Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

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Pressure-robustness and discrete Helmholtz projectors in mixed finite element methods for the incompressible Navier–Stokes equations

Alexander Linke, Christian Merdon

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Weierstraß-Institut
Mohrenstr. 39
10117 Berlin
Germany
email: alexander.linke@wias-berlin.de
christian.merdon@wias-berlin.de

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Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax:	+49 30 20372-303
E-Mail:	preprint@wias-berlin.de
World Wide Web:	http://www.wias-berlin.de/

Abstract. Recently, it was understood how to repair a certain L^2-orthogonality of discretely-divergence-free vector fields and gradient fields such that the velocity error of inf-sup stable discretizations for the incompressible Stokes equations becomes pressure-independent. These new 'pressure-robust' Stokes discretizations deliver a small velocity error, whenever the continuous velocity field can be well approximated on a given grid. On the contrary, classical inf-sup stable Stokes discretizations can guarantee a small velocity error only, when both the velocity and the pressure field can be approximated well, simultaneously.

In this contribution, 'pressure-robustness' is extended to the time-dependent Navier–Stokes equations. In particular, steady and time-dependent potential flows are shown to build an entire class of benchmarks, where pressure-robust discretizations can outperform classical approaches significantly. Speedups will be explained by a new theoretical concept, the 'discrete Helmholtz projector' of an inf-sup stable discretization. Moreover, di erent discrete nonlinear convection terms are discussed, and skew-symmetric pressure-robust dis-cretizations are proposed.

1. Introduction

Though inf-sup stable mixed finite elements for the incompressible Stokes equations [BF91, GR86, Lay08] are a seemingly mature research field, the concept of a *pressure-robust* mixed method [JLM+16] was introduced only recently. A pressure-robust mixed method for the incompressible Stokes equations

$$\begin{aligned} -\nu \Delta \mathbf{u} + \nabla p &= \mathbf{f}, \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned}$$

— with velocity field $\mathbf{u}$, pressure p, an exterior forcing $\mathbf{f}$ and a kinematic viscosity $\nu > 0$ — denotes a convergent discretization, whose velocity error *does not depend on the continuous pressure*. In fact, the velocity errors of classical inf-sup stable mixed methods like the non-conforming Crouzeix–Raviart element or the H^1-conforming Taylor–Hood element depend on a (for $\nu \to 0$ possibly arbitrarily large) pressure-dependent error contribution [OR04]

$$\frac{1}{\nu} \inf_{q_h \in Q_h} \|p - q_h\|_{L^2},$$

where Q_h denotes the discrete pressure space. Therefore, classical inf-sup stable mixed finite elements are optimally convergent, but only deliver a useful velocity error in the case $\nu = \mathcal{O}(1)$, when the continuous velocity $\mathbf{u}$ and the pressure p are simultaneously well-resolved on a given grid. In the literature, this lack of robustness is sometimes called *poor mass conservation* [GLRW12, MNO+11], and is traditionally mitigated by *grad-div stabilization* [OR04, CELR11, JJLR14, OLHL09, GLOS05, BBJL07]. *Pressure-robust* schemes instead, deliver a velocity error, which is independent of the continuous pressure p (and of the size of the kinematic viscosity ν). Only the velocity has to be resolved well, in order to deliver a small velocity error.

This contribution now applies *pressure-robust* finite element Stokes discretisations to the time-dependent incompressible Navier–Stokes equations

$$\begin{aligned} \mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f}, \\ \nabla\cdot\mathbf{u} &= 0, \end{aligned}$$

and elaborates on the question, when these schemes outperform classical inf-sup stable discretizations. In order to answer this question, a first order (Bernardi–Raugel element) and a second order (P_2^+-P_1^{disc} element) inf-sup stable discretization are compared to some recent pressure-robust variants [LMT16, Lin14], which have the same degrees of freedom for velocity and pressure. The corresponding Stokes discretizations are constructed by modifying the L^2 scalar product in the discretization of the exterior force $\mathbf{f}$ by

$$\int_D \mathbf{f}\cdot\mathbf{v}_h\,dx \to \int_D \mathbf{f}\cdot\Pi\mathbf{v}_h\,dx, \tag{1.1}$$

which repairs a certain L^2 orthogonality between discretely divergence-free vector fields and gradient fields. Here, $\Pi\mathbf{v}_h$ denotes an $\mathcal{O}(h)$ approximation of $\mathbf{v}_h$, whose *weak divergence* $\nabla\cdot\Pi\mathbf{v}_h$ coincides with the *discrete divergence* of $\mathbf{v}_h$. In the time-dependent incompressible Navier–Stokes case, this contribution shows that similar modifications of the L^2 scalar products in the nonlinear convection term $(\mathbf{u}_h\cdot\nabla)\mathbf{u}_h$, and the (approximative) time derivative $\frac{\mathbf{u}_h^{n+1}-\mathbf{u}_h^n}{\Delta t}$ can lead to remarkably more accurate velocity approximations. This will be practically demonstrated focusing on a classical class of benchmarks flows: potential flows. For potential flows $\mathbf{u} = \nabla h$ with h being a (maybe time-dependent) harmonic potential, the nonlinear convection term and the time derivative are gradient fields: $(\mathbf{u}\cdot\nabla)\mathbf{u} = \frac{1}{2}\nabla(\mathbf{u}^2)$, $\mathbf{u}_t = \nabla(h_t)$. Therefore, they are balanced by the pressure gradient ∇p in the momentum balance, which makes the pressure p comparably large in these benchmarks, though the exterior force equals $\mathbf{f} = \mathbf{0}$. In a nutshell, potential flows show that *pressure-robust schemes* merit their name also in the time-dependent Navier–Stokes problem, — since they outperform classical inf-sup stable schemes due to large pressures.

Mathematically, this reasoning can be made more precise by looking at the Helmholtz projector $\mathbb{P}(\mathbf{f})$ of a vector field $\mathbf{f} \in L^2(D)^d$, which denotes its divergence-free part in the sense of the Helmholtz decomposition. For potential flows, the identities $\mathbb{P}((\mathbf{u}\cdot\nabla)\mathbf{u}) = \mathbf{0}$ and $\mathbb{P}(\mathbf{u}_t) = \mathbf{0}$ hold, since the time derivative and the convection term are irrotational. Similarly, in this contribution a discrete Helmholtz projector $\mathbb{P}_h$ will be introduced for any inf-sup stable (Navier–)Stokes discretization. For *pressure-robust* mixed methods it will be shown that $\mathbb{P}_h(\nabla\phi) = \mathbf{0}$ holds for all $\phi \in H^1(D)$, while for classical inf-sup stable methods it only holds $\mathbb{P}_h(\nabla\phi) = \mathcal{O}(h^{k+1})$ with k being the approximation order of the discrete pressure space. Then, the discrete Helmholtz projector is applied in the numerical analysis of the steady Navier–Stokes problem with a small data assumption and a one-step analysis of the time-dependent Stokes problem.

The outline of the paper is as follows. Section 2 states the incompressible Navier-Stokes equations and recalls the Helmholtz decomposition. Section 3 elaborates on classical finite element methods, its accompanying discrete Helmholtz projector and consequences for a priori error estimates. Section 4 explains the concept of pressure-robust finite element methods and how to fix classical schemes with a variational crime. Section 5 introduces the class of irrotational flows as a nice study subject to verify the sharpness of the a priori error estimates and the benefits of pressure-robust methods. Finally, Section 6 shows the results of some numerical experiments with irrotational and other flows.

2. Incompressible Navier–Stokes equations and Fundamentals

The sections recalls the incompressible Navier–Stokes equations and the Helmholtz decomposition.

2.1. Incompressible Navier–Stokes equations.

The incompressible Navier-Stokes equations in some time interval $[0, T]$ with right-hand side $\mathbf{f} \in L^2((0,T) \times D; \mathbb{R}^d)$ and Dirichlet data $\mathbf{u}_D \in L^2((0,T) \times D; \mathbb{R}^d)$ and some initial condition $\mathbf{u}_0 \in L^2(D; \mathbb{R}^d)$ for some d-dimensional ($d = 2, 3$) bounded Lipschitz domain D with polygonal boundary ∂D read

$$\mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} \qquad \text{in } [0, T] \times D, \tag{2.1}$$

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{in } [0, T] \times D, \tag{2.2}$$

$$\mathbf{u} = \mathbf{u}_D \qquad \text{along } [0, T] \times \partial D, \tag{2.3}$$

$$\mathbf{u}(0) = \mathbf{u}_0 \qquad \text{in } D. \tag{2.4}$$

For simplicity we study the weak formulation after the application of some implicit Euler time discretisation rule with arbitrary time steps. The weak formulation employs the multilinear forms

$$a(\mathbf{u}, \mathbf{v}) := \int_{\mathcal{T}} \nu \nabla\mathbf{u} : \nabla\mathbf{v}\, dx, \qquad b(\mathbf{u}, q) := -\int_D q\nabla \cdot \mathbf{u}\, dx,$$

$$c(\mathbf{a}, \mathbf{u}, \mathbf{v}) := \int_D ((\mathbf{a} \cdot \nabla)\,\mathbf{u}) \cdot \mathbf{v}\, dx, \qquad d(\mathbf{u}, \mathbf{v}) := \frac{1}{t_n - t_{n-1}} \int_D \mathbf{u} \cdot \mathbf{v}\, dx,$$

$$F(\mathbf{v}) := \int_D \mathbf{f} \cdot \mathbf{v}\, dx$$

and the usual Sobolev spaces $H^1(D)$, their vector-valued version $H^1(D; \mathbb{R}^d)$ (or $H^1_0(D; \mathbb{R}^d)$ with zero boundary conditions) and $L^2_0(D) := \{q \in L^2(D) : \int_D q\, dx = 0\}$.

After the initial state $\mathbf{u}^0 = \mathbf{u}(0) = \mathbf{u}_0$ for time $t_0 = 0$, the weak solution $(\mathbf{u}(t_n), p(t_n)) = (\mathbf{u}^n, p^n) \in H^1(D; \mathbb{R}^d) \times L^2_0(D)$ for some time step $t_n > 0, n > 0$ is characterised by $\mathbf{u}|_{\partial D} = \mathbf{u}_D(t_n)$ and, for all $\mathbf{v} \in V := H^1_0(D; \mathbb{R}^d)$ and $q \in Q := L^2_0(\times D)$,

$$d(\mathbf{u}^n, \mathbf{v}) + a(\mathbf{u}^n, \mathbf{v}) + c(\mathbf{u}^n, \mathbf{u}^n, \mathbf{v}) + b(\mathbf{v}, p) = F(\mathbf{v}) + d_h(\mathbf{u}^{n-1}, \mathbf{v}) \tag{2.5}$$

$$b(\mathbf{u}^n, q) = 0. \tag{2.6}$$

The divergence constraint can be cast into the set of divergence-free functions

$$V_0 := \{\mathbf{v} \in H^1_0(D) : \nabla \cdot \mathbf{v} = 0\}.$$

Then, the exact velocity satisfies $\mathbf{u}^n \in \mathbf{u}_D(t_n) + V_0$ (where $\mathbf{u}_D$ denotes a divergence-free H^1-conforming continuation of the boundary data) and

$$d(\mathbf{u}^n, \mathbf{v}) + a(\mathbf{u}^n, \mathbf{v}) + c(\mathbf{u}^n, \mathbf{u}^n, \mathbf{v}) = F(\mathbf{v}) + d_h(\mathbf{u}^{n-1}, \mathbf{v}) \qquad \text{for all } \mathbf{v} \in V := V_0.$$

2.2. Helmholtz projection.

This section introduces the continuous Helmholtz projection operator $\mathbb{P}$. Every vector field $\mathbf{f} \in L^2(D; \mathbb{R}^d)$ can be uniquely decomposed into a gradient of some $\alpha \in H^1(D)$ and a divergence-free remainder $\mathbb{P}(\mathbf{f}) := \beta \in L^2_\sigma(D) := \{\mathbf{w} \in H(\mathrm{div}, D) : \nabla \cdot \mathbf{w} = 0, \beta \cdot \mathbf{n} = 0 \text{ along } \partial D\}$ [GR86], i.e.

$$\mathbf{f} = \nabla\alpha + \beta.$$

Due to

$$\int_D \mathbf{w} \cdot \nabla q\, dx = 0 \quad \text{for all } \mathbf{w} \in L^2_\sigma(D), q \in H^1(D), \tag{2.7}$$

the decomposition is L^2 orthogonal. Hence, it holds $\mathbb{P}(\nabla q) = 0$ for any $q \in H^1(D)$. The divergence-free part is the Helmholtz projector and can also be characterized by the best-approximation in L^2_σ, i.e.

$$\mathbb{P}(\mathbf{f}) = \mathbb{P}(\nabla\alpha + \beta) = \beta := \operatorname*{argmin}_{\beta\in L^2_\sigma} \|\mathbf{f} - \beta\|_{L^2(D)}\,.$$

As we will see, the fundamental orthogonality (2.7) is violated in most classical finite element methods (as a bargain for inf-sup stability), but can be restored by an astonishingly simple variational crime.

3. Classical inf-sup stable finite element discretisation of the Navier–Stokes equations

This section briefly recalls some standard finite element methods for the Navier–Stokes equations. After that, the discrete Helmholtz projector is introduced and connected to the well-known a priori error estimates.

3.1. **Standard finite element discretisations.** Given a regular triangulation $\mathcal{T}$ of the domain (at time step t_n) with the set of vertices $\mathcal{N}$ and the set of faces $\mathcal{F}$, the discrete velocity ansatz space V_h and the discrete pressure ansatz space Q_h of standard finite element methods are defined as piecewise polynomials with respect to $\mathcal{T}$. To ensure solvability of the discrete problem, an inf-sup condition has to be satisfied, which results in certain limitations in the choice of V_h and Q_h [GR86].

The Taylor–Hood (TH) finite element method employs piecewise quadratic and continuous velocity ansatz functions $V_h := P_2(\mathcal{T};\mathbb{R}^d) \cap H^1(D;\mathbb{R}^d)$ and piecewise linear and continuous pressure ansatz functions $Q_h = P_1(\mathcal{T}) \cap H^1(D) \cap L^2_0(D)$.

The lowest order Bernardi–Raugel (BR) finite element method enriches the piecewise linear velocity ansatz functions with normal-weighted face bubble functions, $b_F\mathbf{n}_F$ for all faces $F \in \mathcal{F}$, which allows a coupling with piecewise constant discontinuous pressure ansatz functions [BR85]. In other words $V_h := P_1(\mathcal{T};\mathbb{R}^d) \cap H^1(D;\mathbb{R}^d) + \{b_F\mathbf{n}_F : F \in \mathcal{F}\}$, where $b_F := \prod_{z\in\mathcal{N}(F)} \varphi_z$ is the product of all d nodal basis functions for nodes $z \in \mathcal{N}(F)$ that are nonzero on $F \in F$, and $Q_h := P_0(\mathcal{T}) \cap L^2_0(D)$.

The second-order P_2^+ finite element method in 2D employs $V_h := P_2(\mathcal{T};\mathbb{R}^d) \cap H^1(D;\mathbb{R}^d) + \{b_T : T \in \mathcal{T}\}^d$ where b_T are cell bubbles $b_T := \prod_{z\in\mathcal{N}(T)} \varphi_z$. This allows to use piecewise linear pressure ansatz functions $Q_h := P_1(\mathcal{T}) \cap L^2_0(D)$. In 3D additional face bubble functions $\{b_F : F \in \mathcal{F}\}$ have to be added to V_h to maintain inf-sup stability.

Given the solution $\mathbf{u}^{n-1} \in V_h$ for time $t_{n-1} \in [0, T]$ (for $n = 1$ $\mathbf{u}_h^0$ is some appropriate approximation of $\mathbf{u}^0$) and an approximation $\mathbf{u}_{n,D}$ of the Dirichlet data $u_D(t_n)$ at the next time step $t_n \in [0, T]$, the next solution $(\mathbf{u}_h^n, p_h^n) \in \left((u_{h,D}^n + V_h) \times Q_h\right)$ statisfies, for all $\mathbf{v}_h \in V_h$ and $q_h \in Q_h$,

$$\begin{aligned} d_h(\mathbf{u}_h^n, \mathbf{v}_h) + a(\mathbf{u}_h^n, \mathbf{v}_h) + c_h(\mathbf{u}_h^n, \mathbf{u}_h^n, \mathbf{v}_h) + b(\mathbf{v}_h, p_h^n) &= F(\mathbf{v}_h) + d_h(\mathbf{u}_h^{n-1}, \mathbf{v}_h),\\ b(\mathbf{u}_h^n, q_h) &= 0 \end{aligned} \tag{3.1}$$

where d_h originates from the implicit Euler scheme

$$d_h(\mathbf{u}_h, \mathbf{v}_h) := \frac{1}{t_n - t_{n-1}} \int_D \mathbf{u}_h \cdot \mathbf{v}_h \, dx,$$

and c_h is a discretisation of c, see Section 3.3.

3.2. **Discrete divergence and Helmholtz projection.** For classical mixed inf-sup stable finite element methods, the discrete divergence is defined by

$$\nabla_h \cdot \mathbf{v}_h := \pi_{Q_h}(\nabla \cdot \mathbf{v}_h),$$

where π_{Q_h} denotes the L^2 best-approximation into the discrete pressure space Q_h. The set of discretely divergence-free test functions is given by

$$V_{0,h} := \{\mathbf{v}_h \in V_h : \nabla_h \cdot \mathbf{v}_h = 0\}.$$

In the discrete world we can define a discrete Helmholtz projector by the L^2 best-approximation in $V_{0,h}$, i.e.

$$\mathbb{P}_h(\mathbf{f}) := \operatorname*{argmin}_{\beta_h \in V_{0,h}} \|f - \beta_h\|_{L^2(D)}. \tag{3.2}$$

This discrete Helmholtz projector has a severe flaw for most classical finite element methods: it does not vanish when applied to gradients as for the continuous Helmholtz projector, i.e.

$$\mathbb{P}_h(\nabla q) \neq 0 \quad \text{in general}$$

which is in fact a consequence of the incomplete L^2 orthogonality

$$\int_D q_h(\nabla \cdot \mathbf{v}_h)\, dx = 0 \quad \text{only for } q_h \in Q_h. \tag{3.3}$$

The following Lemma rewrites this lack of pressure-robustness in terms of dual norms which appear in two forms in the paper:

$$\|\mathbf{f}\|_{V_{0,h}^\star} := \sup_{\mathbf{v}_h \in V_{0,h}} \frac{\int_D \mathbf{f} \cdot \mathbf{v}_h\, dx}{\|\nabla \mathbf{v}_h\|_{L^2(D)}} \quad \text{for any } \mathbf{f} \in L^2(D; \mathbb{R}^d)$$

or

$$\|L\|_{V_{0,h}^\star} := \sup_{\mathbf{v}_h \in V_{0,h}} \frac{L(\mathbf{v}_h)}{\|\nabla \mathbf{v}_h\|_{L^2(D)}} \quad \text{for any } L \in V_{0,h}^\star.$$

Lemma 3.1. For some function with Helmholtz decomposition $\mathbf{f} = \nabla q + \mathbb{P}(\mathbf{f})$, it holds

$$\|\mathbb{P}(\mathbf{f}) - \mathbb{P}_h(\mathbf{f})\|_{V_{0,h}^\star} = \|\mathbb{P}_h(\nabla q)\|_{V_{0,h}^\star} \leqslant \min_{q_h \in Q_h} \|q - q_h\|_{L^2(D)}.$$

Proof. The first identity follows from linearity and

$$\int_D \mathbb{P}_h(\mathbb{P}\mathbf{f}) \cdot \mathbf{v}_h\, dx = \int_D \mathbb{P}\mathbf{f} \cdot \mathbf{v}_h\, dx \quad \text{for all } \mathbf{v}_h \in V_{0,h}.$$

The best-approximation $\mathbb{P}_h(\nabla q) \in V_{0,h}$ of ∇q is characterised by

$$\int_D \mathbb{P}_h(\nabla q) \cdot \mathbf{v}_h\, dx = \int_D \nabla q \cdot \mathbf{v}_h\, dx \quad \text{for all } \mathbf{v}_h \in V_{0,h}$$

An integration by parts and (3.3) for any $q_h \in Q_h$ yield

$$\int_D \mathbb{P}_h(\nabla q) \cdot \mathbf{v}_h\, dx = -\int_D (q - q_h)(\nabla \cdot \mathbf{v}_h)\, dx \leqslant \min_{q_h \in Q_h} \|q - q_h\|_{L^2(D)} \|\nabla \cdot \mathbf{v}_h\|_{L^2(D)}.$$

A division by $\|\nabla \cdot \mathbf{v}_h\|_{L^2(D)} \leqslant \|\nabla \mathbf{v}_h\|_{L^2(D)}$ concludes the proof. $\square$

Remark 3.2 (Divergence-free finite element methods)**.** Finite element methods, whose discretely divergence-free functions are really divergence-free, i.e. $\nabla \cdot \mathbf{v}_h = \nabla_h \cdot \mathbf{v}_h = 0$ for all $\mathbf{v}_h \in V_{0,h}$, have a flawless discrete Helmholtz projector in the sense $\mathbb{P}(\nabla q) = 0$ for all $q \in H^1(D)$. However, due to the importance of the inf-sup property, these methods are rare or very difficult to construct. One example is the Scott-Vogelius finite element method, which in d dimensions needs $V_h = P_d(\mathrm{bary}(\mathcal{T})) \cap H^1(D;\mathbb{R}^d)$ and $Q_h = P_{d-1}(\mathrm{bary}(\mathcal{T}))$ on special barycentric-refined triangulation to satisfy the inf-sup property [SV85, Zha05]. This results in a computationally very expensive method, especially in 3D, which only in problems with very difficult pressures is competitive with other finite element methods like Taylor-Hood.

3.3. **Different discretisations for the nonlinear convection term.** The nonlinear convection term can be discretised in its convective form

$$c_h(\mathbf{a}_h, \mathbf{u}_h, \mathbf{v}_h) := \int_D ((\mathbf{a}_h \cdot \nabla)\mathbf{u}_h) \cdot \mathbf{v}_h \, dx, \tag{3.4}$$

in the skew-symmetric convective form

$$c_h(\mathbf{a}_h, \mathbf{u}_h, \mathbf{v}_h) := \frac{1}{2}\int_D ((\mathbf{a}_h \cdot \nabla)\mathbf{u}_h) \cdot \mathbf{v}_h \, dx - \frac{1}{2}\int_D ((\mathbf{a}_h \cdot \nabla)\mathbf{v}_h) \cdot \mathbf{u}_h \, dx, \tag{3.5}$$

or in the skew-symmetric rotational form

$$c_h(\mathbf{a}_h, \mathbf{u}_h, \mathbf{v}_h) := \int_D (\mathrm{rot}\mathbf{u}_h \times \mathbf{a}_h) \cdot \mathbf{v}_h \, dx - \frac{1}{2}\int_D \mathbf{u}_h \cdot \mathbf{a}_h (\nabla_h \cdot \mathbf{v}_h) \, dx. \tag{3.6}$$

In our implementations the second term on the right-hand side is omitted. This results in a perturbed discrete pressure p^n that can be corrected a posteriori by

$$\widetilde{p}_h := p_h + \frac{1}{2} \operatorname*{argmin}_{q_h \in Q_h} \left\| q_h - |\mathbf{u}_h|^2 \right\|_{L^2(D)}.$$

Remark 3.3. As another alternative for the handling of the pressure in the rotational form in a time-dependent setting, one could use the form

$$c_h(\mathbf{a}_h, \mathbf{u}_h^n, \mathbf{v}_h) := \int_D (\mathrm{rot}\mathbf{u}_h^n \times \mathbf{a}_h) \cdot \mathbf{v}_h \, dx - \frac{1}{2}\int_D (\mathbf{u}_h^{n-1} \cdot \mathbf{u}_h^n)(\nabla_h \cdot \mathbf{v}_h) \, dx.$$

This has the advantage that the seond term is linear in $\mathbf{u}_h^n$ and therefore easy to handle in a Newton scheme. Secondly, the pressure does not have to be corrected a posteriori. And third, c_h is still skew-symmetric.

3.4. **A priori Stokes error estimates.** This subsection collects some known a priori error estimates with a stronger emphasis on the role of the discrete Helmholtz projector $\mathbb{P}_h$ from (3.2).

Theorem 3.4 (A priori stationary Stokes error estimates for classical FEMs)**.** If $u \in H^2(D)$ and $p \in H^1(D)$ and $\mathbf{f} = -\nu\Delta\mathbf{u} + \nabla p \in L^2(D)$, it holds

$$\|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{L^2(D)} \leqslant 2 \inf_{\mathbf{v}_h \in \mathbf{u}_h + V_{0,h}} \|\nabla(\mathbf{u} - \mathbf{v}_h)\|_{L^2(D)} + \frac{1}{\nu} \|\mathbb{P}_h(\nabla p)\|_{V_{0,h}^\star}.$$

Proof. The point of departure is Strang's Lemma, namely

$$\nu \|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{L^2(D)} \leqslant 2\nu \inf_{\mathbf{v}_h \in \mathbf{u}_h + V_{0,h}} \|\nabla(\mathbf{u} - \mathbf{v}_h)\|_{L^2(D)} + \sup_{\mathbf{v}_h \in V_{0,h}\setminus\{0\}} \frac{|a(\mathbf{u}, \mathbf{v}_h) - F(\mathbf{v}_h)|}{\|\nabla \mathbf{v}_h\|_{L^2(D)}}.$$

The best-approximation space $\mathbf{u}_h + V_{0,h}$ in the first term on the right-hand side was chosen to properly handle inhomogeneous Dirichlet boundary data. Insertion of $\mathbf{f} = -\nu\Delta\mathbf{u} + \nabla p$ and an integration by parts reveal

$$a(\mathbf{u}, \mathbf{v}_h) - F(\mathbf{v}_h) = \int_D \nabla p \cdot \mathbf{v}_h \, dx = \int_D \mathbb{P}_h(\nabla p) \cdot \mathbf{v}_h \, dx \leqslant \|\mathbb{P}_h(\nabla p)\|_{V_{0,h}^\star} \|\nabla \mathbf{v}_h\|_{L^2(D)} .$$

A division by $\|\nabla \mathbf{v}_h\|_{L^2(D)}$ concludes the proof. □

The rest of this section deals with the a priori estimate for the spatial error in each time step of the time-dependent problem. The additional term depends only on the error in the previous time step.

Theorem 3.5 (A priori instationary Stokes error estimates for classical FEMs)**.** If $\mathbf{u}(t_n)$, $\mathbf{u}(t_{n-1}) \in H^2(D)$ and $p(t_n), p(t_{n-1}) \in H^1(D)$ and $\mathbf{f} = -\nu\Delta\mathbf{u} + \nabla p + (t_n - t_{n-1})^{-1}(\mathbf{u}(t_n) - \mathbf{u}(t_{n-1})) \in L^2(D)$, it holds

$$|||\mathbf{u}^n - \mathbf{u}_h^n||| \leqslant 2 \inf_{\mathbf{v}_h \in \mathbf{u}_h^n + V_{0,h}} |||\mathbf{u}^n - \mathbf{v}_h||| + \|\mathbb{P}_h(\nabla p)\|_{(V_{0,h},\widetilde{a})^\star} + \left\| d(\mathbf{u}(t_{n-1}) - \mathbf{u}_h^{n-1}, \bullet) \right\|_{(V_{0,h},\widetilde{a})^\star}$$

with the energy norm $||| \bullet |||^2 := \widetilde{a}(\bullet, \bullet) := a(\bullet, \bullet) + d(\bullet, \bullet)$ and the corresponding dual norm $\|\bullet\|_{(V_{0,h},\widetilde{a})^\star}$.

Proof. This time Strang's Lemma is applied to the bilinear form $\widetilde{a} := a + d$, i.e.

$$|||\mathbf{u}^n - \mathbf{u}_h^n||| \leqslant 2 \inf_{\mathbf{v}_h \in \mathbf{u}_h^n + V_{0,h}} |||\mathbf{u}^n - \mathbf{v}_h||| + \sup_{\mathbf{v}_h \in V_{0,h} \setminus \{0\}} \frac{|\widetilde{a}(\mathbf{u}(t_n), \mathbf{v}_h) - F(\mathbf{v}_h) - d(\mathbf{u}_h^{n-1}, \mathbf{v}_h)|}{|||\mathbf{v}_h|||}.$$

Insertion of $\mathbf{f} = -\nu\Delta\mathbf{u} + \nabla p + (t_n - t_{n-1})^{-1}(\mathbf{u}(t_n) - \mathbf{u}(t_{n-1}))$ and an integration by parts reveal

$$\begin{aligned}
&\widetilde{a}(\mathbf{u}(t_n), \mathbf{v}_h) - F(\mathbf{v}_h) - d(\mathbf{u}_h^{n-1}, \mathbf{v}_h)\\
&\qquad = a(\mathbf{u}(t_n), \mathbf{v}_h) + d(\mathbf{u}(t_n), \mathbf{v}_h) - F(\mathbf{v}_h) - d(\mathbf{u}(t_{n-1}), \mathbf{v}_h) + d(\mathbf{u}(t_{n-1}) - \mathbf{u}_h^{n-1}, \mathbf{v}_h)\\
&\qquad = \int_D \nabla p \cdot \mathbf{v}_h \, dx + d(\mathbf{u}(t_{n-1}) - \mathbf{u}_h^{n-1}, \mathbf{v}_h)\\
&\qquad \leqslant \left(\|\mathbb{P}_h(\nabla p)\|_{(V_{0,h},\widetilde{a})^\star} + \left\| d(\mathbf{u}(t_{n-1}) - \mathbf{u}_h^{n-1}, \bullet) \right\|_{(V_{0,h},\widetilde{a})^\star} \right) |||\mathbf{v}_h|||.
\end{aligned}$$

A division by $|||\mathbf{v}_h|||$ concludes the proof. □

Remark 3.6. The notation in Theorem 3.5 hides the dependency on the parameters. However, elementary calculations show

$$\begin{aligned}
\|\mathbb{P}_h(\nabla p)\|_{(V_{0,h},\widetilde{a})^\star} &\leqslant \frac{1}{\nu^{1/2}} \|\mathbb{P}_h(\nabla p)\|_{V_{0,h}^\star} ,\\
\left\| d(\mathbf{u}(t_{n-1}) - \mathbf{u}_h^{n-1}, \bullet) \right\|_{(V_{0,h},\widetilde{a})^\star} &\leqslant \min\left\{ \frac{1}{(t_n - t_{n-1})^{1/2}}, \frac{C_F}{\nu^{1/2}(t_n - t_{n-1})} \right\} \left\| \mathbb{P}_h(\mathbf{u}(t_{n-1}) - \mathbf{u}_h^{n-1}) \right\|_{L^2(D)}
\end{aligned}$$

and hence

$$\begin{aligned}
\|\nabla(\mathbf{u}^n - \mathbf{u}_h^n)\|_{L^2(D)} &\leqslant \frac{1}{\nu^{1/2}} |||\mathbf{u}^n - \mathbf{u}_h^n||| \leqslant \frac{2}{\nu^{1/2}} \inf_{\mathbf{v}_h \in V_{0,h}} |||\mathbf{u}^n - \mathbf{v}_h|||\\
&+ \frac{1}{\nu} \|\mathbb{P}_h(\nabla p)\|_{V_{0,h}^\star} + \min\left\{ \frac{1}{\nu^{1/2}(t_n - t_{n-1})^{1/2}}, \frac{C_F}{\nu(t_n - t_{n-1})} \right\} \left\| \mathbb{P}_h(\mathbf{u}(t_{n-1}) - \mathbf{u}_h^{n-1}) \right\|_{L^2(D)} .
\end{aligned}$$

In other words, the gradient error again depends on the pressure contribution weighted with $1/\nu$. Moreover, compared to the normal Stokes estimate, an error from the last time step appears and also scales (for certain time step sizes) with $1/\nu$. even if $(\mathbf{u}(t_{n-1}) - \mathbf{u}_h^{n-1})$ is a gradient its discrete Helmholtz projector does not vanish in general.

3.5. **A priori Navier–Stokes error estimates.** In the a priori error estimates for the stationary Navier–Stokes equations another term appears that is related to the nonlinear convection. However, the first problem is to guarantee existence and uniqueness of solutions. A common sufficient assumption then is the small data ssumption by [Lay08] that reads

$$0 \leqslant \nu^{-2} M \|\mathbf{f}\|_{V_{0,h}^\star} := \alpha < 1 \tag{3.7}$$

where

$$M := \sup_{\mathbf{u},\mathbf{v},\mathbf{w} \in V} \frac{c_h(\mathbf{u},\mathbf{v},\mathbf{w})}{\|\nabla \mathbf{u}\|_{L^2(D)} \|\nabla \mathbf{v}\|_{L^2(D)} \|\nabla \mathbf{w}\|_{L^2(D)}} < \infty$$

is the continuity constant of the trilinear form c_h. Moreover, under the assumption that c_h is skew-symmetric, one can show that

$$\|\nabla \mathbf{u}\|_{L^2(D)} \leqslant \|\mathbf{f}\|_{V_{0,h}^\star} \quad \text{and} \quad \|\nabla \mathbf{u}_h\|_{L^2(D)} \leqslant \|\mathbf{f}\|_{V_{0,h}^\star}. \tag{3.8}$$

Note, that the small data assumption (3.7) is meant for problems with homogeneous Dirichlet boundary data.

Remark 3.7 (Remarks on the small data assumption)**.** In case of inhomogeneous boundary data $\mathbf{u} = \mathbf{u_D}$ along $\partial\Omega$ a perturbed small data assumption can be shown. If the Dirichlet data can be represented by some smooth and divergence-free function $u_D \in H^2(\Omega; \mathbb{R}^d)$, it can be subtracted from the system leading to a modified right-hand side. Eventually, one arrives at the small data assumption

$$0 \leqslant \nu^{-2} M \|\mathbf{f} + \Delta \mathbf{u}_D - (\mathbf{u}_D \cdot \nabla)\mathbf{u}_D\|_{V_{0,h}^\star} := \alpha < 1.$$

Theorem 3.8 (A priori stationary Navier–Stokes error estimates for classical FEMs)**.** *If $u \in H^2(D)$ and $p \in H^1(D)$ and $\mathbf{f} = -\nu\Delta\mathbf{u} + \nabla p + (\mathbf{u}\cdot\nabla)\mathbf{u} \in L^2(D)$ and under the small data assumption (3.7) with homogeneous Dirichlet boundary conditions $\mathbf{u}_D = 0$, and under the assumption that c_h is skew-symmetric, it holds*

(1)
$$\begin{aligned} |c_h(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) - c_h(\mathbf{u}, \mathbf{u}, \mathbf{v}_h)| &\leqslant M \|\nabla \mathbf{v}_h\|_{L^2(D)}^2 \|\nabla \mathbf{u}_h\|_{L^2(D)} \\ &\quad + M \|\nabla \mathbf{v}_h\|_{L^2(D)} \|\nabla(\mathbf{u} - \mathbf{u}_h + \mathbf{v}_h)\|_{L^2(D)} \left(\|\nabla \mathbf{u}\|_{L^2(D)} + \|\nabla \mathbf{u}_h\|_{L^2(D)}\right), \end{aligned}$$

(2)
$$\|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{L^2(D)} \leqslant \frac{2}{1-\alpha} \inf_{\mathbf{v}_h \in V_{0,h}} \|\nabla(\mathbf{u} - \mathbf{v}_h)\|_{L^2(D)} + \frac{1}{\nu(1-\alpha)} \|\mathbb{P}_h(\nabla p)\|_{V_{0,h}^\star}.$$

When c_h is the (skew-symmetric) rotational form (3.6), the exact pressure p relates to the corrected pressure. Morevoer, note that $(1-\alpha)$ strongly depends on ν.

Proof. The proof of (2) follows the steps in [Lay08] where (1) is one of the main estimates. The complete proof is repeated here for completeness and to show where the dual norms appear and where differences in the modified scheme appear.

Proof of (1). The Sobolev embedding theorem for $\mathbf{v} \in H_0^1(D)^d$ implies $\mathbf{v} \in L^6(D)^d$ (also in 3D) with $\|\mathbf{v}\|_{L^6(D)} \leqslant C \|\nabla \mathbf{v}\|_{L^2(D)}$ [Tem91]. This and a Hölder inequality with $1/6 + 1/6 + 1/6 + 1/2 = 1$ show

$$\begin{aligned} c_h(\mathbf{u}_h, \mathbf{v}_h, \mathbf{w}_h) &\leqslant \|1\|_{L^6(D)} \|\mathbf{u}_h\|_{L^6(D)} \|\nabla \mathbf{v}_h\|_{L^2(D)} \|\mathbf{w}_h\|_{L^6(D)} \\ &\leqslant M \|\nabla \mathbf{u}_h\|_{L^2(D)} \|\nabla \mathbf{v}_h\|_{L^2(D)} \|\nabla \mathbf{w}_h\|_{L^2(D)} \end{aligned}$$

This and the skew-symmetry of c_h show, for any $\mathbf{v}_h \in V_{0,h}$,

$$\begin{aligned} &c_h(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) - c_h(\mathbf{u}, \mathbf{u}, \mathbf{v}_h) \\ &\qquad = c_h(\mathbf{u}_h - \mathbf{u}, \mathbf{u}_h, \mathbf{v}_h) + c_h(\mathbf{u}, \mathbf{u}_h - \mathbf{u}, \mathbf{v}_h) \\ &\qquad = c_h(\mathbf{v}_h, \mathbf{u}_h, \mathbf{v}_h) - c_h(\mathbf{u}, \mathbf{u} - \mathbf{u}_h + \mathbf{v}_h, \mathbf{v}_h) - c_h(\mathbf{u} - \mathbf{u}_h + \mathbf{v}_h, \mathbf{u}_h, \mathbf{v}_h) \\ &\qquad \leqslant M \|\nabla \mathbf{v}_h\|_{L^2(D)} \|\nabla(\mathbf{u} - \mathbf{u}_h + \mathbf{v}_h)\|_{L^2(D)} \Big(\|\nabla \mathbf{u}\|_{L^2(D)} + \|\nabla \mathbf{u}_h\|_{L^2(D)} \Big) \\ &\qquad\qquad + M \|\nabla \mathbf{v}_h\|^2_{L^2(D)} \|\nabla \mathbf{u}_h\|_{L^2(D)} . \end{aligned}$$

Proof of (2). The point of departure is the identity

$$\begin{aligned} a(\mathbf{u}_h - \mathbf{w}_h, \mathbf{v}_h) &= a(\mathbf{u} - \mathbf{w}_h, \mathbf{v}_h) + a(\mathbf{u}_h - \mathbf{u}, \mathbf{v}_h) \\ &= a(\mathbf{u} - \mathbf{w}_h, \mathbf{v}_h) - (a(\mathbf{u}, \mathbf{v}_h) + c_h(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) - F(\mathbf{v}_h)) \end{aligned} \tag{3.9}$$

that holds for every $\mathbf{v}_h \in V_{0,h}$ and $\mathbf{w}_h \in \mathbf{u}_h + V_{0,h}$.

The identity $\mathbf{f} = -\nu\Delta\mathbf{u} + \nabla p + (\mathbf{u}\cdot\nabla)\mathbf{u}$, an integration by parts show and (1) for the now fixed $\mathbf{v}_h := \mathbf{u}_h - \mathbf{w}_h$ yield

$$\begin{aligned} &a(\mathbf{u}, \mathbf{v}_h) + c_h(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) - F(\mathbf{v}_h) \\ &\quad = a(\mathbf{u}, \mathbf{v}_h) + c_h(\mathbf{u}, \mathbf{u}, \mathbf{v}_h) - F(\mathbf{v}_h) + (c_h(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) - c_h(\mathbf{u}, \mathbf{u}, \mathbf{v}_h)) \\ &\quad = \int_D \nabla p \cdot \mathbf{v}_h \, dx + (c_h(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) - c_h(\mathbf{u}, \mathbf{u}, \mathbf{v}_h)) \\ &\quad \leqslant \|\mathbb{P}_h(\nabla p)\|_{V^\star_{0,h}} \|\nabla \mathbf{v}_h\|_{L^2(D)} + |c_h(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) - c_h(\mathbf{u}, \mathbf{u}, \mathbf{v}_h)| \\ &\quad \leqslant \|\mathbb{P}_h(\nabla p)\|_{V^\star_{0,h}} \|\nabla \mathbf{v}_h\|_{L^2(D)} + M \|\nabla \mathbf{v}_h\|^2_{L^2(D)} \|\nabla \mathbf{u}_h\|_{L^2(D)} \\ &\qquad + M \|\nabla \mathbf{v}_h\|_{L^2(D)} \|\nabla(\mathbf{u} - \mathbf{w}_h)\|_{L^2(D)} \Big(\|\nabla \mathbf{u}\|_{L^2(D)} + \|\nabla \mathbf{u}_h\|_{L^2(D)} \Big) . \end{aligned}$$

The last estimate and (3.9) result in

$$A \|\nabla(\mathbf{u}_h - \mathbf{w}_h)\|^2_{L^2(D)} \leqslant B \|\nabla(\mathbf{u} - \mathbf{w}_h)\|_{L^2(D)} \|\nabla(\mathbf{u}_h - \mathbf{w}_h)\|_{L^2(D)} + \|\mathbb{P}_h(\nabla p)\|_{V^\star_{0,h}} \|\nabla(\mathbf{u}_h - \mathbf{w}_h)\|_{L^2(D)}$$

with coefficients $A := \nu - M \|\nabla \mathbf{u}_h\|_{L^2(D)}$ and $B := \nu + M(\|\nabla \mathbf{u}_h\|_{L^2(D)} + \|\nabla \mathbf{u}\|_{L^2(D)})$. Division by $\|\nabla(\mathbf{u}_h - \mathbf{w}_h)\|_{L^2(D)}$ and a triangle inequality show

$$\|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{L^2(D)} \leqslant \|\nabla(\mathbf{u} - \mathbf{w}_h)\|_{L^2(D)} + BA^{-1} \|\nabla(\mathbf{u} - \mathbf{w}_h)\|_{L^2(D)} + A^{-1} \|\mathbb{P}_h(\nabla p)\|_{V^\star_{0,h}} .$$

Since $\mathbf{w}_h$ is arbitrary, it holds

$$\|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{L^2(D)} \leqslant (1 + BA^{-1}) \inf_{\mathbf{w}_h \in V_{0,h}} \|\nabla(\mathbf{u} - \mathbf{w}_h)\|_{L^2(D)} + A^{-1} \|\mathbb{P}_h(\nabla p)\|_{V^\star_{0,h}} .$$

The small data assumption (3.7) and (3.8) show

$$0 < A^{-1} \leqslant \nu^{-1}(1 - \alpha)^{-1} < \infty \quad \text{and} \quad 1 \leqslant B \leqslant \nu(1 + 2\alpha) \leqslant 3\nu$$

and hence $1 + BA^{-1} \leqslant 1 + (1 + 2\alpha)/(1 - \alpha) \leqslant 2/(1 - \alpha)$. □

4. Pressure-robust finite element discretisations for the Navier–Stokes equations

This section explains the terminology pressure-robustness and its implications. Moreover, two novel pressure-robust finite element methods are presented.

4.1. **Pressure-robustness by repairing the discrete Helmholtz decompositions.** In this subsection the Helmholtz projector is repaired, such that the Helmholtz projection is zero when applied to gradients. The idea behind this is based on [Lin14, BLMS15, LMT16] and employs a reconstruction operator Π that maps discretely divergence-free test functions onto divergence-free test functions, i.e. $\Pi V_{0,h} \subset V_0$. The 'repaired' Helmholtz projector than maps onto $\Pi V_{0,h}$ and reads

$$\mathbb{P}_h^\star(\mathbf{f}) := \operatorname*{argmin}_{\beta_h \in \Pi V_{0,h}} \|\mathbf{f} - \Pi \beta_h\|_{L^2(D)}.$$

Behind this definition is the idea to avoid the application of the divergence opreator on discretely divergence-free test function and to replace it by the discrete divergence operator. In fact, the projection Π satisfies

$$-\int \nabla q \cdot (\Pi \mathbf{v}_h)\, dx = \int_\Omega q(\nabla \cdot \Pi \mathbf{v}_h)\, dx = \int_\Omega q(\nabla_h \cdot \mathbf{v}_h)\, dx$$

and so 'repairs'

$$-\int \nabla q \cdot \mathbf{v}_h\, dx = \int_\Omega q(\nabla \cdot \mathbf{v}_h)\, dx \neq \int_\Omega q(\nabla_h \cdot \mathbf{v}_h)\, dx \quad \text{for } q \in H^1(D)\backslash Q_h.$$

In other words, the reconstruction $\Pi \mathbf{v}_h$ of a discretely divergence-free test function $\mathbf{v}_h$ is orthogonal in L^2 to gradients of arbitrary H^1-functions.

Lemma 4.1. For some function with Helmholtz decomposition $\mathbf{f} = \nabla q + \mathbb{P}(\mathbf{f})$, it holds

$$\mathbb{P}_h^\star(\nabla q) = \mathbb{P}(\nabla q) = 0 \quad \text{for all } q \in H^1(D) \quad \text{and} \quad \|(\mathbb{P}(\mathbf{f}) - \mathbb{P}_h^\star(\mathbf{f})) \circ \Pi\|_{V_{0,h}^\star} = 0.$$

Proof. Direct consequences of $\int_D \Pi \mathbf{v}_h \cdot \nabla q = 0$ for all $\mathbf{v}_h \in V_{0,h}$. □

The second property of Lemma 4.1 implies that, for any right-hand side $\mathbf{f} = \nabla q + \mathbb{P}(\mathbf{f}) \in L^2(D)$, testing with $\Pi V_{0,h}$ is pressure-robust but causes a consistency error

$$F(\Pi \mathbf{v}_h) = \int_D \mathbb{P}_h^\star(\mathbf{f}) \cdot (\Pi \mathbf{v}_h)\, dx = \int_D \mathbb{P}(\mathbf{f}) \cdot (\Pi \mathbf{v}_h)\, dx = \int_D \mathbb{P}(\mathbf{f}) \cdot \mathbf{v}_h\, dx + \int_D \mathbb{P}(\mathbf{f}) \cdot (\Pi \mathbf{v}_h - \mathbf{v}_h)\, dx$$

whereas testing with $V_{0,h}$ is not pressure-robust (only if $q \in Q_h$)

$$F(\mathbf{v}_h) = \int_D \mathbb{P}_h(\mathbf{f}) \cdot \mathbf{v}_h\, dx = \int_D \mathbb{P}(\mathbf{f}) \cdot \mathbf{v}_h\, dx + \int_D \mathbb{P}_h(\nabla q) \cdot \mathbf{v}_h\, dx.$$

This observation motivates the formulation of pressure-robust finite element methods where crucial terms are tested with the reconstructed testfunctions to remove the influence from irrotational parts in their Helmholtz decomposition.

4.2. **Pressure-robust finite element discretisations by divergence-free reconstruction of test functions.** To make use of the modified discrete Helmholtz projector on the discrete level, the crucial terms in the discrete equations have to be tested with divergence-free test functions (while trial functions remain unchanged).

The modified method at the time step $t_n \in [0, T]$ seeks $(\mathbf{u}_h^n, p_h^n) \in ((u_{n,D} + V_h) \times Q_h)$ such that, for all $\mathbf{v}_h \in V_h$ and $q_h \in Q_h$,

(4.1)

$$\begin{aligned} d_h(\Pi \mathbf{u}_h^n, \Pi \mathbf{v}_h) + a(\mathbf{u}_h^n, \mathbf{v}_h) + c_h(\Pi \mathbf{u}_h^n, \mathbf{u}_h^n, \Pi \mathbf{v}_h) + b(\mathbf{v}_h, p_h^n) &= F(\Pi \mathbf{v}_h) + d_h(\Pi \mathbf{u}_h^{n-1}, \Pi \mathbf{v}_h), \\ b(\mathbf{u}_h^n, q_h) &= 0. \end{aligned}$$

In (4.1), for the convection term c_h we can use the forms (3.4) or (3.6). The form (3.5) is not possible, since $\Pi \mathbf{v}_h$ is not in H^1.

Luckily, for the design of Π in case of finite element methods with discontinuous pressure spaces, standard interpolation operators into BDM or Raviart-Thomas functions

$$\mathrm{BDM}_k(\mathcal{T}) := H(\mathrm{div}, D) \cap P_k(\mathcal{T})^d$$
$$\mathrm{RT}_k(\mathcal{T}) := \{v \in H(\mathrm{div}, D) : \forall T \in \mathcal{T} \exists a \in P_k(\mathcal{T})^d, b \in P_k(T), v(\mathbf{x}) = a(\mathbf{x}) + b(\mathbf{x})\mathbf{x}\}$$

have exactly the desired properties [LMT16]. Note, that $\mathrm{RT}_{k-1}(\mathcal{T}) \subset \mathrm{BDM}_k(\mathcal{T})$. Details on the interpolation and its properties can be found in textbooks [BF91].

The BDM_1 or RT_0 standard interpolation can be employed in case of the Bernardi–Raugel finite element method and the BDM_2 or RT_1 standard interpolation in case of the P_2^+ finite element method. An advantageous property of the interpolation operator into BDM_k is that it is exact when applied to a continuous vector-valued piecewise polynomial of order k, i.e. $\Pi \mathbf{v}_h = \mathbf{v}_h$ for all $\mathbf{v}_h \in P_k(\mathcal{T}; \mathbb{R}^d) \cap H(\mathrm{div}, D)$. Hence only the additional face bubbles of the Bernardi–Raugel or the additional cell bubbles (plus face bubbles in 3D) of the P_2^+ finite element method are modified by Π and their interpolation comes at very low computational costs. In the case of the cell bubble reconstruction into BDM_2 only basis functions with zero normal component along the boundary of the cell are involved. Similarly, for a face bubble reconstruction into $BDM_1(\mathcal{T})$ only the basis functions that are nonzero on this face have to be considered.

Remark 4.2. To further reduce the costs it is possible to decompose the test functions into a space V_h^k with the property $\nabla \cdot V_h^k \subseteq Q_h$ and a remainder $V_h^\star := V_h \backslash V_h^k$ (which consists of the bubble functions in case of the Bernardi–Raugel or P_2^+ finite element methods). Then, the $\mathrm{RT}_{k-1}(\mathcal{T})$ reconstruction operator is used only on test functions from $V_h^\star$. This still satisfies all the needed properties from the framework given in [LMT16].

4.3. **Pressure-robust a priori Stokes error estimates.** The variational crime causes a consistency error in the a priori estimates that can be handled with the following lemma.

Lemma 4.3 (Consistency error)**.** For some function $\mathbf{u} \in H^{k+1}$ and the standard interpolation Π into RT_{k-1} or BDM_k, it holds

$$\|\Delta \mathbf{u} \circ (1 - \Pi)\|_{V_{0,h}^\star} := \sup_{\mathbf{v}_h \in V_{0,h}} \frac{\int_D \Delta \mathbf{u} \cdot (\mathbf{v}_h - \Pi \mathbf{v}_h)\, dx}{\|\nabla \mathbf{v}_h\|_{L^2(D)}} \leqslant C |h_T^k \mathbf{u}|_{k+1}.$$

Proof. The proof uses standard interpolation estimates and can be found in [LMT16, Lemma 2.2]. In fact this lemma holds also with respect to $V^\star$, but is only needed in $V_{0,h}$. □

Theorem 4.4 (A priori stationary Stokes error estimates for pressure-robust FEMs)**.** If $u \in H^2(D)$ and $p \in H^1(D)$ and $\mathbf{f} = -\nu \Delta \mathbf{u} + \nabla p \in L^2(D)$, it holds

$$\|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{L^2(D)} \leqslant 2 \inf_{\mathbf{v}_h \in \mathbf{u}_h + V_{0,h}} \|\nabla(\mathbf{u} - \mathbf{v}_h)\|_{L^2(D)} + \|\Delta \mathbf{u} \circ (1 - \Pi)\|_{V_{0,h}^\star}.$$

Proof. The point of departure is Strang's Lemma, namely

$$\nu \|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{L^2(D)} \leqslant 2\nu \inf_{\mathbf{v}_h \in \mathbf{u}_h + V_{0,h}} \|\nabla(\mathbf{u} - \mathbf{v}_h)\|_{L^2(D)} + \sup_{\mathbf{v}_h \in V_{0,h} \backslash \{0\}} \frac{|a(\mathbf{u}, \mathbf{v}_h) - F(\Pi \mathbf{v}_h)|}{\|\nabla \mathbf{v}_h\|_{L^2(D)}}.$$

Insertion of $\mathbf{f} = -\nu \Delta \mathbf{u} + \nabla p$, $\int_D \nabla p \cdot \Pi \mathbf{v}_h \, dx = 0$ and an integration by parts yield

$$a(\mathbf{u}, \mathbf{v}_h) - F(\Pi \mathbf{v}_h) = -\nu \int_D \Delta \mathbf{u} \cdot (\mathbf{v}_h - \Pi \mathbf{v}_h)\, dx \leqslant \nu \|\Delta \mathbf{u} \circ (1 - \Pi)\|_{V_{0,h}^\star} \|\nabla \mathbf{v}_h\|_{L^2(D)}.$$

A division by $\|\nabla \mathbf{v}_h\|_{L^2(D)}$ concludes the proof. □

Remark 4.5. The pressure term $\|\mathbb{P}_h(\nabla p)\|_{V_{0,h}^\star}/\nu$ from the a priori error estimate in Theorem 3.4 for classical finite element methods vanishes in the a priori error estimate in Theorem 4.4 for the modified pressure-robust finite element methods. However, one has to pay with the consistency error term $\|\Delta \mathbf{u} \circ (1-\Pi)\|_{V_{0,h}^\star}$. The latter term is advantageous. It is pressure-independent which ensures the important invariance property that if the right-hand side is changed by a gradient the velocity stays unchanged. Furthermore, the consistency error does not scale with $1/\nu$ like the pressure-dependent term in the classical estimate. The new methods are therefore robust in problems with large Reynolds numbers and complicated pressures. Last but not least, it has the same convergence order (see Lemma 4.3) as the velocity best-approximation error. The error estimate of Theorem 3.4 (in particular the pressure-dependence therein) is sharp, which can be seen in various simple benchmark examples [LM16]. Further examples are presented below.

Remark 4.6. Theorem 4.4 shows that for avoiding *poor mass conservation*, i.e., the pressure-dependent velocity error contribution $\frac{1}{\nu}\inf_{q_h \in Q_h}\|p-q_h\|_{L^2(D)}$, there is actually no need for any kind of *artificial viscosity* (the pressure-robust method has the same stiffness matrix like the classical inf-sup stable method !). This is contrary to traditional attempts like grad-div stabilization [OR04, CELR11, JJLR14], which try to improve the velocity error by adding a stabilization term that allows for an 'improved' L^2-control of the divergence. In our opinion, 'poor mass conservation' is not a stability problem, but related to the lack of L^2 orthogonality between divergence-free and irrotational forces [Lin14].

The rest of this section deals with the a priori estimate for the spatial error in each time step of the time-dependent problem.

Theorem 4.7 (A priori instationary Stokes error estimates for pressure-robust FEMs)**.** If $\mathbf{u}(t_n), \mathbf{u}(t_{n-1}) \in H^2(D)$ and $p(t_n), p(t_{n-1}) \in H^1(D)$ and $\mathbf{f} = -\nu\Delta\mathbf{u} + \nabla p + (t_n - t_{n-1})^{-1}(\mathbf{u}(t_n) - \mathbf{u}(t_{n-1})) \in L^2(D)$, it holds

$$|\!|\!|\mathbf{u}^n - \mathbf{u}_h^n|\!|\!| \leqslant 2 \inf_{\mathbf{v}_h \in \mathbf{u}_h^n + V_{0,h}} |\!|\!|\mathbf{u}^n - \mathbf{v}_h|\!|\!| + \|\Delta\mathbf{u} \circ (1-\Pi)\|_{(V_{0,h},\tilde{a})^\star} + \left\|d(\mathbf{u}(t_{n-1}) - \mathbf{u}_h^{n-1}, \Pi\bullet)\right\|_{(V_{0,h},\tilde{a})^\star}$$

with the energy norm $|\!|\!| \bullet |\!|\!|^2 := \tilde{a}(\bullet,\bullet) := a(\bullet,\bullet) + d(\bullet,\bullet)$.

Proof. The proof is a combination of arguments from Theorem 4.4 and Theorem 3.5. □

Remark 4.8. Theorem 4.7 implies

$$\begin{aligned}\|\nabla(\mathbf{u}^n - \mathbf{u}_h^n)\|_{L^2(D)} \leqslant \frac{1}{\nu^{1/2}}|\!|\!|\mathbf{u}^n - \mathbf{u}_h^n|\!|\!| \leqslant \frac{2}{\nu^{1/2}} \inf_{\mathbf{v}_h \in V_{0,h}} |\!|\!|\mathbf{u}^n - \mathbf{v}_h|\!|\!| \\ + \|\Delta\mathbf{u} \circ (1-\Pi)\|_{V_{0,h}^\star} + \nu^{-1/2}\left\|d(\mathbf{u}(t_{n-1}) - \mathbf{u}_h^{n-1}, \Pi\bullet)\right\|_{(V_{0,h},\tilde{a})^\star}\end{aligned}$$

with the pressure- and ν-independent consistency error term from Lemma 4.3 that was already discussed in Remark 4.5. The latter term can be further estimated by

$$\begin{aligned}\left\|d(\mathbf{u}(t_{n-1}) - \mathbf{u}_h^{n-1}, \Pi\bullet)\right\|_{(V_{0,h},\tilde{a})^\star} \leqslant \frac{\nu^{1/2}}{(t_n - t_{n-1})}\left\|\mathbb{P}(\mathbf{u}(t_{n-1}) - \mathbf{u}_h^{n-1}) \circ (1-\Pi)\right\|_{V_{0,h}^\star} \\ + \min\left\{\frac{1}{(t_n - t_{n-1})^{1/2}}, \frac{C_F}{\nu^{1/2}(t_n - t_{n-1})}\right\}\left\|\mathbb{P}(\mathbf{u}(t_{n-1}) - \mathbf{u}_h^{n-1})\right\|_{L^2(D)}.\end{aligned}$$

Since only the continuous Helmholtz projector appears, the error from the last time step vanishes whenever $\mathbf{u}(t_{n-1}) - \mathbf{u}_h^{n-1}$ is a gradient.

4.4. **Pressure-robust a priori Navier–Stokes error estimates.** In case of the pressure-robust scheme and homogeneous Dirichlet boundary conditions, the small data assumption (3.7) can be relaxed to

$$\nu^{-2}\widetilde{M}\,\|\mathbb{P}(\mathbf{f})\|_{(\Pi V_{0,h})^\star} := \alpha < 1 \tag{4.2}$$

with the continuity constant $\widetilde{M}$ from Theorem 4.10.(1) below. For a right-hand side with a large gradient in their Helmholtz decomposition this actually might result in a larger range for ν in which one can show unique existence of solutions. For completeness the crucial parts of the proof are given in Lemma 4.9. Note, that the dual norm with respect to $\Pi V_{0,h}$ can be bound essentially by the dual norm with respect to $V_{0,h}$ by the interpolation properties of Π in the sense that

$$\|\mathbb{P}(\mathbf{f})\|_{(\Pi V_{0,h})^\star} \leqslant \|\mathbb{P}(\mathbf{f})\|_{V_{0,h}^\star} + Ch\,\|\mathbb{P}(\mathbf{f})\|_{L^2(D)}\,.$$

Lemma 4.9 (Unique existence and stability)**.** Under the assumption (4.2), $\mathbf{u}_D = 0$, and for skew-symmetric d_h, it exists a unique solution $\mathbf{u}_h$ of the pressure-robust finite element method for the stationary Navier–Stokes equations that satisfies the stability estimate

$$\nu\,\|\nabla\mathbf{u}_h\|_{L^2(D)} \leqslant \|\mathbb{P}(\mathbf{f})\|_{(\Pi V_{0,h})^\star}\,.$$

Proof. For existence of solutions see e.g. [Lay08]. To prove the stability estimate, the choice $\mathbf{v}_h = \mathbf{u}_h$ in (4) and the skew-symmetry of d_h yield

$$\nu\,\|\nabla\mathbf{u}_h\|^2_{L^2(D)} = F(\Pi\mathbf{u}_h) = \int_D \mathbf{f}\cdot(\Pi\mathbf{u}_h)\,dx = \int_D \mathbb{P}(\mathbf{f})\cdot(\Pi\mathbf{u}_h)\,dx \leqslant \|\mathbb{P}(\mathbf{f})\|_{(\Pi V_{0,h})^\star}\,\|\nabla\mathbf{u}_h\|_{L^2(D)}\,.$$

To prove uniqueness, assume that two discrete solutions $\mathbf{u}_h$ and $\tilde{\mathbf{u}}_h$ exist. Then, it holds

$$a(\mathbf{u}_h - \tilde{\mathbf{u}}_h, \mathbf{v}_h) + d_h(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) - d_h(\tilde{\mathbf{u}}_h, \tilde{\mathbf{u}}_h, \mathbf{v}_h) = 0 \quad \text{for all } \mathbf{v}_h \in V_{0,h}.$$

The choice $\mathbf{v}_h = \mathbf{u}_h - \tilde{\mathbf{u}}_h$ and the skew-symmetry of c_h show

$$\begin{aligned}\nu\,\|\nabla(\mathbf{u}_h - \tilde{\mathbf{u}}_h)\|^2_{L^2(D)} = -d_h(\mathbf{u}_h - \tilde{\mathbf{u}}_h, \tilde{\mathbf{u}}_h, \mathbf{u}_h - \tilde{\mathbf{u}}_h) &\leqslant M\,\|\nabla\tilde{\mathbf{u}}_h\|_{L^2(D)}\,\|\nabla(\mathbf{u}_h - \tilde{\mathbf{u}}_h)\|^2_{L^2(D)}\\ &\leqslant M\nu^{-1}\,\|\mathbb{P}(\mathbf{f})\|_{(\Pi V_{0,h})^\star}\,\|\nabla(\mathbf{u}_h - \tilde{\mathbf{u}}_h)\|^2_{L^2(D)}\end{aligned}$$

and hence

$$(1 - M\nu^{-2}\,\|\mathbb{P}(\mathbf{f})\|_{V_{0,h}^\star})\,\|\nabla(\mathbf{u}_h - \tilde{\mathbf{u}}_h)\|^2_{L^2(D)} \leqslant 0.$$

The small data assumption (4.2) guarantees that the factor on the left-hand side is positive which implies $\mathbf{u}_h = \tilde{\mathbf{u}}_h$ and concludes the proof. □

Theorem 4.10 (A priori stationary Navier–Stokes error estimates for pressure-robust FEMs)**.** If $u \in H^2(D)$ and $p \in H^1(D)$ and $\mathbf{f} = -\nu\Delta\mathbf{u} + \nabla p + (\mathbf{u}\cdot\nabla)\mathbf{u} \in L^2(D)$ and under the modified small data assumption (4.2) with homogeneous Dirichlet boundary conditions $\mathbf{u}_D = 0$, and under the assumption that c_h is skew-symmetric, it holds

$$\begin{aligned}(1)\qquad |c_h(\Pi\mathbf{u}_h, \mathbf{u}_h, \Pi\mathbf{v}_h) - c_h(\mathbf{u}, \mathbf{u}, \Pi\mathbf{v}_h)| &\leqslant \widetilde{M}\,\|\nabla\mathbf{v}_h\|^2_{L^2(D)}\,\|\nabla\mathbf{u}_h\|_{L^2(D)}\\ &\quad + \widetilde{M}\,\|\nabla\mathbf{v}_h\|_{L^2(D)}\,\|\nabla(\mathbf{u} - \mathbf{u}_h) + \mathbf{v}_h\|_{L^2(D)}\left(\|\nabla\mathbf{u}\|_{L^2(D)} + \|\nabla\mathbf{u}_h\|_{L^2(D)}\right),\end{aligned}$$

$$(2)\qquad \|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{L^2(D)} \leqslant \frac{2}{1-\alpha}\inf_{\mathbf{v}_h\in V_{0,h}}\|\nabla(\mathbf{u} - \mathbf{v}_h)\|_{L^2(D)} + \frac{1}{1-\alpha}\,\|\Delta\mathbf{u}\circ(1-\Pi)\|_{V_{0,h}^\star}\,.$$

Proof. The proof of (1) uses the same arguments as the proof of Theorem 3.8.(1) plus the additional argument that $\|\Pi \mathbf{v}_h\|_{L^6(D)} \leqslant C \|\nabla \mathbf{v}_h\|_{L^2(D)}$. This follows from a triangle inequality and standard interpolation estimates for Π, i.e.

$$\|\Pi \mathbf{v}_h\|_{L^6(D)} \leqslant \|\Pi \mathbf{v}_h - \mathbf{v}_h\|_{L^6(D)} + \|\mathbf{v}_h\|_{L^6(D)} \leqslant C \|\nabla \mathbf{v}_h\|_{L^2(D)}.$$

The proof of (2) is a combination of arguments from Theorem 3.8 and Theorem 4.4. Note, that the term $\nu \|\Delta \mathbf{u} \circ (1 - \Pi)\|_{V_{0,h}^\star}$ plays the same role as the term $\|\mathbb{P}_h(\nabla p)\|_{V_{0,h}^\star}$ in Theorem 3.8. □

5. Irrotational Flows

This section introduces another prominent class of examples that complement the benchmarks of [LM16] to show that the given a priori estimates are sharp, and in particular that the estimates for the classicial methods are not pessimistic.

5.1. **Properties of Irrotational Flows.** Irrotational flows are special solutions of the Navier–Stokes equations based on harmonic functions. Given a harmonic function $\chi \in H^2(D)$, i.e. $\Delta\chi = 0$, its gradient has the following properties.

Lemma 5.1 (Properties of irrotational flows)**.** For any (possibly time-dependent) smooth harmonic potential χ, the vector field $\mathbf{u} := \nabla\chi$ satisfies

(a) $\mathbf{u}$ and $p := 0$ solve the stationary Stokes equations with zero right-hand side, i.e.,

$$-\nu\Delta\mathbf{u} + \nabla p = 0 \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0.$$

(b) $\mathbf{u}$ and $p := -|\mathbf{u}|^2/2$ solve the stationary Navier–Stokes equations, i.e.,

$$-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = 0 \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0.$$

(c) $\mathbf{u}$ and $p := -|\mathbf{u}|^2/2 - \chi_t$ solve the instationary Navier–Stokes equations, i.e.,

$$\mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = 0 \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0.$$

Proof. The proof uses only elementary calculations. For instance, (a) follows directly from $-\Delta\mathbf{u} = \nabla \times (\nabla \times (\nabla\chi)) - \nabla(\Delta\chi) = 0$. Elementary calculations and $\nabla \times \nabla\chi = 0$ show

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = (\nabla \times \mathbf{u}) \times \mathbf{u} + \nabla(|\mathbf{u}|^2)/2 = \nabla(|\mathbf{u}|^2)/2.$$

This together with (a) implies (b). The last assertion (c) follows from $\mathbf{u}_t = \nabla(\chi_t)$. □

5.2. **Numerical velocity errors from the time derivative.** Lemma 5.1.(c) reveals an interesting property of potential flows that are scaled in time. Their time-derivative goes completely into the pressure and the reason for this is the mentioned L^2 orthogonality between gradients (recall that $\mathbf{u}_t = \nabla\chi_t$ is a gradient) and divergence-free functions. Therefore, it holds

$$d(\mathbf{u}^\star, \mathbf{v}) = \frac{\alpha(t_n) - \alpha(t_{n-1})}{t_n - t_{n-1}} \int_D \nabla\chi \cdot \mathbf{v}\, dx = 0 \quad \text{for all } \mathbf{v} \in V_0.$$

If the discretisation of the time-derivative does not preserve this property, even polynomial potential flows in the velocity ansatz space cannot be computed exactly. If however $\nabla\chi(t_n)$ with inhomogeneous Dirichlet boundary conditions lies in the finite element velocity ansatz space, then a pressure-robust finite element method will deliver the exact solution in this timestep due to Theorem 4.7 and Remark 4.8.

5.3. **Numerical velocity errors from the nonlinear convection term.** The same holds true for the nonlinear convection term. The proof of Lemma 5.1.(b) shows that, for every potential flow $\mathbf{u}$, $(\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla(|\mathbf{u}|^2)/2$ is a gradient and therefore, it holds

$$c(\mathbf{u}, \mathbf{u}, \mathbf{v}) = \frac{1}{2} \int_D \nabla(|\mathbf{u}|^2) \cdot \mathbf{v} \, dx = 0 \quad \text{for all } \mathbf{v} \in V_0.$$

If $\mathbf{u}$ is a polynomial of order k, $\nabla(|\mathbf{u}|^2)$ has order $2k - 1$ in general. Thus, the problem here is even more severe than for the time derivative.

Remark 5.2. From potential flows one can learn that the nonlinear term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ may excite at high Reynolds numbers two different kinds of velocity errors:

- dominant advection, whenever $\mathbb{P}((\mathbf{u} \cdot \nabla)\mathbf{u})$ is large,
- the *lack of pressure-robustness*, which was sometimes phenomenologically denoted as *poor mass conservation* [Lin09], whenever $(\mathbf{u} \cdot \nabla)\mathbf{u} - \mathbb{P}((\mathbf{u} \cdot \nabla)\mathbf{u})$ is large.

This discrimination between these two sources of velocity errors at high Reynolds numbers is in our opinion important for the further improvement of discretization schemes for the incompressible Navier–Stokes equations.

6. Numerical examples

This section studies the described sources of velocity errors numerically. We also discuss the L^2 error between the discrete pressure and the L^2 best approximation, i.e.

$$p_{\text{best}} := \underset{q_h \in Q_h}{\operatorname{argmin}} \|p - q_h\|_{L^2(D)} \, .$$

All examples have been calculated on unstructured regular triangulations.

6.1. **Example 1 - time-dependent Stokes potential flow.** This example studies the potential flow derived from the time-dependent harmonic function $\chi(x, t) := \min(t, 1)(x^3 - 3xy^2)$ and the viscosity $\nu = 1/20$ in the time interval $[0, 2]$ and the domain $D := (-1, 1)^2$. This leads to the velocity field $\mathbf{u}$ and the pressure p from Lemma 5.1.(c). The nonlinear convection term is omitted in this example to concentrate on the influence of the time discretisation. The step size is fixed to $dt = 0.01$.

Table 6.1 shows the error of the unmodified P_2^+ finite element method in 10 equidistant time steps. Although the exact solution is quadratic and therefore in the velocity ansatz space, there are relatively large errors in the time steps in the interval $[0, 1]$. After $t = 1$, the exact solution becomes stationary and the time derivative disappears. Even then, the discrete solution, due to the errors in the previous time step, still sees changes in time and converges only slowly to the exact solution. Contrary, the results for the modified P_2^+ finite element method (also shown in Table 6.1) are perfect in every time step up to numerical precision. Since the pressure is a polynomial of order 4 it cannot be computed exactly (in fact this causes the errors in the velocity for the unmodified method). However, even the error in the pressure is smaller for the modified finite element method. In fact, it equals the best approximation error of the pressure in the pressure ansatz space and is in this sense optimal. Asymptotically for $t \to \infty$ both methods deliver the exact solution as expected.

After $t = 1$ the time-dependent pressure contribution disappears and the pressure becomes trivial, i.e. $p(t) = 0$ for $t > 1$. Accordingly, the pressure error for the modified method also becomes zero, while the pressure error for the unmodified method is not. Even after $t = 2$ seconds, the errors of the unmodified methods are still large and the convergence to the stationary solution is very slow.

t	$\Vert(\mathbf{u}-\mathbf{u}_h)(t)\Vert_{L^2(D)}$	$\Vert\nabla(\mathbf{u}-\mathbf{u}_h)(t)\Vert_{L^2(D)}$	$\Vert(p_h-p_{\mathrm{best}})(t)\Vert_{L^2(D)}$	$\Vert\nabla\cdot\mathbf{u}_h(t)\Vert_{L^2(D)}$
0.20	2.6910e-03/4.4707e-15	6.0592e-01/1.7409e-12	1.7152e-02/1.6974e-14	3.6573e-02/1.0606e-13
0.40	2.8001e-03/4.9061e-15	6.1244e-01/1.8838e-12	1.7137e-02/3.4901e-14	3.7007e-02/1.1424e-13
0.60	2.8325e-03/4.5923e-15	6.1253e-01/1.5345e-12	1.7131e-02/5.3307e-14	3.7008e-02/9.1677e-14
0.80	2.8479e-03/5.3264e-15	6.1244e-01/1.7943e-12	1.7129e-02/9.8341e-14	3.7000e-02/1.0804e-13
1.00	2.8563e-03/6.9049e-15	6.1239e-01/2.2975e-12	1.7128e-02/6.7768e-14	3.6996e-02/1.3980e-13
1.20	6.8253e-04/6.6368e-15	1.6042e-02/1.1277e-12	2.5935e-04/2.6757e-14	8.9850e-04/6.9843e-14
1.40	3.7981e-04/8.4006e-15	4.2960e-03/1.4756e-12	8.5689e-05/2.2165e-14	1.8095e-04/9.2226e-14
1.60	2.4933e-04/8.7068e-15	2.1634e-03/1.3395e-12	4.5360e-05/3.5255e-14	8.1357e-05/8.4434e-14
1.80	1.8143e-04/9.3948e-15	1.2759e-03/1.4089e-12	2.6948e-05/2.5870e-14	4.3224e-05/8.7051e-14
2.00	1.4170e-04/9.1326e-15	8.3112e-04/8.7159e-13	1.7217e-05/4.6744e-14	2.4904e-05/5.3867e-14
$\vdots$				
∞	1.8018e-15/2.4747e-15	3.6315e-13/4.2315e-13	2.7708e-15/8.4975e-15	2.3344e-14/2.6434e-14

TABLE 6.1. Errors for the unmodified/modified P_2^+ finite element method in Example 1.

t	$\Vert(\mathbf{u}-\mathbf{u}_h)(t)\Vert_{L^2(D)}$	$\Vert\nabla(\mathbf{u}-\mathbf{u}_h)(t)\Vert_{L^2(D)}$	$\Vert(p_h-p_{\mathrm{best}})(t)\Vert_{L^2(D)}$	$\Vert\nabla\cdot\mathbf{u}_h(t)\Vert_{L^2(D)}$
0.20	5.9939e-02/1.1851e-02	4.0665e+00/1.0627e+00	1.0133e-01/6.7264e-03	4.7342e-01/9.7465e-02
0.40	7.6058e-02/2.3841e-02	4.9593e+00/2.1730e+00	7.7654e-02/1.2124e-02	5.6120e-01/1.9785e-01
0.60	8.5266e-02/3.6074e-02	5.5304e+00/3.2966e+00	7.4426e-02/1.7510e-02	6.1716e-01/2.9898e-01
0.80	9.3472e-02/4.8470e-02	6.1607e+00/4.4240e+00	7.5262e-02/2.2929e-02	6.7596e-01/4.0026e-01
1.00	1.0199e-01/6.0974e-02	6.8962e+00/5.5526e+00	7.7348e-02/2.8373e-02	7.4217e-01/5.0159e-01
1.20	7.1557e-02/6.2145e-02	5.6048e+00/5.6263e+00	4.6596e-02/2.7898e-02	5.1401e-01/5.0621e-01
1.40	6.6765e-02/6.2779e-02	5.6205e+00/5.6435e+00	2.7755e-02/2.7820e-02	5.0646e-01/5.0696e-01
1.60	6.5378e-02/6.3164e-02	5.6381e+00/5.6478e+00	2.7108e-02/2.7752e-02	5.0669e-01/5.0705e-01
1.80	6.4832e-02/6.3420e-02	5.6447e+00/5.6489e+00	2.7327e-02/2.7705e-02	5.0704e-01/5.0706e-01
2.00	6.4578e-02/6.3598e-02	5.6475e+00/5.6493e+00	2.7450e-02/2.7673e-02	5.0720e-01/5.0707e-01
$\vdots$				
∞	6.4173e-02/6.4173e-02	5.6499e+00/5.6499e+00	2.7590e-02/2.7590e-02	5.0720e-01/5.0720e-01

TABLE 6.2. Errors for the unmodified/modified Bernardi–Raugel finite element method in Example 1.

Table 6.2 displays the results for the unmodified and the modified Bernardi–Raugel finite element method. Also for this method there are some smaller but still significant improvements by the modification. Asymptotically for $t \to \infty$ both methods deliver the same stationary solution as expected.

6.2. **Example 2 - irrotational flow with nonlinear convection.** This example studies the influence of the nonlinear convection term and its discretisation by rotational or convective form. This time, we consider the stationary solution from Example 1 without time derivative but with nonlinear convection term and viscosity parameter $\nu = 1/20$ on several consecutive refinement levels.

Table 6.3 displays the results for the P_2^+ finite element method in case of a convective discretisation of the nonlinear term. Although the exact velocity is in the velocity ansatz space, the discrete solution of the unmodified method shows errors. These errors disappear when using the rotational form of the nonlinear term (see Table 6.4) or when using the test function modifications in any form of the nonlinear term.

The same comparison for the Bernardi–Raugel method Table 6.5 and Table 6.6 shows that the rotational form is superior compared to the convective form in the example as predicted by the theory for potential flows. The L^2 error of the velocity is improved by a factor 5. The same factor is reached by the modified Bernardi–Raugel method. In the special case of a potential flow the unmodified method with rotational form gives slightly better results then the modified method due to the additional consistency errors.

ndof	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(D)}$	$\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{L^2(D)}$	$\|p_h - p_{\text{best}}\|_{L^2(D)}$	$\|\nabla\cdot\mathbf{u}_h\|_{L^2(D)}$
1200	1.6010e-02/1.6935e-14	3.8424e+00/4.9696e-12	7.0968e-02/5.5260e-14	2.3628e-01/2.8388e-13
4529	1.9067e-03/2.0753e-14	1.0329e+00/1.5620e-11	1.3858e-02/1.0990e-13	6.3991e-02/9.8909e-13
18175	2.4999e-04/2.1977e-14	2.6717e-01/3.7412e-11	3.7823e-03/1.7093e-13	1.6705e-02/2.3667e-12
71847	3.3861e-05/2.3933e-14	7.0547e-02/9.2368e-11	9.2512e-04/2.8032e-13	4.4435e-03/5.9195e-12
	2.9611/exact	1.9723/exact	2.0857/exact	1.9615/exact

TABLE 6.3. Errors for the unmodified/modified P_2^+ finite element method in Example 2 with nonlinear convection (convective form).

ndof	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(D)}$	$\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{L^2(D)}$	$\|p_h - p_{\text{best}}\|_{L^2(D)}$	$\|\nabla\cdot\mathbf{u}_h\|_{L^2(D)}$
1200	2.4353e-15/2.7630e-15	4.3055e-13/3.8745e-13	1.4266e-14/1.3747e-14	2.7122e-14/2.4870e-14
4529	1.5380e-15/1.4581e-15	4.5658e-13/4.3840e-13	1.5912e-14/1.6321e-14	2.9016e-14/2.8166e-14
18175	1.4589e-15/1.5453e-15	7.6389e-13/7.4916e-13	2.4669e-14/2.2352e-14	5.0441e-14/4.9466e-14
71847	1.6415e-15/1.5395e-15	1.0911e-12/1.0851e-12	3.4217e-14/3.3817e-14	7.3189e-14/7.3270e-14
	exact/exact	exact/exact	exact/exact	exact/exact

TABLE 6.4. Errors for the unmodified/modified P_2^+ finite element method in Example 2 with nonlinear convection (skew-symmetric rotational form).

ndof	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(D)}$	$\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{L^2(D)}$	$\|p_h - p_{\text{best}}\|_{L^2(D)}$	$\|\nabla\cdot\mathbf{u}_h\|_{L^2(D)}$
491	7.6846e-01/1.5266e-01	4.1488e+01/8.0158e+00	1.0094e+00/2.5310e-01	4.2562e+00/8.2654e-01
1808	1.6024e-01/2.1543e-02	1.8449e+01/3.2436e+00	2.7847e-01/4.2950e-02	2.0754e+00/3.2881e-01
7161	4.0297e-02/4.3692e-03	1.0282e+01/1.4828e+00	1.2940e-01/1.0178e-02	1.1417e+00/1.3591e-01
28123	1.0823e-02/1.1113e-03	5.2187e+00/7.2493e-01	4.8291e-02/3.6182e-03	5.8256e-01/6.6258e-02
	1.9472/2.0279	1.0045/1.0599	1.4600/1.5319	0.9966/1.0641

TABLE 6.5. Errors for the unmodified/modified Bernardi–Raugel finite element method in Example 2 with nonlinear convection (convective form).

ndof	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(D)}$	$\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{L^2(D)}$	$\|p_h - p_{\text{best}}\|_{L^2(D)}$	$\|\nabla\cdot\mathbf{u}_h\|_{L^2(D)}$
491	9.5389e-02/1.0653e-01	6.9980e+00/7.4593e+00	1.0197e-01/1.1565e-01	6.5807e-01/7.1574e-01
1808	1.8847e-02/1.8771e-02	3.1710e+00/3.1416e+00	2.9111e-02/2.9507e-02	3.1963e-01/3.1192e-01
7161	4.2442e-03/4.2378e-03	1.4666e+00/1.4659e+00	7.8215e-03/7.7830e-03	1.3505e-01/1.3429e-01
28123	1.1088e-03/1.1078e-03	7.2532e-01/7.2491e-01	3.3535e-03/3.3632e-03	6.6189e-02/6.6109e-02
	1.9882/1.9872	1.0428/1.0431	1.2544/1.2428	1.0563/1.0497

TABLE 6.6. Errors for the unmodified/modified Bernardi–Raugel finite element method in Example 2 with nonlinear convection (rotational form).

6.3. **Example 3 - rigid body rotation with nonlinear convection.** This example from [LM16] is revisited on a square $D := (-1,1)^2$ to study what happens with the different discretisations of the nonlinear term which is discretised in convective form. The exact solution is a circular flow $\mathbf{u}(x,y) := (-y,x)$ with quadratic pressure $p(x,y) := (x^2+y^2)/2 - 1/4$ balancing the nonlinear convection term, i.e. $\nabla p + (\mathbf{u}\cdot\nabla)\mathbf{u} = 0$.

This example shows that the rotational form in general does not lead to better results for standard finite element methods. In fact, the rotational form is inferior to the convective form by about a factor 2 in the L^2 error of the velocity in this example, see Tables 6.7-6.10. In practise it could be tedious to find the optimal discretisation of the nonlinear term for each problem and the present example shows that neither may give good results compared to the modified methods. To summarize, the pressure-robust method always gives optimal results whichever discretisation of the nonlinear term is used. Since in CFD usually the convective form is preferred to the rotational form, we will only investigate the convective form in the following numerical examples.

ndof	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(D)}$	$\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{L^2(D)}$	$\|p_h-p_{\text{best}}\|_{L^2(D)}$	$\|\nabla\cdot\mathbf{u}_h\|_{L^2(D)}$
1200	4.0272e-04/9.2449e-16	1.2423e-01/3.6157e-13	1.0601e-03/2.8583e-15	7.9580e-03/2.1652e-14
4529	5.7096e-05/1.3097e-15	3.4937e-02/1.0610e-12	2.4937e-04/5.7437e-15	2.2722e-03/6.8847e-14
18175	6.8230e-06/1.2445e-15	8.4184e-03/2.4199e-12	5.6836e-05/8.9405e-15	5.4726e-04/1.5522e-13
71847	9.0455e-07/1.4507e-15	2.2007e-03/5.6558e-12	1.4136e-05/1.5062e-14	1.4296e-04/3.6677e-13
	2.9929/exact	1.9872/exact	2.0610/exact	1.9883/exact

TABLE 6.7. Errors for the unmodified/modified P_2^+ finite element method in Example 3 with nonlinear convection (convective form).

ndof	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(D)}$	$\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{L^2(D)}$	$\|p_h-p_{\text{best}}\|_{L^2(D)}$	$\|\nabla\cdot\mathbf{u}_h\|_{L^2(D)}$
1200	8.1682e-04/1.6779e-15	2.5249e-01/6.8647e-13	2.1220e-03/5.3397e-15	1.6209e-02/4.1183e-14
4529	1.1431e-04/2.3815e-15	7.0079e-02/2.1059e-12	4.9878e-04/1.1242e-14	4.5502e-03/1.3680e-13
18175	1.3644e-05/2.5785e-15	1.6835e-02/4.8281e-12	1.1366e-04/1.7671e-14	1.0945e-03/3.0946e-13
71847	1.8095e-06/2.6522e-15	4.4027e-03/1.1302e-11	2.8271e-05/3.0109e-14	2.8600e-04/7.3282e-13
	2.9923/exact	1.9866/exact	2.0609/exact	1.9878/exact

TABLE 6.8. Errors for the unmodified/modified P_2^+ finite element method in Example 3 with nonlinear convection (rotational form).

ndof	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(D)}$	$\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{L^2(D)}$	$\|p_h-p_{\text{best}}\|_{L^2(D)}$	$\|\nabla\cdot\mathbf{u}_h\|_{L^2(D)}$
491	3.9897e-02/1.0746e-15	1.8322e+00/6.7352e-14	3.1336e-02/1.1252e-15	2.1406e-01/7.2026e-15
1808	1.1620e-02/1.3589e-15	1.0809e+00/1.6705e-13	1.2782e-02/1.9298e-15	1.2683e-01/1.7727e-14
7161	2.6582e-03/1.3883e-15	5.6523e-01/2.7689e-13	5.0257e-03/2.6782e-15	6.3692e-02/2.7345e-14
28123	6.8098e-04/2.2803e-15	2.9069e-01/8.7453e-13	2.2556e-03/9.5695e-15	3.2940e-02/8.6137e-14
	2.0172/exact	0.9850/exact	1.1867/exact	0.9767/exact

TABLE 6.9. Errors for the unmodified/modified Bernardi–Raugel finite element method in Example 3 with nonlinear convection (convective form).

ndof	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(D)}$	$\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{L^2(D)}$	$\|p_h-p_{\text{best}}\|_{L^2(D)}$	$\|\nabla\cdot\mathbf{u}_h\|_{L^2(D)}$
491	7.7621e-02/1.6685e-15	3.7193e+00/1.2426e-13	6.2003e-02/2.3589e-15	4.3560e-01/1.3725e-14
1808	2.2229e-02/2.1272e-15	2.1970e+00/3.1392e-13	2.4992e-02/3.7814e-15	2.5451e-01/3.3635e-14
7161	5.3066e-03/1.8231e-15	1.1294e+00/5.0067e-13	1.0066e-02/7.3986e-15	1.2734e-01/4.9802e-14
28123	1.3665e-03/3.1990e-15	5.8167e-01/1.6780e-12	4.5086e-03/1.9326e-14	6.5922e-02/1.6518e-13
	2.0096/exact	0.9828/exact	1.1897/exact	0.9752/exact

TABLE 6.10. Errors for the unmodified/modified Bernardi–Raugel finite element method in Example 3 with nonlinear convection (rotational form).

6.4. **Example 4 - test case with higher polynomial.** This example studies a potential flow of higher polynomial degree, i.e. $\mathbf{u} := \nabla h$ with $h := y^5 + 5x^4y - 10x^2y^3$ on the square domain $D := (-0.5, 0.5)^2$ in presence of the nonlinear convection term which is discretised in convective form. This time the exact solution is not in the ansatz space of any finite element method under consideration. However, the effect of pressure-robustness is clearly visible in the numbers of Table 6.13 and 6.14 for the Bernardi–Raugel and the P_2^+ finite element method, respectively. The numbers in these tables show by what factor the error of the unmodified method is larger compared to the modified method for different refinement levels and different chocies of ν. On the finer meshes the factor almost scales with $1/\nu$, in other words there the error is indeed dominated by the influence of the bad resolved pressure. The factors even seem to increase on finer meshes where the influence of the dominant convection reduces. For the smallest ν and finest mesh in the presented study, an improvement factor of about 73 is attained for the Bernardi–Raugel finite element method. If one assume linear convergence of the L^2 gradient error norm, one would need more than 6 uniform mesh refinements to get a similar error with the unmodified method. For the P2 bubble finite

ν	ndof					
	491	1808	7161	28123	112212	446447
$1e+05$	$2.8392e+00$	$1.4233e+00$	$7.0681e-01$	$3.5651e-01$	$1.7544e-01$	$8.7360e-02$
$1e+01$	$2.8392e+00$	$1.4233e+00$	$7.0684e-01$	$3.5651e-01$	$1.7545e-01$	$8.7361e-02$
$1e+00$	$2.8391e+00$	$1.4229e+00$	$7.0716e-01$	$3.5655e-01$	$1.7548e-01$	$8.7373e-02$
$1e-01$	$2.8502e+00$	$1.4257e+00$	$7.1629e-01$	$3.5994e-01$	$1.7752e-01$	$8.8389e-02$
$1e-02$	$4.2000e+00$	$2.0085e+00$	$1.2193e+00$	$6.0641e-01$	$3.1379e-01$	$1.5888e-01$
$2e-03$	-	-	$5.7239e+00$	$2.4883e+00$	$1.3011e+00$	$6.6753e-01$
$1e-03$	-	-	-	$5.3208e+00$	$2.6212e+00$	$1.3273e+00$
$5e-04$	-	-	-	-	$5.7847e+00$	$2.6825e+00$
$2e-04$	-	-	-	-	-	$8.1808e+00$

TABLE 6.11. L^2 gradient errors of the unmodified Bernardi–Raugel finite element method for different refinement levels and different choices of ν in Example 4. A '-' indicates that the unmodified method did not converge on this refinement level.

ν	ndof					
	491	1808	7161	28123	112212	446447
$1e+05$	$2.8392e+00$	$1.4233e+00$	$7.0681e-01$	$3.5651e-01$	$1.7544e-01$	$8.7360e-02$
$1e+01$	$2.8391e+00$	$1.4233e+00$	$7.0680e-01$	$3.5651e-01$	$1.7544e-01$	$8.7360e-02$
$1e+00$	$2.8381e+00$	$1.4236e+00$	$7.0676e-01$	$3.5650e-01$	$1.7544e-01$	$8.7360e-02$
$1e-01$	$2.8340e+00$	$1.4266e+00$	$7.0639e-01$	$3.5642e-01$	$1.7545e-01$	$8.7362e-02$
$1e-02$	$3.2267e+00$	$1.4963e+00$	$7.0964e-01$	$3.5683e-01$	$1.7564e-01$	$8.7392e-02$
$2e-03$	-	$2.4736e+00$	$8.0866e-01$	$3.7655e-01$	$1.7917e-01$	$8.7898e-02$
$1e-03$	-	-	$1.0031e+00$	$4.1765e-01$	$1.8781e-01$	$8.9221e-02$
$5e-04$	-	-	$1.9980e+00$	$5.2075e-01$	$2.1132e-01$	$9.3376e-02$
$2e-04$	-	-	-	$1.8380e+00$	$3.1187e-01$	$1.1174e-01$

TABLE 6.12. L^2 gradient errors of the modified Bernardi–Raugel finite element method for different refinement levels and different choices of ν in Example 4. A '-' indicates that the modified method did not converge on this refinement level.

element method the maximal observed improvement factor is about 139, which still equals more than three refinement levels under the assumption of quadratic convergence of the L^2 gradient error. Note, that the factors will most certainly increase for smaller ν. Tables 6.11 and 6.12 list the absolute L^2 gradient errors of the classical and the modified Bernardi–Raugel finite element methods. The numbers on the finest mesh in the last columns in these tables show that the solutions of the modified method (see Table 6.12) are stable even for rather small $\nu \geqslant 10^{-3}$ before larger velocity errors appear. The reason for these velocity errors is that the divergence-free part of the discrete nonlinear term $\mathbb{P}(\frac{1}{\nu}(\mathbf{u}_h \cdot \nabla)\mathbf{u}_h))$ can be rather large on coarse grids, though the divergence-free part of the continuous nonlinear term $\mathbb{P}(\frac{1}{\nu}(\mathbf{u}\cdot\nabla)\mathbf{u}))$ vanishes. In some sense, one could call this effect as some kind of pseudo-dominant convection, which can occur on too coarse grids. On the contrary, the solutions of the standard method are only stable up to $\nu \geqslant 10^{-1}$ (see Table 6.11). Of course, also for the standard method the velocity errors are due to this pseudo-dominant convection. However, here the velocity errors are larger than in the modified method, since the discrete Helmholtz projector of the standard method $\mathbb{P}_h(\frac{1}{\nu}(\mathbf{u}_h \cdot \nabla)\mathbf{u}_h))$ is larger. Example 4 clearly shows that the pressure-related errors can become the dominant source for the velocity error.

6.5. **Example 5 - 3d potential flow.** This example studies the three-dimensional potential flow $\mathbf{u} = \nabla h$ for the potential $h(x, y, z) = xyz$ on the cube $\Omega := (1.0, 1.1)^3$ with nonlinear convection term in its convective form.

ν	ndof					
	491	1808	7161	28123	112212	446447
$1e+05$	1.00	1.00	1.00	1.00	1.00	1.00
$1e+01$	1.00	1.00	1.00	1.00	1.00	1.00
$1e+00$	1.00	1.00	1.00	1.00	1.00	1.00
$1e-01$	1.01	1.00	1.01	1.01	1.01	1.01
$1e-02$	1.30	1.34	1.72	1.70	1.79	1.82
$2e-03$	-	-	7.08	6.61	7.26	7.59
$1e-03$	-	-	-	12.74	13.96	14.88
$5e-04$	-	-	-	-	27.37	28.73
$2e-04$	-	-	-	-	-	73.21

Table 6.13. Multiplicative factor between the L^2 gradient errors of the unmodified/modified Bernardi–Raugel finite element method for different refinement levels and different choices of ν in Example 4. A '-' indicates that the unmodified or both methods did not converge on this refinement level.

ν	ndof						
	304	1200	4529	18175	71847	287593	1146124
$1e+00$	1.00	1.00	1.00	1.00	1.00	1.00	1.00
$1e-01$	1.01	1.00	1.00	1.00	1.01	1.01	1.01
$1e-02$	1.31	1.34	1.21	1.41	1.44	1.45	1.47
$2e-03$	-	3.67	3.28	4.72	5.03	5.31	5.48
$1e-03$	-	-	5.12	8.48	9.39	10.20	10.72
$2e-04$	-	-	-	22.46	33.79	41.26	47.57
$1e-04$	-	-	-	-	49.01	68.30	84.17
$5e-05$	-	-	-	-	-	-	139.19

Table 6.14. Multiplicative factor between the L^2 gradient errors of the unmodified/modified P2 bubble finite element method for different refinement levels and different choices of ν in Example 4. A '-' indicates that the unmodified or both methods did not converge on this refinement level.

ν	ndof			
	884	5124	36555	277056
$1e+00$	1.01	1.01	1.02	1.03
$1e-01$	1.63	2.19	2.42	2.58
$1e-02$	12.31	19.46	21.95	23.84
$2e-03$	35.69	71.82	97.94	114.61
$1e-03$	40.19	93.73	156.28	208.62
$5e-04$	38.57	102.71	203.99	328.12
$2e-04$	-	-	133.33	441.19

Table 6.15. Multiplicative factor between the L^2 gradient errors of the unmodified/modified Bernardi–Raugel finite element method for different refinement levels and different choices of ν in Example 5. A '-' indicates that the unmodified or both methods did not converge on this refinement level.

Table 6.15 shows that in this three-dimensional test problem an improvement factor of about 441 is possible on the finest mesh with the smallest viscosity parameter under consideration. If a linear convergence of the L^2 gradient error is assumed, this corresponds to more than 8 uniform refinements! Since, the number of degrees of freedom multiplies by 8 for each refinement, one needs more than 16 million times more degrees of freedom for the unmodified method to get the same error that the modified method computed.

6.6. **Example 6 - transient higher-order potential flow.** This example studies the time-dependent exact velocity $\mathbf{u}(t) := \min(\max(t,0),1)\nabla h$ for $h := (x^3y - y^3x)$ similar to the first example but with a cubic velocity and we also include the nonlinear convection term discretised in convective form.

t	$\Vert(\mathbf{u}-\mathbf{u}_h)(t)\Vert_{L^2(D)}$	$\Vert\nabla(\mathbf{u}-\mathbf{u}_h)(t)\Vert_{L^2(D)}$	$\Vert(p_h-p_{\text{best}})(t)\Vert_{L^2(D)}$	$\Vert\nabla\cdot\mathbf{u}_h(t)\Vert_{L^2(D)}$
0.20	3.7906e-03/4.8096e-05	4.3356e+00/7.2055e-02	5.2881e-03/3.3700e-05	4.9134e-01/6.6274e-03
0.40	5.3027e-03/9.9701e-05	5.9597e+00/1.4865e-01	1.1126e-02/8.3382e-05	6.6878e-01/1.3934e-02
0.60	9.7365e-03/1.5633e-04	1.0524e+01/2.3196e-01	2.1485e-02/1.5711e-04	1.1704e+00/2.2176e-02
0.80	1.7642e-02/2.1819e-04	1.8070e+01/3.2301e-01	3.9046e-02/2.5131e-04	1.9866e+00/3.1439e-02
1.00	2.9185e-02/2.8595e-04	2.8780e+01/4.2237e-01	6.6886e-02/3.6577e-04	3.1189e+00/4.1747e-02
1.20	3.1087e-02/2.9656e-04	2.8635e+01/4.2337e-01	5.6894e-02/3.6530e-04	3.1003e+00/4.1875e-02
1.40	3.1840e-02/3.0333e-04	2.8640e+01/4.2336e-01	5.6705e-02/3.6375e-04	3.1007e+00/4.1874e-02
1.60	3.2438e-02/3.0737e-04	2.8648e+01/4.2334e-01	5.6385e-02/3.6014e-04	3.1014e+00/4.1873e-02
1.80	3.2768e-02/3.0924e-04	2.8662e+01/4.2333e-01	5.6311e-02/3.5637e-04	3.1024e+00/4.1872e-02
2.00	3.2953e-02/3.1010e-04	2.8673e+01/4.2332e-01	5.6336e-02/3.5444e-04	3.1032e+00/4.1871e-02

Table 6.16. Errors for the unmodified/modified Bernardi–Raugel finite element method in Example 6 for $\nu = 1/200$, $dt = 1/100$ and 112212 degrees of freedom.

t	$\Vert(\mathbf{u}-\mathbf{u}_h)(t)\Vert_{L^2(D)}$	$\Vert\nabla(\mathbf{u}-\mathbf{u}_h)(t)\Vert_{L^2(D)}$	$\Vert(p_h-p_{\text{best}})(t)\Vert_{L^2(D)}$	$\Vert\nabla\cdot\mathbf{u}_h(t)\Vert_{L^2(D)}$
0.20	8.6985e-04/7.8502e-06	8.0572e-01/1.0355e-02	3.1188e-01/3.1188e-01	4.7676e-02/5.3133e-04
0.40	9.2879e-04/1.7693e-05	9.0666e-01/2.0794e-02	3.9485e-01/3.9492e-01	5.4454e-02/1.1022e-03
0.60	1.0442e-03/3.0185e-05	1.1581e+00/3.2181e-02	4.4030e-01/4.4037e-01	7.0122e-02/1.7410e-03
0.80	1.4293e-03/4.5527e-05	1.6982e+00/4.4579e-02	4.5948e-01/4.5954e-01	1.0263e-01/2.4523e-03
1.00	2.0697e-03/6.4122e-05	2.4536e+00/5.8131e-02	5.0075e-01/5.0080e-01	1.4782e-01/3.2392e-03
1.20	2.4214e-03/7.3304e-05	2.6540e+00/6.1244e-02	4.3572e-01/4.3571e-01	1.5990e-01/3.4613e-03
1.40	2.6154e-03/7.9184e-05	2.6823e+00/6.2296e-02	4.3572e-01/4.3571e-01	1.6169e-01/3.5286e-03
1.60	2.7954e-03/8.4353e-05	2.6866e+00/6.2715e-02	4.3572e-01/4.3571e-01	1.6204e-01/3.5571e-03
1.80	2.9711e-03/8.9147e-05	2.6883e+00/6.2858e-02	4.3572e-01/4.3571e-01	1.6212e-01/3.5691e-03
2.00	3.1422e-03/9.3704e-05	2.6908e+00/6.2904e-02	4.3572e-01/4.3571e-01	1.6221e-01/3.5737e-03

Table 6.17. Errors for the unmodified/modified P_2^+ finite element method in Example 6 for $\nu = 1/1000$, $dt = 1/100$ and 18175 degrees of freedom.

Table 6.16 shows the errors at 10 equidistant time steps in the interval $[0, 2]$ for the Bernardi–Raugel finite element method. The smaller ν compared to the other nonlinear examples causes a much larger pressure-dependent error term and so increases the gap in the errors between the unmodified method and its pressure-robust modification. At time $t = 1.0$ the L^2 velocity gradient error of the modified method is about 42 times smaller than the corresponding error of the unmodified method. Since the L^2 velocity gradient error in this smooth setting converges with linear speed with respect to the mesh width, this factor corresponds to a saving of more than 5 refinement levels.

Table 6.17 shows the results for the P_2^+ finite element method for $\nu = 1/1000$ with similar conclusions. Here, the improvement in the L^2 velocity gradient error also comes with a factor of about 42, which is still more than two refinement levels assuming the optimal quadratic convergence rate.

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