Snapshots of modern mathematics
from Oberwolfach

№ 3/2015

Zero-dimensional symmetry

George Willis

This snapshot is about *zero-dimensional symmetry*. Thanks to recent discoveries we now understand such symmetry better than previously imagined possible. While still far from complete, a picture of zero-dimensional symmetry is beginning to emerge.

1 An introduction to symmetry: spinning globes and infinite wallpapers

Let's begin with an example. Think of a sphere, for example a globe's surface. Every point on it is specified by two parameters, longitude and latitude. This makes the sphere a *two-dimensional surface.*

You can rotate the globe along an axis through the center; the object you obtain after the rotation still looks like the original globe (although now maybe New York is where the Mount Everest used to be), meaning that the sphere has *rotational symmetry*. Each rotation is prescribed by the latitude and longitude where the axis cuts the southern hemisphere, and by an angle through which it rotates the sphere.[1] These three parameters specify all rotations of the sphere, which thus has *three-dimensional rotational symmetry*.

In general, a symmetry may be viewed as being a transformation (such as a rotation) that leaves an object looking unchanged. When one transformation is followed by a second, the result is a third transformation that is called the *product* of the other two. The collection of symmetries and their product

[1] Note that we include the rotation through the angle 0, that is, the case where the globe actually does not rotate at all.

operation forms an algebraic structure called a *group*[2]. Mathematicians have used groups to classify symmetry since the nineteenth century, with some of the most influential ones being Évariste Galois (1811–1832), Felix Klein (1849–1925), and Sophus Lie (1842–1899), see [4, 3].

In the example of rotations of the globe, one could move the axis just a little or vary the angle of rotation ever so slightly. Mathematicians say that the rotational symmetries can be *continuously parametrized* or even *smoothly parametrized* (meaning that there are no abrupt changes or "peaks" in the parametrization). *Zero-dimensional symmetry* occurs when this is not the case. The sphere again provides an example because its symmetries include not just rotations, but also *reflections* about some bisecting plane. Reflections reverse the orientation, but rotations don't – if you are right-handed, then your reflection in a mirror is left-handed, whereas simply turning around won't make you a left-hander. The same is true for two-dimensional beings that are drawn on the globe. It is therefore not possible to make a smooth transition from a rotation to a reflection. The full symmetry group of a sphere (including both rotations and reflections) is called $O(3)$. Let's choose a bisecting plane (say, the equatorial plane, but this doesn't really matter). You can try to convince yourself that every symmetry of the sphere can be generated by first rotating the sphere and then (if necessary) reflecting it about the equatorial plane. In other words: every element of $O(3)$ can be written as a product of an element of the group of rotations called $SO(3)$ and the equatorial reflection (or the neutral element that leaves every point fixed); we say that the group $O(3)$ *factors into* $SO(3)$ and the group that consists of the equatorial reflection and the neutral element.

The symmetries of any object always have a smooth factor and a zero-dimensional factor.[3] Sometimes, though, the smooth factor is *trivial* (containing only the neutral element), and then we see zero-dimensional symmetry groups in isolation. For example, a cube has only 48 symmetries, and it is not possible to make a smooth transition from one to a different one – a vertex of the cube is either rotated or reflected to another vertex or it isn't.

Symmetries of wallpaper patterns are also examples of zero-dimensional groups: imagine a regular pattern on an infinite wall[4] and the ways in which it

---

[2] A *group* is a set of objects (in the introductory example the rotations) that comes with a product operation and one specific *neutral element* (in our case, the rotation through angle 0) where every element is *invertible* (this means in our case that you can cancel the effect of one rotation by applying a "backwards" rotation – what you obtain is the neutral element that leaves the globe fixed), and where associativity of the product operation holds.

[3] The neutral element that leaves every point fixed is always an element of both factors.

[4] Here regular means that after you have decided what the pattern looks like on one tile and how to combine the tiles, you have already chosen what the whole wall looks like; the first tile just is repeated over and over using translations, reflections, rotations, and so-called *glide-*

Figure 1: A ceiling in an egyptian tomb (left) and a wall in the Alhambra, exhibiting different symmetry groups.

may be translated, rotated, or reflected and still look the same.[5] There are 17 distinct symmetry groups of the plane (17 ways to generate a regular pattern using essentially just one tile), and all of their corresponding patterns have been used in wall designs, see [8].

Symmetries of crystal structures give further examples. A crystal of salt has sodium and chlorine atoms arrayed in a regular fashion and, imagining the pattern extending infinitely, there is a certain group of translations, rotations, and reflections of the crystal that preserve the pattern. It may be shown mathematically that there are only 32 crystallographic groups, That puts a limit on the number of crystal shapes that may occur in nature [7].

2 Understanding zero-dimensional, non-discrete symmetries: ancestral trees

All of the examples of symmetry mentioned so far are well-understood. The next one is, too, but it is representative of a type that is not generally understood and is the subject of current research. The example is the “tree” of ancestors of an individual: a person at the “root” (let’s call her Polly) has two parents (Fred and Martha), each of those has two parents, and so on, see Figure 2.

The ancestral tree has many symmetries. We might for example exchange the positions of the Polly’s parents Fred and Martha (and also of all their respective ancestors, that is, Grandpa George swaps places with Grandpa Gerd,

---

reflections (that is, combinations of translations and reflections). The tile is not necessarily rectangular, it may as well be a parallelogram or a rhombus.

[5] You can play with symmetries of wallpapers and draw beautiful patterns using the free software MORENAMENTS (downloadable from http://imaginary.org/de/program/morenaments). Explanations are available here: http://imaginary.org/imaginary-entdeckerbox.

and Grannies Gita and Gertrude swap places as well), or we might leave the parents and grandparents in place and exchange the positions of the Grandpa Gerd's parents (and their respective ancestors).

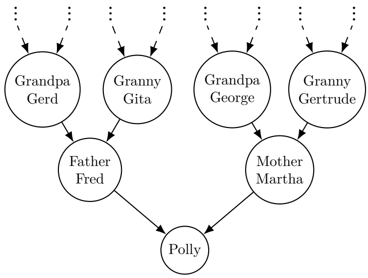

Figure 2: Polly and her infinitely many ancestors.

What distinguishes this example from the previous ones is that we can fix the ancestral tree for any number of generations back, but flexibility remains to move still earlier generations by symmetries, while spheres and crystals are so rigid that once three or four points are fixed by a symmetry, so is the rest of the sphere or crystal.

Although zero-dimensional, the symmetry of the ancestral tree shares with higher dimensional symmetry groups the property of being *non-discrete*. An object has a *discrete* symmetry group if all symmetries that are **close** to fixing the points of the object in fact **do** fix all points. The group of symmetries of a salt crystal is an example of a discrete group: it is not possible to move a sodium atom just a little bit in the crystal, it must be moved to the place of another sodium atom if it moves at all; and any symmetry that fixes four atoms (not all in the same plane) must already fix the whole crystal. The three-dimensional group of rotations of the sphere $SO(3)$ is non-discrete because rotations through small angles go close to fixing all points of the sphere while not doing so. The symmetry group of the ancestral tree, on the other hand, is non-discrete because a symmetry can fix any number of generations without fixing the whole tree. It is the only one among the examples we've seen that is both zero-dimensional and non-discrete.

Recent discoveries appear to show us a path to an understanding of non-discrete zero-dimensional symmetry that will match what we know about higher-dimensional symmetry. Since any symmetry group has higher-dimensional and zero-dimensional factors, that would complete our understanding of all non-discrete symmetry. We will see next where that path might lead and some of the difficulties that lie ahead.

3 Local theory

The parameters that determine the dimension of a higher-dimensional symmetry group also allow us to analyse it. For example, the group $SO(2)$ of rotations of the circle in the plane may be parametrized by the angle of rotation, and a rotation through an angle of ϑ followed by a rotation through an angle of φ has the same effect as a rotation through an angle of $\vartheta + \varphi$. The product in $SO(2)$ therefore corresponds to adding the angles. In $SO(3)$, however, the dependence of the product on the three parameters describing the axes and angles of the rotations is more complicated. It turns out, though, that for rotations through small angles (in the *locality* of the neutral element), adding the parameters gives a good approximation to the product. (If the instructions on a treasure map say "walk 10 metres north from a certain point in Bermuda and then 10 metres west", you will still find the treasure if you walk west first and then north; if the instructions are to travel 1,000 kilometres north and then 1,000 kilometres west, you must follow them in that order if you wish to arrive at the treasure.) Geometrically, this approximation corresponds to finding a flat tangent space to a three-dimensional curved space. In the language of calculus, it corresponds to taking a derivative. The same idea applies to any higher-dimensional symmetry group and gives rise to a *local theory* of these groups. The local theory was initiated by S. Lie [3, 5], and for this reason the higher-dimensional groups are called *Lie groups*.

One of the main reasons why the local theory is useful is that it detects important features of a symmetry such as the axis of a rotation. This may be seen more clearly by considering the example of the *projective symmetries of the plane*. These symmetries preserve straight lines and parallelism of lines, although not necessarily angles, and allow scaling and shearing of the plane. We might for example fix a point p in the plane and contract all distances by a scaling factor of 2 in the horizontal direction from p and expand all distances in the vertical direction by a scaling factor of $\sqrt{3}$.

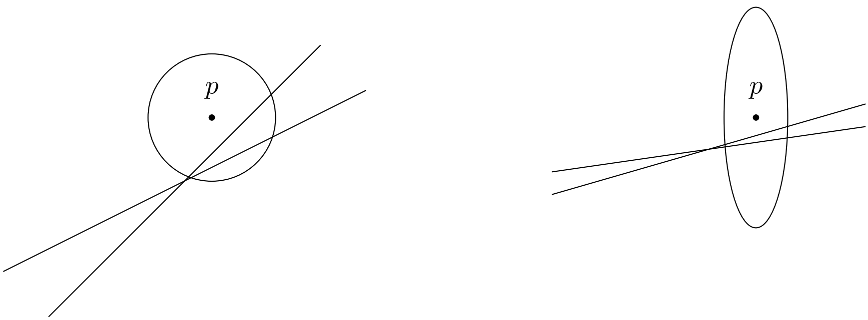

Figure 3: A circle centred at p and two straight lines under the effect of a horizontal contraction with scaling factor 2 and a vertical expansion with scaling factor $\sqrt{3}$.

All points on the horizontal line through p become twice as close to p under this symmetry and all points on the vertical line through p become $\sqrt{3}$ times as far away. Observe that this is a local property – we don't need to look at the whole plane to see the scaling factors, a small neighbourhood of p suffices.

There cannot be a local theory for zero-dimensional symmetry groups in the same sense, since in this case we cannot meaningfully say that two symmetries are close if their parameters differ only slightly. However, a local theory of the non-discrete groups is currently being developed based on the idea that for zero-dimensional groups "close" doesn't mean "rotating by a small angle" or "translating by a small distance" but "fixing a large part of the symmetric object", see [2]. This local theory considers symmetries that fix parts of the object and studies problems like this: if the object is divided into halves A and B, is it possible to fix A and not B with one symmetry and to fix B but not A with a second symmetry?

A notion of *scaling* non-discrete zero-dimensional symmetry groups has been developed separately, see [9]. The effect of the symmetry being zero-dimensional is that the factor of expansion or contraction is always a positive integer, it cannot be $\sqrt{3}$ as in the example of the previous paragraph. If it is equal to 1, the symmetry is like a rotation, and if it is bigger than 1, there is something analogous to an axis of scaling. One of the challenges for research is to understand how these ideas can be combined: in Lie groups the scaling follows from the local theory, but in zero-dimensional groups local theory and scaling methods are independent as far as we know today.

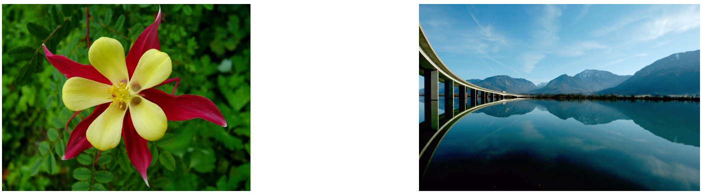

Figure 4: Zero-dimensional, discrete symmetries in nature.

Another promising lead given by recent research is that we have learned how to break down zero-dimensional symmetry groups into smaller factors, which again has parallels with higher-dimensional symmetry. Symmetry groups may often have more factors than just one higher-dimensional and one zero-dimensional factor. Any translation of the plane, for example, may be obtained by first translating by a certain distance in the horizontal direction and then by a distance in the vertical direction: the group of translations of the plane thus has the group of horizontal translations and the group of vertical translations as factors. When a higher-dimensional group has factors in this way, its dimension is equal to the sum of the dimensions of the factors. Since the dimension is always a positive integer (or zero), there must be symmetry groups that cannot be factored. The one-dimensional groups of translations of the line and rotations of the circle are obviously two such groups, but there are others with dimension greater than 1. The group of rotations of the sphere $SO(3)$ is a three-dimensional example for these so-called *simple* groups. Every higher-dimensional group may be broken down into simple or one-dimensional factors, and the simple groups have been classified and completely enumerated.

Until recently, it was difficult to see how to break zero-dimensional groups down into simple pieces in a similar way because $0 + 0 = 0$. However, new ideas have now shown how general zero-dimensional groups may be decomposed into simple pieces, see [1, 6]. Work is also being done to classify the simple groups and, although enough examples have been found to show that there are too many to enumerate them all, methods from the local theory are helping to separate them into types. There is hope that we can learn enough about simple groups to verify facts about general zero-dimensional symmetry by checking the simple cases and then combining the information.

This is an exciting time for research on zero-dimensional symmetry. It is more complicated than higher dimensional symmetry, but what we have learned in recent years opens the prospect that complication can be turned into rich understanding.

Image credits

Fig. 1, left: “Wallpaper group-cmm-2”. Author: Dmharvey. Licensed under Creative Commons Attribution-Share Alike 3.0 via Wikimedia Commons, http://commons.wikimedia.org/wiki/File:Wallpaper_group-cmm-2.jpg, [Online; accessed 12-January-2015]

Fig. 1, right: “Alhambra-p3-closeup”. Author: Dmharvey. Licensed under Creative Commons Attribution-Share Alike 3.0 via Wikimedia Commons, http://commons.wikimedia.org/wiki/File:Alhambra-p3-closeup.jpg, [Online; accessed 12-January-2015]

Fig. 2: Provided by Daniel Kronberg, member of the Snapshots team.

Fig. 4, left: “2007-07-09Aquilegia01”. Author: Wildfeuer. Licensed under Creative Commons Attribution-Share Alike 3.0 via Wikimedia Commons, http://commons.wikimedia.org/wiki/File:2007-07-09Aquilegia01.jpg, [Online; accessed 12-January-2015]

Fig. 4, right: "Ferlacher Stausee Matzen Halouz Singerberg 24012008 76". Author: Johann Jaritz. Licensed under Creative Commons Attribution-Share Alike 3.0 via Wikimedia Commons, http://commons.wikimedia.org/wiki/File:Ferlacher_Stausee_Matzen_Halouz_Singerberg_24012008_76.jpg, [Online; accessed 12-January-2015]

References

[1] P. E. Caprace and N. Monod, *Decomposing locally compact groups into simple pieces*, Math. Proc. Cambridge Philos. Soc. **150** (2011), no. 1, 97–128.

[2] P. E. Caprace, C.D. Reid, and G.A. Willis, *Locally normal subgroups of simple locally compact groups*, Comptes Rendus – Mathématique **351** (2013), no. 17, 657–661.

[3] S. Helgason, *Sophus Lie and the role of Lie groups in mathematics*, http://ocw.mit.edu/courses/mathematics/18-755-introduction-to-lie-groups-fall-2004/readings/helga_sophusl1_1.pdf, 2013.

[4] F. Klein, *Vergleichende Betrachtungen über neuere geometrische Forschungen ('A comparative review of recent researches in geometry')*, Mathematische Annalen **43**.

[5] S. Lie and F. Engel, *Theorie der Transformations-gruppen I, II, III*, Leipzig, 1888, 1890, 1893.

[6] Phillip Wesolek, *Elementary totally disconnected locally compact groups*, arxiv.org/abs/1405.4851v3, 2014.

[7] Wikipedia, *Crystallographic point group*, http://en.wikipedia.org/wiki/Crystallographic_point_group, [Online; accessed 24-December-2014].

[8] ______, *Wallpaper group*, http://en.wikipedia.org/wiki/Wallpaper_group, [Online; accessed 24-December-2014].

[9] G. A. Willis, *The structure of totally disconnected, locally compact groups*, Mathematische Annalen **300**.

George Willis *is a professor of mathematics at the University of Newcastle (Australia).*

Mathematical subjects
Algebra and Number Theory, Geometry and Topology

Connections to other fields
Chemistry and Earth Science, Computer Science, Physics

DOI
10.14760/SNAP-2015-003-EN

Snapshots of modern mathematics from Oberwolfach are written by participants in the scientific program of the Mathematisches Forschungsinstitut Oberwolfach (MFO). The snapshot project is designed to promote the understanding and appreciation of modern mathematics and mathematical research in the general public worldwide. It is part of the mathematics communication project "Oberwolfach meets IMAGINARY" funded by the Klaus Tschira Foundation and the Oberwolfach Foundation. All snapshots can be found on www.imaginary.org and on www.mfo.de/snapshots.

Junior Editor
Sophia Jahns
junior-editors@mfo.de

Senior Editor
Carla Cederbaum
senior-editor@mfo.de

Mathematisches Forschungsinstitut
Oberwolfach gGmbH
Schwarzwaldstr. 9 – 11
77709 Oberwolfach
Germany

Director
Gerhard Huisken