

# 1 General Information

DFG Reference No.: STE 792/3-2

Project number: 445863039

Project title: Factors in graphs

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Name of the cooperation partners(s): no cooperation partners

Reporting period: 01.10.2023 to 31.10.2024

## 2 Summary

**Zusammenfassung:** Kantenfärbungen und Faktoren von Graphen sind klassische Gebiete der Graphentheorie. Frühe und die Graphentheorie prägende Sätze, wie z.B. der Satz von König (1916) oder der Satz von Petersen (1891) machen Aussagen über Kantenfärbungen und Faktoren von Graphen. Besonderes Interesse gilt Faktoren von regulären Graphen.

Vizing (1965) zeigte, dass die minimale Anzahl von Farben, mit denen die Kanten eines einfachen Graphen mit maximalem Eckengrad  $k$  gefärbt werden können entweder gleich  $k$  oder  $k+1$  ist. Das Ergebnis teilt die Klasse der einfachen Graphen in zwei Klassen ein;  $G$  ist in Klasse 1, falls  $G$   $k$ -kantenfärbbar ist und andernfalls in Klasse 2. Für keine der beiden Klassen gibt es Charakterisierungen und es ist ein *NP*-vollständiges Problem zu entscheiden, ob ein Graph mit  $k$  Farben färbbar ist. Es ist wenig über die Struktur von Graphen der Klasse 2 bekannt.

Neben dem Studium von Faktoren in kantenfärbungskritischen Graphen bildete die Untersuchung von  $r$ -Graphen der Klasse 2 einen wesentlichen Schwerpunkt des Projektes.

Jeder  $r$ -Graph hat ein perfektes Matching und jeder  $r$ -kantenfärbbare  $r$ -Graph enthält  $r$  paarweise disjunkte perfekte Matchings. Somit lässt sich die Menge der  $r$ -Graphen in Mengen  $R_k$  ( $k \in \{1, \dots, r\}$ ) partitionieren, wobei  $G \in R_k$ , falls  $k$  die maximale Anzahl paarweise disjunkter perfekter Matchings in  $G$  ist.

Es wurde u.a. das Verhältnis zwischen Kantenzusammenhang und der Existenz von paarweise disjunkten perfekten Matchings untersucht. Sei  $m(t, r)$  die maximale Zahl  $s$ , so dass jeder  $t$ -kantenzusammenhängende  $r$ -Graph  $s$  paarweise disjunkte perfekte Matchings enthält. Wir haben obere Schranken für diesen Parameter bewiesen, die darauf hinweisen, dass  $r$ -Graphen mit nur wenigen paarweise disjunkten perfekten Matchings einen kleinen Kantenzusammenhang haben.

Das Konzept der  $H$ -Färbung für  $r$ -Graphen ist ein vereinheitlichender Ansatz zur Untersuchung von Fragen zu Zyklen und Matchings in  $r$ -Graphen. Dieses Konzept wurde von Jaeger (1985) zunächst für kubische Graphen eingeführt. Sei  $\mathcal{H}_r$  eine minimale Menge von  $r$ -Graphen, so dass es für jeden  $r$ -Graphen  $G$  einen Graphen  $H \in \mathcal{H}_r$  gibt, der  $G$  färbt. Jaeger vermutete, dass  $\mathcal{H}_3$  nur den Petersengraphen enthält. Diese Vermutung ist offen, und wir betrachten die Frage, ob  $\mathcal{H}_r$  eine endliche Menge ist. Wir zeigen, dass dies nur dann der Fall ist, falls  $r = 3$  und Jaegers Vermutung wahr ist.

**Summary:** Edge coloring and factors of graphs are classical areas of graph theory. Early and fundamental theorems of graph theory, such as König’s theorem (1916) or Petersen’s theorem (1891) make statements about edge colorings and factors of graphs. Factors of regular graphs are of particular interest.

Vizing (1965) showed that the minimum number of colors, which are needed to properly color the edges of a simple graph with maximum vertex degree  $k$  is equal to  $k$  or to  $k + 1$ . This result divides the class of simple graphs into two classes; a graph is in class 1 if it is edge colorable with  $k$  colors and in class 2 otherwise. For neither class there are characterizations and it is an *NP*-complete problem to decide whether a graph is colorable with  $k$  colors. Little is known about the structure of graphs of class 2.

In addition to the study of factors in edge-chromatic critical graphs the investigation of  $r$ -graphs of class 2 was a major focus of the project. Many hard graph-theoretical problems can be reduced to  $r$ -graphs. Every  $r$ -graph has a perfect matching and every  $r$ -edge colorable  $r$ -graph contains  $r$  pairwise disjoint perfect matchings. Thus, the set of  $r$ -graphs can be partitioned into sets  $R_k$  ( $k \in \{1, \dots, r\}$ ), where  $G \in R_k$ , if  $k$  is the maximum number of pairwise disjoint perfect matchings in  $G$ .

Among other things, the relationship between edge-connectivity and the existence of pairwise disjoint perfect matchings was investigated. Let  $m(t, r)$  be the maximum number  $s$  such that every  $t$ -edge connected  $r$ -graph contains  $s$  pairwise disjoint perfect matchings. We have proved upper bounds for this parameter, which indicate that  $r$ -graphs with only a few pairwise disjoint perfect matchings have small edge connectivity.

The concept of  $H$ -coloring for  $r$ -graphs is a unifying approach for studying questions on matchings and cycles in  $r$ -graphs. This concept was first introduced by Jaeger (1985) for cubic graphs. Let  $\mathcal{H}_r$  be an inclusion-wise minimal set of  $r$ -graphs such that for every  $r$ -graph  $G$  there is a graph  $H \in \mathcal{H}_r$  that colors  $G$ . Jaeger conjectured that  $\mathcal{H}_3$  only contains the Petersen graph. This conjecture is open and we consider the question of whether  $\mathcal{H}_r$  is a finite set. We show that this can only be the case if  $r = 3$  and Jaeger’s conjecture is true.

### 3 Progress Report

This report is on the results of the one-year extension of the project *Factors in graphs*. The results achieved in the first three-year funding phase were reported in the extension proposal. Isaak H. Woilf (doctoral candidate) and Yulai Ma (postdoc) were employed in the extension phase of the project.

The overall objective of the research project was to gain a better understanding of structural properties of edge-chromatic critical graphs and regular graphs, especially  $r$ -graphs. The main results of the first phase have been published in [16, 18, 26, 27]. On the basis of the results achieved in the first phase of the project the second phase of the project was mainly devoted to the study of factors in (regular) graphs. We will summarize the major findings of the project.

#### Perfect matchings in $r$ -graphs

We summarize the major results of [17, 19, 28]. The following two conjectures had been in the focus of our research in this field.

**Conjecture 3.1** (Seymour’s exact conjecture [24, 25]). *Every planar  $r$ -graph is a class 1 graph.*

**Conjecture 3.2** (Generalized Berge-Fulkerson Conjecture [24]). *Every  $r$ -graph has  $2r$  perfect matchings such that every edge is in precisely two of them.*

For  $r = 3$ , Conjecture 3.1 is true by the 4-Color-Theorem. For  $r \in \{4, 5\}$  it is proved by Guenin [8], for  $r = 6$  by Dvořák, Kawarabayashi and Král' [5] and for  $r = 7$  and  $r = 8$  by Chudnovsky, Edwards, Kawarabayashi, Seymour [2] and by Chudnovsky, Edwards, Seymour [3], respectively. Some of these cases are proved by using sophisticated discharging. For  $r \geq 9$ , Conjecture 3.1 is open.

Informally, three main lines of research can be identified in this subject area. First, the introduction and study of structural parameters for  $r$ -graphs which allow to obtain partial results. Second,  $H$ -coloring of  $r$ -graphs which induces an order structure on the set of  $r$ -graphs and allows to transfer properties from "smaller"  $r$ -graphs to "larger" ones. Third, the formulation of the problems as combinatorial optimization problems; see e.g. [1] for recent results in this line.

The project was intended to follow the first two lines and we intensively worked on  $H$ -coloring of  $r$ -graphs. Let  $G$  and  $H$  be graphs. An  $H$ -coloring of  $G$  is a mapping  $f: E(G) \rightarrow E(H)$  such that

- if  $e_1, e_2 \in E(G)$  are adjacent, then  $f(e_1) \neq f(e_2)$ ,
- for every  $v \in V(G)$  there exists a vertex  $u \in V(H)$  with  $f(\partial_G(v)) = \partial_H(u)$ .

If such a mapping exists, then we say  $H$  colors  $G$ . Some substructures of  $H$  induce substructures in  $G$ . For instance, if  $H_1$  is a  $k$ -regular subgraph of  $H$ , then  $G[f^{-1}(E(H_1))]$  is a  $k$ -regular subgraph of  $G$ , or if  $H_2$  is a spanning subgraph of  $H$ , then  $G[f^{-1}(E(H_2))]$  is a spanning subgraph of  $G$ . Furthermore, if  $G$  and  $H$  are cubic graphs and  $H$  has a Berge-Fulkerson cover or a 5-cycle double cover, then  $G$  has these properties as well, see [31]. Starting point are Petersen coloring of cubic graphs which had been introduced by Jaeger [11].

**Conjecture 3.3** (Jaeger [11]). *Every cubic bridgeless graph can be colored by the Petersen graph.*

Conjecture 3.3, which is commonly called the Petersen coloring conjecture, is a starting point for research in several directions. Various aspects of it are studied and partial results are proved, e.g., in [4, 9, 10, 12, 20, 22, 23]. The definition of  $H$ -coloring indicates a way to generalize it to regular graphs. Let  $\mathcal{H}_r$  be an inclusion-wise minimal set of  $r$ -graphs, such that for each  $r$ -graph there is  $H \in \mathcal{H}_r$  which colors  $G$ . Mazzuocolo et al. [21] asked whether for all  $r > 3$ , there exists a connected  $r$ -graph  $H$  which colors all  $r$ -graphs, i.e. whether  $|\mathcal{H}_r| = 1$ . Indeed it was even unclear whether  $\mathcal{H}_r$  is a finite set. We completely answer this question in [17]. First we characterize  $r$ -graphs which can only be colored by themselves.

**Theorem 3.4** ([17]). *Let  $r \geq 3$  and let  $G$  be a connected  $r$ -graph. The following statements are equivalent.*

- 1)  $G \in \mathcal{H}_r$ .
- 2) The only connected  $r$ -graph coloring  $G$  is  $G$  itself.
- 3)  $G$  cannot be colored by a smaller  $r$ -graph.

The set of the smallest class 2  $r$ -graphs is denoted by  $\mathcal{T}_r$ . In [17], the set  $\mathcal{T}_r$  is determined for each  $r \geq 3$  and it is shown that  $\mathcal{T}_r \subseteq \mathcal{H}_r$ . Then the following theorem is deduced by the use of an appropriate (recursive) construction of elements of  $\mathcal{H}_r$ . A similar result is proved for simple  $r$ -graphs.

**Theorem 3.5.** [17] *Either  $\mathcal{H}_3 = \{P\}$  or  $\mathcal{H}_3$  is a unique infinite set. Moreover, if  $r \geq 4$ , then  $\mathcal{H}_r$  is a unique infinite set.*

We also studied the reduction of some notoriously difficult conjectures to specific sub-classes of graphs. An *automorphism* of a graph  $G$  is a mapping  $\alpha: V(G) \rightarrow V(G)$ , such that for every two vertices  $u, v \in V(G)$  the number of edges between  $u$  and  $v$  is the same as the number of edges between  $\alpha(u)$  and  $\alpha(v)$ . For every  $v \in V(G)$ , the smallest positive integer  $k$  such that  $\alpha^k(v) = v$  is denoted by  $d_\alpha(v)$ .

Let  $v$  be a vertex of a tree  $T$ . An automorphism  $\alpha$  of  $T$  is *rotational* with respect to  $v$ , if  $d_\alpha(v) = 1$  and  $d_\alpha(u) = d_G(v)$  for every  $u \in V(T) \setminus \{v\}$ . The unique tree with vertex degrees in  $\{1, r\}$  and a vertex  $v$  with distance  $i$  to every leaf is denoted by  $T_i^r$ . Vertex  $v$  is unique and it is called the *root* of  $T_i^r$ .

An  $r$ -regular graph  $G$  is a  $T_i^r$ -*graph*, if  $G$  has a spanning tree  $T$  isomorphic to  $T_i^r$ . If, additionally,  $G$  has an automorphism that is rotational on  $T$  (with respect to the root), then  $G$  is a *rotation  $T_i^r$ -graph*. Note that  $G$  can be embedded in the plane (crossings allowed) such that the embedding has a  $\frac{2\pi}{r}$ -rotation symmetry fixing the root. A *rotation  $r$ -graph* is an  $r$ -graph that is a rotation  $T_i^r$ -graph for some integer  $i$ . In [28] we prove that Conjecture 3.2, the generalized Fan-Raspaud and Tutte's Flow conjectures can be reduced to rotation  $r$ -graphs.

**Theorem 3.6.** [28] *Let  $r$  be a positive odd integer. For every  $r$ -graph  $G$  there is a simple rotation  $r$ -graph  $G'$ , such that  $G$  can be obtained from  $G'$  by a finite number of 2-cut reductions.*

The following corollary is a direct consequence of Theorem 3.6.

**Corollary 3.7.** [28] *Let  $r$  be a positive odd integer and let  $A$  be a graph-property that is preserved under 2-cut reduction. Every  $r$ -graph has property  $A$  if and only if every simple rotation  $r$ -graph has property  $A$ .*

**Corollary 3.8.** [28] *Let  $r$  be an odd integer. The following statements are equivalent:*

1. *For  $r \geq 1$ , every  $r$ -graph  $G$  has a collection of  $2r$  perfect matchings such that every edge of  $G$  is in precisely two of them.*
2. *For  $r \geq 1$ , every simple rotation  $r$ -graph  $G$  has a collection of  $2r$  perfect matchings such that every edge of  $G$  is in precisely two of them.*
3. *For  $r \geq 1$ , every  $r$ -graph  $G$  has a collection of  $2r - 1$  perfect matchings such that every edge of  $G$  is in at least one of them.*
4. *For  $r \geq 1$ , every simple rotation  $r$ -graph  $G$  has a collection of  $2r - 1$  perfect matchings such that every edge of  $G$  is in at least one of them.*

The approach of reducing or extending variants of open conjectures to other graph classes was pursued further in [19]. A  $(t, r)$ -PM of an  $r$ -graph  $G$  is a collection of  $t \cdot r$  perfect matchings  $M_1, \dots, M_{tr}$  of  $G$  such that every edge of  $G$  is contained in exactly  $t$  of them. Trivially, if  $G$  is in class 1, then it has a  $(t, r)$ -PM for every  $t \geq 1$ . The following two conjectures are weak versions of Conjecture 3.2.

**Conjecture 3.9.** *There is a  $t \geq 1$  such that for all  $r \geq 1$  every  $r$ -graph has a  $(t, r)$ -PM.*

**Conjecture 3.10.** *For every  $r \geq 1$  there is a  $t_r \geq 1$  such that every  $r$ -graph has a  $(t_r, r)$ -PM.*

Clearly, Conjecture 3.2 implies Conjecture 3.9, which implies Conjecture 3.10. The reductions of these two conjectures to planar  $r$ -graphs or to Petersen-minor-free  $r$ -graphs are also open. The main result of [19] is the following statement.

**Theorem 3.11.** *For any  $t \geq 1$  and  $r \geq 1$ , the following statements are equivalent.*

1. *Every planar  $r$ -graph has a  $(t, r)$ -PM.*
2. *Every  $K_5$ -minor-free  $r$ -graph has a  $(t, r)$ -PM.*
3. *Every  $K_{3,3}$ -minor-free  $r$ -graph has a  $(t, r)$ -PM.*
4. *Every  $r$ -graph  $G$  with  $cr(G_s) \leq 1$  has a  $(t, r)$ -PM.*

### Lonely edges in cubic graphs and transversal factors

We summarize the major results of [6, 7]. In question on multi-packing perfect matchings in  $r$ -graphs, edges which are only contained in few perfect matchings are of particular interest. Isaak H. Wolf had two short research stays in the group of Giuseppe Mazzuocolo (Università degli Studi di Modena e Reggio Emilia, Italy) where they studied cubic graphs which have edges which are in precisely one perfect matching [6]. Such edge are called lonely edges. For a positive integer  $k$  let  $U_k$  be the class of cubic graphs with precisely  $k$  lonely edges. It is shown that  $U_k = \emptyset$  if  $k \geq 7$  and for  $k \in \{3, 4, 5, 6\}$  the graphs of  $U_k$  are characterized. First results for the classes  $U_1$  and  $U_2$  are obtained.

In a second project they study factors of regular graphs which transverse all elements of predefined set  $\mathcal{C}$  of circuits. In [7] they prove a novel extension of a result of Kardős et al. [14]. They study the problem under what conditions there exist  $t$  and  $r$  such that an  $r$ -graph  $G$  has a  $t$ -factor  $F$  which contains an edge of any predefined set of circuits in  $G$ . Some instances of this problem are proven, particularly if  $t = \frac{r}{3}$  and if  $t$  is even and  $t = \frac{r}{2}$ .

### Factors in graphs with bounded isolated toughness

We summarize the major result of [30], which studies graphs with isolated toughness smaller than 1. Kano, Lu and Yu [13] stated the following problem: If  $n, m$  are positive integers and  $G$  is a simple graph such that  $iso(G - S) \leq \frac{n}{m}|S|$  for all  $\emptyset \neq S \subseteq V(G)$ , what factors does  $G$  have. Isaak H. Wolf answers the question if  $n > m$  by characterizing  $\frac{m}{n}$ -tough graphs in terms of their component factors.

### Reconfiguration graphs

Yulai Ma was also involved in a project on reconfiguration graphs. In [15] they give a detailed classification of the connectivity of the reconfiguration graphs concerning the  $t$ -chromatic  $P_5$ -free graphs for  $t = 3$ , and for  $t \geq 4$  and  $t + 1 \leq k \leq \binom{t}{2}$ .

### Talks

Eckhard Steffen was invited to organize a session Graph Coloring on the 25th Int. Symposium on Mathematical Programming (ISMP 2024) in Montréal, Canada.

- Yulai Ma, Edge-connectivity and pairwise disjoint perfect matchings in regular graphs, 11th Symposium on Graph Theory and Combinatorial Optimization Frontiers, Tianjin, China, October 2024
- Yulai Ma, Sets of  $r$ -graphs that color all  $r$ -graphs, 2nd Symposium on Graph Coloring and Related Problems, Tianjin, China, November 2024
- Yulai Ma, Pairwise disjoint perfect matchings in  $r$ -graphs, Discrete Math Seminar, Institute for Basic Science (IBS), Daejeon, South Korea, December 2024  
(Video on <https://www.youtube.com/watch?v=DdKE6VYHK5g>)
- Eckhard Steffen, Sets of  $r$ -graphs that color all  $r$ -graphs. ISMP 2024, Montréal, Canada, July 2024
- Isaak H. Wolf, Pairwise disjoint perfect matchings in  $r$ -graphs. ISMP 2024, Montréal, Canada, July 2024

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## 4 Published Project Results

This section lists the published project results of the entire project (first and second phase).

### 4.1 Publications with scientific quality assurance

- [16] Y. MA, D. MATTIOLO, E. STEFFEN, I. H. WOLF, Pairwise disjoint perfect matchings in  $r$ -edge-connected  $r$ -regular graphs, *SIAM Journal on Discrete Mathematics* **37**(3) (2023) 1548-1565 <https://doi.org/10.1137/22M1500654>
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- [26] E. STEFFEN, I. H. WOLF, Even factors in edge-chromatic-critical graphs with a small number of divalent vertices, *Graphs and Combinatorics* **38** (2022) 1 - 11  
<https://doi.org/10.1007/s00373-022-02506-x>
- [27] E. STEFFEN, I. H. WOLF, Bounds for the chromatic index of signed multigraphs, *Discrete Applied Mathematics* **337** (2023) 185-189 <https://doi.org/10.1016/j.dam.2023.05.008>
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