty,M}\, \zeta_M(u,v) \times \\
\times \left(\|\kappa_M(P_M \nabla u)\|_{0,4,M} + \|\kappa_M(P_M \nabla v)\|_{0,4,M}\right) \|\kappa_M(P_M \nabla z)\|_{0,4,M}\,.
\end{aligned}
\tag{33}
$$

If $|u|_{1,M} = 0$ or $|v|_{1,M} = 0$, then (33) is a particular case of (32). Thus, it suffices to consider the case $|u|_{1,M} \neq 0$, $|v|_{1,M} \neq 0$. Denoting $\xi(x) = |x|\, x$, one obtains

$$
\begin{aligned}
d_M(u;u,z) - d_M(v;v,z) = \frac{\beta\, h_M^{1+d/2}\, |\boldsymbol{b}_M|}{|u|_{1,M}} \big(\xi(\kappa_M(P_M \nabla u)) - \xi(\kappa_M(P_M \nabla v)), \kappa_M(P_M \nabla z)\big)_M \\
+ \beta\, h_M^{1+d/2}\, |\boldsymbol{b}_M| \left(\frac{1}{|u|_{1,M}} - \frac{1}{|v|_{1,M}}\right) \big(\xi(\kappa_M(P_M \nabla v)), \kappa_M(P_M \nabla z)\big)_M\,.
\end{aligned}
\tag{34}
$$

The integral terms on M possess the same structure as the term $N_M(u,v,z)$ in the proof of Lemma 1 (the second term corresponds to $N_M(0,v,z)$). They are estimated using the same technique, only with a different Hölder inequality. Then, (16) is applied to $\|\kappa_M(P_M \nabla(u-v))\|_{0,M}$ resp. $\|\kappa_M(P_M \nabla v)\|_{0,M}$. Furthermore, the first inequality from (18) is employed. To finish the estimate of the second term in (34), the triangle inequality is used. One obtains

$$
\begin{aligned}
|d_M(u;u,z) - d_M(v;v,z)| \le C\, h_M^{1+d/2}\, \|\boldsymbol{b}\|_{0,\infty,M}\, \frac{|u-v|_{1,M}}{|u|_{1,M}} \times \\
\times \left(\|\kappa_M(P_M \nabla u)\|_{0,4,M} + \|\kappa_M(P_M \nabla v)\|_{0,4,M}\right) \|\kappa_M(P_M \nabla z)\|_{0,4,M}\,.
\end{aligned}
$$

The same type of inequality follows by interchanging u and v. Then, using the sharper of these two estimates and $\min\{|u|_{1,M}^{-1}, |v|_{1,M}^{-1}\} \le 2/(|u|_{1,M} + |v|_{1,M})$ gives (33). □

The properties of the operator T_h, namely its monotonicity and local Lipschitz continuity, follow now by the results of the two previous lemmas and (24).

Lemma 3. *If $\tilde{\tau}_M$ is defined by (21), then the operator T_h defined in (25) is locally Lipschitz-continuous and strongly monotone, i.e., it satisfies*

$$
(T_h w_h - T_h z_h, w_h - z_h) \ge \|w_h - z_h\|_{\mathrm{LPS}}^2 + \frac{1}{7} \sum_{M \in \mathscr{M}_h} \tilde{\tau}_M\, \|\kappa_M(P_M \nabla(w_h - z_h))\|_{0,3,M}^3\,,
\tag{35}
$$

for all $w_h, z_h \in V_h$. If $\tilde{\tau}_M$ is defined by (22), then the operator T_h is Lipschitz-continuous and it satisfies

$$
(T_h z_h, z_h) \ge \frac{1}{2}\, \|z_h\|_{\mathrm{LPS}}^2 - C_0 \left(\|\widetilde{u}_{bh}\|_{0,\Omega}^2 + \|f\|_{0,\Omega}^2\right),
\tag{36}
$$

for all $z_h \in V_h$, where $C_0 > 0$ depends on ε, $\boldsymbol{b}$, c, σ_0, h, and W_h but not on z_h.

Proof. Let us define the operators $A_h, N_h : V_h \to V_h$ by

$$(A_h z_h, v_h) = a(z_h, v_h) + s_h(z_h, v_h) \qquad \forall\ z_h, v_h \in V_h\,,$$
$$(N_h z_h, v_h) = d_h(z_h + \widetilde{u}_{bh}; z_h + \widetilde{u}_{bh}, v_h) \qquad \forall\ z_h, v_h \in V_h\,.$$

Then, for any $w_h, z_h \in V_h$, there holds

$$T_h w_h - T_h z_h = A_h(w_h - z_h) + N_h w_h - N_h z_h\,.$$

The operator A_h is linear on a finite-dimensional space and hence it is Lipschitz continuous. Thus, the (local) Lipschitz-continuity of T_h follows from (27), (31), and the equivalence of norms on finite-dimensional spaces. The strong monotonicity (35) follows from (24) and (26). Finally, in view of (24), it holds

$$(T_h z_h, z_h) = \|z_h\|^2_{\mathrm{LPS}} + d_h(z_h + \widetilde{u}_{bh}; z_h, z_h) + a(\widetilde{u}_{bh}, z_h) + s_h(\widetilde{u}_{bh}, z_h) + d_h(z_h + \widetilde{u}_{bh}; \widetilde{u}_{bh}, z_h) - (f, z_h).$$

Now, (36) follows from (30), (10), (16), (18), (4), the equivalence of norms on finite-dimensional spaces, the Cauchy-Schwarz inequality, and the Young inequality. □

To prove that the discrete problem (19) has at least one solution, we shall use the following simple consequence of Brouwer's fixed-point theorem, whose proof can be found in [30, p. 164, Lemma 1.4].

Lemma 4. *Let X be a finite-dimensional Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let $P : X \to X$ be a continuous mapping and $K > 0$ a real number such that $(Px, x) > 0$ for any $x \in X$ with $\|x\| = K$. Then there exists $x \in X$ such that $\|x\| \le K$ and $Px = 0$.*

Collecting the previous results, the main result of this section can be stated now, namely, the well-posedness of the problem (19).

Theorem 5. *If $\tilde{\tau}_M$ is defined by (21) or (22), then the problem (19) has a solution. If $\tilde{\tau}_M$ is defined by (21), the solution of (19) is unique.*

Proof. If $\tilde{\tau}_M$ is defined by (21), then it follows from (35) that, for any $z_h \in V_h$,

$$(T_h z_h, z_h) \ge \|z_h\|^2_{\mathrm{LPS}} + (T_h 0, z_h) \ge \sigma_0 \|z_h\|^2_{0,\Omega} - \|T_h 0\|_{0,\Omega} \|z_h\|_{0,\Omega}\,.$$

Thus, using Young's inequality one gets

$$(T_h z_h, z_h) \ge C_1 \|z_h\|^2_{0,\Omega} - C_2\,,$$

where C_1, C_2 are positive constants that depend on h and the data of (1) but not on z_h. According to (36), the same inequality holds if $\tilde{\tau}_M$ is defined by (22). Thus, in view of Lemma 4 with any $K > \sqrt{C_2/C_1}$, the operator T_h has a zero and hence the problem (19) has a solution. The uniqueness in the case that $\tilde{\tau}_M$ is defined by (21) follows from the strong monotonicity (35). $\square$

3.2. **Error estimates.** For the analysis of the methods introduced in Section 3, we will need an appropriate interpolation operator. An important tool for the construction of such an operator is provided by the following result, whose proof can be found in [23, Lemma 1].

Lemma 6. *Let us suppose (9) to be satisfied. Then, there exists an operator $\varrho_h : L^2(\Omega) \to V_h$ such that, for any $v, w \in L^2(\Omega)$, the estimates*

$$|(v - \varrho_h v, w)| \leq C \sum_{M \in \mathscr{M}_h} \|v\|_{0,M} \, \|\kappa_M w\|_{0,M} \,, \tag{37}$$

$$|\varrho_h v|_{1,M}^2 + h_M^{-2} \, \|\varrho_h v\|_{0,M}^2 \leq C \sum_{\substack{M' \in \mathscr{M}_h, \\ M \cap M' \neq \emptyset}} h_{M'}^{-2} \, \|v\|_{0,M'}^2 \qquad \forall \; M \in \mathscr{M}_h \,, \tag{38}$$

are valid. Consequently, for any $\alpha \in \mathbb{R}$, it holds

$$\sum_{M \in \mathscr{M}_h} h_M^\alpha \left(|\varrho_h v|_{1,M}^2 + h_M^{-2} \, \|\varrho_h v\|_{0,M}^2 \right) \leq C \sum_{M \in \mathscr{M}_h} h_M^{\alpha-2} \, \|v\|_{0,M}^2 \,, \tag{39}$$

where the constant C is independent of v and h but can depend on α.

With the operators i_h and ϱ_h, an operator $r_h \in \mathscr{L}(H^2(\Omega), W_h) \cap \mathscr{L}(H^2(\Omega) \cap H_0^1(\Omega), V_h)$ is defined by

$$r_h v := i_h v + \varrho_h (v - i_h v) \,. \tag{40}$$

To formulate the interpolation properties of r_h, it is convenient to introduce the mesh dependent norm

$$\|v\|_{1,h} = \left(\sum_{M \in \mathscr{M}_h} \{ |v|_{1,M}^2 + h_M^{-2} \, \|v\|_{0,M}^2 \} \right)^{1/2} .$$

Then, using (38), (4), (5), and (11), one obtains

$$\|v - r_h v\|_{1,h} \leq C \, \|v - i_h v\|_{1,h} \leq \tilde{C} \, h^k \, |v|_{k+1,\Omega} \qquad \forall \; v \in H^{k+1}(\Omega), \;\; k = 1, \ldots, l \,, \tag{41}$$

and consequently

$$|v - r_h v|_{1,\Omega} + h^{-1} \, \|v - r_h v\|_{0,\Omega} \leq C \, h^k \, |v|_{k+1,\Omega} \qquad \forall \; v \in H^{k+1}(\Omega), \;\; k = 1, \ldots, l \,. \tag{42}$$

The derivation of the error estimates will be based on the following two lemmas. The first one states an interpolation error estimate and the second one states a bound on the nonlinear form d_h.

Lemma 7. *Let $u \in H^{k+1}(\Omega)$ for some $k \in \{1,\dots,l\}$, and let $\eta := u - r_h u$. Then, for any $v_h \in V_h \setminus \{0\}$, the following estimate holds*

$$\|\eta\|_{\mathrm{LPS}} + \frac{a(\eta,v_h) + s_h(\eta,v_h) - s_h(u,v_h)}{\|v_h\|_{\mathrm{LPS}}} \le C\left(\varepsilon + h\,\|\boldsymbol{b}\|_{0,\infty,\Omega} + h^2\,\|\sigma\|_{0,\infty,\Omega} + h^2\,|\boldsymbol{b}|^2_{1,\infty,\Omega}\,\sigma_0^{-1}\right)^{1/2} h^k\,|u|_{k+1,\Omega}\,. \tag{43}$$

Proof. Since, in view of (5), (16), (18), and (20)

$$\|v\|_{\mathrm{LPS}} \le C\left(\varepsilon + h\,\|\boldsymbol{b}\|_{0,\infty,\Omega} + h^2\,\|\sigma\|_{0,\infty,\Omega}\right)^{1/2} \|v\|_{1,h} \qquad \forall\ v \in H^1(\Omega)\,,$$

it follows from (41) that

$$\|\eta\|_{\mathrm{LPS}} \le C\left(\varepsilon + h\,\|\boldsymbol{b}\|_{0,\infty,\Omega} + h^2\,\|\sigma\|_{0,\infty,\Omega}\right)^{1/2} h^k\,|u|_{k+1,\Omega}\,.$$

Next, for any $v_h \in V_h \setminus \{0\}$, integration by parts gives

$$(\boldsymbol{b}\cdot\nabla\eta, v_h) = -(\eta, \boldsymbol{b}\cdot\nabla v_h) - ((\nabla\cdot\boldsymbol{b})\,\eta, v_h)\,.$$

Thus, applying the Cauchy-Schwarz inequality and (42), it follows that

$$a(\eta,v_h) + s_h(\eta,v_h) \le \left(\|\eta\|_{\mathrm{LPS}} + C\,|\boldsymbol{b}|_{1,\infty,\Omega}\,\sigma_0^{-1/2}\,h^{k+1}\,|u|_{k+1,\Omega}\right)\,\|v_h\|_{\mathrm{LPS}} - (\eta, \boldsymbol{b}\cdot\nabla v_h)\,.$$

The use of (37), (11), (4) and (5) leads to

$$\begin{aligned}(\eta, \boldsymbol{b}\cdot\nabla v_h) &\le C \sum_{M\in\mathscr{M}_h} \|u - i_h u\|_{0,M}\,\|\kappa_M(\boldsymbol{b}\cdot\nabla v_h)\|_{0,M}\\ &\le C\,h^k\,|u|_{k+1,\Omega}\left(\sum_{M\in\mathscr{M}_h} h_M^2\,\|\kappa_M(\boldsymbol{b}\cdot\nabla v_h)\|^2_{0,M}\right)^{1/2}.\end{aligned}$$

Applying (16), (18), (20), and (8), one derives

$$\begin{aligned}\|\kappa_M(\boldsymbol{b}\cdot\nabla v_h)\|_{0,M} &\le \|\kappa_M((\boldsymbol{b}-\boldsymbol{b}_M)\cdot\nabla v_h)\|_{0,M} + \|\kappa_M(\boldsymbol{b}_M\cdot\nabla v_h)\|_{0,M}\\ &\le C\,|\boldsymbol{b}|_{1,\infty,M}\,\|v_h\|_{0,M} + \tau_0^{-1/2}\,(\varepsilon + h_M\,\|\boldsymbol{b}\|_{0,\infty,M})^{1/2}\,h_M^{-1}\,\tau_M^{1/2}\,\|\kappa_M(\boldsymbol{b}_M\cdot\nabla v_h)\|_{0,M}\,,\end{aligned}$$

which leads to the estimate

$$(\eta, \boldsymbol{b}\cdot\nabla v_h) \le C\left(\varepsilon + h\,\|\boldsymbol{b}\|_{0,\infty,\Omega} + h^2\,|\boldsymbol{b}|^2_{1,\infty,\Omega}\,\sigma_0^{-1}\right)^{1/2} h^k\,|u|_{k+1,\Omega}\,\|v_h\|_{\mathrm{LPS}}\,.$$

Finally, using (17), (18), (20), (4), and (5), one obtains

$$s_h(u,u) \le \sum_{M\in\mathscr{M}_h} \tau_M\,|\boldsymbol{b}_M|^2\,\|\kappa_M\nabla u\|^2_{0,M} \le C\,\|\boldsymbol{b}\|_{0,\infty,\Omega}\,h^{2\,k+1}\,|u|^2_{k+1,\Omega}\,,$$

and hence

$$s_h(u,v_h) \le \sqrt{s_h(u,u)}\,\sqrt{s_h(v_h,v_h)} \le C\,\|\boldsymbol{b}\|^{1/2}_{0,\infty,\Omega}\,h^{k+1/2}\,|u|_{k+1,\Omega}\,\|v_h\|_{\mathrm{LPS}}\,,$$

which completes the proof. ☐

Lemma 8. *For any $w_h \in W_h$ and $u, v \in H^{k+1}(\Omega)$ with $k \in \{1, \dots, l\}$, it holds*

$$d_h(w_h; r_h u, r_h v) \le C\, h^{2k-d/2} \left(\max_{M\in\mathscr{M}_h} \|\tau_M^{\rm sold}(w_h)\|_{0,M} \right) |u|_{k+1,\Omega}\, |v|_{k+1,\Omega}\,. \tag{44}$$

Proof. First, the application of Hölder's inequality and (10) leads to

$$\begin{aligned} d_h(w_h; r_h u, r_h v) &\le \sum_{M\in\mathscr{M}_h} \|\tau_M^{\rm sold}(w_h)\|_{0,M}\, \|\kappa_M(P_M\nabla(r_h u))\|_{0,4,M}\, \|\kappa_M(P_M\nabla(r_h v))\|_{0,4,M} \\ &\le C \sum_{M\in\mathscr{M}_h} \|\tau_M^{\rm sold}(w_h)\|_{0,M}\, h_M^{-d/2}\, \|\kappa_M(P_M\nabla(r_h u))\|_{0,M}\, \|\kappa_M(P_M\nabla(r_h v))\|_{0,M} \\ &\le C \left(\max_{M\in\mathscr{M}_h} \|\tau_M^{\rm sold}(w_h)\|_{0,M} \right) \left(\sum_{M\in\mathscr{M}_h} h_M^{-d/2}\, \|\kappa_M(P_M\nabla(r_h u))\|_{0,M}^2 \right)^{1/2} \\ &\qquad \times \left(\sum_{M\in\mathscr{M}_h} h_M^{-d/2}\, \|\kappa_M(P_M\nabla(r_h v))\|_{0,M}^2 \right)^{1/2} . \end{aligned} \tag{45}$$

Using (16) and (17), for $u \in H^{k+1}(\Omega)$ with $k \in \{1, \dots, l\}$ there holds

$$\begin{aligned} \|\kappa_M(P_M\nabla(r_h u))\|_{0,M} &\le \|\kappa_M(P_M\nabla u)\|_{0,M} + \|\kappa_M(P_M\nabla(u - r_h u))\|_{0,M} \\ &\le C\, h_M^k\, |u|_{k+1,M} + C\, |u - r_h u|_{1,M}\,. \end{aligned} \tag{46}$$

According to (39), one has for any $\alpha \in \mathbb{R}$

$$\begin{aligned} \sum_{M\in\mathscr{M}_h} h_M^\alpha\, |u - r_h u|_{1,M}^2 &\le 2 \sum_{M\in\mathscr{M}_h} h_M^\alpha\, |u - i_h u|_{1,M}^2 + 2 \sum_{M\in\mathscr{M}_h} h_M^\alpha\, |\varrho_h(u - i_h u)|_{1,M}^2 \\ &\le C \sum_{M\in\mathscr{M}_h} h_M^\alpha\, (|u - i_h u|_{1,M}^2 + h_M^{-2}\, \|u - i_h u\|_{0,M}^2)\,, \end{aligned}$$

and hence it follows from (11), (4), and (5) that, for $\alpha \ge -2$,

$$\sum_{M\in\mathscr{M}_h} h_M^\alpha\, \|\kappa_M(P_M\nabla(r_h u))\|_{0,M}^2 \le C\, h^{2k+\alpha}\, |u|_{k+1,\Omega}^2\,. \tag{47}$$

Inserting (47) with $\alpha = -d/2$ into (45), the statement of the lemma is proved. ☐

We are now in position to prove the first error estimate. The following theorem states the error estimate in the case $\tilde\tau_M$ is given by (21).

Theorem 9. *Let $\tilde\tau_M$ be defined by (21). Let the weak solution of (1) satisfy $u \in H^{k+1}(\Omega)$ for some $k \in \{1, \dots, l\}$. Let $\widetilde{u}_b \in H^2(\Omega)$ be an extension of u_b and let $\widetilde{u}_{bh} = i_h \widetilde{u}_b$. Then the* *solution u_h of the local projection discretization (19) satisfies the error estimate*

$$
\begin{aligned}
&\|u-u_h\|_{\mathrm{LPS}} + \left(\sum_{M\in\mathscr{M}_h} \tilde{\tau}_M \,\|\kappa_M(P_M\nabla(u-u_h))\|_{0,3,M}^3\right)^{1/2}\\
&\leq C\Big\{\varepsilon + h\,\|\boldsymbol{b}\|_{0,\infty,\Omega}\,(1+h^{k-d/2}\,|u|_{k+1,\Omega}) + h^2\left(\|\sigma\|_{0,\infty,\Omega} + |\boldsymbol{b}|_{1,\infty,\Omega}^2\,\sigma_0^{-1}\right)\Big\}^{1/2} h^k\,|u|_{k+1,\Omega}\,.
\end{aligned}
$$

If $u\in W^{k+1,\infty}(\Omega)$ with $k\in\{1,\dots,l\}$, then

$$
\begin{aligned}
&\|u-u_h\|_{\mathrm{LPS}} + \left(\sum_{M\in\mathscr{M}_h} \tilde{\tau}_M \,\|\kappa_M(P_M\nabla(u-u_h))\|_{0,3,M}^3\right)^{1/2}\\
&\leq C\Big\{\varepsilon + h\,\|\boldsymbol{b}\|_{0,\infty,\Omega}\,(1+h^{k}\,|u|_{k+1,\infty,\Omega}) + h^2\left(\|\sigma\|_{0,\infty,\Omega} + |\boldsymbol{b}|_{1,\infty,\Omega}^2\,\sigma_0^{-1}\right)\Big\}^{1/2} h^k\,|u|_{k+1,\Omega}\,.
\end{aligned}
$$

Proof. The error $u-u_h$ is split into the interpolation error $\eta := u - r_h u$ and the discrete error $e_h := u_h - r_h u$. Then $e_h\in V_h$ and also $r_h u - \widetilde{u}_{bh}\in V_h$. From the monotonicity (35) follows with (19) and (14)

$$
\begin{aligned}
&\|e_h\|_{\mathrm{LPS}}^2 + \frac{1}{7}\sum_{M\in\mathscr{M}_h} \tilde{\tau}_M\,\|\kappa_M(P_M\nabla e_h)\|_{0,3,M}^3 \leq (T_h(u_h-\widetilde{u}_{bh}) - T_h(r_hu-\widetilde{u}_{bh}), e_h)\\
&= a(u_h,e_h) + s_h(u_h,e_h) + d_h(u_h;u_h,e_h) - a(r_hu,e_h) - s_h(r_hu,e_h) - d_h(r_hu;r_hu,e_h)\\
&= a(\eta,e_h) + s_h(\eta,e_h) - s_h(u,e_h) - d_h(r_hu;r_hu,e_h)\,.
\end{aligned}
$$

The first three terms on the right-hand side can be estimated using (43). To bound the nonlinear term, Hölder's and Young's inequalities are applied to conclude

$$
\begin{aligned}
d_h(r_hu;r_hu,e_h) &\leq \{d_h(r_hu;r_hu,r_hu)\}^{\frac{2}{3}}\,\{d_h(e_h;e_h,e_h)\}^{\frac{1}{3}}\\
&\leq 2\,d_h(r_hu;r_hu,r_hu) + \frac{3}{70}\,d_h(e_h;e_h,e_h)\,.
\end{aligned}
\tag{48}
$$

Then (44), (46), (5), (18), and (42) yield

$$
d_h(r_hu;r_hu,r_hu) \leq C\,\|\boldsymbol{b}\|_{0,\infty,\Omega}\,h^{3\,k+1-d/2}\,|u|_{k+1,\Omega}^3\,. \tag{49}
$$

Therefore,

$$
\begin{aligned}
&\|e_h\|_{\mathrm{LPS}}^2 + \sum_{M\in\mathscr{M}_h} \tilde{\tau}_M\,\|\kappa_M(P_M\nabla e_h)\|_{0,3,M}^3\\
&\leq C\left\{\varepsilon + h\,\|\boldsymbol{b}\|_{0,\infty,\Omega}\,(1+h^{k-d/2}\,|u|_{k+1,\Omega}) + h^2\,\|\sigma\|_{0,\infty,\Omega} + h^2\,|\boldsymbol{b}|_{1,\infty,\Omega}^2\,\sigma_0^{-1}\right\} h^{2\,k}\,|u|_{k+1,\Omega}^2\,.
\end{aligned}
\tag{50}
$$

Next, to estimate the interpolation error, for any $p \in [1,6]$, it follows from (10), (15), and (13) that

$$
\begin{aligned}
(51)\qquad \|\kappa_M(P_M\nabla\eta)\|_{0,p,M} &\le \|\nabla\eta - \pi_M\nabla\eta\|_{0,p,M}\\
&\le \|\nabla(u-i_hu)\|_{0,p,M} + \|\nabla(i_hu - r_hu) - \pi_M\nabla\eta\|_{0,p,M}\\
&\le |u-i_hu|_{1,p,M} + C\,h_M^{\frac{d}{p}-\frac{d}{2}}\,\|\nabla(i_hu-r_hu)-\pi_M\nabla\eta\|_{0,M}\\
&\le |u-i_hu|_{1,p,M} + \tilde C\,h_M^{\frac{d}{p}-\frac{d}{2}}\,\big(|\varrho_h(u-i_hu)|_{1,M} + |u-i_hu|_{1,M}\big)\\
&\le \bar C\,h_M^{k+\frac{d}{p}-\frac{d}{2}}\,|u|_{k+1,M} + \tilde C\,h_M^{\frac{d}{p}-\frac{d}{2}}\,|\varrho_h(u-i_hu)|_{1,M}\,.
\end{aligned}
$$

Then, applying (51), (21), (5), (18), (38), (11), (4), and (6), one derives

$$
(52)\qquad \sum_{M\in\mathscr{M}_h} \tilde\tau_M\,\|\kappa_M(P_M\nabla\eta)\|^3_{0,3,M} \le C\,h\,\|\boldsymbol{b}\|_{0,\infty,\Omega}\sum_{M\in\mathscr{M}_h} h_M^{3k-d/2}\,|u|^3_{k+1,M}\,.
$$

Thus, combining (50), (52), and (43), the first estimate of the theorem follows.

If $u\in W^{k+1,\infty}(\Omega)$ with $k\in\{1,\dots,l\}$, then local norms of Sobolev spaces with $p=2$ can be estimated with norms of Sobolev spaces with $p=\infty$, thereby gaining powers of h from the smallness of the local domain: $|u|_{k+1,M}\le C\,h_M^{d/2}\,|u|_{k+1,\infty,M}$ for any $M\in\mathscr{M}_h$. Hence, it follows from (52), (4) and (5) that

$$
(53)\qquad \sum_{M\in\mathscr{M}_h} \tilde\tau_M\,\|\kappa_M(P_M\nabla\eta)\|^3_{0,3,M} \le C\,\|\boldsymbol{b}\|_{0,\infty,\Omega}\,h^{3k+1}\,|u|_{k+1,\infty,\Omega}\,|u|^2_{k+1,\Omega}\,.
$$

Furthermore, using (38), (11), and (4), one gets

$$
|u-r_hu|_{1,M} \le C \sum_{\substack{M'\in\mathscr{M}_h,\\ M\cap M'\neq\emptyset}} h^k_{M'}\,|u|_{k+1,M'} \le \tilde C\,h^{k+d/2}\,|u|_{k+1,\infty,\Omega} \qquad \forall\ M\in\mathscr{M}_h\,.
$$

Therefore, according to (44) and (46),

$$
(54)\qquad d_h(r_hu;r_hu,r_hu) \le C\,\|\boldsymbol{b}\|_{0,\infty,\Omega}\,h^{3k+1}\,|u|_{k+1,\infty,\Omega}\,|u|^2_{k+1,\Omega}\,,
$$

which implies the second estimate of the theorem. □

Remark. Theorem 9 implies, in particular, the following convergence estimates in the convection-dominated case $\varepsilon < h$: If $u\in H^2(\Omega)$, then

$$
\|u-u_h\|_{\mathrm{LPS}} \le C_0\,h^{2-d/4}\,(h^{(d-2)/4} + |u|^{1/2}_{2,\Omega})\,|u|_{2,\Omega}\,,
$$

where C_0 depends on the data of the problem. If $u\in W^{2,\infty}(\Omega)$, then

$$
\|u-u_h\|_{\mathrm{LPS}} \le C_0\,h^{3/2}\,(1 + h^{1/2}\,|u|^{1/2}_{2,\infty,\Omega})\,|u|_{2,\Omega}\,.
$$

If $u \in H^{k+1}(\Omega)$ with $k \in \{2,\dots,l\}$, then

$$\|u-u_h\|_{\mathrm{LPS}} \le C_0\, h^{k+1/2}\,(1+h^{(2\,k-d)/4}\,|u|_{k+1,\Omega}^{1/2})\,|u|_{k+1,\Omega}\,.$$

We end this section by presenting the error estimate in the case $\tilde\tau_M$ is defined by (22).

Theorem 10. *Let $\tilde\tau_M$ be defined by (22). Let the weak solution of (1) satisfy $u \in H^{k+1}(\Omega)$ for some $k \in \{1,\dots,l\}$. Let $\widetilde u_b \in H^2(\Omega)$ be an extension of u_b and let $\widetilde u_{bh} = i_h\widetilde u_b$. Then the solution u_h of the local projection discretization (19) satisfies the error estimate*

$$\|u-u_h\|_{\mathrm{LPS}} + (d_h(u_h;u-u_h,u-u_h))^{1/2}$$
$$\le C\left(\varepsilon + h\,\|\boldsymbol{b}\|_{0,\infty,\Omega} + h^2\,\|\sigma\|_{0,\infty,\Omega} + h^2\,|\boldsymbol{b}|_{1,\infty,\Omega}^2\,\sigma_0^{-1}\right)^{1/2} h^k\,|u|_{k+1,\Omega}\,.$$

Proof. Set again $\eta := u - r_h u$ and $e_h := u_h - r_h u$. From (19) and (14), it follows that

$$a(e_h,e_h) + s_h(e_h,e_h) + d_h(u_h;u_h,e_h)$$
$$= a(u_h,e_h) + s_h(u_h,e_h) + d_h(u_h;u_h,e_h) - a(r_h u,e_h) - s_h(r_h u,e_h)$$
$$= a(\eta,e_h) + s_h(\eta,e_h) - s_h(u,e_h)\,.$$

Thus, in view of (24), one gets

$$\|e_h\|_{\mathrm{LPS}}^2 + d_h(u_h;e_h,e_h) = a(\eta,e_h) + s_h(\eta,e_h) - s_h(u,e_h) - d_h(u_h;r_h u,e_h)\,.$$

The first three terms on the right-hand side can be estimated using (43). To bound the nonlinear term, Hölder's and Young's inequalities are again applied

$$d_h(u_h;r_hu,e_h) \le \sqrt{d_h(u_h;r_hu,r_hu)}\,\sqrt{d_h(u_h;e_h,e_h)} \le d_h(u_h;r_hu,r_hu) + \frac{1}{4}\,d_h(u_h;e_h,e_h)\,. \tag{55}$$

Using (44), (23), and (5), one obtains

$$d_h(u_h;r_hu,r_hu) \le C\,\|\boldsymbol{b}\|_{0,\infty,\Omega}\,h^{2\,k+1}\,|u|_{k+1,\Omega}^2\,. \tag{56}$$

Therefore,

$$\|e_h\|_{\mathrm{LPS}}^2 + d_h(u_h;e_h,e_h) \le C\left(\varepsilon + h\,\|\boldsymbol{b}\|_{0,\infty,\Omega} + h^2\,\|\sigma\|_{0,\infty,\Omega} + h^2\,|\boldsymbol{b}|_{1,\infty,\Omega}^2\,\sigma_0^{-1}\right) h^{2\,k}\,|u|_{k+1,\Omega}^2\,.$$

Note that an application of the triangle inequality gives

$$d_h(u_h;u-u_h,u-u_h) \le 2\,d_h(u_h;\eta,\eta) + 2\,d_h(u_h;e_h,e_h)\,. \tag{57}$$

It follows from Hölder's inequality, (23), (51), (39) with $\alpha = 0$, (11), (4), and (5), that

$$d_h(u_h;\eta,\eta) \le \sum_{M\in\mathscr{M}_h} \|\tau_M^{\mathrm{sold}}(u_h)\|_{0,M}\,\|\kappa_M(P_M\nabla\eta)\|_{0,4,M}^2 \le C\,\|\boldsymbol{b}\|_{0,\infty,\Omega}\,h^{2\,k+1}\,|u|_{k+1,\Omega}^2\,. \tag{58}$$

Finally, using the triangle inequality and the estimate (43), the statement of the theorem follows. □

Remark. Theorems 9 and 10 prove the convergence of the method in the LPS norm plus an extra term involving the crosswind derivative of the error. Hence, these estimates give, essentially, an extra control of the whole gradient of the error.

4. The time-dependent problem

We now move on to the study of the time-dependent problem (3). A weak form of problem (3) reads as follows: Find $u \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$ such that $u = u_b$ on $[0,T] \times \partial\Omega$, $u(0,\cdot) = u_0$ and

$$(u_t, v) + a(u,v) = (f,v) \quad \forall\, v \in H_0^1(\Omega), \quad \text{for almost every } t \in (0,T]. \tag{59}$$

To avoid technicalities in the analysis, it is assumed that the boundary condition does not depend on time, $u_b(t,\cdot) = u_b$. The initial condition u_0 is assumed to satisfy $u_0|_{\partial\Omega} = u_b$ and it is approximated by a function $u_h^0 \in W_h$ such that $u_h^0 - \widetilde{u}_{bh} \in V_h$.

To perform the discretization of the time derivative, the time interval $[0,T]$ is divided into N_T equi-distant strips of length $\delta t = T/N_T$. The nodes are denoted by $t^n = n\,\delta t$ for $n = 0,1,\ldots,N_T$ and the abbreviations $u^n := u(t^n,\cdot)$, $f^n := f(t^n,\cdot)$, etc. are used. Since this section studies the LPS method with nonlinear crosswind diffusion in combination with a one-step θ-scheme as temporal discretization, from now on, the superscript $n+\theta$ denotes for all functions which are defined in $[0,T]$ the values at time $t^{n+\theta} := \theta\, t^{n+1} + (1-\theta)\, t^n$ with any $n \in \{0,\ldots,N_T-1\}$ and $\theta \in [0,1]$, e.g. $\boldsymbol{b}^{n+\theta} = \boldsymbol{b}(t^{n+\theta},\cdot)$. For functions, which are defined only at the discrete times t^n and t^{n+1}, it denotes the linear interpolation, e.g. $u_h^{n+\theta} = \theta\, u_h^{n+1} + (1-\theta)\, u_h^n$. Finally, it is convenient to introduce the interpolation operator $\tilde{r}_h^{n+\theta}$ satisfying

$$\tilde{r}_h^{n+\theta} u = \theta\, r_h u^{n+1} + (1-\theta)\, r_h u^n \tag{60}$$

with r_h from (40). Thus, writing α instead of $n+\theta$, functions u^α, u_h^α, $\tilde{r}_h^\alpha u$, etc. are defined for any $\alpha \in [0,N_T]$.

Then, given $\theta \in (0,1]$, the fully discrete problem reads as follows: For $n = 0,1,\ldots,N_T-1$, find $u_h^{n+1} \in W_h$ such that $u_h^{n+1} - \widetilde{u}_{bh} \in V_h$ and

$$\begin{aligned}\left(\frac{u_h^{n+1}-u_h^n}{\delta t}, v_h\right) + a^{n+\theta}(u_h^{n+\theta}, v_h) + s_h^{n+\theta}(u_h^{n+\theta}, v_h) + d_h^{n+\theta}(u_h^{n+\theta}; u_h^{n+\theta}, v_h)&\\ = (f^{n+\theta}, v_h) \qquad \forall\, v_h \in V_h\,.&\end{aligned} \tag{61}$$

For $\theta = 1/2$, the Crank-Nicolson scheme is recovered and for $\theta = 1$, the implicit Euler scheme is obtained.

Remark. To simplify the notation, we will not explicitly indicate at which time instant the functions $\boldsymbol{b}$ and σ in the definition of the norm $\|\cdot\|_{\mathrm{LPS}}$ are evaluated. This will be implicitly determined from the context or by the argument of the norm. Thus, if we write, e.g., $\|u_h^{n+\theta}\|_{\mathrm{LPS}}$, the norm $\|\cdot\|_{\mathrm{LPS}}$ is defined using $\boldsymbol{b}^{n+\theta}$ and $\sigma^{n+\theta}$.

4.1. **Well-posedness and stability.** The well-posedness of (61) can be traced back to the well-posedness of the LPS scheme with crosswind diffusion for the steady-state problem. The discretization of the temporal derivative can be written in the form

$$\left(\frac{u_h^{n+1}-u_h^n}{\delta t}, v_h\right) = \frac{1}{\theta}\left(\frac{u_h^{n+\theta}-u_h^n}{\delta t}, v_h\right).$$

The first part of this term has the form of a reaction term for $u_h^{n+\theta}$. Thus, given u_h^n, the equation at the discrete time t^{n+1} is an equation for $u_h^{n+\theta}$ which has the same form as (19) with the data of the problem at $t^{n+\theta}$ and with a reaction coefficient which has a contribution from the temporal derivative. Thus, defining the operator $\tilde{T}_h^{n+\theta} : V_h \to V_h$ by

$$(\tilde{T}_h^{n+\theta} z_h, v_h) = (T_h^{n+\theta} z_h, v_h) + \frac{1}{\theta\,\delta t}\,(z_h + \widetilde{u}_{bh}, v_h) - \frac{1}{\theta\,\delta t}\,(u_h^n, v_h) \qquad \forall\ z_h, v_h \in V_h\,,$$

it follows that $\tilde{T}_h^{n+\theta}(u_h^{n+\theta} - \widetilde{u}_{bh}) = 0$. Therefore, the existence and uniqueness of a solution $u_h^{n+\theta}$ can be proved in the same way as in the steady-state case, see Section 3.1. This fact is stated in the next result.

Corollary 11. *Let $n \in \{0, 1, \ldots, N_T - 1\}$ and $u_h^n \in W_h$ with $u_h^n|_{\partial\Omega} = \widetilde{u}_{bh}$ be given. If $\tilde{\tau}_M$ is defined by (21) or (22), then the problem (61) possesses a solution u_h^{n+1}. In the case that $\tilde{\tau}_M$ is defined by (21), the solution of (61) is unique. Furthermore, there is a constant $C > 0$ such that the solution of the scheme (61) with $\tilde{\tau}_M$ given by (22) is unique if $\delta t\,\|\boldsymbol{b}^{n+\theta}\|_{0,\infty,M} \le C\,h_M$ for any $M \in \mathscr{M}_h$.*

Proof. The only point remaining to prove is the uniqueness in the case $\tilde{\tau}_M$ is given by (22). For this, let $v_h, w_h \in W_h$ and $z_h := v_h - w_h$. Then, applying (31), (10), (16), $\|P_M^{n+\theta}\|_2 = 1$, and (8), one arrives at

$$|d_h^{n+\theta}(v_h; v_h, z_h) - d_h^{n+\theta}(w_h; w_h, z_h)| \le C \sum_{M\in\mathscr{M}_h} h_M^{-1}\,\|\boldsymbol{b}^{n+\theta}\|_{0,\infty,M}\,\|z_h\|_{0,M}^2\,.$$

Thus, if $v_h, w_h \in V_h$, one obtains

$$(\tilde{T}_h^{n+\theta} v_h - \tilde{T}_h^{n+\theta} w_h, z_h) \ge \sum_{M\in\mathscr{M}_h}\left(\frac{\tilde{C}}{\theta\,\delta t} - \frac{C\,\|\boldsymbol{b}^{n+\theta}\|_{0,\infty,M}}{h_M}\right)\|z_h\|_{0,M}^2 + \|z_h\|_{\mathrm{LPS}}^2\,.$$

Consequently, for δt small enough, the operator $\tilde{T}_h^{n+\theta}$ is strongly monotone and hence the solution to the discrete problem (61) is unique. $\square$

The next result states the stability of the method.

Lemma 12. *Let $\theta \in [1/2, 1]$ be given. Let $\widetilde{u}_h^\alpha := u_h^\alpha - \widetilde{u}_{bh}$ for any $\alpha \in [0, N_T]$. Then any solution of (61) satisfies the following stability estimate for all $N = 1, 2, \ldots, N_T$:*

$$
\begin{aligned}
(62) \qquad & \|\widetilde{u}_h^N\|_{0,\Omega}^2 + (2\,\theta - 1) \sum_{n=0}^{N-1} \|\widetilde{u}_h^{n+1} - \widetilde{u}_h^n\|_{0,\Omega}^2 + \delta t \sum_{n=0}^{N-1} \|\widetilde{u}_h^{n+\theta}\|_{\mathrm{LPS}}^2 \\
& + \delta t \sum_{n=0}^{N-1} d_h^{n+\theta}(\bar{u}_h^{n+\theta}; \widetilde{u}_h^{n+\theta}, \widetilde{u}_h^{n+\theta}) \le \|\widetilde{u}_h^0\|_{0,\Omega}^2 + C\,\delta t \sum_{n=0}^{N-1} \Big\{ \sigma_0^{-1} \|f^{n+\theta}\|_{0,\Omega}^2 \\
& + \Big[\varepsilon + \sigma_0^{-1} \left(\|\boldsymbol{b}^{n+\theta}\|_{0,\infty,\Omega}^2 + \|c^{n+\theta}\|_{0,\infty,\Omega}^2\right) + h\,\|\boldsymbol{b}^{n+\theta}\|_{0,\infty,\Omega}\Big] \|\widetilde{u}_{bh}\|_{1,\Omega}^2 + \mu_h \Big\}\,,
\end{aligned}
$$

where

$$
(63) \qquad \bar{u}_h^{n+\theta} = \widetilde{u}_h^{n+\theta}\,, \qquad \mu_h = h\,\|\boldsymbol{b}^{n+\theta}\|_{0,\infty,\Omega} |\widetilde{u}_{bh}|_{1,3,\Omega}^3 \qquad \textit{if } \tilde{\tau}_M \textit{ is given by (21)},
$$

$$
(64) \qquad \bar{u}_h^{n+\theta} = u_h^{n+\theta}\,, \qquad \mu_h = 0 \qquad \textit{if } \tilde{\tau}_M \textit{ is given by (22)}.
$$

Proof. The proof starts in the usual way by setting $v_h = \widetilde{u}_h^{n+\theta} \in V^h$ in (61) and using that $u_h^{n+1} - u_h^n = \widetilde{u}_h^{n+1} - \widetilde{u}_h^n$, which leads to

$$
\begin{aligned}
(65) \qquad & (\widetilde{u}_h^{n+1} - \widetilde{u}_h^n, \widetilde{u}_h^{n+\theta}) + \delta t\,\|\widetilde{u}_h^{n+\theta}\|_{\mathrm{LPS}}^2 + \delta t\, d_h^{n+\theta}(u_h^{n+\theta}; u_h^{n+\theta}, \widetilde{u}_h^{n+\theta}) \\
& = \delta t\,(f^{n+\theta}, \widetilde{u}_h^{n+\theta}) - \delta t\, a^{n+\theta}(\widetilde{u}_{bh}, \widetilde{u}_h^{n+\theta}) - \delta t\, s_h^{n+\theta}(\widetilde{u}_{bh}, \widetilde{u}_h^{n+\theta})\,.
\end{aligned}
$$

A straightforward computation gives

$$
(66) \qquad (\widetilde{u}_h^{n+1} - \widetilde{u}_h^n, \widetilde{u}_h^{n+\theta}) = \frac{1}{2} \left(\|\widetilde{u}_h^{n+1}\|_{0,\Omega}^2 - \|\widetilde{u}_h^n\|_{0,\Omega}^2\right) + \frac{2\,\theta - 1}{2} \|\widetilde{u}_h^{n+1} - \widetilde{u}_h^n\|_{0,\Omega}^2\,.
$$

Next, the application of the Cauchy-Schwarz inequality, the Young inequality, (16), (18), (20), (4), and (5) yields

$$
\begin{aligned}
(f^{n+\theta}, \widetilde{u}_h^{n+\theta}) &\le \frac{1}{\sigma_0} \|f^{n+\theta}\|_{0,\Omega}^2 + \frac{1}{4} \|\widetilde{u}_h^{n+\theta}\|_{\mathrm{LPS}}^2\,, \\
a^{n+\theta}(\widetilde{u}_{bh}, \widetilde{u}_h^{n+\theta}) &\le 6\left[\varepsilon + \sigma_0^{-1} \left(\|\boldsymbol{b}^{n+\theta}\|_{0,\infty,\Omega}^2 + \|c^{n+\theta}\|_{0,\infty,\Omega}^2\right)\right] \|\widetilde{u}_{bh}\|_{1,\Omega}^2 + \frac{1}{8} \|\widetilde{u}_h^{n+\theta}\|_{\mathrm{LPS}}^2\,, \\
s_h^{n+\theta}(\widetilde{u}_{bh}, \widetilde{u}_h^{n+\theta}) &\le C\,h\,\|\boldsymbol{b}^{n+\theta}\|_{0,\infty,\Omega} |\widetilde{u}_{bh}|_{1,\Omega}^2 + \frac{1}{8} \|\widetilde{u}_h^{n+\theta}\|_{\mathrm{LPS}}^2\,.
\end{aligned}
$$

If $\tilde\tau_M$ is given by (21), then, from (26) and an analog of (48), one obtains

$$\begin{aligned} d_h^{n+\theta}(u_h^{n+\theta};u_h^{n+\theta},\widetilde u_h^{n+\theta}) &\ge \frac{1}{7}\,d_h^{n+\theta}(\widetilde u_h^{n+\theta};\widetilde u_h^{n+\theta},\widetilde u_h^{n+\theta}) + d_h^{n+\theta}(\widetilde u_{bh};\widetilde u_{bh},\widetilde u_h^{n+\theta})\\ &\ge \frac{1}{10}\,d_h^{n+\theta}(\widetilde u_h^{n+\theta};\widetilde u_h^{n+\theta},\widetilde u_h^{n+\theta}) - 2\,d_h^{n+\theta}(\widetilde u_{bh};\widetilde u_{bh},\widetilde u_{bh})\,. \end{aligned}$$

Furthermore, the use of (10), (16), (18), $\|P_M^{n+\theta}\|_2=1$, (4), and (5) leads to

$$d_h^{n+\theta}(\widetilde u_{bh};\widetilde u_{bh},\widetilde u_{bh}) \le C\sum_{M\in\mathscr{M}_h} h_M^{1-d/2}\,\|\boldsymbol{b}^{n+\theta}\|_{0,\infty,M}\,|\widetilde u_{bh}|_{1,M}^3 \le \tilde C\,h\,\|\boldsymbol{b}^{n+\theta}\|_{0,\infty,\Omega}\,|\widetilde u_{bh}|_{1,3,\Omega}^3\,.$$

If $\tilde\tau_M$ is given by (22), then, using an inequality like (55), one gets

$$\begin{aligned} d_h^{n+\theta}(u_h^{n+\theta};u_h^{n+\theta},\widetilde u_h^{n+\theta}) &= d_h^{n+\theta}(u_h^{n+\theta};\widetilde u_h^{n+\theta},\widetilde u_h^{n+\theta}) + d_h^{n+\theta}(u_h^{n+\theta};\widetilde u_{bh},\widetilde u_h^{n+\theta})\\ &\ge \frac{1}{2}\,d_h^{n+\theta}(u_h^{n+\theta};\widetilde u_h^{n+\theta},\widetilde u_h^{n+\theta}) - \frac{1}{2}\,d_h^{n+\theta}(u_h^{n+\theta};\widetilde u_{bh},\widetilde u_{bh})\,. \end{aligned}$$

Applying the Hölder inequality, (23), (10), (16), $\|P_M^{n+\theta}\|_2=1$, (4), and (5), one deduces that

$$\begin{aligned} d_h^{n+\theta}(u_h^{n+\theta};\widetilde u_{bh},\widetilde u_{bh}) &\le C\sum_{M\in\mathscr{M}_h} h_M^{1+d/2}\,\|\boldsymbol{b}^{n+\theta}\|_{0,\infty,M}\,\|\kappa_M(P_M^{n+\theta}\,\nabla\widetilde u_{bh})\|_{0,4,M}^2\\ &\le \tilde C\,h\,\|\boldsymbol{b}^{n+\theta}\|_{0,\infty,\Omega}\,|\widetilde u_{bh}|_{1,\Omega}^2\,. \end{aligned}$$

Now, inserting the above relations into (65) and using the notation (63) and (64), one obtains

$$\begin{aligned} &\frac{1}{2}\,(\|\widetilde u_h^{n+1}\|_{0,\Omega}^2 - \|\widetilde u_h^{n}\|_{0,\Omega}^2) + \frac{2\,\theta-1}{2}\,\|\widetilde u_h^{n+1}-\widetilde u_h^{n}\|_{0,\Omega}^2 + \frac{\delta t}{2}\,\|\widetilde u_h^{n+\theta}\|_{\rm LPS}^2 + \frac{\delta t}{6}\,d_h^{n+\theta}(\bar u_h^{n+\theta};\widetilde u_h^{n+\theta},\widetilde u_h^{n+\theta})\\ &\le \delta t\,\sigma_0^{-1}\,\|f^{n+\theta}\|_{0,\Omega}^2 + C\,\delta t\,\big\{\varepsilon+\sigma_0^{-1}\,(\|\boldsymbol{b}^{n+\theta}\|_{0,\infty,\Omega}^2+\|c^{n+\theta}\|_{0,\infty,\Omega}^2) + h\,\|\boldsymbol{b}^{n+\theta}\|_{0,\infty,\Omega}\big\}\,\|\widetilde u_{bh}\|_{1,\Omega}^2\\ &\quad + C\,\delta t\,\mu_h\,, \end{aligned}$$

and (62) follows by summing up from $n=0$ to $N-1$. □

Remark. The inequality (62) is a proper stability result provided that $\|u_h^0\|_{0,\Omega}$, $\|\widetilde u_{bh}\|_{1,\Omega}$ and, if $\tilde\tau_M$ is given by (21), also $|\widetilde u_{bh}|_{1,3,\Omega}$ are bounded when $h\to0$. One may set $u_h^0=I_hu_0$ and $\widetilde u_{bh}=I_h\widetilde u_b$, where $I_h: H^1(\Omega)\to W_h$ is the Scott-Zhang interpolation operator (cf., e.g., [12]) and $\widetilde u_b\in H^1(\Omega)$ is an extension of u_b. Then $\|u_h^0\|_{0,\Omega}\le C\,\|u_0\|_{1,\Omega}$ and $\|\widetilde u_{bh}\|_{1,\Omega}\le C\,\|\widetilde u_b\|_{1,\Omega}$. If $\widetilde u_b\in W^{1,3}(\Omega)$ (requiring the stronger assumption $u_b\in W^{2/3,3}(\partial\Omega)$), then also $|\widetilde u_{bh}|_{1,3,\Omega}\le C\,\|\widetilde u_b\|_{1,3,\Omega}$. It is important that I_h preserves homogeneous boundary conditions since one has to assure that u_h^0 and $\widetilde u_{bh}$ coincide on the boundary of Ω. If $u_0\in H^2(\Omega)$ and $u_b\in H^{3/2}(\partial\Omega)$, which are the minimal regularity assumptions for deriving the error estimates in the next section, one may use the operator i_h from Section 2 instead of I_h. Now $\widetilde u_b\in H^2(\Omega)$ and, according to (11) and (13), one has $\|u_h^0\|_{0,\Omega}\le C\,\|u_0\|_{2,\Omega}$ and $\|\widetilde u_{bh}\|_{1,\Omega}+|\widetilde u_{bh}|_{1,3,\Omega}\le C\,\|\widetilde u_b\|_{2,\Omega}$.

Remark. It is worth remarking that, for the homogeneous case $u_b = 0$, instead of the direct proof presented in this manuscript, an analysis completely analogous to the one given in [8] leads to the following stability result

$$\frac{1}{2}\,\|u_h^N\|_{0,\Omega}^2 + \delta t\,\sum_{n=0}^{N-1}\{\|u_h^{n+\theta}\|_{\mathrm{LPS}}^2 + d_h^{n+\theta}(u_h^{n+\theta};u_h^{n+\theta},u_h^{n+\theta})\}$$
$$\le e^{\frac{T}{T-\delta t}}\,\{T\,\delta t\,\sum_{n=0}^{N-1}\,\|f^{n+\theta}\|_{0,\Omega}^2 + \frac{1}{2}\,\|u_h^0\|_{0,\Omega}^2\}\,.$$

A similar analysis could also be carried out for the non-homogeneous case, but in that case the presence of u_b makes the constants dependent on σ_0^{-1}.

Also, if u_b would be supposed time dependent, then in the first line of the proof of stability there holds $u_h^{n+1} - u_h^n = \widetilde{u}_h^{n+1} - \widetilde{u}_h^n + \widetilde{u}_{bh}^{n+1} - \widetilde{u}_{bh}^n$, thus creating an extra right-hand side depending on the time derivative of u_b.

4.2. **Error estimates.** In this section, error estimates are derived for the solution of the discrete problem (61) with $\theta \in [1/2, 1]$. The error will be analyzed essentially in the quantity which is given by the stability estimate (62). Let us denote the error by $e^\alpha := u^\alpha - u_h^\alpha$ with $\alpha \in [0, N_T]$. Furthermore, to simplify the presentation of our results, we introduce the quantities

$$E^N = \|e^N\|_{0,\Omega} + \left(\delta t\sum_{n=0}^{N-1}\,\|e^{n+\theta}\|_{\mathrm{LPS}}^2\right)^{1/2},$$

$$Q^N = h\left(|u_0|_{k+1,\Omega} + |u^N|_{k+1,\Omega} + \sigma_0^{-1/2}\,\|u_t\|_{L^2(0,t^N;H^{k+1}(\Omega))}\right) + \left(\delta t\sum_{n=0}^{N-1}\left(\varepsilon + h\,\|\boldsymbol{b}^{n+\theta}\|_{0,\infty,\Omega}\right.\right.$$
$$\left.\left. + h^2\,\|\sigma^{n+\theta}\|_{0,\infty,\Omega} + h^2\,\sigma_0^{-1}\,|\boldsymbol{b}^{n+\theta}|_{1,\infty,\Omega}^2\right)\left(|u^n|_{k+1,\Omega}^2 + |u^{n+1}|_{k+1,\Omega}^2\right)\right)^{1/2},$$

$$R^N = \left(\delta t\sum_{n=0}^{N-1}\,h^{k+1-d/2}\,\|\boldsymbol{b}^{n+\theta}\|_{0,\infty,\Omega}\left(|u^n|_{k+1,\Omega}^3 + |u^{n+1}|_{k+1,\Omega}^3\right)\right)^{1/2},$$

$$S^N = \left(\delta t\sum_{n=0}^{N-1}\,h^{k+1}\,\|\boldsymbol{b}^{n+\theta}\|_{0,\infty,\Omega}\left(|u^n|_{k+1,\infty,\Omega} + |u^{n+1}|_{k+1,\infty,\Omega}\right)\left(|u^n|_{k+1,\Omega}^2 + |u^{n+1}|_{k+1,\Omega}^2\right)\right)^{1/2},$$

$$
\begin{aligned}
X^N &= \max_{n=0,\dots,N-1} \big(\varepsilon + h\,\|\boldsymbol{b}^{n+\theta}\|_{0,\infty,\Omega} + \|\sigma^{n+\theta}\|_{0,\infty,\Omega} \\
&\qquad\qquad + \sigma_0^{-1}\,\|\boldsymbol{b}^{n+\theta}\|_{0,\infty,\Omega}^2 + \sigma_0^{-1}\,\|c^{n+\theta}\|_{0,\infty,\Omega}^2\big)^{1/2}, \\
Y^N &= h^{1/2} \max_{n=0,\dots,N-1} \|\boldsymbol{b}^{n+\theta}\|_{0,\infty,\Omega}^{1/2},
\end{aligned}
$$

where $N = 1, 2, \dots, N_T$.

Theorem 13. *Let $\theta \in [1/2, 1]$ be given. Let the weak solution of (3) satisfy $u, u_t \in L^2(0,T;H^{k+1}(\Omega))$ for some $k \in \{1,\dots,l\}$ and let $u_{tt} \in L^2(0,T;L^2(\Omega))$. Let $\widetilde{u}_b \in H^2(\Omega)$ be an extension of u_b and let $\widetilde{u}_{bh} = i_h \widetilde{u}_b$. Let $u_0 \in H^{k+1}(\Omega)$ and let $u_h^0 = i_h u_0$. Let $\{u_h^n\}_{n=0}^{N_T}$ be the solution of the local projection discretization (61). If $\tilde{\tau}_M$ is defined by (21) and $u_t \in L^3(0,T;W^{1,3}(\Omega))$, then the error estimate*

$$
\begin{aligned}
E^N &+ \left(\delta t \sum_{n=0}^{N-1} \sum_{M\in\mathscr{M}_h} \tilde{\tau}_M \|\kappa_M(P_M^{n+\theta}\nabla e^{n+\theta})\|_{0,3,M}^3\right)^{1/2} \\
&\le C\,h^k\,Q^N + C\,h^k\,R^N + C\,\delta t\,X^N\,\|u_t\|_{L^2(0,t^N;H^1(\Omega))} \\
&\qquad + C\,(\delta t)^{3/2}\,Y^N\,\|u_t\|_{L^3(0,t^N;W^{1,3}(\Omega))}^{3/2} + C\,\delta t\,\sigma_0^{-1/2}\,\|u_{tt}\|_{L^2(0,t^N;L^2(\Omega))}
\end{aligned}
$$

is satisfied for $N = 1, 2, \dots, N_T$. Moreover, if $\theta = 1/2$, $u_{tt} \in L^3(0,T;W^{1,3}(\Omega))$, and $u_{ttt} \in L^2(0,T;L^2(\Omega))$, then

$$
\begin{aligned}
E^N &+ \left(\delta t \sum_{n=0}^{N-1} \sum_{M\in\mathscr{M}_h} \tilde{\tau}_M \|\kappa_M(P_M^{n+\theta}\nabla e^{n+\theta})\|_{0,3,M}^3\right)^{1/2} \\
&\le C\,h^k\,Q^N + C\,h^k\,R^N + C\,(\delta t)^2\,X^N\,\|u_{tt}\|_{L^2(0,t^N;H^1(\Omega))} \\
&\qquad + C\,(\delta t)^3\,Y^N\,\|u_{tt}\|_{L^3(0,t^N;W^{1,3}(\Omega))}^{3/2} + C\,(\delta t)^2\,\sigma_0^{-1/2}\,\|u_{ttt}\|_{L^2(0,t^N;L^2(\Omega))}\,.
\end{aligned}
$$

If $u \in L^2(0,T;W^{k+1,\infty}(\Omega))$, then, in both estimates, R^N can be replaced by S^N.

If $\tilde{\tau}_M$ is defined by (22) and $u_t \in L^4(0,T;W^{1,4}(\Omega))$, then the following error estimate holds

$$
\begin{aligned}
E^N + \left(\delta t \sum_{n=0}^{N-1} d_h^{n+\theta}(u_h^{n+\theta}; e^{n+\theta}, e^{n+\theta})\right)^{1/2} &\le C\,h^k\,Q^N + C\,\delta t\,X^N\,\|u_t\|_{L^2(0,t^N;H^1(\Omega))} \\
&\quad + C\,\delta t\,T^{1/4}\,Y^N\,\|u_t\|_{L^4(0,t^N;W^{1,4}(\Omega))} + C\,\delta t\,\sigma_0^{-1/2}\,\|u_{tt}\|_{L^2(0,t^N;L^2(\Omega))}\,.
\end{aligned}
$$

Moreover, if $\theta = 1/2$, $u_{tt} \in L^4(0,T;W^{1,4}(\Omega))$, and $u_{ttt} \in L^2(0,T;L^2(\Omega))$, then

$$
\begin{aligned}
E^N + \left(\delta t \sum_{n=0}^{N-1} d_h^{n+\theta}(u_h^{n+\theta}; e^{n+\theta}, e^{n+\theta})\right)^{1/2} &\le C\,h^k\,Q^N + C\,(\delta t)^2\,X^N\,\|u_{tt}\|_{L^2(0,t^N;H^1(\Omega))} \\
&\quad + C\,(\delta t)^2\,T^{1/4}\,Y^N\,\|u_{tt}\|_{L^4(0,t^N;W^{1,4}(\Omega))} + C\,(\delta t)^2\,\sigma_0^{-1/2}\,\|u_{ttt}\|_{L^2(0,t^N;L^2(\Omega))}\,.
\end{aligned}
$$

Proof. Analogously to the steady-state case, the error will be split into an interpolation error and a remainder which belongs to the finite element space. The decomposition of the error e^α with any $\alpha \in [0, N_T]$ has the form

$$e^\alpha = \eta^\alpha - e_h^\alpha \qquad \text{with} \qquad \eta^\alpha := u^\alpha - \bar{r}_h^\alpha\,, \quad e_h^\alpha := u_h^\alpha - \bar{r}_h^\alpha \in V_h\,,$$

where we use the abbreviation $\bar{r}_h^\alpha = \tilde{r}_h^\alpha u$ with $\tilde{r}_h^\alpha$ given by (60). Using this decomposition, one obtains with the triangle inequality and with (57)

$$\begin{aligned} \|e^N\|_{0,\Omega}^2 &+ \delta t \sum_{n=0}^{N-1} \|e^{n+\theta}\|_{\mathrm{LPS}}^2 + \delta t \sum_{n=0}^{N-1} d_h^{n+\theta}(\gamma_0^{n+\theta}; e^{n+\theta}, e^{n+\theta}) \\ &\le 4\left[\|\eta^N\|_{0,\Omega}^2 + \delta t \sum_{n=0}^{N-1} \|\eta^{n+\theta}\|_{\mathrm{LPS}}^2 + \delta t \sum_{n=0}^{N-1} d_h^{n+\theta}(\gamma_1^{n+\theta}; \eta^{n+\theta}, \eta^{n+\theta})\right] \\ &\quad + 4\left[\|e_h^N\|_{0,\Omega}^2 + \delta t \sum_{n=0}^{N-1} \|e_h^{n+\theta}\|_{\mathrm{LPS}}^2 + \delta t \sum_{n=0}^{N-1} d_h^{n+\theta}(\gamma_2^{n+\theta}; e_h^{n+\theta}, e_h^{n+\theta})\right], \end{aligned} \tag{67}$$

where $\gamma_0^{n+\theta} = e^{n+\theta}$, $\gamma_1^{n+\theta} = \eta^{n+\theta}$, $\gamma_2^{n+\theta} = e_h^{n+\theta}$ if $\tilde{\tau}_M$ is defined by (21) and $\gamma_0^{n+\theta} = \gamma_1^{n+\theta} = \gamma_2^{n+\theta} = u_h^{n+\theta}$ if $\tilde{\tau}_M$ is defined by (22).

First let us estimate the interpolation errors. The starting point is the identity

$$\eta^{n+\theta} = u^{n+\theta} - \theta\, u^{n+1} - (1-\theta)\, u^n + \theta\,(u^{n+1} - r_h u^{n+1}) + (1-\theta)\,(u^n - r_h u^n)\,. \tag{68}$$

One has

$$u^{n+\theta} - \theta\, u^{n+1} - (1-\theta)\, u^n = (1-\theta) \int_{t^n}^{t^{n+\theta}} u_t(t)\, \mathrm{d}t - \theta \int_{t^{n+\theta}}^{t^{n+1}} u_t(t)\, \mathrm{d}t\,, \tag{69}$$

which, in view of (42), leads to

$$\begin{aligned} \|\eta^{n+\theta}\|_{0,\Omega} &\le C\, h^{k+1}\,(|u^n|_{k+1,\Omega} + |u^{n+1}|_{k+1,\Omega}) + \sqrt{\delta t}\, \|u_t\|_{L^2(t^n,t^{n+1};L^2(\Omega))}\,, \\ |\eta^{n+\theta}|_{1,\Omega} &\le C\, h^{k}\,(|u^n|_{k+1,\Omega} + |u^{n+1}|_{k+1,\Omega}) + \sqrt{\delta t}\, \|u_t\|_{L^2(t^n,t^{n+1};H^1(\Omega))}\,. \end{aligned}$$

Using Taylor's formula with integral remainder or applying successively integration by parts gives

$$u^n = u^{n+\theta} - \theta\, \delta t\, u_t^{n+\theta} + \int_{t^{n+\theta}}^{t^n} u_{tt}(t)\,(t^n - t)\, \mathrm{d}t\,, \tag{70}$$

$$u^{n+1} = u^{n+\theta} + (1-\theta)\, \delta t\, u_t^{n+\theta} + \int_{t^{n+\theta}}^{t^{n+1}} u_{tt}(t)\,(t^{n+1} - t)\, \mathrm{d}t\,. \tag{71}$$

This may be used to derive improved interpolation estimates with respect to the time step provided that $u_{tt} \in L^2(0,T;H^1(\Omega))$. Indeed,

$$u^{n+\theta} - \theta\, u^{n+1} - (1-\theta)\, u^n = -(1-\theta) \int_{t^n}^{t^{n+\theta}} u_{tt}(t)\,(t-t^n)\,\mathrm{d}t - \theta \int_{t^{n+\theta}}^{t^{n+1}} u_{tt}(t)\,(t^{n+1}-t)\,\mathrm{d}t\,, \tag{72}$$

which leads to

$$\begin{aligned}
\|\eta^{n+\theta}\|_{0,\Omega} &\le C\,h^{k+1}\,(|u^n|_{k+1,\Omega} + |u^{n+1}|_{k+1,\Omega}) + (\delta t)^{3/2}\,\|u_{tt}\|_{L^2(t^n,t^{n+1};L^2(\Omega))}\,,\\
|\eta^{n+\theta}|_{1,\Omega} &\le C\,h^{k}\,(|u^n|_{k+1,\Omega} + |u^{n+1}|_{k+1,\Omega}) + (\delta t)^{3/2}\,\|u_{tt}\|_{L^2(t^n,t^{n+1};H^1(\Omega))}\,.
\end{aligned}$$

Now let us estimate the norms of the interpolation error in (67). In view of (60), (42), (16), (18), (5), and (4), one has

$$\begin{aligned}
&\|\eta^N\|_{0,\Omega} = \|u^N - r_h u^N\|_{0,\Omega} \le C\,h^{k+1}\,|u^N|_{k+1,\Omega}\,,\\
&\|\eta^{n+\theta}\|_{\mathrm{LPS}} \le \big(\varepsilon + C\,h\,\|\boldsymbol{b}^{n+\theta}\|_{0,\infty,\Omega}\big)^{1/2}\,|\eta^{n+\theta}|_{1,\Omega} + \|\sigma^{n+\theta}\|_{0,\infty,\Omega}^{1/2}\,\|\eta^{n+\theta}\|_{0,\Omega}\,.
\end{aligned}$$

Furthermore, analogously as in (51), for any $p \in [2,6]$, one obtains

$$\begin{aligned}
\|\kappa_M(P_M^{n+\theta}\nabla\eta^{n+\theta})\|_{0,p,M} &\le C\,|u^{n+\theta} - \theta\, i_h u^{n+1} - (1-\theta)\, i_h u^n|_{1,p,M}\\
&\quad + C\,h_M^{\frac{d}{p}-\frac{d}{2}}\,\big(|\varrho_h(u^n - i_h u^n)|_{1,M} + |\varrho_h(u^{n+1} - i_h u^{n+1})|_{1,M}\big)\,.
\end{aligned} \tag{73}$$

If $\tilde\tau_M$ is defined by (21), this inequality implies that

$$d_h^{n+\theta}(\eta^{n+\theta};\eta^{n+\theta},\eta^{n+\theta}) \le C\,(I + II)\,,$$

where

$$\begin{aligned}
I &:= h\,\|\boldsymbol{b}^{n+\theta}\|_{0,\infty,\Omega} \sum_{M\in\mathscr{M}_h} |u^{n+\theta} - \theta\, u^{n+1} - (1-\theta)\, u^n|_{1,3,M}^3\\
II &:= h\,\|\boldsymbol{b}^{n+\theta}\|_{0,\infty,\Omega} \sum_{M\in\mathscr{M}_h} \big(|u^{n+1} - i_h u^{n+1}|_{1,3,M}^3 + |u^n - i_h u^n|_{1,3,M}^3\big)\\
&\quad + h\,\|\boldsymbol{b}^{n+\theta}\|_{0,\infty,\Omega} \sum_{M\in\mathscr{M}_h} h_M^{-\frac{d}{2}}\,\big(|\varrho_h(u^n - i_h u^n)|_{1,M}^3 + |\varrho_h(u^{n+1} - i_h u^{n+1})|_{1,M}^3\big)\,.
\end{aligned}$$

Using (69) and (72), one obtains

$$I \le C\,h\,(\delta t)^2\,\|\boldsymbol{b}^{n+\theta}\|_{0,\infty,\Omega}\,\|u_t\|_{L^3(t^n,t^{n+1};W^{1,3}(\Omega))}^3\,,$$

resp.

$$I \le C\,h\,(\delta t)^5\,\|\boldsymbol{b}^{n+\theta}\|_{0,\infty,\Omega}\,\|u_{tt}\|_{L^3(t^n,t^{n+1};W^{1,3}(\Omega))}^3\,.$$

Furthermore, it follows from (13), (38), (11), (6), and (4) that

$$II \le C\,h\,\|\boldsymbol{b}^{n+\theta}\|_{0,\infty,\Omega} \sum_{M\in\mathscr{M}_h} h_M^{3\,k-d/2}\,(|u^n|^3_{k+1,M} + |u^{n+1}|^3_{k+1,M})\,, \tag{74}$$

which implies in view of (4) and (5) that

$$II \le C\,h^{3\,k+1-d/2}\,\|\boldsymbol{b}^{n+\theta}\|_{0,\infty,\Omega}\,(|u^n|^3_{k+1,\Omega} + |u^{n+1}|^3_{k+1,\Omega})\,.$$

If $u \in L^2(0,T;W^{k+1,\infty}(\Omega))$, the inequality (74) together with (4) and (5) implies that

$$II \le C\,h^{3\,k+1}\,\|\boldsymbol{b}^{n+\theta}\|_{0,\infty,\Omega}\,(|u^n|_{k+1,\infty,\Omega}\,|u^n|^2_{k+1,\Omega} + |u^{n+1}|_{k+1,\infty,\Omega}\,|u^{n+1}|^2_{k+1,\Omega})\,.$$

If $\tilde\tau_M$ is defined by (22), then, proceeding analogously as when deriving (58), but with (73) instead of (51), and applying (13) in addition, one gets

$$d_h^{n+\theta}(u_h^{n+\theta};\eta^{n+\theta},\eta^{n+\theta}) \le C\,\tilde I + C\,\|\boldsymbol{b}^{n+\theta}\|_{0,\infty,\Omega}\,h^{2\,k+1}\,(|u^n|^2_{k+1,\Omega} + |u^{n+1}|^2_{k+1,\Omega})\,,$$

where

$$\tilde I := h\,\|\boldsymbol{b}^{n+\theta}\|_{0,\infty,\Omega} \sum_{M\in\mathscr{M}_h} h_M^{d/2}\,|u^{n+\theta} - \theta\,u^{n+1} - (1-\theta)\,u^n|^2_{1,4,M}\,.$$

Similarly as above, one obtains

$$\tilde I \le C\,h\,(\delta t)^{3/2}\,\|\boldsymbol{b}^{n+\theta}\|_{0,\infty,\Omega}\,\|u_t\|^2_{L^4(t^n,t^{n+1};W^{1,4}(\Omega))}\,,$$

resp.

$$\tilde I \le C\,h\,(\delta t)^{7/2}\,\|\boldsymbol{b}^{n+\theta}\|_{0,\infty,\Omega}\,\|u_{tt}\|^2_{L^4(t^n,t^{n+1};W^{1,4}(\Omega))}\,.$$

Now let us estimate the norms of the discrete part of the error on the right-hand side of (67). To derive an equation for this part of the error, the weak formulation (59) at $t = t^{n+\theta}$ is subtracted from (61) with $v = v_h = e_h^{n+\theta}$. Then, using the fact that $u_h^\alpha = e_h^\alpha + \bar r_h^\alpha$, one deduces that

$$\begin{aligned}(e_h^{n+1} - e_h^n, e_h^{n+\theta}) + \delta t\,\|e_h^{n+\theta}\|^2_{\mathrm{LPS}} + \delta t\,d_h^{n+\theta}(u_h^{n+\theta};u_h^{n+\theta},e_h^{n+\theta})&\\ = \delta t\left[\left(u_t^{n+\theta} - \frac{\bar r_h^{n+1} - \bar r_h^n}{\delta t}, e_h^{n+\theta}\right) + a^{n+\theta}(\eta^{n+\theta},e_h^{n+\theta}) - s_h^{n+\theta}(\bar r_h^{n+\theta},e_h^{n+\theta})\right]&.\end{aligned} \tag{75}$$

Furthermore, one obtains

$$d_h^{n+\theta}(u_h^{n+\theta};u_h^{n+\theta},e_h^{n+\theta}) \ge \frac{1}{7}\,d_h^{n+\theta}(\gamma_2^{n+\theta};e_h^{n+\theta},e_h^{n+\theta}) + d_h^{n+\theta}(\gamma_3^{n+\theta};\bar r_h^{n+\theta},e_h^{n+\theta})\,, \tag{76}$$

where $\gamma_3^{n+\theta} = \bar r_h^{n+\theta}$ if $\tilde\tau_M$ is defined by (21) and $\gamma_3^{n+\theta} = u_h^{n+\theta}$ if $\tilde\tau_M$ is defined by (22) ($\gamma_2^{n+\theta}$ was defined below (67)). This follows from (26) if $\tilde\tau_M$ is defined by (21) and simply by writing the second argument of $d_h^{n+\theta}$ as $e_h^{n+\theta}+\bar{r}_h^{n+\theta}$ and using the fact that $d_h^{n+\theta}(u_h^{n+\theta};e_h^{n+\theta},e_h^{n+\theta})\geq 0$ if $\tilde{\tau}_M$ is defined by (22). Since $\theta\geq 1/2$, it follows from (66) with $\widetilde{u}$ replaced by e that

$$(e_h^{n+1}-e_h^n,e_h^{n+\theta})\geq\frac{1}{2}\left(\|e_h^{n+1}\|_{0,\Omega}^2-\|e_h^n\|_{0,\Omega}^2\right). \tag{77}$$

Substituting (76) and (77) into (75) and summing up over the discrete times yields an upper bound for the discrete part of the estimate (67)

$$\begin{aligned}\|e_h^N\|_{0,\Omega}^2&+\delta t\sum_{n=0}^{N-1}\|e_h^{n+\theta}\|_{\mathrm{LPS}}^2+\delta t\sum_{n=0}^{N-1}d_h^{n+\theta}(\gamma_2^{n+\theta};e_h^{n+\theta},e_h^{n+\theta})\\&\leq\frac{7}{2}\|e_h^0\|_{0,\Omega}^2+7\,\delta t\sum_{n=0}^{N-1}\left[\left(u_t^{n+\theta}-\frac{\bar{r}_h^{n+1}-\bar{r}_h^n}{\delta t},e_h^{n+\theta}\right)+a^{n+\theta}(\eta^{n+\theta},e_h^{n+\theta})\right.\\&\qquad\left.-s_h^{n+\theta}(\bar{r}_h^{n+\theta},e_h^{n+\theta})-d_h^{n+\theta}(\gamma_3^{n+\theta};\bar{r}_h^{n+\theta},e_h^{n+\theta})\right].\end{aligned} \tag{78}$$

Using (39), (11), (5), and (4), one obtains

$$\|e_h^0\|_{0,\Omega}=\|i_hu^0-r_hu^0\|_{0,\Omega}=\|\varrho_h(u^0-i_hu^0)\|_{0,\Omega}\leq C\,h^{k+1}\,|u^0|_{k+1,\Omega}.$$

Applying the Cauchy-Schwarz and Young inequalities gives

$$\left(u_t^{n+\theta}-\frac{\bar{r}_h^{n+1}-\bar{r}_h^n}{\delta t},e_h^{n+\theta}\right)\leq\frac{1}{\sigma_0}\left\|u_t^{n+\theta}-\frac{\bar{r}_h^{n+1}-\bar{r}_h^n}{\delta t}\right\|_{0,\Omega}^2+\frac{1}{4}\|e_h^{n+\theta}\|_{\mathrm{LPS}}^2.$$

The last term can be hidden in the left-hand side of (78). The first term is a mixture of discretization errors in time and space. Elimination of $u^{n+\theta}$ from (70) and (71) yields

$$u_t^{n+\theta}=\frac{u^{n+1}-u^n}{\delta t}-\frac{1}{\delta t}\int_{t^n}^{t^{n+\theta}}u_{tt}(t)\,(t^n-t)\;\mathrm{d}t-\frac{1}{\delta t}\int_{t^{n+\theta}}^{t^{n+1}}u_{tt}(t)\,(t^{n+1}-t)\;\mathrm{d}t\,.$$

Since interpolation in space and differentiation in time commute, one has

$$u^{n+1}-\bar{r}_h^{n+1}-(u^n-\bar{r}_h^n)=\int_{t^n}^{t^{n+1}}(u_t-r_hu_t)(t)\,\mathrm{d}t\,.$$

Thus, applying the Cauchy-Schwarz inequality, one derives

$$\left\|u_t^{n+\theta}-\frac{\bar{r}_h^{n+1}-\bar{r}_h^n}{\delta t}\right\|_{0,\Omega}^2\leq\frac{2}{\delta t}\|u_t-r_hu_t\|_{L^2(t^n,t^{n+1};L^2(\Omega))}^2+2\,\delta t\,\|u_{tt}\|_{L^2(t^n,t^{n+1};L^2(\Omega))}^2\,.$$

The first term on the right-hand side can be bounded using (42).

Assuming $u_{ttt} \in L^2(0,T;L^2(\Omega))$ and replacing (70) and (71) by

$$u^n = u^{n+\theta} - \theta\,\delta t\, u_t^{n+\theta} + \frac{\theta^2}{2}\,(\delta t)^2\, u_{tt}^{n+\theta} + \frac{1}{2}\int_{t^{n+\theta}}^{t^n} u_{ttt}(t)\,(t^n - t)^2\;\mathrm{d}t\,,$$

$$u^{n+1} = u^{n+\theta} + (1-\theta)\,\delta t\, u_t^{n+\theta} + \frac{(1-\theta)^2}{2}\,(\delta t)^2\, u_{tt}^{n+\theta} + \frac{1}{2}\int_{t^{n+\theta}}^{t^{n+1}} u_{ttt}(t)\,(t^{n+1} - t)^2\;\mathrm{d}t\,,$$

one obtains

$$\begin{aligned} u_t^{n+\theta} &= \frac{u^{n+1}-u^n}{\delta t} + \frac{\delta t}{2}\,[\theta^2 - (1-\theta)^2]\,u_{tt}^{n+\theta} \\ &\quad - \frac{1}{2\,\delta t}\int_{t^n}^{t^{n+\theta}} u_{ttt}(t)\,(t^n-t)^2\;\mathrm{d}t - \frac{1}{2\,\delta t}\int_{t^{n+\theta}}^{t^{n+1}} u_{ttt}(t)\,(t^{n+1}-t)^2\;\mathrm{d}t\,, \end{aligned}$$

which shows that an improved estimate with respect to δt follows for $\theta = 1/2$, i.e., for the Crank-Nicolson scheme. Indeed, one gets

$$\left\| u_t^{n+1/2} - \frac{\bar r_h^{n+1} - \bar r_h^n}{\delta t}\right\|_{0,\Omega}^2 \le \frac{2}{\delta t}\,\|u_t - r_h u_t\|^2_{L^2(t^n,t^{n+1};L^2(\Omega))} + (\delta t)^3\,\|u_{ttt}\|^2_{L^2(t^n,t^{n+1};L^2(\Omega))}\,.$$

Now let us consider the remaining three terms on the right-hand side of (78). According to (68) and (60), one has

$$\begin{aligned} a^{n+\theta}(\eta^{n+\theta}, e_h^{n+\theta}) - s_h^{n+\theta}(\bar r_h^{n+\theta}, e_h^{n+\theta}) &= a^{n+\theta}(u^{n+\theta} - \theta\, u^{n+1} - (1-\theta)\, u^n, e_h^{n+\theta}) \\ &\quad + \theta\left[a^{n+\theta}(u^{n+1} - r_h u^{n+1}, e_h^{n+\theta}) - s_h^{n+\theta}(r_h u^{n+1}, e_h^{n+\theta})\right] \\ &\quad + (1-\theta)\left[a^{n+\theta}(u^n - r_h u^n, e_h^{n+\theta}) - s_h^{n+\theta}(r_h u^n, e_h^{n+\theta})\right]. \end{aligned}$$

The last two terms can be estimated by (43) and the estimation of the first term on the right-hand side is performed using

$$\|u^{n+\theta} - \theta\, u^{n+1} - (1-\theta)\, u^n\|^2_{1,\Omega} \le \delta t\,\|u_t\|^2_{L^2(t^n,t^{n+1};H^1(\Omega))}\,,$$

resp.

$$\|u^{n+\theta} - \theta\, u^{n+1} - (1-\theta)\, u^n\|^2_{1,\Omega} \le (\delta t)^3\,\|u_{tt}\|^2_{L^2(t^n,t^{n+1};H^1(\Omega))}\,,$$

which follows from (69), resp. (72). Finally, the last term on the right-hand side of (78) can be estimated analogously as (49), (54), and (56): if $\tilde\tau_M$ is defined by (21), one derives

$$d_h^{n+\theta}(\bar r_h^{n+\theta}; \bar r_h^{n+\theta}, \bar r_h^{n+\theta}) \le C\,\|\boldsymbol{b}^{n+\theta}\|_{0,\infty,\Omega}\, h^{3\,k+1-d/2}\,(|u^n|^3_{k+1,\Omega} + |u^{n+1}|^3_{k+1,\Omega})\,,$$

if, in addition, $u \in L^2(0,T;W^{k+1,\infty}(\Omega))$, then

$$\begin{aligned} &d_h^{n+\theta}(\bar r_h^{n+\theta}; \bar r_h^{n+\theta}, \bar r_h^{n+\theta}) \\ &\qquad \le C\,\|\boldsymbol{b}^{n+\theta}\|_{0,\infty,\Omega}\, h^{3\,k+1}\,(|u^n|_{k+1,\infty,\Omega} + |u^{n+1}|_{k+1,\infty,\Omega})(|u^n|^2_{k+1,\Omega} + |u^{n+1}|^2_{k+1,\Omega})\,, \end{aligned}$$

and, if $\tilde{\tau}_M$ is defined by (22), then

$$d_h^{n+\theta}(u_h^{n+\theta};\bar{r}_h^{n+\theta},\bar{r}_h^{n+\theta}) \le C\,\|\boldsymbol{b}^{n+\theta}\|_{0,\infty,\Omega}\,h^{2\,k+1}\,(|u^n|_{k+1,\Omega}^2+|u^{n+1}|_{k+1,\Omega}^2)\,.$$

These estimates together with analogs of (48) and (55) lead to an estimate of the term $d_h^{n+\theta}(\gamma_3^{n+\theta};\bar{r}_h^{n+\theta},e_h^{n+\theta})$.

Collecting all the above estimates proves the theorem. □

5. Examples of spaces and partitions satisfying the hypotheses

This section is devoted to the presentation of some examples of spaces W_h and D_M and partitions $\mathscr{M}_h$ satisfying the hypotheses from Section 2. For simplicity, the discussion is restricted to the two-dimensional case. In three dimensions, the spaces can be constructed analogously. Throughout this section, $\{\mathscr{T}_h\}_{h>0}$ stands for a regular family of triangulations of $\overline{\Omega}$. This family is formed either by triangles or by convex quadrilaterals K with diameters h_K and one has $h=\max_{K\in\mathscr{T}_h} h_K$. In what follows, $\widehat{K}$ stands for a reference mesh cell, which is either a triangle or a square, depending on the type of elements in $\mathscr{T}_h$. For any $K\in\mathscr{T}_h$, there exists a bijective mapping $F_K:\widehat{K}\to K$ that maps $\widehat{K}$ onto K and is affine if $\widehat{K}$ is a triangle and bilinear if $\widehat{K}$ is a square. For any integer $l\ge 0$, we denote by P_l the space of polynomials of total degree at most l and by Q_l the space of polynomials of degree at most l in each variable. Finally, we set $R_l(\widehat{K})=P_l(\widehat{K})$ if $\widehat{K}$ is a triangle and $R_l(\widehat{K})=Q_l(\widehat{K})$ if $\widehat{K}$ is a square.

i) *The two-level approach.* This is the approach considered in the original local projection stabilization method (cf. [2, 3]). The starting point is $\{\mathscr{M}_h\}_{h>0}$, a shape regular family of triangulations of $\overline{\Omega}$. Then, each triangle is divided into three triangles by connecting its vertices with the barycenter and each quadrilateral is divided into four quadrilaterals by connecting midpoints of opposite edges. The resulting triangulation is denoted by $\mathscr{T}_h$. Then, given an integer $l\ge 1$, the spaces W_h and D_M are given by

$$W_h := \{v_h\in C(\overline{\Omega})\,;\ v_h|_K\circ F_K\in R_l(\widehat{K})\ \ \forall\,K\in\mathscr{T}_h\}\,,\qquad D_M := P_{l-1}(M)\,. \tag{79}$$

The inf-sup condition (9) is proved for this pair in [28].

Alternatively, for the quadrilateral case, the space D_M could be defined as the space of mapped polynomials. More precisely, we can present the following two alternative definitions for D_M:

$$D_M^1 := \{v\in L^2(M)\,;\ v\circ F_M\in P_{l-1}(\widehat{M})\}\,,$$
$$D_M^2 := \{v\in L^2(M)\,;\ v\circ F_M\in Q_{l-1}(\widehat{M})\}\,,$$

where $\widehat{M}$ is a reference macro-cell and F_M is the analog of F_K. Both definitions lead to different methods (both different from the one presented so far) and have the advantage that the computations can be done directly on the reference element, leading to simpler implementations. All the approximation and stability assumptions hold for D^2_M, but for D^1_M the approximation property (12) holds only on uniformly refined meshes (see [29, pp. 345–346] for a discussion on the topic).

ii) *The one-level approach.* This alternative was introduced in [28] and assumes $\mathscr{M}_h = \mathscr{T}_h$. Introducing a polynomial bubble function $b_{\widehat{K}} \in H^1_0(\widehat{K}) \setminus \{0\}$ (cubic if $\widehat{K}$ is a triangle and biquadratic if $\widehat{K}$ is a square), the spaces are given by

$$W_h := \{v_h \in C(\overline{\Omega})\,;\ v_h|_K \circ F_K \in R_l(\widehat{K}) + b_{\widehat{K}} \cdot R_{l-1}(\widehat{K})\ \ \forall\, K \in \mathscr{T}_h\}\,, \qquad D_M := P_{l-1}(M)\,.$$

The inf-sup condition (9) is proved for this pair in [28].

iii) *The overlapping method.* Let $x_1, \dots, x_{N_h}$ be the inner vertices of the triangulation $\mathscr{T}_h$, define the neighborhoods $M_i := \bigcup_{K \in \mathscr{T}_h, x_i \in K} K$, and set $\mathscr{M}_h := \{M_i\}_{i=1}^{N_h}$. The spaces W_h and D_M are given by (79). The inf-sup condition (9) is proved for this pair in [22].

In all of the examples above, i_h can be chosen to be the Lagrange interpolation operator and j_M to be the orthogonal L^2 projection of $L^2(M)$ onto D_M (see, e.g., [12]). The validity of the geometrical hypotheses (4)–(7) follows from the mesh regularity. The inverse inequality (8) arises from a local inverse inequality (cf. [12]) and the mesh regularity. Finally, if F_K is linear for any $K \in \mathscr{T}_h$, then the space G_M consists of functions that are polynomial on the mesh cells included in M and the inverse inequality (10) is standard (cf. [12]).

Note that if the set $\mathscr{M}_h$ consists of nonoverlapping sets M, which is the case for both the one-level and two-level methods, then (significantly) more degrees of freedom are used for constructing the space W_h than in case of the method with overlapping sets M. This increase of the number of degrees of freedom is either due to an enrichment by bubble functions (in the one-level method) or due to a refinement of the given triangulation (in the two-level method). On the other hand, given a triangulation $\mathscr{T}_h$ of $\overline{\Omega}$ and using $\mathscr{M}_h$ consisting of overlapping sets M, the space W_h can be defined as a standard finite element space consisting of piecewise polynomials of degree l on $\mathscr{T}_h$, like in the Galerkin discretization.

6. Numerical illustrations

In this section, the theory of this paper is illustrated by the results of numerical computations performed for the steady-state problem (1). From the three possibilities for spaces and partitions proposed in the preceding section, we have chosen the overlapping version of the LPS method. This is mainly due to the fact that, as shown in [22], the overlapping version

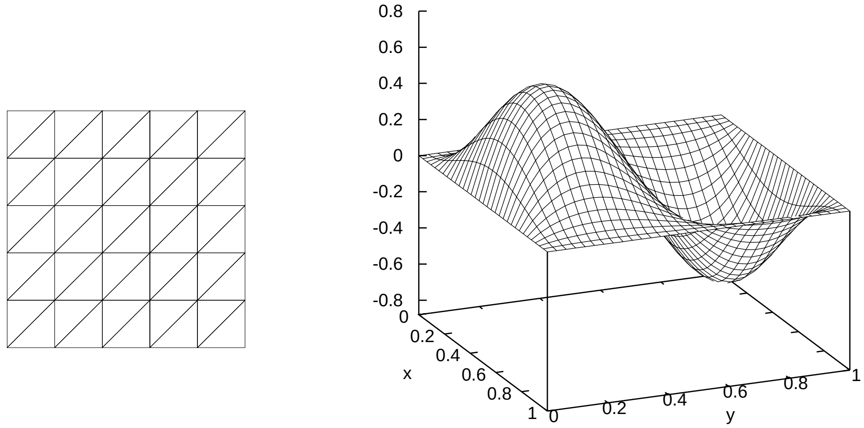

FIGURE 1. Type of the triangulations used in numerical computations (left) and exact solution for Example 1 (right).

is more robust with respect to the stabilization parameter than both the one- and two-level approaches. The overlapping version was applied with triangular meshes and conforming piecewise linear approximation spaces W_h (thus $l = 1$). The solution of the nonlinear system was performed using a fixed point iteration with damping (treating the stabilization parameter $\tau_M^{\rm sold}(u_h)$ explicitly), as proposed in [18]. Both possible definitions (21) and (22) of $\tilde{\tau}_M(u_h)$ were considered.

In the below examples, $\Omega = (0,1)^2$ and Friedrichs-Keller triangulations of the type depicted in Fig. 1 were used. It is worth mentioning that the mesh is not aligned with the convection.

Example 1. *Smooth polynomial solution* [20]*, support of error estimates.* We consider problem (1) with $\varepsilon = 10^{-8}$, $\boldsymbol{b} = (3,2)^T$, $c = 2$ and $u_b = 0$. The right-hand side f is chosen such that

$$u(x,y) = 100\,x^2\,(1-x)^2\,y\,(1-y)\,(1-2\,y)\,,$$

is the solution of (1), see Fig. 1.

In the stabilization parameters, the values $\tau_0 = 0.02$ and $\beta = 0.1$ were used. Table 1 shows errors of the discrete solutions measured in various norms for various mesh sizes. The notation $\|\cdot\|_{0,\infty,h}$ is used for the discrete L^∞ norm defined as the maximum of the errors at the vertices of the respective triangulation. The convergence orders were computed using values from the two finest triangulations. One can observe that the convergence order with respect to the LPS norm is 3/2, as predicted by the theory, and that in other norms one obtains the usual optimal convergence orders.

TABLE 1. Example 1, errors of the discrete solutions.

	parameter (21)				parameter (22)			
h	$\|\cdot\|_{\mathrm{LPS}}$	$\|\cdot\|_{0,\Omega}$	$\lvert\cdot\rvert_{1,\Omega}$	$\|\cdot\|_{0,\infty,h}$	$\|\cdot\|_{\mathrm{LPS}}$	$\|\cdot\|_{0,\Omega}$	$\lvert\cdot\rvert_{1,\Omega}$	$\|\cdot\|_{0,\infty,h}$
8.84−2	4.74−2	1.83−2	4.20−1	6.46−2	4.30−2	1.47−2	4.00−1	5.04−2
4.42−2	1.48−2	3.54−3	1.88−1	1.52−2	1.41−2	2.93−3	1.84−1	1.13−2
2.21−2	5.02−3	7.24−4	9.02−2	3.40−3	4.93−3	6.57−4	8.96−2	2.44−3
1.10−2	1.76−3	1.58−4	4.45−2	7.63−4	1.75−3	1.57−4	4.44−2	5.57−4
5.52−3	6.19−4	3.63−5	2.21−2	1.77−4	6.18−4	3.83−5	2.21−2	1.44−4
order	1.50	2.12	1.01	2.11	1.50	2.03	1.01	1.95

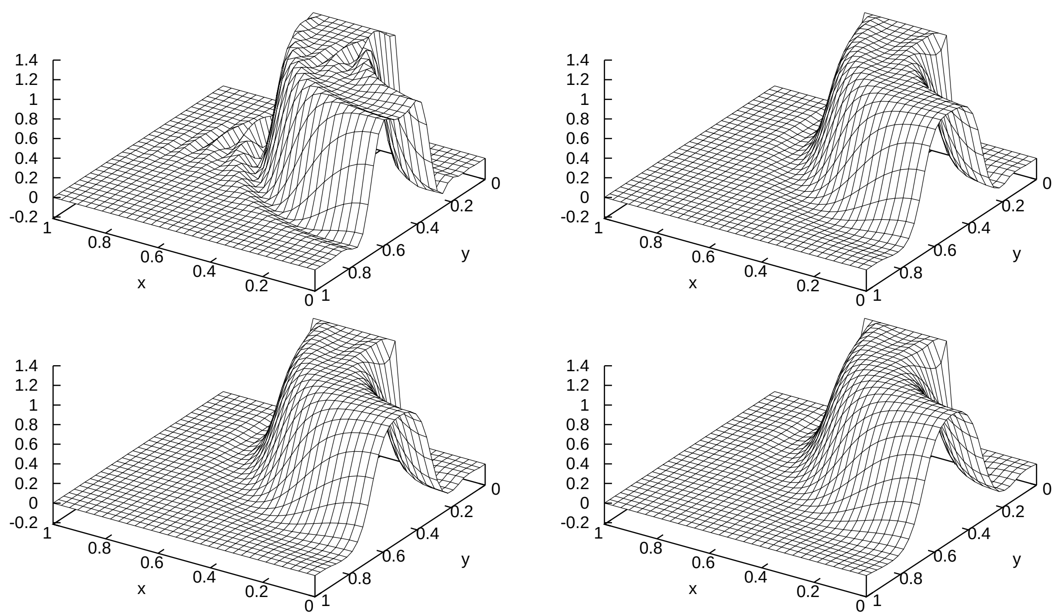

FIGURE 2. Example 2: solutions for the parameter (22) with $\tau_0 = 0.02$ and $\beta = 0$, $\beta = 0.03$, $\beta = 0.05$, $\beta = 0.1$, left to right, top to bottom.

Example 2. *Solution with two interior layers* [25], *reduction of spurious oscillations.* Equation (1) was considered with $\varepsilon = 10^{-8}$, $\boldsymbol{b}(x,y) = (-y,x)^T$, $c = f = 0$, and the boundary conditions

$$u = u_b \quad \text{on } \Gamma^D, \qquad \frac{\partial u}{\partial \boldsymbol{n}} = 0 \quad \text{on } \Gamma^N,$$

where $\Gamma^N = \{0\} \times (0,1)$, $\Gamma^D = \partial\Omega \setminus \overline{\Gamma^N}$, $\boldsymbol{n}$ is the outward pointing unit normal vector to the boundary of Ω, and

$$u_b(x,y) = \begin{cases} 1 & \text{for } (x,y) \in (1/3,2/3) \times \{0\}, \\ 0 & \text{else on } \Gamma^D. \end{cases}$$

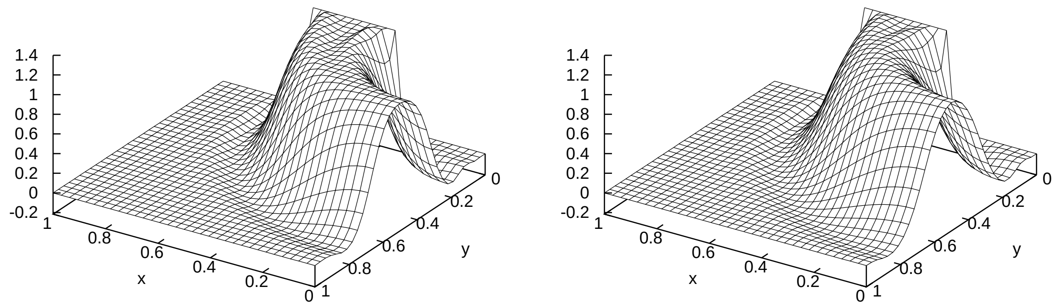

FIGURE 3. Example 2: solutions for the parameter (21) with $\tau_0 = 0.02$, $\beta = 0.03$ (left) and $\tau_0 = 0.02$, $\beta = 0.1$ (right).

Results are presented that were obtained on the triangulation of the type from Fig. 1 having 33×33 vertices. Figure 2 shows results for the LPS method with the nonlinear crosswind diffusion term d_h defined using the parameter (22). One can observe that the crosswind diffusion term manages to reduce the oscillations appearing in the solution of the linear LPS method. An increase of the parameter β does not only reduce the oscillations but also increases the smearing appearing at the layers. In this respect, the method behaves as expected. Two results obtained for d_h defined using the parameter (21) are shown in Fig. 3. A detailed comparison of the results in Figs. 2 and 3 reveals that the method with the parameter (21) is less successful in suppressing spurious oscillations whereas it leads to a more pronounced smearing.

From the discussion of the preceding paragraph, the choice of the stabilization parameter β appears as an important issue. A good choice of user-chosen parameters in stabilized finite element methods is an open problem for all methods. In general, the parameters need to be chosen not as constant but as function (see [18] for the construction of an example). A non-constant choice, done automatically like in [19], will be the subject of future research.

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