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# Asymptotic analyses and error estimates for a Cahn–Hilliard type phase field system modelling tumor growth

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**Abstract**

This paper is concerned with a phase field system of Cahn–Hilliard type that is related to a tumor growth model and consists of three equations in terms of the variables order parameter, chemical potential and nutrient concentration. This system has been investigated in the recent papers [7] and [9] from the viewpoint of well-posedness, long time behavior and asymptotic convergence as two positive viscosity coefficients tend to zero at the same time. Here, we continue the analysis performed in [9] by showing two independent sets of results as just one of the coefficents tends to zero, the other remaining fixed. We prove convergence results, uniqueness of solutions to the two resulting limit problems, and suitable error estimates.

# 1 Introduction

In this paper, we deal with a system of partial differential equations related to a model for tumor growth that was recently proposed in [16] (cf. also [17] and [23]) and further studied analytically in [7, 9, 14]. The system reads

$$\alpha\partial_t\mu + \partial_t\varphi - \Delta\mu = p(\varphi)(\sigma - \gamma\mu) \tag{1.1}$$

$$\mu = \beta\partial_t\varphi - \Delta\varphi + F'(\varphi) \tag{1.2}$$

$$\partial_t\sigma - \Delta\sigma = -p(\varphi)(\sigma - \gamma\mu) \tag{1.3}$$

and it is complemented with the boundary and initial conditions

$$\partial_\nu\mu = \partial_\nu\varphi = \partial_\nu\sigma = 0 \tag{1.4}$$

$$\mu(0) = \mu_0, \quad \varphi(0) = \varphi_0 \quad \text{and} \quad \sigma(0) = \sigma_0\,. \tag{1.5}$$

Each of the partial differential equations (1.1)–(1.3) is meant to hold in a three-dimensional bounded domain $\Omega$ endowed with a smooth boundary $\Gamma$ and for every positive time, and $\partial_\nu$ in (1.4) stands for the outward normal derivative on $\Gamma$. Moreover, $\alpha$ and $\beta$ are nonnegative parameters, strictly positive in principle, and $\gamma$ is a positive constant. Furthermore, $p$ is a nonnegative function and $F$ is a nonnegative double-well potential. Finally, $\mu_0$, $u_0$ and $\sigma_0$ are given initial data defined in $\Omega$.

As sketched in [7] (see also its reference list), the physical context is that of a tumor-growth model. The unknown function $\varphi$ is an order parameter which is close to two values in the regions of nearly pure phases, say, $\varphi \simeq 1$ in the tumorous phase and $\varphi \simeq -1$ in the healthy cell phase; the second unknown $\mu$ is the related chemical potential, specified by (1.2) as in the case of the viscous Cahn–Hilliard or Cahn–Hilliard equation, depending on whether $\beta > 0$ or $\beta = 0$ (see [3, 12, 13]); the third unknown $\sigma$ stands for the nutrient concentration, typically, $\sigma \simeq 1$ in a nutrient-rich extracellular water phase and $\sigma \simeq 0$ in a nutrient-poor extracellular water phase.

In the paper [9], the asymptotic analysis as both parameters $\alpha$ and $\beta$ tend to zero at the same time has been performed, while the cases when each parameter tends to zero separately have been left unanswered. The present paper addresses these questions; in both cases, the convergence results, the uniquenes for the limit problems, as well as the error estimates for the difference of solutions in suitable norms are proved. In particular, let us detail here the main mathematical difficulties one encounters in the two passages to the limit.

**Passage to the limit as $\beta \searrow 0$.** The passage to the limit as $\beta$ tends to zero works under quite general assumptions on the proliferation function $p$ and on the interaction potential $F$: $p$ is required to be a nonnegative, bounded and Lipschitz continuous function, while $F$ is the sum of a convex (possibly multivalued) potential $\widehat{B}$ and a possibly nonconvex but smooth part $\widehat{\pi}$. These assumptions are satisfied in quite a large number of physically meaningful cases like the classical double-well potential and the logarithmic potential, defined by

$$F_{cl}(r) := \tfrac{1}{4}(r^2-1)^2 = \tfrac{1}{4}((r^2-1)^+)^2 + \tfrac{1}{4}((1-r^2)^+)^2 \quad \text{for } r \in \mathbb{R} \tag{1.6}$$

$$F_{log}(r) := (1-r)\ln(1-r) + (1+r)\ln(1+r) + \kappa(1-r^2)^+ \quad \text{for } |r| < 1\,, \tag{1.7}$$

where the decomposition $F = \widehat{B} + \widehat{\pi}$ is written explicitly. In (1.7), $\kappa$ is a positive constant which does or does not provide a double well depending on its value, and the definition of the logarithmic part of $F_{log}$ is extended by continuity to $\pm 1$ and by $+\infty$ outside $[-1,1]$. Moreover, another possible choice is the following:

$$F(r) := I(r) + ((1-r^2)^+)^2 \quad \text{for } r \in \mathbb{R}\,, \tag{1.8}$$

where $I$ is the indicator function of $[-1,1]$, taking the value $0$ in $[-1,1]$ and $+\infty$ elsewhere. Regarding the function $p$, in the original model studied in [16] it was set as proportional to $\sqrt{F(\varphi)}$ for $|\varphi| < 1$ and zero elsewhere. Here we can allow such a behavior as well as more general ones. Moreover, we can show the uniqueness of the solution to the limit problem with $\alpha > 0$ and $\beta = 0$, and an error estimate of the expected order $1/2$ in $\beta$, under a condition of smallness of the fixed coefficient $\alpha$: the bigger is the Lipschitz constant $L$ of the function $\pi$, the smaller has to be $\alpha$. About this concern, we construct an example of severe non-uniqueness for the limit problem (Example 2.4, where $\alpha L = 1$).

**Passage to the limit as $\alpha \searrow 0$.** In this second case, when $\beta$ is kept fixed and $\alpha$ tends to zero, we obtain similar results, but in a less general setting. Indeed, in this case, due to the difficulties in estimating the mean value of the chemical potential $\mu$ (cf. (4.1)), we need to assume $F$ to be defined on the whole of $\mathbb{R}$ and to satisfy proper growth conditions (cf. (2.42)), which are, in particular, fulfilled by the classical double-well potential (1.6) as well as by more general polynomially or exponentially growing potentials. Uniqueness and the error estimate can be obtained only under more restrictive conditions on $p$, which must substantially be constant, and $F$, for which it is required a polynomial growth with power four (cf. (1.6)). This is mainly due to the fact that we need to differentiate in time equation (1.2) in order to get these last results (cf. Section 4 for more details). However, these results can be compared with [9, Thms. 2.6 and 2.7] where similar (but less restirictive) conditions have to be assumed on $F$.

We think that the methods used in our asymptotic analyses could be useful also in other situations. In particular, let us point out that in the case of the choice $p \equiv 0$ (admitted by our assumptions (2.4) and (2.48)) our system (1.1)–(1.3) decouples and (1.1)–(1.2) reduces to a well-known phase field system of Caginalp type which can be seen as a (doubly) viscous approximation of the Cahn–Hilliard system. To this concern, let us quote the paper [20], where similar asymptotic analyses are carried out in the case $p \equiv 0$ with the choice (1.6) for $F$, as both the pameters tend to 0 or just $\alpha$ goes to $0$ with $\beta > 0$ fixed. To the best of our knowledge, we do not know of other investigations as $\beta \searrow 0$: this contribution by us seems to be new also in the case $p \equiv 0$. Other examples of rigorous asymptotic analyses with respect to parameters intervening on phase field models can be found, e.g., in [4–6, 8, 10, 11, 15, 19, 21, 22].

**Plan of the paper.** Our paper is organized as follows. In the next section, we state our assumptions and results on the mathematical problem. The last two sections are devoted to the corresponding proofs.

## 2 Statement of the problem and results

In this section, we make precise assumptions and state our results. As in the Introduction, $\Omega \subset \mathbb{R}^3$ is the domain where the evolution takes place, and $\Gamma$ is its boundary. We assume $\Omega$ to be open, bounded and connected, and $\Gamma$ to be smooth. Moreover, the symbol $\partial_\nu$ denotes the outward normal derivative on $\Gamma$. Given a final time $T$, let

$$Q := \Omega \times (0,T) \quad \text{and} \quad \Sigma := \Gamma \times (0,T). \tag{2.1}$$

Moreover, we set for brevity

$$V := H^1(\Omega), \quad H := L^2(\Omega) \quad \text{and} \quad W := \{v \in H^2(\Omega) : \ \partial_\nu v = 0 \text{ on } \Gamma\} \tag{2.2}$$

and endow these spaces with their standard norms. For the norm in a generic Banach space $X$ (or a power of it), we use the symbol $\|\cdot\|_X$ with the following exceptions: we simply write $\|\cdot\|_p$ and $\|\cdot\|_*$ if $X = L^p(\Omega)$ or $X = L^p(Q)$ for $p \in [1,+\infty]$ and $X = V^*$, the dual space of $V$, respectively. Finally, it is understood that $H \subset V^*$ as usual, i.e., in order that $\langle u, v\rangle = \int_\Omega uv$ for every $u \in H$ and $v \in V$, where $\langle\,\cdot\,,\,\cdot\,\rangle$ stands for the duality pairing between $V^*$ and $V$.

As far as the structure of the system is concerned, we are given two constants $\alpha$ and $\beta$ and three functions $p$, $\widehat{B}$ and $\widehat{\pi}$ satisfying the conditions listed below

$$\alpha,\ \beta \in (0,1) \tag{2.3}$$

$$p : \mathbb{R} \to \mathbb{R} \text{ is nonnegative, bounded and Lipschitz continuous} \tag{2.4}$$

$$\widehat{B} : \mathbb{R} \to [0,+\infty] \text{ is convex, proper, lower semicontinuous} \tag{2.5}$$

$$\widehat{\pi} \in C^1(\mathbb{R}) \text{ is nonnegative and } \pi := \widehat{\pi}\,' \text{ is Lipschitz continuous.} \tag{2.6}$$

We also define the potential $F : \mathbb{R} \to [0,+\infty]$ and the graph $B$ in $\mathbb{R}\times\mathbb{R}$ by

$$F := \widehat{B} + \widehat{\pi} \quad \text{and} \quad B := \partial\widehat{B} \tag{2.7}$$

and denote by $D(B)$ and $D(\widehat{B})$ the effective domains of $B$ and $\widehat{B}$, respectively. It is well known that $B$ and the operators (denoted with the same symbol $B$) induced by $B$ on $L^2$ spaces are maximal monotone (see, e.g., [2, Ex. 2.3.4, p. 25]).

We notice that, among many others, the most important and typical examples of potentials fit our assumptions. Namely, we can take as $F$ the classical double-well potential or the logarithmic potential defined in (1.6) and (1.7), respectively. Another possible choice is the nonsmooth potential (1.8). In the case of a so irregular potential, its subdifferential is multi-valued and the precise statement of problem (1.1)–(1.5) has to introduce a selection $\xi$ of $B(u)$.

As far as the initial data of our problem are concerned, we assume that

$$\mu_0\,,\ \sigma_0 \in H, \quad \varphi_0 \in V \quad \text{and} \quad F(\varphi_0) \in L^1(\Omega)\,, \tag{2.8}$$

while the regularity properties we pretend for the solution are the following:

$$\mu, \sigma \in H^1(0,T;V^*) \cap L^2(0,T;V) \tag{2.9}$$
$$\varphi \in H^1(0,T;H) \cap L^2(0,T;W) \tag{2.10}$$
$$\xi \in L^2(0,T;H). \tag{2.11}$$

We notice that (2.9)–(2.10) imply $\mu, \sigma \in C^0([0,T];H)$ and $\varphi \in C^0([0,T];V)$. At this point, we consider the problem of finding a quadruplet $(\mu, \varphi, \sigma, \xi)$ with the above regularity in order that $(\mu, \varphi, \sigma, \xi)$ and the related function

$$R = p(\varphi)(\sigma - \mu) \tag{2.12}$$

satisfy the system

$$\alpha\langle \partial_t \mu, v\rangle + \int_\Omega \partial_t\varphi\, v + \int_\Omega \nabla\mu \cdot \nabla v = \int_\Omega Rv$$
$$\text{for every } v \in V, \text{ a.e. in } (0,T) \tag{2.13}$$
$$\mu = \beta\partial_t\varphi - \Delta\varphi + \xi + \pi(\varphi) \quad \text{and} \quad \xi \in B(\varphi) \quad \text{a.e. in } Q \tag{2.14}$$
$$\langle \partial_t\sigma, v\rangle + \int_\Omega \nabla\sigma \cdot \nabla v = -\int_\Omega Rv$$
$$\text{for every } v \in V, \text{ a.e. in } (0,T) \tag{2.15}$$
$$\mu(0) = \mu_0, \quad \varphi(0) = \varphi_0 \quad \text{and} \quad \sigma(0) = \sigma_0\,. \tag{2.16}$$

This is a weak formulation of the boundary value problem (1.1)–(1.5) described in the Introduction. The homogeneous Neumann boundary condition for $\varphi$ is contained in (2.10) (see (2.2) for the definition of $W$), while the analogous ones for $\mu$ and $\sigma$ are meant in a generalized sense through the variational equations (2.13) and (2.15). We notice once and for all that the addition of (2.13) and (2.15) yields

$$\langle \partial_t\big(\alpha\mu + \varphi + \sigma\big), v\rangle + \int_\Omega \nabla(\mu+\sigma)\cdot\nabla v = 0 \tag{2.17}$$

for every $v \in V$, a.e. in $(0,T)$.

Well-posedness for the above problem is ensured by [9, Thm. 2.2], which states

**Theorem 2.1.** *Assume* (2.4)–(2.7) *and* (2.8)*. Then, for every* $\alpha, \beta \in (0,1)$*, there exists a unique quadruplet* $(\mu, \varphi, \sigma, \xi)$ *satisfying* (2.9)–(2.11) *and solving problem* (2.12)–(2.16)*.*

In the same paper [9], the authors study the asymptotic analysis of the above problem as both the parameters $\alpha$ and $\beta$ tend to zero simultaneously and prove an error estimate for the difference of the solution to problem (2.12)–(2.16) and the one of the expected limit problem under further assumptions on the potential $F$ (see [9, Thms. 2.5 and 2.6]). Namely, it is assumed that $F$ is everywhere defined and smooth and satisfies some growth condition. In particular, the classical double-well potential (1.6) is allowed.

In the present paper, we discuss the analogous problems obtained by letting just one of the parameters tend to zero while keeping the other fixed. The results that deal with the possible cases are presented at once.

In the first case, we keep $\alpha$ fixed and let $\beta$ tend to zero. The limit problem one expects is the following:

$$\langle \partial_t(\alpha\mu + \varphi), v \rangle + \int_\Omega \nabla\mu \cdot \nabla v = \int_\Omega R\, v \qquad \forall\, v \in V, \text{ a.e. in } (0,T) \tag{2.18}$$

$$\mu = -\Delta\varphi + \xi + \pi(\varphi) \quad \text{and} \quad \xi \in B(\varphi) \qquad \text{a.e. in } Q \tag{2.19}$$

$$\langle \partial_t \sigma, v \rangle + \int_\Omega \nabla\sigma \cdot \nabla v = -\int_\Omega R\, v \qquad \forall\, v \in V, \text{ a.e. in } (0,T) \tag{2.20}$$

$$(\alpha\mu + \varphi)(0) = \alpha\mu_0 + \varphi_0 \quad \text{and} \quad \sigma(0) = \sigma_0 \qquad \text{in } \Omega\,, \tag{2.21}$$

where $R$ is defined by (2.12). For its solution, we require that

$$\mu \in L^\infty(0,T;H) \cap L^2(0,T;V) \tag{2.22}$$

$$\varphi \in L^\infty(0,T;V) \cap L^2(0,T;W) \tag{2.23}$$

$$\alpha\mu + \varphi \in H^1(0,T;V^*) \tag{2.24}$$

$$\sigma \in H^1(0,T;V^*) \cap L^2(0,T;V) \tag{2.25}$$

$$\xi \in L^2(0,T;H)\,. \tag{2.26}$$

Our result on the asymptotic behavior as $\beta \searrow 0$ holds under the assumption that $\alpha$ is sufficiently small.

**Theorem 2.2.** *Assume* (2.4)–(2.7) *on the structure and* (2.8) *on the initial data. Then, there exists* $\alpha_0 \in (0,1)$ *such that, for* $\alpha \in (0,\alpha_0)$ *and* $\beta \in (0,1)$*, the unique solution* $(\mu_{\alpha,\beta}, \varphi_{\alpha,\beta}, \sigma_{\alpha,\beta}, \xi_{\alpha,\beta})$ *to problem* (2.12)–(2.16)*, with the regularity* (2.9)–(2.11)*, satisfies*

$$\mu_{\alpha,\beta} \to \mu_\alpha \qquad \textit{weakly star in } L^\infty(0,T;H) \cap L^2(0,T;V) \tag{2.27}$$

$$\varphi_{\alpha,\beta} \to \varphi_\alpha \qquad \textit{weakly star in } L^\infty(0,T;V) \cap L^2(0,T;W) \tag{2.28}$$

$$\sigma_{\alpha,\beta} \to \sigma_\alpha \qquad \textit{weakly in } H^1(0,T;V^*) \cap L^2(0,T;V) \tag{2.29}$$

$$\partial_t(\alpha\mu_{\alpha,\beta} + \varphi_{\alpha,\beta}) \to \partial_t(\alpha\mu_\alpha + \varphi_\alpha) \qquad \textit{weakly in } L^2(0,T;V^*) \tag{2.30}$$

$$\beta\varphi_{\alpha,\beta} \to 0 \qquad \textit{strongly in } H^1(0,T;H) \cap L^2(0,T;W) \tag{2.31}$$

$$\xi_{\alpha,\beta} \to \xi_\alpha \qquad \textit{weakly in } L^2(0,T;H) \tag{2.32}$$

*as* $\beta$ *tends to zero, at least for a subsequence, and every limiting quadruplet* $(\mu_\alpha, \varphi_\alpha, \sigma_\alpha, \xi_\alpha)$ *solves problem* (2.18)–(2.21)*. In particular, problem* (2.18)–(2.21) *has at least a solution satisfying* (2.22)–(2.26)*.*

Moreover, we can prove uniqueness for the limit problem and an error estimate under an additional restriction on $\alpha$.

**Theorem 2.3.** *Assume* (2.4)–(2.7) *and* (2.8)*. Then, there exists* $\alpha_{00} \in (0,\alpha_0)$ *such that, for* $\alpha \in (0,\alpha_{00})$*, the following conclusions hold:*

$i)$ *the solution* $(\mu_\alpha, \varphi_\alpha, \sigma_\alpha, \xi_\alpha)$ *to problem* (2.18)–(2.21) *satisfying* (2.22)–(2.26) *is unique;*

$ii)$ *if* $(\mu_{\alpha,\beta}, \varphi_{\alpha,\beta}, \sigma_{\alpha,\beta}, \xi_{\alpha,\beta})$ *is the unique solution to problem* (2.12)–(2.16) *with the regularity specified by* (2.9)–(2.11) *for* $\beta \in (0,1)$*, the estimate*

$$\begin{aligned} &\|\mu_{\alpha,\beta} - \mu_\alpha\|_{L^2(0,T;H)} + \|\varphi_{\alpha,\beta} - \varphi_\alpha\|_{L^2(0,T;V)} + \|\sigma_{\alpha,\beta} - \sigma_\alpha\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \\ &\quad + \|(\alpha\mu_{\alpha,\beta} + \varphi_{\alpha,\beta} + \sigma_{\alpha,\beta}) - (\alpha\mu_\alpha + \varphi_\alpha + \sigma_\alpha)\|_{L^\infty(0,T;V^*)} \le C_\alpha\, \beta^{1/2} \end{aligned} \tag{2.33}$$

*holds true for* $\beta \in (0,1)$ *with a constant* $C_\alpha$ *that depends only on* $\alpha$*,* $\Omega$*,* $T$*, the structure of the system, and the norms of the initial data related to assumptions* (2.8)*.*

**Example 2.4.** As said in the Introduction, we can construct an example of severe non-uniqueness for the limit problem. We take $p = 0$, so that the third equation is decoupled. Moreover, we take the indicator function of $[-1, 1]$ as $\widehat{B}$ and, given $L > 0$, we choose $\widehat{\pi}$ smooth, nonnegative and such that

$$|\pi'(r)| \le L \quad \text{for every } r \in \mathbb{R} \quad \text{and} \quad \pi(r) = -Lr \quad \text{for } r \in [-1, 1].$$

Finally, we take

$$\varphi_0 = \mu_0 = \sigma_0 = 0.$$

Then, for any function $\psi \in L^\infty(0, T)$ satisfying $|\psi(t)| \le 1$ for a.a. $t \in (0, T)$, the definition

$$\mu(x, t) := -L\psi(t), \quad \varphi(x, t) := \psi(t), \quad \sigma(x, t) := 0 \quad \text{and} \quad \xi(x, t) := 0$$

provides a solution if $\alpha L = 1$. Indeed, the initial and boundary conditions are trivially satisfied as well as the equations, since we have

$$\alpha\mu + \varphi = -\alpha L\psi + \psi = 0, \quad \mu = -L\psi = -L\varphi = \pi(\varphi) \quad \text{and} \quad \xi = 0 \in B(\psi) = B(\varphi).$$

In the second case, we keep $\beta$ fixed and study the asymptotics with respect to the parameter $\alpha$ and the corresponding expected limit problem, namely

$$\langle \partial_t\varphi, v\rangle + \int_\Omega \nabla\mu \cdot \nabla v = \int_\Omega R\, v \qquad \forall\, v \in V, \text{ a.e. in } (0, T) \tag{2.34}$$
$$\mu = \beta\partial_t\varphi - \Delta\varphi + \xi + \pi(\varphi) \quad \text{and} \quad \xi \in B(\varphi) \qquad \text{a.e. in } Q \tag{2.35}$$
$$\langle \partial_t\sigma, v\rangle + \int_\Omega \nabla\sigma \cdot \nabla v = -\int_\Omega R\, v \qquad \forall\, v \in V, \text{ a.e. in } (0, T) \tag{2.36}$$
$$\varphi(0) = \varphi_0 \quad \text{and} \quad \sigma(0) = \sigma_0 \qquad \text{in } \Omega\,, \tag{2.37}$$

where $R$ is defined by (2.12). For its solution, we require the following regularity:

$$\mu \in L^2(0, T; V) \tag{2.38}$$
$$\varphi \in H^1(0, T; H) \cap L^2(0, T; W) \tag{2.39}$$
$$\sigma \in H^1(0, T; V^*) \cap L^2(0, T; V) \tag{2.40}$$
$$\xi \in L^2(0, T; H)\,. \tag{2.41}$$

Our results hold in a less general setting. We need a first restriction on the potential $F$ in order to prove an asymptotic result and the existence of a solution to the limit problem. Namely, we also assume that

$$D(\widehat{B}) = \mathbb{R} \quad \text{and} \quad |B^0(r)| \le C_B\big(\widehat{B}(r) + 1\big) \quad \text{for every } r \in \mathbb{R}\,, \tag{2.42}$$

where $B^0(r)$ is the element of $B(r)$ having minimum modulus and $C_B$ is a given constant. Notice that the classical potential (1.6) and similar potentials with polynomial or exponential growth satisfy this assumption. Moreover, (2.42) still allows $B$ to be multi-valued. We have the following result:

**Theorem 2.5.** *Assume* (2.4)–(2.7) *and* (2.42) *on the structure and* (2.8) *on the initial data. Moreover, for* $\alpha, \beta \in (0, 1)$*, let* $(\mu_{\alpha,\beta}, \varphi_{\alpha,\beta}, \sigma_{\alpha,\beta}, \xi_{\alpha,\beta})$ *be the unique to problem* (2.12)–(2.16) *satisfying* (2.9)–(2.11)*. Then we have that*

$$\mu_{\alpha,\beta} \to \mu_\beta \quad \textit{weakly in } L^2(0, T; V) \tag{2.43}$$
$$\varphi_{\alpha,\beta} \to \varphi_\beta \quad \textit{weakly in } H^1(0, T; H) \cap L^2(0, T; W) \tag{2.44}$$
$$\sigma_{\alpha,\beta} \to \sigma_\beta \quad \textit{weakly in } H^1(0, T; V^*) \cap L^2(0, T; V) \tag{2.45}$$
$$\xi_{\alpha,\beta} \to \xi_\beta \quad \textit{weakly in } L^2(0, T; H) \tag{2.46}$$
$$\alpha\mu_{\alpha,\beta} \to 0 \quad \textit{weakly in } H^1(0, T; V^*) \textit{ and strongly in } L^2(0, T; V) \tag{2.47}$$

*as* $\alpha$ *tends to zero, at least for a subsequence, and every limiting quadruplet* $(\mu_\beta, \varphi_\beta, \sigma_\beta, \xi_\beta)$ *solves problem* (2.34)–(2.37)*. In particular, the limit problem* (2.34)–(2.37) *has at least a solution satisfying* (2.38)–(2.41).

Both the uniqueness for the limit problem and the error estimates look difficult to prove. We can overcome such a difficulty only in a particular case. Moreover, we have to take $p$ constant and make stronger conditions on the potential $F$ (which, however, still allow the choice of the classical potential (1.6)) and slightly reinforce those on the initial data. Namely, we also assume that

$$p \text{ is a nonnegative constant} \tag{2.48}$$

$$F \text{ is a } C^2 \text{ function on } \mathbb{R} \text{ satisfying } |F''(r)| \leq C(r^2+1) \\ \text{for every } r \in \mathbb{R} \text{ and for some constant } C > 0. \tag{2.49}$$

We observe that in this setting the last condition in (2.8) is a consequence of $\varphi_0 \in V$ and (2.49). Indeed, these assumptions ensure that $F(r) = O(r^4)$ as $|r|$ tends to $+\infty$ and that $\varphi_0 \in L^4(\Omega)$, thanks to the Sobolev inequality.

**Theorem 2.6.** *Assume* (2.4)–(2.7)*,* (2.42) *and* (2.48)–(2.49) *on the structure, and* (2.8) *on the initial data, and let $\beta \in (0,1)$. Then, the following conclusions hold true:*

$i)$ *the solution $(\mu_\beta, \varphi_\beta, \sigma_\beta, \xi_\beta)$ to problem* (2.34)–(2.37) *satisfying* (2.38)–(2.41) *is unique;*

$ii)$ *if $(\mu_{\alpha,\beta}, \varphi_{\alpha,\beta}, \sigma_{\alpha,\beta}, \xi_{\alpha,\beta})$ is the unique solution to problem* (2.12)–(2.16) *with the regularity specified by* (2.9)–(2.11) *for $\alpha \in (0,1)$, the estimate*

$$\|\mu_{\alpha,\beta} - \mu_\beta\|_{L^2(0,T;V)} + \|\varphi_{\alpha,\beta} - \varphi_\beta\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \\ + \|\sigma_{\alpha,\beta} - \sigma_\beta\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \leq C_\beta\, \alpha^{1/2} \tag{2.50}$$

*holds true for $\alpha \in (0,1)$ with a constant $C_\beta$ that depends only on $\beta$, $\Omega$, $T$, the structure of the system, and the norms of the initial data related to assumptions* (2.8)*.*

Now, we list some facts. We repeatedly make use of the notation

$$Q_t := \Omega \times (0,t) \quad \text{for } t \in [0,T] \tag{2.51}$$

and of well-known inequalities, namely, the elementary Young inequality

$$ab \leq \delta a^2 + \frac{1}{4\delta}\, b^2 \quad \text{for every } a, b \geq 0 \text{ and } \delta > 0\,, \tag{2.52}$$

the Hölder inequality, and its consequences. Moreover, as $\Omega$ is bounded and smooth, we can owe to the Poincaré and Sobolev–type inequalities, namely,

$$\|v\|_V \leq C\Big(\|\nabla v\|_H + \big|\textstyle\int_\Omega v\big|\Big) \quad \text{for every } v \in V \tag{2.53}$$

$$V \subset L^q(\Omega) \quad \text{and} \quad \|v\|_q \leq C\|v\|_V \quad \text{for every } v \in V \text{ and } 1 \leq q \leq 6 \tag{2.54}$$

$$L^q(\Omega) \subset V^* \quad \text{and} \quad \|v\|_* \leq C\|v\|_q \quad \text{for every } v \in L^q(\Omega) \text{ and } q \geq 6/5\,. \tag{2.55}$$

In (2.53)–(2.55), $C$ only depends on $\Omega$. Furthermore, we denote by $\mathcal{A}$ the Riesz isomorphism from $V$ onto $V^*$ associated to the standard inner product of $V$, i.e., $\mathcal{A} : V \to V^*$ is defined by

$$\langle \mathcal{A}u, v\rangle := (u,v)_V = \int_\Omega \big(\nabla u \cdot \nabla v + uv\big) \quad \text{for } u, v \in V. \tag{2.56}$$

We notice that $\mathcal{A}u = -\Delta u + u$ if $u \in W$. We also remark that

$$\langle \mathcal{A}u, \mathcal{A}^{-1}v^*\rangle = \langle v^*, u\rangle \quad \text{for every } u \in V \text{ and } v^* \in V^* \tag{2.57}$$

$$\langle v^*, \mathcal{A}^{-1}u^*\rangle = \langle u^*, \mathcal{A}^{-1}v^*\rangle = (u^*, v^*)_* \quad \text{for every } u^*, v^* \in V^*\,, \tag{2.58}$$

where $(\,\cdot\,,\,\cdot\,)_*$ denotes the dual scalar product in $V^*$ associated to the standard one in $V$, and recall that $\langle v^*, u\rangle = \int_\Omega v^* u$ if $v^* \in H$. As a consequence of (2.58), we have

$$\frac{d}{dt}\,\|v^*\|_*^2 = 2\langle \partial_t v^*, \mathcal{A}^{-1} v^*\rangle \quad \text{for every } v^* \in H^1(0,T;V^*). \tag{2.59}$$

Finally, in order to simplify the notation, we follow a general rule in performing our a priori estimates. The small-case italic $c$ without any subscript stands for different constants, which may only depend on $\Omega$, $T$, the shape of the nonlinearities and the norms of the initial data related to assumptions at hand. A notation like $c_\delta$ signals a constant that also depends on the parameter $\delta$. We point out that $c$ and $c_\delta$ do not depend on $\alpha$ and $\beta$ and that their meaning might change from line to line and even within the same chain of inequalities. On the contrary, constants that are later referred to are always denoted by different symbols, e.g., by a capital letter.

The starting point for the proofs given in the next sections is the following general result (see [9, Thm. 2.3]):

**Theorem 2.7.** *Assume* (2.4)–(2.7) *and* (2.8)*. Then, for some constant $\widehat{C}$ which only depends on $\Omega$, $T$ and the shape of the nonlinearities, the following holds true: for every $\alpha, \beta \in (0,1)$, the solution $(\mu, \varphi, \sigma, \xi)$ to problem* (2.12)–(2.16) *with the regularity specified by* (2.9)–(2.11) *satisfies*

$$\begin{aligned}
&\alpha^{1/2}\|\mu\|_{L^\infty(0,T;H)} + \|\nabla\mu\|_{L^2(0,T;H)} \\
&\quad + \beta^{1/2}\|\partial_t\varphi\|_{L^2(0,T;H)} + \|\varphi\|_{L^\infty(0,T;V)} + \|F(\varphi)\|_{L^\infty(0,T;L^1(\Omega))}^{1/2} \\
&\quad + \|\partial_t(\alpha\mu+\varphi)\|_{L^2(0,T;V^*)} + \|\sigma\|_{H^1(0,T;V^*)\cap L^\infty(0,T;H)\cap L^2(0,T;V)} \\
&\le \widehat{C}\left(\alpha^{1/2}\|\mu_0\|_H + \|\varphi_0\|_V + \|F(\varphi_0)\|_{L^1(\Omega)}^{1/2} + \|\sigma_0\|_H\right)
\end{aligned} \tag{2.60}$$

*as well as*

$$\begin{aligned}
&\|\mu\|_{L^2(0,T;V)} + \|\varphi\|_{L^2(0,T;W)} + \|\xi\|_{L^2(0,T;H)} \\
&\le \widehat{C}\left(\alpha^{1/2}\|\mu_0\|_H + \|\varphi_0\|_V + \|F(\varphi_0)\|_{L^1(\Omega)}^{1/2} + \|\sigma_0\|_H + \|\mu\|_{L^2(0,T;H)} + 1\right).
\end{aligned} \tag{2.61}$$

# 3 Proofs of Theorems 2.2 and 2.3

It is understood that the assumptions (2.4)–(2.7) and (2.8) are in force.

**Proof of Theorem 2.2.** We start from the uniform estimates stated in Theorem 2.7. As $\alpha$ is fixed, (2.60) ensures that even $\|\mu_{\alpha,\beta}\|_{L^\infty(0,T;H)}$ is bounded, so that (2.61) provides a bound for its left-hand side. Therefore the convergences specified in (2.27)–(2.32) hold for a subsequence. Moreover, as $\eta_{\alpha,\beta} := \alpha\mu_{\alpha,\beta} + \varphi_{\alpha,\beta}$ converges to $\eta_\alpha := \alpha\mu_\alpha + \varphi_\alpha$ weakly in $H^1(0,T;V^*) \cap L^2(0,T;V)$, by applying the Aubin-Lions lemma (see, e.g., [18, Thm. 5.1, p. 58]) we deduce that

$$\eta_{\alpha,\beta} \to \eta_\alpha \quad \text{strongly in } L^2(0,T;H) \text{ as } \beta \searrow 0. \tag{3.1}$$

Now, we prove that (3.1) implies that both

$$\mu_{\alpha,\beta} \to \mu_\alpha \quad \text{and} \quad \varphi_{\alpha,\beta} \to \varphi_\alpha \quad \text{strongly in } L^2(0,T;H) \tag{3.2}$$

provided $\alpha$ is small enough. We show (3.2) by using a Cauchy sequence argument: we write (2.14) for the solutions corresponding to $\beta$ and $\beta'$ and take the difference; then, we multiply by $\alpha$, sum

$\varphi_{\alpha,\beta} - \varphi_{\alpha,\beta'}$ to both sides, rearrange and finally test by $\varphi_{\alpha,\beta} - \varphi_{\alpha,\beta'}$. We obtain

$$\begin{aligned}
&\int_\Omega \bigl((\eta_{\alpha,\beta} - \eta_{\alpha,\beta'}) - \alpha(\beta\,\partial_t\varphi_{\alpha,\beta} - \beta'\,\partial_t\varphi_{\alpha,\beta'})\bigr)\,(\varphi_{\alpha,\beta} - \varphi_{\alpha,\beta'}) \\
&= \|\varphi_{\alpha,\beta} - \varphi_{\alpha,\beta'}\|_H^2 + \alpha \int_\Omega |\nabla(\varphi_{\alpha,\beta} - \varphi_{\alpha,\beta'})|^2 \\
&\quad + \alpha \int_\Omega (\xi_{\alpha,\beta} - \xi_{\alpha,\beta'})(\varphi_{\alpha,\beta} - \varphi_{\alpha,\beta'}) + \alpha \int_\Omega (\pi(\varphi_{\alpha,\beta}) - \pi(\varphi_{\alpha,\beta'}))(\varphi_{\alpha,\beta} - \varphi_{\alpha,\beta'})
\end{aligned} \tag{3.3}$$

a.e. in $(0,T)$. Then we integrate over $(0,T)$ and observe that the resulting left-hand side tends to zero as $\beta, \beta' \searrow 0$, due to the strong convergence given in (3.1) and (2.31) coupled with the weak convergence (2.28). The term $\alpha \int_Q (\xi_{\alpha,\beta} - \xi_{\alpha,\beta'})(\varphi_{\alpha,\beta} - \varphi_{\alpha,\beta'})$ is nonegative thanks to the monotonicity of $B$; the last term can be treated using the Lipschitz continuity of $\pi$, namely,

$$\alpha \int_Q (\pi(\varphi_{\alpha,\beta}) - \pi(\varphi_{\alpha,\beta'}))(\varphi_{\alpha,\beta} - \varphi_{\alpha,\beta'}) \ge -\alpha L \|\varphi_{\alpha,\beta} - \varphi_{\alpha,\beta'}\|_2^2,$$

where $L$ denotes a Lipschitz constant for $\pi$. Hence, from (3.3) and the subsequent remarks we deduce that $\{\varphi_{\alpha,\beta}\}$ is a Cauchy sequence in $L^2(Q) \equiv L^2(0,T;H)$ provided $\alpha L < 1$, so that (2.28) entails that $\varphi_{\alpha,\beta}$ converges to $\varphi_\alpha$ strongly in $L^2(0,T;H)$ and (3.2) is completely proved.

From (3.2) it follows that $\pi(\varphi_{\alpha,\beta})$ and $p(\varphi_{\alpha,\beta})$ converge to $\pi(\varphi_\alpha)$ and $p(\varphi_\alpha)$, respectively, strongly in $L^2(0,T;H)$. Therefore, we can identify the limits of the nonlinear terms $\xi_{\alpha,\beta}$ and $R_{\alpha,\beta}$. For the former, we can apply, e.g., [1, Cor. 2.4, p. 41]. For the latter, we note that $R_{\alpha,\beta}$ converges to $p(\varphi_\alpha)(\sigma_\alpha - \mu_\alpha)$ strongly in $L^1(Q)$ since (cf. (2.29)) we also have a strong convergence of $\sigma_{\alpha,\beta}$ to $\sigma_\alpha$ in $L^2(Q)$. At this point, we can write the integrated–in–time version of problem (2.13)–(2.15) for the approximating solution with time dependent test functions and take the limit as $\alpha$ tends to zero. We obtain the analogous systems for $(\mu_\alpha, \varphi_\alpha, \sigma_\alpha, \xi_\alpha)$, and this implies (2.18)–(2.21) for such a quadruplet. □

**Proof of Theorem 2.3.** In order to show Theorem 2.3, we do not follow the order of the statement. Indeed, assume for a while that the part $ii)$ has been proved for every solution to problem (2.18)–(2.21) (with a constant $C_\alpha$ that might depend on the solution we are considering), provided that $\alpha$ is small. From this, we derive the uniqueness part $i)$. Indeed, let $(\mu_i, \varphi_i, \sigma_i, \xi_i)$, $i = 1, 2$, be two solutions. If $\alpha$ is small, inequality (2.33) holds for both of them with a common constant $C_\alpha$. Hence, we immediately derive, e.g.,

$$\begin{aligned}
&\|\mu_1 - \mu_2\|_{L^2(0,T;H)} + \|\varphi_2 - \varphi_2\|_{L^2(0,T;V)} \\
&\quad + \|\sigma_1 - \sigma_2\|_{L^2(0,T;V)} \le 2C_\alpha\,\beta^{1/2} \quad \text{for every } \beta \in (0,1).
\end{aligned}$$

This implies that $\mu_1 = \mu_2$, $\varphi_1 = \varphi_2$ and $\sigma_1 = \sigma_2$. Then, $\xi_1 = \xi_2$ by comparison in (2.19).

Now, we show that the error estimate (2.33) holds for every solution $(\mu_\alpha, \varphi_\alpha, \sigma_\alpha, \xi_\alpha)$ to the problem (2.18)–(2.21) (with a constant $C_\alpha$ that might depend on the solution we are considering). To this end, we present equations (2.14)–(2.15), (2.19)–(2.20), (2.17) and its analogue obtained by summing (2.18) and (2.20) in a slightly different form. Namely, in each equation, we add the same function to both sides and make the Riesz ismorphism $\mathcal{A}$ appear (see (2.56)). In doing so, we owe to the regularity conditions (2.9)–(2.11) and (2.38)–(2.41). If $(\overline{\mu}, \overline{\varphi}, \overline{\sigma}, \overline{\xi})$ stands for the solution to the limit problem, we have

$$\partial_t(\alpha\mu_{\alpha,\beta} + \varphi_{\alpha,\beta} + \sigma_{\alpha,\beta}) + \mathcal{A}(\mu_{\alpha,\beta} + \sigma_{\alpha,\beta}) = \mu_{\alpha,\beta} + \sigma_{\alpha,\beta} \tag{3.4}$$

$$\mu_{\alpha,\beta} = \beta\partial_t\varphi_{\alpha,\beta} + \mathcal{A}\varphi_{\alpha,\beta} + \xi_{\alpha,\beta} + \pi(\varphi_{\alpha,\beta}) - \varphi_{\alpha,\beta} \tag{3.5}$$

$$\partial_t\sigma_{\alpha,\beta} + \mathcal{A}\sigma_{\alpha,\beta} = -R_{\alpha,\beta} + \sigma_{\alpha,\beta} \tag{3.6}$$

$$\partial_t(\alpha\overline{\mu} + \overline{\varphi} + \overline{\sigma}) + \mathcal{A}(\overline{\mu} + \overline{\sigma}) = \overline{\mu} + \overline{\sigma} \tag{3.7}$$

$$\overline{\mu} = \mathcal{A}\overline{\varphi} + \overline{\xi} + \pi(\overline{\varphi}) - \overline{\varphi} \tag{3.8}$$

$$\partial_t\overline{\sigma} + \mathcal{A}\overline{\sigma} = -\overline{R} + \overline{\sigma}\,, \tag{3.9}$$

where $R_{\alpha,\beta}$ and $\overline{R}$ are defined by (2.12) according to the equations we are considering. All these equations are meant in the framework of the Hilbert triplet $(V, H, V^*)$, a.e. in $(0,T)$. However, the explicite versions of (3.5) and (3.8) also hold a.e. in $Q$. Moreover, we have to add the conditions $\xi_{\alpha,\beta} \in B(\varphi_{\alpha,\beta})$ and $\overline{\xi} \in B(\overline{\varphi})$ a.e. in $Q$ as well as the initial conditions (2.16) and (2.21), respectively. Now, we take the differences between (3.4)–(3.6) and (3.7)–(3.9) and have

$$\begin{aligned}
&\partial_t(\alpha\mu + \varphi + \sigma) + \mathcal{A}(\mu + \sigma) = \mu + \sigma \\
&\mu = \beta\partial_t\varphi_{\alpha,\beta} + \mathcal{A}\varphi + \xi_{\alpha,\beta} - \overline{\xi} + \pi(\varphi_{\alpha,\beta}) - \pi(\overline{\varphi}) - \varphi \\
&\partial_t\sigma + \mathcal{A}\sigma = -(R_{\alpha,\beta} - \overline{R}) + \sigma\,,
\end{aligned}$$

where we have set, for convenience,

$$\mu := \mu_{\alpha,\beta} - \overline{\mu}, \quad \varphi := \varphi_{\alpha,\beta} - \overline{\varphi} \quad \text{and} \quad \sigma := \sigma_{\alpha,\beta} - \overline{\sigma}.$$

At this point, we write these equations at time $s \in (0,T)$ and test them by

$$\mathcal{A}^{-1}(\alpha\mu + \varphi + \sigma)(s), \quad -\varphi(s) \quad \text{and} \quad \sigma(s)\,,$$

respectively. Next, we sum up and integrate over $(0,t)$ with respect to $s$, for an arbitrary $t \in (0,T)$. Then, we recall and use (2.56)–(2.59); by rearranging a little and omitting the evaluation at $s$ inside integrals for brevity, we obtain that

$$\begin{aligned}
&\frac{1}{2}\,\|(\alpha\mu + \varphi + \sigma)(t)\|_*^2 + \int_{Q_t} (\mu + \sigma)(\alpha\mu + \varphi + \sigma) + \int_0^t \|\varphi\|_V^2\,ds \\
&\quad + \int_{Q_t} (\xi_{\alpha,\beta} - \overline{\xi})\varphi - \int_{Q_t} \mu\varphi + \frac{1}{2}\int_\Omega |\sigma(t)|^2 + \int_0^t \|\sigma\|_V^2\,ds \\
&= \int_0^t \bigl(\mu + \sigma, \alpha\mu + \varphi + \sigma\bigr)_*\,ds - \beta\int_{Q_t} \partial_t\varphi_{\alpha,\beta}\,\varphi + \int_{Q_t} \bigl\{\varphi - \bigl(\pi(\varphi_{\alpha,\beta}) - \pi(\overline{\varphi})\bigr)\bigr\}\varphi \\
&\quad - \int_{Q_t} (R_{\alpha,\beta} - \overline{R})\sigma + \int_{Q_t} |\sigma|^2\,.
\end{aligned} \tag{3.10}$$

The terms on the left-hand side not having a definite sign are treated simultaneously in the following way:

$$\begin{aligned}
&\int_{Q_t} (\mu + \sigma)(\alpha\mu + \varphi + \sigma) - \int_{Q_t} \mu\varphi = \int_{Q_t} \bigl(\alpha|\mu|^2 + \mu\sigma + \sigma(\alpha\mu + \varphi + \sigma)\bigr) \\
&\geq \alpha\int_{Q_t} |\mu|^2 + \int_{Q_t} \mu\sigma - \int_0^t \|\sigma\|_V\,\|\alpha\mu + \varphi + \sigma\|_*\,ds \\
&\geq \frac{\alpha}{2}\int_{Q_t} |\mu|^2 - c\int_{Q_t} |\sigma|^2 - \delta\int_0^t \|\sigma\|_V^2\,ds - c_\delta\int_0^t \|\alpha\mu + \varphi + \sigma\|_*^2\,ds.
\end{aligned}$$

Now, we deal with the right-hand side of (3.10). For the first integral, we observe that the norm of the embedding $H \subset V^*$ is $1$ (since the norms of $V$ and $H$ are the standard ones), and we have the estimate

$$\int_0^t \bigl(\mu+\sigma, \alpha\mu+\varphi+\sigma\bigr)_* \, ds \le \frac{\alpha}{8} \int_0^t \bigl(\|\mu\|_*^2 + \|\sigma\|_*^2\bigr)\, ds + c \int_0^t \|\alpha\mu+\varphi+\sigma\|_*^2 \, ds$$
$$\le \frac{\alpha}{8} \int_{Q_t} |\mu|^2 + \frac{\alpha}{8} \int_{Q_t} |\sigma|^2 + c \int_0^t \|\alpha\mu+\varphi+\sigma\|_*^2 \, ds \, .$$

Next, we have that

$$-\beta \int_{Q_t} \partial_t \varphi_{\alpha,\beta} \, \varphi \le \delta \int_{Q_t} |\varphi|^2 + c_\delta \, \beta^2 \int_{Q_t} |\partial_t \varphi_{\alpha,\beta}|^2 \le \delta \int_0^t \|\varphi\|_V^2 \, ds + c_\delta \beta \, ,$$

the last inquality being due to (2.60) for $\partial_t \varphi_{\alpha,\beta}$. For the next term, we use the Lipschitz continuity of $\pi$ and still denote by $L$ the Lipschitz constant of $\pi$. Then, we obtain that

$$\int_{Q_t} \bigl\{\varphi - \bigl(\pi(\varphi_{\alpha,\beta}) - \pi(\overline{\varphi})\bigr)\bigr\}\varphi \le (1+L) \int_{Q_t} |\varphi|^2$$
$$= (1+L) \left( \int_{Q_t} \varphi\bigl(\alpha\mu+\varphi+\sigma\bigr) - \alpha \int_{Q_t} \varphi\mu - \int_{Q_t} \varphi\sigma \right)$$
$$\le \delta \int_0^t \|\varphi\|_V^2 + c_\delta \int_0^t \|\alpha\mu+\varphi+\sigma\|_*^2 \, ds$$
$$\quad + \frac{\alpha}{8} \int_{Q_t} |\mu|^2 + (1+L)^2 \alpha \int_0^t \|\varphi\|_V^2 \, ds + \delta \int_0^t \|\varphi\|_V^2 \, ds + c_\delta \int_{Q_t} |\sigma|^2$$

and we assume at once $\alpha$ to be small in order that $(1+L)^2 \alpha < 1$. Finally, there is one more term to treat on the right-hand side of (3.10), namely,

$$-\int_{Q_t} (R_{\alpha,\beta} - \overline{R})\sigma = -\int_{Q_t} \bigl\{p(\varphi_{\alpha,\beta})(\sigma_{\alpha,\beta} - \mu_{\alpha,\beta}) - p(\overline{\varphi})(\overline{\sigma} - \overline{\mu})\bigr\}\sigma$$
$$\le \int_{Q_t} |p(\varphi_{\alpha,\beta}) - p(\overline{\varphi})| \, |\sigma_{\alpha,\beta} - \mu_{\alpha,\beta}| \, |\sigma| + \int_{Q_t} |p(\overline{\varphi})| \, |\sigma - \mu| \, |\sigma|$$
$$\le c \int_{Q_t} |\varphi| \, |\sigma_{\alpha,\beta} - \mu_{\alpha,\beta}| \, |\sigma| + c \int_{Q_t} |\sigma - \mu| \, |\sigma| \, .$$

Note that the last inequality holds true since $p$ is Lipschitz continuous and bounded. On the other hand, we can use the Hölder and Sobolev inequalities to obtain

$$c \int_{Q_t} |\varphi| \, |\sigma_{\alpha,\beta} - \mu_{\alpha,\beta}| \, |\sigma| \le c \int_0^t \|\varphi\|_4 \, \|\sigma_{\alpha,\beta} - \mu_{\alpha,\beta}\|_4 \, \|\sigma\|_2 \, ds$$
$$\le c \int_0^t \|\varphi\|_V \, \|\sigma_{\alpha,\beta} - \mu_{\alpha,\beta}\|_V \, \|\sigma\|_H \, ds \le \delta \int_0^t \|\varphi\|_V^2 \, ds + c_\delta \int_0^t \|\sigma_{\alpha,\beta} - \mu_{\alpha,\beta}\|_V^2 \, \|\sigma\|_H^2 \, ds$$

and we observe at once that the function $s \mapsto \|(\sigma_{\alpha,\beta} - \mu_{\alpha,\beta})(s)\|_V^2$ is bounded in $L^1(0,T)$, thanks to (2.60) for $\sigma_{\alpha,\beta}$ and (2.60)–(2.61) for $\mu_{\alpha,\beta}$. Finally, we have that

$$c \int_{Q_t} |\sigma - \mu| \, |\sigma| \le \frac{\alpha}{8} \int_{Q_t} |\mu|^2 + c \int_{Q_t} |\sigma|^2.$$

At this point, we collect (3.10) and all the inequalities we have obtained, rearrange and infer that

$$\begin{aligned}
&\frac{1}{2}\,\|(\alpha\mu+\varphi+\sigma)(t)\|_*^2+\frac{\alpha}{8}\int_{Q_t}|\mu|^2\\
&\quad+\big(1-(1+L)^2\alpha-4\delta\big)\int_0^t\|\varphi\|_V^2\,ds+\frac{1}{2}\int_\Omega|\sigma(t)|^2+(1-\delta)\int_0^t\|\sigma\|_V^2\,ds\\
&\leq c_\delta\int_0^t\big(1+\|\sigma_{\alpha,\beta}-\mu_{\alpha,\beta}\|_V^2\big)\;\|\sigma\|_H^2\,ds+c_\delta\int_0^t\|\alpha\mu+\varphi+\sigma\|_*^2\,ds+c_\delta\beta.
\end{aligned}$$

Then, we choose $\delta$ small enough (let us recall that we are assuming $(1+L)^2\alpha<1$) and apply the Gronwall lemma. This yields (2.33), and the proof is complete. □

# 4 Proof of Theorems 2.5 and 2.6

It is understood that assumptions (2.4)–(2.7) and (2.42) on the structure and (2.8) on the initial data are in force. We first prove Theorem 2.5. The main tool is Theorem 2.7, applied to the solution $(\mu_{\alpha,\beta},\varphi_{\alpha,\beta},\sigma_{\alpha,\beta},\xi_{\alpha,\beta})$ to problem (2.12)–(2.16). However, we need a preliminary estimate that has already been performed in [9]. Nevertheless, for the reader's convenience, we repeat the core of the argument here.

**Auxiliary a priori estimate.** We omit the indices $\alpha$ and $\beta$ for a while. We observe that the growth condition (2.42) formally implies (see [9, formulas (2.25)–(2.26) and Rem. 2.5])

$$\int_\Omega|\xi(t)|\leq c\int_\Omega\big(\widehat{B}(\varphi(t))+1\big)\quad\text{for a.a. } t\in(0,T). \tag{4.1}$$

Now, we simply integrate (2.14) over $\Omega$ and use the homogeneous Neumann boundary condition for $\varphi$. We deduce that

$$\int_\Omega\mu(t)=\int_\Omega\beta\partial_t\varphi(t)+\int_\Omega\xi(t)+\int_\Omega\pi(\varphi(t))\quad\text{for a.a. } t\in(0,T).$$

As $\pi$ is Lipschitz continuous and $\widehat{\pi}$ is nonnegative, the above identity and (4.1) imply that

$$\Big|\int_\Omega\mu(t)\Big|\leq\int_\Omega\beta|\partial_t\varphi(t)|+c\int_\Omega F(\varphi(t))+c\int_\Omega|\varphi(t)|+c. \tag{4.2}$$

By taking advantage of (2.60), we deduce that the function $t\mapsto\int_\Omega\mu(t)$ is estimated in $L^2(0,T)$. Therefore, by using (2.60) once more and the Poincaré inequality (2.53), we conclude that

$$\|\mu\|_{L^2(0,T;V)}\leq c\,. \tag{4.3}$$

**Proof of Theorem 2.5.** Now, we explicitly write the indices $\alpha$ and $\beta$. First, (4.3) ensures that the left-hand sides of both (2.60) and (2.61) are bounded, so that, since $\beta$ is fixed, we deduce (2.43)–(2.47), at least for a subsequence: note that (2.47) follows from (2.43), (2.44) and the bound for $\{\alpha\partial_t\mu_{\alpha,\beta}+\partial_t\varphi_{\alpha,\beta}\}$ in $L^2(0,T;V^*)$. Now, let $(\mu_\beta,\varphi_\beta,\sigma_\beta,\xi_\beta)$ be any limiting quadruplet. Then, (2.44)–(2.45) imply that the initial conditions for $(\varphi_\beta,\sigma_\beta)$ are satisfied. Moreover, the Aubin-Lions lemma (see, e.g., [18, Thm. 5.1, p. 58]) ensures that $\varphi_{\alpha,\beta}$ converges to $\varphi_\beta$ strongly in $L^2(0,T;H)$ (even better, of course), whence, $\pi(\varphi_{\alpha,\beta})$ and $p(\varphi_{\alpha,\beta})$ converge to $\pi(\varphi_\beta)$ and $p(\varphi_\beta)$, respectively, strongly in

$L^2(0,T;H)$. Therefore, we can identify the limits of the nonlinear terms $\xi_{\alpha,\beta}$ and $R_{\alpha,\beta}$. For the former, we can apply, e.g., [1, Cor. 2.4, p. 41]. For the latter, we note that $R_{\alpha,\beta}$ converges to $p(\varphi_\beta)(\sigma_\beta - \mu_\beta)$ (at least) weakly in $L^1(Q)$. At this point, we can write the integrated–in–time version of problem (2.13)–(2.15) for the approximating solution with time dependent test functions and take the limit as $\alpha$ tends to zero. We obtain the analogous systems for $(\mu_\beta, \varphi_\beta, \sigma_\beta, \xi_\beta)$, and this implies (2.34)–(2.36) for such a quadruplet. □

Now, we assume that (2.48)–(2.49) hold, in addition, and start proving Theorem 2.6. However, as in the previous section, we do not follow the order of the statement. Indeed, let us assume for a while that its part $ii)$ has been proved in the following modified version: the error estimate (2.50) holds for every solution to the limit problem (2.34)–(2.37) (with a constant $C_\beta$ that might depend on the solution we are considering). From this, we derive the uniqueness stated in $i)$ in the following way: let $(\mu_i, \varphi_i, \sigma_i, \xi_i)$, $i = 1, 2$, be two solutions. Then inequality (2.50) holds for both of them with a common constant $C_\beta$. Hence, we immediately derive, e.g.,

$$\|\mu_1 - \mu_2\|_{L^2(0,T;V)} + \|\varphi_1 - \varphi_2\|_{L^2(0,T;V)} + \|\sigma_1 - \sigma_2\|_{L^2(0,T;V)} \le C_\beta\, \alpha^{1/2}$$

for every $\alpha \in (0,1)$. We deduce that $\mu_1 = \mu_2$, $\varphi_1 = \varphi_2$ and $\sigma_1 = \sigma_2$, whence also $\xi_1 = \xi_2$ since $B$ is single-valued.

So, it remains to prove that the error estimate (2.50) holds true for every solution to the limit problem (2.34)–(2.37) (with a constant $C_\beta$ that might depend on the solution we are considering). To this end, we need a further estimate on the solution $(\mu_{\alpha,\beta}, \varphi_{\alpha,\beta}, \sigma_{\alpha,\beta})$ to problem (2.12)–(2.16).

**A new a priori estimate.** We proceed formally in order not to be too technical. Moreover, let us omit the indices $\alpha$ and $\beta$. We consider equation (2.13), as well as the variational equation one obtains by formally differentiating (2.14) with respect to time. Then, we present them in the form of abstract equations by introducing the Riesz operator $\mathcal{A}$ defined by (2.56), i.e.,

$$\begin{aligned}
&\partial_t\big(\alpha\mu + \varphi\big) + \mathcal{A}\mu = p(\sigma - \mu) + \mu \\
&\partial_t\mu = \beta\partial_t^2\varphi + \mathcal{A}\partial_t\varphi + F''(\varphi)\partial_t\varphi - \partial_t\varphi.
\end{aligned}$$

These equations are meant in the sense of the Hilbert triplet $(V, H, V^*)$, a.e. in $(0,T)$. Now, we test them by $\mathcal{A}^{-1}\partial_t\mu$ and $-\mathcal{A}^{-1}\partial_t\varphi$, respectively: we sum up and integrate over $(0,t)$. By omitting the evaluation point inside integrals for brevity, we obtain

$$\begin{aligned}
&\alpha \int_0^t \langle \partial_t\mu, \mathcal{A}^{-1}\partial_t\mu\rangle\, ds + \int_0^t \langle \partial_t\varphi, \mathcal{A}^{-1}\partial_t\mu\rangle\, ds + \int_0^t \langle \mathcal{A}\mu, \mathcal{A}^{-1}\partial_t\mu\rangle\, ds \\
&\quad + p\int_0^t \langle \mu, \mathcal{A}^{-1}\partial_t\mu\rangle\, ds - \int_0^t \langle \partial_t\mu, \mathcal{A}^{-1}\partial_t\varphi\rangle\, ds + \beta\int_0^t \langle \partial_t^2\varphi, \mathcal{A}^{-1}\partial_t\varphi\rangle\, ds \\
&\quad + \int_0^t \langle \mathcal{A}\partial_t\varphi, \mathcal{A}^{-1}\partial_t\varphi\rangle\, ds + \int_0^t \langle F''(\varphi)\partial_t\varphi, \mathcal{A}^{-1}\partial_t\varphi\rangle\, ds - \int_0^t \langle \partial_t\varphi, \mathcal{A}^{-1}\partial_t\varphi\rangle\, ds \\
&= p\int_0^t \langle \sigma, \mathcal{A}^{-1}\partial_t\mu\rangle\, ds + \int_0^t \langle \mu, \mathcal{A}^{-1}\partial_t\mu\rangle\, ds.
\end{aligned}$$

Now, let us recall (2.56)–(2.59). Two terms cancel out by (2.58) and we have

$$\begin{aligned}
&\alpha\int_0^t \|\partial_t\mu\|_*^2\, ds + \frac{1}{2}\|\mu(t)\|_H^2 + \frac{p}{2}\|\mu(t)\|_*^2 + \frac{\beta}{2}\|\partial_t\varphi(t)\|_*^2 + \int_0^t \|\partial_t\varphi\|_H^2\, ds \\
&= -\int_0^t \big(F''(\varphi)\partial_t\varphi, \partial_t\varphi\big)_*\, ds + \int_0^t \|\partial_t\varphi\|_*^2\, ds + p\int_0^t \big(\sigma, \partial_t\mu\big)_*\, ds \\
&\quad + \frac{1}{2}\|\mu(t)\|_*^2 - \frac{1}{2}\|\mu_0\|_*^2 + \frac{1}{2}\|\mu_0\|_H^2 + \frac{p}{2}\|\mu_0\|_*^2 + \frac{\beta}{2}\|\partial_t\varphi(0)\|_*^2\,.
\end{aligned} \tag{4.4}$$

Therefore, we just have to deal with the right-hand side. In order to control the integral involving $F''$, we account for the dual Sobolev inequality (2.55), the growth condition (2.49), the Hölder and Sobolev inequalities, the estimate (2.60) for $\varphi$, and have

$$
\begin{aligned}
&- \int_0^t \big(F''(\varphi)\partial_t\varphi, \partial_t\varphi\big)_* \, ds \le \int_0^t \|F''(\varphi)\partial_t\varphi\|_* \|\partial_t\varphi\|_* \, ds \\
&\le c \int_0^t \|F''(\varphi)\partial_t\varphi\|_{6/5} \, \|\partial_t\varphi\|_* \, ds \le c \int_0^t \|1+\varphi^2\|_3 \, \|\partial_t\varphi\|_2 \, \|\partial_t\varphi\|_* \, ds \\
&\le c \int_0^t \big(1+\|\varphi\|_6^2\big) \, \|\partial_t\varphi\|_2 \, \|\partial_t\varphi\|_* \, ds \le c \int_0^t \big(1+\|\varphi\|_V^2\big) \, \|\partial_t\varphi\|_H \, \|\partial_t\varphi\|_* \, ds \\
&\le \frac{1}{2} \int_0^t \|\partial_t\varphi\|_H^2 \, ds + c \int_0^t \|\partial_t\varphi\|_*^2 \, ds \, .
\end{aligned}
$$

Another term to treat is

$$
\begin{aligned}
& p \int_0^t \big(\sigma, \partial_t\mu\big)_* \, ds = -p \int_0^t \big(\partial_t\sigma, \mu\big)_* \, ds + p\,\big(\sigma(t), \mu(t)\big)_* - p\,\big(\sigma_0, \mu_0\big)_* \\
&\quad \le \frac{p}{4} \int_0^t \|\mu\|_*^2 \, ds + p\, \|\partial_t\sigma\|_{L^2(0,T;V^*)}^2 + \frac{p}{4} \|\mu(t)\|_*^2 + p\, \|\sigma\|_{L^\infty(0,T;V^*)}^2 + c,
\end{aligned}
$$

the last inequality being due to (4.3) (which holds true under the assumptions of Theorem 2.5, thus also in the present case) and (2.8). The next term on the right-hand side of (4.4) is $(1/2)\|\mu(t)\|_*^2$. We observe once more that the norm of the embedding $H \subset V^*$ is $1$ since we are using the standard norms in $V$ and $H$. Therefore, we have

$$
\frac{1}{2} \, \|\mu(t)\|_*^2 \le \frac{1}{2} \, \|\mu(t)\|_H^2 \, .
$$

Finally, we consider the norms of the initial values of the time derivatives. We formally have

$$
\beta\partial_t\varphi(0) = \mu_0 - \mathcal{A}\varphi_0 + \varphi_0 - F'(\varphi_0) \, ,
$$

which is a fixed element of $V^*$ due to (2.8). Indeed, for $F'(\varphi_0)$, we make the following observation. Assumption (2.49) implies $|F'(r)| \le c(|r|^3 + 1)$ for every $r \in \mathbb{R}$. As $\varphi_0 \in V$, we have $\varphi_0 \in L^6(\Omega)$ by the Sobolev inequality. We infer that $\varphi^3 \in L^2(\Omega)$, whence $F'(\varphi_0) \in H$. At this point, we can collect (4.4) and all the inequalities we have proved, then apply the Gronwall lemma. We conclude that

$$
\alpha^{1/2} \|\partial_t\mu_{\alpha,\beta}\|_{L^2(0,T;V^*)} + p^{1/2} \|\mu_{\alpha,\beta}\|_{L^\infty(0,T;V^*)} + \|\partial_t\varphi_{\alpha,\beta}\|_{L^\infty(0,T;V^*)} \le c \, , \tag{4.5}
$$

where we have used the full notation with indices.

**Conclusion of the proof of Theorem 2.6.** We prove that (2.50) holds for every solution $(\mu_\beta, \varphi_\beta, \sigma_\beta)$ to problem (2.34)–(2.37) (with a constant $C_\beta$ that might depend on the solution we are considering). We often omit writing the evaluation point, explicitly, in order to simplify the notation. We take the difference between (2.13)–(2.15), written for $(\mu_{\alpha,\beta}, \varphi_{\alpha,\beta}, \sigma_{\alpha,\beta})$, and (2.34)–(2.36), written for $(\mu_\beta, \varphi_\beta, \sigma_\beta)$, where $\xi_{\alpha,\beta} = B(\varphi_{\alpha,\beta})$ and $\xi_\beta = B(\varphi_\beta)$. By setting, for convenience,

$$
\mu := \mu_{\alpha,\beta} - \mu_\beta \, , \quad \varphi := \varphi_{\alpha,\beta} - \varphi_\beta \quad \text{and} \quad \sigma := \sigma_{\alpha,\beta} - \sigma_\beta \, ,
$$

and recalling that $p$ is a nonnegative constant (cf. (2.48)), we have

$$
\begin{aligned}
&\int_\Omega \partial_t\varphi \, v + \int_\Omega \nabla\mu \cdot \nabla v = -\alpha \langle \partial_t\mu_{\alpha,\beta}, v \rangle + p \int_\Omega (\sigma - \mu) v \\
&\int_\Omega \mu v = \beta \int_\Omega \partial_t\varphi \, v + \int_\Omega \nabla\varphi \cdot \nabla v + \int_\Omega (\xi_{\alpha,\beta} - \xi_\beta) v + \int_\Omega \big(\pi(\varphi_{\alpha,\beta}) - \pi(\varphi_\beta)\big) v \\
&\langle \partial_t\sigma, v \rangle + \int_\Omega \nabla\sigma \cdot \nabla v = p \int_\Omega (\mu - \sigma) v \, ,
\end{aligned}
$$

where each equality holds true for every $v \in V$ and a.e. in $(0,T)$. Now, we take the test function $v$ equal to

$$\varphi + \beta\mu, \quad \mu - \varphi \quad \text{and} \quad \sigma\,,$$

respectively. Next, we integrate with respect to time, exploit the initial conditions $\varphi(0) = 0$ and $\sigma(0) = 0$, add the volume integrals of $p\beta|\mu|^2$ and of $p|\sigma|^2$ to both sides for convenience, and rearrange a little. We have, for every $t \in [0,T]$,

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\varphi(t)|^2 + \beta\int_{Q_t} \partial_t\varphi\,\mu + \int_{Q_t} \nabla\mu\cdot\nabla\varphi + \beta\int_{Q_t} |\nabla\mu|^2 \\
&\quad + (1+p\beta)\int_{Q_t} |\mu|^2 + \frac{\beta}{2}\int_\Omega |\varphi(t)|^2 - \beta\int_{Q_t} \partial_t\varphi\,\mu + \int_{Q_t} |\nabla\varphi|^2 - \int_{Q_t} \nabla\varphi\cdot\nabla\mu \\
&\quad + \int_{Q_t} (\xi_{\alpha,\beta} - \xi_\beta)\varphi + \frac{1}{2}\int_\Omega |\sigma(t)|^2 + \int_{Q_t} |\nabla\sigma|^2 + p\int_{Q_t} |\sigma|^2 \\
&= \int_{Q_t} \mu\varphi - \int_{Q_t} \bigl(\pi(\varphi_{\alpha,\beta}) - \pi(\varphi_\beta)\bigr)\varphi + \int_{Q_t} \bigl(F'(\varphi_{\alpha,\beta}) - F'(\varphi_\beta)\bigr)\mu \\
&\quad - \alpha\int_0^t \langle \partial_t\mu_{\alpha,\beta}, \varphi + \beta\mu\rangle\,ds + p\int_{Q_t} \bigl\{(\sigma-\mu)(\varphi+\beta\mu-\sigma) + \beta|\mu|^2 + |\sigma|^2\bigr\}. \qquad (4.6)
\end{aligned}$$

Four terms on the left-hand side cancel out, and the other ones are nonnegative. Now, we estimate each integral on the right-hand side, separately. For the first two of them, we have

$$\int_{Q_t} \mu\varphi - \int_{Q_t} \bigl(\pi(\varphi_{\alpha,\beta}) - \pi(\varphi_\beta)\bigr)\varphi \le \frac{1}{4}\int_{Q_t} |\mu|^2 + c\int_{Q_t} |\varphi|^2$$

since $\pi$ is Lipschitz continuous. In order to deal with the next integral, we observe that

$$|F'(\varphi_{\alpha,\beta}) - F'(\varphi_\beta)| \le \bigl(|\varphi_{\alpha,\beta}|^2 + |\varphi_\beta|^2 + 1\bigr)|\varphi| \quad \text{a.e. in } Q\,,$$

due to the mean value theorem and (2.49). Hence, applying the Hölder and Sobolev inequalities, and owing to estimate (2.60) for $\varphi_{\alpha,\beta}$ and the regularity (2.39) of $\varphi_\beta$, we can infer that

$$\begin{aligned}
&\int_{Q_t} \bigl(F'(\varphi_{\alpha,\beta}) - F'(\varphi_\beta)\bigr)\mu \le c\int_0^t \bigl\||\varphi_{\alpha,\beta}|^2 + |\varphi_\beta|^2 + 1\bigr\|_3 \,\|\varphi\|_2\,\|\mu\|_6\,ds \\
&\le c\int_0^t \bigl(\|\varphi_{\alpha,\beta}\|_6^2 + \|\varphi_\beta\|_6^2 + 1\bigr)\,\|\varphi\|_2\,\|\mu\|_6\,ds \\
&\le c\int_0^t \|\varphi\|_H\,\|\mu\|_V\,ds \le \delta\int_0^t \|\mu\|_V^2\,ds + c_\delta\int_{Q_t} |\varphi|^2\,.
\end{aligned}$$

Next, we take advantage of (4.5) for $\partial_t\mu_{\alpha,\beta}$ and deduce that

$$\begin{aligned}
&-\alpha\int_0^t \langle \partial_t\mu_{\alpha,\beta}, \varphi+\beta\mu\rangle\,ds \le \delta\int_0^t \bigl(\|\varphi\|_V^2 + \|\mu\|_V^2\bigr)\,ds + c_\delta\,\alpha^2\int_0^t \|\partial_t\mu_{\alpha,\beta}\|_*^2\,ds \\
&\le \delta\int_0^t \bigl(\|\varphi\|_V^2 + \|\mu\|_V^2\bigr)\,ds + c_\delta\,\alpha\,.
\end{aligned}$$

Finally, the last term in (4.6) can be simplified and estimated in the following way:

$$\begin{aligned}
&p\int_{Q_t} \bigl\{(\sigma-\mu)(\varphi+\beta\mu-\sigma) + \beta|\mu|^2 + |\sigma|^2\bigr\} \\
&\le p\int_{Q_t} \bigl(\sigma\varphi - \mu\varphi + (\beta+1)\sigma\mu\bigr) \le \delta\int_{Q_t} |\mu|^2 + c_\delta\int_{Q_t} \bigl(|\varphi|^2 + |\sigma|^2\bigr).
\end{aligned}$$

By combining (4.6) and these inequalities, choosing $\delta$ small enough and applying the Gronwall lemma, we obtain (2.50).

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