Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 0946 – 8633

Modeling and analysis of a phase field system for damage and phase separation processes in solids

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submitted: September 19, 2013

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No. 1841
Berlin 2013

2010 *Mathematics Subject Classification.* 35K85, 35K55, 49J40, 49S05, 35J50, 74A45, 74G25, 34A12, 82B26, 82C26, 35K92, 35K65, 35K35;

Key words and phrases. Cahn-Hilliard system, phase separation, elliptic-parabolic systems, doubly nonlinear differential inclusions, complete damage, existence results, energetic solutions, weak solutions, linear elasticity, rate-dependent systems.

This project is supported by the DFG Research Center "Mathematics for Key Technologies" MATHEON in Berlin (Germany) and by GNAMPA (Indam, Italy).

Edited by
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Abstract

In this work, we analytically investigate a multi-component system for describing phase separation and damage processes in solids. The model consists of a parabolic diffusion equation of fourth order for the concentration coupled with an elliptic system with material dependent coefficients for the strain tensor and a doubly nonlinear differential inclusion for the damage function. The main aim of this paper is to show existence of weak solutions for the introduced model, where, in contrast to existing damage models in the literature, different elastic properties of damaged and undamaged material are regarded. To prove existence of weak solutions for the introduced model, we start with an approximation system. Then, by passing to the limit, existence results of weak solutions for the proposed model are obtained via suitable variational techniques.

1 Introduction

The ongoing miniaturization in the area of micro-electronics leads to higher demands on strength and lifetime of the materials, while the structural size is continuously being reduced. Materials, which enable the functionality of technical products, change the microstructure over time. Phase separation, coarsening phenomena and damage processes take place. The complete failure of electronic devices like motherboards or mobile phones often results from micro–cracks in solder joints. Therefore, the knowledge of the mechanisms inducing phase separation, coarsening and damage phenomena is of great importance for technological applications. A uniform distribution of the original materials is aimed to guarantee evenly distributed material properties of the sample. For instance, mechanical properties, such as the strength and the stability of the material, depend on how finely regions of the original materials are mixed. The control of the evolution of the microstructure and, therefore, of the lifetime of materials relies on the ability to understand phase separation, coarsening and damage processes. Hence, a major aim is to develop reliable mathematical models for describing such effects.

Phase separation and coarsening phenomena are usually described by phase–field models of Cahn–Hilliard type. The evolution is modeled by a parabolic diffusion equation for the phase fractions. To include elastic effects, resulting from stresses caused by different elastic properties of the phases, Cahn-Hilliard systems are coupled with an elliptic equation, describing quasi-static balance of forces. Such coupled Cahn-Hilliard systems with elasticity are also called Cahn-Larché systems. Since in general the mobility, stiffness and surface tension coefficients depend on the phases (see for instance [BDM07] and [BDDM07] for the explicit structure deduced by the embedded atom method), the mathematical analysis of the coupled problem is very complex. Existence results were derived for special cases in [Gar00, CMP00, BP05] (constant mobility, stiffness and surface tension coefficients), in [BCD$^+$02] (concentration dependent mobility, two space dimensions), [SP12, SP13] (concentration dependent surface tension and nonlinear diffusion) and in [PZ08] in an abstract measure-valued setting (concentration dependent mobility and surface tension tensors). For numerical results and simulations we refer e.g. to [Wei01, Mer05, BM10].

From a microscopic point of view, damage behavior originates from breaking atomic links in the material whereas a macroscopic theory may specify damage in the isotropic case by a scalar variable related to the proportion of damaged bonds in the micro-structure of the material with respect to the undamaged ones. According to the latter perspective, phase-field models are quite common to model smooth transitions between damaged and undamaged material states. Such phase-field models have been mainly investigated for incomplete damage which means that damaged material cannot loose all its elastic energy.

A first local in time existence result for a 3D damage model has been introduced in [BS04], where irreversibility of the damage evolution is accounted for. Damage for viscoelastic materials, in which also viscosity degenerates during the damage process, is investigated in [BSS05]. Damage models are also analytically investigated in [MT10, KRZ11] and, there, existence, uniqueness and regularity properties are shown. These models do not account for temperature effect. A local in time existence result for a complete dissipative damage model with the evolving of temperature can be found in [BB08]. A *coupled* system describing incomplete *damage*, linear elasticity and *phase separation* appeared firstly in [HK11, HK13b]. There, existence of weak solutions has been proven under mild assumptions, where, for instance, the stiffness tensor may be material-dependent and the chemical free energy may be of polynomial or logarithmic type. All these works are based on the gradient-of-damage model proposed by Frémond and Nedjar [FN96] (see also [Fré02]) which describes damage as a result from microscopic movements in the solid. The distinction between a balance law for the microscopic forces and constitutive relations of the material yield a satisfying derivation of an evolution law for the damage propagation from the physical point of view. In particular, the gradient of the damage variable enters the resulting equations and serves as a regularization term for the mathematical analysis as well as it ensures the structural size effect. Internal constraints are ensured by the presence of non-smooth operators (subdifferential operators) in the evolution equations. Hence, in the case that the evolution of the damage is assumed to be uni-directional, i.e. the damage process is irreversible, the microforce balance law becomes a doubly-nonlinear differential inclusion.

For a non-gradient approach of damage models for brittle materials we refer to [FG06, GL09, Bab11]. There, the damage variable z takes on two distinct values, i.e. $\{0, 1\}$, in contrast to the gradient approach, where $z \in [0, 1]$. In addition, the mechanical properties are described in [FG06, GL09, Bab11] differently. They choose a z-mixture of a linearly elastic strong and weak material with two different elasticity tensors. A non-gradient model for incomplete damage in the framework of Young measures is considered in [FKS12].

Damage modeling is an active field in the engineering community since the 1970s. We do not actually detail literature. For some recent works we refer to [Car86, DPO94, Mie95, MK00, MS11, Fré02, JL05, GUE$^+$07, VSL11]. A variational approach to fracture and crack propagation models can be found for instance in [BFM08, CFM09, CFM10, Neg10, LT11].

The reason why incomplete damage models are more feasible for mathematical investigations is that a coercivity assumption on the elastic energy prevents the material from a complete degeneration. Typically, the following form is chosen:

$$W^{\mathrm{el}}(e, z) = \frac{1}{2}(\Phi(z) + \delta)\, \mathbb{C}e : e, \qquad \delta > 0 \text{ small}, \tag{1}$$

where $\Phi : [0, 1] \to \mathbb{R}_+$ is a continuous and monotonically increasing function with $\Phi(0) = 0$. The symbol $\mathbb{C}$ denotes the stiffness tensor and e is the strain tensor.

Dropping $\delta > 0$ in (1) may lead to serious troubles. However, in the case of viscoelastic materials, the inertia terms circumvent this kind of problem in the sense that the deformation field still exists on the whole domain accompanied with a loss of spatial regularity (cf. [RR12]). Unfortunately, this result cannot be expected in the case of quasi-static mechanical equilibrium (see for instance [BMR09]). Mathematical works dealing with complete models and covering global-in-time existence are rare and are mainly focused on purely *rate-independent systems* [MR06, BMR09, MRZ10, Mie11] by using Γ-convergence techniques to recover energetic properties in the limit. Very recently, global-in-time existence results are also obtained for *rate-dependent systems* in [HK12, HK13a] by considering the damage process on a time-dependent domain. Alternatively, in [BFS13] the problem of understanding complete damage is tackled using some defect measures which are conjectured to concentrate on the (complete) damaged portions of the material. This theoretical prediction is supported by numerical simulations.

The main aim of this work is to show existence of weak solutions of a unified model for phase separation and damage processes, where, in contrast to the existing incomplete damage models in the literature [MT10, KRZ11, HK11, HK13b] or local in time damage evolution [BS04, BSS05], different elastic properties of damaged and undamaged material are regarded. More precisely, we choose an elastic energy density W^{el} of the form

$$W^{\mathrm{el}}(e,c,z) = \Phi(z)\, W_1^{\mathrm{el}}(e,c) + \big(1-\Phi(z)\big) W_2^{\mathrm{el}}(e,c), \tag{2}$$

where $\Phi : [0,1] \to [0,1]$ is a continuously differentiable and monotonically increasing function with $\Phi(0) = \Phi'(0) = 0$, $\Phi(1) = 1$, $W_1^{\mathrm{el}} \geq W_2^{\mathrm{el}}$ and c is the concentration field. This means that for undamaged material the elastic energy density W_1^{el} is stored, whereas in the completely damaged case $z = 0$ the energy W_2^{el} is stored. For the elastic energy W_1^{el} we assume an H^1-coercivity condition for u and for W_2^{el} a weaker $W^{1,p}$-coercivity condition, $1 < p < 2$.

Our highly nonlinear model covers the intermediate case between incomplete and complete damage which takes care for different deformation properties of damaged and undamaged material. It consists of a parabolic diffusion equation of fourth order for the concentration coupled with an elliptic system with material dependent coefficients for the strain tensor and a doubly nonlinear differential inclusion for the damage function, see Definition (S_0) on page 5.

The paper is organized as follows: In Section 2, we start with introducing the model formally and stating the notation and assumptions. Then, we introduce an appropriate notion of weak solutions for our introduced system in Subsection 2.4. To handle the differential inclusion rigorously, we adapt the concept of weak solutions which has been proposed in [HK11] for phase separation systems coupled with rate-dependent damage processes. The main result is stated in Subsection 2.5. Section 3 is devoted to the existence proof of the proposed model. The proof is based on an approximation-a priori estimates-passage to the limit procedure. In particular, the limit analysis relies on the monotone structure of the system.

2 Modeling

We consider an N-component alloy occupying a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^3$. To account for deformation, phase separation and damage processes, a state of the system at a fixed time point is specified by the triple $q = (u,c,z)$. The displacement field $u : \Omega \to \mathbb{R}^3$ determines the current position $x + u(x)$ of an undeformed material point x. Throughout this paper, we will work with the linearized strain tensor $e(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$, which is an adequate assumption only when small strains occur in the material. However, this assumption is justified for phase separation processes in alloys since the deformation usually has a small gradient. The vector-valued function $c : \Omega \to \mathbb{R}^N$ describes the chemical concentration of the N-components, which satisfies the normalized condition $\sum_{j=1}^N c_j = 1$ in Ω. To account for damage effects, we choose a scalar damage variable $z : \Omega \to \mathbb{R}$, which models the reduction of the effective volume of the material due to void nucleation, growth, and coalescence. The damage process is modeled unidirectional, i.e. damage may only increase. In particular, self-healing processes in the material are forbidden. No damage at a material point $x \in \Omega$ is described by $z(x) = 1$, whereas $z(x) = 0$ stands for a completely damaged material point $x \in \Omega$.

2.1 Energies and evolutionary equations

Here, we qualify our model formally and postpone a rigorous treatment to Section 2.4. The presented model is based on two functionals, i.e. a generalized Ginzburg-Landau free energy functional $\mathcal{E}$ and a damage pseudo-dissipation potential $\mathcal{R}$ (in the sense by Moreau). The free energy density φ of the system is given by

$$\varphi(e(u), c, \nabla c, z, \nabla z) := \frac{\gamma}{2}\mathbf{\Gamma}\nabla c : \nabla c + \frac{\delta}{2}|\nabla z|^2 + W^{\mathrm{ch}}(c) + W^{\mathrm{el}}(e(u), c, z), \qquad \gamma, \delta > 0, \tag{3}$$

where the gradient terms penalize spatial changes of the variables c and z, W^{ch} denotes the chemical energy density and W_{el} is the elastically stored energy density accounting for elastic deformations and damage effects. For simplicity of notation, we set $\gamma = \delta = 1$.

The *chemical free energy density* W^{ch} depends on temperature, which is convex above a critical temperature value and non-convex below. Therefore, if an alloy is cooled down below the critical temperature, spinodal decomposition and coarsening phenomena occur due to the several local minimizers of W^{ch}. We assume that the chemical energy is of polynomial type. More precisely, we need the assumptions (A13)-(A14) of Section 2.3 for a rigorous treatment.

The *elastically stored energy density* W_1^{el} in (2) due to stresses and strains, which occur in the material, is typically of quadratic form, i.e.

$$W_1^{\mathrm{el}}(e(u), c) = \frac{1}{2}\big(e(u) - e^*(c)\big) : \mathbb{C}(c)\big(e(u) - e^*(c)\big). \tag{4}$$

Here, $e^*(c)$ denotes the *eigenstrain*, which is usually linear in c, and $\mathbb{C}(c) \in \mathcal{L}(\mathbb{R}^{n\times n}_{\mathrm{sym}})$ is a fourth order stiffness tensor, which may depend on the concentration. The stiffness tensor is assumed to be symmetric and positive definite. Note that we are not restricted to homogeneous elasticity.

To incorporate the effect of damage on the elastic response of the material, we choose an elastic energy density W^{el} of the form (2), i.e.

$$W^{\mathrm{el}}(e(u), c, z) = \Phi(z)\, W_1^{\mathrm{el}}(e(u), c) + \big(1 - \Phi(z)\big) W_2^{\mathrm{el}}(e(u), c), \tag{5}$$

where $\Phi : [0, 1] \to \mathbb{R}_+$ is a continuously differentiable and monotonically increasing function with $\Phi(0) = \Phi'(0) = 0$, $\Phi(1) = 1$ and $W_1^{\mathrm{el}} \geq W_2^{\mathrm{el}}$. This means that in the undamaged case the material accumulates the elastic energy density W_1^{el}, whereas in the completely damaged case only the lower energy W_2^{el} is stored. Hence, in particular, different elastic properties of damaged and undamaged material can be modeled.

We assume that W_1^{el} is of quadratic growth in e, whereas W_2^{el} only has to satisfy a lower p-growth condition, $1 < p < 2$. This means that the displacement field for damaged material only need to be an element of $L^p(\Omega)$, $1 < p < 2$. The complete growth conditions for W^{el} can be found in Section 2.3.

The overall free energy $\mathcal{E}$ of Ginzburg-Landau type has the following structure:

$$\begin{aligned} \mathcal{E}(u, c, z) &:= \tilde{\mathcal{E}}(u, c, z) + \int_\Omega I_{[0,\infty)}(z)\,\mathrm{d}x, \\ \tilde{\mathcal{E}}(u, c, z) &:= \int_\Omega \varphi(e(u), c, \nabla c, z, \nabla z)\,\mathrm{d}x. \end{aligned} \tag{6}$$

Here, $I_{[0,\infty)}$ signifies the indicator function of the subset $[0, \infty) \subseteq \mathbb{R}$, i.e. $I_{[0,\infty)}(x) = 0$ for $x \in [0, \infty)$ and $I_{[0,\infty)}(x) = \infty$ for $(-\infty, 0)$.

We assume that the energy dissipation for the damage process is triggered by a dissipation potential $\mathcal{R}$ of the form

$$\begin{aligned} \mathcal{R}(\dot{z}) &:= \tilde{\mathcal{R}}(\dot{z}) + \int_\Omega I_{(-\infty,0]}(\dot{z})\,\mathrm{d}x, \\ \tilde{\mathcal{R}}(\dot{z}) &:= \int_\Omega \Big(-\alpha\dot{z} + \frac{1}{2}\beta\dot{z}^2\Big)\,\mathrm{d}x \quad \text{for } \alpha \geq 0 \text{ and } \beta > 0. \end{aligned} \tag{7}$$

Due to $\beta > 0$, the dissipation potential is referred to as *rate-dependent*. In the case $\beta = 0$, which is not considered in this work, $\mathcal{R}$ is called *rate-independent*. We refer for rate-independent processes to [EM06, MT99, MR06, MRZ10, Rou10] and in particular to [Mie05] for a survey.

The governing evolutionary equations for a system state $q = (u, c, z)$ can be expressed by virtue of the functionals (6) and (7). The evolution is driven by the following elliptic-parabolic system of differential equations and differential inclusion:

$$\left.\begin{cases} \textit{Diffusion}: & \partial_t c &= \mathrm{div}(\mathbb{M}\nabla w) \\ & w &= \mathbb{P}\big((-\mathrm{div}(\mathbf{\Gamma}\nabla c) + W^{\mathrm{ch}}_{,c}(c) + W^{\mathrm{el}}_{,c}(e(u), c, z)\big) \\ \textit{Balance of forces}: & \mathrm{div}\,\sigma &= f \\ \textit{Damage evolution}: & 0 &\in \partial_z \mathcal{E}(u, c, z) + \partial_{\dot z}\mathcal{R}(\partial_t z) \end{cases}\right\} \qquad (S_0)$$

where $\sigma = \sigma(e, c, z) := \partial_e \varphi(e, c, \nabla c, z, \nabla z)$ denotes the Cauchy stress tensor, w is the chemical potential given by $w = w(u, c, z) := \partial_c \varphi(e, c, \nabla c, z, \nabla z) - \mathrm{div}(\partial_{\nabla c}\varphi(e, c, \nabla c, z, \nabla z))$ and $-f$ stands for the exterior volume force applied to the body. The matrix $\mathbb{P}$ denotes the orthogonal projection of $\mathbb{R}^N$ onto the tangent space $T\Sigma = \big\{x \in \mathbb{R}^N \mid \sum_{k=1}^N x_k = 0\big\}$ of the affine plane $\Sigma := \big\{x \in \mathbb{R}^N \mid \sum_{l=1}^N x_k = 1\big\}$. The diffusion equation is a fourth order quasi-linear parabolic equation of Cahn-Hilliard type and models phase separation processes for the concentration c while the balance of forces is described by an elliptic equation for u. The doubly nonlinear differential inclusion specifies the flow rule of the damage profile according to the constraints $0 \leq z \leq 1$ and $\partial_t z \leq 0$ (in space and time). Actually, we have $z \leq 1$ combining the two constraints $z \geq 0$ and $\partial_t z \leq 0$ (irreversible damage), once the initial datum is lower than 1. The inclusion has to be read in terms of generalized subdifferentials.

We need to impose some restrictions on the mobility matrix $\mathbb{M}$. We assume that $\mathbb{M}$ is symmetric and positive definite on the tangent space $T\Sigma$. In addition, due to the constraint $\sum_{k=1}^N c_k = 1$, $\mathbb{M}$ has to satisfy the property $\sum_{l=1}^N \mathbb{M}_{kl} = 0$ for all $k = 1, \ldots, N$. Note, that $\mathbb{M} = \mathbb{M}\,\mathbb{P}$. The gradient tensor Γ is assumed to be symmetric and positive definite.

Let $D \subset \partial\Omega$ with $\mathcal{H}^{n-1}(D) > 0$ ($\mathcal{H}^n$: n-dimensional Hausdorff measure) denote the portion of the boundary $\partial\Omega$ on which we prescribe Dirichlet boundary conditions. We set $D_T := (0, T) \times D$ and $(\partial\Omega)_T := (0, T) \times \partial\Omega$. The initial-boundary conditions of our system are summarized as follows:

Initial conditions

$$\begin{aligned} &c(0) = c^0 \text{a.e. in } \Omega \quad \text{ and } \quad c^0 \in \Sigma \text{a.e. in } \Omega, \\ &0 \leq z(0) = z^0 \leq 1 \text{a.e. in } \Omega. \end{aligned}$$

Boundary conditions (IBC)

$$\begin{aligned} &u = b \text{ on } D_T, \quad \sigma \cdot \overrightarrow{\nu} = 0 \text{ on } (\partial\Omega)_T \setminus D_T. \\ &\nabla z \cdot \overrightarrow{\nu} = 0 \text{ on } (\partial\Omega)_T, \quad \mathbf{\Gamma}\nabla c \cdot \overrightarrow{\nu} = 0 \text{ on } (\partial\Omega)_T, \qquad \mathbb{M}\nabla w \cdot \overrightarrow{\nu} = 0 \text{ on } (\partial\Omega)_T, \end{aligned}$$

where $\overrightarrow{\nu}$ stands for the unit normal on $\partial\Omega$ pointing outward and b is the boundary value function on the Dirichlet boundary D, which can be suitably extended to a function on $\overline{\Omega_T}$.

To show existence of weak solutions for the system (S_0), we first consider a regularized version for the displacement field:

Regularized energy

$$\tilde{\mathcal{E}}_\varepsilon(u, c, z) := \int_\Omega \Big(\frac{1}{2}\mathbf{\Gamma}\nabla c : \nabla c + \frac{1}{2}|\nabla z|^2 + W^{\mathrm{ch}}(c) + W^{\mathrm{el}}(e, c, z) + \frac{\varepsilon}{4}|\nabla u|^4\Big)\,\mathrm{d}x,$$

$$\mathcal{E}_\varepsilon(u,c,z) := \tilde{\mathcal{E}}_\varepsilon(u,c,z) + \int_\Omega I_{[0,\infty)}(z)\,\mathrm{d}x.$$

Evolution system

$$\left.\begin{cases} \textit{Diffusion}: & \partial_t c \quad = \operatorname{div}(\mathbb{M}\nabla w) \\ & w \quad = \mathbb{P}(-\operatorname{div}(\mathbf{\Gamma}\nabla c) + W^{\mathrm{ch}}_{,c}(c) + W^{\mathrm{el}}_{,c}(e(u),c,z)) \\ \textit{Balance of forces}: & \operatorname{div}\sigma + \varepsilon\operatorname{div}(|\nabla u|^2\nabla u) = f \\ \textit{Damage evolution}: & 0 \quad \in \partial_z\mathcal{E}_\varepsilon(u,c,z) + \partial_{\dot z}\mathcal{R}(\partial_t z) \end{cases}\right\} \tag{S_ε}$$

Initial-boundary conditions

$$\text{(IBC) with } (\sigma + \varepsilon|\nabla u|^2\nabla u)\cdot\overrightarrow{\nu} = 0 \quad \text{instead of} \quad \sigma\cdot\overrightarrow{\nu} = 0 \quad \text{on } (\partial\Omega)_T. \tag{IBC$_\varepsilon$}$$

2.2 Notation

The notation, we will use throughout this paper, is collected in the following list.

Spaces and sets.

$W^{1,r}(\Omega;\mathbb{R}^n)$	standard Sobolev space
$W^{1,r}_+(\Omega)$	functions of $W^{1,r}(\Omega)$ which are non-negative almost everywhere
$W^{1,r}_-(\Omega)$	functions of $W^{1,r}(\Omega)$ which are non-positive almost everywhere
$W^{1,r}_D(\Omega;\mathbb{R}^n)$	functions of $W^{1,r}(\Omega;\mathbb{R}^n)$ which vanish on $D\subseteq\partial\Omega$ in the sense of traces
G_T	$(0,T)\times G$
$\mathbb{R}_+$	$\{x\in\mathbb{R} : x\geq 0\}$

Functions, operations and measures.

I_M	indicator function of a subset $M\subseteq X$
$W_{,e}$	classical partial derivative of a function W with respect to the variable e
$\langle g^\star, f\rangle$	dual pairing of $g^\star\in(W^{1,r}(\Omega;\mathbb{R}^n))^\star$ and $f\in W^{1,r}(\Omega;\mathbb{R}^n)$
$\mathrm{d}E$	Gâteaux differential of E
$p^\star$	Sobolev critical exponent $\frac{np}{n-p}$ for $n>p$
$\mathcal{H}^n$	Hausdorff measure of dimension n
$\mathcal{L}^n$	Lebesgue measure of dimension n
Σ	$\left\{x\in\mathbb{R}^N \mid \sum_{l=1}^N x_k = 1\right\}$
$T\Sigma$	$\left\{x\in\mathbb{R}^N \mid \sum_{l=1}^N x_k = 0\right\}$

2.3 Assumptions

The general setting, the growth assumptions and the assumptions on the coefficient tensors which are mandatory for the existence theorem are summarized below.

(i) *Setting*

Space dimension	$n \in \mathbb{N}$,
Components in the alloy	$N \in \mathbb{N}$ with $N \geq 2$,
Regularization exponent	$1 < p < 2$,
Conjugate exponent	$p' = \dfrac{p}{p-1}$,
Growth exponent	$s < \dfrac{n(p-1)}{n-p}$,
Viscosity factors	$\alpha, \beta > 0$,
Domain	$\Omega \subseteq \mathbb{R}^n$ bounded Lipschitz domain,
Dirichlet boundary	$D \subseteq \partial\Omega$ with $\mathcal{H}^{n-1}(D) > 0$,
Time interval	$[0,T]$ with $T > 0$,
External volume force	$f \in W^{1,1}(0,T;L^{p'}(\Omega;\mathbb{R}^n))$ with $f(0) = f^0 \in L^{p'}(\Omega;\mathbb{R}^n)$,
Constant	$C > 0$ (context dependent)

(ii) *Energy densities*

$\Phi \in \mathcal{C}^1([0,1];[0,1])$ monotonically increasing with $\Phi(0) = \Phi'(0) = 0$ and $\Phi(1) = 1$.

Elastic energy density W_1^{el}: $W_1^{\mathrm{el}} \in \mathcal{C}^1(\mathbb{R}^{n\times n} \times \mathbb{R}^N;\mathbb{R})$ with

$$W_1^{\mathrm{el}}(e,c) = W_1^{\mathrm{el}}(e^t,c), \tag{A1}$$

$$|W_1^{\mathrm{el}}(e,c)| \leq C(|e|^2 + |c|^2 + 1), \tag{A2}$$

$$C|e_1 - e_2|^2 \leq \big(W_{1,e}^{\mathrm{el}}(e_1,c) - W_{1,e}^{\mathrm{el}}(e_2,c)\big) : (e_1 - e_2), \tag{A3}$$

$$|W_{1,e}^{\mathrm{el}}(e_1 + e_2, c)| \leq C(W_1^{\mathrm{el}}(e_1,c) + |e_2| + 1), \tag{A4}$$

$$|W_{1,c}^{\mathrm{el}}(e,c)| \leq C(|e|^2 + |c|^2 + 1) \tag{A5}$$

for any $e_1, e_2 \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ and $c \in \Sigma$,

$h_c(\cdot) = W_{1,e}^{\mathrm{el}}(\cdot,c) - W_{1,e}^{\mathrm{el}}(0,c)$ is positively 1-homogeneous, i.e.

$$h_c(\lambda e) = \lambda h_c(e) \text{ for any } \lambda > 0 \text{ and all } e \in \mathbb{R}^{n\times n}_{\mathrm{sym}}. \tag{A6}$$

Elastic energy density W_2^{el}: $W_2^{\mathrm{el}} \in \mathcal{C}^1(\mathbb{R}^{n\times n} \times \mathbb{R}^N;\mathbb{R})$ with

$$W_2^{\mathrm{el}}(e,c) = W_2^{\mathrm{el}}(e^t,c), \tag{A7}$$

$$W_2^{\mathrm{el}}(e,c) \leq W_1^{\mathrm{el}}(e,c), \tag{A8}$$

$$|W_2^{\mathrm{el}}(e,c)| \leq C(|e|^p + |c|^s + 1), \tag{A9}$$

$$C|e_1 - e_2|^p \leq (W_{2,e}^{\mathrm{el}}(e_1,c) - W_{2,e}^{\mathrm{el}}(e_2,c)) : (e_1 - e_2), \tag{A10}$$

$$|W_{2,e}^{\mathrm{el}}(e_1 + e_2, c)| \leq C(W_2^{\mathrm{el}}(e_1,c) + |e_2|^{p-1} + 1), \tag{A11}$$

$$|W_{2,c}^{\mathrm{el}}(e,c)| \leq C(|e|^p + |c|^s + 1) \tag{A12}$$

for any $e_1, e_2 \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ and $c \in \Sigma$.

Chemical energy density: $W^{\mathrm{ch}} \in \mathcal{C}^1(\mathbb{R}^N;\mathbb{R})$ with $W^{\mathrm{ch}} \geq -C$, (A13)

$$|W^{\mathrm{ch}}_{,c}(c)| \le C(|c|^{2^\star/2}+1) \tag{A14}$$
for any $c \in \Sigma$.

(iii) *Tensors*

Mobility tensor	$\mathbb{M} \in \mathbb{R}^{N\times N}$ symmetric and positive definite on $T\Sigma$ and $\sum_{l=1}^{N} \mathbb{M}_{kl} = 0$ for all $k = 1, \ldots, N$.
Energy gradient tensor	$\mathbf{\Gamma} \in \mathcal{L}(\mathbb{R}^{N\times n}; \mathbb{R}^{N\times n})$ constant, symmetric and positive definite fourth order tensor.

Note that (A3), (A4), (A10) and (A11) imply the growth conditions

$$W_1(e,c) \ge C_1|e|^2 - C_2(|c|^4+1) \qquad \text{and} \qquad W_2^{\mathrm{el}}(e,c) \ge C_1|e|^p - C_2(|c|^{sp'}+1) \tag{8}$$

for all $c \in \Sigma$ and $e \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$.

Let us point out that the above properties are satisfied in the case we choose W_1 as in (4) and for W_2 we may take, for instance,

$$W_2^{\mathrm{el}}(c, e(u)) = \frac{1}{2}\big((e(u)-\hat{e}(c)) : \hat{\mathbb{C}}(c)(e(u)-\hat{e}(c))\big)^{p/2} - C, \quad 1 < p < 2,$$

where $C \ge 0$ is some constant.

2.4 Weak formulation

In this subsection, we state the notion of weak solutions for our proposed system and its regularized version. We use the concept of weak solutions introduced in [HK11] which consists of an energy inequality and a variational inequality for the doubly nonlinear differential inclusion.

The next Proposition (see [HK11, HK13b]) collects the basic properties of this concept of weak solution. In particular, note that the sole condition (*ii*) is weaker than the usual variational inequality that characterizes the doubly nonlinear inclusion (*i*).

Proposition 2.1 *Let $(u, c, w, z) \in \mathcal{C}^2(\Omega_T; \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R})$ satisfy the diffusion equation and the equation of balance of forces of (S_0) with initial-boundary conditions* (IBC). *Then the following two properties are equivalent for all $t \in [0, T]$:*

(i) $0 \in \partial_z \mathcal{E}(u(t), c(t), z(t)) + \partial_{\dot z}\mathcal{R}(\dot z(t))$,

(ii) Energy inequality

$$\begin{aligned}\mathcal{E}(u(t), c(t), z(t)) &+ \int_0^t \langle \mathrm{d}_{\dot z}\tilde{\mathcal{R}}(\partial_t z), \partial_t z\rangle \,\mathrm{d}s + \int_{\Omega_t} \nabla w : \mathbb{M}\nabla w \,\mathrm{d}x\mathrm{d}s - \int_\Omega f(t)\cdot u(t)\,\mathrm{d}x\mathrm{d}x \\ &\le \mathcal{E}(u(0), c(0), z(0)) + \int_{\Omega_t} W^{\mathrm{el}}_{,e}(e(u), c, z) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}s - \int_{\Omega_t} \partial_t f\cdot u\,\mathrm{d}x\mathrm{d}s \\ &\qquad - \int_\Omega f(0)\cdot u(0)\,\mathrm{d}x\end{aligned}$$

and the variational inequality

$$0 \le \Big\langle \mathrm{d}_z\tilde{\mathcal{E}}(u(t), c(t), z(t)) + r(t) + \mathrm{d}_{\dot z}\tilde{\mathcal{R}}(\partial_t z(t)), \zeta\Big\rangle \tag{9}$$

for all $\zeta \in H^1_-(\Omega) \cap L^\infty(\Omega)$ and $r(t) \in \partial I(H^1_+(\Omega)\cap L^\infty(\Omega); z(t))$.

Note that if one of the two properties are satisfied then we even obtain the equation of balance of energy:

$$\begin{aligned}
&\mathcal{E}(u(t),c(t),z(t)) + \int_0^t \langle \mathrm{d}_{\dot z}\tilde{\mathcal{R}}(\partial_t z),\partial_t z\rangle \,\mathrm{d}s + \int_{\Omega_t} \nabla w : \mathbb{M}\nabla w \,\mathrm{d}x\mathrm{d}s - \int_\Omega f(t)\cdot u(t)\,\mathrm{d}x \\
&\quad = \mathcal{E}(u(0),c(0),z(0)) + \int_{\Omega_t} W^{\mathrm{el}}_{,e}(e(u),c,z) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}s - \int_{\Omega_t} \partial_t f\cdot u\,\mathrm{d}x\mathrm{d}s - \int_\Omega f(0)\cdot u(0)\,\mathrm{d}x
\end{aligned}$$

We would like to emphasize that the statement of Proposition 2.1 is also true for the diffusion equation and the equation of balance of forces (S_ε) with initial-boundary conditions $(\mathrm{IBC}_\varepsilon)$ if we replace $\mathcal{E}$ by $\mathcal{E}_\varepsilon$.

Definition 2.2 (Weak solutions for the regularized system (S_ε)) *A quadruple $q_\varepsilon = (u_\varepsilon, c_\varepsilon, w_\varepsilon, z_\varepsilon)$ is called a weak solution of the regularized system (S_ε) with the initial-boundary conditions $(\mathrm{IBC}_\varepsilon)$ if the following properties are satisfied:*

(i) Spaces
The components of q_ε are in the following spaces:

$$\begin{aligned}
&u_\varepsilon \in L^\infty(0,T;W^{1,4}(\Omega;\mathbb{R}^n)),\ u_\varepsilon|_{D_T} = b|_{D_T},\\
&c_\varepsilon \in L^\infty(0,T;H^1(\Omega;\mathbb{R}^N)) \cap H^1(0,T;(H^1(\Omega;\mathbb{R}^N))'),\ c_\varepsilon \in \Sigma \textit{ a.e. in } \Omega_T,\\
&z_\varepsilon \in L^\infty(0,T;H^1_+(\Omega)) \cap H^1(0,T;L^2(\Omega)),\ z_\varepsilon(0) = z^0,
\end{aligned}$$

and

$$w_\varepsilon \in L^2(0,T;H^1(\Omega;\mathbb{R}^N)).$$

(ii) Diffusion
For all $\zeta \in H^1(\Omega;\mathbb{R}^N)$ and for a.e. $t\in[0,T]$:

$$\int_{\Omega_T} \partial_t c_\varepsilon(t)\cdot\zeta\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} \mathbb{M}\nabla w_\varepsilon(t) : \nabla\zeta\,\mathrm{d}x\,\mathrm{d}t \tag{10}$$

For all $\zeta \in H^1(\Omega;\mathbb{R}^N)$ and for a.e. $t\in[0,T]$:

$$\begin{aligned}
\int_\Omega w_\varepsilon(t)\cdot\zeta\,\mathrm{d}x = &\int_\Omega \Big(\mathbb{P}\mathbf{\Gamma}\nabla c_\varepsilon(t) : \nabla\zeta + \mathbb{P}W^{\mathrm{ch}}_{,c}(c_\varepsilon(t))\cdot\zeta\Big)\,\mathrm{d}x \\
&+ \int_\Omega \mathbb{P}W^{\mathrm{el}}_{,c}(e(u_\varepsilon(t)),c_\varepsilon(t),z_\varepsilon(t))\cdot\zeta\,\mathrm{d}x
\end{aligned} \tag{11}$$

(iii) Balance of forces
For all $\zeta \in W^{1,4}_D(\Omega;\mathbb{R}^n)$ and for a.e. $t\in[0,T]$:

$$\int_\Omega W^{\mathrm{el}}_{,e}(e(u_\varepsilon(t)),c_\varepsilon(t),z_\varepsilon(t)) : e(\zeta)\,\mathrm{d}x + \varepsilon\int_\Omega |\nabla u_\varepsilon(t)|^2\nabla u_\varepsilon(t) : \nabla\zeta\,\mathrm{d}x = \int_\Omega f(t)\cdot\zeta\,\mathrm{d}x \tag{12}$$

(iv) Damage variational inequality
For all $\zeta \in H^1_-(\Omega)$ and for a.e. $t\in[0,T]$:

$$0 \le \int_\Omega \big(\nabla z_\varepsilon(t)\cdot\nabla\zeta + (W^{\mathrm{el}}_{,z}(e(u_\varepsilon(t)),c_\varepsilon(t),z_\varepsilon(t)) - \alpha + \beta(\partial_t z_\varepsilon(t)))\zeta\big)\,\mathrm{d}x, \tag{13}$$

$$0 \le z_\varepsilon(t), \tag{14}$$

$$0 \ge \partial_t z_\varepsilon(t). \tag{15}$$

(v) *Energy inequality*
For a.e. $t \in [0,T]$*:*

$$\begin{aligned}\mathcal{E}_\varepsilon(u_\varepsilon(t),c_\varepsilon(t),z_\varepsilon(t)) + \int_\Omega \alpha(z^0 - z_\varepsilon(t))\,\mathrm{d}x + \int_{\Omega_t} \beta|\partial_t z_\varepsilon|^2\,\mathrm{d}x\mathrm{d}s + \int_{\Omega_t} \nabla w_\varepsilon : \mathbb{M}\nabla w_\varepsilon \mathrm{d}x\mathrm{d}s&\\ - \int_\Omega f(t)\cdot u_\varepsilon(t)\,\mathrm{d}x&\\ \leq \mathcal{E}_\varepsilon(u_\varepsilon^0,c^0,z^0) + \int_{\Omega_t} W^{\mathrm{el}}_{,e}(e(u_\varepsilon),c_\varepsilon,z_\varepsilon) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}s + \varepsilon\int_{\Omega_t} |\nabla u_\varepsilon|^2\nabla u_\varepsilon : \nabla\partial_t b\,\mathrm{d}x\mathrm{d}s&\\ - \int_{\Omega_t}\partial_t f\cdot u_\varepsilon\,\mathrm{d}x\mathrm{d}s - \int_\Omega f(0)\cdot u_\varepsilon(0)\,\mathrm{d}x,&\end{aligned} \tag{16}$$

where u_ε^0 *is the unique minimizer of* $\mathcal{E}_\varepsilon(\,.\,,c^0,z^0) - \int_\Omega f(0)\cdot(\,.\,)\,\mathrm{d}x$ *in* $W^{1,4}(\Omega;\mathbb{R}^n)$ *with trace* $u^0|_D = b(0)|_D$*.*

Note that we can choose $r=0$ in (9) due to $\Phi(0)=\Phi'(0)=0$, see Lemma 3.7 and Remark 3.8 in [HK13b] for details.

Definition 2.3 (Weak solution for the limit system (S_0)) *A quadruple* $q=(u,c,w,z)$ *is called a weak solution of the system* (S_0) *with the initial-boundary conditions* (IBC) *if the following properties are satisfied:*

(i) *Spaces*
The components of q *are in the following spaces:*

$$\begin{aligned}&u\in L^\infty(0,T;W^{1,p}(\Omega;\mathbb{R}^n)),\ u|_{D_T}=b|_{D_T},\\ &c\in L^\infty(0,T;H^1(\Omega;\mathbb{R}^N))\cap H^1(0,T;(H^1(\Omega;\mathbb{R}^N))'),\ c\in\Sigma \text{ a.e. in } \Omega_T,\\ &z\in L^\infty(0,T;H^1_+(\Omega))\cap H^1(0,T;L^2(\Omega)),\ z(0)=z^0,\end{aligned}$$

and

$$w\in L^2(0,T;H^1(\Omega;\mathbb{R}^N)).$$

(ii) *Diffusion*
For all $\zeta\in L^2(0,T;H^1(\Omega;\mathbb{R}^N))$ *with* $\partial_t\zeta\in L^2(\Omega_T;\mathbb{R}^N)$ *and* $\zeta(T)=0$*:*

$$\int_{\Omega_T}(c-c^0)\cdot\partial_t\zeta\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T}\mathbb{M}\nabla w:\nabla\zeta\,\mathrm{d}x\,\mathrm{d}t \tag{17}$$

For all $\zeta\in H^1(\Omega;\mathbb{R}^N)\cap L^\infty(\Omega;\mathbb{R}^N)$ *and for a.e.* $t\in[0,T]$*:*

$$\begin{aligned}\int_\Omega w(t)\cdot\zeta\,\mathrm{d}x = \int_\Omega \big(\mathbb{P}\boldsymbol{\Gamma}\nabla c(t):\nabla\zeta + \mathbb{P}W^{\mathrm{ch}}_{,c}(c(t))\cdot\zeta\big)\,\mathrm{d}x&\\ + \int_\Omega \mathbb{P}W^{\mathrm{el}}_{,c}(e(u(t)),c(t),z(t))\cdot\zeta\,\mathrm{d}x&\end{aligned} \tag{18}$$

(iii) *Balance of forces*
For all $\zeta\in W^{1,p}_D(\Omega;\mathbb{R}^n)$ *and for a.e.* $t\in[0,T]$*:*

$$\int_\Omega W^{\mathrm{el}}_{,e}(e(u(t)),c(t),z(t)):e(\zeta)\,\mathrm{d}x = \int_\Omega f(t)\cdot\zeta\,\mathrm{d}x \tag{19}$$

(iv) *Damage variational inequality*
For all $\zeta \in H^1_-(\Omega) \cap L^\infty(\Omega)$ *and for a.e.* $t \in [0,T]$*:*

$$0 \leq \int_\Omega \big(\nabla z(t) \cdot \nabla \zeta + (W^{\mathrm{el}}_{,z}(e(u(t)), c(t), z(t)) - \alpha + \beta(\partial_t z(t)))\zeta\big)\,\mathrm{d}x, \tag{20}$$

$$0 \leq z(t), \tag{21}$$

$$0 \geq \partial_t z(t). \tag{22}$$

(v) *Energy inequality*
For a.e. $t \in [0,T]$*:*

$$\begin{aligned}\mathcal{E}(u(t), c(t), z(t)) + \int_\Omega \alpha(z^0 - z(t))\,\mathrm{d}x + \int_{\Omega_t} \beta|\partial_t z|^2\,\mathrm{d}x\mathrm{d}s + \int_{\Omega_t} \nabla w : \mathbb{M}\nabla w\,\mathrm{d}x\mathrm{d}s \\ - \int_\Omega f(t) \cdot u(t)\,\mathrm{d}x \\ \leq \mathcal{E}(u^0, c^0, z^0) + \int_{\Omega_t} W^{\mathrm{el}}_{,e}(e(u), c, z) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}s - \int_{\Omega_t} \partial_t f \cdot u\,\mathrm{d}x\mathrm{d}s - \int_\Omega f(0) \cdot u(0)\,\mathrm{d}x,\end{aligned} \tag{23}$$

where u^0 *is the unique minimizer of* $\mathcal{E}(\,.\,, c^0, z^0) - \int_\Omega f(0) \cdot (\,.\,)\,\mathrm{d}x$ *in* $W^{1,p}(\Omega;\mathbb{R}^n)$ *with trace* $u^0|_D = b(0)|_D$*.*

Note that both notions of weak solutions imply mass conservation, i.e.

$$\int_\Omega c(t)\,\mathrm{d}x \equiv \mathit{const}.$$

2.5 Main results

The main result of this work is the following theorem.

Theorem 2.4 (Existence theorem) *Let the assumptions of Section 2.3 be satisfied. Then for every*

$$\begin{aligned} &b \in W^{1,1}(0,T; W^{1,\infty}(\Omega;\mathbb{R}^n)),\\ &f \in W^{1,1}(0,T; L^{p'}(\Omega;\mathbb{R}^n)) \text{ with } f^0 = f(0) \in L^{p'}(\Omega;\mathbb{R}^n),\\ &c^0 \in H^1(\Omega;\mathbb{R}^N) \text{ with } c^0 \in \Sigma \text{ a.e. in } \Omega,\\ &z^0 \in H^1(\Omega) \text{ with } 0 \leq z^0 \leq 1 \text{ a.e. in } \Omega,\end{aligned}$$

there exists a weak solution q *of the system* (S_0) *in the sense of Definition 2.3 with the initial-boundary conditions* (IBC)*.*

3 Existence of weak solutions of $(\boldsymbol{S_0})$

By slight modifications of the proof of Theorem 2.5 in [HK13b], we can establish the following existence theorem.

Theorem 3.1 (Existence theorem, cf. [HK13b]) *Let the assumptions of Section 2.3 be satisfied. Then for every*

$$
\begin{aligned}
&b \in W^{1,1}(0,T;W^{1,\infty}(\Omega;\mathbb{R}^n)),\\
&f \in W^{1,1}(0,T;L^{p'}(\Omega;\mathbb{R}^n)) \text{ with } f^0 = f(0) \in L^{p'}(\Omega;\mathbb{R}^n),\\
&c^0 \in H^1(\Omega;\mathbb{R}^N) \text{ with } c^0 \in \Sigma \text{ a.e. in } \Omega,\\
&z^0 \in H^1(\Omega) \text{ with } 0 \le z^0 \le 1 \text{ a.e. in } \Omega,
\end{aligned}
$$

there exists a weak solution q_ε of the regularized system (S_ε) in the sense of Definition 2.2 with the initial-boundary conditions $(\mathrm{IBC}_\varepsilon)$.

Next, we will show that an appropriate subsequence of the regularized solutions q_ε for $\varepsilon \in (0,1]$ of Definition 2.2 converges in "some sense" to q which satisfies the limit equations given in Definition 2.3. For each $\varepsilon \in (0,1]$, we denote with $q_\varepsilon = (u_\varepsilon, c_\varepsilon, w_\varepsilon, z_\varepsilon)$ a solution according to Theorem 3.1.

Lemma 3.2 *For a.e. $t \in [0,T]$, $t = 0$ and every $\varepsilon \in (0,1]$:*

$$
\begin{aligned}
\mathcal{E}_\varepsilon(u_\varepsilon(t), c_\varepsilon(t), z_\varepsilon(t)) + \int_0^t \int_\Omega \big(-\alpha \partial_t z_\varepsilon + \beta |\partial_t z_\varepsilon|^2 \big)\,\mathrm{d}x\mathrm{d}s + \int_0^t \int_\Omega \nabla w_\varepsilon : \mathbb{M} \nabla w_\varepsilon \,\mathrm{d}x\mathrm{d}s&\\
\le C(\mathcal{E}_1(u_1^0, c^0, z^0) + 1).&
\end{aligned}
\tag{24}
$$

Proof. In the following, $C > 0$ denotes a context-dependent constant independently of t and ε. By means of (A4) and (A11), we estimate for $s \in [0,T]$:

$$
\begin{aligned}
&\int_\Omega \partial_e W^{\mathrm{el}}(e(u_\varepsilon(s), c_\varepsilon(s), z_\varepsilon(s)) : e(\partial_t b(s))\,\mathrm{d}x\\
&\qquad \le C\|\nabla \partial_t b(s)\|_{L^\infty(\Omega)} \int_\Omega \Big(W^{\mathrm{el}}(e(u_\varepsilon(s)), c_\varepsilon(s), z_\varepsilon(s)) + 1 \Big)\,\mathrm{d}x &(25)\\
&\qquad \le C\|\nabla \partial_t b(s)\|_{L^\infty(\Omega)} \big(\mathcal{E}_\varepsilon(e(u_\varepsilon(s)), c_\varepsilon(s), z_\varepsilon(s)) + 1 \big). &(26)
\end{aligned}
$$

In addition, for $s \in [0,T]$,

$$
\begin{aligned}
&\varepsilon \int_\Omega |\nabla u_\varepsilon(s)|^2 \nabla u_\varepsilon(s) : \nabla \partial_t b(s)\,\mathrm{d}x\\
&\qquad \le \varepsilon \|\nabla \partial_t b(s)\|_{L^\infty(\Omega)} \int_\Omega |\nabla u_\varepsilon(s)|^3\,\mathrm{d}x\\
&\qquad \le \varepsilon C \|\nabla \partial_t b(s)\|_{L^\infty(\Omega)} \Big(\int_\Omega |\nabla u_\varepsilon(s)|^4\,\mathrm{d}x + 1 \Big)\\
&\qquad \le C \|\nabla \partial_t b(s)\|_{L^\infty(\Omega)} \Big(\mathcal{E}_\varepsilon(e(u_\varepsilon(s)), c_\varepsilon(s), z_\varepsilon(s)) + 1 \Big) &(27)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_\Omega \partial_t f(s) \cdot u_\varepsilon(s)\,\mathrm{d}x &\le C \|\partial_t f(s)\|_{L^{p'}(\Omega)} \|u_\varepsilon(s)\|_{L^p(\Omega)}\\
&\le C \|\partial_t f(s)\|_{L^{p'}(\Omega)} \Big(\mathcal{E}_\varepsilon(e(u_\varepsilon(s)), c_\varepsilon(s), z_\varepsilon(s)) + 1 \Big), &(28)
\end{aligned}
$$

$$
\int_\Omega f(0) \cdot u_\varepsilon(0)\,\mathrm{d}x \le C \|f(0)\|_{L^{p'}(\Omega)} \Big(\mathcal{E}_\varepsilon(e(u_\varepsilon^0), c^0, z^0) + 1 \Big), \tag{29}
$$

$$\int_\Omega f(s)\cdot u_\varepsilon(s)\,\mathrm{d}x \le C\|f(s)\|^{p'}_{L^{p'}(\Omega)} + \frac{1}{2}\Big(\mathcal{E}_\varepsilon(e(u_\varepsilon(s)),c_\varepsilon(s),z_\varepsilon(s))+1\Big), \tag{30}$$

where the last inequality follows by the general Young's inequality. To simplify notation, we define the functions

$$\gamma_\varepsilon(t) := \frac{1}{2}\mathcal{E}_\varepsilon(e(u_\varepsilon(t)),c_\varepsilon(t),z_\varepsilon(t)) + \int_\Omega \alpha(z^0-z_\varepsilon(t))\,\mathrm{d}x + \int_{\Omega_t}\beta|\partial_t z_\varepsilon|^2\,\mathrm{d}x\mathrm{d}s + \int_{\Omega_t}\nabla w_\varepsilon : \mathbb{M}\nabla w_\varepsilon\,\mathrm{d}x\mathrm{d}s$$

and

$$h(s) := \|\nabla\partial_t b(s)\|_{L^\infty(\Omega)} + \|\partial_t f(s)\|_{L^{p'}(\Omega)}.$$

Using (25)–(29), the energy inequality (16) of the regularized system can be estimated for a.e. $t\in[0,T]$ as follows:

$$\begin{aligned}\gamma_\varepsilon(t) &\le \mathcal{E}_\varepsilon(e(u^0_\varepsilon),c^0,z^0) + C + C\int_0^t h(s)\,\mathcal{E}_\varepsilon(e(u_\varepsilon(s)),c_\varepsilon(s),z_\varepsilon(s))\,\mathrm{d}s \\ &\qquad + C\|f(0)\|_{L^{p'}(\Omega)}\mathcal{E}_\varepsilon(e(u^0_\varepsilon),c^0,z^0) \\ &\le C\mathcal{E}_\varepsilon(e(u^0_\varepsilon),c^0,z^0) + C + C\int_0^t h(s)\,\mathcal{E}_\varepsilon(e(u_\varepsilon(s)),c_\varepsilon(s),z_\varepsilon(s))\,\mathrm{d}s \\ &\le C\mathcal{E}_\varepsilon(e(u^0_\varepsilon),c^0,z^0) + C + C\int_0^t h(s)\,\gamma_\varepsilon(s)\,\mathrm{d}s\end{aligned}$$

Since $\mathcal{E}_\varepsilon(u^0_\varepsilon,c^0,z^0) - \int_\Omega f(0)\cdot u^0_\varepsilon\mathrm{d}x \le \mathcal{E}_\varepsilon(u^0_1,c^0,z^0) - \int_\Omega f(0)\cdot u^0_1\mathrm{d}x \le \mathcal{E}_1(u^0_1,c^0,z^0) - \int_\Omega f(0)\cdot u^0_1\mathrm{d}x$ Gronwall's inequality shows for a.e. $t\in[0,T]$ and every $\varepsilon\in(0,1]$:

$$\begin{aligned}\gamma_\varepsilon(t) &\le C + C\mathcal{E}_\varepsilon(e(u^0_\varepsilon),c^0,z^0) \\ &\qquad + C\int_0^t (C + C\mathcal{E}_\varepsilon(e(u^0_\varepsilon),c^0,z^0))h(s)\exp\left(\int_s^t h(l)\,\mathrm{d}l\right)\mathrm{d}s \\ &\le C(\mathcal{E}_1(u^0_1,c^0,z^0)+1).\end{aligned}$$

■

Lemma 3.3 (A-priori estimates) *There exists some constant $C>0$ independently of $\varepsilon>0$ such that for all $\varepsilon\in(0,1]$:*

(i) $\|u^0_\varepsilon\|_{W^{1,p}(\Omega;\mathbb{R}^n)} \le C$,
$\|u_\varepsilon\|_{L^\infty(0,T;W^{1,p}(\Omega;\mathbb{R}^n))} \le C$,

(ii) $\varepsilon^{1/4}\|u_\varepsilon\|_{L^\infty(0,T;W^{1,4}(\Omega;\mathbb{R}^n))} \le C$,

(iii) $\|\sqrt{\Phi(z^0)}\,e(u^0_\varepsilon)\|_{L^2(\Omega;\mathbb{R}^{n\times n})} \le C$,
$\|\sqrt{\Phi(z_\varepsilon)}\,e(u_\varepsilon)\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^{n\times n}))} \le C$,

(iv) $\|c_\varepsilon\|_{L^\infty(0,T;H^1(\Omega;\mathbb{R}^N))} \le C$,

(v) $\|\partial_t c_\varepsilon\|_{L^2(0,T;(H^1(\Omega;\mathbb{R}^N))')} \le C$,

(vi) $\|z_\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} \le C$,

(vii) $\|\partial_t z_\varepsilon\|_{L^2(\Omega_T)} \le C$,

(viii) $\|w_\varepsilon\|_{L^2(0,T;H^1(\Omega;\mathbb{R}^N))} \le C$.

Proof. According to Lemma 3.2, we immediately obtain (vi) and (vii). Due to $\int_\Omega c_\varepsilon(t)\,\mathrm{d}x =$ const. and the boundedness of $\|\nabla c_\varepsilon(t)\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^{N\times n}))}$, Poincaré's inequality yields (iv). In addition, (ii) follows from Poincaré's inequality.

From the growth conditions for W_1^{el} and W_2^{el}, Lemma 3.2, Young's inequality, $1 < p < 2$, $z \in [0,1]$ a.e. in Ω_T, $s < \frac{n(p-1)}{n-p}$ and (iv), we infer for $t = 0$, a.e. $t \in [0,T]$ and any $\varepsilon > 0$:

$$
\begin{aligned}
&\int_\Omega \big(\Phi(z_\varepsilon(t))|e(u_\varepsilon(t))|^2 + |e(u_\varepsilon(t))|^p\big)\,\mathrm{d}x\\
&\quad\leq C_1 \int_\Omega \Big(\Phi(z_\varepsilon(t))\big(W_1^{\mathrm{el}}(e(u_\varepsilon(t)), c_\varepsilon(t)) + |c_\varepsilon(t)|^4 + 1\big) + W_2^{\mathrm{el}}(e(u_\varepsilon(t)), c_\varepsilon(t)) + |c_\varepsilon(t)|^{sp'} + 1\Big)\,\mathrm{d}x\\
&\quad\leq C_1 \int_\Omega \big(W^{\mathrm{el}}(e(u_\varepsilon(t)), c_\varepsilon(t), z_\varepsilon(t)) + \Phi(z_\varepsilon(t))W_2^{\mathrm{el}}(e(u_\varepsilon(t)), c_\varepsilon(t))\big)\,\mathrm{d}x + C_2\\
&\quad\leq C_1 \int_\Omega W^{\mathrm{el}}(e(u_\varepsilon(t)), c_\varepsilon(t), z_\varepsilon(t))\,\mathrm{d}x + C_3 \int_\Omega \Phi(z_\varepsilon(t))\big(|e(u_\varepsilon(t))|^p + |c_\varepsilon(t)|^s + 1\big)\,\mathrm{d}x + C_2\\
&\quad\leq C_4 + C_3 \int_\Omega \Phi(z_\varepsilon(t))|e(u_\varepsilon(t))|^p\,\mathrm{d}x\\
&\quad\leq C_4 + C_3 \int_\Omega \big(\Phi(z_\varepsilon(t))\big)^{1-\frac{p}{2}}\big(\Phi(z_\varepsilon(t))\big)^{\frac{p}{2}}|e(u_\varepsilon(t))|^p\,\mathrm{d}x\\
&\quad\leq C_4 + C_3\,\varepsilon \int_\Omega \Phi(z_\varepsilon(t))|e(u_\varepsilon(t))|^2\,\mathrm{d}x + C(\varepsilon)
\end{aligned}
\tag{31}
$$

Hence, we attain (i) by using the generalized Korn's inequality, see for instance [Nit81] and [KO88], and (iii).

Due to (11) we obtain boundedness of $\int_\Omega w_\varepsilon(t)\,\mathrm{d}x$. Since $\|\nabla w_\varepsilon(t)\|_{L^2(\Omega_T;\mathbb{R}^{N\times n})}$ is also bounded, Poincaré's inequality yields (viii).

Finally, we know from the boundedness of $\{\nabla w_\varepsilon\}$ in $L^2(\Omega_T;\mathbb{R}^{N\times n})$ that $\{\partial_t c_\varepsilon\}$ is also bounded in $L^2(0,T;(H^1(\Omega;\mathbb{R}^N))')$ with respect to ε by equation (10). Therefore, (v) is satisfied. ∎

Lemma 3.4 (Convergence properties) *There exists a subsequence $\{q_{\varepsilon_k}\}$ with $\varepsilon_k \searrow 0$ and a tuple $q = (u,c,w,z)$, satisfying (i) of Definition (2.3), $0 \leq z \leq 1$ and $\partial_t z \leq 0$ a.e. in Ω_T, such that*

(i) $c_{\varepsilon_k} \rightharpoonup c$ *in* $H^1(0,T;(H^1(\Omega;\mathbb{R}^N))')$,
$c_{\varepsilon_k}(t) \rightharpoonup c(t)$ *in* $H^1(\Omega;\mathbb{R}^N)$ *a.e.* $t \in [0,T]$,
$c_{\varepsilon_k} \to c$ *a.e. in* Ω_T,

(ii) $z_{\varepsilon_k} \overset{\star}{\rightharpoonup} z$ *in* $L^\infty(0,T;H^1(\Omega))$,
$z_{\varepsilon_k}(t) \rightharpoonup z(t)$ *in* $H^1(\Omega)$ *a.e.* $t \in [0,T]$,
$z_{\varepsilon_k} \to z$ *a.e. in* Ω_T,
$z_{\varepsilon_k} \rightharpoonup z$ *in* $H^1(0,T;L^2(\Omega))$,
$z_{\varepsilon_k} \to z$ *in* $L^{\hat{p}}(\Omega_T)$ *for* $\hat{p} \in [1,\infty)$,

(iii) $u_{\varepsilon_k} \overset{\star}{\rightharpoonup} u$ *in*$L^\infty(0,T;W^{1,p}(\Omega;\mathbb{R}^n))$,
$u^0_{\varepsilon_k} \rightharpoonup u^0$ *in* $W^{1,p}(\Omega;\mathbb{R}^n)$,
$\sqrt{\Phi(z_{\varepsilon_k})}\, e(u_{\varepsilon_k}) \overset{\star}{\rightharpoonup} \sqrt{\Phi(z)}\, e(u)$ *in* $L^\infty(0,T;L^2(\Omega;\mathbb{R}^{n\times n}))$,
$\sqrt{\Phi(z^0)}\, e(u^0_{\varepsilon_k}) \rightharpoonup \sqrt{\Phi(z^0)}\, e(u^0)$ *in* $L^2(\Omega;\mathbb{R}^{n\times n})$,

(iv) $w_{\varepsilon_k} \rightharpoonup w$ *in* $L^2(0,T;H^1(\Omega;\mathbb{R}^N))$

as $k \to \infty$.

Proof.

(i) Properties (iv) and (v) of Lemma 3.3 show that $\{c_{\varepsilon_k}\}$ converges strongly to an element c in $L^2(\Omega_T)$ for a subsequence by a compactness result due to J. P. Aubin and J. L. Lions (see [Sim86]). This allows us to extract a further subsequence (still denoted by $\{\varepsilon_k\}$) such that $c_{\varepsilon_k}(t) \to c(t)$ in $L^2(\Omega;\mathbb{R}^N)$ for a.e. $t \in [0,T]$ as $k \to \infty$. Taking also the boundedness of $\{c_{\varepsilon_k}\}$ in $L^\infty(0,T;H^1(\Omega;\mathbb{R}^N))$ into account, we obtain a subsequence with $c_{\varepsilon_k}(t) \rightharpoonup c(t)$ in $H^1(\Omega;\mathbb{R}^N)$ for a.e. $t \in [0,T]$. Moreover, $c_{\varepsilon_k} \to c$ a.e. in Ω_T with $c \in \Sigma$ as well as $c_{\varepsilon_k} \overset{\star}{\rightharpoonup} c$ in $H^1(0,T;(H^1(\Omega;\mathbb{R}^N))')$ as $k \to \infty$.

(ii) These properties follow from the same argumentation as in (i) and the boundedness of $\{z_{\varepsilon_k}\}$ in $H^1(0,T;L^2(\Omega))$. The function z derived in this way is monotonically decreasing with respect to t, i.e. $\partial_t z \le 0$ a.e. in Ω_T and $z \in [0,1]$ a.e.. By compact embeddings, we obtain the strong convergence results.

(iii) Because of the boundedness of $\{u_{\varepsilon_k}\}$ and $\{u^0_{\varepsilon_k}\}$ in $L^\infty(0,T;W^{1,p}(\Omega;\mathbb{R}^n))$ and $W^{1,p}(\Omega;\mathbb{R}^n)$, respectively, we obtain the first two properties. The other properties follow from the previous two, the boundedness of $\{\sqrt{\Phi(z_{\varepsilon_k})}e(u_{\varepsilon_k})\}$ and $\{\sqrt{\Phi(z^0)}e(u^0_{\varepsilon_k})\}$ in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^{n\times n}))$ and $L^2(\Omega;\mathbb{R}^{n\times n})$, respectively, and (ii).

(iv) This property follows from the boundedness of $\{w_{\varepsilon_k}\}$ in $L^2(0,T;H^1(\Omega;\mathbb{R}^N))$. ■

Lemma 3.5 *There exist sequences* $\{q_{\varepsilon_k}\}$ *and* $\{q^0_{\varepsilon_k}\}$ *with* $\varepsilon_k \searrow 0$ *such that the following properties are satisfied:*

(i) There exist $\theta_{u;c;z} \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^{n\times n}))$ *and* $\theta^0_{u;c;z} \in L^2(\Omega;\mathbb{R}^{n\times n})$ *with*

$$\sqrt{\Phi(z_{\varepsilon_k})}W^{\mathrm{el}}_{1,e}(e(u_{\varepsilon_k}), c_{\varepsilon_k}) \overset{\star}{\rightharpoonup} \theta_{u;c;z} \qquad \text{in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^{n\times n}))$$

and

$$\sqrt{\Phi(z^0)}W^{\mathrm{el}}_{1,e}(e(u^0_{\varepsilon_k}), c^0) \rightharpoonup \theta^0_{u;c;z} \qquad \text{in } L^2(\Omega;\mathbb{R}^{n\times n}).$$

In particular,

$$\Phi(z_{\varepsilon_k})W^{\mathrm{el}}_{1,e}(e(u_{\varepsilon_k}), c_{\varepsilon_k}) \overset{\star}{\rightharpoonup} \sqrt{\Phi(z)}\,\theta_{u;c;z} \qquad \text{in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^{n\times n})) \tag{32}$$

and

$$\Phi(z^0)W^{\mathrm{el}}_{1,e}(e(u^0_{\varepsilon_k}), c^0) \rightharpoonup \sqrt{\Phi(z^0)}\,\theta^0_{u;c;z} \qquad \text{in } L^2(\Omega;\mathbb{R}^{n\times n}). \tag{33}$$

(ii)

$$\liminf_{k\to\infty}\int_{\Omega_T}\Phi(z_{\varepsilon_k})W^{\mathrm{el}}_{1,e}(e(u_{\varepsilon_k}),c_{\varepsilon_k}):e(u_{\varepsilon_k})\,\mathrm{d}x\mathrm{d}t\ge\int_{\Omega_T}\sqrt{\Phi(z)}\,\theta_{u;c;z}:e(u)\,\mathrm{d}x\mathrm{d}t$$

and

$$\liminf_{k\to\infty}\int_{\Omega}\Phi(z^0)W^{\mathrm{el}}_{1,e}(e(u^0_{\varepsilon_k}),c^0):e(u^0_{\varepsilon_k})\,\mathrm{d}x\ge\int_{\Omega}\sqrt{\Phi(z^0)}\,\theta^0_{u;c;z}:e(u^0)\,\mathrm{d}x.$$

Proof. To (i): Since

$$\|\sqrt{\Phi(z_{\varepsilon_k})}W^{\mathrm{el}}_{1,e}(e(u_{\varepsilon_k}),c_{\varepsilon_k})\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^{n\times n}))}\le C$$

there exists some $\theta_{u;c;z}\in L^\infty(0,T;L^2(\Omega;\mathbb{R}^{n\times n}))$ such that

$$\sqrt{\Phi(z_{\varepsilon_k})}W^{\mathrm{el}}_{1,e}(e(u_{\varepsilon_k}),c_{\varepsilon_k})\overset{\star}{\rightharpoonup}\theta_{u;c;z}\qquad\text{in }L^\infty(0,T;L^2(\Omega;\mathbb{R}^{n\times n})).$$

In consequence,

$$\Phi(z_{\varepsilon_k})\,W^{\mathrm{el}}_{1,e}(e(u_{\varepsilon_k}),c_{\varepsilon_k})\overset{\star}{\rightharpoonup}\sqrt{\Phi(z)}\,\theta_{u;c;z}\qquad\text{in }L^\infty(0,T;L^2(\Omega;\mathbb{R}^{n\times n})).$$

In the same way, we obtain the result for $\sqrt{\Phi(z^0)}W^{\mathrm{el}}_{1,e}(e(u^0_{\varepsilon_k}),c^0)$.
To (ii): Since $h_{\hat c}(\cdot)=W^{\mathrm{el}}_{1,e}(\cdot,\hat c)-W^{\mathrm{el}}_{1,e}(0,\hat c)$ is one homogeneous we obtain by means of the uniform convexity assumption

$$\begin{aligned}&\Big(\sqrt{\Phi(z)}\big(W^{\mathrm{el}}_{1,e}(e(u),c_{\varepsilon_k})-W^{\mathrm{el}}_{1,e}(0,c_{\varepsilon_k})\big)-\sqrt{\Phi(z_{\varepsilon_k})}\big(W^{\mathrm{el}}_{1,e}(e(u_{\varepsilon_k}),c_{\varepsilon_k})-W^{\mathrm{el}}_{1,e}(0,c_{\varepsilon_k})\big)\Big)\\&\qquad:\big(\sqrt{\Phi(z)}\,e(u)-\sqrt{\Phi(z_{\varepsilon_k})}\,e(u_{\varepsilon_k})\big)\\&=\Big(\big(W^{\mathrm{el}}_{1,e}(\sqrt{\Phi(z)}e(u),c_{\varepsilon_k})-W^{\mathrm{el}}_{1,e}(0,c_{\varepsilon_k})\big)-\big(W^{\mathrm{el}}_{1,e}(\sqrt{\Phi(z_{\varepsilon_k})}e(u_{\varepsilon_k}),c_{\varepsilon_k})-W^{\mathrm{el}}_{1,e}(0,c_{\varepsilon_k})\big)\Big)\\&\qquad:\big(\sqrt{\Phi(z)}\,e(u)-\sqrt{\Phi(z_{\varepsilon_k})}\,e(u_{\varepsilon_k})\big)\\&=C\big|\sqrt{\Phi(z)}\,e(u)-\sqrt{\Phi(z_{\varepsilon_k})}\,e(u_{\varepsilon_k})\big|^2\ge0\end{aligned}$$

Therefore,

$$\begin{aligned}\liminf_{k\to\infty}\int_{\Omega_T}\sqrt{\Phi(z_{\varepsilon_k})}\,W^{\mathrm{el}}_{1,e}(e(u_{\varepsilon_k}),c_{\varepsilon_k}):\sqrt{\Phi(z)}e(u)\,\mathrm{d}x\mathrm{d}t\le\liminf_{k\to\infty}\int_{\Omega_T}\Big(\Phi(z)\,W^{\mathrm{el}}_{1,e}(e(u),c_{\varepsilon_k}):e(u)\\-\sqrt{\Phi(z)}\,W^{\mathrm{el}}_{1,e}(e(u),c_{\varepsilon_k}):\sqrt{\Phi(z_{\varepsilon_k})}\,e(u_{\varepsilon_k})+\Phi(z_{\varepsilon_k})\,W^{\mathrm{el}}_{1,e}(e(u_{\varepsilon_k}),c_{\varepsilon_k}):e(u_{\varepsilon_k})\Big)\,\mathrm{d}x\mathrm{d}t\end{aligned}$$

since

$$\lim_{k\to\infty}\int_{\Omega_T}\Big(\sqrt{\Phi(z)}\,W^{\mathrm{el}}_{1,e}(0,c_{\varepsilon_k})-\sqrt{\Phi(z_{\varepsilon_k})}\,W^{\mathrm{el}}_{1,e}(0,c_{\varepsilon_k})\Big):\big(\sqrt{\Phi(z)}\,e(u)-\sqrt{\Phi(z_{\varepsilon_k})}\,e(u_{\varepsilon_k})\big)\,\mathrm{d}x\mathrm{d}t=0.$$

By (i), Lemma 3.4, the growth assumptions on $W^{\mathrm{el}}_{1,e}$ and the generalized Lebesgue's convergence theorem, we can pass to the limit:

$$\int_{\Omega_T}\sqrt{\Phi(z)}\,\theta_{u;c;z}:e(u)\,\mathrm{d}x\mathrm{d}t\le\liminf_{k\to\infty}\int_{\Omega_T}\Phi(z_{\varepsilon_k})\,W^{\mathrm{el}}_{1,e}(e(u_{\varepsilon_k}),c_{\varepsilon_k}):e(u_{\varepsilon_k})\,\mathrm{d}x\mathrm{d}t$$

Analogously, we obtain the second assertion of (ii). ■

Lemma 3.6 *There exist sequences* $\{q_{\varepsilon_k}\}$ *and* $\{q^0_{\varepsilon_k}\}$ *with* $\varepsilon_k \searrow 0$ *such that the following properties are satisfied:*

(i) There exist an $\eta_{u;c} \in L^\infty(0,T;L^{p'}(\Omega;\mathbb{R}^{n\times n}))$ *and* $\eta^0_{u;c} \in L^{p'}(\Omega;\mathbb{R}^{n\times n})$ *with*

$$W^{\mathrm{el}}_{2,e}(e(u_{\varepsilon_k}),c_{\varepsilon_k}) \overset{\star}{\rightharpoonup} \eta_{u;c} \qquad \text{in } L^\infty(0,T;L^{p'}(\Omega;\mathbb{R}^{n\times n}))$$

and

$$W^{\mathrm{el}}_{2,e}(e(u^0_{\varepsilon_k}),c^0) \rightharpoonup \eta^0_{u;c} \qquad \text{in } L^{p'}(\Omega;\mathbb{R}^{n\times n}).$$

In particular,

$$(1-\Phi(z_{\varepsilon_k}))W^{\mathrm{el}}_{2,e}(e(u_{\varepsilon_k}),c_{\varepsilon_k}) \overset{\star}{\rightharpoonup} (1-\Phi(z))\,\eta_{u;c} \qquad \text{in } L^\infty(0,T;L^{p'}(\Omega;\mathbb{R}^{n\times n})) \tag{34}$$

and

$$(1-\Phi(z^0))W^{\mathrm{el}}_{2,e}(e(u^0_{\varepsilon_k}),c^0) \rightharpoonup (1-\Phi(z^0))\,\eta_{u;c} \qquad \text{in } L^{p'}(\Omega;\mathbb{R}^{n\times n}). \tag{35}$$

Furthermore,

$$\liminf_{k\to\infty}\int_{\Omega_T}(1-\Phi(z_{\varepsilon_k}))W^{\mathrm{el}}_{2,e}(e(u_{\varepsilon_k}),c_{\varepsilon_k}):e(u_{\varepsilon_k})\,\mathrm{d}x\mathrm{d}t \geq \int_{\Omega_T}(1-\Phi(z))\,\eta_{u;c}:e(u)\,\mathrm{d}x\mathrm{d}t$$

and

$$\liminf_{k\to\infty}\int_{\Omega}(1-\Phi(z^0))W^{\mathrm{el}}_{2,e}(e(u^0_{\varepsilon_k}),c^0):e(u^0_{\varepsilon_k})\,\mathrm{d}x \geq \int_{\Omega}(1-\Phi(z^0))\,\eta^0_{u;c}:e(u^0)\,\mathrm{d}x.$$

(ii) For any $\zeta \in L^1(0,T;W^{1,4}_D(\Omega;\mathbb{R}^n))$*:*

$$\begin{aligned}\lim_{k\to\infty}\int_{\Omega_T}\Big(&\Phi(z_{\varepsilon_k})W^{\mathrm{el}}_{1,e}(e(u_{\varepsilon_k}),c_{\varepsilon_k}):\nabla\zeta\\ &+(1-\Phi(z_{\varepsilon_k}))\,W^{\mathrm{el}}_{2,e}(e(u_{\varepsilon_k}),c_{\varepsilon_k}):\nabla\zeta+\varepsilon_k\,|\nabla u_{\varepsilon_k}|^2\,\nabla u_{\varepsilon_k}:\nabla\zeta\Big)\,\mathrm{d}x\mathrm{d}t\\ &=\int_{\Omega_T}\Big(\sqrt{\Phi(z)}\,\theta_{u;c;z}:\nabla\zeta+(1-\Phi(z))\,\eta_{u;c}:\nabla\zeta\Big)\,\mathrm{d}x\mathrm{d}t\end{aligned}$$

For any $\zeta \in W^{1,4}_D(\Omega;\mathbb{R}^n)$*:*

$$\begin{aligned}\lim_{k\to\infty}\int_{\Omega}\Big(&\Phi(z^0)W^{\mathrm{el}}_{1,e}(e(u^0_{\varepsilon_k}),c^0):\nabla\zeta\\ &+(1-\Phi(z^0))\,W^{\mathrm{el}}_{2,e}(e(u^0_{\varepsilon_k}),c^0):\nabla\zeta+\varepsilon_k\,\big|\nabla u^0_{\varepsilon_k}\big|^2\,\nabla u^0_{\varepsilon_k}:\nabla\zeta\Big)\,\mathrm{d}x\\ &=\int_{\Omega}\Big(\sqrt{\Phi(z^0)}\,\theta^0_{u;c;z}:\nabla\zeta+(1-\Phi(z^0))\,\eta^0_{u;c}:\nabla\zeta\Big)\,\mathrm{d}x\end{aligned}$$

(iii) For any $\zeta \in L^1(0,T;W^{1,p}_D(\Omega;\mathbb{R}^n))$*:*

$$\int_{\Omega_T}\Big(\sqrt{\Phi(z)}\,\theta_{u;c;z}:\nabla\zeta+(1-\Phi(z))\,\eta_{u;c}:\nabla\zeta\Big)\,\mathrm{d}x\mathrm{d}t=\int_{\Omega_T}f\cdot\zeta\,\mathrm{d}x\mathrm{d}t$$

For any $\zeta \in W^{1,p}_D(\Omega;\mathbb{R}^n)$*:*

$$\int_{\Omega}\Big(\sqrt{\Phi(z)}\,\theta^0_{u;c;z}:\nabla\zeta+(1-\Phi(z^0))\,\eta^0_{u;c}:\nabla\zeta\Big)\,\mathrm{d}x=\int_{\Omega}f^0\cdot\zeta\,\mathrm{d}x$$

Proof. To (i): Since

$$\|W^{\mathrm{el}}_{2,e}(e(u_{\varepsilon_k}), c_{\varepsilon_k})\|_{L^\infty(0,T;L^{p'}(\Omega;\mathbb{R}^{n\times n}))} \le C$$

there exists an $\eta_{u;c} \in L^\infty(0,T;L^{p'}(\Omega;\mathbb{R}^{n\times n}))$ such that

$$W^{\mathrm{el}}_{2,e}(e(u_{\varepsilon_k}), c_{\varepsilon_k}) \overset{\star}{\rightharpoonup} \eta_{u;c} \qquad \text{in } L^\infty(0,T;L^{p'}(\Omega;\mathbb{R}^{n\times n})).$$

In consequence,

$$(1-\Phi(z_{\varepsilon_k}))\, W^{\mathrm{el}}_{2,e}(e(u_{\varepsilon_k}), c_{\varepsilon_k}) \overset{\star}{\rightharpoonup} (1-\Phi(z))\, \eta_{u;c} \qquad \text{in} L^\infty(0,T;L^{p'}(\Omega;\mathbb{R}^{n\times n})).$$

In the same way, we obtain the result for $W^{\mathrm{el}}_{2,e}(e(u^0_{\varepsilon_k}), c^0)$.
The convexity condition for W^{el}_2 implies

$$\begin{aligned}\int_{\Omega_T}(1-\Phi(z_{\varepsilon_k}))W^{\mathrm{el}}_2(e(u_{\varepsilon_k}), c_{\varepsilon_k})\,\mathrm{d}x\mathrm{d}t \ge{}& \int_{\Omega_T}(1-\Phi(z_{\varepsilon_k}))W^{\mathrm{el}}_2(e(u), c_{\varepsilon_k})\,\mathrm{d}x\mathrm{d}t\\ &+\int_{\Omega_T}(1-\Phi(z_{\varepsilon_k}))W^{\mathrm{el}}_{2,e}(e(u), c_{\varepsilon_k}) : (e(u_{\varepsilon_k})-e(u))\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Since, for a suitable sequence, $(1-\Phi(z_{\varepsilon_k}))W^{\mathrm{el}}_2(e(u), c_{\varepsilon_k}) \to (1-\Phi(z))W^{\mathrm{el}}_2(e(u), c)$ strongly in $L^1(\Omega_T)$ and $(1-\Phi(z_{\varepsilon_k}))W^{\mathrm{el}}_{2,e}(e(u), c_k) \to (1-\Phi(z))W^{\mathrm{el}}_{2,e}(e(u), c)$ strongly in $L^{p'}(\Omega_T)$ by Lebesgue's generalized convergence theorem, and $e(u_{\varepsilon_k}) \rightharpoonup e(u)$ in $L^p(\Omega_T)$ we obtain

$$\liminf_{k\to\infty}\int_{\Omega_T}(1-\Phi(z_{\varepsilon_k}))W^{\mathrm{el}}_2(e(u_{\varepsilon_k}), c_{\varepsilon_k})\,\mathrm{d}x\mathrm{d}t \ge \int_{\Omega_T}(1-\Phi(z))W^{\mathrm{el}}_2(e(u), c)\,\mathrm{d}x\mathrm{d}t. \tag{36}$$

From the convexity condition for W^{el}_2 we further deduce

$$\begin{aligned}\int_{\Omega_T}(1-\Phi(z_{\varepsilon_k}))W^{\mathrm{el}}_2(e(u), c_{\varepsilon_k})\,\mathrm{d}x\mathrm{d}t \ge{}& \int_{\Omega_T}(1-\Phi(z_{\varepsilon_k}))W^{\mathrm{el}}_2(e(u_{\varepsilon_k}), c_{\varepsilon_k})\,\mathrm{d}x\mathrm{d}t\\ &+\int_{\Omega_T}(1-\Phi(z_{\varepsilon_k}))W^{\mathrm{el}}_{2,e}(e(u_{\varepsilon_k}), c_{\varepsilon_k}) : (e(u)-e(u_{\varepsilon_k}))\,\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{37}$$

Equation (37) may be rewritten as

$$\begin{aligned}\int_{\Omega_T}(1-\Phi(z_{\varepsilon_k}))W^{\mathrm{el}}_{2,e}(e(u_{\varepsilon_k}), c_{\varepsilon_k}) : e(u_{\varepsilon_k})\,\mathrm{d}x\mathrm{d}t \ge{}& \int_{\Omega_T}(1-\Phi(z_{\varepsilon_k}))W^{\mathrm{el}}_2(e(u_{\varepsilon_k}), c_{\varepsilon_k})\,\mathrm{d}x\mathrm{d}t\\ -\int_{\Omega_T}(1-\Phi(z_{\varepsilon_k}))W^{\mathrm{el}}_2(e(u), c_{\varepsilon_k})\,\mathrm{d}x\mathrm{d}t &+ \int_{\Omega_T}(1-\Phi(z_{\varepsilon_k}))W^{\mathrm{el}}_{2,e}(e(u_{\varepsilon_k}), c_{\varepsilon_k}) : e(u)\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Applying the lim inf on both sides and taking (36) and (34) into account gives

$$\begin{aligned}\liminf_{k\to\infty}&\int_{\Omega_T}(1-\Phi(z_{\varepsilon_k}))W^{\mathrm{el}}_{2,e}(e(u_{\varepsilon_k}), c_{\varepsilon_k}) : e(u_{\varepsilon_k})\,\mathrm{d}x\mathrm{d}t\\ &\ge \liminf_{k\to\infty}\int_{\Omega_T}(1-\Phi(z_{\varepsilon_k}))W^{\mathrm{el}}_2(e(u_{\varepsilon_k}), c_{\varepsilon_k})\,\mathrm{d}x\mathrm{d}t\\ &\quad -\int_{\Omega_T}(1-\Phi(z))W^{\mathrm{el}}_2(e(u), c)\,\mathrm{d}x\mathrm{d}t + \int_{\Omega_T}(1-\Phi(z))\eta_{u;c} : e(u)\,\mathrm{d}x\mathrm{d}t\\ &\ge \int_{\Omega_T}(1-\Phi(z))\eta_{u;c} : e(u)\,\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{38}$$

By similar arguments, we derive the claim with the initial data.
To (ii): Let $\zeta \in L^1(0,T; W_D^{1,4}(\Omega;\mathbb{R}^n))$ be arbitrary. By Lemma 3.4, Lemma 3.5 and (i), we can pass to the limit in equation (12). More precisely, we obtain

$$\begin{aligned} 0 &= \lim_{k\to\infty} \bigg(\int_{\Omega_T} \Big(\Phi(z_{\varepsilon_k}) W^{\mathrm{el}}_{1,e}(e(u_{\varepsilon_k}), c_{\varepsilon_k}) : e(\zeta) + (1-\Phi(z_{\varepsilon_k}))\, W^{\mathrm{el}}_{2,e}(e(u_{\varepsilon_k}), c_{\varepsilon_k}) : e(\zeta) \Big) \,\mathrm{d}x\mathrm{d}t \\ &\qquad + \varepsilon_k \int_{\Omega_T} |\nabla u_{\varepsilon_k}|^2\, \nabla u_{\varepsilon_k} : \nabla \zeta \,\mathrm{d}x\mathrm{d}t \bigg) - \int_{\Omega_T} f\cdot \zeta \,\mathrm{d}x\mathrm{d}t \\ &= \int_{\Omega_T} \Big(\sqrt{\Phi(z)}\, \theta_{u;c;z} : \nabla\zeta + (1-\Phi(z))\, \eta_{u;c} : \nabla \zeta \Big) \,\mathrm{d}x\mathrm{d}t - \int_{\Omega_T} f\cdot\zeta\,\mathrm{d}x\mathrm{d}t \end{aligned} \tag{39}$$

by noticing

$$\left| \int_{\Omega_T} \varepsilon_k |\nabla u_\varepsilon|^2 \nabla u_{\varepsilon_k} : \nabla\zeta \,\mathrm{d}x\mathrm{d}t \right| \le \varepsilon_k \|u_{\varepsilon_k}\|^3_{L^\infty(0,T;W^{1,4}(\Omega;\mathbb{R}^n))} \|\zeta\|_{L^1(0,T;W^{1,4}(\Omega;\mathbb{R}^n))} \to 0.$$

Now let $\zeta \in W_D^{1,4}(\Omega;\mathbb{R}^n)$ be arbitrary. By Lemma 3.4, Lemma 3.5, (i) and the fact that u_ε^0 is a minimizer of $\mathcal{E}_\varepsilon(\,.\,, c^0, z^0) - \int_\Omega f\cdot(\,.\,)\,\mathrm{d}x$ we deduce

$$\begin{aligned} 0 &= \lim_{k\to\infty} \bigg(\int_{\Omega} \Big(\Phi(z^0) W^{\mathrm{el}}_{1,e}(e(u^0_{\varepsilon_k}), c^0) : e(\zeta) + (1-\Phi(z^0))\, W^{\mathrm{el}}_{2,e}(e(u^0_{\varepsilon_k}), c^0) : e(\zeta) \Big) \,\mathrm{d}x \\ &\qquad + \varepsilon_k \int_{\Omega} \left|\nabla u^0_{\varepsilon_k}\right|^2 \nabla u^0_{\varepsilon_k} : \nabla \zeta \,\mathrm{d}x \bigg) - \int_{\Omega} f^0\cdot \zeta \,\mathrm{d}x \\ &= \int_{\Omega} \Big(\sqrt{\Phi(z^0)}\, \theta^0_{u;c;z} : \nabla\zeta + (1-\Phi(z^0))\, \eta^0_{u;c} : \nabla \zeta \Big) \,\mathrm{d}x - \int_{\Omega} f^0\cdot\zeta\,\mathrm{d}x\,. \end{aligned}$$

To (iii): Since $f \in L^\infty(0,T;L^{p'}(\Omega;\mathbb{R}^n))$ and $(1-\Phi(z))\,\eta_{u;c} \in L^\infty(0,T;L^{p'}(\Omega;\mathbb{R}^{n\times n}))$ we obtain from (39) the claim by a density argument. Analogously, we derive the second claim for $q^0_{\varepsilon_k} = (u^0_{\varepsilon_k}, c^0, z^0)$. ■

Lemma 3.7 *There exist sequences $\{q_{\varepsilon_k}\}$ and $\{q^0_{\varepsilon_k}\}$ with $\varepsilon_k \searrow 0$ such that*

(i)

$$\lim_{k\to\infty} \int_{\Omega_T} \Phi(z_{\varepsilon_k})\, W^{\mathrm{el}}_{1,e}(e(u_{\varepsilon_k}), c_{\varepsilon_k}) : e(u_{\varepsilon_k}) \,\mathrm{d}x\mathrm{d}t = \int_{\Omega_T} \sqrt{\Phi(z)}\, \theta_{u;c;z} : e(u)\,\mathrm{d}x\mathrm{d}t, \tag{40}$$

$$\lim_{k\to\infty} \int_{\Omega_T} (1-\Phi(z_{\varepsilon_k}))\, W^{\mathrm{el}}_{2,e}(e(u_{\varepsilon_k}), c_{\varepsilon_k}) : e(u_{\varepsilon_k}) \,\mathrm{d}x\mathrm{d}t = \int_{\Omega_T} (1-\Phi(z))\, \eta_{u;c} : e(u)\,\mathrm{d}x\mathrm{d}t, \tag{41}$$

$$\lim_{k\to\infty} \int_{\Omega_T} \varepsilon_k |\nabla u_{\varepsilon_k}|^4 \,\mathrm{d}x\mathrm{d}t = 0, \tag{42}$$

(ii)

$$\lim_{k\to\infty} \int_{\Omega} \Phi(z^0)\, W^{\mathrm{el}}_{1,e}(e(u^0_{\varepsilon_k}), c^0) : e(u^0_{\varepsilon_k}) \,\mathrm{d}x = \int_{\Omega} \sqrt{\Phi(z^0)}\ \ \theta^0_{u;c;z} : \nabla u^0 \,\mathrm{d}x,$$

$$\lim_{k\to\infty}\int_\Omega (1-\Phi(z^0))\, W^{\mathrm{el}}_{2,e}(e(u^0_{\varepsilon_k}),c^0) : e(u^0_{\varepsilon_k})\,\mathrm{d}x = \int_\Omega (1-\Phi(z^0))\,\eta^0_{u;c}\,\mathrm{d}x,$$
$$\lim_{k\to\infty}\int_\Omega \varepsilon_k|\nabla u^0_{\varepsilon_k}|^4\,\mathrm{d}x = 0.$$

Proof. We obtain by (12) and Lemma 3.6 (iii)

$$\begin{aligned}
&\lim_{k\to\infty}\Big(\int_{\Omega_T}\Big(\Phi(z_{\varepsilon_k})W^{\mathrm{el}}_{1,e}(e(u_{\varepsilon_k}),c_{\varepsilon_k}) : e(u_{\varepsilon_k}-b)\\
&\qquad + (1-\Phi(z_{\varepsilon_k}))\,W^{\mathrm{el}}_{2,e}(e(u_{\varepsilon_k}),c_{\varepsilon_k}) : e(u_{\varepsilon_k}-b)\Big)\,\mathrm{d}x\mathrm{d}t + \int_{\Omega_T}\varepsilon_k\,|\nabla u_{\varepsilon_k}|^2\,\nabla u_{\varepsilon_k} : \nabla(u_{\varepsilon_k}-b)\,\mathrm{d}x\mathrm{d}t\Big)\\
&\qquad = \lim_{k\to\infty}\int_{\Omega_T} f\cdot(u_{\varepsilon_k}-b)\,\mathrm{d}x\mathrm{d}t = \int_{\Omega_T} f\cdot(u-b)\,\mathrm{d}x\mathrm{d}t\\
&\qquad = \int_{\Omega_T}\sqrt{\Phi(z)}\theta_{u;c;z} : \nabla(u-b) + (1-\Phi(z))\,\eta_{u;c} : \nabla(u-b)\,\mathrm{d}x\mathrm{d}t\\
&\qquad = \int_{\Omega_T}\sqrt{\Phi(z)}\theta_{u;c;z} : e(u-b) + (1-\Phi(z))\,\eta_{u;c} : e(u-b)\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Because of Lemma 3.3 (ii), Lemma 3.4, Lemma 3.5 and Lemma 3.6

$$\lim_{k\to\infty}\int_{\Omega_T}\Phi(z_{\varepsilon_k})W^{\mathrm{el}}_{1,e}(e(u_{\varepsilon_k}),c_{\varepsilon_k}) : e(b)\,\mathrm{d}x\mathrm{d}t = \int_{\Omega_T}\sqrt{\Phi(z)}\,\theta_{u;c;z} : e(b)\,\mathrm{d}x\mathrm{d}t,$$
$$\lim_{k\to\infty}\int_{\Omega_T}(1-\Phi(z_{\varepsilon_k}))\,W^{\mathrm{el}}_{2,e}(e(u_{\varepsilon_k}),c_{\varepsilon_k}) : e(b)\,\mathrm{d}x\mathrm{d}t = \int_{\Omega_T}(1-\Phi(z))\,\eta_{u;c} : \nabla b\,\mathrm{d}x\mathrm{d}t,$$
$$\lim_{k\to\infty}\int_{\Omega_T}\varepsilon_k\,|\nabla u_{\varepsilon_k}|^2\,\nabla u_{\varepsilon_k} : \nabla b\,\mathrm{d}x\mathrm{d}t = 0.$$

Hence,

$$\begin{aligned}
&\lim_{k\to\infty}\Big(\int_{\Omega_T}\Big(\Phi(z_{\varepsilon_k})W^{\mathrm{el}}_{1,e}(e(u_{\varepsilon_k}),c_{\varepsilon_k}) : e(u_{\varepsilon_k}) + (1-\Phi(z_{\varepsilon_k}))\,W^{\mathrm{el}}_{2,e}(e(u_{\varepsilon_k}),c_\varepsilon) : e(u_{\varepsilon_k})\Big)\,\mathrm{d}x\mathrm{d}t\\
&\qquad\qquad + \int_{\Omega_T}\varepsilon_k\,|\nabla u_{\varepsilon_k}|^4\,\mathrm{d}x\mathrm{d}t\Big) \qquad (43)\\
&\qquad = \int_{\Omega_T}\Big(\sqrt{\Phi(z)}\,\theta_{u;c;z} : e(u) + (1-\Phi(z))\,\eta_{u;c} : \nabla u\Big)\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

The lower semicontinuity of all three terms on the left hand side in (43) implies the claim. The assertion for $\{q^0_{\varepsilon_k}\}$ can be derived by slight modifications. ■

Lemma 3.8 *There exist subsequences $\{q_{\varepsilon_k}\}$ and $\{q^0_{\varepsilon_k}\}$ with $\varepsilon_k \searrow 0$ such that*

(i)

$$\sqrt{\Phi(z_{\varepsilon_k})}e(u_{\varepsilon_k}) \to \sqrt{\Phi(z)}e(u) \qquad \text{in } L^2(\Omega_T;\mathbb{R}^{n\times n}), \tag{44}$$
$$(1-\Phi(z_{\varepsilon_k}))^{1/p}e(u_{\varepsilon_k}) \to (1-\Phi(z))^{1/p}e(u) \qquad \text{in } L^p(\Omega_T;\mathbb{R}^{n\times n}), \tag{45}$$
$$\nabla u_{\varepsilon_k} \to \nabla u \qquad \text{in } L^p(\Omega_T;\mathbb{R}^{n\times n}),$$
$$\nabla u_{\varepsilon_k} \to \nabla u \qquad \text{a.e. in } \Omega_T.$$

(ii)

$$
\begin{aligned}
\sqrt{\Phi(z^0)}e(u^0_{\varepsilon_k}) &\to \sqrt{\Phi(z^0)}e(u^0) && \textit{in } L^2(\Omega;\mathbb{R}^{n\times n}),\\
(1-\Phi(z^0))^{1/p}e(u^0_{\varepsilon_k}) &\to (1-\Phi(z^0))^{1/p}e(u^0) && \textit{in } L^p(\Omega;\mathbb{R}^{n\times n}),\\
\nabla u^0_{\varepsilon_k} &\to \nabla u^0 && \textit{in } L^p(\Omega;\mathbb{R}^{n\times n}),\\
\nabla u^0_{\varepsilon_k} &\to \nabla u^0 && \textit{a.e. in } \Omega_T.
\end{aligned}
$$

Proof. Because of the uniform convexity condition for W_1^{el} and the one homogeneity of $h_{\hat{c}}(\cdot) = W_{1,e}^{\mathrm{el}}(\cdot,\hat{c}) - W_{1,e}^{\mathrm{el}}(0,\hat{c})$ we get

$$
\begin{aligned}
\limsup_{k\to\infty}\int_{\Omega_T} C|\sqrt{\Phi(z)}e(u) - \sqrt{\Phi(z_{\varepsilon_k})}\,e(u_{\varepsilon_k})|^2\,\mathrm{d}x\mathrm{d}t \le \limsup_{k\to\infty}\int_{\Omega_T}\Big(\sqrt{\Phi(z)}\,W_{1,e}^{\mathrm{el}}(e(u),c_{\varepsilon_k})\\
- \sqrt{\Phi(z_{\varepsilon_k})}\,W_{1,e}^{\mathrm{el}}(e(u_{\varepsilon_k}),c_{\varepsilon_k})\Big) : \big(\sqrt{\Phi(z)}\,e(u) - \sqrt{\Phi(z_{\varepsilon_k})}\,e(u_{\varepsilon_k})\big)\,\mathrm{d}x\mathrm{d}t
\end{aligned}
$$

as

$$
\lim_{k\to\infty}\int_{\Omega_T}\Big(\sqrt{\Phi(z)}\,W_{1,e}^{\mathrm{el}}(0,c_{\varepsilon_k}) - \sqrt{\Phi(z_{\varepsilon_k})}\,W_{1,e}^{\mathrm{el}}(0,c_{\varepsilon_k})\Big) : \big(\sqrt{\Phi(z)}\,e(u) - \sqrt{\Phi(z_{\varepsilon_k})}\,e(u_{\varepsilon_k})\big)\,\mathrm{d}x\mathrm{d}t = 0.
$$

Since, for a suitable sequence (also denoted by $\{\varepsilon_k\}$),

$$
\begin{aligned}
&\lim_{k\to\infty}\int_{\Omega_T}\sqrt{\Phi(z_{\varepsilon_k})}\,W_{1,e}^{\mathrm{el}}(e(u_{\varepsilon_k}),c_{\varepsilon_k}) : \sqrt{\Phi(z)}e(u)\,\mathrm{d}x\mathrm{d}t = \int_{\Omega_T}\sqrt{\Phi(z)}\,\theta_{u;c;z} : e(u)\,\mathrm{d}x\mathrm{d}t,\\
&\lim_{k\to\infty}\int_{\Omega_T}\Phi(z)\,W_{1,e}^{\mathrm{el}}(e(u),c_{\varepsilon_k}) : e(u)\,\mathrm{d}x\mathrm{d}t = \int_{\Omega_T}\Phi(z)\,W_{1,e}^{\mathrm{el}}(e(u),c) : e(u)\,\mathrm{d}x\mathrm{d}t,\\
&\lim_{k\to\infty}\int_{\Omega_T}\sqrt{\Phi(z)}\,W_{1,e}^{\mathrm{el}}(e(u),c_{\varepsilon_k}) : \sqrt{\Phi(z_{\varepsilon_k})}\,e(u_{\varepsilon_k})\,\mathrm{d}x\mathrm{d}t = \int_{\Omega_T}\sqrt{\Phi(z)}\,W_{1,e}^{\mathrm{el}}(e(u),c) : \sqrt{\Phi(z)}\,e(u)\,\mathrm{d}x\mathrm{d}t,\\
&\lim_{k\to\infty}\int_{\Omega_T}\Phi(z_{\varepsilon_k})\,W_{1,e}^{\mathrm{el}}(e(u_{\varepsilon_k}),c_{\varepsilon_k}) : e(u_{\varepsilon_k})\,\mathrm{d}x\mathrm{d}t = \int_{\Omega_T}\sqrt{\Phi(z)}\,\theta_{u;c;z} : e(u)\,\mathrm{d}x\mathrm{d}t,
\end{aligned}
$$

we obtain the first assertion.Due to the convexity condition (A10), Lemma 3.4, Lemma 3.6 and Lemma 3.7, we infer

$$
\begin{aligned}
&\lim_{k\to\infty}\int_{\Omega_T}(1-\Phi(z_{\varepsilon_k}))\,|e(u)-e(u_{\varepsilon_k})|^p\,\mathrm{d}x\mathrm{d}t\\
&\quad\le \lim_{k\to\infty}\int_{\Omega_T}(1-\Phi(z_{\varepsilon_k}))\big(W_{2,e}^{\mathrm{el}}(e(u),c_{\varepsilon_k}) - W_{2,e}^{\mathrm{el}}(e(u_{\varepsilon_k}),c_{\varepsilon_k})\big) : (e(u)-e(u_{\varepsilon_k}))\,\mathrm{d}x\mathrm{d}t\\
&\quad= \int_{\Omega_T}(1-\Phi(z))\,W_{2,e}^{\mathrm{el}}(e(u),c) : e(u)\,\mathrm{d}x\mathrm{d}t - \int_{\Omega_T}(1-\Phi(z))\,\eta_{u;c} : e(u)\,\mathrm{d}x\mathrm{d}t\\
&\qquad - \int_{\Omega_T}(1-\Phi(z))\,W_{2,e}^{\mathrm{el}}(e(u),c) : e(u)\,\mathrm{d}x\mathrm{d}t + \int_{\Omega_T}(1-\Phi(z))\,\eta_{u;c} : e(u)\,\mathrm{d}x\mathrm{d}t\\
&\quad= 0.
\end{aligned}
$$

In consequence,

$$
\big(1-\Phi(z_{\varepsilon_k})\big)^{1/p}|e(u)-e(u_{\varepsilon_k})| \to 0 \qquad \text{in } L^p(\Omega_T). \tag{46}
$$

We estimate

$$
\begin{aligned}
&\int_{\Omega_T} \Big|\big(1-\Phi(z_{\varepsilon_k})\big)^{1/p} e(u_{\varepsilon_k}) - \big(1-\Phi(z)\big)^{1/p} e(u)\Big|^p \,\mathrm{d}x\mathrm{d}t \\
&\qquad \le \int_{\Omega_T} \Big(\big|\big(1-\Phi(z_{\varepsilon_k})\big)^{1/p}\big(e(u_{\varepsilon_k})-e(u)\big)\big| + \big|\big(\big(1-\Phi(z_{\varepsilon_k})\big)^{1/p} - \big(1-\Phi(z)\big)^{1/p}\big)e(u)\big|\Big)^p \,\mathrm{d}x\mathrm{d}t \\
&\qquad \le C\int_{\Omega_T} \big(1-\Phi(z_{\varepsilon_k})\big)\big|\big(e(u_{\varepsilon_k})-e(u)\big)\big|^p \,\mathrm{d}x\mathrm{d}t \\
&\qquad\qquad\qquad\qquad\qquad + C\int_{\Omega_T} \big|\big(1-\Phi(z_{\varepsilon_k})\big)^{1/p} - \big(1-\Phi(z)\big)^{1/p}\big|^p \big|e(u)\big|^p \,\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

The first term on the right hand side converges to zero in view of (46). Since $z_{\varepsilon_k} \to z$ a.e. in Ω_T for a suitable subsequence, we obtain

$$
\int_{\Omega} \big|\big(1-\Phi(z_{\varepsilon_k})\big)^{1/p} - \big(1-\Phi(z)\big)^{1/p}\big|^p \big|e(u)\big|^p \,\mathrm{d}x\mathrm{d}t \to 0
$$

by the generalized Lebesque's convergence theorem and, therefore, equation (45) follows. Due to (44), (45) and $z_{\varepsilon_k} \to z$ a.e. in Ω for a subsequence $\{\varepsilon_k\}$, we may extract a subsequence (still denoted by $\{\varepsilon_k\}$) such that

$$
\Phi(z_{\varepsilon_k})\, e(u_{\varepsilon_k}) \to \Phi(z)\, e(u) \qquad \text{a.e. in } \Omega_T, \tag{47}
$$

$$
\big(1-\Phi(z_{\varepsilon_k})\big)^{1/p} e(u_{\varepsilon_k}) \to \big(1-\Phi(z)\big)^{1/p} e(u) \qquad \text{a.e. in } \Omega_T. \tag{48}
$$

From (47) we obtain for $\Omega_{1,T} := \{(t,x) \in \Omega_T : \Phi(z) > \frac{1}{2}\}$

$$
e(u_{\varepsilon_k}) \to e(u) \qquad \text{a.e. in } \Omega_{1,T}.
$$

Similarly, by (48) we get for $\Omega_{2,T} := \{(t,x) \in \Omega_T : \Phi(z) \le \frac{1}{2}\}$

$$
e(u_{\varepsilon_k}) \to e(u) \qquad \text{a.e. in } \Omega_{2,T}.
$$

Since

$$
e(u_{\varepsilon_k}) \le \sqrt{2}\sqrt{\Phi(z_{\varepsilon_k})}\, e(u_{\varepsilon_k}) \qquad \text{in } \Big\{(t,x) \in \Omega_T : \Phi(z_{\varepsilon_k}) > \frac{1}{2}\Big\}
$$

and

$$
e(u_{\varepsilon_k}) \le \sqrt[p]{2}\sqrt[p]{\big(1-\Phi(z_{\varepsilon_k})\big)}\, e(u_{\varepsilon_k}) \qquad \text{in } \Big\{(t,x) \in \Omega_T : \Phi(z_{\varepsilon_k}) \le \frac{1}{2}\Big\}
$$

we conclude from (44), (45) and the generalized Lebesque's convergence theorem

$$
e(u_{\varepsilon_k}) \to e(u) \quad \text{in } L^p(\Omega_T). \tag{49}
$$

The generalized Korn's inequality, in turn, implies

$$
\nabla u_{\varepsilon_k} \to \nabla u \qquad \text{in } L^p(\Omega_T)
$$

and therefore for a subsequence (still denoted by $\{\varepsilon_k\}$):

$$
\nabla u_{\varepsilon_k} \to \nabla u \quad \text{a.e. in } \Omega_T. \tag{50}
$$

By similar arguments, we derive the properties of (ii) for $\{q^0_{\varepsilon_k}\}$. ■

Lemma 3.9 *Let $\zeta \in H^1_+(\Omega)$. Then there exists a sequence $\{q_{\varepsilon_k}\}$ with $\varepsilon_k \searrow 0$ such that for a.e. $s \in [0,T]$*

$$\int_\Omega W^{\mathrm{el}}_{,z}(e(u(s)), c(s), z(s))\zeta \,\mathrm{d}x \le \liminf_{k\to\infty} \int_\Omega W^{\mathrm{el}}_{,z}(e(u_{\varepsilon_k}(s)), c_{\varepsilon_k}(s), z_{\varepsilon_k}(s))\zeta \,\mathrm{d}x.$$

In addition, $W^{\mathrm{el}}_{,z}(e(u), c, z)$ in $L^2(0,T;L^1(\Omega))$.

Proof. We abbreviate

$$g(c,z) := \Phi(z)\, C_2\big(|c|^4 + 1\big) + \big(1 - \Phi(z)\big) C_2\big(|c|^{sp'} + 1\big).$$

Note that due to (8)

$$W^{\mathrm{el}}_{,z}(e(u), c, z) + g(c,z) \ge 0\,.$$

In addition,

$$z_{\varepsilon_k} \to z, \qquad c_{\varepsilon_k} \to c \qquad \text{and} \quad \nabla u_{\varepsilon_k} \to \nabla u \qquad a.e. \text{ in } \Omega_T$$

for a subsequence as $\varepsilon_k \to 0$ and for a.e. $s \in [0,T]$

$$\int_\Omega \big|g(c_{\varepsilon_k}(s), z_{\varepsilon_k}(s))\big| \,\mathrm{d}x \to \int_\Omega \big|g(c(s), z(s))\big| \,\mathrm{d}x.$$

Therefore, we obtain the first assertion by Fatou's lemma.
Moreover, the first assertion combined with (13) tested by $\zeta = -1$ yields for a.e. $s \in [0,T]$:

$$\begin{aligned}
&\int_\Omega \big|W^{\mathrm{el}}_{,z}(e(u(s)), c(s), z(s)) + g(c(s), z(s))\big| \,\mathrm{d}x \\
&\qquad \le \liminf_{k\to\infty} \int_\Omega W^{\mathrm{el}}_{,z}(e(u_{\varepsilon_k}(s)), c_{\varepsilon_k}(s), z_{\varepsilon_k}(s)) \,\mathrm{d}x + \int_\Omega g(c(s), z(s)) \,\mathrm{d}x \\
&\qquad \le \liminf_{k\to\infty} \int_\Omega (\alpha - \beta \partial_t z_{\varepsilon_k}(s)) \,\mathrm{d}x + \int_\Omega g(c(s), z(s)) \,\mathrm{d}x \\
&\qquad \le C\Big(\liminf_{k\to\infty} \|\partial_t z_{\varepsilon_k}(s)\|_{L^1(\Omega)} + \|g(c(s), z(s))\|_{L^1(\Omega)} + 1\Big)
\end{aligned}$$

Hence, we obtain by Lemma 3.3 (vii) and Fatou's lemma

$$\begin{aligned}
||W^{\mathrm{el}}_{,z}(e(u), c, z)||_{L^2(0,T;L^1(\Omega))} &\le C\Big(\liminf_{k\to\infty} \|\partial_t z_{\varepsilon_k}\|_{L^2(0,T;L^1(\Omega))} + \|g(c,z)\|_{L^2(0,T;L^1(\Omega))} + 1\Big) \\
&\le C < \infty
\end{aligned}$$

and the second assertion follows. ■

Proof of Theorem 2.4. We establish items (i)-(v) of Definition 2.3.

(i) These space and regularity properties immediately follow from Lemma 3.4.

(ii) Let $\zeta \in L^2(0,T;H^1(\Omega;\mathbb{R}^N))$ with $\partial_t \zeta \in L^2(\Omega_T;\mathbb{R}^N)$ and $\zeta(T) = 0$. Then, equation (10) can be rewritten as

$$\int_{\Omega_T} (c_{\varepsilon_k} - c^0) \cdot \partial_t \zeta \,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} \mathbb{M} \nabla w_{\varepsilon_k} : \nabla \zeta \,\mathrm{d}x\,\mathrm{d}t\,. \tag{51}$$

In view of Lemma 3.4, we can pass to the limit and obtain (17).
Now, let $\zeta \in L^2(0,T;H^1(\Omega;\mathbb{R}^N)) \cap L^\infty(\Omega_T;\mathbb{R}^N)$. Integration from $t=0$ to $t=T$ of equation (11) yields

$$\begin{aligned}\int_{\Omega_T} w_{\varepsilon_k}\cdot\zeta\,\mathrm{d}x\mathrm{d}t = &\int_{\Omega_T} \mathbb{P}\boldsymbol{\Gamma}\nabla c_{\varepsilon_k} : \nabla\zeta\,\mathrm{d}x\mathrm{d}t\\ &+\int_{\Omega_T}(\mathbb{P}W^{\mathrm{ch}}_{,c}(c_{\varepsilon_k}) + \mathbb{P}W^{\mathrm{el}}_{,c}(e(u_{\varepsilon_k}),c_{\varepsilon_k},z_{\varepsilon_k}))\cdot\zeta\,\mathrm{d}x\mathrm{d}t\,.\end{aligned} \tag{52}$$

Due to Lemma 3.4, the growth conditions for $W^{\mathrm{ch}}_{,c}$ and $W^{\mathrm{el}}_{,c}$, Lemma 3.8 and the generalized Lebesgue's convergence theorem, we can pass to the limit in (52):

$$\int_{\Omega_T} w\cdot\zeta\,\mathrm{d}x\mathrm{d}t = \int_{\Omega_T}\mathbb{P}\boldsymbol{\Gamma}\nabla c:\nabla\zeta + (\mathbb{P}W^{\mathrm{ch}}_{,c}(c) + \mathbb{P}W^{\mathrm{el}}_{,c}(e(u),c,z))\cdot\zeta\,\mathrm{d}x\mathrm{d}t.$$

Hence, we obtain for a.e. $t\in[0,T]$ equation (18).

(iii) This is a direct consequence of Lemma 3.6 (ii) and (iii), Lemma 3.8 and the generalized Lebesgue's convergence theorem.

(iv) From Lemma 3.4 and Lemma 3.9, we infer the damage variational inequality (20). The inequalities (21) and (22) are obvious due to Lemma 3.4.

(v) Weakly semi-continuity arguments lead to

$$\begin{aligned}\liminf_{k\to\infty}&\Big(\mathcal{E}_{\varepsilon_k}(u_{\varepsilon_k}(t),c_{\varepsilon_k}(t),z_{\varepsilon_k}(t)) + \int_{\Omega_t}\alpha|\partial_t z_{\varepsilon_k}| + \beta|\partial_t z_{\varepsilon_k}|^2 + |\nabla w_{\varepsilon_k}|^2\,\mathrm{d}x\mathrm{d}s\Big)\\ &\geq \mathcal{E}(u(t),c(t),z(t)) + \int_{\Omega_t}\alpha|\partial_t z| + \beta|\partial_t z|^2 + |\nabla w|^2\,\mathrm{d}x\mathrm{d}s.\end{aligned}$$

Due to Lemma 3.8 and $\lim_{k\to\infty}\int_\Omega \varepsilon_k|\nabla u^0_{\varepsilon_k}|^4\,\mathrm{d}x = 0$ we can pass to the limit in (16) and obtain (23). ■

Literatur

[Bab11] J.-F. Babadjian. A quasistatic evolution model for the interaction between fracture and damage. *Arch. Ration. Mech. Anal.*, 200(3):945–1002, 2011.

[BB08] E. Bonetti and G. Bonfanti. Well-posedness results for a model of damage in thermoviscoelastic materials. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 25(6):1187–1208, 2008.

[BCD+02] E. Bonetti, P. Colli, W. Dreyer, G. Gilardi, G. Schimperna, and J. Sprekels. On a model for phase separation in binary alloys driven by mechanical effects. *Physica D*, 165:48–65, 2002.

[BDDM07] T. Böhme, W. Dreyer, F. Duderstadt, and W. Müller. A higher gradient theory of mixtures for multi-component materials with numerical examples for binary alloys. WIAS–Preprint No. 1268, Weierstrass Institute for Applied Analysis and Stochastics, Berlin, 2007.

[BDM07] T. Böhme, W. Dreyer, and W. Müller. Determination of stiffness and higher gradient coefficients by means of the embedded atom method: An approach for binary alloys. *Contin. Mech. Thermodyn.*, 18:411–441, 2007.

[BFM08] B. Bourdin, G.A. Francfort, and J.-J. Marigo. The variational approach to fracture. *J. Elasticity*, 91(1-3):5–148, 2008.

[BFS13] E. Bonetti, F. Freddi, and A. Segatti. . *in preparation*, 2013.

[BM10] S. Bartels and R. Müller. A posteriori error controlled local resolution of evolving interfaces for generalized Cahn–Hilliard equations. *Interfaces and Free Boundaries*, 12(1):45–73, 2010.

[BMR09] G. Bouchitte, A. Mielke, and T. Roubícek. A complete-damage problem at small strains. *ZAMP Z. Angew. Math. Phys.*, 60:205–236, 2009.

[BP05] L. Bartkowiak and I. Pawlow. The Cahn-Hilliard-Gurtin system coupled with elasticity. *Control and Cybernetics*, 34:1005–1043, 2005.

[BS04] E. Bonetti and G. Schimperna. Local existence for Frémond's model of damage in elastic materials. *Contin. Mech. Thermodyn.*, 16(4):319–335, 2004.

[BSS05] E. Bonetti, G. Schimperna, and A. Segatti. On a doubly nonlinear model for the evolution of damaging in viscoelastic materials. *J. of Diff. Equations*, 218(1):91–116, 2005.

[Car86] A. Carpinteri. *Mechanical damage and crack growth in concrete. Plastic collapse to brittle fracture.* Springer, Netherlands, 1986.

[CFM09] A. Chambolle, G.A. Francfort, and J.-J. Marigo. When and how do cracks propagate? *J. Mech. Phys. Solids*, 57(9):1614–1622, 2009.

[CFM10] A. Chambolle, G.A. Francfort, and J.-J. Marigo. Revisiting energy release rates in brittle fracture. *J. Nonlinear Sci.*, 20(4):395–424, 2010.

[CMP00] M. Carrive, A. Miranville, and A. Piétrus. The Cahn-Hilliard equation for deformable elastic media. *Adv. Math. Sci. App.*, 10:539–569, 2000.

[DPO94] E.A. DeSouzaNeto, D. Peric, and D.R.J. Owen. A phenomenological three-dimensional rate-independent continuum damage model for highly filled polymers: Formulation and computational aspects. *J. Mech. Phys. Solids*, 42:1533–1550, 1994.

[EM06] M. A. Efendiev and A. Mielke. On the rate-independent limit of systems with dry friction and small viscosity. *J. Convex Analysis*, 13:151–167, 2006.

[FG06] G.A. Francfort and A. Garroni. A variational view of partial brittle damage evolution. *Arch. Ration. Mech. Anal.*, 182(1):125–152, 2006.

[FKS12] A. Fiaschi, D. Knees, and U. Stefanelli. Young-measure quasi-static damage evolution. *Arch. Ration. Mech. Anal.*, 203(2):415–453, 2012.

[FN96] M. Frémond and B. Nedjar. Damage, gradient of damage and principle of virtual power. *Int. J. Solids Structures*, 33(8):1083–1103, 1996.

[Fré02] M. Frémond. *Non-smooth thermomechanics.* Berlin: Springer, 2002.

[Gar00] H. Garcke. *On mathematical models for phase separation in elastically stressed solids*. Habilitation thesis, University Bonn, 2000.

[GL09] A. Garroni and C. Larsen. Threshold-based quasi-static brittle damage evolution. *Arch. Ration. Mech. Anal.*, 194(2):585–609, 2009.

[GUE+07] M.G.D. Geers, R.L.J.J. Ubachs, M. Erinc, M.A. Matin, P.J.G. Schreurs, and W.P. Vellinga. Multiscale Analysis of Microstructura Evolution and Degradation in Solder Alloys. *Internatilnal Journal for Multiscale Computational Engineering*, 5(2):93–103, 2007.

[HK11] C. Heinemann and C. Kraus. Existence of weak solutions for Cahn-Hilliard systems coupled with elasticity and damage. *Adv. Math. Sci. Appl.*, 21(2):321–359, 2011.

[HK12] C. Heinemann and C. Kraus. Complete damage in linear elastic materials - modeling, weak formulation and existence results. 2012.

[HK13a] C. Heinemann and C. Kraus. 1759: Heinemann, christian; kraus, christiane a degenerating cahn–hilliard system coupled with complete damage processes. 2013.

[HK13b] C. Heinemann and C. Kraus. Existence results for diffuse interface models describing phase separation and damage. *European J. Appl. Math.*, 24:179–211, 2013.

[JL05] R. Desmorat J. Lemaitre. *Engineering Damage Mechanics: Ductile, Creep, Fatigue and Brittle Failures.* Springer-Verlag, Berlin, 2005.

[KO88] N. Kikuchi and J.T. Oden. *Contact problems in elasticity: A study of variational inequalities and finite element methods.* Philadelphia, PA: SIAM, 1988.

[KRZ11] D. Knees, R. Rossi, and C. Zanini. *A Vanishing Viscosity Approach to a Rate-independent Damage Model.* Preprint No. 1633. WIAS, 2011.

[LT11] G. Lazzaroni and R. Toader. A model for crack propagation based on viscous approximation. *Math. Models Methods Appl. Sci.*, 21(10):2019–2047, 2011.

[Mer05] T. Merkle. The Cahn-Larché system: a model for spinodal decomposition in eutectic solder; modelling, analysis and simulation. Phd-thesis, Universität Stuttgart, Stuttgart, 2005.

[Mie95] C. Miehe. Discontinuous and continuous damage evolution in Ogden-type large-strain elastic materials. *Eur. J. Mech.*, 14:697–720, 1995.

[Mie05] A. Mielke. Evolution in rate-independent systems. *Handbook of Differential Equations: Evolutionary Equations*, 2:461–559, 2005.

[Mie11] A. Mielke. Complete-damage evolution based on energies and stresses. *Discrete Contin. Dyn. Syst., Ser. S*, 4(2):423–439, 2011.

[MK00] C. Miehe and J. Keck. Superimposed finite elastic-viscoelastic-plastoelastic stress response with damage in filled rubbery polymers. Experiments, modelling and algorithmic implementation. *J. Mech. Phys. Solids 48*, 48:323–365, 2000.

[MR06] A. Mielke and T. Roubícek. Rate-independent damage processes in nonlinear elasticity. *Mathematical Models and Methods in Applied Sciences*, 16:177–209, 2006.

[MRZ10] A. Mielke, T. Roubícek, and J. Zeman. Complete Damage in elastic and viscoelastic media. *Comput. Methods Appl. Mech. Engrg*, 199:1242–1253, 2010.

[MS11] A. Menzel and P. Steinmann. A theoretical and computational framework for anisotropic continuum damage mechanics at large strains. *Int. J. Solids Struct.*, 38:9505–9523, 2011.

[MT99] A. Mielke and F. Theil. A mathematical model for rate-independent phase transformations with hysteresis. In R. Balean H.-D. Alber and R. Farwig, editors, *Models of Continuum Mechanics in Analysis and Engineering*, pages 117–129, Aachen, 1999. Shaker Verlag.

[MT10] A. Mielke and M. Thomas. Damage of nonlinearly elastic materials at small strain — Existence and regularity results. *ZAMM Z. Angew. Math. Mech.*, 90:88–112, 2010.

[Neg10] M. Negri. From rate-dependent to rate-independent brittle crack propagation. *J. Elasticity*, 98(2):159–187, 2010.

[Nit81] J.A. Nitsche. On Korn's second inequality. *RAIRO, Anal. Numér.*, 15:237–248, 1981.

[PZ08] I. Pawlow and W. M. Zajączkowski. Measure-valued solutions of a heterogeneous Cahn-Hilliard system in elastic solids. *Colloquium Mathematicum*, 112 No.2, 2008.

[Rou10] T. Roubícek. Thermodynamics of rate-independent processes in viscous solids at small strains. *SIAM J. Math. Anal.*, 42 No. 1, 2010.

[RR12] E. Rocca and R. Rossi. A degenerating PDE system for phase transitions and damage. *arXiv:1205.3578v1*, 2012.

[Sim86] J. Simon. Compact sets in the space $L^p(0,T;B)$. *Annali di Matematica Pura ed Applicata*, 146:65–96, 1986.

[SP12] G. Schimperna and I. Pawłow. On a class of Cahn-Hilliard models with nonlinear diffusion. *arXiv:1106.1581*, 2012.

[SP13] G. Schimperna and I. Pawłow. A Cahn-Hilliard equation with singular diffusion. *J. Differ. Equations*, 254(2):779–803, 2013.

[VSL11] G. Z. Voyiadjis, A. Shojaei, and G. Li. A thermodynamic consistent damage and healing model for self healing materials. *Int. J. Plast.*, 27(7):1025–1044, 2011.

[Wei01] U. Weikard. Numerische Lösungen der Cahn-Hilliard-Gleichung und der Cahn-Larché-Gleichung. Phd-thesis, Universität Bonn, Bonn, 2001.