Mathematisches Forschungsinstitut Oberwolfach

Report No. 50/2018

DOI: 10.4171/OWR/2018/50

Combinatorial Optimization

Organised by
Jesus De Loera, Davis
Satoru Iwata, Tokyo
Martin Skutella, Berlin

4 November – 10 November 2018

Abstract. Combinatorial Optimization is an active research area that developed from the rich interaction among many mathematical areas, including combinatorics, graph theory, geometry, optimization, probability, theoretical computer science, and many others. It combines algorithmic and complexity analysis with a mature mathematical foundation and it yields both basic research and applications in manifold areas such as, for example, communications, economics, traffic, network design, VLSI, scheduling, production, computational biology, to name just a few. Through strong inner ties to other mathematical fields it has been contributing to and benefiting from areas such as, for example, discrete and convex geometry, convex and nonlinear optimization, algebraic and topological methods, geometry of numbers, matroids and combinatorics, and mathematical programming. Moreover, with respect to applications and algorithmic complexity, Combinatorial Optimization is an essential link between mathematics, computer science and modern applications in data science, economics, and industry.

Mathematics Subject Classification (2010): 90C10 Integer programming, 90C27 Combinatorial optimization.

Introduction by the Organisers

The workshop *Combinatorial Optimization* was organized by Jesús A. De Loera (Davis), Satoru Iwata (Tokyo), and Martin Skutella (Berlin). It was well attended, with 51 participants from a broad geographic and thematic representation within the field (e.g., some participants came from Mathematics departments, some came from CS departments, some came from Economics or Operations Research schools). We are particularly proud of the gender diversity of participants and speakers. Moreover, in compiling the list of invitees we put particular emphasis on excellent young scientists.

Recent advances in Combinatorial Optimization have seen an acceleration due to the fact that the subject seats at the intersections of several mathematical and computational methods (e.g., Combinatorics, Graph Theory, Convex Analysis, Game Theory, Probability, Discrete Geometry, etc.) and it is required in several modern application fields such as data science, energy logistics, and transportation. Our workshop brought together scholars from different perspectives, e.g., people interested in structural results, approximation algorithms, mixed integer optimization, and algebraic techniques. Together they exchanged experience and identified promising topics for future research.

The workshop's program reflected the diversity of methods in our subject. On the other hand, there was enough overlap among individual expertise to generate new ideas and obtain input from other directions. In particular, the flash introductory talks (2 minutes per speaker to present their key research concerns), the 27 regular talks, together with two problem brainstorming session on Monday and Wednesday evening provided the basic impulse for stimulating discussions covering a broad spectrum of topics. More than ten problems are posed and at least two of them received partial answers from the participants during the workshop.

The workshop was extremely successful and this is evident from the results presented. We highlight just a few here.

The past two decades have seen notable progress in developing structural insight and improving algorithmic capabilities in solving global optimization problems with non-linear constraints. The incorporation of computational algebraic geometry to address non-convexity and integrality has become very exciting. Methods of real algebraic geometry allows us to deal with non-convex optimization problems such as minimizing a polynomial over a compact convex set. Monique Laurent presented some exciting recent results on this direction where she showed that such problems can be reduced to eigenvalue problem computation. Another talk that touched on the topic of non-convex problems, with integer variables, was the presentation by Michele Conforti who discussed some exciting new construction on cutting planes for compact non-convex sets based on lexicographic variable orders (close related to the theory of Gröbner bases, a core subject in algebraic geometry).

The most famous problem in Combinatorial Optimization is the Traveling Salesperson Problem (TSP). In particular, the TSP has been as a great source of inspiration for every generation of researchers to develop new techniques and algorithms, from the early days of the field until today. Jens Vygen gave a survey lecture on several recent breakthrough results concerning the algorithmic approximability of the TSP as well as the integrality gap of well-known linear programming relaxations of the problem. Ola Svensson presented the first known constant-factor approximation algorithm for the asymmetric version of the problem (ATSP), which is considerably more difficult than the classical symmetric TSP and whose approximability with constant performance guarantee had been open for decades. Finally, Vera Traub, a young PhD student from Bonn, gave a remarkable presentation of her recent result that the integrality gap of the asymmetric traveling salesman path LP is constant.

Our subject is closely related to famous problems in Computational Complexity of Theoretical Computer Science. Parameterized complexity is a branch of computational complexity theory that focuses on classifying computational problems according to their inherent difficulty with respect to multiple parameters of the input or output. The complexity of a problem is then measured as a function of those parameters. This allows the classification of NP-hard problems on a finer scale than in the classical setting, where the complexity of a problem is only measured by the number of bits in the input. During our workshop, at least two presentations dealt with parametrizing the problems. First was the talk of Robert Weismantel who showed us that for an integer optimization problem (IP), one important data parameter is the maximum absolute value, Δ, among all square submatrices of the constraint matrix. He showed how the running times and complexity depend on Δ.

Another example of parametrized analysis was the presentation of Shmuel Onn. In his account he explained how optimality certificates involving Graver sets for block-structured integer programs can be used to prove a strongly polynomial result on parametrized integer programming in terms of the combinatorial complexity of the system (tree-depth). Recent results on improved complexity estimates for augmentation algorithms with Graver-type improving vectors make this the only such method of proof.

A notable subject of the workshop was related to computation of mathematical equilibria on markets. László Végh presented a strongly polynomial time algorithm for market equilibrium problems generalizing famous results of Éva Tardos for network flow problems. He showed that convex separable optimization over networks can be done in strongly polynomial time. Neil Olver also presented his joint work with László Végh that derives a simpler and faster strongly polynomial algorithm for the maximum generalized flow problem.

Submodularity is a key concept that has played a fundamental role in combinatorial optimization. In fact, the talk of Rico Zenklusen showed that one can minimize a submodular function over all subsets of a specified nonzero cardinality modulo a constant prime power. The technique comes from purely combinatorial arguments that establish the non-existence of a certain type of set systems. Yu Yokoi presented a generalization of Galvin's list edge coloring theorem for bipartite graphs to the framework of supermodular coloring introduced by Lex Schrijver in the 1980s.

The submodularity is recognized as a discrete counterpart of convexity. Extending this viewpoint has led to the theory of Discrete Convex Analysis that deals with discrete convex functions over the integer lattice, which can be viewed as a direct product of paths. Hiroshi Hirai gave a survey talk on his recent results that provided further extensions. His approach deals with discrete convex functions over more general combinatorial structures. Concrete outcomes include a dichotomy theorem on the multifacility location problems. The talk nicely demonstrated the utility of novel geometric concepts such as Euclidean building and CAT(0) space in combinatorial optimization.

Finally the growing interest in data-driven analysis stimulates the study of combinatorial optimization techniques too. Among the aspects discussed at the workshop were the study of solution of sparse regression problems presented by Santanu Dey.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1641185, "US Junior Oberwolfach Fellows". The organizers and participants are truly grateful to the MFO staff for making this a productive and enjoyable meeting.

Workshop: Combinatorial Optimization

Table of Contents

Abstracts

Integer optimization from the perspective of subdeterminants

Robert Weismantel

We discuss several recent developments in establishing a refined theory of integer optimization. The goal is to understand complexity questions for instances of linear integer optimization problems based on the dimension and a data parameter that corresponds to the largest absolute value of the determinant of a submatrix of the constraint matrix. The talk will in particular focus on two questions:

(a) Given an optimal solution of the linear programming relaxation. How close is an optimal integer solution measured in l_1- or l_∞-norm?

(b) Given a system of linear equations with nonnegativity conditions for the variables and a linear objective function. How sparse is an optimal integer solution?

This talk is based on the papers [1, 2, 3].

References

[1] F. Eisenbrand, R. Weismantel, *Proximity results and faster algorithms for integer Programming using the Steinitz Lemma*. Proceeding of the 29th Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2018, 806–816.

[2] I. Aliev, J. De Loera, F. Eisenbrand, T. Oertel, R. Weismantel, *The support of integer optimal solutions*. SIAM J.Optim. 28, 2018, 2152–2157.

[3] J. Paat, R. Weismantel, S. Weltge, *Distances between optimal solutions of mixed integer programs*. Mathematical Programming 2018, to appear.

The packing property and intersecting clutters

Gérard Cornuéjols
(joint work with Ahmad Abdi, Dabeen Lee)

A clutter is *intersecting* if the members do not have a common element yet every two members intersect. It has been conjectured that for clutters without an intersecting minor, total primal integrality and total dual integrality of the corresponding set covering linear system must be equivalent. In this paper, we provide a polynomial characterization of clutters without an intersecting minor.

One important class of intersecting clutters comes from projective planes, namely the *deltas*, while another comes from graphs, namely the *blockers of extended odd holes*. Using similar techniques, we will provide a polynomial algorithm for finding a delta or the blocker of an extended odd hole minor in a given clutter. This result is quite surprising as the same problem is NP-hard if the input were the blocker instead of the clutter.

References

[1] A. Abdi, G. Cornuéjols, D. Lee, *Intersecting restrictions in clutters.* Working paper, Carnegie Mellon University, October 2018.

A Friendly Smoothed Analysis of the Simplex Method

Daniel Dadush
(joint work with Sophie Huiberts)

Explaining the excellent practical performance of the simplex method for linear programming has been a major topic of research for over 50 years. One of the most successful frameworks for understanding the simplex method was given by Spielman and Teng [2], who developed the notion of smoothed analysis. Starting from an arbitrary linear program (LP) with d variables and n constraints, Spielman and Teng analyzed the expected runtime over random perturbations of the LP, known as the smoothed LP, where variance σ^2 Gaussian noise is added to the LP data. In particular, they gave a two-stage shadow vertex simplex algorithm which uses an expected $\widetilde{O}(d^{55}n^{86}\sigma^{-30} + d^{70}n^{86})$ number of simplex pivots to solve the smoothed LP. Their analysis and runtime was substantially improved by Deshpande and Spielman [3] and later Vershynin [4]. The fastest current algorithm, due to Vershynin, solves the smoothed LP using an expected $O\big(\log^2 n \cdot \log\log n \cdot (d^3\sigma^{-4} + d^5 \log^2 n + d^9 \log^4 d)\big)$ number of pivots, improving the dependence on n from polynomial to logarithmic.

While the original proof of Spielman and Teng has now been substantially simplified, the resulting analyses are still quite long and complex and the parameter dependencies far from optimal. In this work, we make substantial progress on this front, providing an improved and simpler analysis of shadow simplex methods, where our main algorithm requires an expected

$$O(d^2\sqrt{\log n}\ \sigma^{-2} + d^5 \log^{3/2} n)$$

number of simplex pivots. We obtain our results via an improved *shadow bound*, key to earlier analyses as well, combined with algorithmic techniques of Borgwardt [1] and Vershynin [4]. As an added bonus, our analysis is completely *modular*, allowing us to obtain non-trivial bounds for perturbations beyond Gaussians, such as Laplace perturbations.

References

[1] K.-H. Borgwardt, *The average number of pivot steps required by the simplex-method is polynomial*, Z. Oper. Res. Ser. A-B, 26(5):A157–A177, 1982.
[2] D.A. Spielman, S.-H. Teng, *Smoothed analysis of algorithms: why the simplex algorithm usually takes polynomial time.* J. ACM, 51(3):385–463 (electronic), 2004.

[3] A. Deshpande, D.A. Spielman, *Improved smoothed analysis of the shadow vertex simplex method*. Proceedings of the 46th Annual IEEE Symposium on Foundations of Computer Science, FOCS '05, pages 349–356, 2005.
[4] R. Vershynin, *Beyond Hirsch conjecture: walks on random polytopes and smoothed complexity of the simplex method*. SIAM J. Comput., 39(2):646–678, 2009. Preliminary version in FOCS '06.

Proximity results and faster algorithms for Integer Programming using the Steinitz Lemma

Friedrich Eisenbrand
(joint work with Robert Weismantel)

We consider integer programming problems in standard form $\max\{c^T x : Ax = b,\, x \geq 0,\, x \in \mathbb{Z}^n\}$ where $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$ and $c \in \mathbb{Z}^n$. We show that such an integer program can be solved in time $(m\Delta)^{O(m)} \cdot \|b\|_\infty^2$, where Δ is an upper bound on each absolute value of an entry in A. This improves upon the longstanding best bound of Papadimitriou [1] of $(m \cdot \Delta)^{O(m^2)}$, where in addition, the absolute values of the entries of b also need to be bounded by Δ. Our result relies on a lemma of Steinitz that states that a set of vectors in $\mathbb{R}^m$ that is contained in the unit ball of a norm and that sum up to zero can be ordered such that all partial sums are of norm bounded by m. We also use the Steinitz lemma to show that the ℓ_1-distance of an optimal integer and fractional solution, also under the presence of upper bounds on the variables, is bounded by $m \cdot (2\,m \cdot \Delta + 1)^m$. Here Δ is again an upper bound on the absolute values of the entries of A. The novel strength of our bound is that it is independent of n. We provide evidence for the significance of our bound by applying it to general knapsack problems where we obtain structural and algorithmic results that improve upon the recent literature.

References

[1] C.H. Papadimitriou, *On the complexity of integer programming*. Journal of the ACM (JACM) **28.4** (1981), 765-768.

Approximation algorithms for the symmetric (path) TSP

Jens Vygen

We survey the recent progress on approximation algorithms for the (symmetric) traveling salesman problem and its path variant: Given a finite metric space (V, c) and $s, t \in V$, find a sequence $s = v_0, v_1, \ldots, v_{n-1}, v_n = t$ containing all elements of V and minimizing $\sum_{i=1}^{n} c(v_{i-1}, v_i)$. We also study the integrality ratio of the standard LP relaxation. The worst known examples are graph instances (where $c(v, w) = \mathrm{dist}_G(v, w)$ for an undirected unweighted graph G); this is why this special case received a lot of interest. Here is the state of the art:

		$s = t$	$s \neq t$
general	integrality ratio $\in$	$[\frac{4}{3}, \frac{3}{2}]$ [1]	$[\frac{3}{2}, 1.5284]$ [4]
	approximation ratio $\leq$	$\frac{3}{2}$ [2,3]	$\frac{3}{2}$ [5]
in graphs	integrality ratio $\in$	$[\frac{4}{3}, \frac{7}{5}]$ [6]	$\frac{3}{2}$ [6]
	approximation ratio $\leq$	$\frac{7}{5}$ [6]	1.497 [7]

References

[1] L.A. Wolsey, *Heuristic analysis, linear programming and branch and bound.* Mathematical Programming Study 13 (1980), 121–134.

[2] N. Christofides, *Worst-case analysis of a new heuristic for the traveling salesman problem.* Technical Report 388, Graduate School of Industrial Administration, Carnegie-Mellon University, Pittsburgh 1976.

[3] A.I. Serdjukov, *Some extremal bypasses in graphs* [in Russian]. Upravlyaemye Sistemy 17 (1978), 76–79.

[4] V. Traub, J. Vygen, *An improved upper bound on the integrality ratio for the s-t-path TSP.* ArXiv:1808.10734.

[5] R. Zenklusen, *A 1.5-approximation for path TSP.* ArXiv:1805.04131; to appear in SODA 2019.

[6] A. Sebő, J. Vygen, textitShorter tours by nicer ears: 7/5-approximation for graphic TSP, 3/2 for the path version, and 4/3 for two-edge-connected subgraphs. Combinatorica 34 (2014), 597–629.

[7] V. Traub, J. Vygen, *Beating the integrality ratio for s-t-tours in graphs.* ArXiv:1804.03112. Preliminary extended abstract in the Proceedings of the 59th Annual IEEE Symposium on Foundations of Computer Science (FOCS 2018), 766–777.

Efficient Submodular Minimization under Congruency Constraints

Rico Zenklusen

(joint work with Martin Nägele, Benny Sudakov)

Submodular function minimization (SFM) is a fundamental and efficiently solvable problem class in Combinatorial Optimization with various applications. Surprisingly, only very little is known about constraint classes under which SFM remains efficiently solvable. The arguably most influential such constraint class are parity constraints. They capture classical combinatorial optimization problems like the odd-cut problem, and they are a key tool in a recent technique to efficiently solve integer programs with a constraint matrix whose subdeterminants are within $\{-2, -1, 0, 1, 2\}$.

By introducing a new approach combining techniques from Combinatorial Optimization, Combinatorics, and Number Theory, we show that efficient SFM is possible even over all sets of cardinality r mod m, for m being a constant prime power. This covers generalizations of the odd-cut problem with relevance in the context of integer programming with constraint matrices with subdeterminants bounded by m. Moreover, our results settle two open questions raised by Geelen and Kapadia [Combinatorica, 2017] in the context of computing the girth and cogirth of certain types of binary matroids.

A Constant-Factor Approximation Algorithm for the Asymmetric Traveling Salesman Problem

Ola Svensson
(joint work with Jakub Tarnawski, László A. Végh)

The traveling salesman problem is one of the most fundamental optimization problems. Given n cities and pairwise distances, it is the problem of finding a tour of minimum total distance that visits each city once. In spite of significant research efforts, current techniques seem insufficient for settling the approximability of the traveling salesman problem. The gap in our understanding is especially large in the general asymmetric setting where the distance from city i to j is *not* assumed to equal the distance from j to i.

Indeed, until recently, it remained an open problem to design an algorithm with *any* constant approximation guarantee. This status is particularly intriguing as the standard linear programming relaxation is believed to give a constant-factor approximation algorithm, where the constant may in fact be as small as 2.

In this talk, we will give an overview of old and new approaches for settling this question. We shall, in particular, talk about our new approach that gives the first constant-factor approximation algorithm for the asymmetric traveling salesman problem. Our approximation guarantee is analyzed with respect to the standard LP relaxation, and thus our result confirms the conjectured constant integrality gap of that relaxation. The main idea of our approach is to first give a generic reduction to structured instances and on those instances we then solve an easier problem (but equivalent in terms of constant-factor approximation) obtained by relaxing the general connectivity requirements into local connectivity conditions.

References

[1] O. Svensson, J. Tarnawski, L.A. Végh, *A Constant-Factor Approximation Algorithm for the Asymmetric Traveling Salesman Problem.* Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing, STOC, pages 204–213, 2018.

The asymmetric traveling salesman path LP has constant integrality ratio

Vera Traub
(joint work with Anna Köhne, Jens Vygen)

The path version of the asymmetric traveling salesman problem is the variant in which the endpoints of the tour are given and distinct. In a recent breakthrough, Svensson, Tarnawski, and Végh [6] found the first constant-factor approximation algorithm for the asymmetric TSP (ATSP), and they also proved that its standard LP relaxation has constant integrality ratio.

Feige and Singh [1] showed that any α-approximation algorithm for ATSP implies a $(2\alpha + \epsilon)$-approximation algorithm for its path version (for any $\epsilon > 0$). However, their proof does not imply any bound on the integrality ratio of the LP.

Nagarajan and Ravi [5] proved that the integrality ratio of the LP relaxation for the path version of ATSP is $O(\sqrt{n})$ (where n is the number of vertices). This bound was improved to $O(\log n)$ by Friggstad, Salavatipour, and Svitkina [3] and to $O(\log n / \log\log n)$ by Friggstad, Gupta, and Singh [2].

We show that the LP relaxation of the asymmetric traveling salesman path problem has constant integrality ratio [4]. To this end we give a black-box reduction to ATSP: if ρ denotes the integrality ratio for ATSP, then the integrality ratio for its path version is at most $4\rho - 3$.

References

[1] U. Feige, M. Singh, *Improved approximation algorithms for traveling salesperson tours and paths in directed graphs.* Proceedings of the 10th International Workshop on Approximation Algorithms for Combinatorial Optimization Problems; LNCS 4627 (M. Charikar, K. Jansen, O. Reingold, J.D.P. Rolim, eds.), Springer, Berlin 2007, 104–118.

[2] Z. Friggstad, A. Gupta, M. Singh, *An improved integrality gap for asymmetric TSP paths.* Mathematics of Oper.ations Research 41 (2016), 745–757.

[3] Z. Friggstad, M.R. Salavatipour, Z. Svitkina, *Asymmetric traveling salesman path and directed latency problems.* SIAM Journal on Computing 42 (2013), 1596–1619.

[4] A. Köhne, V. Traub, J. Vygen, *The asymmetric traveling salesman path LP has constant integrality ratio.* ArXiv:1808.06542.

[5] V. Nagarajan, R. Ravi, *The directed minimum latency problem.* Proceedings of the 11th International Workshop on Approximation Algorithms for Combinatorial Optimization Problems; LNCS 5171 (A. Goel, K. Jansen, J.D.P. Rolim, R. Rubinfeld, eds.), Springer, Berlin 2008, pp. 193–206.

[6] O. Svensson, J. Tarnawski, L.A. Végh, *A constant-factor approximation algorithm for the asymmetric traveling salesman problem.* Proceedings of the 50th Annual ACM Symposium on Theory of Computing (STOC 2018), 204–213.

Blended Conditional Gradients

Sebastian Pokutta
(joint work with Gábor Braun, Dan Tu, Stephen Wright)

We present a blended conditional gradient algorithm [2] for minimizing a smooth convex function over a polytope P, that combines gradient projection steps with conditional gradient steps, achieving linear convergence for strongly convex functions. It does not make use of away steps or pairwise steps, but retains all favorable properties of conditional gradient algorithms, most notably not requiring projections onto P and maintaining iterates as sparse convex combinations of extreme points. The algorithm decreases measures of optimality (primal and dual gaps) rapidly, both in the number of iterations and in wall-clock time, outperforming even the efficient lazified conditional gradient algorithms of [1]. We also present a streamlined algorithm for the special case in which P is a probability simplex, called simplex gradient descent.

References

[1] G. Braun, S. Pokutta, D. Zink, *Lazifying Conditional Gradient Algorithms.* Proceedings of ICML (2017).
[2] G. Braun, S. Pokutta, D. Tu, S. Wright, *Blended Conditional Gradients: the unconditioning of conditional gradients.* Preprint arXiv:1805.07311 (2018).

Gomory's mixed-integer cuts are optimal

Marco Di Summa
(joint work with Amitabh Basu, Michele Conforti, Giacomo Zambelli)

Among many families of cutting planes for integer programming proposed in the literature, Gomory mixed-integer cuts seem to stand out for at least two reasons: (i) they can be derived via a simple closed formula from the optimal tableau of the continuous relaxation; (ii) in practice, they tend to perform better than other types of general-purpose cutting planes. However, a formal justification for this behavior has not been given up to now.

We give a rigorous theoretical explanation for the empirical superiority of Gomory mixed-integer cuts by working in the context of the pure integer infinite group relaxation proposed by Gomory and Johnson [1, 2], which is an infinite-dimensional model that encompasses all possible integer programming problems at the same time. We show that for this model, Gomory mixed-integer cuts are the valid inequalities that cut off the maximum volume from the nonnegative orthant, and therefore can be seen as the optimal cutting planes if the volume cut off is chosen as a criterion to measure the strength of a cutting plane.

References

[1] R.E. Gomory, E.L. Johnson, *Some continuous functions related to corner polyhedra, I.* Mathematical Programming **3** (1972), 23–85.

[2] R.E. Gomory, E.L. Johnson, *Some continuous functions related to corner polyhedra, II.* Mathematical Programming **3** (1972), 359–389.

Integer Programming in Parameter-Tractable Strongly-Polynomial Time

Shmuel Onn

We consider the general integer programming problem in standard form:

$$\text{IP}: \quad \max\{wx \,:\, Ax = b\,,\; l \leq x \leq u\,,\; x \in \mathbb{Z}^n\}\;.$$

We consider two parameters which control the matrix A defining the program: its "numeric complexity" $a = \|A\|_\infty := \max\{|A_{i,j}|\}$, and its "combinatorial complexity" $d = \min\{\text{td}(A), \text{td}(A^T)\}$, where $\text{td}(A)$ is the *tree-depth* of the matrix A.

We prove the following.

Theorem. *The integer programming problem IP can be solved in parameter-tractable strongly-polynomial number of arithmetic operations $f(a,d)\text{poly}(n)$ for some function f of the parameters a, d and some polynomial in the dimension n.*

We discuss some of the applications of this theorem, which extends, improves, unifies and simplifies many results of the last decade. The talk is based on [1, 2].

References

[1] M. Koutecký, A. Levin, S. Onn, *A Parameterized Strongly Polynomial Algorithm for Block Structured Integer Programs.* International Colloquium on Automata, Languages, and Programming, 45th ICALP Proceedings, Leibniz International Proceedings in Informatics, 107-85:1–14, 2018, Schloss Dagstuhl.

[2] F. Eisenbrand, C. Hunkenschröder, K.M. Klein, M. Koutecký, A. Levin, S. Onn, *An Algorithmic Theory of Integer Programming in Variable Dimension.* In preparation.

Representability of optimization models

Amitabh Basu
(joint work with Christopher T. Ryan, Sriram Sankaranarayanan)

We review representability theorems for mixed-integer linear and convex optimization. We then describe recent new results on mixed-integer bilevel optimization. *Mixed-integer bilevel linear* (MIBL) programs of the form

$$
\begin{aligned}
\max_{x,y} \quad & c^\top x + d^\top y \\
\text{s.t.} \quad & Ax + By \le b \\
& y \in \arg\max_y \{ f^\top y : Cx + Dy \le g, y_i \in \mathbb{Z} \text{ for } i \in \mathcal{I}_F \} \\
& x_i \in \mathbb{Z} \text{ for } i \in \mathcal{I}_L
\end{aligned} \tag{1}
$$

where x and y are finite-dimensional real decision vectors, b, c, d, f and g are finite-dimensional rational data vectors and the constraint matrices A, B, C, and D have conforming dimensions (notation for the dimensions of these objects is given below). The decision-maker who determines x is called the *leader*, while the decision-maker who determines y is called the *follower*. The sets $\mathcal{I}_L$ and $\mathcal{I}_F$ are subsets of the index sets of x and y (respectively) that determine which leader and follower decision variables are integers.

Bilevel programming has a long history, with traditions in theoretical economics (see, for instance, [12], which originally appeared in 1975) and operations research (see, for instance, [5, 9]). While much of the research community's attention has focused on the continuous case, there is a growing literature on bilevel programs with integer variables, starting with early work in the 1990s by Bard and Moore [13, 2] through a more recent surge of interest. Research has largely focused on algorithmic concerns, with a recent emphasis on leveraging advancements in cutting plane techniques. Typically, these algorithms restrict how variables appear in the problem. For instance, [14] consider the setting where all variables are integer-valued. [7] allow for continuous variables but restrict the leader's continuous variables from entering the follower's problem. Only very few papers have studied questions of computational complexity in the mixed-integer setting, and also often with restricting the appearance of integer variables (see, for instance, [10]).

To our knowledge, a thorough study of general MIBL programs with no additional restrictions on the variables and constraints has not been undertaken in the literature. The contribution of this paper is to ask and answer a simple question: what types of sets can be modeled as feasible regions (or possibly projections of feasible regions) of such general MIBL programs? Or put in the standard terminology of the optimization literature: what sets are *MIBL-representable*?

Separate from the design of algorithms and questions of computational complexity, studying representability shows the reach of a modeling framework. The classical paper of [8] provides a characterization of sets that can be represented by mixed-integer linear feasible regions. They show that a set is the projection of the feasible region of a mixed-integer linear problem (termed *MILP-representable*) if and only if it is the Minkowski sum of a polytope and a finitely-generated integer monoid (concepts more carefully defined below). This result is the gold standard in the theory of representability, as it answers a long-standing question on the limits of mixed-integer programming as a modeling framework. Jeroslow and Lowe's result also serves as inspiration for recent interest in the representability of a variety of problems.

To our knowledge, questions of representability have not even been explicitly asked of *continuous bilevel linear* (CBL) programs where $\mathcal{I}_L = \mathcal{I}_F = \emptyset$ in (1). Accordingly, our initial focus concerns characterizations of CBL-representable sets. In the first key result of our paper, we show that every CBL-representable set can also be modeled as the feasible region of a linear complementarity (LC) problem (in the sense of [6]). Indeed, we show that both CBL-representable sets and LC-representable sets are precisely finite unions of polyhedra. Our proof method works through a connection to superlevel sets of piecewise linear convex functions (what we term *polyhedral reverse-convex sets*) that alternately characterize finite unions of polyhedra. In other words, an arbitrary finite union of polyhedra can be modeled as a continuous bilevel program, a linear complementarity problem, or an optimization problem over a polyhedral reverse-convex set.

A natural question arises: how should we relate CBL-representability and MILP-representability? Despite some connections between CBL programs and MILPs (see, for instance, [1]), the collection of sets they represent are incomparable. The Jeroslow-Lowe characterization of MILP-representability as the finite union of polytopes summed with a finitely-generated monoid has a fundamentally different geometry than CBL-representability as a finite union of polyhedra. It is thus natural to conjecture that MIBL-representability should involve some combination of the two geometries. We will see that this intuition is *roughly* correct, with an important caveat.

A distressing fact about MIBL programs, noticed early on in [13], is that the feasible region of a MIBL program may not be *topologically closed* (maybe the simplest example illustrating this fact is Example 1.1 of [10]). This throws a wrench in the classical narrative of representability that has largely focused on closed sets. Indeed, the recent work of [11] is careful to study representability by *closed* convex sets. This focus is entirely justified. Closed sets are indeed of most interest to the working optimizer and modeler, since sets that are not closed may fail to have desirable optimality properties (such as nonexistence of optimal solutions). Accordingly, we aim our investigation on *closures* of MIBL-representable sets. In fact, we provide a complete characterization of these sets as unions of finitely many MILP-representable sets. This is our second key result on MIBL-representablility. The result conforms to the rough intuition of the last paragraph. MIBL-representable sets are indeed finite unions of other objects, but instead of these objects being polyehdra as in the case of CBL-programs, we now take unions of MILP-representable sets, reflecting the inherent integrality of MIBL programs.

To prove this second key result on MIBL-representability we develop a generalization of Jeroslow and Lowe's theory to mixed integer sets in *generalized polyhedra*, which are finite intersections of closed *and* open halfspaces. Indeed, it is the non-closed nature of generalized polyhedra that allows us to study the non-closed feasible regions of MIBL-programs. Specifically, these tools arise when we take the value function approach to bilevel programming. Here, we leverage the characterization of [4] of the value function of the mixed-integer program in the lower level problem. Blair's characterization involves analyzing superlevel and sublevel sets of *Chvátal functions* when incorporated into the value function approach. A Chvátal function is (roughly speaking) a linear function with integer rounding (a more formal definition later). [3] show that superlevel sets of Chvátal functions are MILP-representable. Sublevel sets are trickier, but for a familiar reason — they are, in general, not closed. This is not an accident. The non-closed nature of mixed-integer bilevel sets, generalized polyhedra, and sublevel sets of Chvátal functions are all tied together in a key technical result that shows that sublevel sets of Chvátal functions are precisely finite unions of generalized mixed-integer linear representable (GMILP-representable) sets. This result is the key to establishing our second main result on MIBL-representability.

In fact, showing that the sublevel set of a Chvátal function is the finite union of GMILP-representable sets is a corollary of a more general result. Namely, we show that the collection of sets that are finite unions of GMILP-representable sets forms an algebra (closed under unions, intersections, and complements). We believe this result is of independent interest.

In summary, we make the following contributions. We provide geometric characterizations of CBL-representability and MIBL-representability (where the latter is up to closures) in terms of finite unions of polyhedra and finite unions of MILP-representable sets, respectively. In the process of establishing these main results, we also develop a theory of representability of generalized mixed-integer polyhedra and show that finite unions of GMILP-representable sets form an algebra. This last result has the implication that finite unions of MILP-representable sets also form an algebra, up to closures.

References

[1] C. Audet, P. Hansen, B. Jaumard, G. Savard, *Links between linear bilevel and mixed 0–1 programming problems.* Journal of Optimization Theory and Applications, 93(2):273–300, 1997.

[2] J.F. Bard, J.T. Moore, *An algorithm for the discrete bilevel programming problem.* Naval Research Logistics, 39(3):419–435, 1992.

[3] A. Basu, K. Martin, C.T. Ryan, G. Wang, *Mixed-integer linear representability, disjunctions, and variable elimination.* International Conference on Integer Programming and Combinatorial Optimization, pages 75–85. Springer, 2017.

[4] C. Blair, *A closed-form representation of mixed-integer program value functions.* Mathematical Programming, 71(2):127–136, 1995.

[5] W. Candler, R. Townsley, *A linear two-level programming problem.* Computers & Operations Research, 9(1):59–76, 1982.
[6] R. Cottle, J.-S. Pang, R.E. Stone, *The Linear Complementarity Problem.* SIAM, 2009.
[7] M. Fischetti, I. Ljubić, M. Monaci, M. Sinnl, *A new general-purpose algorithm for mixed-integer bilevel linear programs.* Operations Research, 65(6):1615–1637, 2017.
[8] R.G. Jeroslow, J.K. Lowe, *Modelling with integer variables.* Mathematical Programming at Oberwolfach II. Springer, Berlin, Heidelberg, 1984. 167-184.
[9] R.G. Jeroslow, *The polynomial hierarchy and a simple model for competitive analysis.* Mathematical Programming, 32(2):146–164, 1985.
[10] M. Köppe, M. Queyranne, C.T. Ryan, *Parametric integer programming algorithm for bilevel mixed integer programs.* Journal of Optimization Theory and Applications, 146(1):137–150, jul 2010.
[11] M. Lubin, I. Zadik, J.P. Vielma, *Mixed-integer convex representability.* International Conference on Integer Programming and Combinatorial Optimization, pages 392–404. Springer, 2017.
[12] J.A. Mirrlees, *The theory of moral hazard and unobservable behaviour: Part I.* The Review of Economic Studies, 66(1):3–21, 1999.
[13] J.T. Moore, J.F. Bard, *The mixed integer linear bilevel programming problem.* Operations Research, 38(5):911–921, 1990.
[14] L. Wang, P. Xu, *he watermelon algorithm for the bilevel integer linear programming problem.* SIAM Journal on Optimization, 27(3):1403–1430, jan 2017.

Lattice Reformulation Cuts

Karen Aardal

(joint work with Frederik von Heymann, Andrea Lodi, Andrea Tramontani, Laurence A. Wolsey)

Here we consider the question whether the lattice reformulation of a linear integer program can be used to produce effective cutting planes. We consider integer programs (IP) in the form $\max\{cx \mid Ax = b, x \in \mathbb{Z}^n_+\}$, where the reformulation takes the form $\max\{cx^0 + cQ\mu \mid Q\mu \geq -x^0,\ \mu \in \mathbb{Z}^{n-m}\}$, where Q is an $n \times (n-m)$ integer matrix. Working on an optimal LP tableau in the μ-space allows us to generate $n-m$ Gomory mixed-integer inequalities (GMIs) in addition to the m GMIs associated with the optimal tableau in the x-space. These provide new cuts that can be seen as GMIs associated to $n-m$ non-elementary split directions associated with the reformulation matrix Q. On the other hand it turns out that the corner polyhedra associated to an LP basis and the GMI or split closures are the same whether working in the x- or μ-spaces. Computationally we show that the effectiveness of the cuts generated by this approach depends on the quality of the reformulation obtained by the reduced basis algorithm used to generate Q and that it is worthwhile to generate several rounds of such cuts. However, the effectiveness of the cuts deteriorates as the number of constraints is increased.

Acknowledgements. The research has been financed in part by The Netherlands Organisation for Scientific Research, NWO, grant number 613.000.801, which we gratefully acknowledge.

References

[1] K. Aardal, F. von Heymann, A. Lodi, A. Tramontani, L.A. Wolsey, *Lattice Reformulation Cuts*. 2018.

List Supermodular Coloring

Yu Yokoi
(joint work with Satoru Iwata)

In 1995, Galvin [4] provided an elegant proof for the list edge coloring conjecture for bipartite graphs. That is, he showed that the list edge chromatic number of any bipartite graph equals its edge chromatic number. A surprising aspect of Galvin's proof is that it utilizes a famous result of Gale and Shapley [3] on the existence of stable matchings in bipartite graphs.

We generalize Galvin's result to the setting of supermodular coloring, introduced by Schrijver [6]. In the proof, we utilize the monochromatic path theorem of Sands, Sauer and Woodrow [5], which is shown by Fleiner [1] to be a generalization of the result of Gale and Shapley. Our result can be extended to the setting of skew-supermodular coloring [2]. Also, our proof naturally suggests an efficient algorithm for finding a list supermodular coloring.

References

[1] T. Fleiner, *A fixed-point approach to stable matchings and some applications*. Mathematics of Operations Research, 28 (2003), pp. 103–126.

[2] A. Frank, T. Király, *A survey on covering supermodular functions*. Research Trends in Combinatorial Optimization (W.J. Cook, L. Lovász, J. Vygen, eds.), Springer-Verlag,, 2009, pp. 87–126.

[3] D. Gale, L. S. Shapley, *College admissions and the stability of marriage*. American Mathematical Monthly, 69 (1962), pp. 9–15.

[4] F. Galvin, *The list chromatic index of a bipartite multigraph*. Journal of Combinatorial Theory, Series B, 63 (1995), pp. 153–158.

[5] B. Sands, N. Sauer, R. Woodrow, *On monochromatic paths in edge-coloured digraphs*. Journal of Combinatorial Theory, Series B, 33 (1982), pp. 271–275.

[6] A. Schrijver, *Supermodular colourings*. Matroid Theory (L. Lovász, A. Recski, eds.), North-Holland, Amsterdam, 1985, pp. 327–343.

Progress on Seymour's Flowing Conjecture

Bertrand Guenin
(joint work with Ahmad Abdi)

A clutter $\mathcal{C}$ is a family of sets over a ground set $E(\mathcal{C})$ with the property that no set properly contains another. The *blocker* $b(\mathcal{C})$ of clutter $\mathcal{C}$ is the set of all inclusion-wise minimal sets that intersect every set in $\mathcal{C}$. A clutter $\mathcal{C}$ is *binary* if for every set $S \in \mathcal{C}$ and $B \in b(\mathcal{C})$, $|S \cap B|$ is odd. Let

$$P(\mathcal{C}) := \left\{ x \in \mathbb{R}_+^{E(\mathcal{C})} : \sum (x_e : e \in S) \geq 1, \text{for all } S \in \mathcal{C} \right\}.$$

A clutter $\mathcal{C}$ is *ideal*, if $P(\mathcal{C})$ is an integral polyhedron. The polyhedron obtained from $P(\mathcal{C})$ by setting some variables to zero and projecting variables is another set covering polyhedron $P(\mathcal{D})$ for some clutter $\mathcal{D}$. We say that $\mathcal{D}$ is a *minor* of $\mathcal{C}$. A clutter is *minimally non-ideal* if it is non-ideal, but every minor is ideal. $\mathcal{O}_5$ is the clutter corresponding to triangles and pentagons of K_5 and $\mathcal{L}_7$ is the clutter corresponding to the lines of the Fano matroid. If a clutter is ideal (resp. binary) then so is any minor and so is its blocker. In 1977, Seymour [1] proposed the Flowing Conjecture,

Conjecture 1. *If $\mathcal{C}$ is a minimally non-ideal binary clutter, then $\mathcal{C}$ or $b(\mathcal{C})$ is $\mathcal{O}_5$ or $\mathcal{L}_7$.*

A *triangle* in clutter $\mathcal{F}$ is a set of cardinality three. Since both $\mathcal{O}_5$ and $\mathcal{L}_7$ have a triangle, a special case of the Flowing Conjecture is given by the,

Conjecture 2. *If $\mathcal{C}$ is a minimally non-ideal binary clutter, then $\mathcal{C}$ or $b(\mathcal{C})$ has a triangle.*

With Ahmad Abdi we proved that both conjectures are in fact equivalent. Namely, we proved that,

Theorem 1. *The only minimally non-ideal binary clutters that have a triangle are $\mathcal{L}_7$ and $\mathcal{O}_5$.*

The result will appear in Combinatorica [2].

The *two-point Fano clutter*, is defined as follows,

$$\mathcal{D}_7 = \big\{\{1,2,6\},\{3,4,5,7\},\{1,3,5\},\{2,4,6,7\},\\ \{2,3,4\},\{1,5,6,7\},\{4,5,6\},\{1,2,3,7\}\big\}.$$

It can be viewed as the clutter whose sets are the lines and their complements of the Fano plane, that intersect two fixed elements exactly once. We showed that,

Theorem 2. *If $\mathcal{C}$ is a minimally non-ideal binary clutter, then $\mathcal{C}$ or $b(\mathcal{C})$ is $\mathcal{O}_5$ or $\mathcal{L}_7$ or $\mathcal{D}_7$.*

An extended abstract has appeared in the Integer Programming and Combinatorial Optimization conference [3]. A full version of the paper will appear in Combinatorica [4].

References

[1] P.D. Seymour, *The matroids with the Max-Flow Min-Cut property.* J. Combin. Theory Ser. B, 23:189–222, 1977.

[2] A. Abdi, B. Guenin, *The minimally non-ideal binary clutters with a triangle.* To appear in Combinatorica.

[3] A. Abdi, B. Guenin, *The two-point Fano and ideal binary clutters.* Integer Programming and Combinatorial Optimization, IPCO 2017. Lecture Notes in Computer Science, vol 10328, Springer, 2017.

[4] A. Abdi, B. Guenin, *The two-point Fano and ideal binary clutters.* To appear in Combinatorica.

Convergence analysis of Lasserre measure-based approximations for polynomial optimization

Monique Laurent
(joint work with Etienne de Klerk)

We consider the optimization problem

$$p_{\min} = \min_{x \in K} p(x), \tag{1}$$

asking to minimize an n-variate polynomial $p \in \mathbb{R}[x]$ over a compact set $K \subseteq \mathbb{R}^n$. This is a hard problem, already for simple regions K like the standard simplex Δ_n, the cube $[0,1]^n$, a ball, or a sphere. For instance, given a graph $G = (V, E)$ the hard combinatorial optimization problem asking to compute the maximum cardinality $\alpha(G)$ of a stable set in G can be reformulated via the following polynomial optimization problems:

$$\frac{1}{\alpha(G)} = \min_{x \in \Delta_n} x^T(I_n + A_G)x, \quad \alpha(G) = \max_{x \in [0,1]^n} \sum_{i \in V} x_i - \sum_{\{i,j\} \in E} x_i x_j,$$

$$\frac{2\sqrt{2}}{3\sqrt{3}} \sqrt{1 - \frac{1}{\alpha(G)}} = \max_{y \in \mathbb{R}^n, z \in \mathbb{R}^m} \left\{ 2 \sum_{\{i,j\} \in \overline{E}} z_{ij} y_i y_j : (y, z) \in S_{n+m} \right\}$$

(as shown by Motzkin-Straus [8] and Nesterov [9]).

Lasserre [7] has introduced a hierarchy of tractable upper bounds for $p_{\min}$. The starting point is to reformulate problem (1) as the problem

$$p_{\min} = \min_{\mu} \left\{ \int_K p(x) d\mu(x) : \mu \text{ is a probability measure on } K \right\}, \tag{2}$$

which follows from the fact that an optimal measure is the Dirac measure δ_a at a global minimizer a of f over K.

Lasserre [7] shows that we may restrict in (2) to the measures having a sum-of-squares density function:

$$p_{\min} = \inf_h \left\{ \int_K p(x) h(x) d\mu_0(x) \text{ s.t. } h \in \Sigma, \int_K h(x) d\mu_0(x) \right\}, \tag{3}$$

where μ_0 is a given reference measure whose support is K, and Σ denotes the set of sums of squares of polynomials. If we impose a degree bound $2r$ on the sum-of-squares densities we get the parameters

$$p^{(r)} = \inf_h \left\{ \int_K p(x) h(x) d\mu_0(x) : h \in \Sigma, \ \deg(h) \leq 2r, \int_K h(x) d\mu_0(x) \right\}, \tag{4}$$

which provide a hierarchy of upper bounds: $p_{\min} \leq p^{(r+1)} \leq p^{(r)}$ converging to $p_{\min}$ as $r \to \infty$. For each integer r, the parameter $p^{(r)}$ can be computed via semidefinite programming and, in fact, it reduces to a generalized eigenvalue computation (since the semidefinite program has just one affine constraint) [7].

We discuss some known results about the convergence rate of the hierarchy $(p^{(r)} - p_{\min})_{r \geq 0}$ and give a few hints about the proof techniques.

(1) Assume K is a compact set which satisfies an interior cone condition (see [3]) and μ_0 is the Lebesgue measure. Then it is shown in [3] that

$$p^{(r)} - p_{\min} = O\left(\frac{1}{\sqrt{r}}\right).$$

Roughly speaking the condition we need on K is that there is a global minimizer of p in K which is not a cusp point. The key idea is to construct a sum-of-squares polynomial h with degree $2r$ that 'looks like' the delta function at a global minimizer a of p in K. For this consider the Gaussian distribution

$$G(x) = \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(\frac{-\|x-a\|^2}{2\sigma^2}\right),$$

and the polynomial

$$h(x) = \frac{1}{(2\pi\sigma^2)^{n/2}} \sum_{k=0}^{2r} \frac{(-1)^k}{k!} \left(\frac{-\|x-a\|^2}{2\sigma^2}\right)^k$$

obtained by considering the Taylor expansion of e^{-t} truncated at degree $2r$. Since this truncation gives a univariate polynomial nonnegative on $\mathbb{R}$ it is a sum of squares and thus h is a sum of squares. When selecting $\sigma \sim 1/r$ one can show the desired rate of convergence.

(2) Assume K is a convex body and μ_0 is the Lebesgue measure. Then it is shown in [1] that

$$p^{(r)} - p_{\min} = O\left(\frac{1}{r}\right).$$

This relies on a link to the simulated annealing bounds for convex optimization established in [6]. Namely, instead of the Gaussian distribution one uses now the Boltzman distribution

$$H(x) = \frac{\exp(-p(x)/T)}{\int_K \exp(-p(x)/T)dx},$$

with 'temperature' parameter $T \sim 1/r$, and its truncated Taylor expansion as sum of squares density h. By replacing p by a convex upper estimator one can use the analysis in [6] to derive the desired convergence result.

(3) Assume K is the unit sphere and p is a homogeneous polynomial. Then it is shown in [4] that

$$p^{(r)} - p_{\min} = O\left(\frac{1}{r}\right).$$

This relies on combining tools about spherical harmonics and a quantum analogue of the classical De Fineti theorem.

(4) Assume $K = [-1,1]^n$ and μ_0 is the measure $\prod_{i=1}^n (1-x_i^2)^{-1/2} dx_i$. Then it is shown in [2] that

$$p^{(r)} - p_{\min} = O\left(\frac{1}{r^2}\right).$$

This relies on using the reformulation of the parameter $p^{(r)}$ as smallest eigenvalue of the associated matrix

$$A_p(r) = \left(\int_K p(x) b_\alpha(x) b_\beta(x) d\mu_0 \right)_{|\alpha|,|\beta| \leq r},$$

where $\{b_\alpha(x)\}$ is an orthonormal basis of polynomials with respect to the measure μ_0. In the univariate case $K = [-1, 1]$ when $p(x) = x$ it turns out that the parameter $p^{(r)}$ coincides with the smallest root of the polynomial b_{r+1} [5]. When selecting for μ_0 a Chebyshev (or Jacobi) type measure then the rate of convergence to -1 of this smallest root is known to be in $\Theta(1/r^2)$, which settles the rate of convergence of the bounds $p^{(r)}$. This also permits to settle the case of a univariate quadratic polynomial and finally the general case which can be shown to be reducible to this situation.

Understanding the exact regime for the rate of convergence of the bounds $p^{(r)}$ for general convex bodies remains an open question.

References

[1] E. de Klerk, M. Laurent, *Comparison of Lasserre's measure-based bounds for polynomial optimization to bounds obtained by simulated annealing.* To appear in Mathematics of Operations Research, arXiv:1703.00744 (2017).

[2] E. de Klerk, M. Laurent, *Worst-case examples for Lasserre's measure–based hierarchy for polynomial optimization on the hypercube.* To appear in Mathematics of Operations Research, arXiv:1804.05524 (2018).

[3] E. de Klerk, M. Laurent, Z. Sun, *Convergence analysis for Lasserre's measure-based hierarchy of upper bounds for polynomial optimization.* Mathematical Programming Series A, 162(1):363–392 (2017).

[4] A.C. Doherty, S. Wehner, *Convergence of SDP hierarchies for polynomial optimization on the hypersphere.* arXiv:1210.5048v2 (2013).

[5] C.F. Dunkl, Y. Xu, *Orthogonal Polynomials of Several Variables.* Cambridge University Press (2001).

[6] A.T. Kalai, S. Vempala, *Simulated annealing for convex optimization.* Mathematics of Operations Research, 31(2):253–266 (2006).

[7] J.B. Lasserre, *A new look at nonnegativity on closed sets and polynomial optimization.* SIAM Journal on Optimization, 21(3):864–885 (2011).

[8] T.S. Motzkin, E.G. Sraus, *Maxima for graphs and a new proof of a theorem of Túran.* Canadian J. Math., 17:533–540 (1965).

[9] Y. Nesterov, *Random walk in a simplex and quadratic optimization over convex polytopes.* CORE Discussion Paper 2003/71, CORE-UCL, Louvain-La-Neuve (2003).

Strongly polynomial algorithms for market equilibrium computation

László Végh

(joint work with Jugal Garg)

Most known strongly polynomial algorithms are for special classes of linear programs, and only few examples are known in nonlinear optimization. The talk will give an overview of two such results. The first result [1] gives a strongly polynomial algorithm for some instances of flows with separable convex objectives, including separable convex quadratic objectives, as well as market equilibrium in linear Fisher markets. The second, more recent result [2] provides the first strongly polynomial algorithm for exchange markets with linear utilities. These results can be obtained by extending the classical technique of variable fixing from linear programs to the convex settings. The main progress in both algorithms is gradually identifying edges in the support of the optimal solutions.

References

[1] L.A. Végh, *Strongly polynomial algorithm for a class of minimum-cost flow problems with separable convex objectives.* SIAM Journal on Computing 45(5) (2016), 1729-1761.

[2] J. Garg, L.A. Végh, *A strongly polynomial algorithm for linear exchange markets.* Arxiv:1809.06266.

A faster and simpler strongly polynomial algorithm for generalized flow maximization

Neil Olver

(joint work with László Végh)

Consider linear programming:

$$\min c^T x \quad \text{subject to} \quad Ax \le b \qquad A \in \mathbb{R}^{m \times n}, c \in \mathbb{R}^n, b \in \mathbb{R}^m.$$

One of the major open problems in optimization concerns the existence of a *strongly polynomial* algorithm for this, that is, a polynomial time algorithm in which the number of arithmetic operations is polynomially bounded by the dimensions of the problem. Progress has been made on special cases: for example, Tardos [1] gave an algorithm that is strongly polynomial as long as the entries of the matrix A are integral and bounded, with no restriction on the vectors b and c.

A further special case was recently solved by Végh [3]. It concerns the case where A has at most two nonzero entries per column, and where the cost vector c has only a single nonzero entry. This yields the *maximum generalized flow problem.* Given is a directed graph $G = (V, E)$, node demands $b : E \to \mathbb{R}$, a sink node $t \in V$, and also *gains* $\gamma : E \to \mathbb{R}_{>0}$. The goal is to maximize the amount of flow arriving at t, while ensuring that the net flow arriving at any other node matches its demands. But unlike with the standard maximum flow problem, flow traversing arc $e \in E$ is scaled by a factor γ_e; if flow f_e enters the arc, $\gamma_e f_e$ will leave.

I will discuss a dramatically simpler, and also faster, strongly polynomial algorithm for this problem. Even for small numerical parameter values, our running time bound is comparable to the best weakly polynomial algorithms. The algorithm maintains a primal solution f and a dual solution μ satisfying certain complementary slackness conditions. The key new technical idea is the relaxation of primal feasibility conditions (crucial in all previous approaches) in a novel way. This allows us to maintain an integral "relabelled" flow throughout the main steps of the algorithm. Our algorithm has a clean and natural primal-dual structure: primal updates augment the integral relabelled flow by discrete units, and dual updates scale μ uniformly on a subset of nodes. The integrality we maintain is quite surprising, since the optimal solution itself does not satisfy strong integrality properties.

References

[1] É. Tardos, *A strongly polynomial algorithm to solve combinatorial linear programs.* Operations Research, 34(2) 250–256, 1986.

[2] N. Olver, L.A. Végh, *A Simpler and Faster Strongly Polynomial Algorithm for Generalized Flow Maximization.* Proceedings of the 49th ACM Symposium on Theory of Computation (STOC), pp. 100–111, 2017.

[3] L.A. Végh, *A strongly polynomial algorithm for generalized flow maximization.* Mathematics of Operations Research, 42(1) 179–211, 2017.

Discrete Convex Analysis beyond $\mathbf{Z}^n$

Hiroshi Hirai

Discrete Convex Analysis (DCA) [14], developed by Murota and his collaborators, is a theory of "convex" functions on integer lattice $\mathbf{Z}^n$, which provides a unified theoretical framework to well-solvable combinatorial optimization problems related to network flow and submodular/matroid optimization. One of main concepts of DCA is L-convexity, which generalizes the submodularity to the integer lattice $\mathbf{Z}^n$. An *$L^\natural$-convex function* is a function $g : \mathbf{Z}^n \to \mathbf{R} \cup \{\infty\}$ satisfying

$$g(x) + g(y) \geq g(\lfloor (x+y)/2 \rfloor) + g(\lceil (x+y)/2 \rceil) \quad (x, y \in \mathbf{Z}^n),$$

where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ are componentwise rounding up and down operators, respectively.

The central of my recent research is generalizations of L-convexity to more general structures beyond $\mathbf{Z}^n$ (= grid graph) and its applications to combinatorial optimization problems that had not been captured by DCA. In [5, 7, 6, 9], I presented results concerning two new classes of discrete convex functions, *submodular functions on modular semilattices* and *L-convex functions on oriented modular graphs*. Here a modular semilattice is a semilattice generalization of a modular lattice, and an oriented modular graph is a kind of an amalgamation of modular lattices and semilattices. The highlight of the results is summarized as follows:

- Our submodular function is minimizable under valued-CSP model. For special modular semilattices, our submodular functions are identical to several other submodular-type functions, such as bisubmodular, k-submodular, and α-bisubmodular functions.
- Our L-convex functions have several properties analogous to that L$^\natural$-convex functions in DCA have, and coincide with L$^\natural$-convex functions if the underlying graph is a grid graph. Analogous to L$^\natural$-convex functions, our L-convex functions are minimized by the steepest descent algorithm (SDA), where each descend step is the minimization of a submodular function on a modular semilattice.

- Our theory is motivated by the complexity classification of the *multifacility location problem* (a.k.a. *minimum 0-extension problem*) on graph G:

$$\begin{array}{ll} \text{Min.} & \displaystyle\sum_{1\le i\le n}\sum_{v\in V(G)} b_{iv}d_G(v,x_i) + \sum_{1\le i<j\le n} c_{ij}d_G(x_i,x_j) \\ \text{s.t.} & (x_1,x_2,\ldots,x_n)\in (V(G))^n. \end{array}$$

  This classification problem was raised by Karzanov [13], who showed that the multifacility location problem is NP-hard if the underlying graph G is not orientable modular. In [7], I showed the converse: if G is orientable modular, then the multifacility location problem on G is an L-convex function minimization, and is polynomially solvable. This establishes the desired complexity classification.
- Dual objectives arising from classes of well-behaved multicommodity flow problems are submodular or L-convex in suitable sense. The DCA-oriented algorithm design (based on SDA) leads to efficient combinatorial polynomial time algorithms for some classes of minimum-cost and node-capacitated multiflow problems [5, 6], where such algorithms had not been known before.
- The underlying structures, modular semilattices and oriented modular graphs, have rich connections to other fields of mathematics that include incidence geometries (*projective and polar spaces*), *Euclidean building*, and metric spaces of global nonpositive curvature (*CAT(0) spaces*) [1]. For some cases, our submodular/L-convex function is naturally extended to a function on a CAT(0) space via an analogue of the Lovász extension, and is characterized by its convexity relative to the CAT(0)-metric [9].

See also survey [8]. The last property enables us to apply continuous optimization methods on CAT(0)-spaces to submodular optimization on modular lattice.

In [4], we applied this idea to the following optimization problem (*Maximum Vanishing Subspace Problem; MVSP*).

$$\begin{array}{ll} \text{Max.} & \dim X + \dim Y \\ \text{s.t.} & A_i(X,Y) = \{0\}\ (i=0,1,2,\ldots,m), \\ & X,Y\subseteq \mathbf{K}^n \text{ (vector subspaces)}, \end{array}$$

where A_i are given $n\times n$ matrices over field $\mathbf{K}$ viewed as bilinear form $\mathbf{K}^n \times \mathbf{K}^n \to \mathbf{K}$. This problem arises from the dual of the *non-commutative rank (nc-rank)* [2, 11] of symbolic matrix $A_0 + A_1x_1 + \cdots + A_mx_m$, where variables x_i are non-commutative, i.e., $x_ix_j \neq x_jx_i$. For this problem, Garg et al [3] and Ivanyos et al [11, 12] gave polynomial time algorithms. MVSP is viewed as a submodular optimization on the modular lattice of vector subspaces, and has a continuous relaxation, via Lovász extension, that is a convex optimization on a CAT(0)-space. Based on this, [4] developed a conceptually-simple polynomial time algorithm for MVSP.

In [10], I consider a weighted generalization of nc-rank, i.e., the degree of the *Dieudonne determinant* Det of $A_0 + A_1x_1 + \cdots + A_mx_m$, where A_i is a matrix over $\mathbf{K}[t]$ and variables x_i are non-commutative. I established a duality theorem in which the dual is viewed as an L-convex optimization on the modular lattice of certain modules (= Euclidean building of $\mathrm{SL}(\mathbf{K}(t)^n)$), and developed a DCA-oriented algorithm to compute $\deg \mathrm{Det}$.

References

[1] J. Chalopin, V. Chepoi, H. Hirai, D. Osajda, *Weakly modular graphs and nonpositive curvature.* Memoirs of the AMS, to appear.

[2] M. Fortin, C. Reutenauer, *Commutative/non-commuative rank of linear matrices and subspaces of matrices of low rank.* Séminaire Lotharingien de Combinatoire, 52 (2004), B52f.

[3] A. Garg, L. Gurvits, R. Oliveira, A. Wigderson, *Operator scaling: theory and applications.* ArXiv:1511.03730 (2015), the conference version appeared in FOCS' 2016.

[4] M. Hamada, H. Hirai, *Maximum vanishing subspace problem, CAT(0)-space relaxation, and block-triangularization of partitioned matrix.* ArXiv:1705.020602017, 2017.

[5] H. Hirai, *L-extendable functions and a proximity scaling algorithm for minimum cost multiflow problem.* Discrete Optimization, 18 (2015), 1–37.

[6] H. Hirai, *A dual descent algorithm for node-capacitated multiflow problems and its applications.* ACM Transactions on Algorithms, to appear.

[7] H. Hirai, *Discrete convexity and polynomial solvability in minimum 0-extension problems.* Mathematical Programming, Series A, 155 (2016), 1–55.

[8] H. Hirai, *Discrete Convex Functions on Graphs and Their Algorithmic Applications.* T. Fukunaga and K. Kawarabayashi (eds.), Combinatorial Optimization and Graph Algorithms, Communications of NII Shonan Meetings, Springer Nature, Singapore, Springer (2017), pp. 67–101.

[9] H. Hirai, *L-convexity on graph structures.* Journal of the Operations Research Society of Japan, 61 (2018), 71–109.

[10] H. Hirai, *Computing degree of determinant via discrete convex optimization over Euclidean building.* ArXiv:1805.112452018, 2018.

[11] G. Ivanyos, Y. Qiao, K.V. Subrahmanyam, *Non-commutative Edmonds' problem and matrix semi-invariants.* Computational Complexity 26 (2017), 717–763.

[12] G. Ivanyos, Y. Qiao, K.V. Subrahmanyam, *Constructive noncommutative rank computation in deterministic polynomial time over fields of arbitrary characteristics.* Computational Complexity, to appear.

[13] A.V. Karzanov, *Minimum 0-extensions of graph metrics.* European Journal of Combinatorics 19 (1998), 71–101.

[14] K. Murota, *Discrete Convex Analysis.* SIAM, Philadelphia, 2003.

A Faster Way to Determine the Minimal Feasible Time Horizon of a Quickest Transshipment Problem

Miriam Schlöter

We study a classical flow over time problem that captures the essence of evacuation planning: Given a dynamic network $\mathcal{N} = (D = (V, A), u, \tau, S^+, S^-)$ consisting of a directed graph D with capacities u and transit times τ on the arcs and sets of sources S^+ and sinks S^- with integral supplies and demands b, respectively, the *quickest transshipment problem* $(\mathcal{N}, b)$ is the problem of computing a flow over time f that fulfills all supplies and demands as quickly as possible.

The concept of flows over time has already been introduced by Ford and Fulkerson [1, 2] in the 1950s. Ford and Fulkerson also came up with a way to reduce flow over time problems to classical static flow problems by getting rid of the time-dependent aspect of the dynamic network – the transit times. Instead of a solving flow over time problem directly they showed that instead one can also solve a static flow problem in the so-called *time-expanded network*, a network that consists of one copy of the original dynamic network per time unit. The time-expanded network thus gives us a way to solve quickest transshipment problems. However, the size of the time-expanded network grows linearly with the time horizon and hence one can only achieve algorithms with pseudo-polynomial running time by using time-expansion.

The first strongly polynomial time algorithm for the quickest transshipment problem is due to Hoppe and Tardos [3]. A slight improvement over the algorithm of Hoppe and Tardos has only recently been achieved by Schlöter and Skutella [4]. Both algorithms have in common that they work in two parts. The first part is the computation of the *minimal feasible time horizon* T of a given quickest transshipment problem $(\mathcal{N}, b)$, while in the second part the actual quickest transshipment f is computed. Here, we only concentrate on the first part. One important object in the context of quickest transshipments is the following parametrized submodular function o^θ that was first introduced in [3]. For each $\theta \geq 0$ the set function $o^\theta \colon 2^{S^+ \cup S^-} \to \mathbb{R}_+$ is defined as follows,

$$o^\theta(X) := \begin{array}{l} \text{max. amount of flow that can be sent from} \\ S^+ \cap X \text{ towards } S^- \setminus X \text{ until time } \theta, \end{array}$$

for all $X \subseteq S^+ \cup S^-$. Hoppe and Tardos showed that o^θ is submodular for all $\theta \geq 0$ [3] and it is due to a result by Ford and Fulkerson [1, 2] that the function o^θ can be evaluated at every set $X \subseteq S^+ \cup S^-$ by one minimum-cost flow computation and thus in strongly polynomial time. Together with a feasibility criterion by Klinz [5] – *a time horizon $\theta \geq 0$ is feasible for a given quickest transshipment problem $(\mathcal{N}, b)$ if and only if $o^\theta(X) \geq b(X)$ for all $X \subseteq S^+ \cup S^-$* – and the fact that submodular functions can be minimized in strongly polynomial time if they can be evaluated in strongly polynomial time, these results imply that we can check in strongly polynomial time whether a given time horizon $\theta \geq 0$ is feasible for a quickest transshipment problem by one submodular function minimization. The so far only known way to determine the minimal feasible time horizon T of a quickest transshipment problem $(\mathcal{N}, b)$ in strongly polynomial time is coupling submodular function minimization with Megiddo's parametric search framework [6]. However, using parametric search to determine T can in the worst case dramatically increase the running time of the algorithm used for submodular function minimization. Moreover, Megiddo's search framework is hard to implement and thus not applicable in practice. These two facts make it an interesting task to come up with new ways to determine T that do not rely on parametric search.

We present a new strongly polynomial time algorithm for determining T for the special cases of quickest transshipment problems in dynamic networks with only

Algorithm 1: Algorithm to determine the minimal feasible time horizon T for a quickest transshipment problem $(\mathcal{N}, b)$ in a dynamic network with a single sink t

Input : A dynamic network $\mathcal{N} = (D = (V, A), u, \tau, S^+, t)$ and a supply/demand function b

Output: The minimal feasible time horizon T for the quickest transshipment problem $(\mathcal{N}, b)$

$i \leftarrow 0$, $S_i \leftarrow S^+$, $\theta_i \leftarrow 0$

while $S_i \neq \emptyset$ **do**

  $\theta_{i+1} \leftarrow$ Minimal value with $o_{S_i}^{\theta_{i+1}}(S_i) - b(S_i) = 0$

  $S_{i+1} \leftarrow$ Minimal minimizer of $o_{S_i}^{\theta_{i+1}} - b$

  $i \leftarrow i + 1$

end

return $T := \theta_i$, $S^* := S_i$

a single source *or* only a single sink that gets completely rid of using parametric search (see Algorithm 1) and as a consequence achieves a huge running time improvement compared to the old approach. Here, we only consider the case of networks with a single sink. The single source case is completely symmetric. Our main result is the following theorem.

Theorem 1. *Algorithm 1 returns the minimal feasible time horizon T of a given quickest transshipment problem $(\mathcal{N}, b)$ in a dynamic network with only a single sink and it terminates after at most $|S^+|$ many iterations.*

The proof of Theorem 1 especially exploits the fact that the parametrized submodular functions o^θ are connected by a so-called *strong map*. For networks with only a single sink this was shown by Baumann and Skutella [7], while for networks with only a single source this is a new results. Using a result by Topkis [8], this implies that the minimal minimizers S_i computed during the course of the algorithm are strictly contained in one another which then yields that the algorithm terminates after at most $|S^+|$ iterations. That the algorithm computes the correct minimal feasible time horizon immediately follows from Klinz' feasibility criterion [5]. Since each step of the algorithm can be computed in strongly polynomial time (minimizing a submodular function and solving a *quickest flow problem* [9]), it follows from Theorem 1 that in the considered special case Algorithm 1 enables us to determine the minimal feasible time horizon T in strongly polynomial time without using parametric search. A more careful analysis shows that Algorithm 1 achieved at least a running time improvement of $|S^+|^2 \log |S^+|$ compared to the parametric search approach.

References

[1] L.R. Ford, D.R. Fulkerson, *Constructing Maximal Dynamic Flows from Static Flows.* Operations Research 6(3) (1958), 419–433.
[2] L.R. Ford, D.R. Fulkerson, *Flows in Networks.* Princeton University Press (1962).
[3] B. Hoppe, É. Tardos, *The Quickest Transshipment Problem.* Mathematics of Operations Research 25(1) (2000), 36–62.
[4] M.Schlöter, M. Skutella, *Fast and Memory-Efficient Algorithms for Evacuation Problems.* Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms (2017), 821–840.
[5] B. Klinz, Cited as personal communication (1994) in [3].
[6] N. Megiddo, *Combinatorial Optimization with Rational Objective Functions.* Mathematics of Operations Research 4(4) (1979), 414–424.
[7] N. Baumann, S. Skutella, *Earliest Arrival Flows with Multiple Sources.* Mathematics of Operations Research 34(2) (2009), 499–512.
[8] D.M. Topkis, *Minimizing a Submodular Function on a Lattice.* Operations Research 26(2) (1978), 305–321.
[9] M. Saho, M. Shigeno, *Cancel-and-Tighten Algorithm for Quickest Flow Problems.* Networks 69(2) (2017), 179–188.

When do Gomory-Hu Subtrees Exist?

Bruce Shepherd
(joint work with Guyslain Naves)

Gomory-Hu (GH) Trees are a classical sparsification technique for graph connectivity [1]. It is one of the fundamental models in combinatorial optimization which also continually finds new applications, most recently in social network analysis. For any edge-capacitated undirected graph $G = (V, E)$ and any subset of *terminals* $Z \subseteq V$, a Gomory-Hu Tree is an edge-capacitated tree $T = (Z, E(T))$ such that for every $u, v \in Z$, the value of the minimum capacity uv cut in G is the same as in T. Moreover, the minimum cuts in T directly identify those in G. It is well-known that there does not always exist a GH tree which is a subgraph of G. For instance, every GH tree for the vertices of $K_{3,3}$ is a 5-star. We characterize those graph and terminal pairs (G, Z) which always admit such a tree. We show that these are the graphs which have no *terminal-$K_{2,3}$ minor*. That is, no $K_{2,3}$ minor whose vertices correspond to terminals in Z. We also show that the family of pairs (G, Z) which forbid such $K_{2,3}$ "Z-minors" arises, roughly speaking, from so-called Okamura-Seymour instances. More precisely, they are subgraphs of *Z-webs*. A Z-web is built from planar graphs with one outside face which contains all the terminals and each inner face is a triangle which may contain an arbitrary graph. This characterization yields an additional consequence for multiflow problems. Fix a graph G and a subset $Z \subseteq V(G)$ of terminals. Call (G, Z) *cut-sufficient* if the cut condition is sufficient to characterize the existence of a multiflow for any demands between vertices in Z, and any edge capacities on G. Then (G, Z) is cut-sufficient if and only if it is terminal-$K_{2,3}$ free.

References

[1] R.E. Gomory, T.C. Hu, *Multi-terminal network flows.* Journal of the Society for Industrial and Applied Mathematics, 9 (1961), 551–570.

Synchronized Traveling Salesman Problem

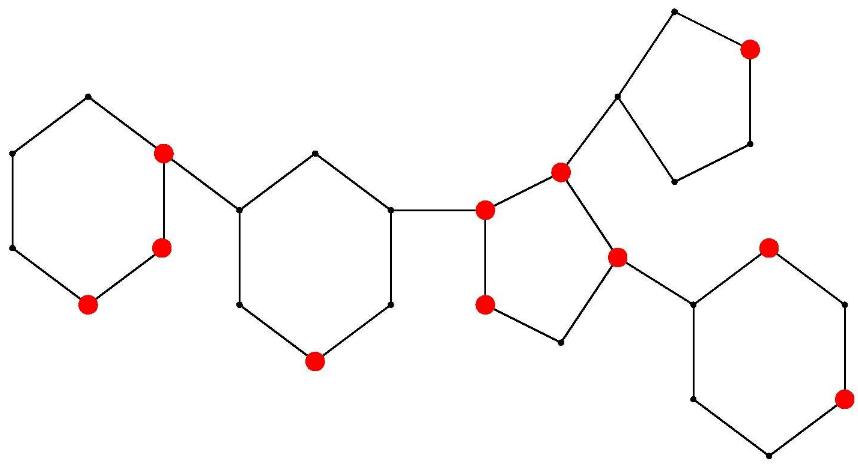
Gyula Pap
(joint work with József Varnyú)

We consider a variation of the well-known traveling salesman problem in which there are multiple agents who all have to tour the whole set of nodes of the same graph, while obeying node- and edge-capacity constraints that require no two agents may "crash". We consider the simplest model in which the input is an undirected graph with all edge lengths and capacities equal to one. A solution to the synchronized traveling salesman problem is called an "agency", which maps the respective agents with time-indexed traveling salesman tours. Our model puts the synchronized traveling salesman problem in a similar relation with the traveling salesman problem as the so-called evacuation problem, or the well-known dynamic flow (flow-over-time) problem is in relation with the minimum cost flow problem. We measure the strength of an agency in terms of number of agents which should be as large as possible, and the time horizon which should be as small as possible.

Let $G = (V, E)$ be an undirected graph with $n = |V|$. A sequence $v(0), v(1), v(2), \cdots, v(T)$ of nodes $v(t) \in V$ is called a **traveling salesman tour (with parking)**, or a tour, for short, if $v(0) = v(T)$, for all $t = 0, 1, \cdots, T-1$ we have either $v(t) = v(t+1)$ or $v(t)v(t+1) \in E$, and all nodes of V appear in this sequence at least once. T is called the **time horizon**, as is commonly referred to in the theory of flows over time. A similar definition may be given for a traveling salesman tour without parking, or in case of a directed graph. In the synchronized traveling salesman problem, we consider an **"agency"** of a number of salesmen so that each one of them has to do a tour with the same time horizon, though they need to start from different initial nodes, and must not "crash" into each other. This may be thought of as a unit capacity for each node or each edge. More precisely, we define an agency as follows.

Definition 1. *Let $k, T \in \mathbb{Z}_+$ be positive integers. k denotes the number of salesmen, or* **agents***, and T denotes the joint time horizon. Let $a_i(t) \in V$ be the node where agent i is supposed to be at time t, where $i = 1, 2, \cdots, k$, and $t = 0, 1, \cdots, T$. This is called a (feasible)* **agency** *with time horizon T and k agents if for any fixed i, $a_i(t)$ is a tour, and there is no value of t and agents $i \neq j$ that crash in a node i.e. $a_i(t) = a_j(t)$, or crash in an edge i.e. $a_i(t) = a_j(t+1), a_i(t+1) = a_j(t)$. In practical terms, this may be understood as a set of agents moving along the unit-length edges of the graph so that they avoid crashing into each other, each of them manages to visit every node at least once, before finally arriving at their respective nodes of origin.*

We may investigate various question related to this notion of an agency – the one that we would consider in this talk are, given an input graph, to maximize k and minimize T. Note that $k \le n \le T$, so our goal may be to try to get both k and T as close to T as possible. There may be various objectives: 1. minimize T/k, 2. maximize k, 3. minimize $\alpha(G) := max\{T/n, n/k\} \ge 1$. It is a hard problem to solve any of these optimization problems, because of the following easy observation.



Theorem 1. *For a given graph there is an agency with $\alpha = 1$ if and only if there is a Hamiltonian cycle.*

Because of the following observation, any bound on T/k or α will also imply a bound on the value of the other one.

Claim 1. *For any graph and any agency we have $\sqrt{T/k} \le \alpha \le T/k$.*

We may consider the question of bounding T/k or α if the graph is assumed to be a tree, and obtain the following lower bound.

Theorem 2. *If G is a tree, then for any feasible agency we have $T/k \ge 4$ and $\alpha \ge 2$.*

Theorem 3. *If G is a tree, then for a feasible agency with $T = 2n - 2$ we have $T/k \ge 5$ and $\alpha \ge \sqrt{5}$.*

An infinite family of trees, and an agency constructed show that these bounds on k/T are tight. Those examples will give a nice upper bound on α, too, but not a tight bound: in case of Theorem 2 the examples only achieve $\alpha = \frac{16}{7} + o(1) \approx 2.285$ which is greater than the lower bound of 2, and in case of Theorem 3 the examples only achieve $\alpha = 2.5 + o(1)$ which is greater than the lower bound of $\sqrt{5} \approx 2.236$.

Based on a result of Boyd, Iwata, Takazawa, 2013, if $n \ge 5$ then in a 3-regular 3-connected graph there is a 2-factor with only cycles of length at least 5. Using this result we can prove the following constructive bound, which holds for all 3-regular 3-connected graphs, but is in no way tight, it is just to claim that an agency with this many agents and that value of time horizon may be constructed in polynomial time.

Theorem 4. *If $G = (V, E)$ is a 3-regular 3-edge-connected graph, then there is an agency (with parking) such that $T/k = 4$, $T = 2n, k = n/2$ and $\alpha = \alpha_1 = \alpha_2 = 2$.*

Further, we may consider optimization problems, that is, given an input graph, optimize either of the above mentioned objectives. We would mention the following two observations that are to find out if the extreme case of $k = n$ may be achieved for the given input graph, either with parking, or without, and without any bound on the time horizon.

Theorem 5. *There is a feasible agency with parking of $k = n$ agents if and only if G is 2-edge-connected.*

Theorem 6. *There is a feasible agency without parking of $k = n$ agents if and only if the edges of G which are contained in a 2-factor form a connected spanning subgraph.*

Both of these properties can be checked for in polynomial time.

There are many open questions concerning exact optimization and approximation of α or T/k, including the following ones. For any input graph, minimize α, or minimize T/k. Do this under the assumption that the input graph is a tree, or under some connectivity constraint. Find a constant lower bound on T/k for trees with no nodes of degree 2. (Actually, as of time of writing of this report, this latter question seems to have been resolved during the Oberwolfach workshop.) Can we say anything about the similar definition for directed graphs?

On bounds on the Shannon capacity

Alexander Schrijver
(joint work with Sven Polak)

In our lecture, we survey and discuss old and new results of Shannon, Rosenfeld, Lovász, Haemers, Blasiak, Shanmugan, Asteris, Dimakis, Zuiddam, Bukh, and Cox on the Shannon capacity, and consider a few observations and questions.

The *Shannon capacity* $\Theta(G)$ of an (undirected) graphs is defined as follows ([8]). For graphs G and H, the graph $G \boxtimes H$ has vertex set $V(G) \times V(H)$, where two distinct vertices (u, v) and (u', v') are adjacent in $G \boxtimes H$ if and only if u and u' are equal or adjacent in G and v and v' are equal or adjacent in H. Then

$$\Theta(G) := \sup_{d \in \mathbb{N}} \sqrt[d]{\alpha(G^d)},$$

where G^d is the d-th power of G with respect to $\boxtimes$.

Trivially, $\alpha(G) \leq \Theta(G)$ for any graph G. Shannon [8] showed that $\Theta(G)$ is upper bounded by the *clique cover number* $\overline{\chi}(G)$ of G, which is the minimum number of cliques needed to cover $V(G)$. Rosenfeld [6] showed that the bound

$\overline{\chi}(G)$ can be sharpened to the *fractional clique cover number* $\overline{\chi}^*(G)$ of G, which is defined as

$$\overline{\chi}^*(G) = \min\{\sum_{C \text{ clique}} y(C) \mid y : \{\text{cliques}\} \to \mathbb{R}_+; \forall v \in V(G) : \sum_{C \ni v} y(C) \geq 1\}.$$

So $\alpha(G) \leq \Theta(G) \leq \overline{\chi}^*(G)$. Since $\alpha(G) = \overline{\chi}(G)$ for each perfect graph G, we know $\Theta(G) = \alpha(G)$ for each perfect graph G. For the smallest imperfect graph C_5 the facts that $\alpha(C_5^2) = 5$ and $\overline{\chi}^*(C_5) = \frac{5}{2}$ then yield

$$\sqrt{5} \leq \Theta(C_5) \leq \frac{5}{2}.$$

It was Lovász [4] who showed that in fact $\Theta(C_5) = \sqrt{5}$, by introducing the following upper bound $\vartheta(G)$ on $\Theta(G)$:

$$\Theta(G) \leq \vartheta(G) := \max\{\mathbf{1}^T X \mathbf{1} \mid X \in \mathbb{R}^{V(G)\times V(G)} \text{ positive semidefinite}; \\ \operatorname{tr}(X) = 1; \forall uv \in E(G) : X_{u,v} = 0\}.$$

As Lovász also showed that $\vartheta(C_5) = \sqrt{5}$, we know $\Theta(C_5) = \sqrt{5}$.

Lovász [4] left as open question whether $\Theta(G) = \vartheta(G)$ would hold for each graph G. Haemers [3] answered this question negatively. He introduced, for any field $\mathbb{F}$, the following upper bound $h_{\mathbb{F}}$ on $\Theta(G)$:

$$h_{\mathbb{F}}(G) := \min\{\operatorname{rank}(X) \mid X \in \mathbb{F}^{V(G)\times V(G)}; \forall v : X_{v,v} = 1; \forall uv \notin E(G) : X_{u,v} = 0\}.$$

Haemers showed that if G is the *Schläfli graph* (a strongly regular graph on 27 vertices), then

$$\Theta(G) \leq h_{\mathbb{R}}(G) = 7 < 9 = \vartheta(G).$$

Blasiak [1] (cf. [7]) sharpened Haemers' bound by introducing the *fractional Haemers bound* $h^*_{\mathbb{F}}(G)$:

$$\Theta(G) \leq h^*_{\mathbb{F}}(G) = \inf\{\frac{\operatorname{rank}(X)}{k} \mid k \in \mathbb{N}; X \in (\mathbb{F}^{k\times k})^{V(G)\times V(G)}; \\ \forall v \in V(G) : X_v = I_k; \forall uv \notin E(G) : X_{u,v} = \mathbf{0}\}.$$

So X is a block matrix with each block being a $k \times k$ matrix. The rank of X then is the rank of the corresponding $(k \times V(G)) \times (k \times V(G))$ matrix.

Zuiddam [10] introduced a collection Δ of upper bounds for the Shannon capacity, proving that for each graph G, at least one of these upper bounds is tight. More precisely, Zuiddam derived the following 'sup-min' relation from Strassen's semiring theorem [9]:

$$\Theta(G) = \min_{f\in\Delta} f(G),$$

where

$$\Delta := \{f : \{\text{graphs}\} \to \mathbb{R} \mid \forall \text{ graphs } G, H : \\ f(G \sqcup H) = f(G) + f(H), \\ f(G \boxtimes H) = f(G)f(H), \\ \text{if } \exists \text{ homomorphism } \overline{G} \to \overline{H}, \text{ then} f(G) \leq f(h), \\ f(K_1) = 1\}.$$

Here $G \sqcup H$ is the disjoint union of G and H. Moreover, an *homomorphism* $\overline{G} \to \overline{H}$ is a function $h : V(G) \to V(H)$ such that if u and v are distinct nonadjacent vertices of G, then $h(u)$ and $h(v)$ are distinct nonadjacent vertices of H.

With LP-duality it can be shown that the function $\overline{\chi}^*$ indeed belongs to Δ. Using SDP-duality, Lovász [4] showed that ϑ also has the properties that make ϑ belong to Δ. With as clever argument, Bukh and Cox [2] showed that for each field $\mathbb{F}$ the function $h^*_{\mathbb{F}}$ belongs to Δ (basically using the duality of 'maximally independent' and 'minimally spanning' in linear spaces).

Note that the set Δ forms a metrizable compact space, so that each sequence of its elements has a convergent subsequence. For instance, the sequence $h^*_{\mathbb{F}_2}$, $h^*_{\mathbb{F}_3}$, $h^*_{\mathbb{F}_5}$, ... has a convergent subsequence. (It was shown by Bukh and Cox [2] that these $h^*_{\mathbb{F}_p}$ are all distinct for distinct primes p.) Similarly with other sequences we might obtain new elements of Δ.

Another topological space is obtained by defining the following semimetric d on the set {graphs}:

$$d(G,H) := \max_{f\in\Delta} |f(G) - f(H)|$$

for graphs G and H. The completion of this semimetric space might create 'graph limits'. We were however not able to construct a nonconvergent Cauchy sequence of graphs in this semimetric space. Potential candidates to yield nonconvergent Cauchy sequences might be the following graphs $C_{k,n}$, for $k,n \in \mathbb{N}$: $C_{k,n}$ has vertex set $V(C_n)$ (where C_n is the circuit with n vertices), with vertices $u,v \in V(C_n)$ being adjacent in $C_{k,n}$ if and only if the distance of u and v in C_n is less than k. One can show that $d(C_{k,n}, C_{k',n'}) = 0 \iff \frac{n}{k} = \frac{n'}{k'}$. (This implies that $\Theta(C_{k,n}) = \Theta(C_{k',n'})$ and $\vartheta(C_{k,n}) = \vartheta(C_{k',n'})$ if $\frac{n}{k} = \frac{n'}{k'}$.) Now the question is whether the function $\frac{n}{k} \mapsto C_{k,n}$ from $\{r \in \mathbb{Q} \mid r \geq 2\}$ to the space of graphs is continuous. If so, any convergent sequence $\frac{n_1}{k_1}, \frac{n_2}{k_2}, \frac{n_3}{k_3}, \ldots$ of rationals will give a Cauchy sequence C_{k_1,n_1}, C_{k_2,n_2}, C_{k_3,n_3}, ... If $\frac{n_1}{k_1}, \frac{n_2}{k_2}, \frac{n_3}{k_3}, \ldots$ has an irrational limit γ, the sequence C_{k_1,n_1}, C_{k_2,n_2}, C_{k_3,n_3}, ... has no limit in {graphs}, so we create a 'graph limit' being not a graph (as $\overline{\chi}^*(G)$ is rational for each graph G, while $\overline{\chi}^*(C_{k_i,n_i}) = \frac{n_i}{k_i} \to \gamma$).

So far, we [5] could only prove that the function $\frac{n}{k} \mapsto C_{k,n}$ is continuous at integer values of $\frac{n}{k} \geq 3$, by showing that for each integer $q \geq 3$ and each $d \in \mathbb{N}$:

$$\max_{\frac{n}{k}<q} \alpha(C^d_{k,n}) = \frac{q-2}{q-1}q^d + \frac{1}{q-1}.$$

This implies $\sup_{\frac{n}{k}<q} \Theta(C_{k,n}) = q$, which can be seen as the main ingredient in proving continuity at q.

We finally mention some (of the many) open problems. Is Δ connected? Find $f \in \Delta$ with $f(C_7) = \Theta(C_7)$. Is $\lim_{p\to\infty} h^*_{\mathbb{F}_p} = h^*_{\mathbb{Q}}$? Is $\lim_{p\to\infty} h^*_{\overline{\mathbb{F}_p}} = h^*_{\overline{\mathbb{Q}}}$? (p running over primes.) Is the function $\frac{n}{k} \mapsto \Theta(C_{k,n})$ continuous? Is the function $\frac{n}{k} \mapsto \vartheta(C_{k,n})$ continuous? Similar questions can be asked for Kneser graphs.

References

[1] A. Blasiak, *A Graph-Theoretic Approach to Network Coding.* Dissertation, Cornell University, Ithaca, N.Y., 2013 (`https://ecommons.cornell.edu/handle/1813/34147`).

[2] B. Bukh, C. Cox, *On a fractional version of Haemers' bound.* ArXiv:1802.00476, 2018.

[3] W. Haemers, *On some problems of Lovász concerning the Shannon capacity of a graph.* IEEE Transactions on Information Theory IT-25 (1979), 231–232.

[4] L. Lovász, *On the Shannon capacity of a graph.* IEEE Transactions on Information Theory IT-25 (1979), 1–7.

[5] S. Polak, A. Schrijver, *New lower bound on the Shannon capacity of C_7 from circular graphs.* ArXiv:1808.07438, 2018 (Information Processing Letters, to appear).

[6] M. Rosenfeld, *On a problem of C.E. Shannon in graph theory.* Proceedings of the American Mathematical Society 18 (1967), 315–319.

[7] K. Shanmugam, M. Asteris, A.G. Dimakis, *On approximating the sum-rate for multiple-unicasts.* 2015 IEEE International Symposium on Information Theory (ISIT), pp. 381–385, 2015 (arXiv:1504.05294).

[8] C.E. Shannon, *The zero error capacity of a noisy channel.* IRE Transactions on Information Theory IT-2 (1956), 8–19.

[9] V. Strassen, *The asymptotic spectrum of tensors.* Journal für die reine und angewandte Mathematik 384 (1988), 102–152.

[10] J. Zuiddam, *The asymptotic spectrum of graphs and the Shannon capacity.* ArXiv:1807.00169, 2018.

A Tale of Santa Claus, Hypergraphs and Matroids

Thomas Rothvoss

(joint work with Sami Davies, Yihao Zhang)

A well-known problem in scheduling and approximation algorithms is the Santa Claus problem. Suppose that Santa Claus has a set of gifts, and he wants to distribute them among a set of children so that the least happy child is made as happy as possible. Here, the value that a child i has for a present j is of the form $p_{ij} \in \{0, p_j\}$. The only known polynomial time algorithm by Annamalai et al. gives a 12.33-approximation algorithm and is based on a modification of Haxell's hypergraph matching argument. This factor compares to the value of an exponential size *configuration LP*.

Formally, the *Santa Claus* problem takes as input a set M of children, a set J of gifts, and values $p_{ij} \in \{0, p_j\}$ for all $i \in M$ and $j \in J$. In other words, a child is only interested in a particular subset of the gifts, but then its value only depends on the gift itself. The goal is to find an assignment $\sigma : J \to M$ of gifts to children so that $\min_{i \in M} \sum_{j \in \sigma^{-1}(i)} p_{ij}$ is maximized.

The first major progress on this problem is due to Bansal and Sviridenko [4], who showed a $O(\log\log n / \log\log\log n)$-approximation based on rounding a *configuration LP*. The authors of [4] also realized that in order to obtain a $O(1)$-approximation, it suffices to answer a purely combinatorial problem: show that in a uniform bipartite hypergraph with equal degrees on all sides, there is a left-perfect matching that selects a constant fraction of nodes from original edges. This question was affirmatively answered by Feige [5] who proved a large unspecified constant using the Lovász Local Lemma repeatedly. Then Asadpour, Feige and Saberi [1] showed that one can answer the question of [4] by using a beautiful theorem on hypergraph matchings due to Haxell [6]; in fact their bound of 4 on the integrality gap of the configuration LP for Santa Claus has not been improved[1].

A *hypergraph* $\mathcal{H} = (X\dot{\cup}W, \mathcal{E})$ is called *bipartite* if $|e \cap X| = 1$ for all hyperedges $e \in \mathcal{E}$. A *(left-) perfect matching* is a set of hyperedges $F \subseteq \mathcal{E}$ that are disjoint but cover each node in X. In general, finding perfect matchings in even bipartite hypergraphs is **NP**-hard, but there is an intriguing sufficient condition:

Theorem 1 (Haxell [6])**.** *Let $\mathcal{H} = (X\dot{\cup}W, \mathcal{E})$ be a bipartite hypergraph with $|e| \leq r$ for all $e \in \mathcal{E}$. Then either $\mathcal{H}$ contains a left-perfect matching or there is a subset $C \subseteq X$ and a subset $U \subseteq W$ so that all hyperedges incident to C intersect U and $|U| \leq (2r-3) \cdot (|C|-1)$.*

Unlike Hall's Theorem, Haxell's proof is non-constructive and based on a possibly exponential time augmentation argument. Annamalai, Kalaitzis and Svensson [3] gave a non-trivially modified version of Haxell's argument for Santa Claus, which runs in polynomial time and gives a 12.33-approximation[2]. In fact our algorithm will borrow a lot from [3]. However, through a much cleaner argument we obtain a result that works in a more general matroid setting and implies a better approximation of $6 + \varepsilon$ for Santa Claus.

Our contributions. Let $\mathcal{M} = (X, \mathcal{I})$ be a *matroid* with *groundset* X and a family of *independent sets* $\mathcal{I} \subseteq 2^X$. Recall that a matroid is characterized by three properties:

(i) *Non-emptyness*: $\emptyset \in \mathcal{I}$;
(ii) *Monotonicity*: For $Y \in \mathcal{I}$ and $Z \subseteq Y$ one has $Z \in \mathcal{I}$;
(iii) *Exchange property*: For all $Y, Z \in \mathcal{I}$ with $|Y| < |Z|$ there is an element $z \in Z \setminus Y$ so that $Y \cup \{z\} \in \mathcal{I}$.

The *bases* $\mathcal{B}(\mathcal{M})$ of the matroid are all inclusion-wise maximal independent sets. The cardinalities of all bases are identical, with size denoted as $\mathrm{rank}(\mathcal{M})$. The convex hull of all bases is called the *base polytope*, that is $P_{\mathcal{B}(\mathcal{M})} := \mathrm{conv}\{\chi(S) \in \{0,1\}^X \mid S \text{ is basis}\}$, where $\chi(S)$ is the *characteristic vector* of S.

Now consider a bipartite graph $G = (X\dot{\cup}W, E)$ with the ground set X on one side and a set of *resources* W on the other side; each resource $j \in W$ has a *size* $p_j \geq 0$. In a problem that we call *Matroid Max-Min Allocation*, the goal is to find a basis $S \in \mathcal{B}(\mathcal{M})$ and an assignment $\sigma : W \to S$ with $(\sigma(j), j) \in E$ so that $\min_{i \in S} \sum_{j \in \sigma^{-1}(i)} p_j$ is maximized. To the best of our knowledge, this problem has not been studied before. In particular if $T \geq 0$ is the target objective function value, then we can define a linear programming relaxation $Q(T)$ as the set of

[1]Note that the conference version of [1] provides a factor of 5, which in the journal version [2] has been improved to 4.

[2]To be precise they obtain a $(6 + 2\sqrt{10} + \varepsilon)$-approximation in time $n^{O(\frac{1}{\varepsilon^2}\log(\frac{1}{\varepsilon}))}$.

vectors $(x, y) \in \mathbb{R}^X_{\geq 0} \times \mathbb{R}^E_{\geq 0}$ satisfying the constraints

$$x \in P_{\mathcal{B}(\mathcal{M})}; \qquad \sum_{j \in N(i)} p_j y_{ij} \geq T \cdot x_i \ \forall i \in X; \quad y(\delta(j)) \leq 1 \ \forall j \in W; \quad y_{ij} \leq x_i \ \forall (i,j) \in E.$$

Here, the decision variable x_i expresses whether element i should be part of the basis, and y_{ij} expresses whether resource j should be assigned to element i. We abbreviate $N(i)$ as the neighborhood of i and $y(\delta(j))$ is shorthand for $\sum_{i:(i,j)\in E} y_{ij}$. Then our main technical result is:

Theorem 2. *Suppose $Q(T) \neq \emptyset$. Then for any $\varepsilon > 0$ one can find $(x, y) \in Q((\frac{1}{3} - \varepsilon) \cdot T)$ so that $x \in \{0,1\}^X$ in time $n^{\Theta_\varepsilon(1)}$, where $n := |X| + |W|$. This assumes that membership in the matroid can be tested in time polynomial in n.*

Previously this result was not even known with non-constructive methods. Note that the vector y in the claim might be fractional. Using standard arguments one can then show the following:

Theorem 3. *Suppose $Q(T) \neq \emptyset$. Then for any $\varepsilon > 0$ one can find $(x, y) \in Q\big((\frac{1}{3} - \varepsilon) \cdot T - \max_{j\in W} p_j\big)$ with both x and y integral in time $n^{\Theta_\varepsilon(1)}$, where $n := |X| + |W|$. This assumes that membership in the matroid can be tested in time polynomial in n.*

Although, this algorithm does not necessarily imply a multiplicative approximation for Matroid Max-Min Allocation, we see it is still useful by applying it to the Santa Claus problem:

Theorem 4. *The Santa Claus problem admits a $(6 + \varepsilon)$-approximation algorithm in time $n^{\Theta_\varepsilon(1)}$.*

In fact, once we have Theorem 3 the reasoning is simple: for a suitable threshold $0 < \delta < 1$, call a gift j *large* if $p_j \geq \delta \cdot OPT$ and *small* otherwise. Then the family of sets of children that can get assigned large gifts forms a *matchable set matroid*. We apply Theorem 3 to the *co-matroid* of the matchable set matroid. Then we obtain a basis $S := \{i \in M \mid x_i = 1\}$, which contains the children *not* receiving a large gift. These children can receive small gifts of total value $(\frac{1}{3} - \varepsilon - \delta) \cdot OPT$. Setting $\delta := \frac{1}{6}$ implies the claim. Note the approximation factor $6 + \varepsilon$ will be with respect to a natural, compact linear program with $O(n^2)$ many variables and constraints. The smallest LP that was previously known to have a constant integrality gap was the $O(n^3)$-size LP of [7].

References

[1] A. Asadpour, U. Feige, A. Saberi, *Santa claus meets hypergraph matchings.* Approximation, Randomization and Combinatorial Optimization, Algorithms and Techniques, 11th International Workshop, APPROX 2008, and 12th International Workshop, RANDOM 2008, Boston, MA, USA, August 25-27, 2008. Proceedings, pages 10–20, 2008.

[2] A. Asadpour, U. Feige, A. Saberi, *Santa claus meets hypergraph matchings.* ACM Trans. Algorithms, 8(3):24:1–24:9, July 2012.

[3] C. Annamalai, C. Kalaitzis, O. Svensson, *Combinatorial algorithm for restricted max-min fair allocation.* Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2015, San Diego, CA, USA, January 4-6, 2015, pages 1357–1372, 2015.

[4] N. Bansal, M. Sviridenko, *The santa claus problem.* Proceedings of the Thirty-eighth Annual ACM Symposium on Theory of Computing, STOC '06, pages 31–40, New York, NY, USA, 2006. ACM.

[5] U. Feige, *On allocations that maximize fairness.* Proceedings of the Nineteenth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA '08, pages 287–293, Philadelphia, PA, USA, 2008. Society for Industrial and Applied Mathematics.

[6] P.E. Haxell, *A condition for matchability in hypergraphs.* Graphs and Combinatorics, 11(3):245–248, 1995.

[7] K. Jansen, L. Rohwedder, *Compact LP relaxations for allocation problems.* 1st Symposium on Simplicity in Algorithms, SOSA 2018, January 7-10, 2018, New Orleans, LA, USA, pages 11:1–11:19, 2018.

Sticky Brownian Rounding and its Applications to Constraint Satisfaction Problems

Mohit Singh

(joint work with Abbas-Zadeh, Nikhil Bansal, Guru Guruganesh, Sasho Nikolov, Roy Schwartz)

Sticky Brownian Rounding and its Applications to Optimization Problems We present a new general and simple method for rounding semi-definite programs, based on Brownian motion. Our approach is inspired by recent results in algorithmic discrepancy theory. We develop and present tools for analyzing our new rounding algorithms, utilizing mathematical machinery from the theory of Brownian motion, complex analysis, and partial differential equations. We will present our method to several classical problems, including Max-Cut, Max-di-cut and Max-2-SAT, and derive new algorithms that are competitive with the best known results. In particular, we show that the basic algorithm achieves 0.861-approximation for Max-cut and a natural variant of the algorithm achieve 0.878-approximation, matching the famous Goemans-Williamson algorithm [1] up to first three decimal digits.

References

[1] M.X. Goemans, D.P. Williamson, *Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming.* Journal of the ACM (JACM) 42.6 (1995), 1115-1145.

LP Relaxation and Tree Packing for Minimum k-Cut

Chandra Chekuri
(joint work with Kent Quanrud, Chao Xu)

Karger used spanning tree packings to derive a near linear-time randomized algorithm for the global minimum cut problem as well as a bound on the number of approximate minimum cuts. This is a different approach from his well-known random contraction algorithm. Thorup developed a fast deterministic algorithm for the minimum k-cut problem via greedy recursive tree packings. We revisit properties of an LP relaxation for k-cut proposed by Naor and Rabani. We show that the dual of the LP yields a tree packing, that when combined with an upper bound on the integrality gap for the LP, easily and transparently extends Karger's analysis for mincut to the k-cut problem. We will also discuss a simple proof of the integrality gap of LP relaxation and some of its other properties.

Subset selection in sparse matrices

Santanu S. Dey
(joint work with Alberto Del Pia, Robert Weismantel)

In statistics, subset selection is also known as feature selection, attribute selection, variable selection or variable subset selection. It is the problem of selecting a subset of relevant variables (or features) to recover a predictor variable. Formally, *subset selection* is a nonlinear optimization problem of the following form:

$$
\begin{array}{ll}
\min & \|Mx + c\mu - b\|_2 \\
\text{s.t.} & x \in \mathbb{R}^n, \ \mu \in \mathbb{R} \\
& |\text{supp(x)}| \leq \sigma.
\end{array}
\tag{1}
$$

In this formulation, x is the d-vector of unknowns and μ is a scalar variable. The remaining characters stand for data in the problem instance: M is an $m \times d$ matrix, b and c are m-vectors, and σ is a natural number. Finally, $\|\cdot\|_2$ denotes the Euclidean norm. Note that in standard formulations of the subset selection problem it is often assumed that c is the vector of all ones. It is often assumed that the columns of M and b are mean-centered (i.e., the sum of entries in the columns of M and in b is zero), in which case it can be shown that the optimal value of μ is zero. Since we want to exploit the sparsity structure of M, we do not assume mean-centering and therefore explicitly retain the μ variable.

We are interested in identifying sparsity conditions on the original data matrix M that allow us to solve subset selection in polynomial time. We consider matrix M obtained by adding a fixed number of extra columns (k) to a block diagonal matrix, where each block (the A^i matrices below) involves a fixed number of variables (θ), i.e. a matrix of the following form:

$$M = \left(\begin{array}{ccc|ccc} A^1 & & & | & & | \\ & \ddots & & c_1 & \cdots & c_k \\ & & A^h & | & & | \end{array} \right). \tag{2}$$

Formally our result is then stated as follows.

Theorem 1. *Problem* (1) *with matrix of the form* (2) *can be solved in polynomial time for varying n (the total number of variables), provided that k and θ are fixed numbers.*

Our approach relies on tools from discrete geometry and an analysis of the proximity of optimal solutions with respect to two consecutive "support-conditions" $|\mathrm{supp}(x)| \leq s$ and $|\mathrm{supp}(x)| \leq s+1$.

Scanning integer points with lex-cuts: A finite cutting plane algorithm for integer programming over compact constraint set

Michele Conforti

The area of nonlinear integer programming is rich in applications but quite challenging from a computational point of view. The tools that are mainly used are sophisticated techniques that exploit relaxations, constraint enforcement (e.g., cutting planes) and convexification of the feasible set. Reformulations in an extended space and cutting planes for nonlinear integer programs have been investigated and proposed for some time. This line of research mostly provides a nontrivial extension of the theory of disjunctive programming to the nonlinear case. To the best of our knowledge, these results are obtained under some restrictive conditions: typically, convexity of the feasible set S, or $S \subseteq \{0,1\}^n$ (these cases cover some important areas of application).

In this paper we focus on *linear* inequalities that we use as cuts. As the convex hull of $S \cap \mathbb{Z}^n$ is a polytope when $S \subseteq \mathbb{R}^n$ is compact, a finite number of linear inequalities suffices for its characterization, and only n such inequalities determine an optimal point. Furthermore, some relaxations are polyhedral: most notably, Dadush, Dey and Vielma proved that if S is a compact and convex set, then its Chvátal closure is a polytope (whereas this is not the case for the split closure of S).

However, nonlinear inequalities are fundamental in the characterization of the convex hull of some nonlinear sets that strengthen the original formulation. For instance, Burer and Kılınç-Karzan, extending several results, show that the convex hull of the intersection of a second-order-cone representable set and a single homogeneous quadratic inequality can be described by adding a single nonlinear inequality, defining an additional second-order-cone representable set.

In this paper we present a finite cutting plane algorithm for problems of the form

$$\min\{cx : x \in S \cap \mathbb{Z}^n\}, \tag{1}$$

where S is a compact subset of $\mathbb{R}^n$ (not necessarily convex or connected) and $c \in \mathbb{Z}^n$. This algorithm uses a new family of cutting planes which includes the Chvátal–Gomory cuts, but neither it contains nor is contained in the family of split cuts. Furthermore, these cuts define a natural but nontrivial polyhedral relaxation of $S \cap \mathbb{Z}^n$.

The cutting planes employed in our algorithm are obtained as follows. We consider the integer points in a unimodular cone K, ordered by a lexicographic rule, associated with a lattice basis. To each integer point x in K, we associate a family of inequalities (lex-cuts) that defines the convex hull of the integer points in K that are not lexicographically smaller than x.

Our algorithm recursively solves optimization problems of the form $\min\{cx : x \in S \cap P\}$, where P is a polyhedron, and we assume that an algorithm for problems of this type is available as a black box. Note that when S is a convex set, this is a convex program that is (in principle) efficiently solvable. To the best of our knowledge, our work represents the first attempt to define a finite cutting plane algorithm for the general problem (1) with S compact.

Deriving a finite cutting plane algorithm that uses a well defined family of inequalities does not seem to be straightforward. The oldest and most notable example is Gomory's finite cutting plane algorithm for bounded integer programs based on fractional cuts. Balas, Ceria and Cornuéjols give a finite cutting plane algorithm for mixed 0/1 problems based on lift-and-project cuts. In those algorithms, as well as in the method proposed here, crucial to the detection of a cutting plane is the computation of a lexicographically optimal solution.

Open Problem: Maximum weighted assignment problem

Amitabh Basu

We consider the maximum weighted assignment problem on the complete bipartite graph with n vertices. The weight matrix will be denoted by $W \in \mathbb{R}^{n\times n}$. The Hungarian Algorithm solves the problem in $O(n^3)$ time. The question is whether one can do better if the rank of W is small. In particular, if the rank of W is 1, i.e., $W = uv^T$ for vectors $u, v \in \mathbb{R}^n$, then the problem can be solved in $O(n \log n)$ time. This is because the weight of any edge between nodes i and j is given by $u_i v_j$. Thus, to maximize the sum $\sum_{i=1}^n u_i v_{\sigma(i)}$ over all permutations σ of $\{1, \dots, n\}$, one simply sorts the u_i and v_j values and matches the values in descending order. We do not know whether a better running time can be achieved even in the case when the rank of W is 2. We wonder if there is an algorithm that runs in time $O(nr^2 \log n)$.

Open Problem: Approximation and Integrality Gap for Linear k-Cut

Chandra Chekuri

In the Linear-k-Cut problem we are given a directed graph $G = (V, E)$ and k distinct terminal nodes $s_1, s_2, \ldots, s_k$. The goal is to remove the minimum number of edges from G to ensure that s_i does not have a path to s_j for all $i < j$. Note that $k = 2$ to corresponds to the standard s-t mincut problem and is polynomial-time solvable. Linear k-Cut is a special case of the well-known Multicut problem. When $k = 3$ the problem is already NP-Hard and APX-Hard. Interestingly the approximability of the $k = 3$ case, assuming the Unique Games Conjecture (UGC) is $\sqrt{2}$ [2]. For arbitrary k an $O(\log k)$ approximation can be easily derived and this algorithm also establishes an $O(\log k)$ upper bound on the integrality gap of the natural LP relaxation.

The main open question is whether the integrality gap of the LP relaxation is $O(1)$ for all k or whether it is $\omega(1)$. Under UGC [1] shows that for all fixed k, the integrality gap of the LP coincides with the hardness of approximation. See [1] for additional discussion and related results including its connection to the Skew-Multicut problem.

References

[1] C. Chekuri, V. Madan, *Approximating Multicut and the Demand Graph.* Proc. of ACM-SIAM Symposium on Discrete Algorithms (SODA), 2017.

[2] K. Bérczi, K. Chandrasekaran, T. Király, V. Madan, *A tight-approximation for Linear 3-Cut.* Proc. of ACM-SIAM Symposium on Discrete Algorithms (SODA), 2018.

Open Problem: Diameter Bounds for Well-Conditioned Polyhedra

Daniel Dadush

A major open question in polyhedral theory is whether the diameter of the 1-skeleton of a polyhedron is bounded as a polynomial in the number of facets n and dimension d, known as the polynomial Hirsch conjecture. Recall that this diameter here is the usual graph diameter of the graph induced by the vertices and edges of a polyhedron. The current best worst-case bounds we have for this problem are the $O(2^d n)$ bound of Barnette and Larman [1, 2], and the quasi-polynomial bound of $n^{O(\log d)}$ by Kalai and Kleitman [3] which was recently improved by Todd [6] and Sukegawa [8].

To make progress on this question, a useful line of research has been to examine classes of "well-conditioned" polyhedra. For polyhedra with totally unimodular constraint matrices Dyer and Frieze [4] gave the first polynomial bound. These bounds have recently been extended and improved to the cases where the subdeterminants of the constraint matrix are all bounded in absolute value by Δ [5, 7], where the best known bound is $O(d^3 \Delta^2 \ln(d\Delta))$ [7]. This last bound was derived as a consequence of a more general theorem, which says that if the normal cones of a d-dimensional polyhedron are all τ-wide, then the diameter is bounded by $O(d/\tau \ln(1/\tau))$. A normal cone is τ-wide if it contains a ball of radius τ centered on the unit sphere. The τ-wideness condition intuitively asks for the facets of the polyhedron to always meet at "sharp angles". The proof of this diameter bound is in fact constructive. More precisely, for two vertices v, w, one first picks objective c, d of unit length, respectively maximized at v, w and which are at distance τ from the boundary of the respective normal cones. One now picks an additional uniformly chosen objective e on the unit sphere and uses it to construct the path from v to w as follows: follow the sequence of maximizers obtained as one linearly interpolates $c \to e$, followed by the same for $e \to d$. This path is the union of two random paths followed by the so-called shadow vertex simplex method, which in [7] was shown to induce a short expected path when the normal cones are all wide.

A natural question to ask is whether one really needs all the normal cones to be wide in the above setting. Indeed, one would expect that a suitable notion of the normal cones being wide on "average" should suffice. Thus, the open problem is to derive such a notion. One natural target, which seems to encapsulate this problem, is to prove a diameter bound for random Gaussian polytopes. That is, for a random polytope of the form $Ax \leq 1$, where $A \in \mathbb{R}^{n \times d}$ is random matrix with i.i.d. $N(0,1)$ entries.

References

[1] D.G. Larman, *Paths of polytopes.* Proc. London Math. Soc. (3), 20:161–178, 1970.

[2] D. Barnette, *An upper bound for the diameter of a polytope.* Discrete Math., 10:9–13, 1974.

[3] G. Kalai, D.J. Kleitman, *A quasi-polynomial bound for the diameter of graphs of polyhedra.* Bull. Amer. Math. Soc. (N.S.), 26(2), 315–316, 1992.

[4] M. Dyer, A. Frieze, *Random walks, totally unimodular matrices, and a randomised dual simplex algorithm.* Math. Programming, 64(1, Ser. A), 1–16, 1994.

[5] N. Bonifas, M. Di Summa, F. Eisenbrand, N. Hähnle, M. Niemeier, *On sub-determinants and the diameter of polyhedra.* Discrete Comput. Geom., 52(1), 102–115, 2014. Preliminary version in SOCG 12.

[6] M.J. Todd, *An improved Kalai–Kleitman bound for the diameter of a polyhedron.* SIAM Journal on Discrete Mathematics, 28(4), 1944–1947, 2014.

[7] D. Dadush, N. Hähnle, *On the shadow simplex method for curved polyhedra.* Discrete Comput. Geom., 56(4), 882–909, 2016. Preliminary version in SOCG '15.

[8] N. Sukegawa, *An asymptotically improved upper bound on the diameter of polyhedra.* Discrete & Computational Geometry, pages 1–10, Jun 2018.

Open Problem: Arborescences and NP-hardness

Volker Kaibel

Problem 1. (Why) is the following problem NP-hard?

Given an undirected graph $G = (V, E)$ and for each node $r \in V$ a cost vector $c^r \in \mathbb{R}^A$ (where A is the set of all $2|E|$ directed versions of edges of G), find a spanning tree $T \subseteq E$ that minimizes $\sum_{r \in V} c^r(T^r)$, where $T^r \subseteq A$ is the arborescence rooted at r induced by T.

Motivation: This optimization problem is equivalent to the integer optimization problem over Martin's extended formulation of the spanning tree polytope.

Problem 2. (Why) is the following problem NP-hard?

Given a directed graph $D = (V, A)$, a root node $r \in V$, and for each node $v \in V \setminus \{r\}$ a cost vector $c^v \in \mathbb{R}^A$, find an arborescence $T \subseteq A$ that minimizes $\sum_{v \in V \setminus \{r\}} c^v(P^v)$, where $P^v \subseteq T$ is the r-v-path in T.

Motivation: This optimization problem is equivalent to the integer optimization problem over Wong's extended formulation of the spanning tree polytope.

Open Problem: Sums of powers

Neil Olver

Let $r_1, r_2, \ldots, r_n$ be given integers. Is there an algorithm running in time polynomial in the input size (which is $O(\sum_{i=1}^n \log(|r_i| + 1))$ to decide if

$$\sum_{i=1}^{n} \left(\tfrac{2}{3}\right)^{r_i} > 1 \ ?$$

If $\frac{2}{3}$ is replaced by an integer or the inverse of an integer, this can be done easily. More generally, one can ask whether

$$\sum_{i=1}^{n} \beta_i \alpha^{r_i} > 0 \ ,$$

for α a fixed rational, and $\beta_1, \ldots, \beta_n$ being rationals given as part of the input.

Open Problem: Optimization over degree sequences

Shmuel Onn

Given k, n, m and univariate functions $f_1, \ldots, f_n$, consider the problem of finding a k-hypergraph on $[n]$ with m edges maximizing $\sum_{i=1}^n f_i(d_i)$ where d_i is the degree of vertex i. In a recent paper (Optimization over Degree Sequences, by Deza, Levin, Meesum, Onn, SIAM Journal on Discrete Mathematics 32:2067–2079, 2018), we prove (among other results) that over graphs (that is, for $k = 2$), if all the functions are the same (that is, $f_1 = \cdots = f_n = f$ for some f), then the problem is solvable in polynomial time. What is the complexity over graphs if the functions are not necessarily identical?

Open Problem: Matroids with cardinality constraints on the intersection

Britta Peis

(joint work with Stefan Lendl, Björn Tauer)

We consider the following problem:

- **Given:** Two matroids $M_1 = (E, \mathcal{B}_1)$, and $M_2 = (E, \mathcal{B}_2)$ on a common ground set E with base sets $\mathcal{B}_1$ and $\mathcal{B}_2$, two cost functions $c_1, c_2 : E \to \mathbb{R}$ on E, and some integer k.
- **Task:** Find two bases $X \in \mathcal{B}_1$ and $Y \in \mathcal{B}_2$ minimizing $c_1(X) + c_2(Y)$ subject to the constraint $|X \cap Y| = k$. (Here, as usual, $c_1(X) = \sum_{e \in X} c_1(e)$.)

Remark 1. *If constraint $|X \cap Y| = k$ is replaced by either $|X \cap Y| \leq k$, or $|X \cap Y| \geq k$, the associated optimization problems can be reduced to max-weight matroid intersection.*

Remark 2. *The special case where $\mathcal{B} = \mathcal{B}_1 = \mathcal{B}_2$ and lower bound constraint $|X \cap Y| \geq k$, i.e.,*

$$\min\{c_1(X) + c_2(Y) \mid X, Y \in \mathcal{B},\ |X \cap Y| \geq k\}$$

is known and studied under the name "recoverable robust matroid basis problem under interval uncertainties" [1, 2].

Can the problem above, i.e., the one with equality constraint $|X \cap Y| = k$, be solved in (strongly) polynomial time, or is it NP-hard?

References

[1] M. Hradovich, A. Kasperski, P. Zielinski, *The recoverable robust spanning tree problem with interval costs is polynomially solvable.* Optimization Letters (2017).

[2] M. Hradovich, A. Kasperski, P. Zielinski, *Recoverable robust spanning tree problem under interval represenations.* Journal of Combinatorial Optimization 34 (2017), 554-573.

Open Problem: Rainbow paths in edge-coloured graphs

Alexander Schrijver

Conjecture. *Each simple k-regular properly k-edge-coloured graph has a rainbow path of length $k-1$.*

Here a path is a *rainbow path* if all edges on it have different colours. For $k \geq 3$, the value $k-1$ is best possible: for each $k \geq 3$ there exists a simple k-regular properly k-edge-coloured graph with no rainbow path of length k.

The conjecture is true for each $k \leq 11$. This is shown as follows. Call a (not necessarily regular) properly k-edge-coloured graph H a *rainbow-bar* if (i) $|V(H)| = k-1$, (ii) H has a rainbow Hamilton path, and (iii) for each rainbow Hamilton path P in H, each end vertex v of P, and each colour c missing on P, H has an edge of colour c incident with v.

It can be shown by computer that no rainbow-bar H exists with $|V(H)| \leq 10$ (and of course 'by hand' for small values of $|V(H)|$). This implies the conjecture for $k \leq 11$. Indeed, by induction (by deleting all edges of one of the colours), G has a rainbow path P of length $k-2$. Let H be the subgraph of G induced by $V(P)$. So $|V(H)| = |V(P)| = k-1$ and P is a rainbow Hamilton path in H. Since $|V(H)| \leq 10$, H is not a rainbow-bar. So there exist a rainbow Hamilton path Q in H, an end vertex v of Q, and a colour c missing on Q such that v is not incident with a c-coloured edge contained in H. So the c-coloured edge of G incident with v extends Q to a rainbow path in G of length $k-1$.

We finally mention that the conjecture with $k-1$ replaced by $\lceil \frac{2}{3}k \rceil$ holds true for all k, as was shown by Babu, Sunil Chandran, and Rajendraprasad [1] (see [2] for a short proof).

References

[1] J. Babu, L. Sunil Chandran, D. Rajendraprasad, *Heterochromatic paths in edge colored graphs without small cycles and heterochromatic-triangle-free graphs.* European Journal of Combinatorics 48 (2015), 110–126.

[2] D. Johnston, C. Palmer, A. Sarkar, *Rainbow Turán problems for paths and forests of stars.* Electronic Journal of Combinatorics 24 (2017), no. 1, Paper 1.34, 15 pp.

Open Problem: Longest K_t-sequence Problem

Shin-ichi Tanigawa

(joint work with Katie Clinch, Bill Jackson)

For a positive integer n, let K_n be the complete graph on n vertices. Also, for a graph G, let $V(G)$ and $E(G)$ denote the vertex set and the edge set, respectively. We consider a sequence $\mathcal{C} = (C_1, \ldots, C_k)$ of edge sets in K_n. Such a sequence $\mathcal{C}$ is said to be *proper* if $C_i \not\subseteq \bigcup_{j=1}^{i-1} C_j$ for every i, and $\mathcal{C}$ is said to be a *K_t-sequence* if each C_i is the edge set of a copy of K_t.

The longest K_t-sequence problem is formulated as follows:

Problem 1. *Given a positive integer n and $F \subseteq E(K_n)$, compute*

$$r_t(F) := \min \left\{ |F \cup \bigcup_{i=1}^{k} C_i| - k : \text{a proper } K_t\text{-sequence } \{C_1, \ldots, C_k\} \text{ in } K_n \right\}$$

in polynomial time.

A related question is the following:

Conjecture 1 (Clinch-Jackson-Tanigawa). *r_t is submodular, and $(E(K_n), r_t)$ forms a matroid.*

We are especially interested in the case when $t = 5$. For $t = 3, 4$ the conjecture is true. (If $t = 2$, it gives a trivial matroid in the sense that $\emptyset$ is the base.)

Fact 1. *$(E(K_n), r_3)$ is the graphic matroid of K_n. $(E(K_n), r_4)$ is the generic 2-rigidity matroid of K_n.*

Our ambitious conjecture is that $(E(K_n), r_5)$ is the generic 3-rigidity matroid.

Open Problem: A special case of submodular function minimization

László Végh

Let r be the rank function on a ground set of a linear matroid on a ground set V, provided via an explicit representation over a field, and let $z : V \to \mathbb{R}$. Solving

$$\min_{S \subseteq V} r(S) - z(S)$$

is a special case of submodular function minimization. Is there a more efficient algorithm for this special setting, by making use of the linear representation?

Open Problem: Linearly representable functions

László Végh

Let f be a polymatroid function on a ground set V: that is, integer valued, monotone increasing, and submodular. Such a function is called *linearly representable*, if for a field $\mathbb{F}$ and some integer n, there are linear subspaces $T_v \subseteq \mathbb{F}^n$ for each $v \in V$, such that for any $S \subseteq V$, $f(S) = \mathrm{rk}(\cup_{v \in S} T_v)$.

Is the following statement true or false? Every linearly representable polymatroid function f can be written as a nonnegative linear combination of linear matroid rank functions over the same field.

Open Problem: Rainbow Arborescence Problem

Yu Yokoi

For a directed graph $G = (V, A)$, an *arborescence* is a directed spanning tree $F \subseteq A$ such that there is a special node r, called *root*, that has in-degree 0 and all other nodes have in-degree 1. Our question is described as follows.

For a graph $G = (V, A)$ with $|V| = k + 1$, suppose that A is partitioned into k arborescences $F_1, F_2, \ldots, F_k$. ($|F_i| = k$ for each $i = 1, 2, \ldots, k$.) Is there always an arborescence $F^* \subseteq A$ such that $|F^* \cap F_i| = 1$ for every $i = 1, 2, \ldots, k$?

Note that the assumption implies $|A| = k^2$ and there exist parallel arcs. The statement is true if the roots of all arborescences F_i are the same, which can be shown by a simple extension algorithm.

Reporter: Sarah Morell, Yu Yokoi