Weierstraß-Institut
für Angewandte Analysis und Stochastik
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Error estimates for nonlinear reaction-diffusion systems involving different diffusion length scales

Sina Reichelt

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Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: Sina.Reichelt@wias-berlin.de

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Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

We derive quantitative error estimates for coupled reaction-diffusion systems, whose coefficient functions are quasi-periodically oscillating modeling microstructure of the underlying macroscopic domain. The coupling arises via nonlinear reaction terms and we allow for different diffusion length scales, i.e. whereas some species have characteristic diffusion length of order 1, other species may diffuse much slower, namely, with order of the characteristic microstructure-length scale. We consider an effective system, which is rigorously obtained via two-scale convergence, and we prove that the error of its solution to the original solution is of order $\varepsilon^{1/2}$.

1 Introduction

Many mathematical models arising from biological, physical or engineering problems involve effects on microscopic scales, e.g. spatial inhomogeneities of the underlying material. In view of numerical simulations as well as more profound structural insight, we are interested in finding effective, or homogenized, models. From the analytical perspective, we ask for a rigorous justification of the effective model and, if available, error estimates describing the difference to the original microscopic model.

We refer to the books [BLP78, JKO94, MaK06, Tar09] for a general survey of homogenization theory. An important step in the theory of periodic homogenization was the introduction of two-scale convergence in [Ngu89, All92], which allows to rigorously treat systems involving different diffusion length scales, see e.g. [HJM94, Pet07, MeM10]. So far, the notion of two-scale convergence is a weak convergence. The periodic unfolding technique, introduced in [CDG02], allows for a natural definition of strong two-scale convergence and, hence, the treatment of nonlinear problems, cf. [Vis04, Vis06, Vis08, MiT07, NeJ07, PtR10, Han11]. Based on this strong notion of convergence, one can ask for quantitative error estimates, see e.g. [Gri04, OnV07, FMP12, Muv13], as well as for numerical simulations, see e.g. [MaS02, Eck05, CFM10, ChM12] .

The objective of this contribution are coupled reaction-diffusion systems of the following type

$$\begin{array}{rlcl} u_t^\varepsilon = & \operatorname{div}(\mathbb{D}_1(x,\tfrac{x}{\varepsilon})\nabla u^\varepsilon) & + & F_1(x,\tfrac{x}{\varepsilon},u^\varepsilon,v^\varepsilon) \\ v_t^\varepsilon = & \operatorname{div}(\varepsilon^2\mathbb{D}_2(x,\tfrac{x}{\varepsilon})\nabla v^\varepsilon) & + & F_2(x,\tfrac{x}{\varepsilon},u^\varepsilon,v^\varepsilon) \end{array} \quad \text{in } \Omega \tag{1.1}$$

supplemented with homogeneous Neumann boundary conditions and initial conditions. Here, $(u^\varepsilon,v^\varepsilon):[0,T]\times\Omega\to\mathbb{R}^{m_1+m_2}$ denote the concentrations of m_1 "classically" diffusing species with characteristic diffusion length of order $O(1)$ and m_2 slowly diffusing species of order $O(\varepsilon)$. Moreover, $\mathbb{D}_i:\Omega\times\mathcal{Y}\to\mathbb{R}^{(m_i\times d)\times(m_i\times d)}$ denotes the diffusion coefficients and $F_i:\Omega\times\mathcal{Y}\times\mathbb{R}^{m_1+m_2}\to\mathbb{R}^{m_i}$ the nonlinear reaction terms and both, $\mathbb{D}_i$ and F_i, are assumed to be periodic in $y=x/\varepsilon$ w.r.t. a prescribed microstructure, cf. Section 2.1.

It was shown in [MRT14] that the solutions $(u^\varepsilon,v^\varepsilon)$ converge for $\varepsilon\to 0$ to a limit (u,V) that decomposes into a one-scale function $u(t,x)$ and a two-scale function $V(t,x,y)$, which solve the effective system

$$\begin{array}{rlcll} u_t = & \operatorname{div}(\mathbb{D}_{\text{eff}}(x)\nabla u) & + & ⨍_{\mathcal{Y}} F_1(x,y,u(x),V(x,y))\,\mathrm{d}y & \text{in } \Omega \\ V_t = & \operatorname{div}_y(\mathbb{D}_2(x,y)\nabla_y V) & + & F_2(x,y,u,V) & \text{in } \Omega\times\mathcal{Y} \end{array} \tag{1.2}$$

In order to install the limit passage (1.1) $\to$ (1.2), we employ the technique of two-scale convergence via periodic unfolding, cf. (2.7). This involves the periodic unfolding operator

$\mathcal{T}_\varepsilon : L^1(\Omega) \to L^1(\Omega \times \mathcal{Y})$, the folding operator $\mathcal{F}_\varepsilon : L^1(\Omega \times \mathcal{Y}) \to L^1(\Omega)$ and the gradient folding operators $\mathcal{G}_\varepsilon^0$ resp. $\mathcal{G}_\varepsilon^1$, cf. Section 2.2. With this method, the strong two-scale convergence of the slowly diffusing species v^ε, i.e. $\max_{0\leq t\leq T} \| \mathcal{T}_\varepsilon v^\varepsilon(t) - V(T)\|_{L^2(\Omega\times\mathcal{Y})} \to 0$, was proved in [MRT14], cf. Section 3.1, whereas the strong convergence $u^\varepsilon \to u$ follows immediately from the compact embedding $H^1(\Omega) \subset L^2(\Omega)$. This result was obtained under the assumption of L^∞-regularity of the coefficients and global Lipschitz continuity of the reaction terms, cf. (3.6.A1)–(3.6.A4). One major analytical difficulty to overcome is the *periodicity defect* [Gri04] or *$\mathcal{T}_\varepsilon$-property of recovered periodicity* [MRT14], i.e.

$$\text{for all } u^\varepsilon \in H^1(\Omega): \quad \begin{aligned} &\mathcal{T}_\varepsilon u^\varepsilon \in L^2(\Omega; H^1(Y)) \not\subseteq L^2(\Omega; H^1(\mathcal{Y})), \text{ but} \\ &\text{w-}\lim_{\varepsilon\to 0} \mathcal{T}_\varepsilon u^\varepsilon \in L^2(\Omega; H^1(\mathcal{Y})), \text{ if the limit exists.} \end{aligned} \tag{1.3.PD}$$

The aim of this paper is to derive in Theorem 3.2 the error estimate

$$\max_{0\leq t\leq T} \left\{ \| \mathcal{T}_\varepsilon v^\varepsilon(t) - V(t)\|_{L^2(\Omega\times\mathcal{Y})} + \|u^\varepsilon(t) - u(t)\|_{L^2(\Omega)} \right\} \leq \varepsilon^{1/2} C. \tag{1.4}$$

Therefore, we assume additional spatial regularity w.r.t. the macroscopic scale $x \in \Omega$ of the given data (3.6.A5), i.e. $\nabla_x \mathbb{D}_i$, $\nabla_x F_i \in L^\infty(\Omega \times \mathcal{Y})$, and the effective solution (u, V) (3.6.A6), i.e. $u \in H^2(\Omega)$, $V \in H^1(\Omega; H^1(\mathcal{Y}))$. Further, for the proof of Theorem 3.2, the domain Ω is assumed to be of rectangular shape, cf. (2.1.ReSh), which significantly simplifies the notations and definitions in Section 2. But, we assume neither additional spatial regularity of the original solutions $(u^\varepsilon, v^\varepsilon)$ nor of the corrector functions.

The same convergence rate (1.4) has been obtained in [Eck05] for phase transition problems in binary mixtures. Therein, the method of asymptotic expansion is employed and, for the derivation of error estimates, additional regularity and also continuity w.r.t. the x and y variable is assumed for all involved coefficients and functions.

In [FMP12], a reaction-diffusion system predicting concrete corrosion is considered, but the system does not include slowly diffusing species v^ε. Nevertheless, for the classically diffusing species u^ε the convergence rate $\varepsilon^{1/2}$ is rigorously proved by the method of periodic unfolding. The result in [FMP12] is obtained under the same assumptions, cf. (2.1.ReSh) and (3.6), but only accounts for exactly periodic coefficients, i.e. $a_\varepsilon(x) = a(x/\varepsilon)$.

The distinctive feature of this contribution is the nonlinear coupling of the classically and slowly diffusing species combined with the periodic unfolding method, which allows to avoid any assumption of spatial continuity. Our proof to (1.4), in the first part, follows along the lines of [MRT14] and we derive the Gronwall-type estimate

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \left(||| \mathcal{T}_\varepsilon v^\varepsilon - V|||^2 + \|u^\varepsilon - u\|^2 \right) \leq C \left(||| \mathcal{T}_\varepsilon v^\varepsilon - V|||^2 + \|u^\varepsilon - u\|^2 \right) + \Delta^{v^\varepsilon} + \Delta^{u^\varepsilon}, \tag{1.5}$$

where $|||\cdot||| := \|\cdot\|_{L^2(\Omega\times\mathcal{Y})}$ and $\|\cdot\| := \|\cdot\|_{L^2(\Omega)}$ and $\Delta^{v^\varepsilon}, \Delta^{u^\varepsilon}$ comprise errors terms. In [MRT14], it was shown that these errors vanish as $\varepsilon \to 0$. The novelty of the contribution, the second part of the proof, is the quantification of their convergence, namely $|\Delta^{u^\varepsilon} + \Delta^{v^\varepsilon}| \leq \varepsilon C$. In order to quantify those error terms, we have to find, in particular, error estimates for the folding and unfolding operators, see the lemmas 3.3, 3.4, and 3.5 in Section 3.3, which heavily rely on the improved regularity w.r.t. $x \in \Omega$ and ideas from [Gri04]. Moreover, we use a quantification result for the periodicity defect (1.3.PD) from [Gri04], see Lemma 3.6.

The structure of the paper is the following: in Section 2, we introduce basic notations, definitions, and results concerning periodic unfolding (Sec. 2.1 & 2.2) and two-scale convergence (Sec. 2.3). In Section 3, we consider the coupled systems (1.1)–(1.2) and derive the error estimate (1.4). Therefore, we list our assumptions and recall the existing convergence result (Sec. 3.1), state our Main Theorem (Thm. 3.2), explain the structure of its proof (Sec. 3.2), and we derive preparatory error estimates (Sec. 3.3). Finally, we give the proof of Theorem 3.2 (Sec. 3.4).

2 Two-scale convergence

Here, and throughout this paper, x denotes the macroscopic variable and the microscopic variable y captures periodic oscillations in x/ε. In order to describe the convergence from (1.1) to (1.2), we introduce the concept of two-scale convergence, which is designed for problems with underlying periodic microstructure, see Section 2.1 for the latter. The definition of two-scale convergence (2.7), introduced in Section 2.3, is based on the periodic unfolding technique, described in Section 2.2, and with this it reduces to the notion of classical weak and strong convergence in the two-scale space $L^2(\Omega \times \mathcal{Y})$.

2.1 Microstructure and the periodicity cell

Following [CDG02, CDG08, MiT07], let $\Omega \subset \mathbb{R}^d$ be a bounded domain and let $Y = [-\frac{1}{2}, \frac{1}{2})^d$ denote the *unit cell* so that $\mathbb{R}^d$ is the disjoint union of translated cells $\lambda + Y$, where $\lambda \in \mathbb{Z}^d$. Identifying opposite faces of $\overline{Y}$ gives the *periodicity cell* $\mathcal{Y}$, i.e. the torus

$$\mathcal{Y} := \mathbb{R}^d/_{\mathbb{Z}^d}.$$

But, in notation, we will not distinguish between elements of the unit cell $y \in Y$ and the ones of the periodicity cell $y \in \mathcal{Y}$. Using the mappings $[\cdot]_Y : \mathbb{R}^d \to \mathbb{Z}^d$ and $\{\cdot\}_Y : \mathbb{R}^d \to Y$, we have the unique decomposition

$$\text{for all } x \in \mathbb{R}^d: \; x = [x]_Y + \{x\}_Y, \quad \text{where } [x]_Y \in \mathbb{Z}^d \text{ and } \{x\}_Y \in Y.$$

A function $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ is called Y*-periodic*, if $f(x) = f(\{x\}_Y)$ for a.a. $x \in \mathbb{R}^d$. Then, we can identify every periodic function f with a function $\widetilde{f}$ on $\mathcal{Y}$. Introducing the small length-scale parameter $\varepsilon > 0$, we define the sets

$$\Lambda_\varepsilon := \left\{\lambda \in \varepsilon\mathbb{Z}^d \,|\, (\lambda + \varepsilon Y) \subseteq \overline{\Omega}\right\} \quad \text{and} \quad \hat{\Omega}_\varepsilon := \operatorname{int}\left(\bigcup_{\lambda_i \in \Lambda_\varepsilon} \lambda_i + \varepsilon Y\right).$$

We assume throughout this paper that the domain Ω is of *rectangular shape*, i.e.

$$\text{for all } \varepsilon \in (0,1]: \quad \Omega = \hat{\Omega}_\varepsilon. \tag{2.1.ReSh}$$

This assumption significantly simplifies the definitions of the unfolding and folding operators $\mathcal{T}_\varepsilon$ and $\mathcal{F}_\varepsilon$, see (2.2) and (2.4), and of two-scale convergence, see (2.7). Hence, we write the nodes of the microscopic cells as

$$\mathcal{N}_\varepsilon(x) := \varepsilon \left[\tfrac{x}{\varepsilon}\right]_Y \in \Lambda_\varepsilon \quad \text{for all } x \in \Omega,$$

which describe the macroscopic scale. The microscopic scale is given by $y = \{x/\varepsilon\}_Y \in Y$ so that we obtain for all $x \in \Omega$ the decomposition $x = \mathcal{N}_\varepsilon(x) + \varepsilon y$.

2.2 Periodic unfolding, folding, and gradient folding operators

The *periodic unfolding operator* $\mathcal{T}_\varepsilon : L^1(\Omega) \to L^1(\Omega \times \mathcal{Y})$ is defined via, cf. [CDG02],

$$(\mathcal{T}_\varepsilon u)(x, y) := u(\mathcal{N}_\varepsilon(x) + \varepsilon y). \tag{2.2}$$

Moreover, we have for all $u, v \in L^2(\Omega)$ the crucial identities

$$\mathcal{T}_\varepsilon(uv) = (\mathcal{T}_\varepsilon u)(\mathcal{T}_\varepsilon v) \in L^1(\Omega \times \mathcal{Y}) \quad \text{and} \quad \int_\Omega u \, \mathrm{d}x = \int_{\Omega\times\mathcal{Y}} \mathcal{T}_\varepsilon u \, \mathrm{d}x \, \mathrm{d}y. \tag{2.3}$$

For the reverse operation, we define the *folding operator* $\mathcal{F}_\varepsilon : L^1(\Omega \times \mathcal{Y}) \to L^1(\Omega)$ via

$$(\mathcal{F}_\varepsilon U)(x) := \fint_{\mathcal{N}_\varepsilon(x)+\varepsilon Y} U(\xi, \{\tfrac{x}{\varepsilon}\}_Y) \, \mathrm{d}\xi. \tag{2.4}$$

Even for smooth functions $U : \Omega \times \mathcal{Y} \to \mathbb{R}$ the folded function $\mathcal{F}_\varepsilon U$ is only piecewise constant in x, hence $\nabla(\mathcal{F}_\varepsilon U)$ cannot be determined in the classical sense. Therefore we now define a so-called *gradient folding operator* $\mathcal{G}^0_\varepsilon$, resp. $\mathcal{G}^1_\varepsilon$, which suitably regularizes the folded function $\mathcal{F}_\varepsilon U$. The definition of the above mentioned gradient folding operator is taken from [MRT14, Def. 3.7], cf. also [Han11, Prop. 2.11], [Vis04, Thm. 6.1], and [MiT07, Prop. 2.10]. At first, we define the functions with zero average via

$$H^1_{\mathrm{av}}(\mathcal{Y}) := \left\{ u \in H^1(\mathcal{Y}) \mid \int_\mathcal{Y} u(y) \, \mathrm{d}y = 0 \right\}.$$

Definition 2.1 (Gradient folding)**.** $\gamma = 0$: *The gradient folding operator* $\mathcal{G}^0_\varepsilon : H^1(\Omega) \times L^2(\Omega; H^1_{av}(\mathcal{Y})) \to H^1(\Omega)$ *maps a pair of functions* $(u, U) \in H^1(\Omega) \times L^2(\Omega; H^1_{av}(\mathcal{Y}))$ *to* $u^\varepsilon := \mathcal{G}^0_\varepsilon(u, U)$*, where* $u^\varepsilon \in H^1(\Omega)$ *is the unique weak solution of the elliptic problem*

$$\int_\Omega (u^\varepsilon - u) \cdot \varphi + (\nabla u^\varepsilon - \{\nabla u + \mathcal{F}_\varepsilon(\nabla_y U)\}) : \nabla\varphi \, \mathrm{d}x = 0 \quad \textit{for all } \varphi \in H^1(\Omega). \tag{2.5}$$

$\gamma = 1$: *The gradient folding operator* $\mathcal{G}^1_\varepsilon : L^2(\Omega; H^1(\mathcal{Y})) \to H^1(\Omega)$ *maps a two-scale function* $U \in L^2(\Omega; H^1(\mathcal{Y}))$ *to* $u^\varepsilon := \mathcal{G}^1_\varepsilon U$*, where* $u^\varepsilon \in H^1(\Omega)$ *is the unique weak solution of the elliptic problem*

$$\int_\Omega (u^\varepsilon - \mathcal{F}_\varepsilon U) \cdot \varphi + (\varepsilon\nabla u^\varepsilon - \mathcal{F}_\varepsilon(\nabla_y U)) : \varepsilon\nabla\varphi \, \mathrm{d}x = 0 \quad \textit{for all } \varphi \in H^1(\Omega). \tag{2.6}$$

For $\varepsilon > 0$ fixed, the Lax-Milgram lemma yields the existence of a unique weak solution $u^\varepsilon \in H^1(\Omega)$, so that the gradient folding operators are indeed well-defined.

2.3 Weak and strong two-scale convergence

We are now in the position to give the definition of weak and strong two-scale convergence following again [CDG02, CDG08, MiT07]. The notion of two-scale convergence was first introduced in [Ngu89] and coincides for bounded sequences with Definition (2.7a), here below; see [MiT07, Sec. 2.3] for a more detailed comparison of the different definitions.

For $(u^\varepsilon)_\varepsilon \subset L^2(\Omega)$, we say u^ε *weakly* (2.7a)*, resp. strongly* (2.7b)*, two-scale converges* to U in $L^2(\Omega \times \mathcal{Y})$, if

$$u^\varepsilon \xrightharpoonup{2\mathrm{w}} U \quad \text{in } L^2(\Omega \times \mathcal{Y}) \quad :\overset{\text{Def.}}{\Longleftrightarrow} \quad \mathcal{T}_\varepsilon u^\varepsilon \rightharpoonup U \quad \text{in } L^2(\Omega \times \mathcal{Y}), \tag{2.7a}$$

$$u^\varepsilon \xrightarrow{2\mathrm{s}} U \quad \text{in } L^2(\Omega \times \mathcal{Y}) \quad :\overset{\text{Def.}}{\Longleftrightarrow} \quad \mathcal{T}_\varepsilon u^\varepsilon \to U \quad \text{in } L^2(\Omega \times \mathcal{Y}). \tag{2.7b}$$

The unfolding operator $\mathcal{T}_\varepsilon$ is defined for the class of Lebesgue-integrable functions, where boundary values play no role, so that in particular $L^2(\Omega\times\mathcal{Y}) = L^2(\Omega\times Y)$. In view of the *periodicity defect* (1.3.PD), we carefully distinguish the spaces $H^1(Y)$ and $H^1(\mathcal{Y}) = H^1_{\text{per}}(Y)$, where the latter one is a closed subspace of $H^1(Y)$. For brevity, we set

$$\begin{aligned} &X = H^1(\Omega), \quad H = L^2(\Omega), \quad \mathbb{X} = L^2(\Omega; H^1(\mathcal{Y})),\\ &\mathbb{X}_{\text{av}} = L^2(\Omega; H^1_{\text{av}}(\mathcal{Y})), \quad \text{and} \quad \mathbb{H} = L^2(\Omega\times\mathcal{Y}). \end{aligned} \tag{2.8}$$

We have sequential compactness w.r.t. the weak two-scale convergence and it is shown in e.g. [Ngu89], [All92, Prop. 1.14], [Dam05, Thm. 5.2, Thm. 5.4], [PeB08, Thm. 3.4] that bounded sequences of one-scale functions $(u^\varepsilon)_\varepsilon$ admit a weakly two-scale converging subsequence, i.e.

(i) $\|u^\varepsilon\|_H \le C \;\Rightarrow\; \exists\, U \in \mathbb{H}:\; u^{\varepsilon'} \xrightharpoonup{2\text{w}} U$ in $\mathbb{H}$,

(ii) $\|u^\varepsilon\|_H + \varepsilon\|\nabla u^\varepsilon\|_H \le C \;\Rightarrow\; \exists\, U \in \mathbb{X}:\; u^{\varepsilon'} \xrightharpoonup{2\text{w}} U \;\&\; \varepsilon'\nabla u^{\varepsilon'} \xrightharpoonup{2\text{w}} \nabla_y U$ in $\mathbb{H}$,

(ii) $\|u^\varepsilon\|_X \le C \;\Rightarrow\; \exists\, (u,U) \in X\times\mathbb{X}_{\text{av}}:\; u^{\varepsilon'} \rightharpoonup u$ in X and $\nabla u^{\varepsilon'} \xrightharpoonup{2\text{w}} \nabla u + \nabla_y U$ in $\mathbb{H}$.

Since (2.5) (with $\gamma = 1$) implies $\|\,\mathcal{G}^1_\varepsilon\, U\|_H + \varepsilon\|\nabla(\mathcal{G}^1_\varepsilon\, U)\|_H \le C$, (ii) implies the existence of a weakly two-scale convergent subsequence. However, for given $U \in \mathbb{X}$ the gradient folding operator guarantees even *strong* two-scale convergence. So, $(\mathcal{G}^1_\varepsilon\, U)_\varepsilon \subset X$ recovers any function $U \in \mathbb{X}$ via strong two-scale convergence and it is shown in [Han11, Prop. 2.11] that

$$\begin{aligned} &\gamma = 0: \quad \text{for all } (u,U) \in X\times\mathbb{X}_{\text{av}}:\; \mathcal{G}^0_\varepsilon(u,U) \xrightarrow{2\text{s}} u \;\&\; \nabla[\mathcal{G}^0_\varepsilon(u,U)] \xrightarrow{2\text{s}} \nabla u + \nabla_y U \text{ in } \mathbb{H},\\ &\gamma = 1: \quad \text{for all } U \in \mathbb{X}:\; \mathcal{G}^1_\varepsilon\, U \xrightarrow{2\text{s}} U \;\&\; \varepsilon\nabla[\mathcal{G}^1_\varepsilon\, U] \xrightarrow{2\text{s}} \nabla_y U \text{ in } \mathbb{H}. \end{aligned}$$

Convenient commutation relations, such as $\mathcal{F}_\varepsilon(\nabla_y U) = \varepsilon\nabla(\mathcal{F}_\varepsilon\, U)$ or $\mathcal{G}^1_\varepsilon(\nabla_y U) = \varepsilon\nabla(\mathcal{G}^1_\varepsilon\, U)$, cannot be expected, since $\mathcal{F}_\varepsilon\, U \notin X$ and $\nabla_y U \notin \mathbb{X}$. Instead, we have that the different folding operators are comparable in the sense that their difference vanishes, see [MRT14, Prop. 3.9],

$$\begin{aligned} &\gamma = 0: \quad \text{for all } (u,U) \in X\times\mathbb{X}_{\text{av}}:\\ &\qquad\quad \|u - \mathcal{G}^0_\varepsilon(u,U)\|_H + \|\nabla u + \mathcal{F}_\varepsilon(\nabla_y U) - \nabla[\mathcal{G}^0_\varepsilon(u,U)]\|_H \to 0,\\ &\gamma = 1: \quad \text{for all } U \in \mathbb{X}:\; \|\,\mathcal{F}_\varepsilon\, U - \mathcal{G}^1_\varepsilon\, U\|_H + \|\,\mathcal{F}_\varepsilon(\nabla_y U) - \varepsilon\nabla(\mathcal{G}^1_\varepsilon\, U)\|_H \to 0. \end{aligned} \tag{2.9}$$

3 Error estimates for reaction-diffusion systems

We consider a system of two coupled reaction-diffusion systems, where the coupling arises via the nonlinear reaction term $(f_1^\varepsilon, f_2^\varepsilon)$, whereas the diffusion has block structure.

$$\begin{pmatrix} u_t^\varepsilon \\ v_t^\varepsilon \end{pmatrix} = \begin{pmatrix} \operatorname{div}(\mathbb{D}_1^\varepsilon\nabla u^\varepsilon) \\ \operatorname{div}(\varepsilon^2\mathbb{D}_2^\varepsilon\nabla v^\varepsilon) \end{pmatrix} + \begin{pmatrix} f_1^\varepsilon(u^\varepsilon, v^\varepsilon) \\ f_2^\varepsilon(u^\varepsilon, v^\varepsilon) \end{pmatrix} \quad \text{in } [0,T]\times\Omega. \tag{3.1.P$^{\text{cp}}_\varepsilon$}$$

We supplement (3.1.P$^{\text{cp}}_\varepsilon$) with homogenous Neumann boundary conditions on $\partial\Omega$ and prescribed initial values $u^\varepsilon(0) = u_0^\varepsilon$, resp. $v^\varepsilon(0) = v_0^\varepsilon$. In [MRT14] (see Theorem 3.1 below) we proved that $(u^\varepsilon, v^\varepsilon)$ converges for $\varepsilon \to 0$ to a limit (u, V) that decomposes into a one-scale function $u(t,x)$ and a two-scale function $V(t,x,y)$, which solve the effective system

$$\begin{pmatrix} u_t \\ V_t \end{pmatrix} = \begin{pmatrix} \mathrm{div}(\mathbb{D}_{\mathrm{eff}}\nabla u) \\ \mathrm{div}_y(\mathbb{D}_2\nabla_y V) \end{pmatrix} + \begin{pmatrix} f_{\mathrm{eff}}(u,V) \\ F_2(u,V) \end{pmatrix} \quad \text{in } [0,T]\times\Omega\times\mathcal{Y}. \tag{3.2.P$_0^{\mathrm{cp}}$}$$

Here, the effective diffusion tensor $\mathbb{D}_{\mathrm{eff}}$ and the effective u-reaction f_{eff} only depend on the macroscopic variable $x\in\Omega$, while the diffusion tensor $\mathbb{D}_2$ and the V-reaction F_2 depend on the two-scale variables $(x,y)\in\Omega\times\mathcal{Y}$, see (3.6.A1)–(3.6.A2) and (3.3)-(3.5), below. The function-to-function map $f_{\mathrm{eff}}:\Omega\times\mathbb{R}^{m_1}\times L^2(\mathcal{Y};\mathbb{R}^{m_2})\to\mathbb{R}^{m_1}$ is defined as

$$f_{\mathrm{eff}}(x,u,Z) := \int_{\mathcal{Y}} F_1(x,y,u,Z(y))\,\mathrm{d}y. \tag{3.3}$$

The effective diffusion tensor $\mathbb{D}_{\mathrm{eff}}:\Omega\to\mathbb{R}^{(m_1\times d)\times(m_1\times d)}$ is given componentwise via the classical homogenization formula, see e.g. [BLP78, All92, LNW02],

$$\mathbb{D}_{\mathrm{eff}}(x)_{ijkl} := \int_{\mathcal{Y}} \mathbb{D}_1(x,y)_{ijkl} + \sum_{r=1}^{d}\mathbb{D}_1(x,y)_{ijkr}\cdot\partial_{y_r}z(y)_{kl}\,\mathrm{d}y, \tag{3.4}$$

for $i,k=1,...,m_1,\ j,l=1,...,d$, where the so-called correctors $z_{ij}\in H^1_{\mathrm{av}}(\mathcal{Y})$ solve the local problem in the weak sense:

$$\mathrm{div}_y\left(\mathbb{D}_1(x,y)_{ijkl} + \sum_{r=1}^{d}\mathbb{D}_1(x,y)_{ijkr}\cdot\partial_{y_r}z(y)_{kl}\right) = 0 \quad \text{in } \mathcal{Y} \text{ for a.a. } x\in\Omega. \tag{3.5}$$

3.1 Assumptions and existing results

We recall (2.8) and we impose the following assumptions on the given data of (3.1.P$_\varepsilon^{\mathrm{cp}}$)–(3.2.P$_0^{\mathrm{cp}}$), for $i=1,2$:

The diffusion tensor
$\mathbb{D}_i:\Omega\times\mathcal{Y}\to\mathbb{R}^{(m_i\times d)\times(m_i\times d)}$ is uniformly bounded and elliptic, i.e. (3.6.A1)
$\exists\,\mu>0:\ \mathbb{D}_i(x,y)\xi:\xi\geq\mu|\xi|^2$ for all $\xi\in\mathbb{R}^{m_i\times d}, (x,y)\in\Omega\times\mathcal{Y}$.

The reaction term
$F_i:\Omega\times\mathcal{Y}\times\mathbb{R}^{m_1+m_2}\to\mathbb{R}^{m_i}$ is uniformly bounded in $\Omega\times\mathcal{Y}$
and differentiable and globally Lipschitz continuous in $\mathbb{R}^{m_1+m_2}$, i.e. (3.6.A2)
$\exists\,L>0:\ |F_i(x,y,A_1,B_1)-F_i(x,y,A_2,B_2)|\leq L(|A_1-A_2|+|B_1-B_2|)$
for all $(A_i,B_i)\in\mathbb{R}^{m_1+m_2}, (x,y)\in\Omega\times\mathcal{Y}$.

The initial values
satisfy u_0, $\mathrm{div}(\mathbb{D}_{\mathrm{eff}}\nabla u_0)\in H$ and V_0, $\mathrm{div}_y(\mathbb{D}_2\nabla_y V_0)\in\mathbb{H}$. (3.6.A3)

The dependence on ε
$\mathbb{D}_i^\varepsilon := \mathcal{F}_\varepsilon\,\mathbb{D}_i$ and $f_i^\varepsilon(\cdot,A,B) := \mathcal{F}_\varepsilon\,F_i(\cdot,\cdot,A,B)$ for all $(A,B)\in\mathbb{R}^{m_1+m_2}$,
u_0^ε, $\mathrm{div}(\mathbb{D}_1^\varepsilon\nabla u_0^\varepsilon)\in H$ with $u_0^\varepsilon\to u_0$ in H, and (3.6.A4)
v_0^ε, $\mathrm{div}(\varepsilon^2\mathbb{D}_2^\varepsilon\nabla v_0^\varepsilon)\in H$ with $v_0^\varepsilon\xrightarrow{2\mathrm{s}}V_0$ in $\mathbb{H}$.

Spatial Lipschitz continuity of the given data
For $(A, B) \in \mathbb{R}^{m_1+m_2}$ fixed, it holds $\nabla_x \mathbb{D}_i, \nabla_x F_i(A, B) \in L^\infty(\Omega \times \mathcal{Y})$ (3.6.A5)
and we write $C_F := \sup_{(x,y)\in\Omega\times\mathcal{Y}}\{|F(x, y, A, B)| + |\nabla_x F(x, y, A, B)|\}$.

Improved spatial regularity of the effective solutions
$\forall\, t \in [0, T]: \ u(t) \in H^2(\Omega)$ and $V(t) \in H^1(\Omega; H^1(\mathcal{Y}))$, $V_t(t) \in H^1(\Omega; L^2(\mathcal{Y}))$. (3.6.A6)

Convergence rates for the initial values
$\exists\, c > 0: \ \| \mathcal{T}_\varepsilon v_0^\varepsilon - V_0\|_{\mathbb{H}} + \|u_0^\varepsilon - u_0\|_H \le \varepsilon^{1/2} c.$ (3.6.A7)

We obtain the two evolution triples $X \subset H \subset X^*$ and $\mathbb{X} \subset \mathbb{H} \subset \mathbb{X}^*$. The assumptions (3.6.A1)–(3.6.A4) guarantee the existence of unique weak solutions $(u^\varepsilon, v^\varepsilon)$ of (3.1.$\mathrm{P}_\varepsilon^{\mathrm{cp}}$) and (u, V) of (3.2.$\mathrm{P}_0^{\mathrm{cp}}$). Further, the differentiability of the reaction terms and the additional regularity of the initial values (3.6.A4) ensure improved time-regularity of the solutions and the following a priori bounds: there exists $C_b > 0$ independent of ε so that, cf. [MRT14, Thm. 2.1 & Prop. 2.2],

$$\begin{aligned} &\|u^\varepsilon\|_{C^1([0,T];H)} + \|\nabla u^\varepsilon\|_{C^0([0,T];H)} + \|v^\varepsilon\|_{C^1([0,T];H)} + \varepsilon\|\nabla v^\varepsilon\|_{C^0([0,T];H)} \le C_b, \\ &\|u\|_{C^1([0,T];H)} + \|\nabla u\|_{C^0([0,T];H)} + \|V\|_{C^1([0,T];\mathbb{H})} + \|\nabla_y V\|_{C^0([0,T];\mathbb{H})} \le C_b. \end{aligned} \tag{3.7}$$

Moreover, we have the following convergence result.

Theorem 3.1 ([MRT14, Thm. 5.1])**.** *Let the assumptions* (3.6.A1)–(3.6.A4) *be satisfied. The sequence of weak solutions* $(u^\varepsilon, v^\varepsilon)$ *of* (3.1.$\mathrm{P}_\varepsilon^{\mathrm{cp}}$) *converges to the weak solution* (u, V) *of* (3.2.$\mathrm{P}_0^{\mathrm{cp}}$) *in the following sense:*

$$\begin{aligned} &\max\nolimits_{0\le t\le T} \| \mathcal{T}_\varepsilon v^\varepsilon(t) - V(t)\|_{\mathbb{H}} \to 0, \ \varepsilon\nabla v^\varepsilon \xrightarrow{2\mathrm{s}} \nabla_y V \ \textit{in } L^2(0, T; \mathbb{H}), \ \textit{and} \\ &v_t^\varepsilon \xrightharpoonup{2\mathrm{w}} V_t \ \textit{in } L^2(0, T; \mathbb{H}), \ \textit{moreover } \forall\, t \in [0, T]: \ \varepsilon\nabla v^\varepsilon(t) \xrightarrow{2\mathrm{s}} \nabla_y V(t) \ \textit{in } \mathbb{H}; \end{aligned} \tag{3.8a}$$

$$\begin{aligned} &u^\varepsilon \rightharpoonup u \ \textit{in } L^2(0, T; X) \ \textit{and } u_t^\varepsilon \rightharpoonup u_t \ \textit{in } H^1(0, T; X^*), \ \textit{moreover} \\ &\exists\, U \in L^2(0, T; \mathbb{X}_{av}) \ \textit{s.t. } \forall\, t \in [0, T]: \ \nabla u^\varepsilon(t) \xrightharpoonup{2\mathrm{w}} \nabla u(t) + \nabla_y U(t) \ \textit{in } \mathbb{H}. \end{aligned} \tag{3.8b}$$

3.2 Main Theorem and outline of the proof

Under the assumption of additional spatial regularity (3.6.A5)–(3.6.A7), we derive the following error estimates for the strong convergences in (3.8). We emphasize that we do not assume improved spatial regularity for the microscopic solutions $(u_\varepsilon, v_\varepsilon)$.

Theorem 3.2. *Let* $(u^\varepsilon, v^\varepsilon)$*, resp.* (u, V)*, denote the solutions of* (3.1.$\mathrm{P}_\varepsilon^{\mathrm{cp}}$)*, resp.* (3.2.$\mathrm{P}_0^{\mathrm{cp}}$)*, and let the assumptions* (2.1.ReSh) *and* (3.6) *hold true. Then there exists a constant* $C > 0$ *independent of* ε *such that*

$$\max_{0\le t\le T} \{\| \mathcal{T}_\varepsilon v^\varepsilon(t) - V(t)\|_{\mathbb{H}} + \|u^\varepsilon(t) - u(t)\|_H\} \le \varepsilon^{1/2} C, \tag{3.9a}$$

$$\| \mathcal{T}_\varepsilon(\varepsilon\nabla v^\varepsilon) - \nabla_y V\|_{L^2(0,T;\mathbb{H})} + \| \mathcal{T}_\varepsilon(\nabla u^\varepsilon) - \{\nabla u + \nabla_y U\}\|_{L^2(0,T;\mathbb{H})} \le \varepsilon^{1/2} C. \tag{3.9b}$$

Thanks to (3.6.A5), we can equally choose $\mathbb{D}^\varepsilon(x) = \mathbb{D}(x, x/\varepsilon)$ or $\mathbb{D}^\varepsilon = \mathcal{F}_\varepsilon \mathbb{D}$ in (3.6.A4) because we can identify $W^{1,\infty}(\Omega)$ with $C^{0,1}(\Omega)$.

For $U \in L^2(0, T; \mathbb{X}_{\mathrm{av}})$ in (3.8b) we have a.e. in $[0, T]$ the representation $U_i(x, y) = \nabla u_i(x) \cdot z_i(y)$, where the correctors $z_i \in H^1_{\mathrm{av}}(\mathcal{Y})$ solve the local problem (3.5). Since

$u \in H^2(\Omega)$ by (3.6.A6), we obtain immediately $U \in H^1(\Omega; H^1_{\mathrm{av}}(\mathcal{Y}))$ and in particular we do *not* assume any improved regularity for the correctors z_i.

Note, (3.9b) implies the strong two-scale convergence $\nabla u^\varepsilon \xrightarrow{2s} \nabla u + \nabla_y U$ in $L^2(0,T;\mathbb{H})$, which also holds in (3.8b) under the assumptions of Theorem 3.1. In the spirit of (3.8a), we can as well prove pointwise in time estimates for the gradients, but then we obtain the lower convergence rate $\varepsilon^{1/4}$, see (3.52). Moreover, we point out that the estimate $\| \mathcal{T}_\varepsilon(\nabla u^\varepsilon) - \{\nabla u + \nabla_y U\}\|_{L^2(0,T;\mathbb{H})} \leq \varepsilon^{1/2} C$ is equivalent to $\|\nabla u^\varepsilon - \{\nabla u + \mathcal{F}_\varepsilon(\nabla_y U)\}\|_{L^2(0,T;H)} \leq \varepsilon^{1/2} C$ as in [FMP12].

Outline of the proof: The essential idea is to derive the following Gronwall estimate

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\left(\| \mathcal{T}_\varepsilon v^\varepsilon - V\|^2_{\mathbb{H}} + \|u^\varepsilon - u\|^2_H\right) \leq C\left(\| \mathcal{T}_\varepsilon v^\varepsilon - V\|^2_{\mathbb{H}} + \|u^\varepsilon - u\|^2_H + \varepsilon\right). \tag{3.10}$$

Then, Gronwall's lemma yields for all $t \in [0,T]$

$$\| \mathcal{T}_\varepsilon v^\varepsilon(t) - V(t)\|^2_{\mathbb{H}} + \|u^\varepsilon(t) - u(t)\|^2_H \leq C\left(\| \mathcal{T}_\varepsilon v_0^\varepsilon - V_0\|^2_{\mathbb{H}} + \|u_0^\varepsilon - u_0\|^2_H + \varepsilon\right)$$

and using assumption (3.6.A7) gives immediately (3.9a). We derive (3.10) in separate steps, namely

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\|u^\varepsilon - u\|^2_H \leq C\left(\| \mathcal{T}_\varepsilon v^\varepsilon - V\|^2_{\mathbb{H}} + \|u^\varepsilon - u\|^2_H + \varepsilon\right) \text{ in Steps 1–2, and} \tag{3.11}$$

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\| \mathcal{T}_\varepsilon v^\varepsilon - V\|^2_{\mathbb{H}} \leq C\left(\| \mathcal{T}_\varepsilon v^\varepsilon - V\|^2_{\mathbb{H}} + \|u^\varepsilon - u\|^2_H + \varepsilon\right) \text{ in Steps 3–4.} \tag{3.12}$$

1. $\frac{\mathrm{d}}{\mathrm{d}t}\|u^\varepsilon - u\|^2_H$*-estimate:* Following the argumentation in [MRT14, Sect. 4.2/Proof of Thm. 4.1 (Step 2–5)], we derive the Gronwall-type estimate

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\|u^\varepsilon - u\|^2_H \leq C\left(\| \mathcal{T}_\varepsilon v^\varepsilon - V\|^2_{\mathbb{H}} + \|u^\varepsilon - u\|^2_H\right) + \Delta^{u^\varepsilon}, \tag{3.13}$$

$$\begin{aligned} \text{where} \quad \Delta^{u^\varepsilon} = {} & \Delta_1^{u^\varepsilon} \text{ (folding mismatch between } \mathcal{F}_\varepsilon \text{ and } \mathcal{G}_\varepsilon^0 \text{ resp. } \mathcal{F}_\varepsilon \text{ and } \mathcal{G}_\varepsilon^1) \\ & + \Delta_2^{u^\varepsilon} \text{ (periodicity defect of } \mathcal{T}_\varepsilon \text{ cf. (1.3.PD))} \\ & + \Delta_3^{u^\varepsilon} \text{ (approximation error } \mathbb{D}_1^\varepsilon \rightsquigarrow \mathbb{D}_{\mathrm{eff}} \text{ resp. } \mathbb{D}_2^\varepsilon \rightsquigarrow \mathbb{D}_2) \\ & + \Delta_4^{u^\varepsilon} \text{ (approximation error } f_1^\varepsilon \rightsquigarrow f_{\mathrm{eff}} \text{ resp. } f_2^\varepsilon \rightsquigarrow F_2) \\ & + \Delta_5^{u^\varepsilon} \text{ (unfolding error } \|V - \mathcal{T}_\varepsilon \mathcal{F}_\varepsilon V\|_{\mathbb{H}} \text{ resp. } \| \mathcal{T}_\varepsilon u - u\|_{\mathbb{H}}). \end{aligned}$$

Above, $u \in H$ is canonically understood as two-scale function $u \in \mathbb{H}$. The last error term $\Delta_5^{u^\varepsilon}$ (resp. $\Delta_5^{v^\varepsilon}$) does not occur in [MRT14], but is addressed as a one-liner here. Since $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^\varepsilon - u\|^2_H = \int_\Omega (u_t^\varepsilon - u_t)\cdot(u^\varepsilon - u)\,\mathrm{d}x$, we ideally subtract the weak formulations of $(3.1.\mathrm{P}_\varepsilon^{\mathrm{cp}})_1$ and $(3.2.\mathrm{P}_0^{\mathrm{cp}})_1$ (resp. $(3.1.\mathrm{P}_\varepsilon^{\mathrm{cp}})_2$ and $(3.2.\mathrm{P}_0^{\mathrm{cp}})_2$), test with the difference $u^\varepsilon - u$ (resp. $\mathcal{T}_\varepsilon v^\varepsilon - V$) and we obtain (3.13). But, due to the two-scale structure of $(3.2.\mathrm{P}_0^{\mathrm{cp}})$, analytical difficulties arise and we cannot proceed straight forward. We modify this basic idea as follows:
In *Step 1a*, we test $(3.1.\mathrm{P}_\varepsilon^{\mathrm{cp}})_1$ (resp. $(3.1.\mathrm{P}_\varepsilon^{\mathrm{cp}})_2$) with $u^\varepsilon - \mathcal{G}_\varepsilon^0(u,U)$ (resp. $v^\varepsilon - \mathcal{G}_\varepsilon^1 V$) and, then, we *reformulate the* ε*-problem* into a two-scale problem using the unfolding operator $\mathcal{T}_\varepsilon$ and the folding operators $\mathcal{F}_\varepsilon$, $\mathcal{G}_\varepsilon^0$ (resp. $\mathcal{G}_\varepsilon^1$). Due to regularity issues between $\mathcal{F}_\varepsilon$ and $\mathcal{G}_\varepsilon^0$, cf. (2.9), we create the error term $\Delta_1^{u^\varepsilon}$ (resp. $\Delta_1^{v^\varepsilon}$).
In *Step 1b*, due to the periodicity defect (1.3.PD), we test $(3.2.\mathrm{P}_0^{\mathrm{cp}})_1$ (resp. $(3.2.\mathrm{P}_0^{\mathrm{cp}})_2$) only with (u,U) (resp. V). Afterwards, we *reformulate the limit problem* and insert the missing terms u^ε and $\mathcal{T}_\varepsilon(\nabla u^\varepsilon)$ (resp. $\mathcal{T}_\varepsilon v^\varepsilon$ and $\mathcal{T}_\varepsilon(\varepsilon\nabla v^\varepsilon)$) at the cost of creating the error term $\Delta_2^{u^\varepsilon}$ (resp. $\Delta_2^{v^\varepsilon}$).
Finally, in *Step 1c*, we add both reformulations and make further rearrangements in terms of the errors $\Delta_3^{u^\varepsilon}$–$\Delta_5^{u^\varepsilon}$ (resp. $\Delta_3^{v^\varepsilon}$–$\Delta_5^{v^\varepsilon}$) so that we end up with (3.13).

2. *Estimation of* Δ^{u^ε} *and* (3.11)*:* We show $|\Delta^{v^\varepsilon}| \leq \varepsilon C$. In more detail, we apply Lemma 3.5 (with $\gamma = 0$) to $\Delta_1^{u^\varepsilon}$ and we use Lemma 3.6 (with $\gamma = 0$) for $\Delta_2^{u^\varepsilon}$. The remaining error terms $\Delta_3^{u^\varepsilon}$–$\Delta_5^{u^\varepsilon}$ resolve easily with Lemma 3.3 and (3.15).
3. $\frac{\mathrm{d}}{\mathrm{d}t}\| \mathcal{T}_\varepsilon v^\varepsilon - V\|_{\mathbb{H}}^2$*-estimate:* Recalling the arguments in [MRT14, Sect. 4.2/Proof of Thm. 4.1 (Step 2–5)] or proceeding analogously to Step 1, we arrive at
$$\frac{\mathrm{d}}{\mathrm{d}t}\| \mathcal{T}_\varepsilon v^\varepsilon - V\|_{\mathbb{H}}^2 \leq C\left(\| \mathcal{T}_\varepsilon v^\varepsilon - V\|_{\mathbb{H}}^2 + \|u^\varepsilon - u\|_H^2\right) + \Delta^{v^\varepsilon}, \tag{3.14}$$
where $\Delta^{v^\varepsilon} = \sum_{i=1}^5 \Delta_i^{v^\varepsilon}$.
4. *Estimation of* Δ^{v^ε} *and* (3.12)*:* As in Step 2, we use Lemma 3.5 resp. Lemma 3.6 (with $\gamma = 1$) for $\Delta_1^{v^\varepsilon}$ resp. $\Delta_2^{v^\varepsilon}$ as well as Lemma 3.3 and (3.15) for $\Delta_3^{v^\varepsilon}$–$\Delta_5^{v^\varepsilon}$.
5. *Derivation of* (3.9b)*:* We derive error estimates for the gradient terms by following the lines of [MRT14, Proof of Thm. 4.1 (Step 7)].

3.3 Preparatory error estimates

The most important observation in deriving the error estimates (3.9a)–(3.9b) is the quantification of the well-known two-scale property, cf. [MiT07, Prop. 2.4(e)], for every $U \in L^2(\Omega \times \mathcal{Y})$ exists a sequence $(u^\varepsilon)_\varepsilon \subset L^2(\Omega)$ such that $u^\varepsilon \xrightarrow{2s} U$ in $L^2(\Omega \times \mathcal{Y})$. For example, such a sequence is given by $u^\varepsilon = \mathcal{F}_\varepsilon U$. More precisely, based in the explicit definitions of $\mathcal{T}_\varepsilon$ and $\mathcal{F}_\varepsilon$, it holds:

Lemma 3.3. *Let $1 \leq p \leq \infty$. For all $U \in W^{1,p}(\Omega; L^p(\mathcal{Y}))$, there exists a constant $C > 0$, only depending on Y, such that*
$$\|U - \mathcal{T}_\varepsilon \mathcal{F}_\varepsilon U\|_{L^p(\Omega\times\mathcal{Y})} \leq \varepsilon C \|U\|_{W^{1,p}(\Omega;L^p(\mathcal{Y}))}.$$

Proof. Thanks to (2.1.ReSh), we have $\Omega = \bigcup_{\lambda_i\in\Lambda_\varepsilon}(\lambda_i + \varepsilon Y)$ and hence the Poincaré-Wirtinger inequality applied on each cell $\lambda_i + \varepsilon Y$ yields for $1 \leq p < \infty$
$$\begin{aligned}\|U - \mathcal{T}_\varepsilon \mathcal{F}_\varepsilon U\|_{L^p(\Omega\times\mathcal{Y})}^p &= \sum_{\lambda_i\in\Lambda_\varepsilon} \int_{\lambda_i+\varepsilon Y}\int_{\mathcal{Y}} \left(U(x,y) - \fint_{\mathcal{N}_\varepsilon(x)+\varepsilon Y} U(\xi,y)\,\mathrm{d}\xi\right)^p \mathrm{d}x\,\mathrm{d}y \\ &\leq \sum_{\lambda_i\in\Lambda_\varepsilon} C\left(\mathrm{diam}(\lambda_i+\varepsilon Y)\right)^p \|\nabla_x U\|_{L^p(\lambda_i+\varepsilon Y)}^p \leq \varepsilon^p C \|U\|_{W^{1,p}(\Omega;L^p(\mathcal{Y}))}^p.\end{aligned}$$
For $p = \infty$, we can directly exploit the Lipschitz continuity. □

Recall (2.8). As a direct consequence of Lemma 3.3, we have, e.g. [Gri04, Eq. (3.4)],
$$\text{for } u \in X: \quad \| \mathcal{T}_\varepsilon u - u\|_{\mathbb{H}} \leq \varepsilon C \|u\|_X. \tag{3.15}$$
For possibly discontinuous functions $U \in H^1(\Omega; L^2(\mathcal{Y}))$, the "naive folding" $x \mapsto U(x, x/\varepsilon)$ is not well-defined. But, in the proof of Lemma 3.5 below, exactly such a "naive folding" is employed. Therefore, we need a suitable regularization U_ε of U so that $\vartheta_\varepsilon(x) = U_\varepsilon(x, x/\varepsilon)$ is well-defined and the differences $\| \mathcal{F}_\varepsilon U - \vartheta_\varepsilon\|_H$ can be estimated by $\varepsilon\|U\|_{H^1(\Omega;L^2(\mathcal{Y}))}$. Therefore, we use in addition to $\mathcal{G}_\varepsilon^0$ resp. $\mathcal{G}_\varepsilon^1$ another regularization of the folding operator $\mathcal{F}_\varepsilon$, namely, the so-called *scale-splitting operator* $\mathcal{Q}_\varepsilon$, cf. [CDG02, CDG08, Gri04].

For $u \in L^1(\Omega)$, the function $\mathcal{Q}_\varepsilon u$ is the $\mathcal{Q}_1$-Lagrangian interpolant of the discrete function $\mathcal{F}_\varepsilon u$. Observe, $\mathcal{Q}_\varepsilon u \in W^{1,\infty}(\Omega)$ and $\mathcal{F}_\varepsilon u \in L^\infty(\Omega)$. (3.16)

Note, for general functions $u \in L^\infty(\Omega)$ and $z \in L^2(\mathcal{Y})$, the composition $x \mapsto u(x)z(x/\varepsilon)$ lies in $L^2(\Omega)$, see e.g. [LNW02, Thm.4].

Lemma 3.4. *For $w \in X$ and $z \in L^2(\mathcal{Y})$, there exists a constant $C > 0$, only depending on the dimension d, such that*

$$\| (\mathcal{F}_\varepsilon w - \mathcal{Q}_\varepsilon w)\, z(\tfrac{\cdot}{\varepsilon})\|_H \le \varepsilon C \|w\|_X \|z\|_{L^2(\mathcal{Y})}.$$

Proof. The proof is adjusted to [Gri04, Prop. 3.2]. Based on the equality

$$\| (\mathcal{F}_\varepsilon w - \mathcal{Q}_\varepsilon w)\, z(\tfrac{\cdot}{\varepsilon})\|_H^2 = \sum_{\lambda_i \in \Lambda_\varepsilon} \int_{\lambda_i + \varepsilon Y} \left| (\mathcal{F}_\varepsilon w(x) - \mathcal{Q}_\varepsilon w(x))\, z(\tfrac{x}{\varepsilon}) \right|^2 \mathrm{d}x \tag{3.17}$$

we consider in the following only one microscopic cell $\lambda_i + \varepsilon Y$, whereby w.l.o.g. $\lambda_i = 0$. We denote with $\{i_n^+\}_{n=1}^d$ the canonical orthonormal basis in $\mathbb{R}^d$ and we set $i_n^- = -i_n^+$. The cube Y has $2d$ sides and $\{i_n^\pm\}_{n=1}^d$ denote their normal vectors. Thus,

$$\text{for } x \in \varepsilon Y: \quad \mathcal{Q}_\varepsilon w(x) = \sum_{i \in \{i_n^\pm\}_{n=1}^d} (\mathcal{F}_\varepsilon w(0) - \mathcal{F}_\varepsilon w(\varepsilon i))\, \tfrac{x}{\varepsilon} + \mathcal{F}_\varepsilon w(0).$$

With $|x| \le \varepsilon \sqrt{d}$, we obtain

$$\begin{aligned} &\int_{\varepsilon Y} \left| (\mathcal{F}_\varepsilon w(0) - \mathcal{Q}_\varepsilon w(x))\, z(\tfrac{x}{\varepsilon}) \right|^2 \mathrm{d}x \\ &\le 2d \sum_{i \in \{i_n^\pm\}_{n=1}^d} \sup_{x \in \varepsilon Y} \left\{ \left| \tfrac{x}{\varepsilon} \right|^2 \right\} |(\mathcal{F}_\varepsilon w(0) - \mathcal{F}_\varepsilon u(\varepsilon i))|^2 \int_{\varepsilon Y} \left| z(\tfrac{x}{\varepsilon}) \right|^2 \mathrm{d}x \\ &\le 2d^2 \sum_{i \in \{i_n^\pm\}_{n=1}^d} |\mathcal{F}_\varepsilon w(0) - \mathcal{F}_\varepsilon w(\varepsilon i)|^2\, \varepsilon^d \|z\|_{L^2(\mathcal{Y})}^2. \end{aligned} \tag{3.18}$$

Using the fundamental relation (for arbitrary $\xi \in \varepsilon Y$)

$$w(\xi) - w(\varepsilon i + \xi) = \pm \int_{\varepsilon i + \xi}^{\xi} \partial_{x_i} w(\tau)\, \mathrm{d}\tau = \pm \varepsilon \int_0^1 \partial_{x_i} w(\xi t + (\varepsilon i + \xi)(1-t))\, \mathrm{d}t$$

we can continue to estimate the difference

$$\begin{aligned} &|\mathcal{F}_\varepsilon w(0) - \mathcal{F}_\varepsilon w(\varepsilon i)|^2 = \left| \fint_{\varepsilon Y} w(\xi) - w(\varepsilon i + \xi)\, \mathrm{d}\xi \right|^2 \\ &\le \varepsilon^2 \fint_{\varepsilon Y} \int_0^1 |\partial_{x_i} w(\xi t + (\varepsilon i + \xi)(1-t))|^2\, \mathrm{d}t\, \mathrm{d}\xi = \frac{\varepsilon^2}{\varepsilon^d} \int_{\varepsilon Y} |\partial_{x_i} w(s)|^2\, \mathrm{d}s, \end{aligned} \tag{3.19}$$

where $|\mathrm{d}s/\mathrm{d}\xi| = 1$. Then, inserting (3.19) into (3.18) and summing up over all $\lambda_i \in \Lambda_\varepsilon$, gives the desired result in (3.17). □

The next Lemma is applied to the estimation of the *folding mismatch* $\Delta_1^{u^\varepsilon}$ resp. $\Delta_1^{v^\varepsilon}$.

Lemma 3.5 (Quantify (2.9))**.** *For all $(u, U) \in H^1(\Omega) \times H^1(\Omega; H^1_{av}(\mathcal{Y}))$, resp. $U \in H^1(\Omega; H^1(\mathcal{Y}))$, there exists a constant $C \ge 0$ such that*

$$\gamma = 0: \qquad \| \mathcal{G}_\varepsilon^0(u, U) - u\|_H + \|\nabla[\mathcal{G}_\varepsilon^0(u, U)] - \{\nabla u + \mathcal{F}_\varepsilon[\nabla_y U]\}\|_H \le \varepsilon C, \tag{3.20a}$$

$$\gamma = 1: \qquad \| \mathcal{G}_\varepsilon^1 U - \mathcal{F}_\varepsilon U\|_H + \|\varepsilon \nabla[\mathcal{G}_\varepsilon^1 U] - \mathcal{F}_\varepsilon[\nabla_y U]\|_H \le \varepsilon C. \tag{3.20b}$$

Proof. The proof is adjusted to the estimate (3.20b) and it utilizes the gradient folding operator $\mathcal{G}_\varepsilon^1$ in the case $\gamma = 1$. In the case $\gamma = 0$, i.e. (3.20a), we resort to $\mathcal{G}_\varepsilon^0$ and we only point out the differences afterwards, cf. [Han11, Prop. 2.1].

The case $\gamma = 1$: By a density argument, we may assume w.l.o.g. that

$$U(x,y) = w(x)z(y) \quad \text{with} \quad w \in X \text{ and } z \in H^1(\mathcal{Y}).$$

Recalling (2.6) and (3.16), we decompose $u^\varepsilon := \mathcal{G}_\varepsilon^1 U \in X$ as follows

$$u^\varepsilon(x) = \vartheta_\varepsilon(x) + g_\varepsilon(x) \quad \text{with} \quad \vartheta_\varepsilon(x) = \mathcal{Q}_\varepsilon\, w(x)z(\tfrac{x}{\varepsilon}). \tag{3.21}$$

By construction, $\vartheta_\varepsilon \in X$ and $g_\varepsilon \in X$ is defined for each $\varepsilon > 0$ as the solution of the elliptic problem

$$\int_\Omega g_\varepsilon \cdot \varphi + \varepsilon\nabla g_\varepsilon : \varepsilon\nabla\varphi \,\mathrm{d}x = \ell_\varepsilon(\varphi) \quad \text{for all } \varphi \in X, \text{ where} \tag{3.22}$$
$$\ell_\varepsilon(\varphi) = \int_\Omega (\mathcal{F}_\varepsilon U - \vartheta_\varepsilon) \cdot \varphi + (\mathcal{F}_\varepsilon(\nabla_y U) - \varepsilon\nabla\vartheta_\varepsilon) : \varepsilon\nabla\varphi \,\mathrm{d}x.$$

The function g_ε can be estimated as follows

$$\begin{aligned}&\tfrac{1}{2}\left(\|g_\varepsilon\|_H + \|\varepsilon\nabla g_\varepsilon\|_H\right)^2 \le \|g_\varepsilon\|_H^2 + \|\varepsilon\nabla g_\varepsilon\|_H^2 = \ell_\varepsilon(g_\varepsilon)\\ &\le \left(\|\mathcal{F}_\varepsilon U - \vartheta_\varepsilon\|_H + \|\mathcal{F}_\varepsilon(\nabla_y U) - \varepsilon\nabla\vartheta_\varepsilon\|_H\right)\left(\|g_\varepsilon\|_H + \|\varepsilon\nabla g_\varepsilon\|_H\right),\end{aligned} \tag{3.23}$$

which yields $\|g_\varepsilon\|_H + \|\varepsilon\nabla g_\varepsilon\|_H \le 2\left(\|\mathcal{F}_\varepsilon U - \vartheta_\varepsilon\|_H + \|\mathcal{F}_\varepsilon(\nabla_y U) - \varepsilon\nabla\vartheta_\varepsilon\|_H\right)$. Now, we estimate the difference between u^ε and $\mathcal{F}_\varepsilon U$ by adding and subtracting ϑ_ε. Recalling $g_\varepsilon = u^\varepsilon - \vartheta_\varepsilon$ and computing $\varepsilon\nabla\vartheta_\varepsilon = \varepsilon\nabla_x\vartheta_\varepsilon + \nabla_y\vartheta_\varepsilon$, we arrive at

$$\begin{aligned}&\|u^\varepsilon - \mathcal{F}_\varepsilon U\|_H + \|\varepsilon\nabla u^\varepsilon - \mathcal{F}_\varepsilon(\nabla_y U)\|_H\\ &\le \left(\|\vartheta_\varepsilon - \mathcal{F}_\varepsilon U\|_H + \|g_\varepsilon\|_H + \|\varepsilon\nabla\vartheta_\varepsilon - \mathcal{F}_\varepsilon(\nabla_y U)\|_H + \|\varepsilon\nabla g_\varepsilon\|_H\right)\\ &\le 2\left(\|\vartheta_\varepsilon - \mathcal{F}_\varepsilon U\|_H + \|\varepsilon\nabla\vartheta_\varepsilon - \mathcal{F}_\varepsilon(\nabla_y U)\|_H\right)\\ &\le 2\left(\|\vartheta_\varepsilon - \mathcal{F}_\varepsilon U\|_H + \|\nabla_y\vartheta_\varepsilon - \mathcal{F}_\varepsilon(\nabla_y U)\|_H + \varepsilon\|\nabla_x\vartheta_\varepsilon\|_H\right).\end{aligned} \tag{3.24}$$

According to [CDG08, Prop. 4.5] it holds $\|\mathcal{Q}_\varepsilon w\|_X \le C\|w\|_X$ and hence $\|\nabla_x\vartheta_\varepsilon\|_H \le C\|\nabla_x U\|_{\mathbb{H}}$. We proceed by estimating the remaining terms in (3.24) with the help of Lemma 3.4

$$\begin{aligned}&\|\vartheta_\varepsilon - \mathcal{F}_\varepsilon U\|_H + \|\nabla_y\vartheta_\varepsilon - \mathcal{F}_\varepsilon(\nabla_y U)\|_H\\ &= \|(\mathcal{Q}_\varepsilon w - \mathcal{F}_\varepsilon w)z(\cdot/\varepsilon)\|_H + \|(\mathcal{Q}_\varepsilon w - \mathcal{F}_\varepsilon w)\nabla_y z(\cdot/\varepsilon)\|_H \le \varepsilon C\|w\|_X\|z\|_{H^1(\mathcal{Y})}\end{aligned}$$

and thus (3.20b) is proved.

The case $\gamma = 0$*:* In (3.21), we set $u^\varepsilon := \mathcal{G}_\varepsilon^0(u,U)$ and decompose $u^\varepsilon = \eta_\varepsilon + g_\varepsilon$, where $\eta_\varepsilon = u + \varepsilon\vartheta_\varepsilon$ and $\vartheta_\varepsilon(x) = \mathcal{Q}_\varepsilon(\nabla u)(x)z(x/\varepsilon)$.

In (3.22), we use $(g_\varepsilon, \varphi)_X = \ell_\varepsilon(\varphi)$ for all $\varphi \in X$ with $\ell_\varepsilon(\varphi) = \int_\Omega (u - \eta_\varepsilon)\cdot\varphi + ([\nabla u + \mathcal{F}_\varepsilon(\nabla_y U)] - \nabla\eta_\varepsilon) : \nabla\varphi \,\mathrm{d}x$.

As in (3.23), we have $\|g_\varepsilon\|_H + \|\nabla g_\varepsilon\|_H \le 2\left(\|u - \eta_\varepsilon\|_H + \|[\nabla u + \mathcal{F}_\varepsilon(\nabla_y U)] - \nabla\eta_\varepsilon\|_H\right)$.

In (3.24), we have $\nabla\eta_\varepsilon = \nabla u + \varepsilon\nabla_x\vartheta_\varepsilon + \nabla_y\vartheta_\varepsilon$ and hence $\|u^\varepsilon - u\|_H + \|\nabla u^\varepsilon - [\nabla u + \mathcal{F}_\varepsilon(\nabla_y U)]\|_H \le 2(\varepsilon\|\vartheta_\varepsilon\|_H + \|\nabla_y\vartheta_\varepsilon - \mathcal{F}_\varepsilon(\nabla_y U)\|_H + \varepsilon\|\nabla_x\vartheta_\varepsilon\|_H)$. Again, the application of Lemma 3.4 and the improved regularity $(u,U) \in H^2(\Omega)\times H^1(\Omega; H^1_{\mathrm{av}}(\mathcal{Y}))$ give (3.20a). □

Furthermore, we use the following result for the *periodicity defect* in $\Delta_2^{u^\varepsilon}$ resp. $\Delta_2^{v^\varepsilon}$.

Lemma 3.6 ([Gri04, Prop. 3.3 & Thm. 3.4])**.** *For every $u \in X$ with $\|u\|_X \le c$ ($\gamma = 0$) resp. $\|u\|_H + \varepsilon\|\nabla u\|_H \le c$ ($\gamma = 1$), there exists a function $\Psi_\varepsilon \in \mathbb{X}$ resp. $\Psi_\varepsilon \in \mathbb{X}_{av}$ and a constant $C > 0$, only depending on Ω, Y, such that*

$$\begin{aligned}
\gamma = 0: \quad & \|\Psi_\varepsilon\|_{\mathbb{X}} \le C\|u\|_X \ \textit{and} \ \|\mathcal{T}_\varepsilon(\nabla u) - \{\nabla u + \nabla_y \Psi_\varepsilon\}\|_{H^{-1}(\Omega;L^2(Y))} \le \varepsilon C\|u\|_X, \\
\gamma = 1: \quad & \|\Psi_\varepsilon\|_{\mathbb{X}} \le C\left(\|u\|_H + \varepsilon\|\nabla u\|_H\right) \ \textit{and} \\
& \|\mathcal{T}_\varepsilon u - \Psi_\varepsilon\|_{H^{-1}(\Omega;H^1(Y))} \le \varepsilon C\left(\|u\|_H + \varepsilon\|\nabla u\|_H\right).
\end{aligned}$$

3.4 Proof of Theorem 3.2

Proof of Theorem 3.2. By the uniform bounds (3.7), all functions are continuous in time and thus we can restore to work with estimates pointwise for all $t \in [0, T]$.

Step 1: $\frac{\mathrm{d}}{\mathrm{d}t}\|u^\varepsilon - u\|_H^2$-estimate. For simplicity in notation we suppress the index $i = 1$.

Step 1a: Reformulation of $(3.1.\mathrm{P}_\varepsilon^{\mathrm{cp}})_1$. We test the ε-problem

$$\int_\Omega u_t^\varepsilon \cdot \varphi \,\mathrm{d}x = \int_\Omega -\mathbb{D}^\varepsilon \nabla u^\varepsilon : \nabla\varphi + f^\varepsilon(u^\varepsilon, v^\varepsilon) \cdot \varphi \,\mathrm{d}x \quad \text{for all } \varphi \in X$$

with $\varphi = u^\varepsilon - \mathcal{G}_\varepsilon^0(u, U)$, where $(u, U) \in X \times \mathbb{X}_{\mathrm{av}}$ solves $(3.2.\mathrm{P}_0^{\mathrm{cp}})$ uniquely for all $t \in [0, T]$, cf. (3.27). Moreover, applying (2.3), inserting the terms $\pm u$ and $\pm[\nabla u + \nabla_y U]$, and rearranging gives

$$\begin{aligned}
\int_\Omega u_t^\varepsilon \cdot (u^\varepsilon - u) \,\mathrm{d}x = & \int_{\mathbb{R}^d \times \mathcal{Y}} -\mathcal{T}_\varepsilon \mathbb{D}^\varepsilon \mathcal{T}_\varepsilon(\nabla u^\varepsilon) : [\mathcal{T}_\varepsilon(\nabla u^\varepsilon) - \{\nabla u - \nabla_y U\}] \,\mathrm{d}x \,\mathrm{d}y \\
& + \int_\Omega f^\varepsilon(u^\varepsilon, v^\varepsilon) \cdot (u^\varepsilon - u) \,\mathrm{d}x + \Delta_1^{u^\varepsilon}, \qquad (3.25)
\end{aligned}$$

$$\begin{aligned}
\text{where } \Delta_1^{u^\varepsilon} := & \int_\Omega (f^\varepsilon(u^\varepsilon, v^\varepsilon) - u_t^\varepsilon) \cdot (u - \mathcal{G}_\varepsilon^0(u, U)) \,\mathrm{d}x \\
& - \int_{\Omega \times \mathcal{Y}} \mathcal{T}_\varepsilon \mathbb{D}^\varepsilon \mathcal{T}_\varepsilon(\nabla u^\varepsilon) : (\{\nabla u + \nabla_y U\} - \mathcal{T}_\varepsilon[\nabla \mathcal{G}_\varepsilon^0(u, U)]) \,\mathrm{d}x \,\mathrm{d}y. \qquad (3.26)
\end{aligned}$$

Step 1b: Reformulation of $(3.2.\mathrm{P}_0^{\mathrm{cp}})_1$. We reformulate $(3.2.\mathrm{P}_0^{\mathrm{cp}})_1$ with (3.4)–(3.5) and $U(x, y) = \nabla u(x) \cdot z(y)$ into

$$\begin{aligned}
\int_\Omega u_t \cdot \psi \,\mathrm{d}x = \int_{\Omega \times \mathcal{Y}} -\mathbb{D}[\nabla u + \nabla_y U] : [\nabla\psi + \nabla_y \Psi] \,\mathrm{d}x \,\mathrm{d}y + \int_\Omega f_{\mathrm{eff}}(u, V) \cdot \psi \,\mathrm{d}x \\
\text{for all } (\psi, \Psi) \in X \times \mathbb{X}_{\mathrm{av}} \quad (3.27)
\end{aligned}$$

and we test (3.27) with the solution, i.e. $(\psi, \Psi) = (u, U)$. Introducing the terms $\pm u^\varepsilon$ and $\pm\mathcal{T}_\varepsilon(\nabla u^\varepsilon)$ and rearranging gives

$$\begin{aligned}
\int_\Omega u_t \cdot (u - u^\varepsilon) \,\mathrm{d}x = & \int_{\Omega \times \mathcal{Y}} -\mathbb{D}[\nabla u + \nabla_y U] : [\{\nabla u + \nabla_y U\} - \mathcal{T}_\varepsilon(\nabla u^\varepsilon)] \,\mathrm{d}x \,\mathrm{d}y \\
& + \int_\Omega f_{\mathrm{eff}}(u, V) \cdot (u - u^\varepsilon) \,\mathrm{d}x + \Delta_2^{u^\varepsilon}, \qquad (3.28)
\end{aligned}$$

$$\text{where } \Delta_2^{u^\varepsilon} := \int_\Omega (f_{\mathrm{eff}}(u, V) - u_t) \cdot u^\varepsilon \,\mathrm{d}x - \int_{\Omega \times \mathcal{Y}} \mathbb{D}[\nabla u + \nabla_y U] : \mathcal{T}_\varepsilon(\nabla u^\varepsilon) \,\mathrm{d}x \,\mathrm{d}y. \qquad (3.29)$$

Step 1c: Derivation of (3.13). Adding (3.25) + (3.28) yields

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^\varepsilon - u\|_H^2 &= \int_\Omega (u^\varepsilon - u)_t \cdot (u^\varepsilon - u)\,\mathrm{d}x \\ &= \int_{\Omega\times\mathcal{Y}} -\mathcal{T}_\varepsilon\,\mathbb{D}^\varepsilon[\mathcal{T}_\varepsilon(\nabla u^\varepsilon) - \{\nabla u + \nabla_y U\}] : [\mathcal{T}_\varepsilon(\nabla u^\varepsilon) - \{\nabla u + \nabla_y U\}]\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \int_\Omega [f^\varepsilon(u^\varepsilon, v^\varepsilon) - f^\varepsilon(u, \mathcal{F}_\varepsilon V)] \cdot (u^\varepsilon - u)\,\mathrm{d}x + \Delta_*^{u^\varepsilon},\end{aligned} \tag{3.30}$$

where $\Delta_*^{u^\varepsilon} = \sum_{i=1}^4 \Delta_i^{u^\varepsilon}$ with

$$\Delta_3^{u^\varepsilon} := \int_{\Omega\times\mathcal{Y}} (\mathbb{D} - \mathcal{T}_\varepsilon\,\mathbb{D}^\varepsilon)[\nabla u + \nabla_y U] : [\mathcal{T}_\varepsilon(\nabla u^\varepsilon) - \{\nabla u + \nabla_y U\}]\,\mathrm{d}x\,\mathrm{d}y, \tag{3.31}$$

$$\Delta_4^{u^\varepsilon} := \int_\Omega [f^\varepsilon(u, \mathcal{F}_\varepsilon V) - f_{\mathrm{eff}}(u, V)] \cdot (u^\varepsilon - u)\,\mathrm{d}x. \tag{3.32}$$

Exploiting the ellipticity of $\mathcal{T}_\varepsilon\,\mathbb{D}^\varepsilon$, the Lipschitz continuity of f^ε in (3.30) as well as Hölder's and Young's inequality give

$$\begin{aligned}\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|u^\varepsilon - u\|_H^2 &\le -\mu\|\,\mathcal{T}_\varepsilon(\nabla u^\varepsilon) - \{\nabla u + \nabla_y U\}\|_{\mathbb{H}}^2 \\ &\qquad + L\,(\|u^\varepsilon - u\|_H + \|v^\varepsilon - \mathcal{F}_\varepsilon V\|_H)\,\|u^\varepsilon - u\|_H + \Delta_*^{u^\varepsilon} \\ &\le 2L\left(\|u^\varepsilon - u\|_H^2 + \|\,\mathcal{T}_\varepsilon v^\varepsilon - V\|_{\mathbb{H}}^2\right) + \Delta^{u^\varepsilon},\end{aligned} \tag{3.33}$$

where $\Delta^{u^\varepsilon} = \Delta_*^{u^\varepsilon} + \Delta_5^{u^\varepsilon}$ with $\Delta_5^{u^\varepsilon} = 2L\|V - \mathcal{T}_\varepsilon\,\mathcal{F}_\varepsilon V\|_{\mathbb{H}}^2$ and hence (3.13).

Step 2: Estimation of Δ^{u^ε} and (3.11). We derive quantitative estimates of the errors $\Delta_1^{u^\varepsilon}, \ldots, \Delta_5^{u^\varepsilon}$. We estimate the error $\Delta_1^{u^\varepsilon}$ (3.25) with Lemma 3.5 and Lemma 3.3, viz.

$$\begin{aligned}|\Delta_1^{u^\varepsilon}| &= \bigg|\int_\Omega (f^\varepsilon(u^\varepsilon, v^\varepsilon) - u_t^\varepsilon) \cdot (u - \mathcal{G}_\varepsilon^0(u, U))\,\mathrm{d}x \\ &\qquad - \int_{\Omega\times\mathcal{Y}} \mathbb{D}^\varepsilon\,\mathcal{T}_\varepsilon(\nabla u^\varepsilon) : [\nabla u + \nabla_y U - \mathcal{T}_\varepsilon[\nabla\,\mathcal{G}_\varepsilon^0(u, U)]]\,\mathrm{d}x\,\mathrm{d}y\bigg| \\ &\le C(C_b)\left(\|u - \mathcal{G}_\varepsilon^0(u, U)\|_H + \|\nabla u + \nabla_y U - \mathcal{T}_\varepsilon[\nabla\,\mathcal{G}_\varepsilon^0(u, U)]\|_{\mathbb{H}}\right) \le \varepsilon C,\end{aligned} \tag{3.34}$$

where $C = C(C_b, , \|U\|_{H^1(\Omega;H^1(\mathcal{Y}))}, \|u\|_{H^2(\Omega)})$ and we used (3.6.A2) and (3.7) to estimate the first integral. In more detail, we split the last term in (3.34) as follows

$$\begin{aligned}&\|\nabla u + \nabla_y U - \mathcal{T}_\varepsilon[\nabla\,\mathcal{G}_\varepsilon^0(u, U)]\|_{\mathbb{H}} \\ &\le \|\,\mathcal{T}_\varepsilon(\nabla u) + \mathcal{T}_\varepsilon\,\mathcal{F}_\varepsilon(\nabla_y U) - \mathcal{T}_\varepsilon[\nabla\,\mathcal{G}_\varepsilon^0(u, U)]\|_{\mathbb{H}} \\ &\quad + \|\nabla u - \mathcal{T}_\varepsilon(\nabla u)\|_{\mathbb{H}} + \|\nabla_y U - \mathcal{T}_\varepsilon\,\mathcal{F}_\varepsilon(\nabla_y U)\|_{\mathbb{H}} \\ &\le \varepsilon C(\|U\|_{H^1(\Omega;H^1(\mathcal{Y}))}, \|u\|_{H^2(\Omega)}),\end{aligned}$$

where we have applied (2.3) and Lemma 3.5 to the term involving $\mathcal{G}_\varepsilon^0(u, U)$ and Lemma 3.3 resp. (3.15) to the remaining two terms.

We treat the second term $\Delta_2^{u^\varepsilon}$ (3.28) with Lemma 3.6. Recalling (3.27), we find a two-scale function Ψ_ε so that $(u^\varepsilon, \Psi_\varepsilon) \in X \times \mathbb{X}_{\mathrm{av}}$ is an admissible test function and hence

$$0 \equiv \int_\Omega (f_{\mathrm{eff}}(u, V) - u_t) \cdot u^\varepsilon\,\mathrm{d}x - \int_{\Omega\times\mathcal{Y}} \mathbb{D}[\nabla u + \nabla_y U] : [\nabla u^\varepsilon + \nabla_y \Psi_\varepsilon]\,\mathrm{d}x\,\mathrm{d}y. \tag{3.35}$$

Subtracting (3.35) from (3.28) yields with Hölder's inequality and (3.6.A5)–(3.6.A6)

$$\begin{aligned}|\Delta_2^{u^\varepsilon}| &= \left|\int_{\Omega\times\mathcal{Y}} \mathbb{D}[\nabla u + \nabla_y U] : [\mathcal{T}_\varepsilon(\nabla u^\varepsilon) - \{\nabla u^\varepsilon + \nabla_y \Psi_\varepsilon\}]\,\mathrm{d}x\,\mathrm{d}y\right| \\ &\le \|\mathbb{D}[\nabla u + \nabla_y U]\|_{H^1(\Omega;L^2(Y))} \|\mathcal{T}_\varepsilon(\nabla u^\varepsilon) - \{\nabla u^\varepsilon + \nabla_y \Psi_\varepsilon\}\|_{H^{-1}(\Omega;L^2(Y))} \\ &\le \varepsilon C(C_b, \|\mathbb{D}\|_{W^{1,\infty}(\Omega;L^\infty(\mathcal{Y}))}, \|U\|_{H^1(\Omega;H^1(\mathcal{Y}))}).\end{aligned} \tag{3.36}$$

The third term $\Delta_3^{u^\varepsilon}$ (3.30) is treated with Hölder's inequality and Lemma 3.3:

$$\begin{aligned}|\Delta_3^{u^\varepsilon}| &= \left|\int_{\Omega\times\mathcal{Y}} (\mathbb{D} - \mathcal{T}_\varepsilon \mathbb{D}^\varepsilon)[\nabla u + \nabla_y U] : [\nabla u + \nabla_y U - \mathcal{T}_\varepsilon(\nabla u^\varepsilon)]\,\mathrm{d}x\,\mathrm{d}y\right| \\ &\le C(C_b)\|(\mathbb{D} - \mathcal{T}_\varepsilon \mathcal{F}_\varepsilon \mathbb{D})\|_{L^\infty(\Omega\times\mathcal{Y})} \le \varepsilon C(C_b, \|\mathbb{D}\|_{W^{1,\infty}(\Omega;L^\infty(\mathcal{Y}))}).\end{aligned} \tag{3.37}$$

The estimation of $\Delta_4^{u^\varepsilon}$ (3.32) is a little more involved. Applying (2.3) only to the first term in (3.32) yields

$$\Delta_4^{u^\varepsilon} = \int_{\Omega\times\mathcal{Y}} \mathcal{T}_\varepsilon f^\varepsilon(\mathcal{T}_\varepsilon u, \mathcal{T}_\varepsilon \mathcal{F}_\varepsilon V) \cdot \mathcal{T}_\varepsilon(u^\varepsilon - u) - F(u, V) \cdot (u^\varepsilon - u)\,\mathrm{d}x\,\mathrm{d}y.$$

Introducing the terms $\pm F(\mathcal{T}_\varepsilon u, \mathcal{T}_\varepsilon \mathcal{F}_\varepsilon V) \cdot \mathcal{T}_\varepsilon(u^\varepsilon - u)$ & $\pm F(u, v) \cdot \mathcal{T}_\varepsilon(u^\varepsilon - u)$, applying Hölder's inequality, and recalling (3.7) & (3.6.A2) gives

$$\begin{aligned}|\Delta_4^{u^\varepsilon}| &\le \|\mathcal{T}_\varepsilon f^\varepsilon(\mathcal{T}_\varepsilon u, \mathcal{T}_\varepsilon \mathcal{F}_\varepsilon V) - F(\mathcal{T}_\varepsilon u, \mathcal{T}_\varepsilon \mathcal{F}_\varepsilon V)\|_{\mathbb{H}} \|\mathcal{T}_\varepsilon(u^\varepsilon - u)\|_{\mathbb{H}} \\ &\quad + \|F(\mathcal{T}_\varepsilon u, \mathcal{T}_\varepsilon \mathcal{F}_\varepsilon V) - F(u, V)\|_{\mathbb{H}} \|\mathcal{T}_\varepsilon(u^\varepsilon - u)\|_{\mathbb{H}} \\ &\quad + \|F(u, V)\|_{\mathbb{H}} \|\mathcal{T}_\varepsilon(u^\varepsilon - u) - (u^\varepsilon - u)\|_{\mathbb{H}} \\ &\le C(L, C_F, C_b)\,(\|\mathcal{T}_\varepsilon \mathcal{F}_\varepsilon F(\mathcal{T}_\varepsilon u, \mathcal{T}_\varepsilon \mathcal{F}_\varepsilon V) - F(\mathcal{T}_\varepsilon u, \mathcal{T}_\varepsilon \mathcal{F}_\varepsilon V)\|_{\mathbb{H}} && (3.38) \\ &\quad + \|\mathcal{T}_\varepsilon u - u\|_{\mathbb{H}} + \|\mathcal{T}_\varepsilon \mathcal{F}_\varepsilon V - V\|_{\mathbb{H}} + \|\mathcal{T}_\varepsilon(u^\varepsilon - u) - (u^\varepsilon - u)\|_{\mathbb{H}}). && (3.39)\end{aligned}$$

We exploit the Lipschitz continuity of F (3.6.A5) in (3.38) and we apply Lemma 3.3 resp. (3.15) in (3.39) so that we arrive at

$$|\Delta_4^{u^\varepsilon}| \le \varepsilon C(L, C_b, C_F, \|V\|_{H^1(\Omega;L^2(\mathcal{Y}))}). \tag{3.40}$$

For the last error term we have immediately

$$|\Delta_5^{u^\varepsilon}| = 2L\|V - \mathcal{T}_\varepsilon \mathcal{F}_\varepsilon V\|_{\mathbb{H}}^2 \le \varepsilon^2 C(L, \|V\|_{H^1(\Omega;L^2(\mathcal{Y}))}). \tag{3.41}$$

Recalling (3.13), we combine the estimates (3.34), (3.36)–(3.37), (3.40)–(3.41), and hence we obtain (3.11).

Step 3: $\frac{\mathrm{d}}{\mathrm{d}t}\|\mathcal{T}_\varepsilon v^\varepsilon - V\|_{\mathbb{H}}^2$-estimate. For brevity we skip the index $i = 2$ in this step and the following. Proceeding as in Step 1, we arrive at (3.14) with

$$\Delta_1^{v^\varepsilon} := \int_\Omega (f^\varepsilon(u^\varepsilon, v^\varepsilon) - v_t^\varepsilon) \cdot (\mathcal{F}_\varepsilon V - \mathcal{G}_\varepsilon^1 V) - \varepsilon \mathbb{D}^\varepsilon \nabla v^\varepsilon : \left[\mathcal{F}_\varepsilon(\nabla_y V) - \varepsilon \nabla(\mathcal{G}_\varepsilon^1 V)\right]\,\mathrm{d}x, \tag{3.42}$$

$$\Delta_2^{v^\varepsilon} := \int_{\Omega\times\mathcal{Y}} [F(u, V) - V_t] \cdot \mathcal{T}_\varepsilon v^\varepsilon - \mathbb{D}\nabla_y V : \nabla_y(\mathcal{T}_\varepsilon v^\varepsilon)\,\mathrm{d}x\,\mathrm{d}y, \tag{3.43}$$

$$\Delta_3^{v^\varepsilon} := \int_{\Omega\times\mathcal{Y}} (\mathbb{D} - \mathcal{T}_\varepsilon \mathbb{D}^\varepsilon)\nabla_y V : \nabla_y(\mathcal{T}_\varepsilon v^\varepsilon - V)\,\mathrm{d}x\,\mathrm{d}y, \tag{3.44}$$

$$\Delta_4^{v^\varepsilon} := \int_{\Omega\times\mathcal{Y}} [\mathcal{T}_\varepsilon f^\varepsilon(\mathcal{T}_\varepsilon u, V) - F(u, V)] \cdot (\mathcal{T}_\varepsilon v^\varepsilon - V)\,\mathrm{d}x\,\mathrm{d}y, \tag{3.45}$$

$$\Delta_5^{v^\varepsilon} := 2L\|\mathcal{T}_\varepsilon u - u\|_{\mathbb{H}}^2 \tag{3.46}$$

Step 4: Estimation of Δ^{v^ε} and (3.12). Applying Lemma 3.5 to the first error term $\Delta_1^{v^\varepsilon}$ (3.42) yields

$$|\Delta_1^{v^\varepsilon}| \leq C(C_b) \left(\| \mathcal{F}_\varepsilon V - \mathcal{G}_\varepsilon^1 V \|_H + \| \mathcal{F}_\varepsilon(\nabla_y V) - \varepsilon \nabla(\mathcal{G}_\varepsilon^1 V) \|_H \right) \leq \varepsilon C, \tag{3.47}$$

where $C = C(C_b, \|V\|_{H^1(\Omega;H^1(\mathcal{Y}))})$.

For the estimation of $\Delta_2^{v^\varepsilon}$ (3.43), let $\Psi_\varepsilon \in \mathbb{X}$ be as in Lemma 3.6. Then, in particular, Ψ_ε is an admissible test function for $(3.2.\mathrm{P}_0^{\mathrm{cp}})_2$ and hence the application of Hölder's inequality and Lemma 3.6 gives

$$\begin{aligned} |\Delta_2^{v^\varepsilon}| &\leq \| |\mathbb{D}\nabla_y V| + |F(u,V)| + |V_t| \|_{H^1(\Omega;L^2(\mathcal{Y}))} \| \mathcal{T}_\varepsilon v^\varepsilon - \Psi_\varepsilon \|_{H^{-1}(\Omega;H^1(\mathcal{Y}))} \\ &\leq \| |\mathbb{D}\nabla_y V| + |F(u,V)| + |V_t| \|_{H^1(\Omega;L^2(\mathcal{Y}))} \varepsilon C(\Omega) \left(\|v^\varepsilon\|_H + \varepsilon \|\nabla v^\varepsilon\|_H \right) \leq \varepsilon C, \end{aligned} \tag{3.48}$$

where $C = C(C_b, C_F, \|\mathbb{D}\|_{W^{1,\infty}(\Omega;L^\infty(\mathcal{Y}))}, \|V\|_{H^1(\Omega;L^2(\mathcal{Y}))}, \|V_t\|_{H^1(\Omega;L^2(\mathcal{Y}))})$.

Recalling $\mathbb{D}^\varepsilon = \mathcal{F}_\varepsilon \mathbb{D}$ and $f^\varepsilon = \mathcal{F}_\varepsilon F$, the error terms $\Delta_3^{v^\varepsilon}$ (3.44)–$\Delta_5^{v^\varepsilon}$ (3.46) are estimated easily by using Lemma 3.3:

$$|\Delta_3^{v^\varepsilon}| \leq 2C_b \| (\mathbb{D} - \mathcal{T}_\varepsilon \mathbb{D}^\varepsilon) \|_{L^\infty(\Omega\times\mathcal{Y})} \leq \varepsilon C(C_b, \Omega, \|\mathbb{D}\|_{W^{1,\infty}(L^\infty(\mathcal{Y}))}), \tag{3.49}$$

$$|\Delta_4^{v^\varepsilon}| \leq 2C_b \| \mathcal{T}_\varepsilon f^\varepsilon(\mathcal{T}_\varepsilon u, V) - F(u,V) \|_{\mathbb{H}} \leq \varepsilon C(C_b, C_F), \tag{3.50}$$

$$|\Delta_5^{v^\varepsilon}| = 2L \| \mathcal{T}_\varepsilon u - u \|_{\mathbb{H}}^2 \leq \varepsilon^2 C(L, \|u\|_X). \tag{3.51}$$

Overall (3.14) and (3.47)–(3.51) give (3.12) and hence we finish the proof of (3.9a).

Step 5: Derivation of (3.9b). Integrating (3.33) over $[0,T]$ and exploiting (3.9a) as well as the Δ^{u^ε}-estimations in Step 2 yields

$$\begin{aligned} &\mu \| \nabla u + \nabla_y U - \mathcal{T}_\varepsilon(\nabla u^\varepsilon) \|_{L^2(0,T;\mathbb{H})}^2 \\ &\leq \int_0^T -\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t} \|u^\varepsilon - u\|_H^2 + 2L \left(\|u^\varepsilon - u\|_H^2 + \| \mathcal{T}_\varepsilon v^\varepsilon - V \|_{\mathbb{H}}^2 \right) + |\Delta^{u^\varepsilon}| \, \mathrm{d}t \leq T\varepsilon C, \end{aligned}$$

which finishes the proof, since the estimation is analogous for the slowly diffusing species. Moreover, we obtain with

$$\left| \tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t} \|u^\varepsilon - u\|_H^2 \right| = \left| \textstyle\int_\Omega (u^\varepsilon - u)_t \cdot (u^\varepsilon - u) \, \mathrm{d}x \right| \leq C_b \|u^\varepsilon - u\|_H \leq \varepsilon^{1/2} C$$

the pointwise in $[0,T]$ estimate of lower convergence rate

$$\mu \| \nabla u + \nabla_y U - \mathcal{T}_\varepsilon(\nabla u^\varepsilon) \|_{\mathbb{H}}^2 \leq \varepsilon^{1/4} C. \tag{3.52}$$

□

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