**Weierstraß-Institut
für Angewandte Analysis und Stochastik**

**Leibniz-Institut im Forschungsverbund Berlin e. V.**

Preprint ISSN 2198-5855

# Well-posedness and optimal control for a Cahn–Hilliard–Oono system with control in the mass term

***Dedicated to our dear friend Maurizio Grasselli on the occasion of his 60th birthday***

Pierluigi Colli[1], Gianni Gilardi[1], Elisabetta Rocca[1], Jürgen Sprekels[2]

submitted: August 16, 2021

[1] Dipartimento di Matematica "F. Casorati"
Università di Pavia
and Research Associate at the IMATI – C.N.R. Pavia
via Ferrata 5
27100 Pavia
Italy
E-Mail: pierluigi.colli@unipv.it
gianni.gilardi@unipv.it
elisabetta.rocca@unipv.it

[2] Department of Mathematics
Humboldt-Universität zu Berlin
Unter den Linden 6
10099 Berlin
and
Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: juergen.sprekels@wias-berlin.de

No. 2867

Berlin 2021

---

2020 *Mathematics Subject Classification.* 35K52, 35D35, 49J20, 49K30, 35Q92, 35Q93.

*Key words and phrases.* Cahn–Hilliard equation, logarithmic potential, regular potential, optimal control problem, strict separation property, necessary optimality conditions.

This research was supported by the Italian Ministry of Education, University and Research (MIUR): Dipartimenti di Eccellenza Program (2018–2022) – Dept. of Mathematics "F. Casorati", University of Pavia. In addition, PC and ER gratefully mention some other support from the GNAMPA (Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni) of INdAM (Istituto Nazionale di Alta Matematica).

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

# Well-posedness and optimal control for a Cahn–Hilliard–Oono system with control in the mass term

***Dedicated to our dear friend Maurizio Grasselli on the occasion of his 60th birthday***

Pierluigi Colli, Gianni Gilardi, Elisabetta Rocca, Jürgen Sprekels

**Abstract**

The paper treats the problem of optimal distributed control of a Cahn–Hilliard–Oono system in $\mathbb{R}^d$, $1 \leq d \leq 3$, with the control located in the mass term and admitting general potentials that include both the case of a regular potential and the case of some singular potential. The first part of the paper is concerned with the dependence of the phase variable on the control variable. For this purpose, suitable regularity and continuous dependence results are shown. In particular, in the case of a logarithmic potential, we need to prove an ad hoc strict separation property, and for this reason we have to restrict ourselves to the case $d = 2$. In the rest of the work, we study the necessary first-order optimality conditions, which are proved under suitable compatibility conditions on the initial datum of the phase variable and the time derivative of the control, at least in case of potentials having unbounded domain.

## 1 Introduction

In this paper, we study the following PDE system, referred to in the literature as the Cahn–Hilliard–Oono (CHO) system (cf. [40–42]), which is of great interest in the study of pattern formations in phase-separating materials:

$$\partial_t \varphi + \epsilon(\varphi - \widehat{c}) - \Delta\mu = 0 \qquad \text{in } Q := \Omega \times (0,T), \tag{1.1}$$

$$\mu = -\Delta\varphi + f'(\varphi) \qquad \text{in } Q, \tag{1.2}$$

$$\partial_\nu \mu = \partial_\nu \varphi = 0 \qquad \text{on } \partial\Omega \times (0,T), \tag{1.3}$$

$$\varphi(0) = \varphi_0 \qquad \text{in } \Omega. \tag{1.4}$$

Here, $\Omega$ is the domain in $\mathbb{R}^d$, $1 \leq d \leq 3$, where the evolution takes place, and $T$ is some final time. In (1.3), $\partial_\nu$ denotes the outward normal derivative on the boundary $\partial\Omega$. In the above equations, the unknowns are $\varphi$, the order parameter representing the relative monomer concentration difference, and $\mu$, the chemical potential, while $\widehat{c}$ is a prescribed mass average, $\epsilon$ is a given phenomenological constant, and $f'$ is the derivative of a double-well potential $f$. Finally, $\varphi_0$ is a given initial datum.

Typical and important examples for $f$ are the so-called *classical regular potential* and the *logarithmic double-well potential*, which are the semiconvex functions given by

$$f_{reg}(r) := \frac{1}{4}\,(r^2-1)^2\,, \quad r \in \mathbb{R}, \tag{1.5}$$

$$f_{log}(r) := (1+r)\ln(1+r) + (1-r)\ln(1-r) - c_1 r^2\,, \quad r \in (-1,1), \tag{1.6}$$

DOI 10.20347/WIAS.PREPRINT.2867 

where $c_1 > 1$ so that $f_{log}$ is nonconvex. Another example is the *double obstacle potential*, where, with $c_2 > 0$,

$$f_{2obs}(r) := -c_2 r^2 \quad \text{if } |r| \leq 1 \quad \text{and} \quad f_{2obs}(r) := +\infty \quad \text{if } |r| > 1. \tag{1.7}$$

In cases like (1.7), one has to split $f$ into a nondifferentiable convex part (the indicator function of $[-1, 1]$ in the present example) and a smooth perturbation. Accordingly, one has to replace the derivative of the convex part by the subdifferential and read (1.2) as a differential inclusion. The coefficients $c_1$ and $c_2$ are related to the critical absolute temperature at which the phase separation takes place. System (1.1)–(1.4) can be seen as a Cahn–Hilliard equation with reaction and turns out to be useful in several applications such as biological models [33], inpainting algorithms [3], and polymers [1]. Indeed, taking $\epsilon = 0$, the CHO equation reduces to the classical Cahn–Hilliard (CH) equation (see [4, 7, 8]). The main feature of (1.1) is that, contrary to the classical CH equation, it does not imply the conservation of total mass. Indeed, integrating it over $\Omega$ and using the boundary and initial conditions (1.3)–(1.4), one finds that

$$\overline{\varphi}(t) = \widehat{c} + \mathrm{e}^{-\epsilon t}(\overline{\varphi}_0 - \widehat{c}) \quad \text{for any } t \geq 0,$$

where $\overline{\varphi}(t) := \frac{1}{|\Omega|}\int_\Omega \varphi(t)\, dx$ denotes the mean value over $\Omega$ at time $t$. We notice that we have mass conservation only in the case $\overline{\varphi}_0 = \widehat{c}$. In all other cases, the time function $\overline{\varphi}(t)$ converges exponentially fast to $\widehat{c}$ as $t$ goes to $+\infty$. Hence, the reaction term present in (1.1) accounts for long-range interactions. Of course, more general terms, in particular nonlinearities, could be considered (cf., for example, [34], where the nonlocal CH equation with nonlinear reaction term was analyzed). Moreover, let us notice that the interest of including reaction terms in CH-type systems is particularly increasing due to the development of diffuse interface models of tumor growth coupling CH with nutrient diffusion and other equations. Indeed, in this class of models the parameter $\varphi$ should be intended as the concentration of the tumor phase, and so, in this case, it is of particular interest not to have mass conservation, but to observe a possible growth or decrease of the tumor mass, due to the proliferation or death of cells. In this respect, we may quote [9–13, 18, 24–27, 36, 43, 44] and the references therein for results on well-posedness, long-time behavior of solutions, asymptotic analyses, and optimal control, related to tumor growth models. More recently, always in the framework of tumor growth dynamics, in [32] a generalized CHO equation has been coupled with an Hele–Shaw equation, and global well-posedness and regularity results have been proved in the 2D case.

The solvability and the existence of global and exponential attractors for the CHO equation with regular potential (1.5) have been studied in [35], while in [31] the authors investigated the case of the singular logarithmic type potential (1.6), by establishing some regularization properties of the unique solution in finite time. Both in 2D and 3D they proved the existence of a global attractor; moreover, in the 2D case, taking advantage of their proof of a strict separation property, the authors of [31] also showed the existence of an exponential attractor and the convergence to a single equilibrium. Finally, the CHO equation has been coupled with Navier–Stokes systems both in case of regular potentials [5] and of singular potentials [37].

In the present work, we study a distributed control problem for the system (1.1)–(1.4), where the term $\epsilon(\varphi - \widehat{c})$ in (1.1) is replaced by $\varphi - u$, thus taking $\epsilon = 1$ and substituting $\widehat{c}$ by a control $u$ which is allowed to be a function of space and time. Having in mind the application to tumor growth models, the control $u$ could represent a source of therapy, like, for example, a concentration of cytotoxic drugs introduced in the cells in order to minimize the volume of the tumor (which is represented by the integral of $\varphi$ over the domain $\Omega$) or to reach some target tumor distribution at the final time $T$ of the therapy cycle. About optimal control problems for CH systems, let us mention some related work. A very general approach for distributed control problems for possibly fractional equations of CH-type is carried out in the papers [20–22], with an extension of the analysis to double obstacle potentials like $f_{2obs}$ in (1.7) via deep quench approximation. The coupling of CH equations in the bulk with dynamic boundary conditions has been investigated in [14, 15, 23], and the presence of a convective term with the velocity vector taken as control has been dealt with in [16, 17, 19, 30] (see also the references in the quoted contributions).

We now go back to the goal of this paper and observe that, since we are not interested in the longtime behavior of the solution, the constants $\widehat{c}$ and $\epsilon$ here actually do not play any significant role, whence we can assume that $\epsilon = 1$ and absorb $\widehat{c}$ in $u$. Therefore, the state system under investigation is the following:

$$\partial_t\varphi + \varphi - \Delta\mu = u \qquad \text{in } Q, \tag{1.8}$$

$$\mu = -\Delta\varphi + f'(\varphi) \qquad \text{in } Q, \tag{1.9}$$

with the same boundary and initial conditions as before. For our purpose, it is crucial to study the dependence of the solution on the control variable $u$. Hence, a major part of the paper is devoted to the well-posedness of the system and the continuous dependence just mentioned. It must be pointed out that, at least in the case of potentials having a bounded domain, the control $u$ and the initial datum $\varphi_0$ must satisfy proper necessary compatibility conditions in order to guarantee the existence of a solution. For this reason, even well-posedness cannot be deduced from [31] and the existing literature. Then, we study the control problem. Precisely, we want to minimize the tracking-type cost functional

$$\mathcal{J}((\varphi,\mu),u) := \frac{\alpha_1}{2}\int_Q |\varphi-\varphi_Q|^2 + \frac{\alpha_2}{2}\int_\Omega |\varphi(T)-\varphi_\Omega|^2 + \frac{\alpha_3}{2}\int_Q |\mu-\mu_Q|^2 + \frac{\alpha_4}{2}\int_Q |u|^2 \tag{1.10}$$

over the set of admissible controls

$$\mathcal{U}_{ad} := \{u \in H^1(0,T;H)\cap L^\infty(Q) : \ \|u\|_{L^\infty(Q)} \le M, \ \|\partial_t u\|_{L^2(Q)} \le M'\}, \tag{1.11}$$

subject to the system given by (1.8)–(1.9) and (1.3)–(1.4). In (1.10), $\varphi_Q$ and $\mu_Q$ are given functions on $Q$, $\varphi_\Omega$ is a given function on $\Omega$, and $\alpha_i$, $i = 1, \ldots, 4$, are nonnegative constants (not all zero). In (1.11), $M > 0$ and $M' > 0$ are prescribed constants as well. We show the existence of an optimal control. Then the main point is to prove the Fréchet differentiability of the control-to-state operator between suitable functional spaces, which allows us to establish first-order necessary optimality conditions in terms of the solution to the linearized system; the latter is then eliminated by means of the solution to a proper adjoint problem.

The paper is organized as follows. In the next section, we list our assumptions and notations and state our results. The proofs of those regarding the well-posedness of the problem, the continuous dependence of its solution on the control variable, and the regularity, are given in Sections 3–5. The argument for the existence and the regularity of the solution is based on the study of proper approximating problems performed in the first part of Section 4. Finally, the last Section 6 is devoted to the analysis of the control problem and the corresponding first-order necessary optimality conditions.

## 2 Statement of the problem and results

In this section, we state precise assumptions and notations and present our results. First of all, the subset $\Omega \subset \mathbb{R}^d$ with $1 \le d \le 3$ is assumed to be bounded, connected and smooth. As in the

Introduction, $\partial_\nu$ stands for the normal derivative on $\Gamma := \partial\Omega$. Moreover, we set for brevity

$$Q_t := \Omega \times (0,t) \quad \text{and} \quad Q^t := \Omega \times (t,T) \quad \text{for } 0 < t < T, \quad \text{and} \quad Q := \Omega \times (0,T). \tag{2.1}$$

If $X$ is a Banach space, $\|\cdot\|_X$ denotes both its norm and the norm of $X^d$. The only exception from this convention on the norms is given by the spaces $L^p$ $(1 \le p \le \infty)$ constructed on $(0,T)$, $\Omega$ and $Q$, whose norms are often denoted by $\|\cdot\|_p$, and by the space $H$ defined below, whose norm is simply denoted by $\|\cdot\|$. We put

$$H := L^2(\Omega)\,, \quad V := H^1(\Omega) \quad \text{and} \quad W := \{v \in H^2(\Omega) : \partial_\nu v = 0\}. \tag{2.2}$$

Moreover, $V^*$ is the dual space of $V$ and $\langle\,\cdot\,,\,\cdot\,\rangle$ is the dual pairing between $V^*$ and $V$. In the following, we work in the framework of the Hilbert triplet $(V, H, V^*)$. Thus, by also using the symbol $(\,\cdot\,,\,\cdot\,)$ for the standard inner product of $H$, we have $\langle g, v\rangle = (g, v)$ for every $g \in H$ and $v \in V$. We also use the symbol $(\,\cdot\,,\,\cdot\,)$ for the standard inner product in any of the product spaces $H^N$ for $N \in \mathbb{N}$.

Next, we introduce the generalized mean value. We write $|\Omega|$ for the measure of $\Omega$ and set

$$\overline{v}_* := |\Omega|^{-1}\langle v_*, 1\rangle \quad \text{for } v_* \in V^* \tag{2.3}$$

where $1$ denotes the constant function $x \mapsto 1$, $x \in \Omega$. More generally, to simplify the notation, we use the same symbol for the real number $a$ and the associated constant functions on $\Omega$ and $Q$. It is clear that $\overline{v}_*$ is the usual mean value of $v_*$ if $v_* \in H$.

Now, we list our assumptions on the structure of our system and the data. For the potential $f$, we assume that

$$f : \mathbb{R} \to (-\infty, +\infty] \quad \text{can be split as} \quad f = \widehat{\beta} + \widehat{\pi}, \quad \text{where} \tag{2.4}$$

$$\widehat{\beta} : \mathbb{R} \to [0, +\infty] \quad \text{is convex, proper, and l.s.c. with } \widehat{\beta}(0) = 0, \tag{2.5}$$

$$\widehat{\pi} : \mathbb{R} \to \mathbb{R} \quad \text{is of class } C^1\text{, and its derivative is Lipschitz continuous.} \tag{2.6}$$

We set for convenience

$$\beta := \partial\widehat{\beta} \quad \text{and} \quad \pi := \widehat{\pi}\,', \tag{2.7}$$

and notice that $\beta$ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ satisfying $0 \in \beta(0)$. We use the symbols $D(\beta)$ and $\beta^\circ(r)$ for the domain of $\beta$ and the element of $\beta(r)$ (with $r \in D(\beta)$) having minimum modulus. We extend the notations $\widehat{\beta}$, $\beta$, $D(\beta)$ and $\beta^\circ$ to the functionals and the operators induced on $L^2$ spaces.

Even though the control variable $u$ is fixed when dealing with well-posedness, we prepare the possibility of letting $u$ vary by introducing an upper bound $M$ in the data. We assume that

$$u \in L^\infty(Q), \quad M \in [0, +\infty) \quad \text{and} \quad \|u\|_\infty \le M, \tag{2.8}$$

$$\varphi_0 \in W \quad \text{with } \inf \varphi_0 \text{ and } \sup \varphi_0 \text{ belonging to the interior of } D(\beta), \tag{2.9}$$

$$\overline{\varphi}_0 \pm M \quad \text{belong to the interior of } D(\beta). \tag{2.10}$$

Assumption (2.9) is rather strong; its first application and a comment on the last assumption (2.10) are given in the forthcoming Remarks 4.2 and 2.2, respectively.

Let us come to the definition of our notion of solution, which we state in a weak form. Namely, a solution is a triplet $(\varphi, \mu, \xi)$ satisfying the regularity requirements

$$\varphi \in H^1(0,T;V^*) \cap L^\infty(0,T;V), \tag{2.11}$$

$$\mu \in L^2(0,T;V), \tag{2.12}$$

$$\xi \in L^2(0,T;H), \quad \text{and} \quad \xi \in \beta(\varphi) \quad \text{a.e. in } Q, \tag{2.13}$$

and the following variational equations and initial condition:

$$\langle \partial_t \varphi(t), v \rangle + \int_\Omega \varphi(t) v + \int_\Omega \nabla\mu(t) \cdot \nabla v = \int_\Omega u(t) v$$
$$\text{for a.a. } t \in (0,T) \text{ and every } v \in V, \tag{2.14}$$

$$\int_\Omega \mu(t) v = \int_\Omega \nabla\varphi(t) \cdot \nabla v + \int_\Omega \xi(t) v + \int_\Omega \pi(\varphi(t)) v$$
$$\text{for a.a. } t \in (0,T) \text{ and every } v \in V, \tag{2.15}$$

$$\varphi(0) = \varphi_0 \quad \text{a.e. in } \Omega\,. \tag{2.16}$$

**Remark 2.1.** We observe that the above variational equations are equivalent to their time-integrated versions with time dependent test functions, i.e.,

$$\int_0^T \langle \partial_t \varphi(t), v(t) \rangle \, dt + \int_Q \varphi\, v + \int_Q \nabla\varphi \cdot \nabla v = \int_Q uv, \tag{2.17}$$

$$\int_Q \mu v = \int_Q \nabla\varphi \cdot \nabla v + \int_Q \xi v + \int_\Omega \pi(\varphi) v, \tag{2.18}$$

both for every $v \in L^2(0,T;V)$. We also point out that (2.15) can be written as a boundary value problem. Namely, since $\varphi \in L^2(0,T;V)$ and $\mu - \xi - \pi(\varphi) \in L^2(0,T;H)$, it is equivalent to

$$\varphi \in L^2(0,T;W), \quad \text{and} \quad \mu = -\Delta\varphi + \xi + \pi(\varphi) \quad \text{a.e. in } Q, \tag{2.19}$$

by elliptic regularity. On the contrary, the analogue for (2.14) only if $\partial_t \varphi$ were more regular.

**Remark 2.2.** Let us comment on assumption (2.10). By just taking $v = 1/|\Omega|$ in (2.14) and accounting for (2.16), we have that

$$\frac{d}{dt}\,\overline{\varphi}(t) + \overline{\varphi}(t) = \overline{u}(t) \quad \text{for a.a. } t \in (0,T), \quad \text{and} \quad \overline{\varphi}(0) = \overline{\varphi}_0, \tag{2.20}$$

which yield

$$\overline{\varphi}(t) = \overline{\varphi}_0 + \int_0^t e^{-(t-s)} \overline{u}(s)\, ds \quad \text{for every } t \in [0,T]. \tag{2.21}$$

Since this implies that

$$|\overline{\varphi}(t) - \overline{\varphi}_0| \le \|\overline{u}\|_\infty \le \|u\|_\infty \le M,$$

assumption (2.10) ensures that $\overline{\varphi}(t)$ belongs to $D(\beta)$ for every $t \in [0,T]$. We observe that the right-hand side of (2.21) just depends on $\varphi_0$ and $u$, and a necessary condition for the existence of a solution is that it remains in $D(\beta)$ for all times. In particular, if $u = 2$ and $\varphi_0 = 0$, it is given by $2(1 - e^{-t})$ and becomes larger than 1 if $t > \ln 2$, so that no solution can exist on an interval $[0,T]$ with $T > \ln 2$ if $f$ is the logarithmic potential (1.6).

We have the following well-posedness result.

**Theorem 2.3.** *Suppose that the conditions* (2.4)–(2.6) *on the structure of the system and* (2.8)–(2.10) *on the data are fulfilled. Then there exists at least one triplet* $(\varphi, \mu, \xi)$*, with the regularity conditions* (2.11)–(2.13)*, that solves problem* (2.14)–(2.16) *and satisfies the estimate*

$$\|\varphi\|_{H^1(0,T;V^*) \cap L^\infty(0,T;V) \cap L^2(0,T;W)} + \|\mu\|_{L^2(0,T;V)} + \|\xi\|_{L^2(0,T;H)} \le K_1\,, \tag{2.22}$$

*with a constant $K_1 > 0$ that depends only on the structure of the system, $\Omega$, $T$, the initial datum, and $M$.*

*Furthermore, if $u_i \in L^\infty(Q)$, $i = 1, 2$, are given and $(\varphi_i, \mu_i, \xi_i)$ are any corresponding solutions, then the estimate*

$$\|\varphi_1 - \varphi_2\|_{C^0([0,T];V^*)\cap L^2(0,T;V)} \le C_1 \left(\|u_1 - u_2\|_{L^1(0,T;V^*)} + \|\overline{u}_1 - \overline{u}_2\|^{1/2}_{L^1(0,T)}\right) \tag{2.23}$$

*holds true with a constant $C_1 > 0$ that depends only on the structure of the system, $\Omega$, $T$, and an upper bound for $\|\xi_i\|_{L^1(Q)}$, $i = 1, 2$. In particular, the solution component $\varphi$ is uniquely determined. If, in addition, $\beta$ is single-valued, then also $\mu$ and $\xi$ are uniquely determined.*

An improvement of the regularity of the solution can be achieved under stronger assumptions on the data. Namely, we also assume:

the interior of $D(\beta)$ contains $0$
and the restriction of $\beta$ to it is a $C^1$ function, (2.24)

$$u \in H^1(0,T;H), \quad M' \in [0,+\infty) \quad \text{and} \quad \|\partial_t u\|_{L^2(0,T;H)} \le M', \tag{2.25}$$

$$\varphi_0 \in H^3(\Omega). \tag{2.26}$$

Notice that (2.24) does not imply that $\beta$ is single-valued (cf. (1.7)), so that uniqueness for the solution is not guaranteed. However, we can prove the existence of a smoother solution. Of course, if in addition $\beta$ is single-valued, then this regularity is ensured for the unique solution.

**Theorem 2.4.** *In addition to the assumptions of Theorem 2.3, assume that* (2.24)–(2.26) *are fulfilled. Then, there exists at least one solution $(\varphi, \mu, \xi)$ to* (2.14)–(2.16) *that also satisfies both the regularity requirements*

$$\begin{gathered} \varphi \in H^1(0,T;V) \cap L^\infty(0,T;W) \cap C^0(\overline{Q}), \\ \mu \in L^\infty(0,T;V) \quad \text{and} \quad \xi \in L^\infty(0,T;H), \end{gathered} \tag{2.27}$$

*and the estimate*

$$\|\varphi\|_{H^1(0,T;V)\cap L^\infty(0,T;W)\cap C^0(\overline{Q})} + \|\mu\|_{L^\infty(0,T;V)} + \|\xi\|_{L^\infty(0,T;H)} \le K_2, \tag{2.28}$$

*with a constant $K_2$ that depends only on the structure of the system, $\Omega$, $T$, the initial datum, and the upper bounds $M$ and $M'$ appearing in* (2.8) *and* (2.25)*.*

In order to approach the control problem with a possibly singular potential, it is important to ensure that the solution component $\varphi$ takes its values far away from the singularities of $\beta$. If $\beta$ is smooth in the interior of its domain, then it is sufficient that all values of $\varphi$ belong to a compact interval contained in the interior of $D(\beta)$. In the case of an everywhere defined smooth potential like (1.5), the boundedness of $\varphi$ given by Theorem 2.4 is sufficient. On the contrary, if $D(\beta)$ is a bounded open interval as in the case of the logarithmic potential (1.6), then the requested property of $\varphi$ is equivalent to the boundedness of the solution component $\xi$. This is related to the boundedness of $\mu$.

**Proposition 2.5.** *Under the assumptions of Theorem 2.3, let $(\varphi, \mu, \xi)$ be a solution to problem* (2.14)–(2.16)*. Then boundedness of $\varphi$ and $\mu$ implies boundedness of $\xi$ and the estimate*

$$\|\xi\|_\infty \le \|\varphi + \mu - \pi(\varphi)\|_\infty. \tag{2.29}$$

*In particular, if $D(\beta)$ is a bounded open interval, then all of the values of $\varphi$ are attained in a compact interval contained in $D(\beta)$ that depends only on the shape of $\beta$ and an upper bound for the norm of $\xi$ in $L^\infty(Q)$.*

Boundedness of $\mu$ is a consequence of (2.27) if $d = 1$, since $V \subset L^\infty(\Omega)$ in this case. If $d \in \{2, 3\}$, the problem is open, in general. A natural way to ensure that $\mu$ is bounded is to prove that $\partial_t\varphi$ belongs to $L^\infty(0, T; H)$. Indeed, then (2.14) and elliptic regularity would yield that $\mu$ belongs to $L^\infty(0, T; W)$, and thus to $L^\infty(Q)$ since $W \subset L^\infty(\Omega)$. We cannot deal with the general case. However, we can prove this result in the two-dimensional case if $f$ is the logarithmic potential.

**Proposition 2.6.** *Assume that $d = 2$, that $f$ is the logarithmic potential* (1.6)*, and that the data satisfy* (2.8)–(2.10)*,* (2.24)–(2.26)*, as well as*

$$\varphi_0 \in H^4(\Omega) \quad \textit{and} \quad \partial_\nu\Delta\varphi_0 = 0 \quad \textit{on } \Gamma\,. \tag{2.30}$$

*Then the solution $(\varphi, \mu, \xi)$ to problem* (2.14)–(2.16) *also satisfies*

$$\partial_t\varphi \in L^\infty(0, T; H) \cap L^2(0, T; W)\,, \quad \mu \in L^\infty(Q)\,, \quad \xi \in L^\infty(Q)\,, \tag{2.31}$$

*as well as*

$$\|\partial_t\varphi\|_{L^\infty(0,T;H)\cap L^2(0,T;W)} + \|\mu\|_\infty + \|\xi\|_\infty \le K_3\,, \tag{2.32}$$

*with a constant $K_3$ that depends only on $\Omega$, $T$, the initial datum, and the upper bounds $M$ and $M'$ appearing in* (2.8) *and* (2.25)*.*

**Remark 2.7.** We point out that under the assumptions of Proposition 2.6 the second part of Proposition 2.5 provides a compact interval $[a, b] \subset (-1, 1)$, which contains all of the values of $\varphi$ and depends only on $\Omega$, $T$ and the constants $M$ and $M'$.

Whenever the potential is smooth in its domain and all of the values of $\varphi$ belong to a compact interval $[a, b]$ contained in the interior of $D(\beta)$, then the potential $f$ can be replaced a posteriori by a potential having a Lipschitz continuous derivative. In this situation, we can prove a second continuous dependence result.

**Theorem 2.8.** *Assume that the potential $f$ has a Lipschitz continuous derivative. If $u_i \in L^\infty(Q)$, $i = 1, 2$, are two choices of $u$ and $(\varphi_i, \mu_i, \xi_i)$ are the corresponding solutions, then the estimate*

$$\|\varphi_1 - \varphi_2\|_{C^0([0,T];H)\cap L^2(0,T;W)} + \|\mu_1 - \mu_2\|_{L^2(0,T;H)} \le C_2\,\|u_1 - u_2\|_{L^2(0,T;H)} \tag{2.33}$$

*holds true with a constant $C_2$ that depends only on $\Omega$, $T$, and the Lipschitz constant of $f'$.*

The above results prepare the study of the control problem, which consists in minimizing the cost functional (1.10) over the set (1.11) of the admissible controls subject to the state system, as said in the Introduction. We recall the definitions and make precise assumptions. The cost functional is given by

$$\mathcal{J}((\varphi, \mu), u) := \frac{\alpha_1}{2}\int_Q |\varphi - \varphi_Q|^2 + \frac{\alpha_2}{2}\int_\Omega |\varphi(T) - \varphi_\Omega|^2 + \frac{\alpha_3}{2}\int_Q |\mu - \mu_Q|^2 + \frac{\alpha_4}{2}\int_Q |u|^2\,, \tag{2.34}$$

where

$$\begin{aligned} &\varphi_Q,\, \mu_Q \in L^2(Q)\,, \quad \varphi_\Omega \in L^2(\Omega)\,, \\ &\quad \text{and} \quad \alpha_i \in [0, +\infty) \quad \text{are not all zero,} \quad \text{for } i = 1, \dots, 4\,. \end{aligned} \tag{2.35}$$

The set of the admissible controls is given by

$$\mathcal{U}_{ad} := \{u \in H^1(0, T; H) \cap L^\infty(Q) :\ \|u\|_{L^\infty(Q)} \le M,\ \|\partial_t u\|_{L^2(Q)} \le M'\}\,, \tag{2.36}$$

where

$$M,\ M' \in [0, +\infty) \text{ and (2.10) is satisfied.} \tag{2.37}$$

The control problem is the following:

*Minimize the cost functional* (2.34) *over the set* (2.36) *of admissible controls subject to the state system* (2.14)–(2.16). (2.38)

At this point, it is worth noticing that the cost functional in the form (2.34) is well defined only if both solution components $\varphi, \mu$ are uniquely determined. Under the assumptions of Theorem 2.3, this can only be guaranteed if $\beta$ is single-valued; otherwise one has to simplify the cost functional by postulating that $\alpha_3 = 0$. In the situation of Theorem 2.8, however, this problem does not arise. We have the following existence result.

**Theorem 2.9.** *Suppose that the conditions* (2.4)–(2.6), (2.9)*, and* (2.34)–(2.37) *are fulfilled, and assume that either* $\alpha_3 = 0$ *or* $\beta$ *is single-valued. Then the control problem* (2.38) *has at least one solution* $u^*$.

The next effort is to find first-order necessary conditions for optimality. To this end, we have to enlarge $\mathcal{U}_{ad}$ in a proper topology and introduce the control-to-state operator $\mathcal{S}$. We fix $R > 0$ small enough in order that

$$\overline{\varphi}_0 \pm (M + R), \quad \text{belong to the interior of } D(\beta), \tag{2.39}$$

and we set

$$\mathcal{U}_R := \{u \in H^1(0, T; H) \cap L^\infty(Q) :\ \|u\|_{L^\infty(Q)} < M + R,\ \|\partial_t u\|_{L^2(Q)} < M' + R\}. \tag{2.40}$$

This is an open neighborhood of $\mathcal{U}_{ad}$ in the topology of the first of the Banach spaces we introduce for later use:

$$\begin{aligned} &\mathcal{X} := H^1(0, T; H) \cap L^\infty(Q) \\ &\text{and} \quad \mathcal{Y} := (C^0([0, T]; H) \cap L^2(0, T; W)) \times L^2(0, T; H). \end{aligned} \tag{2.41}$$

In order to continue, we need the functional $f$ to be more regular on the range of the first component $\varphi$ of the state corresponding to any element $u \in \mathcal{U}_R$. We first observe that thanks to (2.39) the results stated above hold true with $M$ and $M'$ replaced by $M + R$ and $M' + R$, respectively; in addition, they provide estimates that are uniform with respect to $u \in \mathcal{U}_R$. Next, we notice that in some situations it is ensured that there exists a compact interval $[a, b]$ contained in the interior of the domain of the double well potential $f$ such that, for every $u \in \mathcal{U}_R$, the first component $\varphi$ of the corresponding state attains its values in $[a, b]$. Namely, as we have seen, this is the case either if, under the assumptions of Theorem 2.4, $f$ is an everywhere defined smooth potential in any dimension $d \in \{1, 2, 3\}$ or if the assumptions of Proposition 2.6 are fulfilled (in particular, $d = 2$ and $f$ is the logarithmic potential (1.6)).

Therefore, we take the following starting point: there exist a compact interval $[a, b]$ and a constant $K > 0$ such that:

$$f \text{ is a function of class } C^3 \text{ in a neighborhood of } [a, b]; \tag{2.42}$$

$$\begin{aligned} &\text{it holds } a \le \varphi \le b \quad \text{a.e. in } Q \text{ and } \max_{0 \le i \le 3} \|f^{(i)}(\varphi)\|_\infty \le K, \text{ whenever } (\varphi, \mu, \xi), \\ &\text{with } \xi = \beta(\varphi), \text{ is a solution to (2.14)–(2.16) for some } u \in \mathcal{U}_R. \end{aligned} \tag{2.43}$$

Notice that under the conditions (2.42)–(2.43) also the solution component $\mu$ is uniquely determined so that the control-to-state operator

$$\mathcal{S} : \mathcal{U}_R \to \mathcal{Y}, \quad u \mapsto \mathcal{S}(u) = (\varphi, \mu), \quad \text{where } (\varphi, \mu, \beta(\varphi)) \text{ is the solution to the state system (2.14)–(2.16) corresponding to } u, \tag{2.44}$$

is well defined on $\mathcal{U}_R$.

Based on (2.42)–(2.43), we prove in Theorem 6.3 that $\mathcal{S}$ is Fréchet differentiable at every point $u^* \in \mathcal{U}_R$, where the Fréchet derivative is related to the solution to a linear system obtained from (2.14)–(2.16) by linearization. A standard argument then yields a necessary condition for optimal controls $u^* \in \mathcal{U}_{ad}$ that involves the solution to the linearized system (Proposition 6.4). As usual, we eliminate the solution to the linearized system by means of the adjoint state variables $(p, q)$, which satisfy

$$p \in H^1(0,T;V^*) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \quad \text{and} \quad q \in L^2(0,T;V) \tag{2.45}$$

and solve the adjoint state system

$$-\langle \partial_t p(t), v\rangle + \int_\Omega p(t)v + \int_\Omega \nabla q(t)\cdot\nabla v + \int_\Omega f''(\varphi^*(t))q(t)v$$
$$= \alpha_1 \int_\Omega (\varphi^* - \varphi_Q)(t)v \quad \text{for a.a. } t \in (0,T) \text{ and every } v \in V\,, \tag{2.46}$$

$$\int_\Omega q(t)v = \int_\Omega \nabla p(t)\cdot\nabla v - \alpha_3 \int_\Omega (\mu^* - \mu_Q)(t)v \quad \text{for a.a. } t \in (0,T) \text{ and every } v \in V\,, \tag{2.47}$$

$$p(T) = \alpha_2(\varphi^*(T) - \varphi_\Omega)\,, \tag{2.48}$$

where $(\varphi^*, \mu^*) = \mathcal{S}(u^*)$. Our final result is the following necessary condition for optimality:

**Theorem 2.10.** *Suppose that the conditions* (2.42)–(2.43) *are fulfilled, let $u^* \in \mathcal{U}_{ad}$ be an optimal control with associated state $(\varphi^*, \mu^*) = \mathcal{S}(u^*)$, and assume that*

$$\alpha_1\varphi_Q \in L^2(0,T;V), \quad \alpha_2\varphi_\Omega \in V, \quad \alpha_3\mu_Q \in L^2(0,T;V), \tag{2.49}$$
$$\nabla\varphi^* \in L^\infty(0,T;L^4(\Omega)). \tag{2.50}$$

*Then it holds the variational inequality*

$$\int_Q p(u - u^*) + \alpha_4 \int_Q (u - u^*) \ge 0 \quad \forall\, u \in \mathcal{U}_{ad}\,, \tag{2.51}$$

*where $p$ is the first component of the solution $(p, q)$ to the adjoint problem* (2.46)–(2.48).

**Remark 2.11.** Notice that each of the regularity conditions (2.49) is fulfilled if either the related $\alpha_i$ is equal to $0$ or if the associated datum belongs to $L^2(0,T;V)$ or $V$. In the first case, we have no corresponding tracking of the tumor fraction $\varphi$ or the chemical potential $\mu$, which is inconvenient from the medical viewpoint, while in the second case we assume the same regularity for the respective target function that the optimal state $(\varphi^*, \mu^*)$ is known to have (Theorem 2.4), which seems to be reasonable.

Throughout the paper, we will repeatedly use Young's inequality

$$a\,b \le \delta\,a^2 + \frac{1}{4\delta}\,b^2 \quad \text{for all } a,b \in \mathbb{R} \text{ and } \delta > 0, \tag{2.52}$$

as well as Hölder's inequality and the Sobolev inequality related to the continuous embedding $V \subset L^p(\Omega)$ for $p \in [1,6]$ (since $\Omega$ is $d$-dimensional with $d \le 3$, bounded and smooth). Furthermore, the embeddings $V \subset H$ and $H \subset V^*$ are compact, so that we obtain from Ehrling's lemma the compactness inequality

$$\|v\| \;\le\; \delta\,\|\nabla v\| + C_\delta\,\|v\|_{V^*} \quad \text{for every } v \in V \text{ and } \delta > 0, \tag{2.53}$$

with some $C_\delta > 0$ that depends only on $\Omega$ and $\delta$.

We also utilize a tool that is widely used in the study of Cahn–Hilliard type equations. We define

$$\operatorname{dom}\mathcal{N} := \{v_* \in V^* :\ \overline{v}_* = 0\} \quad \text{and} \quad \mathcal{N} : \operatorname{dom}\mathcal{N} \to \{v \in V :\ \overline{v} = 0\} \tag{2.54}$$

by setting, for every $v_* \in \operatorname{dom}\mathcal{N}$,

$$\begin{aligned} &\mathcal{N}v_* \text{ is the unique element of } V \text{ that satisfies}\\ &\int_\Omega \nabla\mathcal{N}v_* \cdot \nabla v = \langle v_*, v\rangle \quad \text{for every } v \in V \quad \text{and} \quad \overline{\mathcal{N}v_*} = 0\,. \end{aligned} \tag{2.55}$$

As $\Omega$ is bounded, smooth, and connected, it turns out that (2.55) yields a well-defined isomorphism, which satisfies

$$\langle u^*, \mathcal{N}v_*\rangle = \langle v_*, \mathcal{N}u^*\rangle = \int_\Omega (\nabla\mathcal{N}u^*)\cdot(\nabla\mathcal{N}v_*) \quad \text{for every } u^*, v_* \in \operatorname{dom}\mathcal{N}. \tag{2.56}$$

Moreover, by also accounting for Poincaré's inequality

$$\|v\|_V^2 \le C_\Omega(\|\nabla v\|^2 + |\overline{v}|^2) \quad \text{for every } v \in V\,, \tag{2.57}$$

where the constant $C_\Omega > 0$ depends only on $\Omega$, we see that the function $\|\cdot\|_* : V^* \to [0,+\infty)$ defined by the formula

$$\|v_*\|_*^2 := \|\nabla\mathcal{N}(v_* - \overline{v}_*)\|^2 + |\overline{v}_*|^2 = \langle v_*, \mathcal{N}(v_* - \overline{v}_*)\rangle + |\overline{v}_*|^2 \quad \text{for } v_* \in V^*, \tag{2.58}$$

is a Hilbert norm on $V^*$ that is equivalent to the usual dual norm. It follows that

$$|\langle v_*, v\rangle| \le C_\Omega \|v_*\|_* \|v\|_V \quad \text{for every } v_* \in V^* \text{ and } v \in V, \tag{2.59}$$

with the same $C_\Omega$ as in (2.57) without loss of generality. Finally, notice that

$$2\,\langle \partial_t v_*(t), \mathcal{N}v_*(t)\rangle = \frac{d}{dt}\int_\Omega |\nabla\mathcal{N}v_*(t)|^2 = \frac{d}{dt}\,\|v_*(t)\|_*^2 \quad \text{for a.a. } t \in (0,T), \tag{2.60}$$

for every $v_* \in H^1(0,T;V^*)$ satisfying $\overline{v}_* = 0$, i.e., $\overline{v_*(t)} = 0$ for a.a. $t \in (0,T)$. This kind of notation is used throughout the paper for time-dependent functions.

We conclude this section by stating a general rule concerning the constants that appear in the estimates to be performed in the following. The small-case symbol $c$ stands for a generic constant whose actual values may change from line to line and even within the same line and depend only on $\Omega$, on the shape of the nonlinearities, and on the constants and the norms of the functions involved in the assumptions of the statements. In particular, the values of $c$ do not depend on the parameters $\varepsilon$ and $n$ we introduce in the next sections. A small-case symbol with a subscript like $c_\delta$ (in particular, with $\delta = \varepsilon$) indicates that the constant may depend on the parameter $\delta$, in addition. On the contrary, we mark precise constants that we can refer to by using different symbols, e.g., capital letters like in (2.53).

# 3 Continuous dependence and uniqueness

In this section, we give the proof of the second part of Theorem 2.3 and of Theorem 2.8. As for the former, we observe that the uniqueness of the the component $\varphi$ of the solution follows from (2.23) provided that this inequality is proved for every solution $(\varphi_i, \mu_i, \xi_i)$ corresponding to $u_i$ for $i = 1, 2,$ according to the statement. Indeed, (2.23) with $u_1 = u_2$ implies that $\varphi_1 = \varphi_2$. Moreover, if $\beta$ is single-valued, then we also deduce that $\xi_1 = \xi_2$, and a comparison in (2.15), written for both solutions, yields that $\mu_1 = \mu_2$ as well.

## 3.1 Continuous dependence, part I

So we just have to prove the inequality (2.23) by assuming that$(\varphi_i, \mu_i, \xi_i)$ is any solution corresponding to $u_i$, $i = 1, 2$. We set, for convenience,

$$\varphi := \varphi_1 - \varphi_2 \,, \quad \mu := \mu_1 - \mu_2 \,, \quad \xi := \xi_1 - \xi_2 \quad \text{and} \quad u := u_1 - u_2 \,. \tag{3.1}$$

Thus, by noting that (2.14) and (2.20) hold true with the new notations, we can subtract the latter integrated over $\Omega$ from the former and test the resulting equality by $\mathcal{N}(\varphi - \overline{\varphi})$ (recall (2.54)–(2.60) for the definition of $\mathcal{N}$ and its properties). At the same time, we write (2.15) for both solutions and test the difference by $-(\varphi - \overline{\varphi})$. Then, we sum up and integrate over $(0, t)$. Since a cancellation occurs, we obtain that

$$\begin{aligned}
&\frac{1}{2}\, \|(\varphi - \overline{\varphi})(t)\|_*^2 + \int_0^t \|(\varphi - \overline{\varphi})(s)\|_*^2 \, ds + \int_{Q_t} |\nabla(\varphi - \overline{\varphi})|^2 + \int_{Q_t} \xi\, \varphi \\
&= \int_{Q_t} (u - \overline{u})\, \mathcal{N}(\varphi - \overline{\varphi}) - \int_{Q_t} \xi\, \overline{\varphi} - \int_{Q_t} (\pi(\varphi_1) - \pi(\varphi_2))(\varphi - \overline{\varphi}).
\end{aligned}$$

All of the terms on the left-hand side are nonnegative (the last one by monotonicity). As for the first term on the right-hand side, we account for (2.21) and notice that $\|\overline{\varphi}\|_\infty \le \|\overline{u}\|_{L^1(0,T)} \le c\, \|u\|_{L^1(0,T;V^*)}$. Hence, with the help of (2.59) we have that

$$\begin{aligned}
&\int_{Q_t} (u - \overline{u})\, \mathcal{N}(\varphi - \overline{\varphi}) \le C_\Omega \int_0^t \|(u - \overline{u})(s)\|_*\, \|\mathcal{N}(\varphi - \overline{\varphi})(s)\|_V \, ds \\
&\le c \int_0^t \|(u - \overline{u})(s)\|_*\, \|(\varphi - \overline{\varphi})(s)\|_* \, ds \le c\, \|u\|_{L^1(0,T;V^*)} \sup_{0 \le s \le t} \|(\varphi - \overline{\varphi})(s)\|_* \\
&\le \frac{1}{4} \sup_{0 \le s \le t} \|(\varphi - \overline{\varphi})(s)\|_*^2 + c\, \|u\|_{L^1(0,T;V^*)}^2 \,.
\end{aligned}$$

Next, we have that

$$\int_{Q_t} \xi\, \overline{\varphi} \le (\|\xi_1\|_1 + \|\xi_2\|_1)\, \|\overline{\varphi}\|_\infty \le c(\|\xi_1\|_1 + \|\xi_2\|_1)\, \|\overline{u}\|_1 \,.$$

Finally, we owe to the Lipschitz continuity of $\pi$ and the compactness inequality (2.53) to obtain that

$$\begin{aligned}
&- \int_{Q_t} (\pi(\varphi_1) - \pi(\varphi_2))(\varphi - \overline{\varphi}) \le c \int_{Q_t} |\varphi|\, |\varphi - \overline{\varphi}| \\
&\le c \int_{Q_t} |\varphi - \overline{\varphi}|^2 + c \int_{Q_t} |\overline{\varphi}|\, |\varphi - \overline{\varphi}| \le c \int_{Q_t} |\varphi - \overline{\varphi}|^2 + \int_0^t |\overline{\varphi}(s)|^2 \, ds \\
&\le \frac{1}{2} \int_{Q_t} |\nabla(\varphi - \overline{\varphi})|^2 + c \int_0^t \|(\varphi - \overline{\varphi})(s)\|_*^2 \, ds + c\, \|u\|_{L^1(0,T;V^*)}^2 \,.
\end{aligned}$$

By collecting all these estimates, and ignoring some nonnegative terms on the left-hand side, we deduce that

$$\begin{aligned}
&\frac{1}{2}\,\|(\varphi-\overline{\varphi})(t)\|_*^2+\frac{1}{2}\int_{Q_t}|\nabla(\varphi-\overline{\varphi})|^2\\
&\leq \frac{1}{4}\sup_{0\leq s\leq t}\|(\varphi-\overline{\varphi})(s)\|_*^2+c\int_0^t\|(\varphi-\overline{\varphi})(s)\|_*^2\,ds\\
&\quad+c\,\|u\|_{L^1(0,T;V^*)}^2+c(\|\xi_1\|_1+\|\xi_2\|_1)\,\|\overline{u}\|_1\,.
\end{aligned}$$

By applying Gronwall's lemma, we conclude that (2.23) holds true with a constant $C_1$ as in the statement.

### 3.2 Continuous dependence, part II

We are going to prove Theorem 2.8. We still use the notations (3.1) regarding $\varphi$, $\mu$, and $u$, and denote the Lipschitz constant of $f'$ by $L$. We write (2.14) and (2.15) for both solutions and test the differences by $\varphi$ and $\mu$, respectively. Then, when adding, an obvious cancellation occurs, and by Young's inequality we have that

$$\begin{aligned}
&\frac{1}{2}\int_\Omega|\varphi(t)|^2+\int_{Q_t}|\varphi|^2+\int_{Q_t}|\mu|^2\\
&=\int_{Q_t}u\varphi+\int_{Q_t}(f'(\varphi_1)-f'(\varphi_2))\mu\\
&\leq\frac{1}{2}\int_Q|u|^2+\frac{1+L^2}{2}\int_{Q_t}|\varphi|^2+\frac{1}{2}\int_{Q_t}|\mu|^2\,.
\end{aligned}$$

Hence, rearranging and using Gronwall's lemma, we immediately deduce that

$$\|\varphi\|_{C^0([0,T];H)}+\|\mu\|_{L^2(0,T;H)}\leq c\,\|u\|_{L^2(0,T;H)}\,,$$

with a constant $c$ that only depends on $L$ and $T$. By applying elliptic regularity to the difference of (2.15), written for both solutions, we also infer that

$$\begin{aligned}
&\|\varphi\|_{L^2(0,T;W)}\leq c\,\|\mu-(f'(\varphi_1)-f'(\varphi_2))\|_{L^2(0,T;H)}\\
&\leq c(\|\mu\|_{L^2(0,T;H)}+\|\varphi\|_{L^2(0,T;H)})\leq c\,\|u\|_{L^2(0,T;H)}\,,
\end{aligned}$$

where $c$ depends on $\Omega$, in addition.

## 4 Approximation and existence

In this section, we prove the existence part of Theorem 2.3. Our method consists in approximating the problem at hand and using compactness and monotonicity arguments.

## 4.1 Approximation

Here, we construct an approximating problem depending on the parameter $\varepsilon \in (0,1)$. This problem is obtained by modifying the formulation of problem (2.14)–(2.16) by replacing $\widehat{\beta}$ and $\beta$ by their Moreau-Yosida regularizations $\widehat{\beta}_\varepsilon$ and $\beta_\varepsilon$, respectively (see, e.g., [6, pp. 28 and 39]). Owing also to our assumption (2.5) on $\widehat{\beta}$, we have that

$$\beta_\varepsilon \text{ is monotone and Lipschitz continuous with } \beta_\varepsilon(0) = 0, \tag{4.1}$$

$$|\beta_\varepsilon(r)| \le |\beta^\circ(r)| \quad \text{for every } r \in D(\beta), \tag{4.2}$$

$$0 \le \widehat{\beta}_\varepsilon(r) = \int_0^r \beta_\varepsilon(s)\,ds \le \widehat{\beta}(r) \quad \text{for every } r \in \mathbb{R}, \tag{4.3}$$

where we recall that $\beta^\circ(r)$ denotes the element of $\beta(r)$ having minimal modulus. The solution we look for is a pair $(\varphi^\varepsilon, \mu^\varepsilon)$ enjoying the regularity properties

$$\varphi^\varepsilon \in H^1(0,T;V^*) \cap L^\infty(0,T;V), \quad \mu^\varepsilon \in L^2(0,T;V) \tag{4.4}$$

and satisfying

$$\int_\Omega \partial_t \varphi^\varepsilon(t)\, v + \int_\Omega \varphi^\varepsilon(t) v + \int_\Omega \nabla\mu^\varepsilon(t) \cdot \nabla v = \int_\Omega u(t) v$$
$$\text{for a.a. } t \in (0,T) \text{ and every } v \in V, \tag{4.5}$$

$$\int_\Omega \mu^\varepsilon(t) v = \int_\Omega \nabla\varphi^\varepsilon(t) \cdot \nabla v + \int_\Omega \beta_\varepsilon(\varphi^\varepsilon(t)) v + \int_\Omega \pi(\varphi^\varepsilon(t)) v$$
$$\text{for a.a. } t \in (0,T) \text{ and every } v \in V, \tag{4.6}$$

$$\varphi^\varepsilon(0) = \varphi_0\,. \tag{4.7}$$

Our aim is to solve this approximating problem. In this respect, we have the following result.

**Theorem 4.1.** *Assume* (2.4)–(2.6) *on the structure of the system and* (2.8)–(2.10) *on the data. Then, for every $\varepsilon \in (0,1)$, problem* (4.5)–(4.7) *has a unique solution $(\varphi^\varepsilon, \mu^\varepsilon)$ satisfying the regularity properties* (4.4)*.*

Uniqueness follows from Section 3.1, since $\beta_\varepsilon$ satisfies all the properties required for $\beta$ and is single-valued. Hence, only the existence of a solution to problem (4.5)–(4.7) has to be proved. Our argument is based on the discretization of (4.5)–(4.7) by a Faedo–Galerkin scheme and a priori estimates.

**The discrete problem.** We introduce the nondecreasing sequence $\{\lambda_j\}$ of the eigenvalues and the complete orthonormal sequence $\{e_j\}$ of corresponding eigenvectors of the Laplace operator with homogeneous Neumann boundary conditions, that is,

$$-\Delta e_j = \lambda_j e_j \quad \text{in } \Omega \quad \text{and} \quad \partial_\nu e_j = 0 \quad \text{on } \Gamma \quad \text{for all } j \in \mathbb{N}, \quad \text{with} \quad (e_i, e_j) = \delta_{ij}$$
$$\text{and} \quad (\nabla e_i, \nabla e_j) = \lambda_i\,\delta_{ij} \quad \text{for all} \quad i,j \in \mathbb{N}. \tag{4.8}$$

Next, for $n \ge 1$, we introduce the subspace $V_n$ of $V$ by setting

$$V_n := \operatorname{span}\{e_1, \dots, e_n\}. \tag{4.9}$$

Then, the sequence $\{V_n\}$ is nondecreasing, and its union is dense in both $V$ and $H$. We observe at once that

$$\lambda_1 = 0 \text{ and the elements of } V_1 \text{ are the constant functions.} \tag{4.10}$$

At this point, we can introduce the discrete problem. Even though it also depends on $\varepsilon$, we do not stress this in the notation for the solution. We look for a pair $(\varphi_n, \mu_n)$ satisfying

$$\varphi_n,\ \mu_n \in H^1(0,T;V_n), \tag{4.11}$$

$$(\partial_t\varphi_n(t), v) + (\varphi_n(t), v) + (\nabla\mu_n(t), \nabla v) = (u(t), v) \quad \text{for a.a. } t \in (0,T) \text{ and every } v \in V_n, \tag{4.12}$$

$$(\mu_n(t), v) = (\nabla\varphi_n(t), \nabla v) + (\beta_\varepsilon(\varphi_n(t)), v) + (\pi(\varphi_n(t)), v) \quad \text{for every } t \in [0,T] \text{ and } v \in V_n, \tag{4.13}$$

$$(\varphi_n(0), v) = (\varphi_0, v) \quad \text{for every } v \in V_n. \tag{4.14}$$

**Remark 4.2.** We notice that $\varphi_n(0)$ is the $H$-projection of $\varphi_0$ on $V_n$, i.e.,

$$\varphi_n(0) = \sum_{j=1}^{n} (\varphi_0, e_j)\, e_j \quad \text{for } n = 1, 2, \dots. \tag{4.15}$$

This and assumption (2.9) have important consequences. The first one is given by the inequalities

$$\|\varphi_n(0)\| \le \|\varphi_0\| \quad \text{and}$$
$$\|\nabla\varphi_n(0)\|^2 \le c \sum_{j=1}^{n} |\lambda_j^{1/2}(\varphi_0, e_j)|^2 \le c \sum_{j=1}^{\infty} |\lambda_j^{1/2}(\varphi_0, e_j)|^2 \le c\, \|\varphi_0\|_V^2\,. \tag{4.16}$$

Next, since $\varphi_0 \in W$, we have that

$$\sum_{j=1}^{\infty} |\lambda_j(\varphi_0, e_j)|^2 \le c\, \|\varphi_0\|_W^2 \quad \text{and}$$
$$\|\varphi_0 - \varphi_n(0)\|_W^2 \le c \sum_{j=n+1}^{\infty} |\lambda_j(\varphi_0, e_j)|^2 \to 0 \quad \text{as } n \to \infty,$$

so that $\varphi_n(0)$ converges to $\varphi_0$ in $W$, and thus uniformly. By combining with our assumption (2.9) on $\varphi_0$, we deduce that there are elements $r'_\pm$ of the interior of $D(\beta)$ and a natural number $n_0$ only depending on $\beta$ and $\varphi_0$ such that $r'_- \le \varphi_n(0) \le r'_+$ in $\Omega$ for every $n \ge n_0$. By accounting for (4.2)–(4.3), we infer that

$$\widehat{\beta}_\varepsilon(\varphi_n(0)) + |\beta_\varepsilon(\varphi_n(0))| \le \sup_{r'_- \le r \le r'_+} \widehat{\beta}(r) + \sup_{r'_- \le r \le r'_+} |\beta^\circ(r)| \tag{4.17}$$

for every $n \ge n_0$. From now on, it is understood that $n \ge n_0$ so that (4.17) holds true.

The next step is proving that this problem has a unique solution. To this end, we represent $\varphi_n$ and $\mu_n$ in term of the eigenfunctions, i.e.,

$$\varphi_n(t) = \sum_{j=1}^{n} \varphi_n^j(t)\, e_j \quad \text{and} \quad \mu_n(t) = \sum_{j=1}^{n} \mu_n^j(t)\, e_j \quad \text{for every } t \in [0,T],$$

for some functions $\varphi_n^j$ and $\mu_n^j$ belonging to $H^1(0,T)$.Then, if we consider the column vectors $y := (\varphi_n^j)_{j=1,\dots,n}$ and $z := (\mu_n^j)_{j=1,\dots,n}$, then the equations (4.12)–(4.13) take the form

$$y'(t) + y(t) + Az(t) = g(t) \quad \text{and} \quad z(t) = Ay(t) + G(y(t)),$$

where $A$ is the diagonal matrix of the first $n$ eigenvalues, $g(t)$ is the column vector with components $(u(t), e_j)$, and $G : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz continuous function. Notice that $g \in L^2(0,T;\mathbb{R}^n)$. Hence, by replacing $z$ in the first equation with the help of the second one and rearranging, since (4.15) provides an initial condition for $y$, we obtain a well-posed Cauchy problem for $y$ which has a unique solution $y \in H^1(0,T;\mathbb{R}^n)$. Then, the second equation yields that $z \in H^1(0,T;\mathbb{R}^n)$.

## 4.2 Uniform a priori estimates

We perform a number of estimates. We point out that they are based on the properties (4.1)–(4.3) and thus hold if $\beta_\varepsilon$ is a possibly different approximation of $\beta$. We make this remark since the estimates we are going to prove here will be used later on for other purposes with a different $\beta_\varepsilon$. Hence, we recall our general rule: the symbol $c$ denotes (possibly different) constants that do not depend on $\varepsilon$ and $n$.

However, before starting, we remark a consequence of (2.10). We choose $\delta_0 > 0$ such that the interval $[\overline{\varphi}_0 - M - \delta_0, \overline{\varphi}_0 + M + \delta_0]$ is included in the interior of $D(\beta)$. Then, for some $C_0 > 0$, we have the inequality

$$\beta_\varepsilon(r)(r - r_0) \geq \delta_0 |\beta_\varepsilon(r)| - C_0 \quad \text{for every } r \in \mathbb{R},\ r_0 \in [\overline{\varphi}_0 - M, \overline{\varphi}_0 + M] \text{ and } \varepsilon \in (0,1). \tag{4.18}$$

This is a generalization of [38, Appendix, Prop. A.1]. The detailed proof given in [28, p. 908] with a fixed $r_0$ also works in the present case with minor changes.

**Preliminary estimate.** We notice that, by (4.10), the constant functions belong to $V_n$ for every $n$. Hence, we deduce from (4.12) and (4.14) that

$$\frac{d}{dt}\,\overline{\varphi_n}(t) + \overline{\varphi_n}(t) = \overline{u}(t) \quad \text{for a.a. } t \in (0,T) \quad \text{and} \quad \overline{\varphi_n}(0) = \overline{\varphi}_0\,. \tag{4.19}$$

As in Remark 2.2 (both the equation and the initial condition are the same here and there), we conclude that

$$\overline{\varphi}_0 - M \leq \overline{\varphi_n}(t) \leq \overline{\varphi}_0 + M \quad \text{for every } t \in [0,T]. \tag{4.20}$$

**First uniform estimate.** We recall the definition of $\mathcal{N}$ and its properties (see (2.54)–(2.60)) and notice that

$$v \in V_n \quad \text{and} \quad \overline{v} = 0 \quad \text{imply that} \quad \mathcal{N}v \in V_n\,. \tag{4.21}$$

Indeed, both $v$ and $w := \mathcal{N}v$ can be expressed in terms of the eigenfunctions $e_j$, and we have that

$$\sum_{n=1}^{\infty} \lambda_j (w, e_j)\, e_j = -\Delta w = v = \sum_{j=2}^{n} (v, e_j)\, e_j\,.$$

Hence, $(w, e_j) = 0$ for every $j > n$ (since $\lambda_j > 0$ for $j > 1$), i.e., $w \in V_n$. Therefore, we subtract (4.19), multiplied by $\int_\Omega v$, from (4.12), write the resulting equality at the time $s$ and test it by $\mathcal{N}(\varphi_n(s) - \overline{\varphi_n}(s))$. At the same time, we test (4.13) written at the time $s$ by $-(\varphi_n(s) - \overline{\varphi_n}(s))$. Then, we sum up and integrate over $(0,t)$ with respect to $s$. By accounting for the cancellation that occurs, we obtain that

$$\frac{1}{2}\,\|\varphi_n(t) - \overline{\varphi_n}(t)\|_*^2 + \int_0^t \|\varphi_n(s) - \overline{\varphi_n}(s)\|_*^2\, ds$$

$$
\begin{aligned}
&+ \int_{Q_t} |\nabla \varphi_n|^2 + \int_{Q_t} \beta_\varepsilon(\varphi_n)(\varphi_n - \overline{\varphi_n}) \\
&= \frac{1}{2} \|\varphi_n(0) - \overline{\varphi}_0\|_*^2 + \int_{Q_t} (u - \overline{u})\, \mathcal{N}(\varphi_n - \overline{\varphi_n}) - \int_{Q_t} \pi(\varphi_n)(\varphi_n - \overline{\varphi_n})\,.
\end{aligned}
$$

The first three terms on the left-hand side are nonnegative. To treat the last one, we account for (4.18) and (4.20) and have that

$$
\int_\Omega \beta_\varepsilon(\varphi_n(t))(\varphi_n(t) - \overline{\varphi_n}(t)) \geq \delta_0 \,\|\beta_\varepsilon(\varphi_n(t))\|_{L^1(\Omega)} - c \quad \text{for a.a. } t \in (0,T), \tag{4.22}
$$

whence also

$$
\int_{Q_t} \beta_\varepsilon(\varphi_n)(\varphi_n - \overline{\varphi_n}) \geq \delta_0 \|\beta_\varepsilon(\varphi_n)\|_{L^1(Q_t)} - c\,.
$$

Let us come to the right-hand side. The first term is estimated from above by using the $H$ norm. Thus, it is uniformly bounded since $\varphi_n(0)$ is the $H$-projection of $\varphi_0$ on $V_n$. The next term can be dealt with as follows:

$$
\begin{aligned}
&\int_{Q_t} (u - \overline{u})\, \mathcal{N}(\varphi_n - \overline{\varphi_n}) \leq \int_0^t \|u(s) - \overline{u}(s)\|_* \|\varphi_n(s) - \overline{\varphi_n}(s)\|_* ds \\
&\leq c\|u - \overline{u}\|_{L^2(0,T;H)}^2 + \frac{1}{2} \int_0^t \|\varphi_n(s) - \overline{\varphi_n}(s)\|_*^2 ds\,.
\end{aligned}
$$

By adding and subtracting $\pi(\overline{\varphi_n})$ in the first factor inside the last integral, and accounting for the Lipschitz continuity of $\pi$, we have that

$$
- \int_{Q_t} \pi(\varphi_n)(\varphi_n - \overline{\varphi_n}) \leq c \int_{Q_t} |\varphi_n - \overline{\varphi_n}|^2 + c \int_{Q_t} |\overline{\varphi_n}|^2 + c\,.
$$

So, we can treat the first integral on the right-hand side with the help of the compactness inequality (2.53), namely

$$
c \int_{Q_t} |\varphi_n - \overline{\varphi_n}|^2 \leq \delta \int_{Q_t} |\nabla \varphi_n|^2 + c_\delta \int_0^t \|\varphi_n(s) - \overline{\varphi_n}(s)\|_*^2 ds\,,
$$

where $\delta > 0$ is arbitrary, and we can use (4.20) for the second one. At this point, we choose $\delta > 0$ small enough and apply the Gronwall lemma. We conclude that

$$
\|\varphi_n\|_{L^\infty(0,T;V^*) \cap L^2(0,T;V)} + \|\beta_\varepsilon(\varphi_n)\|_{L^1(Q)} \leq c\,. \tag{4.23}
$$

Since $\overline{\mu_n} = \overline{\beta_\varepsilon(\varphi_n)} + \overline{\pi(\varphi_n)}$ by (4.13) tested by $1/|\Omega|$, we deduce that

$$
\|\overline{\mu_n}\|_{L^1(0,T)} \leq c\,. \tag{4.24}
$$

**Second uniform estimate.** We test (4.12) by $\mu_n$ and (4.13) by $-(\partial_t \varphi_n + \varphi_n)$ and sum up. Then, an obvious cancellation occurs, and integration over $(0,t)$ leads to

$$
\begin{aligned}
&\int_{Q_t} |\nabla \mu_n|^2 + \frac{1}{2} \int_\Omega |\nabla \varphi_n(t)|^2 + \int_{Q_t} |\nabla \varphi_n|^2 + \int_\Omega \widehat{\beta}_\varepsilon(\varphi_n(t)) + \int_{Q_t} \beta_\varepsilon(\varphi_n)\varphi_n \\
&= \frac{1}{2} \int_\Omega |\nabla \varphi_n(0)|^2 + \int_\Omega \widehat{\beta}_\varepsilon(\varphi_n(0)) \\
&\quad + \int_{Q_t} u\mu_n - \int_\Omega \widehat{\pi}(\varphi_n(t)) + \int_\Omega \widehat{\pi}(\varphi_n(0)) - \int_{Q_t} \pi(\varphi_n)\varphi_n\,.
\end{aligned}
$$

All of the terms on the left-hand side are nonnegative. The first two terms on the right-hand side can be estimated thanks to (4.16)–(4.17). The same (4.16), along with (2.6), helps in the second term involving $\widehat{\pi}$, while the last integral is bounded due to (4.23). It remains to treat two further terms. By accounting for Young's inequality, the Poincaré inequality (2.57), and (4.24), we have that

$$\begin{aligned}&\int_{Q_t} u\mu_n = \int_{Q_t} u(\mu_n - \overline{\mu_n}) + \int_{Q_t} u\overline{\mu_n}\\ &\leq \frac{1}{2}\int_{Q_t} |\nabla\mu_n|^2 + c\int_{Q_t} |u|^2 + |\Omega|\,\|u\|_\infty\,\|\overline{\mu_n}\|_{L^1(0,T)} \leq \frac{1}{2}\int_{Q_t} |\nabla\mu_n|^2 + c\,.\end{aligned}$$

Finally, we owe to the quadratic growth of $\widehat{\pi}$, the compactness inequality (2.53), and (4.23), to see that

$$-\int_\Omega \widehat{\pi}(\varphi_n(t)) \leq c\int_\Omega |\varphi_n(t)|^2 + c \leq \frac{1}{4}\int_\Omega |\nabla\varphi_n(t)|^2 + c\,\|\varphi_n(t)\|_*^2 + c \leq \frac{1}{4}\int_\Omega |\nabla\varphi_n(t)|^2 + c\,.$$

By combining all these estimates, we conclude that

$$\|\nabla\mu_n\|_{L^2(0,T;H)} + \|\varphi_n\|_{L^\infty(0,T;V)} + \|\widehat{\beta}_\varepsilon(\varphi_n)\|_{L^\infty(0,T;L^1(\Omega))} \leq c\,. \tag{4.25}$$

**Third uniform estimate.** We test (4.13) by $\varphi_n(t) - \overline{\varphi_n}(t)$. We have a.e. in $(0,T)$ that

$$\begin{aligned}&\int_\Omega |\nabla\varphi_n|^2 + \int_\Omega \beta_\varepsilon(\varphi_n)(\varphi_n - \overline{\varphi_n})\\ &= -\int_\Omega \pi(\varphi_n)(\varphi_n - \overline{\varphi_n}) + \int_\Omega \mu_n(\varphi_n - \overline{\varphi_n}).\end{aligned}$$

The second term on the left-hand side is estimated from below by (4.22), and the first term on the right-hand side is uniformly bounded by (4.25) and the Lipschitz continuity of $\pi$. The last term is treated with the help of Poincaré's inequality:

$$\begin{aligned}&\int_\Omega \mu_n(\varphi_n - \overline{\varphi_n}) = \int_\Omega (\mu_n - \overline{\mu_n})(\varphi_n - \overline{\varphi_n})\\ &\leq c\,\|\varphi_n - \overline{\varphi_n}\|\,\|\nabla\mu_n\| \leq c\,\|\nabla\mu_n\|\,.\end{aligned}$$

By combining, we deduce that

$$\delta_0\|\beta_\varepsilon(\varphi_n(t))\|_{L^1(\Omega)} \leq c\,\|\nabla\mu_n(t)\| + c\,, \quad \text{i.e.,} \quad \delta_0^2\|\beta_\varepsilon(\varphi_n(t))\|_{L^1(\Omega)}^2 \leq c\,\|\nabla\mu_n(t)\|^2 + c$$

for a.a. $t \in (0,T)$, and (4.25) yields that

$$\|\beta_\varepsilon(\varphi_n)\|_{L^2(0,T;L^1(\Omega))} \leq c\,\|\nabla\mu_n\|_{L^2(0,T;H)} + c \leq c\,.$$

Consequently, at first the mean value $\overline{\beta_\varepsilon(\varphi_n)}$, and then $\overline{\mu_n}$, are bounded in $L^2(0,T)$. By accounting for (4.25) and the Poincaré inequality (2.57), we then conclude that

$$\|\mu_n\|_{L^2(0,T;V)} \leq c\,. \tag{4.26}$$

### 4.3 Proof of Theorem 4.1

The above estimates provide some weak convergence for both $\varphi_n$ and $\mu_n$. However, no strong convergence can as yet be deduced. Thus, we perform one more a priori estimate, which is uniform at least with respect to $n$.

**An auxiliary estimate.** First, we subtract (4.19), multiplied by $\int_\Omega v$, from (4.12). Then, we test the equality obtained by $\mathcal{N}w$ and (4.13) by $-w$, where $w := \partial_t(\varphi_n - \overline{\varphi_n}) + (\varphi_n - \overline{\varphi_n})$. We integrate the resulting equalities over $(0,t)$, rearrange the second one, and obtain that

$$\begin{aligned}&\int_0^t \|\partial_t(\varphi_n(s) - \overline{\varphi_n}(s)) + (\varphi_n(s) - \overline{\varphi_n}(s))\|_*^2\,ds + \int_{Q_t} \nabla\mu_n \cdot \nabla\mathcal{N}w\\&= \int_{Q_t} (u - \overline{u})\mathcal{N}w\,,\end{aligned}$$

$$\begin{aligned}&\int_\Omega |\nabla\varphi_n(t)|^2 + \int_{Q_t} |\nabla\varphi_n|^2 + \int_\Omega \widehat{\beta}_\varepsilon(\varphi_n(t))\\&= \int_\Omega \widehat{\beta}_\varepsilon(\varphi_n(0)) - \int_\Omega \widehat{\pi}(\varphi_n(t)) + \int_\Omega \widehat{\pi}(\varphi_n(0))\\&\quad + \int_{Q_t} (\beta_\varepsilon(\varphi_n) + \pi(\varphi_n))(-\varphi_n + \partial_t\overline{\varphi_n} + \overline{\varphi_n}) + \int_{Q_t} \mu_n w\,.\end{aligned}$$

At this point, we sum up and notice that a cancellation occurs due to the definition of $\mathcal{N}$. Moreover, we owe to (4.19) on both sides, and account for the quadratic growth of $\widehat{\pi}$, the linear growth of $\beta_\varepsilon + \pi$, and some of the inequalities (4.16) and (4.17). We obtain that

$$\begin{aligned}&\int_0^t \|\partial_t\varphi_n(s) + \varphi_n(s) - \overline{u}(s)\|_*^2\,ds + \int_\Omega |\nabla\varphi_n(t)|^2 + \int_{Q_t} |\nabla\varphi_n|^2 + \int_\Omega \widehat{\beta}_\varepsilon(\varphi_n(t))\\&\le \int_{Q_t} (u - \overline{u})\mathcal{N}w + c\int_\Omega |\varphi_n(t)|^2 + c_\varepsilon \int_{Q_t} (|\varphi_n|^2 + |\overline{u}|^2) + c_\varepsilon\,.\end{aligned}$$

Thanks to (4.25) and the assumptions on $u$, all of the terms on the right-hand side are under control but the first one. In this respect, we have that

$$\begin{aligned}&\int_{Q_t} (u - \overline{u})\mathcal{N}w \le \|u - \overline{u}\|_{L^2(0,T;H)} \|\mathcal{N}w\|_{L^2(0,t;H)} \le c\|w\|_{L^2(0,t;V^*)}\\&= c\,\|\partial_t\varphi_n + \varphi_n - \overline{u}\|_{L^2(0,t;V^*)} \le \frac{1}{2}\int_0^t \|\partial_t\varphi_n(s) + \varphi_n(s) - \overline{u}(s)\|_*^2\,ds + c\,.\end{aligned}$$

By combining, rearranging, and noting that $\varphi_n - \overline{u}$ is obviously bounded in $L^2(0,T;V^*)$, we conclude that

$$\|\partial_t\varphi_n\|_{L^2(0,T;V^*)} \le c_\varepsilon\,. \tag{4.27}$$

**Conclusion.** Now, we can let $n$ tend to infinity in the discrete problem. Indeed, the estimates (4.23) and (4.25)–(4.26) are uniform with respect to $n$, so that, by also applying well-known compactness results (for the strong convergence, see, e.g., [45, Sect. 8, Cor. 4]), we conclude that there exists a pair $(\varphi^\varepsilon, \mu^\varepsilon)$ such that

$$\begin{aligned}&\varphi_n \to \varphi^\varepsilon \qquad \text{weakly star in } H^1(0,T;V^*) \cap L^\infty(0,T;V)\\&\qquad\qquad\qquad \text{and strongly in } C^0([0,T];H)\,,\end{aligned} \tag{4.28}$$

$$\mu_n \to \mu^\varepsilon \qquad \text{weakly in } L^2(0,T;V)\,, \tag{4.29}$$

just for a subsequence, in principle. However, since we prove that the limit $(\varphi^\varepsilon, \mu^\varepsilon)$ is a solution to (4.5)–(4.7) and we already know that uniqueness holds for this problem, the whole sequences converge to $(\varphi^\varepsilon, \mu^\varepsilon)$. By Lipschitz continuity, we infer that

$$\beta_\varepsilon(\varphi_n) + \pi(\varphi_n) \to \beta_\varepsilon(\varphi^\varepsilon) + \pi(\varphi^\varepsilon) \quad \text{strongly in } C^0([0,T];H).$$

Since $\varphi_n(0)$ converges to both $\varphi^\varepsilon(0)$ and $\varphi_0$ in $H$, the initial condition (4.7) is satisfied, and it remains to prove that the equations (4.5)–(4.6) are satisfied as well. It suffices to verify that $(\varphi^\varepsilon, \mu^\varepsilon)$ solves the corresponding integrated versions with time dependent test functions in $L^2(0,T;V)$, i.e.,

$$\int_Q \partial_t\varphi^\varepsilon\, v + \int_Q \varphi^\varepsilon v + \int_Q \nabla\mu^\varepsilon \cdot \nabla v = \int_Q uv \quad \text{for every } v \in L^2(0,T;V)\,, \tag{4.30}$$

$$\int_Q \mu^\varepsilon v = \int_Q \nabla\varphi^\varepsilon \cdot \nabla v + \int_Q \beta_\varepsilon(\varphi^\varepsilon)v + \int_\Omega \pi(\varphi^\varepsilon)v \quad \text{for every } v \in L^2(0,T;V)\,. \tag{4.31}$$

To this end, we fix $m$ for a while and take any $V_m$-valued step function $w$. Then, if $n \geq m$, the choice $v = w(t)$ is admissible in both (4.12) and (4.13). By testing the equations and integrating over $(0,T)$, we obtain (4.30)–(4.31) for $(\varphi_n, \mu_n)$ with this test function, and letting $n$ tend to infinity we have the same for $(\varphi^\varepsilon, \mu^\varepsilon)$.Since $w$ and $m$ are arbitrary, the equations (4.30)–(4.31) also hold with any $V_\infty$-valued step function, where $V_\infty$ is the union of all the subspaces $V_m$. As $V_\infty$ is dense in $V$, the set of these step functions is dense in $L^2(0,T;V)$, and we immediately conclude.

## 4.4 Proof of Theorem 2.3

Since the estimates of Section 4.2 are uniform with respect to $n$ and $\varepsilon$, they are preserved in the limit as $n$ tends to infinity, and we have that

$$\|\varphi^\varepsilon\|_{L^\infty(0,T;V)} + \|\mu^\varepsilon\|_{L^2(0,T;V)} + \|\widehat{\beta}_\varepsilon(\varphi^\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))} \leq c\,. \tag{4.32}$$

However, we need some further estimates.

**Fourth a priori estimate.** From (4.30) it easily follows that

$$\left|\int_Q \partial_t\varphi^\varepsilon\, v\right| \leq \Big(\|\varphi^\varepsilon\|_{L^\infty(0,T;H)} + \|\nabla\mu^\varepsilon\|_{L^2(0,T;H)} + \|u\|_{L^2(0,T;H)}\Big)\|v\|_{L^2(0,T;V)} \quad \text{for every } v \in L^2(0,T;V),$$

whence the estimate (4.32) implies that

$$\|\partial_t\varphi^\varepsilon\|_{L^2(0,T;V^*)} \leq c\,. \tag{4.33}$$

**Fifth a priori estimate.** We test (4.6) by $\beta_\varepsilon(\varphi^\varepsilon(t))$ and integrate over $(0,T)$. We obtain that

$$\int_Q \beta_\varepsilon'(\varphi^\varepsilon)|\nabla\varphi^\varepsilon|^2 + \int_Q |\beta_\varepsilon(\varphi^\varepsilon)|^2 = \int_Q (\mu^\varepsilon - \pi(\varphi^\varepsilon))\beta_\varepsilon(\varphi^\varepsilon)\,.$$

By accounting for (4.3), the Lipschitz continuity of $\pi$, and (4.32), we immediately conclude that

$$\|\beta_\varepsilon(\varphi^\varepsilon)\|_{L^2(0,T;H)} \le c\,. \tag{4.34}$$

**Conclusion.** We deduce that

$$\begin{aligned}
\varphi^\varepsilon \to \varphi \qquad & \text{weakly star in } H^1(0,T;V^*) \cap L^\infty(0,T;V) \\
& \quad \text{and strongly in } C^0([0,T];H)\,, && (4.35)\\
\mu^\varepsilon \to \mu \qquad & \text{weakly in } L^2(0,T;V)\,, && (4.36)\\
\beta_\varepsilon(\varphi^\varepsilon) \to \xi \qquad & \text{weakly in } L^2(0,T;H)\,, && (4.37)
\end{aligned}$$

as $\varepsilon$ tends to $0$ (at least for a subsequence $\varepsilon_k$ tending to $0$), where $(\varphi, \mu, \xi)$ is a solution to problem (2.14)–(2.16). By virtue of the estimates (4.32)–(4.33) and compactness results, we see that (4.35)–(4.37) hold true for some triplet $(\varphi, \mu, \xi)$. Then, in view of (4.7), it turns out that (2.16) is satisfied, and we can take the limit in (4.30)–(4.31) to obtain (2.17)–(2.18). It remains to show that $\xi \in \beta(\varphi)$ a.e. in $Q$, but this follows from the strong convergence of $\varphi^\varepsilon$, the weak convergence of $\beta_\varepsilon(\varphi^\varepsilon)$, and a well-known property of the Yosida approximation (see, e.g., [2, Prop. 2.2, p. 38]). Thus, the proof of Theorem 2.3 is complete.

# 5 Regularity and separation

This section is devoted to the proofs of Theorem 2.4 and Propositions 2.5 and 2.6. Hence, it is understood that the assumptions of the corresponding statements are in force.

## 5.1 Regularity

In order to prove Theorem 2.4, we make a preliminary remark. Since $\beta$ is a $C^1$ function in the interior of its domain, we can prove an estimate for the derivative $\beta'_\varepsilon$ of the Yosida approximation. By the definition of $\beta_\varepsilon$, we have that

$$\beta_\varepsilon(r) \in \beta(r - \varepsilon\beta_\varepsilon(r)) \quad \text{for every } r \in \mathbb{R}. \tag{5.1}$$

Assume now that $r$ belongs to the interior of $D(\beta)$. Then the same is true for $r - \varepsilon\beta_\varepsilon(r)$, and it holds that $|r - \varepsilon\beta_\varepsilon(r)| \le |r|$. Indeed, by supposing, e.g., that $r > 0$, it follows that $\beta_\varepsilon(r) \ge 0$ whence $r - \varepsilon\beta_\varepsilon(r) \le r$. But we also have that $r - \varepsilon\beta_\varepsilon(r) \ge 0$ since $\beta_\varepsilon(0) = 0$ and $\beta_\varepsilon$ is Lipschitz continuous with Lipschitz constant $1/\varepsilon$. This proves the second assertion. The first one follows since $[0, r]$ is included in the interior of $\beta$ (recall (2.24)). Therefore, (5.1) becomes an equation, and $s = \beta_\varepsilon(r)$ is the function implicitly defined by the equation $\beta(r - \varepsilon s) - s = 0$. Since $(\partial/\partial s)(\beta(r - \varepsilon s) - s) \le -1$, $\beta_\varepsilon$ is differentiable and its derivative is given by

$$\beta'_\varepsilon(r) = \frac{\beta'(r - \varepsilon\beta_\varepsilon(r))}{1 + \varepsilon\beta'(r - \varepsilon\beta_\varepsilon(r))}\,.$$

Hence, we deduce that

$$|\beta'_\varepsilon(r)| \le \sup_{s \in [a,b]} |\beta'(s)| \quad \text{for every } r \in [a,b]\,, \tag{5.2}$$

whenever $[a, b]$ contains $0$ and is included in the interior of $D(\beta)$.

**Regularity estimate.** To be completely rigorous, we come back to the Faedo–Galerkin scheme and notice that we can assume $\varphi_n$ and $\mu_n$ to be of class $C^1$. Indeed, $u$ belongs to $H^1(0,T;H)$, and we can suppose that $\beta_\varepsilon$ is of class $C^1$ since it could be replaced by a regularization of it having the same properties in the whole procedure developed so far (see, e.g., [29, formulas (3.5–6)]). We denote the Lipschitz constant of $\pi$ by $L$, set $L' := L + 1$, and test (4.12) by $\partial_t\mu_n + L'\,\partial_t\varphi_n$. At the same time, we differentiate (4.13) and test the resulting equation by $-(\partial_t\varphi_n + \varphi_n)$. Then, we sum up, integrate over $(0,t)$, and notice that a cancellation occurs. Thus, we obtain that

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega |\nabla\mu_n(t)|^2 + L'\int_{Q_t} |\partial_t\varphi_n|^2 + \frac{L'}{2}\int_\Omega |\varphi_n(t)|^2 \\
&\quad + \int_{Q_t} |\nabla\partial_t\varphi_n|^2 + \frac{1}{2}\int_\Omega |\nabla\varphi_n(t)|^2 + \int_{Q_t} \beta_\varepsilon'(\varphi_n)|\partial_t\varphi_n|^2 + \int_{Q_t} \varphi_n\,\beta_\varepsilon'(\varphi_n)\partial_t\varphi_n \\
&= \frac{1}{2}\int_\Omega |\nabla\mu_n(0)|^2 + \frac{L'}{2}\int_\Omega |\varphi_n(0)|^2 + \frac{1}{2}\int_\Omega |\nabla\varphi_n(0)|^2 \\
&\quad + \int_{Q_t} u\,\partial_t\mu_n + L'\int_{Q_t} u\,\partial_t\varphi_n - L'\int_{Q_t} \nabla\mu_n\cdot\nabla\partial_t\varphi_n \\
&\quad - \int_{Q_t} \pi'(\varphi_n)|\partial_t\varphi_n|^2 - \int_{Q_t} \varphi_n\,\pi'(\varphi_n)\partial_t\varphi_n\,. 
\end{aligned}
\tag{5.3}
$$

All of the terms on the left-hand side are nonnegative but the last one. To treat it, we introduce $\gamma$, $\widehat{\gamma} : \mathbb{R} \to \mathbb{R}$ by setting

$$
\gamma(r) := r\,\beta_\varepsilon'(r) \quad \text{and} \quad \widehat{\gamma}(r) := \int_0^r \gamma(s)\,ds \quad \text{for } r \in \mathbb{R},
$$

and notice that $\widehat{\gamma}$ is nonnegative. Hence, we have that

$$
\int_{Q_t} \varphi_n\,\beta_\varepsilon'(\varphi_n)\partial_t\varphi_n = \int_\Omega \widehat{\gamma}(\varphi_n(t)) - \int_\Omega \widehat{\gamma}(\varphi_n(0)) \geq -\int_\Omega \widehat{\gamma}(\varphi_n(0))\,.
$$

On the other hand, by assuming that $n$ is large enough as we did to obtain (4.17), and owing to (5.2), we can estimate, at first $\|\beta_\varepsilon'(\varphi_n(0))\|_\infty$ and then $\|\widehat{\gamma}(\varphi_n(0))\|_\infty$, uniformly with respect to $n$ and $\varepsilon$. Let us come to the right-hand side. While the second and third terms can easily be estimated by means of (4.16), the first one needs some work. We write (4.13) at $t = 0$ and test it by $-\Delta\mu_n(0)$. We obtain that

$$
\int_\Omega |\nabla\mu_n(0)|^2 = \int_\Omega \{\nabla(-\Delta\varphi_n(0)) + (\beta_\varepsilon'(\varphi_n(0)) + \pi'(\varphi_n(0))\nabla\varphi_n(0)\}\cdot\nabla\mu_n(0)\,.
$$

By arguing as in Remark 4.2, we see that $\|\nabla(-\Delta\varphi_n(0))\| \leq c\,\|\varphi_0\|_{H^3(\Omega)}$. Moreover, $\|\beta_\varepsilon'(\varphi_n(0))\|_\infty$ has been already estimated, and $\varphi_n(0)$ is uniformly bounded in $V$ by (4.16). Therefore, by the Schwarz inequality it is clear that

$$
\|\nabla\mu_n(0)\| \leq c\,. \tag{5.4}
$$

Let us proceed and treat the first integral involving $u$. By integrating by parts, and then owing to the Poincaré and Young inequalities and our assumption on $u$, we have for every $\delta > 0$ that

$$
\int_{Q_t} u\,\partial_t\mu_n = \int_\Omega u(t)\mu_n(t) - \int_\Omega u(0)\mu_n(0) - \int_{Q_t} \partial_t u\,\mu_n
$$

$$
\begin{aligned}
&= \int_\Omega u(t)(\mu_n(t) - \overline{\mu_n}(t)) - \int_\Omega u(0)(\mu_n(0) - \overline{\mu_n}(0)) - \int_{Q_t} \partial_t u(\mu_n - \overline{\mu_n}) \\
&\quad + \int_\Omega u(t)\overline{\mu_n}(t) - \int_\Omega u(0)\overline{\mu_n}(0) - \int_{Q_t} \partial_t u \, \overline{\mu_n} \\
&\le \delta \int_\Omega |\nabla \mu_n(t)|^2 + \int_\Omega |\nabla \mu_n(0)|^2 + \int_{Q_t} |\nabla \mu_n|^2 \\
&\quad + M|\Omega| \, |\overline{\mu_n}(t)| + M|\Omega| \, |\overline{\mu_n}(0)| + |\Omega| \int_0^T |\overline{\mu_n}(s)|^2 \, ds + c_\delta \, . \qquad (5.5)
\end{aligned}
$$

Since (5.4) has been established, we just have to estimate $\overline{\mu_n}$ pointwise. We observe that testing (4.13) by $1/|\Omega|$ and using (4.25) yields that

$$
|\overline{\mu_n}(t)| \le |\Omega|^{-1} \|\beta_\varepsilon(\varphi_n(t))\|_1 + c \qquad (5.6)
$$

for every $t \in [0,T]$. On the other hand, we can test (4.13) by $\varphi_n(t) - \overline{\varphi_n}(t)$, and account for (4.22), the Poincaré inequality, and (4.25), to have that

$$
\begin{aligned}
&\delta_0 \, \|\beta_\varepsilon(\varphi_n(t))\|_1 \le \int_\Omega (\mu_n(t) - \overline{\mu_n}(t))(\varphi_n(t) - \overline{\varphi_n}(t)) - \int_\Omega \pi(\varphi_n(t))(\varphi_n(t) - \overline{\varphi_n}(t)) + c \\
&\le c \, \|\nabla \mu_n(t)\| + c \, .
\end{aligned}
$$

By combining, we deduce that

$$
|\overline{\mu_n}(t)| \le c \, \|\nabla \mu_n(t)\| + c \quad \text{for every } t \in [0,T]. \qquad (5.7)
$$

Hence, by also accounting for (4.25) and (5.4), we have that

$$
\begin{aligned}
&M|\Omega| \, |\overline{\mu_n}(t)| \le \delta \int_\Omega |\nabla \mu_n(t)|^2 + c_\delta \, , \\
&|\overline{\mu_n}(0)| \le c \int_\Omega |\nabla \mu_n(0)|^2 + c \le c \quad \text{and} \quad \int_0^T |\overline{\mu_n}(s)|^2 \, ds \le c \int_Q |\nabla \mu_n|^2 + c \le c \, ,
\end{aligned}
$$

and (5.5) yields

$$
\int_{Q_t} u \, \partial_t \mu_n \le 2\delta \int_\Omega |\nabla \mu_n(t)|^2 + c_\delta \, .
$$

Next, we have

$$
L' \int_{Q_t} u \, \partial_t \varphi_n - L' \int_{Q_t} \nabla \mu_n \cdot \nabla \partial_t \varphi_n \le \delta \int_{Q_t} |\partial_t \varphi_n|^2 + \delta \int_{Q_t} |\nabla \partial_t \varphi_n|^2 + c_\delta \, ,
$$

as well as

$$
- \int_{Q_t} \pi'(\varphi_n)|\partial_t \varphi_n|^2 - \int_{Q_t} \varphi_n \, \pi'(\varphi_n) \partial_t \varphi_n \le L \int_{Q_t} |\partial_t \varphi_n|^2 + \delta \int_{Q_t} |\partial_t \varphi_n|^2 + c_\delta \, .
$$

By combining all these estimates and (5.3), recalling that $L' = L + 1$, and choosing $\delta$ small enough, we conclude that

$$
\|\nabla \mu_n\|_{L^\infty(0,T;H)} + \|\partial_t \varphi_n\|_{L^2(0,T;V)} \le c \, . \qquad (5.8)
$$

By also accounting for (5.7), we have that

$$
\|\mu_n\|_{L^\infty(0,T;V)} \le c \, . \qquad (5.9)
$$

**Conclusion.** At this point we can let $n$ tend to infinity and conclude that

$$\|\varphi^\varepsilon\|_{L^\infty(0,T;V)} + \|\partial_t\varphi^\varepsilon\|_{L^2(0,T;V)} + \|\mu^\varepsilon\|_{L^\infty(0,T;V)} \le c\,, \tag{5.10}$$

where $(\varphi^\varepsilon, \mu^\varepsilon)$ is the solution to the approximating problem (4.5)–(4.7) (see Section 4.3). Moreover, testing (4.6) by $\beta_\varepsilon(\varphi^\varepsilon(t))$ for a.a. $t \in (0,T)$, and then using the elliptic regularity theory, we easily infer that

$$\|\beta_\varepsilon(\varphi^\varepsilon(t))\| \le \|\mu^\varepsilon(t) - \pi(\varphi^\varepsilon(t))\| \le c \quad \text{and} \quad \|\varphi^\varepsilon(t)\|_W \le c \quad \text{for a.a. } t \in (0,T),$$

whence

$$\|\beta_\varepsilon(\varphi^\varepsilon)\|_{L^\infty(0,T;H)} + \|\varphi^\varepsilon\|_{L^\infty(0,T;W)} \le c\,.$$

Now, we conclude. Indeed, by letting $\varepsilon$ tend to zero along a proper subsequence, we obtain that the solution $(\varphi, \mu, \xi)$ given by (4.35)–(4.37) has the expected regularity and satisfies estimate (2.28) with a constant $K_2$ as in the statement.

## 5.2 The separation property

In this section, we prove our results regarding the separation property. Our proof of Proposition 2.5 is based on a simple consideration on an elliptic problem. For a given $g \in H$, let us consider the problem of finding a pair $(z, \zeta) \in V \times H$ satisfying

$$\int_\Omega zv + \int_\Omega \nabla z \cdot \nabla v + \int_\Omega \zeta v = \int_\Omega gv \quad \text{for every } v \in V \quad \text{and} \quad \zeta \in \beta(z) \quad \text{a.e. in } \Omega\,. \tag{5.11}$$

Since $\beta$ is maximal monotone, this problem has a unique solution, and its solution is the weak limit in $V \times H$ of $(z_\varepsilon, \beta_\varepsilon(z_\varepsilon))$, where $z_\varepsilon \in V$ solves

$$\int_\Omega z_\varepsilon v + \int_\Omega \nabla z_\varepsilon \cdot \nabla v + \int_\Omega \beta_\varepsilon(z_\varepsilon) v = \int_\Omega gv \quad \text{for every } v \in V\,. \tag{5.12}$$

For the reader's convenience, we give a short detail on our last claim. Since $\beta_\varepsilon$ is monotone and $\beta_\varepsilon(0) = 0$, by testing (5.12) first by $z_\varepsilon$, and then by $\beta_\varepsilon(z_\varepsilon)$, we immediately obtain that

$$\|z_\varepsilon\|_V \le \|g\| \quad \text{and} \quad \|\beta_\varepsilon(z_\varepsilon)\| \le \|g\|\,.$$

Then, letting $\varepsilon$ tend to zero, we have weak convergence in $V$ and strong convergence in $H$ for $z_\varepsilon$, and weak convergence in $H$ for $\beta_\varepsilon(z_\varepsilon)$, so that the limiting pair of $(z_\varepsilon, \beta_\varepsilon(z_\varepsilon))$ solves (5.11).

**Lemma 5.1.** *Assume that $g \in L^\infty(\Omega)$. Then,*

$$\zeta \in L^\infty(\Omega) \quad \textit{and} \quad \|\zeta\|_\infty \le \|g\|_\infty\,. \tag{5.13}$$

*Proof.* For $k \in \mathbb{N}$, we introduce $\zeta_k \in V \cap L^\infty(\Omega)$ by setting

$$\zeta_k := \min\{k, \max\{\beta_\varepsilon(z_\varepsilon), -k\}\}$$

and take $v = |\zeta_k|^{p-2}\zeta_k$ in (5.12), where $p > 2$ is arbitrary. Since $z_\varepsilon\zeta_k \ge 0$, $\nabla z_\varepsilon \cdot \nabla\zeta_k \ge 0$ and $\beta_\varepsilon(z_\varepsilon)\zeta_k \ge |\zeta_k|^2$, by also applying the Hölder and Young inequalities, we obtain that

$$\begin{aligned}\|\zeta_k\|_p^p &\le \int_\Omega z_\varepsilon |\zeta_k|^{p-2}\zeta_k + \int_\Omega \nabla z_\varepsilon \cdot \nabla(|\zeta_k|^{p-2}\zeta_k) + \int_\Omega \beta_\varepsilon(z_\varepsilon)|\zeta_k|^{p-2}\zeta_k = \int_\Omega g|\zeta_k|^{p-2}\zeta_k \\ &\le \|g\|_p \, \||\zeta_k|^{p-2}\zeta_k\|_{p'} \le \frac{1}{p}\,\|g\|_p^p + \frac{1}{p'}\,\||\zeta_k|^{p-2}\zeta_k\|_{p'}^{p'} = \frac{1}{p}\,\|g\|_p^p + \frac{1}{p'}\|\zeta_k\|_p^p\,,\end{aligned}$$

whence $\|\zeta_k\|_p \leq \|g\|_p$. By letting $k$ tend to infinity, we deduce that $\beta_\varepsilon(z_\varepsilon)$ is bounded and that $\|\beta_\varepsilon(z_\varepsilon)\|_\infty \leq \|g\|_\infty$. Thus, $\{\beta_\varepsilon(z_\varepsilon)\}$ has a weak star limit in $L^\infty(\Omega)$ as $\varepsilon \searrow 0$. Since it converges to $\zeta$ weakly in $L^2(\Omega)$, we infer that this limit is $\zeta$. Therefore, due to the weak star lower semicontinuity of the $L^\infty$ norm, we obtain (5.13). $\square$

**Proof of Proposition 2.5.** We rearrange (2.15) and apply the lemma by choosing, for a.a. $t \in (0,T)$, $z = \varphi(t)$ and $g = \varphi(t) + \mu(t) - \pi(\varphi(t))$, so that $\zeta = \xi(t)$. Hence

$$\xi(t) \in L^\infty(\Omega) \quad \text{and} \quad \|\xi(t)\|_\infty \leq \|\varphi(t) + \mu(t) - \pi(\varphi(t))\|_\infty \quad \text{for a.a. } t \in (0,T),$$

that is, (2.29) is valid. The last claim of the statement easily follows. Indeed, if $D(\beta)$ is a bounded open interval, then by maximal monotonicity we have that $\beta^\circ$ tends to infinity at the end-points of $D(\beta)$. Hence, there exists a compact interval $[a,b]$ such that $|\beta^\circ(r)| > \|\xi\|_\infty$ for every $r \in D(\beta) \setminus [a,b]$. Since $\xi \in \beta(\varphi)$ a.e. in $Q$, we also have that $|\xi| \geq |\beta^\circ(\varphi)|$ a.e. in $Q$, whence $\varphi \in [a,b]$ a.e. in $Q$. $\square$

Let us come to the separation property in dimension $d = 2$, i.e., Proposition 2.6, whose assumptions are understood to be in force. In our proof, we employ some ideas and methods from [31], but our argument is rather different. First, we need a new approximation $\beta_\varepsilon$ of $\beta$, since the Yosida regularization does not seem to be suitable for our purpose.

**New regularization.** In the case of the logarithmic potential (1.6), we can choose

$$\beta(r) = \ln \frac{1+r}{1-r} \quad \text{for } r \in D(\beta) = (-1,1),$$

and define $\widehat{\beta}$ accordingly in order that $\widehat{\beta}(0) = 0$. Since $\beta$ is odd, for $\varepsilon \in (0,1)$ we define $\beta_\varepsilon$ to be the odd function on $\mathbb{R}$ that satisfies

$$\begin{aligned} &\beta_\varepsilon(r) = \beta(r) \quad \text{for } 0 \leq r \leq 1-\varepsilon, \\ &\beta_\varepsilon(r) = \beta(1-\varepsilon) + \beta_\varepsilon'(1-\varepsilon)(r-(1-\varepsilon)) \quad \text{for } r > 1-\varepsilon, \end{aligned}$$

so that $\beta_\varepsilon$ is of class $C^1$ and (4.1)–(4.2) are satisfied. We observe that $\beta_\varepsilon$ enjoys some additional regularity: indeed, $\beta_\varepsilon'$ is piecewise Lipschitz continuous and globally continuous, thus Lipschitz continuous. Moreover, if $\widehat{\beta}_\varepsilon$ is the primitive of $\beta_\varepsilon$ that vanishes at the origin, (4.3) is satisfied, too. In the previous sections, we have used one further property of the Yosida regularization. Namely, we had to guarantee that $\xi \in \beta(\varphi)$ by knowing that $\varphi^\varepsilon$ and $\beta_\varepsilon(\varphi^\varepsilon)$ converge to $\varphi$ and $\xi$ strongly in $L^2(Q)$ and weakly in $L^2(Q)$, respectively. This still holds for the new $\beta_\varepsilon$, since for every $v, w \in H$ with $w \in \beta(v)$ (i.e., $|v| < 1$ and $w = \beta(v)$ a.e. in $\Omega$) there exist $\{v_\varepsilon\}$ such that $v_\varepsilon$ and $\beta_\varepsilon(v_\varepsilon)$ strongly converge in $H$ to $v$ and $w$, respectively. Indeed, one can take $v_\varepsilon := \min\{1-\varepsilon, \max\{v_\varepsilon, -1+\varepsilon\}\}$ and observe that the Lebesgue dominated convergence theorem can be applied since it holds a.e. in $\Omega$ that $v_\varepsilon \to v$, $\beta_\varepsilon(v_\varepsilon) \to \beta(v)$, $|v_\varepsilon| \leq |v|$, and $|\beta_\varepsilon(v_\varepsilon)| \leq |\beta(v)|$. All this ensures that the solution (still termed $(\varphi^\varepsilon, \mu^\varepsilon)$, of course) to the new approximating problem (4.5)–(4.7) converges to the unique solution to the original problem in the sense of (4.35)–(4.37) as $\varepsilon$ tends to zero. Hence, new uniform estimates for the new $(\varphi^\varepsilon, \mu^\varepsilon, \beta_\varepsilon(\varphi^\varepsilon))$ provide corresponding estimates for $(\varphi, \mu, \xi)$.

**Two inequalities.** We first prove that

$$\beta_\varepsilon'(r) \leq 2\, e^{|\beta_\varepsilon(r)|} \quad \text{for every } r \in \mathbb{R}. \tag{5.14}$$

It suffices to consider positive values of $r$. If $r < 1-\varepsilon$, then we have that

$$\beta_\varepsilon'(r) = \beta'(r) = \frac{2}{(1-r)(1+r)} \le 2\,\frac{1+r}{1-r} = 2\,e^{\beta(r)} = 2\,e^{\beta_\varepsilon(r)}.$$

If $r \ge 1-\varepsilon$, then we have that

$$\beta_\varepsilon'(r) = \beta'(1-\varepsilon) = \frac{2}{\varepsilon(2-\varepsilon)} \le \frac{2}{\varepsilon} = 2\,e^{-\ln\varepsilon} \le 2\,e^{\ln(2-\varepsilon)-\ln\varepsilon} = 2\,e^{\beta(1-\varepsilon)} \le 2\,e^{\beta_\varepsilon(r)}.$$

Hence, (5.14) is valid.

Now, we prove that, for every $p \ge 1$, there exist two positive constants $\kappa$ and $\kappa'$ such that

$$rs\,e^{ps} \le \frac{1}{2}\,s^2\,e^{ps} + e^{\kappa r} + \kappa' \quad \text{for every } r,s \ge 0. \tag{5.15}$$

To prove this claim, consider the function $\psi : \mathbb{R} \to (-\infty,+\infty]$ defined by

$$\psi(s) := (1+s)\ln(1+s) - s \ \text{ if } \ s > -1$$

and extended by $1$ and $+\infty$ at $s=-1$ and for $s<-1$, respectively. Then, $\psi$ is convex and l.s.c., and a direct computation easily shows that its conjugate function is given by $\psi^*(r) = e^r - r - 1$ for $r \in \mathbb{R}$. Then, the Young inequality $r's' \le \psi(s') + \psi^*(r')$ holds true for every $r', s' \in \mathbb{R}$. In particular, if $r,s \ge 0$ and $\delta > 0$, then we have that

$$rs\,e^{ps} \le \psi(\delta\,se^{ps}) + \psi^*(r/\delta) \le \psi(\delta\,se^{ps}) + e^{r/\delta}\,.$$

On the other hand, we also have that

$$\begin{aligned}
&\psi(\delta s\,e^{ps}) \le (1+\delta s\,e^{ps})\ln(1+\delta s\,e^{ps}) \\
&\le (1+\delta s\,e^{ps})\ln(e^{ps}+\delta s\,e^{ps}) = (1+\delta s\,e^{ps})(ps + \ln(1+\delta\,s)) \\
&\le (1+\delta s\,e^{ps})(ps+\delta s) = s(p+\delta) + \delta(p+\delta)s^2\,e^{ps} \\
&\le \delta\,s^2 + \frac{(p+\delta)^2}{4\delta} + \delta(p+\delta)s^2 e^{ps} \le \delta(1+p+\delta)s^2 e^{ps} + \frac{(p+\delta)^2}{4\delta}\,.
\end{aligned}$$

Then, (5.15) follows if we choose $\delta > 0$ such that $\delta(1+p+\delta) = 1/2$, and set $\kappa := 1/\delta$ and $\kappa' := (p+\delta)^2/(4\delta)$.

**The basic estimate.** Here is the most important change with respect to [31], since we come back to the discrete problem (4.12)–(4.14) instead of directly dealing with problem (2.14)–(2.16). We can take advantage of all of the estimates of Section 4.2. First, we notice that $\beta_\varepsilon + \pi$ is a $C^1$ function and recall that $u \in H^1(0,T;H)$. It follows that $\varphi_n$ and $\mu_n$ are functions in $H^2(0,T;H)$. Then, we can differentiate both (4.12) and (4.13) with respect to time and test the resulting inequalities by $\partial_t\varphi_n$ and $\Delta\partial_t\varphi_n$, respectively. If we sum up and integrate by parts and over $(0,t)$, then a cancellation occurs, and we obtain that

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\partial_t\varphi_n(t)|^2 + \int_{Q_t} |\partial_t\varphi_n|^2 + \int_{Q_t} |\Delta\partial_t\varphi_n|^2 \\
&= \frac{1}{2}\int_\Omega |\partial_t\varphi_n(0)|^2 + \int_{Q_t} \partial_t u\,\partial_t\varphi_n + \int_{Q_t} (\beta_\varepsilon' + \pi')(\varphi_n)\,\partial_t\varphi_n\,\Delta\partial_t\varphi_n\,.
\end{aligned} \tag{5.16}$$

We estimate the first term on the right-hand side later on. The next term can be treated in an obvious way by the Young inequality, while the last one needs some work. We have that

$$\int_{Q_t} (\beta_\varepsilon' + \pi')(\varphi_n)\, \partial_t\varphi_n \, \Delta\partial_t\varphi_n \le \int_0^t \|(\beta_\varepsilon' + \pi')(\varphi_n(s))\|_3 \, \|\partial_t\varphi_n(s)\|_6 \, \|\Delta\partial_t\varphi_n(s)\| \, ds$$
$$\le \frac{1}{4} \int_{Q_t} |\Delta\partial_t\varphi_n|^2 + \int_0^t \|(\beta_\varepsilon' + \pi')(\varphi_n(s))\|_3^2 \, \|\partial_t\varphi_n(s)\|_6^2 \, ds \,,$$

and we have to estimate the last integral. We have a.e. in $(0,T)$ that

$$\|\partial_t\varphi_n\|_6^2 \le c\,\|\partial_t\varphi_n\|_V^2 \le c\, \Big( |\partial_t\overline{\varphi_n}|^2 + \int_\Omega |\nabla\partial_t\varphi_n|^2 \Big)$$
$$\le c + c \Big| \int_\Omega \partial_t\varphi_n(-\Delta\partial_t\varphi_n) \Big| \le c + c\,\|\partial_t\varphi_n\| \, \|\Delta\partial_t\varphi_n\| \,,$$

and we deduce that

$$\int_0^t \|(\beta_\varepsilon' + \pi')(\varphi_n(s))\|_3^2 \, \|\partial_t\varphi_n(s)\|_6^2 \, ds$$
$$\le c \int_0^t \|(\beta_\varepsilon' + \pi')(\varphi_n(s))\|_3^2 \, (c + c\,\|\partial_t\varphi_n(s)\| \, \|\Delta\partial_t\varphi_n(s)\|) \, ds$$
$$\le c \int_0^t \|(\beta_\varepsilon' + \pi')(\varphi_n(s))\|_3^2 \, ds$$
$$\quad + \frac{1}{4} \int_{Q_t} |\Delta\partial_t\varphi_n|^2 + c \int_0^t \|(\beta_\varepsilon' + \pi')(\varphi_n(s))\|_3^4 \, \|\partial_t\varphi_n(s)\|^2 \, ds$$
$$\le c \int_0^t \left( \|\beta_\varepsilon'(\varphi_n(s))\|_3^2 + 1 \right) \, ds$$
$$\quad + \frac{1}{4} \int_{Q_t} |\Delta\partial_t\varphi_n|^2 + c \int_0^t (1 + \|\beta_\varepsilon'(\varphi_n(s))\|_3^4) \|\partial_t\varphi_n(s)\|^2 \, ds \,.$$

By combining these estimates with (5.16) and applying the Gronwall lemma, we conclude that

$$\|\partial_t\varphi_n\|_{L^\infty(0,T;H)}^2 + \|\Delta\partial_t\varphi_n\|_{L^2(0,T;H)}^2$$
$$\le c\,(\|\partial_t\varphi_n(0)\|^2 + \|\partial_t u\|_{L^2(0,T;H)}^2 + \|\beta_\varepsilon'(\varphi_n)\|_{L^2(0,T;L^3(\Omega))}^2 + 1) \times$$
$$\quad \times \exp\Big( c \int_0^T (1 + \|\beta_\varepsilon'(\varphi_n(s)\|_3^4) ds \Big) \,,$$

whence also

$$\|\partial_t\varphi_n\|_{L^\infty(0,T;H)}^2 + \|\Delta\partial_t\varphi_n\|_{L^2(0,T;H)}^2$$
$$\le c\,(\|\partial_t\varphi_n(0)\|^2 + \|\beta_\varepsilon'(\varphi_n)\|_{L^2(0,T;L^3(\Omega))}^2 + 1)\, e^{c\,\|\beta_\varepsilon'(\varphi_n)\|_{L^4(0,T;L^3(\Omega))}^4} \,. \tag{5.17}$$

In order to let $n$ tend to infinity, we have to estimate the $H$ norm of $\partial_t\varphi_n(0)$. To this end, we set $\varepsilon_0 := (1 - \|\varphi_0\|_\infty)/2$, assume that $\varepsilon \le \varepsilon_0$, and recall that $\varphi_n(0)$ converges uniformly to $\varphi_0$ (see also Remark 4.2). Hence, we can assume $n$ large enough in order that $\|\varphi_n(0)\| \le 1 - \varepsilon_0$. On the other hand, $(\beta_\varepsilon + \pi)(r) = f(r)$, the whole logarithmic potential, for $|r| \le 1 - \varepsilon_0$, and $f$ is smooth in $(-1,1)$. Therefore, $(\beta_\varepsilon+\pi)(\varphi_n(0)) = f(\varphi_n(0))$ is as smooth as $\varphi_n(0)$ (i.e., as the eigenfunctions $e_j$), and, for $0\le s\le 4$, the norm of $f(\varphi_n(0))$ in $H^s(\Omega)$ can be uniformly estimated by the corresponding norm of $\varphi_n(0)$, thus by the norm of $\varphi_0$, due to our further assumption (2.30) (see Remark 4.2 once more). At this point, we start estimating. We test (4.12) and (4.13), written with $t=0$, by $\partial_t\varphi_n(0)$ and $\Delta\partial_t\varphi_n(0)$, respectively, and sum up. Since the terms involving $\mu_n(0)$ cancel each other, with the help of some integrations by parts we obtain that

$$\begin{aligned}
&\|\partial_t\varphi_n(0)\|^2 = \int_\Omega |\partial_t\varphi_n(0)|^2 \\
&= \int_\Omega (u(0)-\varphi_n(0))\,\partial_t\varphi_n(0) + \int_\Omega \nabla\varphi_n(0)\cdot\nabla\Delta\partial_t\varphi_n(0) + \int_\Omega f(\varphi_n(0))\Delta\partial_t\varphi_n(0) \\
&= \int_\Omega \{u(0)-\varphi_n(0)-\Delta^2\varphi_n(0)+\Delta(f(\varphi_n(0))\}\partial_t\varphi_n(0) \\
&\le c\,\|\partial_t\varphi_n(0)\|\,.
\end{aligned}$$

At this point, we are ready to let $n$ tend to infinity in (5.17). Indeed, by recalling (4.28) and applying, e.g, [45, Sect. 8, Cor. 4]), we have that $\varphi_n$ converges to $\varphi^\varepsilon$ strongly in $C^0([0,T];L^3(\Omega))$. Since $\beta'_\varepsilon$ is Lipschitz continuous, this implies that $\beta'_\varepsilon(\varphi_n)$ converges to $\beta'_\varepsilon(\varphi^\varepsilon)$ in the same topology. As a consequence, the norm of $\beta'_\varepsilon(\varphi_n)$ in $L^p(0,T;L^3(\Omega))$ converges to the corresponding norm of $\beta'_\varepsilon(\varphi^\varepsilon)$ for $p=2$ and $p=4$, so that, taking the limit in (5.17), we obtain an inequality. Then, by estimating the above norms with the norm in $L^\infty(0,T;L^3(\Omega))$, we conclude that

$$\begin{aligned}
&\|\partial_t\varphi^\varepsilon\|^2_{L^\infty(0,T;H)} + \|\Delta\partial_t\varphi^\varepsilon\|^2_{L^2(0,T;H)} \\
&\le c\,(1+\|\beta'_\varepsilon(\varphi^\varepsilon)\|^2_{L^\infty(0,T;L^3(\Omega))})\,e^{c\,\|\beta'_\varepsilon(\varphi^\varepsilon)\|^4_{L^\infty(0,T;L^3(\Omega))}}\,. \qquad (5.18)
\end{aligned}$$

**Conclusion of the proof of Proposition 2.6.** Our aim is to derive a bound in $L^\infty(0,T;H)$ for $\partial_t\varphi$. To this end, we estimate the right-hand side of (5.18). Here, we follow [31] rather closely. We set $\varphi^\varepsilon_k := \min\{k,\max\{\varphi^\varepsilon,-k\}\}$ and $\Psi_\varepsilon(r) := \beta_\varepsilon(r)e^{3|\beta_\varepsilon(r)|}$ for $r\in\mathbb{R}$ and observe that $v=\Psi_\varepsilon(\varphi^\varepsilon_k(t))$ is admissible in (4.6) for a.a. $t\in(0,T)$. Therefore, a.e. in $(0,T)$ we have that

$$\int_\Omega \Psi'_\varepsilon(\varphi^\varepsilon_k)\nabla\varphi^\varepsilon\cdot\nabla\varphi^\varepsilon_k + \int_\Omega \beta_\varepsilon(\varphi^\varepsilon)\Psi_\varepsilon(\varphi^\varepsilon_k) = \int_\Omega g_\varepsilon\Psi_\varepsilon(\varphi^\varepsilon_k)\,,$$

where, for brevity, we have set $g_\varepsilon=\mu^\varepsilon-\pi(\varphi^\varepsilon)$. Since $\Psi'_\varepsilon$ is nonnegative, $\nabla\varphi^\varepsilon\cdot\nabla\varphi^\varepsilon_k = |\nabla\varphi^\varepsilon_k|^2$ a.e., and $\beta_\varepsilon(\varphi^\varepsilon)\beta_\varepsilon(\varphi^\varepsilon_k)\ge|\beta_\varepsilon(\varphi^\varepsilon_k)|^2$ a.e., we deduce that

$$\int_\Omega |\beta_\varepsilon(\varphi^\varepsilon_k)|^2\,e^{3|\beta_\varepsilon(\varphi^\varepsilon_k)|} \le \int_\Omega g_\varepsilon\Psi_\varepsilon(\varphi^\varepsilon_k)\,.$$

We estimate the right-hand side using the inequality (5.15) with $p=3$, $r=|g_\varepsilon|$, and $s=|\beta_\varepsilon(\varphi^\varepsilon_k)|$, and have that

$$\int_\Omega g_\varepsilon\Psi_\varepsilon(\varphi^\varepsilon_k) \le \int_\Omega |g_\varepsilon|\,|\beta_\varepsilon(\varphi^\varepsilon_k)|e^{3|\beta_\varepsilon(\varphi^\varepsilon_k)|} \le \frac12\int_\Omega |\beta_\varepsilon(\varphi^\varepsilon_k)|^2\,e^{3|\beta_\varepsilon(\varphi^\varepsilon_k)|} + \int_\Omega (e^{\kappa|g_\varepsilon|}+\kappa').$$

On the other hand, by observing that $(1-r^2)e^r\le e$ for every $r\in\mathbb{R}$, we also have that

$$\int_\Omega e^{3|\beta_\varepsilon(\varphi^\varepsilon_k)|} \le 9\int_\Omega |\beta_\varepsilon(\varphi^\varepsilon_k)|^2 e^{3|\beta_\varepsilon(\varphi^\varepsilon_k)|} + c\,.$$

By combining all this with (5.14), we infer that

$$\int_\Omega |\beta'_\varepsilon(\varphi_k^\varepsilon)|^3 \le c \int_\Omega e^{3|\beta_\varepsilon(\varphi_k^\varepsilon)|} \le c \int_\Omega e^{\kappa|g_\varepsilon|} + c\,.$$

At this point, we recall the Trudinger inequality (see, e.g., [39])

$$\int_\Omega e^{|v|} \le C_\Omega\, e^{C_\Omega\,\|v\|_V^2} \quad \text{for every } v \in V\,,$$

which holds since we are supposing that $d = 2$, and where the constant $C_\Omega$ only depends on $\Omega$. Hence, we conclude that

$$\int_\Omega |\beta'_\varepsilon(\varphi_k^\varepsilon)|^3 \le c \int_\Omega e^{\kappa|g_\varepsilon|} + c \le c\, e^{\kappa^2 C_\Omega \|\mu^\varepsilon - \pi(\varphi^\varepsilon)\|_V^2} \quad \text{a.e. in } (0,T)\,.$$

By accounting for (4.32) and (5.10), in addition we infer that

$$\|\beta'_\varepsilon(\varphi_k^\varepsilon)\|_{L^\infty(0,T;L^3(\Omega))} \le c\,, \quad \text{whence also} \quad \|\beta'_\varepsilon(\varphi^\varepsilon)\|_{L^\infty(0,T;L^3(\Omega))} \le c\,.$$

At this point, we can take the limit in (5.18) and deduce that $\partial_t\varphi$ and $\Delta\partial_t\varphi$ belong to $L^\infty(0,T;H)$ and $L^2(0,T;H)$, respectively. Therefore, $\partial_t\varphi$ belongs to $L^2(0,T;W)$. Moreover, elliptic regularity in (2.14) implies that $\mu$ belongs to $L^\infty(0,T;W) \subset L^\infty(Q)$. Since $\pi(\varphi)$ is bounded, too, by applying Proposition 2.5, we obtain that $\xi$ is bounded. This concludes the proof of (2.31). Furthermore, it is clear that (2.32) holds true as well, with a constant $K_3$ as in the statement. Thus, the proof of Proposition 2.6 is complete.

# 6 The control problem

In this section, we investigate the optimal control problem (2.38). We remark that we will use some notations already utilized in the previous sections with a different meaning (e.g., $\xi$ in the linearized system introduced later on). However, no confusion can arise.

## 6.1 Proof of Theorem 2.9

We assume first that $\alpha_3 = 0$ in which case the cost functional is independent of the solution variables $\mu, \xi$. By Theorem 2.3, the mapping $\mathcal{U}_{ad} \ni u \mapsto \varphi$ is well defined, and thus also the cost functional. Now we pick any minimizing sequence $\{((\varphi_n, \mu_n, \xi_n), u_n)\}$ for the optimal control problem, that is, in particular, $(\varphi_n, \mu_n, \xi_n)$, where $\xi_n \in \beta(\varphi_n)$, satisfies (2.14)–(2.16) with right-hand side $u = u_n$ for all $n \in \mathbb{N}$. Owing to (2.36) and the global estimate (2.22), we may without loss of generality assume that there are limits $\varphi, \mu, \xi, u$ such that

$$u_n \to u \qquad \text{weakly star in } H^1(0,T;H) \cap L^\infty(Q), \tag{6.1}$$

$$\varphi_n \to \varphi \qquad \text{weakly star in } H^1(0,T;V^*) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \text{ and strongly in } C^0([0,T];H), \tag{6.2}$$

$$\mu_n \to \mu \qquad \text{weakly in } L^2(0,T;V), \tag{6.3}$$

$$\xi_n \to \xi \qquad \text{weakly in } L^2(0,T;H), \tag{6.4}$$

where the strong convergence in (6.2) follows from, e.g., [45, Sect. 8, Cor. 4]. Obviously, we have that $u \in \mathcal{U}_{ad}$. Now, by also recalling Remark 2.1, we can pass to the limit as $n \to \infty$ in the state system (2.14)–(2.16), written with right-hand side $u_n$, to see that $(\varphi, \mu, \xi)$ solves (2.14)–(2.16) with right-hand side $u$. In view of (6.4) and the strong convergence stated in (6.2), and since the extension of $\beta$ to $L^2(Q)$ is maximal monotone, a standard argument then yields that $\xi \in \beta(\varphi)$. The pair $((\varphi, \mu, \xi), u)$ is thus admissible for (2.38), and the weak sequential lower semicontinuity properties of the cost functional yield that it is a minimizer.

Suppose now that $\beta$ is single-valued, in which case the cost functional is well defined also if $\alpha_3 > 0$. Then the above line of argumentation can obviously be repeated, only that $\xi_n = \beta(\varphi_n)$ and $\xi = \beta(\varphi)$ in this case. The assertion is thus proved. □

## 6.2 Fréchet differentiability of the control-to-state operator

In this section, we show the Fréchet differentiability of the control-to-state operator $\mathcal{S}$ introduced in (2.44). To this end, we recall the definition of $\mathcal{U}_R$ given in (2.40) and the assumptions (2.42)–(2.43) which are assumed to hold throughout the remainder of this section. Observe that they are fulfilled in the case $d = 3$ for everywhere defined smooth potentials and in the case $d = 2$ for the logarithmic potential (1.6) under the assumptions of Proposition 2.6.

Now let some $u^* \in \mathcal{U}_R$ be fixed and $(\varphi^*, \mu^*) = \mathcal{S}(u^*)$ be the associated state. We then consider the corresponding linearized system, that is, given $h \in \mathcal{X}$, we seek a pair $(\xi, \eta)$ satisfying

$$\xi \in H^1(0,T;V^*) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \quad \text{and} \quad \eta \in L^2(0,T;V)\,, \tag{6.5}$$

$$\langle \partial_t \xi(t), v\rangle + \int_\Omega \xi(t) v + \int_\Omega \nabla\eta(t) \cdot \nabla v = \int_\Omega h(t) v$$
$$\quad \text{for a.a. } t \in (0,T) \text{ and every } v \in V\,, \tag{6.6}$$

$$\int_\Omega \eta(t) v = \int_\Omega \nabla\xi(t) \cdot \nabla v + \int_\Omega f''(\varphi^*(t))\xi(t) v$$
$$\quad \text{for a.a. } t \in (0,T) \text{ and every } v \in V\,, \tag{6.7}$$

$$\xi(0) = 0\,. \tag{6.8}$$

We have the following result.

**Lemma 6.1.** *Suppose that the conditions* (2.42)–(2.43) *are fulfilled, and let $u^* \in \mathcal{U}_R$ be given with associated state $(\varphi^*, \mu^*) = \mathcal{S}(u^*)$. Then the system* (6.6)–(6.8) *has for every $h \in \mathcal{X}$ a unique solution $(\xi, \eta)$ with the regularity* (6.5). *Moreover, the linear mapping $h \mapsto (\xi, \eta)$ is continuous from $\mathcal{X}$ into $\mathcal{Y}$.*

*Proof.* The existence proof is similar to that for the state system, and we just provide a sketch, leaving the details to the reader. Indeed, the system (6.6)–(6.8) has the same structure for $(\xi, \eta)$ as the state system (2.14)–(2.16) for $(\varphi, \mu)$, only that we have zero initial conditions here and the term $f''(\varphi^*)\xi$ in place of $\beta(\varphi) + \pi(\varphi)$ (recall that $\beta$ is single-valued). We thus may argue as in Section 4: one approximates the system (6.6)–(6.8) by a Faedo–Galerkin scheme using the eigenfunctions (4.8) and the subspaces (4.9). Since the term $f''(\varphi^*)\xi$, where $f''(\varphi^*) \in L^\infty(Q)$, is much easier to handle than the nonlinearities $\beta$ and $\pi$, similar, but in comparison with the state system considerably simpler, a priori estimates can be performed on the Faedo–Galerkin system, yielding the bound (compare

(2.22))

$$\|\xi_n\|_{H^1(0,T;V^*)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} + \|\eta_n\|_{L^2(0,T;V)} \le c \quad \forall\, n \in \mathbb{N}.$$

Using compactness arguments, we can pass to the limit as $n \to \infty$ in the Faedo–Galerkin system on a subsequence in order to establish the existence result.

We only prove that the mapping $h \mapsto (\xi, \eta)$ has the asserted continuity property. To this end, let $h \in \mathcal{X}$ be fixed and $(\xi, \eta)$ be an associated solution to (6.6)–(6.8) with the regularity (6.5). We then insert $v = \xi$ in (6.6) and $v = \eta$ in (6.7), add the resulting identities, and integrate over $(0, t)$ for arbitrary $t \in (0, T]$. After a cancellation of two terms, we then obtain that

$$\begin{aligned} &\frac{1}{2}\,\|\xi(t)\|^2 + \int_{Q_t} |\xi|^2 + \int_{Q_t} |\eta|^2 = \int_{Q_t} h\,\xi - \int_{Q_t} f''(\varphi^*)\,\xi\,\eta \\ &\le \frac{1}{2}\int_{Q_t} |\eta|^2 + c\int_{Q_t} (|\xi|^2 + |h|^2)\,. \end{aligned}$$

Now observe that, by virtue of continuous embedding, we obviously have that $\xi \in C^0([0,T];H)$. Therefore, Gronwall's lemma yields that

$$\|\xi\|_{C^0([0,t];H)} + \|\eta\|_{L^2(0,t;H)} \le c\,\|h\|_{L^2(0,t;H)} \quad \text{for all } t \in [0,T]. \tag{6.9}$$

Finally, applying elliptic regularity theory to (6.7), we may also conclude that

$$\|\xi\|_{L^2(0,t;W)} \le c\,\|h\|_{L^2(0,t;H)} \quad \text{for all } t \in [0,T]. \tag{6.10}$$

With this, the asserted continuity property is shown. Finally, it is easily seen that the inequality (6.9) also implies the uniqueness of the solution: indeed, by using linearity, if $(\xi, \eta)$ solves (6.6)–(6.8) with $h = 0$, then (6.9) implies that $\xi = \eta = 0$. $\square$

**Remark 6.2.** The existence proof sketched above yields that a unique solution to (6.6)–(6.8) with the regularity (6.5) also exists if we only have $h \in L^2(Q)$ and that also in this situation the continuity property shown above is valid.

We now show the following Fréchet differentiability result.

**Theorem 6.3.** *Suppose that the conditions* (2.42)–(2.43) *are fulfilled, and let $u^* \in \mathcal{U}_R$ be given with associated state $(\varphi^*, \mu^*) = \mathcal{S}(u^*)$. Then the operator $\mathcal{S}$ is Fréchet differentiable at $u^*$ as a mapping from $\mathcal{X}$ into $\mathcal{Y}$, and the Fréchet derivative $D\mathcal{S}(u^*) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ acts as follows: for every $h \in \mathcal{X}$, the value $D\mathcal{S}(u^*)(h)$ is given by the solution $(\xi, \eta)$ to the linearized system* (6.6)–(6.8)*.*

*Proof.* Since $u^* \in \mathcal{U}_R$, it follows that there is some $\Lambda > 0$ such that $u^* + h \in \mathcal{U}_R$ whenever $\|h\|_{\mathcal{X}} \le \Lambda$. In the following, we only consider such perturbations $h \in \mathcal{X}$. In the remainder of this proof, we denote by the small-case symbol $c$ constants that may depend on the data of the system but not on the choice of $h \in \mathcal{X}$ with $\|h\|_{\mathcal{X}} \le \Lambda$. We then define

$$(\varphi^h, \mu^h) := \mathcal{S}(u^* + h), \quad y^h := \varphi^h - \varphi^* - \xi^h, \quad z^h := \mu^h - \mu^* - \eta^h, \tag{6.11}$$

where $(\xi^h, \eta^h)$ denotes the unique solution to the linearized system (6.6)–(6.8). We then have

$$y^h \in H^1(0,T;V^*) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \quad z^h \in L^2(0,T;V). \tag{6.12}$$

Now, by virtue of Lemma 6.1, the linear mapping $h \mapsto (\xi^h, \eta^h)$ is continuous between $\mathcal{X}$ and $\mathcal{Y}$. According to the definition of Fréchet differentiability, it therefore suffices to construct a function $G : (0,\Lambda) \to (0,\infty)$ that satisfies $\lim_{\lambda \searrow 0} G(\lambda)/(\lambda^2) = 0$ and

$$\|(y^h, z^h)\|_{\mathcal{Y}}^2 \,\le\, G(\|h\|_\infty)\,. \tag{6.13}$$

We are going to show that we may choose $G(\lambda) = \widehat{c}\,\lambda^4$ with some sufficiently large $\widehat{c} > 0$.

Next, it is easily seen, using Taylor's theorem with integral remainder, that $(y^h, z^h)$ is a solution to the system

$$\langle \partial_t y^h(t), v\rangle + (y^h(t), v) + (\nabla z^h(t), \nabla v) = 0, \quad \text{for a.a. } t \in (0,T) \text{ and all } v \in V, \tag{6.14}$$

$$(z^h(t), v) = (\nabla y^h(t), \nabla v) + (f''(\varphi^*(t))\, y^h(t), v) + (R^h(t)(\varphi^h(t) - \varphi^*(t))^2, v), \quad \text{for a.a. } t \in (0,T) \text{ and all } v \in V, \tag{6.15}$$

$$y^h(0) = 0 \quad \text{a.e. in } \Omega, \tag{6.16}$$

with the remainder

$$R^h(t) := \int_0^1 (1-s)\, f'''(\varphi^*(t) + \, s\,(\varphi^h(t) - \varphi^*(t)))\, ds\,.$$

Observe that $\varphi^* + s(\varphi^h - \varphi^*) \in [a,b]$ almost everywhere in $Q$ for all $s \in [0,1]$, and thus

$$\|R^h\|_\infty \,\le\, c \quad \text{for all } h \in \mathcal{X} \text{ satisfying } \|h\|_{\mathcal{X}} \le \Lambda. \tag{6.17}$$

Also, we conclude from (2.33) in Theorem 2.8 that the estimate

$$\|\varphi^h - \varphi^*\|_{C^0([0,t];H)\cap L^2(0,t;W)} \,+\, \|\mu^h - \mu^*\|_{L^2(0,t;H)} \,\le\, C_2\, \|h\|_{L^2(0,T;H)} \tag{6.18}$$

is valid for all $t \in (0,T]$ and all admissible perturbations $h$. Furthermore, taking $v = 1$ in (6.14) and accounting for (6.16), we immediately see that

$$\overline{y^h}(t) = 0 \quad \text{for all } t \in [0,T]. \tag{6.19}$$

Therefore, we may insert $v = \mathcal{N}(y^h)(t)$ in (6.14) and $v = -y^h(t)$ in (6.15), add the resulting identities, and integrate over time. Noting an obvious cancellation, we then obtain that

$$\frac{1}{2}\,\|y^h(t)\|_*^2 + \int_0^t \|y^h(s)\|_*^2\, ds \,+\, \int_{Q_t} |\nabla y^h|^2 = -\int_{Q_t} f''(\varphi^*)\, |y^h|^2 - \int_{Q_t} R^h\,(\varphi^h - \varphi^*)^2\, y^h \,=:\, I_1 + I_2, \tag{6.20}$$

with obvious meaning. Now, owing to (2.43) and by virtue of the compactness inequality (2.53), we have that

$$I_1 \,\le\, c\int_0^t \|y^h(s)\|^2\, ds \,\le\, \frac{1}{4}\int_{Q_t} |\nabla y^h|^2 \,+\, c\int_0^t \|y^h(s)\|_*^2\, ds\,. \tag{6.21}$$

Moreover, by also using Young's and Hölder's inequalities, (6.18), and (6.17), we see that

$$\begin{aligned}
I_2 \;&\le\, c\int_0^t \|(\varphi^h - \varphi^*)(s)\|_4\, \|(\varphi^h - \varphi^*)(s)\|_2\, \|y^h(s)\|_4\, ds \\
&\le\, c\,\|\varphi^h - \varphi^*\|_{C^0([0,t];H)} \int_0^t \|(\varphi^h - \varphi^*)(s)\|_V\, \|y^h(s)\|_V\, ds \\
&\le\, c\,\|h\|_{L^2(0,t;H)}\, \|\varphi^h - \varphi^*\|_{L^2(0,t;V)}\, \|y^h\|_{L^2(0,t;V)} \\
&\le\, \frac{1}{4}\int_{Q_t} |\nabla y^h|^2 + c\int_0^t \|y^h(s)\|_*^2\, ds \,+\, c\|h\|_{L^2(0,T;H)}^4\,.
\end{aligned} \tag{6.22}$$

Combining (6.20)–(6.22) with Gronwall's lemma, we thus have shown that

$$\|y^h\|^2_{L^\infty(0,t;V^*)\cap L^2(0,t;V)} \le c\,\|h\|^4_{L^2(0,T;H)} \quad \text{for all } t\in[0,T]. \tag{6.23}$$

The next step is to insert $v = y^h(t)$ in (6.14) and $v = z^h(t)$ in (6.15), add the resulting equations, and integrate over time. This yields the identity

$$\begin{aligned}&\frac{1}{2}\,\|y^h(t)\|^2 + \int_{Q_t} |y^h|^2 + \int_{Q_t} |z^h|^2\\ &= \int_{Q_t} f''(\varphi^*)\,y^h\,z^h + \int_{Q_t} R^h\,(\varphi^h-\varphi^*)^2\,z^h := I_3 + I_4\,. \end{aligned}\tag{6.24}$$

Clearly, by (2.43) and Young's inequality we have that

$$I_3 \le \frac{1}{4}\int_{Q_t} |z^h|^2 + c\int_{Q_t} |y^h|^2\,. \tag{6.25}$$

Moreover, also using (6.17), (6.18), and Hölder's inequality, we find that

$$\begin{aligned} I_4 &\le c\int_0^t \|(\varphi^h-\varphi^*)(s)\|_\infty\,\|(\varphi^h-\varphi^*)(s)\|_2\,\|z^h(s)\|_2\,ds\\ &\le \frac{1}{4}\int_{Q_t}|z^h|^2 + c\,\|\varphi^h-\varphi^*\|^2_{C^0([0,t];H)}\int_0^t \|(\varphi^h-\varphi^*)(s)\|_W^2\,ds\\ &\le \frac{1}{4}\int_{Q_t}|z^h|^2 + c\,\|h\|^4_{L^2(0,T;H)}\,. \end{aligned}\tag{6.26}$$

Now recall that $y^h \in C^0([0,T];H)$. Thus, combining (6.24)–(6.26) with Gronwall's lemma, and invoking (6.23), we have finally shown that

$$\|y^h\|^2_{C^0([0,T];H)\cap L^2(0,T;V)} + \|z^h\|^2_{L^2(0,T;H)} \le c_0\,\|h\|^4_{L^2(0,T;H)}\,,$$

with a sufficiently large constant $c_0 > 0$. Since $\|h\|_{L^2(0,T;H)} \le T^{1/2}\,|\Omega|^{1/2}\,\|h\|_\infty$, the condition (6.13) is fulfilled with $G(\lambda) = \widehat{c}\lambda^4$ and $\widehat{c} = T^2\,|\Omega|^2\,c_0$. The assertion is thus proved. □

## 6.3 Necessary conditions for optimality

In this section, we establish first-order necessary optimality conditions for the optimal control problem. Using Theorem 6.3 and the chain rule, a standard argument yields the following result.

**Proposition 6.4.** *Suppose that the conditions* (2.42)–(2.43) *are fulfilled, and let $u^* \in \mathcal{U}_{ad}$ be an optimal control with associated state $(\varphi^*,\mu^*) = \mathcal{S}(u^*)$. Then it holds the variational inequality*

$$\begin{aligned}&\alpha_1\int_Q(\varphi^*-\varphi_Q)\xi + \alpha_2\int_\Omega(\varphi^*(T)-\varphi_\Omega)\xi(T) + \alpha_3\int_Q(\mu^*-\mu_Q)\eta\\ &+\alpha_4\int_Q u^*(u-u^*) \ge 0 \quad \forall\,u\in\mathcal{U}_{ad}\,,\end{aligned}\tag{6.27}$$

*where $(\xi,\eta)$ is the unique solution to the linearized system* (6.6)–(6.8) *with $h = u - u^*$.*

As usual, we now eliminate the variables $(\xi, \eta)$ from (6.27) by means of the solution $(p, q)$ to the associated adjoint system, which is given by (2.46)–(2.48). We have the following result.

**Lemma 6.5.** *Suppose that the conditions* (2.42)–(2.43) *are fulfilled, let* $u^* \in \mathcal{U}_{ad}$ *be an optimal control with associated state* $(\varphi^*, \mu^*) = \mathcal{S}(u^*)$*, and assume that the regularity conditions* (2.49)–(2.50) *are satisfied. Then the adjoint system* (2.46)–(2.48) *has a unique solution* $(p, q)$ *such that*

$$p \in H^1(0,T;V^*) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \quad q \in L^2(0,T;V). \tag{6.28}$$

*Proof.* Again, we employ a Faedo–Galerkin approximation with the eigenfunctions defined in (4.8) and the $n$-dimensional subspaces $V_n$ from (4.9). We then look for functions

$$p_n(x,t) = \sum_{i=1}^n p_n^i(t) e_i(x), \quad q_n(x,t) = \sum_{i=1}^n q_n^i(t) e_i(x),$$

that solve the final value problem

$$(-\partial_t p_n(t), v) + (p_n(t), v) + (\nabla q_n(t), \nabla v) \;=\; -(f''(\varphi^*(t)) q_n(t), v) + (g_1(t), v)$$
$$\text{for a.a. } t \in (0,T) \text{ and every } v \in V_n\,, \tag{6.29}$$
$$(q_n(t), v) \;=\; (\nabla p_n(t), \nabla v) - (g_3(t), v)$$
$$\text{for a.a. } t \in (0,T) \text{ and every } v \in V_n\,, \tag{6.30}$$
$$(p_n(T), v) = (g_2, v) \quad \text{for every } v \in V_n\,, \tag{6.31}$$

where, owing to the regularity of $(\varphi^*, \mu^*)$ and (2.49),

$$g_1 = \alpha_1(\varphi^* - \varphi_Q) \in L^2(0,T;V), \quad g_2 = \alpha_2(\varphi^*(T) - \varphi_\Omega) \in V,$$
$$g_3 = \alpha_3(\mu^* - \mu_Q) \in L^2(0,T;V). \tag{6.32}$$

Now observe that, by taking $v = e_i$ in (6.30) and recalling (4.8), we have that

$$q_n^i(t) = \lambda_i p_n^i(t) - (g_3(t), e_i) \quad \text{for } 1 \le i \le n.$$

Therefore, if we insert $v = e_i$, $1 \le i \le n$, in (6.29), then we obtain a standard terminal value problem for an explicit linear system of ordinary differential equations in the unknowns $p_n^1, \dots, p_n^n$ in which all of the coefficient functions belong to $L^2(0,T)$. By Carathéodory's theorem (applied backward in time), it has a unique solution in $H^1(0,T;\mathbb{R}^n)$ that specifies $p_n$. At the same time, inserting $e_i$, $1 \le i \le n$, in (6.30), we recover $(q_n^1, \dots, q_n^n) \in L^2(0,T;\mathbb{R}^n)$, which in turn specifies $q_n$. Hence, the system (6.29)–(6.31) has a unique solution $(p_n, q_n) \in H^1(0,T;V_n) \times L^2(0,T;V_n)$.

We now derive a priori estimates. First, we insert $v = p_n$ in (6.29) and $v = q_n$ in (6.30), add the results, and integrate over $(t,T]$ where $t \in [0,T)$. Noting a cancellation of two terms and recalling the notation introduced in (2.1), we then arrive at the identity

$$\frac{1}{2}\,\|p_n(t)\|^2 + \int_{Q^t} |p_n|^2 + \int_{Q^t} |q_n|^2$$
$$= \frac{1}{2}\,\|p_n(T)\|^2 - \int_{Q^t} f''(\varphi^*)\, p_n\, q_n + \int_{Q^t} g_1\, p_n - \int_{Q^t} g_3\, q_n\,. \tag{6.33}$$

Applying Young's inequality, we see that the three integrals on the right-hand side are bounded by an expression of the form

$$c + \frac{1}{2} \int_{Q^t} |q_n|^2 + c \int_{Q^t} |p_n|^2,$$

and the first summand is bounded since $\|p_n(T)\|^2 = (p_n(T), g_2) \le \|p_n(T)\| \, \|g_2\|$, that is, $\|p_n(T)\| \le \|g_2\|$. Therefore, applying Gronwall's lemma backward in time, we obtain from (6.33) the estimate

$$\|p_n\|_{L^\infty(0,T;H)} + \|q_n\|_{L^2(0,T;H)} \le c \quad \forall\, n \in \mathbb{N}. \tag{6.34}$$

In the second estimate, we take $v = -\Delta p_n$ in (6.29) and $v = -\Delta q_n$ in (6.30) to obtain the identity

$$\begin{aligned}
&\frac{1}{2}\|\nabla p_n(t)\|^2 + \int_{Q^t} |\nabla p_n(t)|^2 + \int_{Q^t} |\nabla q_n|^2 \\
&= \frac{1}{2}\|\nabla p_n(T)\|^2 + \int_{Q^t} \nabla g_1 \cdot \nabla p_n - \int_{Q^t} \nabla g_3 \cdot \nabla q_n - \int_{Q^t} \nabla(f''(\varphi^*)\, q_n) \cdot \nabla p_n \\
&:= I_1 + I_2 + I_3 + I_4,
\end{aligned} \tag{6.35}$$

with obvious meaning. Since $g_1, g_3 \in L^2(0,T;V)$, Young's inequality implies that for every $\delta > 0$ (which has yet to be chosen) we have that

$$I_2 + I_3 \le \delta \int_{Q^t} (|\nabla p_n|^2 + |\nabla q_n|^2) + c_\delta. \tag{6.36}$$

Moreover, we have, as seen above, $\|p_n(T)\| \le \|g_2\|$. Also, by the same token, and since $\Delta p_n(T) \in V_n$,

$$\|\nabla p_n(T)\|^2 = (p_n(T), -\Delta p_n(T)) = (g_2, -\Delta p_n(T)) = (\nabla g_2, \nabla p_n(T)),$$

which implies that $\|\nabla p_n(T)\| \le \|g_2\|_V$ and thus $I_1 \le c$.

Finally, using the Young and Hölder inequalities, as well as (2.50), we obtain that

$$\begin{aligned}
I_4 &= -\int_{Q^t} f''(\varphi^*)\, \nabla q_n \cdot \nabla p_n - \int_{Q^t} f'''(\varphi^*)\, q_n \, \nabla\varphi^* \cdot \nabla p_n \\
&\le \delta \int_{Q^t} |\nabla q_n|^2 + c_\delta \int_{Q^t} |\nabla p_n|^2 + c \int_t^T \|q_n(s)\|_4 \, \|\nabla\varphi^*(s)\|_4 \, \|\nabla p_n(s)\|_2 \, ds \\
&\le 2\delta \int_{Q^t} |\nabla q_n|^2 + \delta \int_{Q^t} |q_n|^2 + c_\delta \int_{Q^t} |\nabla p_n|^2.
\end{aligned} \tag{6.37}$$

Combining (6.35)–(6.37), choosing $\delta = 1/4$, and using (6.34) and Gronwall's lemma, we have thus shown that

$$\|p_n\|_{L^\infty(0,T;V)} + \|q_n\|_{L^2(0,T;V)} \le c \quad \forall\, n \in \mathbb{N}. \tag{6.38}$$

But then we may insert $v = -\Delta p_n(t)$ in (6.30), and Young's inequality and elliptic regularity yield that

$$\|p_n\|_{L^2(0,T;W)} \le c \quad \forall\, n \in \mathbb{N}. \tag{6.39}$$

In our last step, we prove that

$$\|\partial_t p_n\|_{L^2(0,T;V^*)} \le c \quad \forall\, n \in \mathbb{N}. \tag{6.40}$$

To this end, we recall that $V_n$ contains the constant functions, so that (6.29) yields

$$-\frac{d}{dt}\,\overline{p_n} + \overline{p_n} = -\overline{f''(\varphi^*) q_n} + \overline{g_1}\,. \tag{6.41}$$

By taking advantage of (6.31) and (6.34) for $q_n$, we deduce that

$$\|\overline{p_n}\|_{H^1(0,T)} \le c \quad \forall\, n \in \mathbb{N}. \tag{6.42}$$

By combining (6.29) with (6.41), and setting for brevity

$$\rho_n := -f''(\varphi^*)q_n + \overline{f''(\varphi^*)q_n} + g_1 - \overline{g_1}\,,$$

we deduce that

$$-(\partial_t(p_n - \overline{p_n}), v) + (p_n - \overline{p_n}, v) + (\nabla q_n, \nabla v) = (\rho_n, v)$$
$$\text{a.e. in } (0,T), \text{ for every } v \in V_n.$$

Recalling (4.21) and (2.55), we can test the above identity by $-\mathcal{N}(\partial_t(p_n - \overline{p_n}))$ and obtain that

$$\int_t^T \|\partial_t(p_n - \overline{p_n})(s)\|_*^2\, ds + \frac{1}{2}\,\|(p_n - \overline{p_n})(t)\|_*^2$$
$$= \frac{1}{2}\,\|(p_n - \overline{p_n})(T)\|_*^2 - \int_t^T \langle \partial_t(p_n - \overline{p_n})(s), q_n(s)\rangle ds - \int_{Q^t} \rho_n\, \mathcal{N}(\partial_t(p_n - \overline{p_n})).$$

On account of (6.31) and (6.38), Young's inequality, and the properties of $\mathcal{N}$, we see that the whole right-hand side is bounded by

$$\frac{1}{2}\int_t^T \|\partial_t(p_n - \overline{p_n})(s)\|_*^2\, ds + c\,,$$

and we we infer that $\{\partial_t(p_n - \overline{p_n})\}$ is bounded in $L^2(0,T;V^*)$. Then, (6.40) follows on account of (6.42).

At this point, we conclude from (6.38)–(6.40) the existence of limit points $(p,q)$ such that, at least for a subsequence labelled again by $n$,

$$p_n \to p \qquad \text{weakly star in } H^1(0,T;V^*) \cap L^\infty(0,T;V) \cap L^2(0,T;W)\,, \tag{6.43}$$
$$q_n \to q \qquad \text{weakly in } L^2(0,T;V)\,. \tag{6.44}$$

It is then a standard matter to conclude that $(p,q)$ is a solution to (2.46)–(2.48). We may allow ourselves to leave this simple argument and the proof of uniqueness to the reader. □

**Proof of Theorem 2.10.** We insert $v = p(t)$ in (6.6) and $v = -q(t)$ in (6.7), add the resulting equations, and integrate over $[0,T]$. Using the properties of the adjoint system (2.46)–(2.48), we then obtain that

$$\begin{aligned}
0 \;&=\; \int_0^T \langle \partial_t \xi(t), p(t)\rangle\, dt + \int_Q (\xi p + \nabla\eta\cdot\nabla p - hp) + \int_Q (-\eta q + f''(\varphi^*)\xi q + \nabla\xi\cdot\nabla q) \\
&= \int_0^T \langle -\partial_t p(t), \xi(t)\rangle\, dt + \int_Q \xi(p + f''(\varphi^*)q) + \int_Q \nabla\xi\cdot\nabla q + \int_Q (-\eta q + \nabla\eta\cdot\nabla p) \\
&\quad - \int_Q hp + \int_\Omega \xi(T)p(T) \\
&= \int_Q (\alpha_1(\varphi^* - \varphi_Q)\xi + \alpha_3(\mu^* - \mu_Q)\eta - hp) + \alpha_2 \int_\Omega (\varphi^*(T) - \varphi_\Omega)\xi(T)\,.
\end{aligned}$$

Substitution of this identity with $h = u - u^*$ in (6.27) yields the variational inequality (2.51). □

## References


[1] M. Bahiana, Y. Oono, Cell dynamical system approach to block copolymers, *Phys. Rev. A* **41** (1990), 6763-6771.

[2] V. Barbu, "Nonlinear Differential Equations of Monotone Type in Banach Spaces", Springer, London, New York, 2010.

[3] A.L. Bertozzi, S. Esedoglu, A. Gillette, Inpainting of binary images using the Cahn–Hilliard equation, *IEEE Trans. Image Process.* **16** (2007), 285-291.

[4] J.F. Blowey, C.M. Elliott, The Cahn–Hilliard gradient theory for phase separation with nonsmooth free energy. I. Mathematical analysis, *European J. Appl. Math.* **2** (1991), 233-280.

[5] S. Bosia, M. Grasselli, A. Miranville, On the long-time behavior of a 2D hydrodynamic model for chemically reacting binary fluid mixtures, *Math. Methods Appl. Sci.* **37** (2014), 726-743

[6] H. Brezis, "Opérateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert", North-Holland Math. Stud. **5**, North-Holland, Amsterdam, 1973.

[7] J.W. Cahn, On spinodal decomposition, *Acta Metall.* **9** (1961), 795-801.

[8] J.W. Cahn, J.E. Hilliard, Free Energy of a Nonuniform System. I. Interfacial Free Energy, *J. Chem. Phys.* **28** (1958), 258-267.

[9] C. Cavaterra, E. Rocca, H. Wu, Long-time dynamics and optimal control of a diffuse interface model for tumor growth, *Appl. Math. Optim.* **83** (2021), 739-787.

[10] P. Colli, G. Gilardi, D. Hilhorst, On a Cahn–Hilliard type phase field system related to tumor growth. *Discret. Cont. Dyn. Syst.* **35** (2015), 2423-2442.

[11] P. Colli, G. Gilardi, E. Rocca, J. Sprekels, Vanishing viscosities and error estimate for a Cahn–Hilliard type phase field system related to tumor growth. *Nonlinear Anal. Real World Appl.* **26** (2015), 93-108.

[12] P. Colli, G. Gilardi, E. Rocca, J. Sprekels, Asymptotic analyses and error estimates for a Cahn–Hilliard type phase field system modelling tumor growth. *Discret. Contin. Dyn. Syst. Ser. S* **10** (2017), 37-54.

[13] P. Colli, G. Gilardi, E. Rocca, J. Sprekels, Optimal distributed control of a diffuse interface model of tumor growth,*Nonlinearity* **30** (2017), 2518-2546.

[14] P. Colli, G. Gilardi, J. Sprekels, On the Cahn–Hilliard equation with dynamic boundary conditions and a dominating boundary potential, *J. Math. Anal. Appl.* **419** (2014), 972-994.

[15] P. Colli, G. Gilardi, J. Sprekels, A boundary control problem for the pure Cahn–Hilliard equation with dynamic boundary conditions, *Adv. Nonlinear Anal.* **4** (2015), 311-325.

[16] P. Colli, G. Gilardi, J. Sprekels, On a Cahn–Hilliard system with convection and dynamic boundary conditions, *Ann. Mat. Pura Appl. (4)* **197** (2018), 1445-1475.

[17] P. Colli, G. Gilardi, J. Sprekels, Optimal velocity control of a viscous Cahn–Hilliard system with convection and dynamic boundary conditions, *SIAM J. Control Optim.* **56** (2018), 1665-1691.

[18] P. Colli, G. Gilardi, J. Sprekels, A distributed control problem for a fractional tumor growth model, *Mathematics* **7** (2019), 792.

[19] P. Colli, G. Gilardi, J. Sprekels, Optimal velocity control of a convective Cahn–Hilliard system with double obstacles and dynamic boundary conditions: a 'deep quench' approach, *J. Convex Anal.* **26** (2019), 485-514.

[20] P. Colli, G. Gilardi, J. Sprekels, Recent results on well-posedness and optimal control for a class of generalized fractional Cahn–Hilliard systems, *Control Cybernet.* **48** (2019), 153-197.

[21] P. Colli, G. Gilardi, J. Sprekels, Optimal distributed control of a generalized fractional Cahn–Hilliard system, *Appl. Math. Optim.* **82** (2020), 551-589.

[22] P. Colli, G. Gilardi, J. Sprekels, Deep quench approximation and optimal control of general Cahn–Hilliard systems with fractional operators and double obstacle potentials, *Discrete Contin. Dyn. Syst. Ser. S* **14** (2021), 243-271.

[23] P. Colli, A. Signori, Boundary control problem and optimality conditions for the Cahn–Hilliard equation with dynamic boundary conditions, *Internat. J. Control* **94** (2021), 1852-1869.

[24] S. Frigeri, M. Grasselli, E. Rocca, On a diffuse interface model of tumor growth, *European J. Appl. Math.* **26** (2015), 215-243.

[25] H. Garcke, K.F. Lam, Well-posedness of a Cahn–Hilliard system modelling tumour growth with chemotaxis and active transport, *European. J. Appl. Math.* **28** (2017), 284–316.

[26] H. Garcke, K.F. Lam, Global weak solutions and asymptotic limits of a Cahn–Hilliard–Darcy system modelling tumour growth, *AIMS Mathematics* **1** (2016), 318–360.

[27] H. Garcke, K.-F. Lam, E. Rocca, Optimal control of treatment time in a diffuse interface model of tumor growth, *Appl. Math. Optim.* **78** (2018), 495-544.

[28] G. Gilardi, A. Miranville, G. Schimperna, On the Cahn–Hilliard equation with irregular potentials and dynamic boundary conditions, *Commun. Pure Appl. Anal.* **8** (2009), 881-912.

[29] G. Gilardi, E. Rocca, Well posedness and long time behaviour for a singular phase field system of conserved type, *IMA J. Appl. Math.* **72** (2007), 498-530.

[30] G. Gilardi, J. Sprekels, Asymptotic limits and optimal control for the Cahn–Hilliard system with convection and dynamic boundary conditions, *Nonlinear Anal.* **178** (2019), 1-31.

[31] A. Giorgini, M. Grasselli, A. Miranville, The Cahn–Hilliard–Oono equation with singular potential, *Math. Models Methods Appl. Sci.* **27** (2017), 2485-2510.

[32] A. Giorgini, K.-F. Lam, E. Rocca, G. Schimperna, On the existence of strong solutions to the Cahn–Hilliard–Darcy system with mass source, preprint arXiv:2009.13344 [math.AP] (2020), pp. 1-30.

[33] E. Khain, L.M. Sander, Generalized Cahn–Hilliard equation for biological applications, *Phys. Rev. E* **77** (2008), 051129-1-051129-7.

[34] S. Melchionna, E. Rocca, On a nonlocal Cahn–Hilliard equation with a reaction term, *Adv. Math. Sci. Appl.* **24** (2014), 461-497.

[35] A. Miranville, Asymptotic behavior of the Cahn–Hilliard–Oono equation, *J. Appl. Anal. Comput.* **1** (2011), 523-536.

[36] A. Miranville, E. Rocca, G. Schimperna, On the long time behavior of a tumor growth model, *J. Differential Equations* **67** (2019), 2616-2642.

[37] A. Miranville, R. Temam, On the Cahn–Hilliard–Oono–Navier–Stokes equations with singular potentials, *Appl. Anal.* **95** (2016), 2609-2624.

[38] A. Miranville, S. Zelik, Robust exponential attractors for Cahn–Hilliard type equations with singular potentials, *Math. Methods Appl. Sci.* **27** (2004), 545-582.

[39] J. Moser, A sharp form of an inequality by N. Trudinger. *Indiana Univ. Math. J.* **20** (1970/71), 1077-1092.

[40] Y. Oono, S. Puri, Computationally efficient modeling of ordering of quenched phases, *Phys. Rev. Lett.* **58** (1987), 836-839.

[41] Y. Oono, S. Puri, Study of phase-separation dynamics by use of cell dynamical systems. I. Modeling, *Phys. Rev. A* **38** (1988), 434-453.

[42] S. Puri, Y. Oono, Study of phase-separation dynamics by use of cell dynamical systems. II. Two-dimensional demonstrations, *Phys. Rev. A* **38** (1988), 1542-1565.

[43] A. Signori, Optimal treatment for a phase field system of Cahn–Hilliard type modeling tumor growth by asymptotic scheme, *Math. Control Relat. Fields* **10** (2020) 305-331.

[44] A. Signori, Vanishing parameter for an optimal control problem modeling tumor growth,*Asymptot. Anal.* **117** (2020) 43-66.

[45] J. Simon, Compact sets in the space $L^p(0,T;B)$, *Ann. Mat. Pura Appl. (4)* **146** (1987), 65-96.