# LONGTIME BEHAVIOR FOR A GENERALIZED CAHN-HILLIARD SYSTEM WITH FRACTIONAL OPERATORS 

Pierluigi Colli ${ }^{a *}$, Gianni Gilardi ${ }^{a}$ and Jürgen Sprekels ${ }^{b}$


#### Abstract

In this contribution, we deal with the longtime behavior of the solutions to the fractional variant of the Cahn-Hilliard system, with possibly singular potentials, that we have recently investigated in the paper Well-posedness and regularity for a generalized fractional Cahn-Hilliard system. More precisely, we study the $\omega$-limit of the phase parameter $y$ and characterize it completely. Our characterization depends on the first eigenvalues $\lambda_{1} \geq 0$ of one of the operators involved: if $\lambda_{1}>0$, then the chemical potential $\mu$ vanishes at infinity and every element $y_{\omega}$ of the $\omega$-limit is a stationary solution to the phase equation; if instead $\lambda_{1}=0$, then every element $y_{\omega}$ of the $\omega$-limit satisfies a problem containing a real function $\mu_{\infty}$ related to the chemical potential $\mu$. Such a function $\mu_{\infty}$ is nonunique and time dependent, in general, as we show by an example. However, we give sufficient conditions for $\mu_{\infty}$ to be uniquely determined and constant.


## 1. Introduction

The paper of Colli et al. (2019) investigates the abstract evolutionary system

$$
\begin{align*}
& \partial_{t} y+A^{2 r} \mu=0  \tag{1.1}\\
& \tau \partial_{t} y+B^{2 \sigma} y+f^{\prime}(y)=\mu+u,  \tag{1.2}\\
& y(0)=y_{0} \tag{1.3}
\end{align*}
$$

where $A^{2 r}$ and $B^{2 \sigma}$, with $r>0$ and $\sigma>0$, denote fractional powers in the spectral sense of the unbounded linear operators $A$ and $B$, respectively, which are supposed to be densely defined in $H:=L^{2}(\Omega)$, with $\Omega \subset \mathbb{R}^{3}$, selfadjoint, and monotone. The above system is a generalization of the Cahn-Hilliard system (namely, the nonviscous system or the viscous one, depending on whether $\tau=0$ or $\tau>0$ ), which models a phase separation process taking place in the container $\Omega$. The unknown functions $y$ and $\mu$ stand for the order parameter and the chemical potential, respectively, while $u$ is a given source term. Moreover, $f$ denotes a double-well potential, for which typical and physically significant examples are the so-called classical regular potential, the logarithmic double-well potential, and the
double obstacle potential, which are given, in this order, by

$$
\begin{align*}
& f_{\text {reg }}(r):=\frac{1}{4}\left(r^{2}-1\right)^{2}, \quad r \in \mathbb{R},  \tag{1.4}\\
& f_{\text {log }}(r):=((1+r) \ln (1+r)+(1-r) \ln (1-r))-c_{1} r^{2}, \quad r \in(-1,1),  \tag{1.5}\\
& f_{\text {2obs }}(r):=-c_{2} r^{2} \quad \text { if }|r| \leq 1 \quad \text { and } \quad f_{2 \text { obs }}(r):=+\infty \quad \text { if }|r|>1 . \tag{1.6}
\end{align*}
$$

Here, the constants $c_{i}$ in (1.5) and (1.6) satisfy $c_{1}>1$ and $c_{2}>0$, so that $f_{\text {log }}$ and $f_{2 \text { obs }}$ are nonconvex. In cases like (1.6), one has to split $f$ into a nondifferentiable convex part $\widehat{\beta}$ (the indicator function of $[-1,1]$, in the present example) and a smooth perturbation $\widehat{\pi}$. Accordingly, one has to replace the derivative of the convex part by the subdifferential and interpret (1.2) as a differential inclusion or, equivalently, as a variational inequality involving $\widehat{\beta}$ rather than its subdifferential. Actually, the latter has been done by Colli et al. (2019), and we do the same in this paper.

Fractional versions of the Cahn-Hilliard system have been considered by several authors and are the subject of a number of recent papers. As for references regarding well-posedness and related problems, a rather large list of citations is given by Colli et al. (2019). Here we recall some literature dealing with the asymptotic behavior of the solutions. Indeed, one can find a number of results in this direction both for the standard Cahn-Hilliard equations and for variants of them. The latter are obtained, e.g., by adding viscosity or memory contributions as well as convective terms; another possibility is coupling with other equations, like heat type equations or fluid dynamics equations, or introducing non-local-in-space terms; finally, one can replace the classical Neumann boundary conditions by other ones, e.g., the dynamic boundary conditions. Without any claim of completeness, by starting from Zheng (1986), we can quote, e.g., Colli et al. (1999), Wu and Zheng (2004), Chill et al. (2006), Abels and Wilke (2007), Gilardi and Rocca (2007), Grasselli et al. (2007), Gilardi et al. (2010), Prüss et al. (2010), Colli et al. (2011), Wang and Wu (2012), Jiang et al. (2015), Akagi et al. (2016), Colli et al. (2018a), and Gilardi and Sprekels (2019) for the study of the trajectories and related topics, and Li and Zhong (1998), Miranville (2001a,b), Efendiev et al. (2002, 2004), Miranville and Zelik (2004, 2005), Gal (2006), Grasselli et al. (2007), Segatti (2007), Gal (2008), Gal and Grasselli (2010, 2011), Miranville (2013), Zhao and Liu (2015), and Gal (2017) for the existence of global or exponential attractors and their properties. However, if nonlocal terms are considered in these papers, they are not defined as fractional powers in the spectral sense of the operators involved. On the contrary, our framework is followed in the paper of Colli and Gilardi (2019), where the longtime behavior of the solutions to a fractional version of the Allen-Cahn equation is studied.

Let us come to the content of this paper. Our aim is studying the $\omega$-limit (in a suitable topology) of the component $y$ of the solution to a proper weak version of problem (1.1)(1.3). The characterization we give (Theorem 2.2) depends on the first eigenvalue $\lambda_{1}$ of the operator $A$. If $\lambda_{1}>0$, then $\mu(t)$ tends to zero as $t$ approaches infinity, and every element $y_{\omega}$ of the $\omega$-limit is a stationary solution, i.e., it solves the equation

$$
\begin{equation*}
B^{2 \sigma} y_{\omega}+f^{\prime}\left(y_{\omega}\right)=u_{\infty} \tag{1.7}
\end{equation*}
$$

at least in a weak sense, where $u_{\infty}$ is the limit of $u(t)$ as $t$ tends to infinity. If, instead, $\boldsymbol{\lambda}_{1}=0$, then the element $y_{\omega}$ satisfies a weaker property, namely, a weak form of the equation

$$
\begin{equation*}
B^{2 \sigma} y_{\omega}+f^{\prime}\left(y_{\omega}\right)=u_{\infty}+\mu_{\infty}(t) \quad \text { for a.a. } t \in(0,+\infty), \tag{1.8}
\end{equation*}
$$

for at least one function $\mu_{\infty} \in L_{l o c}^{\infty}([0,+\infty))$. We also show that, in the general case, the characterization (1.8) is the best possible (see Example 2.3): $\mu_{\infty}$ is nonconstant and nonunique, in general, and $\mu(t)$ does not converge at infinity. On the other hand, we give sufficient conditions on $f$ and on the solution that ensure that the function $\mu_{\infty}$ is unique and constant and that (1.8) holds in the strong sense (see Proposition 2.4).

## 2. Statement of the problem and results

In this section, we state precise assumptions and notations and present our results. Our framework is the same as in the paper of Colli et al. (2019), and we briefly recall it here, for the reader's convenience. First of all, the open set $\Omega \subset \mathbb{R}^{3}$ is assumed to be bounded, connected and smooth. We use the notation

$$
\begin{equation*}
H:=L^{2}(\Omega) \tag{2.1}
\end{equation*}
$$

and denote by $\|\cdot\|$ and $(\cdot, \cdot)$ the standard norm and inner product of $H$. As for the operators involved in our system, we postulate that

$$
\begin{equation*}
A: D(A) \subset H \rightarrow H \quad \text { and } \quad B: D(B) \subset H \rightarrow H \quad \text { are } \tag{2.2}
\end{equation*}
$$

unbounded, monotone, selfadjoint linear operators with compact resolvents.
We denote by $\left\{\lambda_{j}\right\}$ and $\left\{\lambda_{j}^{\prime}\right\}$ the nondecreasing sequences of the eigenvalues and by $\left\{e_{j}\right\}$ and $\left\{e_{j}^{\prime}\right\}$ the (complete) systems of the corresponding orthonormal eigenvectors, that is,

$$
\begin{align*}
& A e_{j}=\lambda_{j} e_{j}, \quad B e_{j}^{\prime}=\lambda_{j}^{\prime} e_{j}^{\prime}, \quad \text { and } \quad\left(e_{i}, e_{j}\right)=\left(e_{i}^{\prime}, e_{j}^{\prime}\right)=\delta_{i j}, \quad \text { for } i, j=1,2, \ldots,  \tag{2.3}\\
& 0 \leq \lambda_{1} \leq \lambda_{2} \leq \ldots \quad \text { and } \quad 0 \leq \lambda_{1}^{\prime} \leq \lambda_{2}^{\prime} \leq \ldots, \quad \text { with } \lim _{j \rightarrow \infty} \lambda_{j}=\lim _{j \rightarrow \infty} \lambda_{j}^{\prime}=+\infty . \tag{2.4}
\end{align*}
$$

The power $A^{r}$ of $A$ with an arbitrary positive real exponent $r$ is given by

$$
\begin{equation*}
A^{r} v=\sum_{j=1}^{\infty} \lambda_{j}^{r}\left(v, e_{j}\right) e_{j} \quad \text { for } v \in V_{A}^{r}, \tag{2.5}
\end{equation*}
$$

where

$$
\begin{equation*}
V_{A}^{r}:=D\left(A^{r}\right)=\left\{v \in H: \sum_{j=1}^{\infty}\left|\lambda_{j}^{r}\left(v, e_{j}\right)\right|^{2}<+\infty\right\} . \tag{2.6}
\end{equation*}
$$

In principle, we could endow $V_{A}^{r}$ with the standard graph norm in order to make $V_{A}^{r}$ a Hilbert space. However, we will choose an equivalent Hilbert structure later on. In the same way, for $\sigma>0$, we define the power $B^{\sigma}$ of $B$. For its domain, we use the notation

$$
\begin{align*}
& V_{B}^{\sigma}:=D\left(B^{\sigma}\right), \quad \text { with the norm }\|\cdot\|_{B, \sigma} \text { associated to the inner product } \\
& (v, w)_{B, \sigma}:=(v, w)+\left(B^{\sigma} v, B^{\sigma} w\right) \quad \text { for } v, w \in V_{B}^{\sigma} . \tag{2.7}
\end{align*}
$$

Accordingly, we introduce a space with a negative exponent. We set

$$
\begin{equation*}
V_{A}^{-r}:=\left(V_{A}^{r}\right)^{*} \quad \text { for } r>0 \tag{2.8}
\end{equation*}
$$

and use the symbol $\langle\cdot, \cdot\rangle_{A, r}$ for the duality pairing between $V_{A}^{-r}$ and $V_{A}^{r}$. We also identify $H$ with a subspace of $V_{A}^{-r}$ in the usual way, i.e., such that

$$
\begin{equation*}
\langle v, w\rangle_{A, r}=(v, w) \quad \text { for every } v \in H \text { and } w \in V_{A}^{r} . \tag{2.9}
\end{equation*}
$$

At this point, we can start listing our assumptions. First of all, $r$ and $\sigma$ are fixed positive real numbers and $\tau \in[0,1]$ is fixed as well.

As for the linear operators, we postulate, besides (2.2), that

$$
\begin{align*}
& \text { either } \lambda_{1}>0 \quad \text { or } \quad 0=\lambda_{1}<\lambda_{2} \text { and } e_{1} \text { is a constant; }  \tag{2.11}\\
& \text { if } \quad \lambda_{1}=0 \text {, then the constant functions belong to } V_{B}^{\sigma} \text {. } \tag{2.12}
\end{align*}
$$

In the paper of Colli et al. (2019) some remarks are given on the above assumptions. Moreover, it is shown that an equivalent Hilbert structure on $V_{A}^{r}$ is obtained by taking the norm defined by

$$
\|v\|_{A, r}^{2}:=\left\{\begin{array}{l}
\left\|A^{r} v\right\|^{2}=\sum_{j=1}^{\infty}\left|\lambda_{j}^{r}\left(v, e_{j}\right)\right|^{2} \quad \text { if } \lambda_{1}>0  \tag{2.13}\\
\left|\left(v, e_{1}\right)\right|^{2}+\left\|A^{r} v\right\|^{2}=\left|\left(v, e_{1}\right)\right|^{2}+\sum_{j=2}^{\infty}\left|\lambda_{j}^{r}\left(v, e_{j}\right)\right|^{2} \quad \text { if } \lambda_{1}=0
\end{array}\right.
$$

We notice that the term $\left(v, e_{1}\right)$ appearing in $(\mathbf{2} .13)$ in the case $\lambda_{1}=0$ is proportional to the mean value of $v$

$$
\begin{equation*}
\operatorname{mean} v:=\frac{1}{|\Omega|} \int_{\Omega} v \tag{2.14}
\end{equation*}
$$

since $e_{1}$ is a constant by (2.11). In particular, we have the Poincaré type inequality

$$
\begin{equation*}
\|v\| \leq C_{P}\left\|A^{r} v\right\| \quad \text { for every } v \in V_{A}^{r} \text { with mean } v=0 \text {, if } \lambda_{1}=0 . \tag{2.15}
\end{equation*}
$$

For the nonlinearity $f$ appearing in our system, we split it as $f=\widehat{\beta}+\widehat{\pi}$ and postulate the following properties (which are fulfilled by all of the important potentials (1.4)-(1.6)):

$$
\begin{equation*}
\widehat{\beta}: \mathbb{R} \rightarrow[0,+\infty] \quad \text { is convex, proper, and 1.s.c., with } \widehat{\beta}(0)=0 \tag{2.16}
\end{equation*}
$$

$\hat{\pi}: \mathbb{R} \rightarrow \mathbb{R}$ is of class $C^{1}$ with a Lipschitz continuous first derivative;

$$
\begin{equation*}
\text { it holds } \liminf _{|s| \gamma+\infty} \frac{\widehat{\beta}(s)+\widehat{\pi}(s)}{s^{2}}>0 . \tag{2.17}
\end{equation*}
$$

We set, for convenience,

$$
\begin{equation*}
\beta:=\partial \widehat{\beta}, \quad \pi:=\hat{\pi}^{\prime}, \quad L_{\pi}=\text { the Lipschitz constant of } \pi, \quad \text { and } \quad L_{\pi}^{\prime}:=L_{\pi}+1 \tag{2.19}
\end{equation*}
$$

Moreover, we term $D(\widehat{\beta})$ and $D(\beta)$ the effective domains of $\widehat{\beta}$ and $\beta$, respectively, and notice that $\beta$ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$.

At this point, we can state the problem under investigation, and we do it on the half-line $t \geq 0$, due to the subject of the present paper. The data are required to satisfy

$$
\begin{align*}
& u \in W_{l o c}^{1,1}([0,+\infty) ; H) \text { and } \partial_{t} u \in L^{1}(0,+\infty ; H) .  \tag{2.20}\\
& y_{0} \in V_{B}^{\sigma} \quad \text { and } \widehat{\beta}\left(y_{0}\right) \in L^{1}(\Omega) .  \tag{2.21}\\
& \text { If } \lambda_{1}=0 \text {, then } m_{0}:=\text { mean } y_{0} \text { belongs to the interior of } D(\beta) . \tag{2.22}
\end{align*}
$$

A solution to our system is a pair $(y, \mu)$ fulfilling the regularity requirements

$$
\begin{align*}
& y \in L^{\infty}\left(0, T ; V_{B}^{\sigma}\right), \quad \partial_{t} y \in L^{2}\left(0, T ; V_{A}^{-r}\right) \quad \text { and } \quad \tau \partial_{t} y \in L^{2}(0, T ; H),  \tag{2.23}\\
& \mu \in L^{2}\left(0, T ; V_{A}^{r}\right),  \tag{2.24}\\
& \widehat{\beta}(y) \in L^{1}(\Omega \times(0, T)), \tag{2.25}
\end{align*}
$$

for every $T>0$, and satisfying the following weak formulation of the equations (1.1)-(1.3):

$$
\begin{align*}
& \left\langle\partial_{t} y(t), v\right\rangle_{A, r}+\left(A^{r} \mu(t), A^{r} v\right)=0 \quad \text { for every } v \in V_{A}^{r} \text { and for a.a. } t \in(0,+\infty)  \tag{2.26}\\
& \left(\tau \partial_{t} y(t), y(t)-v\right)+\left(B^{\sigma} y(t), B^{\sigma}(y(t)-v)\right) \\
& \quad+\int_{\Omega} \widehat{\beta}(y(t))+(\pi(y(t))-u(t), y(t)-v) \leq(\mu(t), y(t)-v)+\int_{\Omega} \widehat{\beta}(v) \\
& \quad \text { for every } v \in V_{B}^{\sigma} \text { and for a.a. } t \in(0,+\infty)  \tag{2.27}\\
& y(0)=y_{0} . \tag{2.28}
\end{align*}
$$

We remark that, if $\lambda_{1}=0$, then $A^{r}(1)=0$ by (2.11), so that (2.26) implies that

$$
\begin{align*}
& \frac{d}{d t} \int_{\Omega} y(t)=0 \quad \text { for a.a. } t \in(0,+\infty), \quad \text { i.e., } \\
& \text { mean } y(t)=m_{0} \quad \text { for every } t \in[0,+\infty) \tag{2.29}
\end{align*}
$$

The well-posedness result stated below was proved by Colli et al. (2019) under a different assumption on $u$. Namely, in studying the problem on the finite time interval $(0, T)$, it was assumed that $u \in H^{1}(0, T ; H)$, while (2.20) only implies that $u \in W^{1,1}(0, T ; H)$. However, we point out that our assumption is sufficient to obtain the same result. We will give some explanation on this in the next section.

Theorem 2.1. Let the assumptions (2.2), (2.10)-(2.12) and (2.16)-(2.18) on the structure of the system, and (2.20)-(2.22) on the data, be fulfilled. Then there exists a pair $(y, \mu)$ satisfying (2.23)-(2.25) and solving problem (2.26)-(2.28). Moreover, the component $y$ of the solution is uniquely determined.

In the paper of Colli et al. (2019, Rem. 4.1), sufficient conditions were given that ensure uniqueness also for $\mu$. However, the aim of this paper is the study of the longtime behavior of the component $y$ alone. The rather weak regularity conditions (2.23) imply that

$$
y:[0,+\infty) \rightarrow V_{B}^{\sigma} \quad \text { is weakly continuous. }
$$

This enables us to the define the following (possibly empty) $\omega$-limit set

$$
\begin{equation*}
\omega=\omega\left(y_{0}, u\right):=\left\{y_{\omega} \in V_{B}^{\sigma}: y\left(t_{n}\right) \rightarrow y_{\omega} \quad \text { weakly in } V_{B}^{\sigma} \text { for some }\left\{t_{n}\right\} \nearrow+\infty\right\} . \tag{2.30}
\end{equation*}
$$

Here is our result, which holds under the additional assumption that $u(t)$ has a limit $u_{\infty}$ as $t$ tends to infinity in the sense of the forthcoming (2.31). The second part of the statement distinguishes two cases regarding the first eigenvalue $\lambda_{1}$ of $A$. If $\lambda_{1}$ is positive, then every element of the $\omega$-limit is a stationary solution in the sense specified below; if instead $\lambda_{1}=0$, then the elements of the $\omega$-limit just satisfy a weaker property.
Theorem 2.2. Let the assumptions (2.2), (2.10)-(2.12) and (2.16)-(2.18) on the structure of the system, and (2.20)-(2.22) on the data, be fulfilled. In addition, assume that there is some $u_{\infty} \in H$ such that

$$
\begin{equation*}
u-u_{\infty} \in L^{2}(0,+\infty ; H) \tag{2.31}
\end{equation*}
$$

and let $(y, \mu)$ be a solution to $\mathbf{( 2 . 2 6 ) - ( \mathbf { 2 . 2 8 } )}$ according to Theorem 2.1. Then the $\omega$-limit (2.30) is nonempty. Moreover, it is characterized as follows:
i) If $\lambda_{1}>0$, then every element $y_{\omega} \in \omega$ satisfies

$$
\begin{align*}
& \left(B^{\sigma} y_{\omega}, B^{\sigma}\left(y_{\omega}-v\right)\right)+\int_{\Omega} \widehat{\beta}\left(y_{\omega}\right)+\left(\pi\left(y_{\omega}\right)-u_{\infty}, y_{\omega}-v\right) \leq \int_{\Omega} \widehat{\beta}(v) \\
& \quad \text { for every } v \in V_{B}^{\sigma} . \tag{2.32}
\end{align*}
$$

[^0]ii) If $\lambda_{1}=0$, then, for every element $y_{\omega} \in \omega$, there exists some $\mu_{\infty} \in L_{\text {loc }}^{\infty}([0,+\infty))$ such that
\[

$$
\begin{align*}
& \left(B^{\sigma} y_{\omega}, B^{\sigma}\left(y_{\omega}-v\right)\right)+\int_{\Omega} \widehat{\beta}\left(y_{\omega}\right)+\left(\pi\left(y_{\omega}\right)-u_{\infty}, y_{\omega}-v\right) \\
& \leq\left(\mu_{\infty}(t), y_{\omega}-v\right)+\int_{\Omega} \widehat{\beta}(v) \\
& \quad \text { for every } v \in V_{B}^{\sigma} \text { and for a.a. } t \in(0,+\infty) . \tag{2.33}
\end{align*}
$$
\]

In (2.31), $u_{\infty}$ obviously denotes the function $[0,+\infty) \ni t \mapsto u_{\infty} \in H$ rather than the element $u_{\infty} \in H$. In the right-hand side of $(\mathbf{2 . 3 3}), \mu_{\infty}(t)$ denotes the constant function $\Omega \ni x \mapsto \mu_{\infty}(t)$ rather than the real value $\mu_{\infty}(t)$. Conventions of this type will be used also in the following.

The part $i i$ ) of the above result seems to be rather poor. Nevertheless, this characterization is the best possible for the general case, that is, one can neither expect uniqueness for $\mu_{\infty}$, nor further properties for it, as the following example shows. Notice that assuming that $A$ and $B$ are particularly good operators does not help at all.

Example 2.3. Let the operators $A$ and $B$ satisfy the hypotheses of Theorem 2.2, and assume that $\lambda_{1}=0$. Moreover, let us choose $\widehat{\pi}=0$ and $\widehat{\beta}$ given by

$$
\widehat{\beta}(r):=r^{2}+|r| \quad \text { for } r \in \mathbb{R} .
$$

Then (2.16)-(2.18) are satisfied. But $\beta$ is multivalued, since $\beta(0)=\operatorname{sign}(0)=[-1,1]$. Thus, if we take $y_{0}=0, u=0$, and any function $\bar{\mu} \in L^{\infty}(0,+\infty)$ satisfying $|\bar{\mu}(t)| \leq 1$ for a.a. $t \in(0,+\infty)$, then a solution $(y, \mu)$ to problem (2.26)-(2.28) is given by the formulas $y(x, t)=0$ and $\mu(x, t)=\bar{\mu}(t)$. Indeed, $(y, \mu)$ trivially solves the first equation (2.26) (since $\mu$ is space independent), as well as (2.28); moreover, the variational inequality (2.27) is solved in the stronger form

$$
\tau \partial_{t} y+B^{2 \sigma} y+\xi+\pi(y)=\mu+u \quad \text { with } \quad \xi \in \beta(y)
$$

since we can take $\xi=\mu$ (we have $\bar{\mu}(t) \in[-1,1]=\beta(0)$, indeed). So, the only element $y_{\omega}$ of the $\omega$-limit is $y_{\omega}=0$, while we have lots of possible $\mu_{\infty}$ 's, namely, the set of such functions coincides with the set of the admissible functions termed $\bar{\mu}$ before.

On the contrary, under further conditions on $\beta$ and on the solution, the characterization in the case $\lambda_{1}=0$ can be improved. Here are the new requirements:

$$
\begin{equation*}
D(\beta) \text { is an open interval, and } \beta \text { is a single-valued } C^{1} \text { function. } \tag{2.34}
\end{equation*}
$$

There exists a compact interval $[a, b] \subset D(\beta)$ such that

$$
\begin{align*}
& \quad y(x, t) \in[a, b] \text { for a.a. }(x, t) \in \Omega \times(0,+\infty) .  \tag{2.35}\\
& V_{B}^{\sigma} \cap L^{\infty}(\Omega) \text { is dense in } V_{B}^{\sigma} . \tag{2.36}
\end{align*}
$$

The above assumptions (with (2.35) only in a given finite time interval $(0, T)$ ) have been introduced in the paper of Colli et al. (2018b). One of the motivations was the derivation of the strong form of (2.27), i.e.,

$$
\begin{equation*}
\tau \partial_{t} y+B^{2 \sigma} y+\beta(y)+\pi(y)=\mu+u \tag{2.37}
\end{equation*}
$$

Precisely, it has been proved that $y \in L^{2}\left(0, T ; V_{B}^{2 \sigma}\right)$ and that (2.37) is satisfied almost everywhere (see Rem. 3.5 and the subsequent lines of Colli et al. (2018b), where some comments on (2.34)-(2.36) were given as well). Here, we point out that the proof of the
derivation of (2.37) also holds true for the half-line $t \geq 0$ if (2.35) is assumed. We use (2.34)-(2.36) in the result stated below.

Proposition 2.4. In addition to the assumptions of Theorem 2.2, suppose that (2.34)-(2.36) are satisfied. Then the function $\mu_{\infty}$ appearing in (2.33) is uniquely determined and constant. Moreover, $y_{\omega} \in V_{B}^{2 \sigma}$, and the pair $\left(y_{\omega}, \mu_{\infty}\right)$ satisfies the equation

$$
\begin{equation*}
B^{2 \sigma} y_{\omega}+\beta\left(y_{\omega}\right)+\pi\left(y_{\omega}\right)=\mu_{\infty}+u_{\infty} \quad \text { a.e. in } \Omega . \tag{2.38}
\end{equation*}
$$

Theorem 2.2 and Proposition 2.4 will be proved in the last section. In the next one, we establish some auxiliary global estimates. To this end, we also recall the approximation and the discretization of problem (2.26)-(2.28) given by Colli et al. (2019).

Notation 2.5. In the remainder of the paper, we will use the same small letter c for (possibly) different constants that depend only on the structure of our system (but $\tau$ ) and on the assumptions on the data. When some final time $T$ is considered, the symbol $c_{T}$ denotes (possibly different) constants that depend on $T$ in addition. On the contrary, precise constants we could refer to are treated in a different way (see, e.g., the forthcoming (3.1), where greek and capital letters are used).

## 3. Global estimates

The proof of Theorem $\mathbf{2 . 2}$ is based on some global-in-time a priori estimates, which we derive in this section by starting from the approximating and discrete problems introduced by Colli et al. (2019). Thus, some recalls are needed.

The approximation of problem (2.26)-(2.28) by a more regular one relies on the use of the Moreau-Yosida regularizations $\widehat{\beta}_{\lambda}$ and $\beta_{\lambda}$ of $\widehat{\beta}$ and $\beta$ at the level $\lambda>0$ (see, e.g., Brezis (1973, p. 28 and p. 39)). We notice that, by accounting for (2.18), the inequalities

$$
\begin{equation*}
\widehat{\beta}_{\lambda}(s)+\widehat{\pi}(s) \geq \alpha s^{2}-C \geq-C^{\prime} \tag{3.1}
\end{equation*}
$$

hold true for some positive constants $\alpha, C, C^{\prime}$, every $s \in \mathbb{R}$, and every sufficiently small $\lambda>0$. In case the reader aims to check (3.1), we suggest the use of the following representation of $\widehat{\beta}_{\lambda}$, namely

$$
\begin{equation*}
\widehat{\beta}_{\lambda}(s):=\inf _{r \in \mathbb{R}}\left\{\frac{1}{2 \lambda}|r-s|^{2}+\widehat{\beta}(r)\right\}=\frac{1}{2 \lambda}\left|s-J_{\lambda}(s)\right|^{2}+\widehat{\beta}\left(J_{\lambda}(s)\right), \tag{3.2}
\end{equation*}
$$

where $J_{\lambda}: \mathbb{R} \rightarrow \mathbb{R}$ denotes the resolvent operator associated to $\beta$, that is, $J_{\lambda}(s)$ is defined as the unique solution to the multi-equation

$$
J_{\lambda}(s)+\lambda \beta\left(J_{\lambda}(s)\right) \ni s \equiv J_{\lambda}(s)+\lambda \beta_{\lambda}(s) \quad \text { for all } s \in \mathbb{R}
$$

Indeed, by combining (2.16)-(2.18), which imply

$$
\widehat{\beta}(s)+\widehat{\pi}(s) \geq 2 \alpha s^{2}-c \quad \text { for all } s \in \mathbb{R}
$$

and for some constant $\alpha>0$, along with (3.2) and the Taylor formula with integral remainder to estimate the difference $\widehat{\pi}(s)-\widehat{\pi}\left(J_{\lambda}(s)\right)$, one can arrive at (3.1).

The approximating problem on any finite time integral $(0, T)$ is obtained by replacing $\widehat{\beta}$ in (2.27) by $\widehat{\beta}_{\lambda}$, namely,

$$
\begin{align*}
& \left\langle\partial_{t} y^{\lambda}(t), v\right\rangle_{A, r}+\left(A^{r} \mu^{\lambda}(t), A^{r} v\right)=0 \quad \text { for every } v \in V_{A}^{r} \text { and for a.a. } t \in(0, T),  \tag{3.3}\\
& \left(\tau \partial_{t} y^{\lambda}(t), y^{\lambda}(t)-v\right)+\left(B^{\sigma} y^{\lambda}(t), B^{\sigma}\left(y^{\lambda}(t)-v\right)\right) \\
& \quad+\int_{\Omega} \widehat{\beta}_{\lambda}\left(y^{\lambda}(t)\right)+\left(\pi\left(y^{\lambda}(t)\right)-u(t), y^{\lambda}(t)-v\right) \\
& \leq\left(\mu^{\lambda}(t), y^{\lambda}(t)-v\right)+\int_{\Omega} \widehat{\beta}_{\lambda}(v) \quad \text { for every } v \in V_{B}^{\sigma} \text { and for a.a. } t \in(0, T),  \tag{3.4}\\
& y^{\lambda}(0)=y_{0} . \tag{3.5}
\end{align*}
$$

In principle, the regularity required for the solution $\left(y^{\lambda}, \mu^{\lambda}\right)$ is still given by (2.23)-(2.25). However, due to the Lipschitz continuity of $\beta_{\lambda}$, (2.25) can be improved. Namely, (2.23) implies $\beta_{\lambda}\left(y^{\lambda}\right) \in L^{2}(0, T ; H)$. Using this and the fact that $\widehat{\beta}_{\lambda}$ is differentiable and $\beta_{\lambda}$ is its derivative, one sees that, in place of (3.4), one can equivalently consider the pointwise variational equation

$$
\begin{equation*}
\left(\tau \partial_{t} y^{\lambda}(t), v\right)+\left(B^{\sigma} y^{\lambda}(t), B^{\sigma} v\right)+\left(\beta_{\lambda}\left(y^{\lambda}(t)\right)+\pi\left(y^{\lambda}(t)\right)-u(t), v\right)=\left(\mu^{\lambda}(t), v\right) \tag{3.6}
\end{equation*}
$$

for every $v \in V_{B}^{\sigma}$ and for a.a. $t \in(0, T)$.
In the paper of Colli et al. (2019), it was shown that the above problem is well-posed and that its unique solution $\left(y^{\lambda}, \mu^{\lambda}\right)$ converges to a solution $(y, \mu)$ to problem (2.26)-(2.28) in the weak topology associated with the regularity requirements, essentially. Moreover, the solution $\left(y^{\lambda}, \mu^{\lambda}\right)$ is obtained as the limit of suitable interpolant functions constructed by starting from the solution to a proper discrete problem. For the reader's convenience, we recall both the notation for the interpolants and the discrete problem.

Let $N$ be a positive integer and $Z$ be one of the spaces $H, V_{A}^{r}, V_{B}^{\sigma}$. We set $h:=T / N$ and $I_{n}:=((n-1) h, n h)$ for $n=1, \ldots, N$. Given $z=\left(z^{0}, z^{1}, \ldots, z^{N}\right) \in Z^{N+1}$, the piecewise constant and piecewise linear interpolants

$$
\bar{z}_{h} \in L^{\infty}(0, T ; Z), \quad \underline{z}_{h} \in L^{\infty}(0, T ; Z) \quad \text { and } \quad \widehat{z}_{h} \in W^{1, \infty}(0, T ; Z)
$$

are defined by setting

$$
\begin{align*}
& \bar{z}_{h}(t)=z^{n} \quad \text { and } \quad \underline{z}_{h}(t)=z^{n-1} \quad \text { for a.a. } t \in I_{n}, n=1, \ldots, N,  \tag{3.7}\\
& \widehat{z}_{h}(0)=z_{0} \quad \text { and } \quad \partial_{t} \widehat{z}_{h}(t)=\frac{z^{n+1}-z^{n}}{h} \quad \text { for a.a. } t \in I_{n}, n=1, \ldots, N . \tag{3.8}
\end{align*}
$$

The discrete problem consists in finding two $(N+1)$-tuples $\left(y^{0}, \ldots, y^{N}\right)$ and $\left(\mu^{0}, \ldots, \mu^{N}\right)$ satisfying

$$
\begin{equation*}
y^{0}=y_{0}, \quad \mu^{0}=0, \quad\left(y^{1}, \ldots, y^{N}\right) \in\left(V_{B}^{2 \sigma}\right)^{N} \quad \text { and } \quad\left(\mu^{1}, \ldots, \mu^{N}\right) \in\left(V_{A}^{2 r}\right)^{N} \tag{3.9}
\end{equation*}
$$

and solving

$$
\begin{align*}
& \frac{y^{n+1}-y^{n}}{h}+\mu^{n+1}+A^{2 r} \mu^{n+1}=\mu^{n}  \tag{3.10}\\
& \tau \frac{y^{n+1}-y^{n}}{h}+\left(L_{\pi}^{\prime} I+B^{2 \sigma}+\beta_{\lambda}+\pi\right)\left(y^{n+1}\right)=L_{\pi}^{\prime} y^{n}+\mu^{n+1}+u^{n+1} \tag{3.11}
\end{align*}
$$

for $n=0,1, \ldots, N-1$, where $I: H \rightarrow H$ is the identity, $L_{\pi}^{\prime}$ is given by (2.19), and

$$
\begin{equation*}
u^{n}:=u(n h) \quad \text { for } n=0,1, \ldots, N . \tag{3.12}
\end{equation*}
$$

Precisely, it has been proved that such a discrete problem is uniquely solvable. Moreover, as just said, some of the interpolants defined above by starting from the discete solution converge to the solution $\left(y^{\lambda}, \mu^{\lambda}\right)$ to the regularized problem (3.3)-(3.5).

Now, we start estimating. It is understood that the assumptions of Theorem 2.2 are in force. In particular, every constant $c$ we introduce will depend only on these assumptions. We closely follow the lines of Colli et al. (2019). However, we modify the argument a little and obtain estimates that are uniform with respect to $T$. In doing this modification, we also avoid using the regularity condition $\partial_{t} u \in L^{2}(0, T ; H)$, which was supposed in the paper of Colli et al. (2019), and just owe to the regularity $\partial_{t} u \in L^{1}(0, T ; H)$ (but uniformly with respect to $T$ in the sense of (2.20) in order to obtain a global-in-time estimate). Since this is the only point of Colli et al. (2019) where the $L^{2}(0, T ; H)$ regularity for $\partial_{t} u$ is accounted for, the well-posedness result in Theorem $\mathbf{2 . 1}$ holds under our assumption (2.20), as announced before its statement.

First uniform estimate. We test (3.10) and (3.11) (by taking the scalar product in $H$ ) by $h \mu^{n+1}$ and $y^{n+1}-y^{n}$, respectively, and add the resulting identities. Noting an obvious cancellation, we obtain the equation

$$
\begin{aligned}
& h\left(\mu^{n+1}-\mu^{n}, \mu^{n+1}\right)+h\left(A^{2 r} \mu^{n+1}, \mu^{n+1}\right)+\frac{\tau}{h}\left\|y^{n+1}-y^{n}\right\|^{2} \\
& +\left(B^{2 \sigma} y^{n+1}, y^{n+1}-y^{n}\right)+\left(\left(L_{\pi}^{\prime} I+\beta_{\lambda}+\pi\right)\left(y^{n+1}\right), y^{n+1}-y^{n}\right) \\
& =L_{\pi}^{\prime}\left(y^{n}, y^{n+1}-y^{n}\right)+\left(u^{n+1}, y^{n+1}-y^{n}\right) .
\end{aligned}
$$

Now, we observe that the function $r \mapsto \frac{L_{\pi}^{\prime}}{2} r^{2}+\widehat{\beta}_{\lambda}(r)+\widehat{\pi}(r)$ is convex on $\mathbb{R}$, since $\widehat{\beta}_{\lambda}$ is convex and $\left|\pi^{\prime}\right| \leq L_{\pi}$. Thus, we have that

$$
\begin{aligned}
& \left(\left(L_{\pi}^{\prime} I+\beta_{\lambda}+\pi\right)\left(y^{n+1}\right), y^{n+1}-y^{n}\right) \\
& \geq \frac{L_{\pi}^{\prime}}{2}\left\|y^{n+1}\right\|^{2}+\int_{\Omega}\left(\widehat{\beta}_{\lambda}\left(y^{n+1}\right)+\widehat{\pi}\left(y^{n+1}\right)\right)-\frac{L_{\pi}^{\prime}}{2}\left\|y^{n}\right\|^{2}-\int_{\Omega}\left(\widehat{\beta}_{\lambda}\left(y^{n}\right)+\widehat{\pi}\left(y^{n}\right)\right) .
\end{aligned}
$$

We easily deduce that

$$
\begin{aligned}
\frac{h}{2} & \left\|\mu^{n+1}\right\|^{2}+\frac{h}{2}\left\|\mu^{n+1}-\mu^{n}\right\|^{2}-\frac{h}{2}\left\|\mu^{n}\right\|^{2}+h\left\|A^{r} \mu^{n+1}\right\|^{2} \\
& +\frac{\tau}{h}\left\|y^{n+1}-y^{n}\right\|^{2}+\frac{1}{2}\left\|B^{\sigma} y^{n+1}\right\|^{2}+\frac{1}{2}\left\|B^{\sigma}\left(y^{n+1}-y^{n}\right)\right\|^{2}-\frac{1}{2}\left\|B^{\sigma} y^{n}\right\|^{2} \\
& +\frac{L_{\pi}^{\prime}}{2}\left\|y^{n+1}\right\|^{2}+\int_{\Omega}\left(\widehat{\beta}_{\lambda}\left(y^{n+1}\right)+\widehat{\pi}\left(y^{n+1}\right)\right)-\frac{L_{\pi}^{\prime}}{2}\left\|y^{n}\right\|^{2}-\int_{\Omega}\left(\widehat{\beta}_{\lambda}\left(y^{n}\right)+\widehat{\pi}\left(y^{n}\right)\right) \\
\leq & -\frac{L_{\pi}^{\prime}}{2}\left(\left\|y^{n}\right\|^{2}-\left\|y^{n+1}\right\|^{2}+\left\|y^{n+1}-y^{n}\right\|^{2}\right)+\left(u^{n+1}, y^{n+1}-y^{n}\right) .
\end{aligned}
$$

Then, we first rearrange and then sum up for $n=0, \ldots, k-1$ with $k \leq N$, employing summation by parts in the last term. We thus arrive at the inequality

$$
\begin{align*}
& \frac{h}{2}\left\|\mu^{k}\right\|^{2}+\sum_{n=0}^{k-1} \frac{h}{2}\left\|\mu^{n+1}-\mu^{n}\right\|^{2}+\sum_{n=0}^{k-1} h\left\|A^{r} \mu^{n+1}\right\|^{2} \\
& \quad+\tau \sum_{n=0}^{k-1} h\left\|\frac{y^{n+1}-y^{n}}{h}\right\|^{2}+\frac{1}{2}\left\|B^{\sigma} y^{k}\right\|^{2}-\frac{1}{2}\left\|B^{\sigma} y_{0}\right\|^{2}+\sum_{n=0}^{k-1} \frac{1}{2}\left\|B^{\sigma}\left(y^{n+1}-y^{n}\right)\right\|^{2} \\
& \quad+\int_{\Omega}\left(\widehat{\beta}_{\lambda}\left(y^{k}\right)+\widehat{\pi}\left(y^{k}\right)\right)-\int_{\Omega}\left(\widehat{\beta}\left(y_{0}\right)+\widehat{\pi}\left(y_{0}\right)\right)+\frac{L_{\pi}^{\prime}}{2} \sum_{n=0}^{k-1}\left\|y^{n+1}-y^{n}\right\|^{2} \\
& \leq\left(u^{k}, y^{k}\right)-\left(u^{1}, y_{0}\right)-\sum_{n=1}^{k-1}\left(u^{n+1}-u^{n}, y^{n}\right) . \tag{3.13}
\end{align*}
$$

Next, we observe that (3.1) implies that

$$
\int_{\Omega}\left(\widehat{\beta}_{\lambda}\left(y^{k}\right)+\widehat{\pi}\left(y^{k}\right)\right) \geq \frac{1}{2} \int_{\Omega}\left(\widehat{\beta}_{\lambda}\left(y^{k}\right)+\widehat{\pi}\left(y^{k}\right)\right)+\frac{\alpha}{2}\left\|y^{k}\right\|^{2}-c
$$

for every sufficiently small $\lambda>0$ and that the integrals are bounded from below. Moreover, we differently deal with the right-hand side of (3.13) with respect to Colli et al. (2019). Namely, we estimate it as follows:

$$
\begin{aligned}
& \left(u^{k}, y^{k}\right)-\left(u^{1}, y_{0}\right)-\sum_{n=1}^{k-1}\left(u^{n+1}-u^{n}, y^{n}\right) \\
& \leq \frac{\alpha}{4}\left\|y^{k}\right\|^{2}+\frac{1}{\alpha}\left\|u^{k}\right\|^{2}+\left\|u^{1}\right\|\left\|y_{0}\right\|+\sum_{n=1}^{k-1} h\left\|\frac{u^{n+1}-u^{n}}{h}\right\|\left\|y^{n}\right\| .
\end{aligned}
$$

At this point, we combine (3.13) with the inequalities just obtained and apply the discrete Gronwall-Bellman lemma given by Yeh (1985, Thm. 1) by observing that

$$
\left\|u^{k}\right\| \leq\|u(0)\|+\left\|\partial_{t} u\right\|_{L^{1}(0,+\infty ; H)}, \quad \text { and } \quad \sum_{n=1}^{k-1} h\left\|\frac{u^{n+1}-u^{n}}{h}\right\| \leq\left\|\partial_{t} u\right\|_{L^{1}(0,+\infty ; H)},
$$

and that the above norm of $\partial_{t} u$ is finite by (2.20). We obtain the estimate

$$
\begin{align*}
& h\left\|\mu^{k}\right\|^{2}+\sum_{n=0}^{k-1} h\left\|\mu^{n+1}-\mu^{n}\right\|^{2}+\sum_{n=0}^{k-1} h\left\|A^{r} \mu^{n+1}\right\|^{2}+\tau \sum_{n=0}^{k-1} h\left\|\frac{y^{n+1}-y^{n}}{h}\right\|^{2} \\
& \quad+\left\|y^{k}\right\|_{B, \sigma}^{2}+\sum_{n=0}^{k-1}\left\|B^{\sigma}\left(y^{n+1}-y^{n}\right)\right\|^{2}+\int_{\Omega}\left(\widehat{\beta}_{\lambda}\left(y^{k}\right)+\widehat{\pi}\left(y^{k}\right)\right)+\sum_{n=0}^{k-1}\left\|y^{n+1}-y^{n}\right\|^{2} \\
& \leq c \text { for } k=0, \ldots, N . \tag{3.14}
\end{align*}
$$

In terms of the interpolants (see also Colli et al. (2019, Prop. 3.9)), by neglecting the first contribution and recalling that $\mu^{0}=0$, we have that

$$
\begin{align*}
& \left\|\bar{\mu}_{h}-\underline{\mu}_{h}\right\|_{L^{2}(0, T ; H)}+\left\|A^{r} \bar{\mu}_{h}\right\|_{L^{2}(0, T ; H)}+\left\|A^{r} \underline{\mu}_{h}\right\|_{L^{2}(0, T ; H)} \\
& \quad+\left\|\underline{y}_{h}\right\|_{L^{\infty}\left(0, T ; V_{B}^{\sigma}\right)}+\left\|\bar{y}_{h}\right\|_{L^{\infty}\left(0, T ; V_{B}^{\sigma}\right)}+\left\|\widehat{y}_{h}\right\|_{L^{\infty}\left(0, T ; V_{B}^{\sigma}\right)} \\
& \quad+h^{-1 / 2}\left\|B^{\sigma}\left(\bar{y}_{h}-\underline{y}_{h}\right)\right\|_{L^{2}(0, T ; H)}+\tau^{1 / 2}\left\|\partial_{t} \widehat{y}_{h}\right\|_{L^{2}(0, T ; H)} \\
& \quad+\left\|\widehat{\beta}_{\lambda}\left(\bar{y}_{h}\right)+\widehat{\pi}\left(\bar{y}_{h}\right)\right\|_{L^{\infty}\left(0, T ; L^{1}(\Omega)\right)}+h^{-1 / 2}\left\|\bar{y}_{h}-\widehat{y}_{h}\right\|_{L^{2}(0, T ; H)} \leq c . \tag{3.15}
\end{align*}
$$

Second uniform estimate. By observing that (3.10) implies $\partial_{t} \widehat{y}_{h}+\bar{\mu}_{h}+A^{2 r} \bar{\mu}_{h}=\underline{\mu}_{h}$, whence also

$$
\begin{aligned}
& \int_{0}^{T}\left(\partial_{t} \widehat{y}_{h}(t), v(t)\right) d t=\int_{0}^{T}\left(\left(\underline{\mu}_{h}-\bar{\mu}_{h}\right)(t), v(t)\right) d t-\int_{0}^{T}\left(A^{r} \bar{\mu}_{h}(t), A^{r} v(t)\right) d t \\
& \leq c\left(\left\|\underline{\mu}_{h}-\bar{\mu}_{h}\right\|_{L^{2}(0, T ; H)}+\left\|A^{r} \bar{\mu}_{h}\right\|_{L^{2}(0, T ; H)}\right)\|v\|_{L^{2}\left(0, T ; V_{A}^{r}\right)}
\end{aligned}
$$

for every $v \in L^{2}\left(0, T ; V_{A}^{r}\right)$, we deduce that

$$
\left\|\partial_{t} \widehat{y}_{h}\right\|_{L^{2}\left(0, T ; V_{A}^{-r}\right)} \leq c\left(\left\|\underline{\mu}_{h}-\bar{\mu}_{h}\right\|_{L^{2}(0, T ; H)}+\left\|A^{r} \bar{\mu}_{h}\right\|_{L^{2}(0, T ; H)}\right) .
$$

Hence, from (3.15) we infer that

$$
\begin{equation*}
\left\|\partial_{t} \widehat{y}_{h}\right\|_{L^{2}\left(0, T ; V_{A}^{-r}\right)} \leq c . \tag{3.16}
\end{equation*}
$$

Basic estimate. We recall that estimates (3.15)-(3.16) hold for every $N>1$, every sufficiently small $\lambda>0$, and every $T>0$. Now, we owe to the convergence results of Colli et al. (2019). We deduce that

$$
\left\|y^{\lambda}\right\|_{L^{\infty}\left(0, T ; ;_{B}^{\sigma}\right)}^{\sigma}+\left\|\partial_{t} y^{\lambda}\right\|_{L^{2}\left(0, T ; V_{A}^{-r}\right)}+\tau^{1 / 2}\left\|\partial_{t} y^{\lambda}\right\|_{L^{2}(0, T ; H)}+\left\|A^{r} \mu^{\lambda}\right\|_{L^{2}(0, T ; H)} \leq c .
$$

Since $c$ is independent of both $\lambda$ and $T$, at the limit as $\lambda \searrow 0$ we conclude that

$$
\begin{gather*}
y \in L^{\infty}\left(0,+\infty ; V_{B}^{\sigma}\right), \quad \partial_{t} y \in L^{2}\left(0,+\infty ; V_{A}^{-r}\right), \quad \text { and } A^{r} \mu \in L^{2}(0,+\infty ; H),  \tag{3.17}\\
\partial_{t} y \in L^{2}(0,+\infty ; H) \quad \text { if } \tau>0 . \tag{3.18}
\end{gather*}
$$

## 4. Longtime behavior

This section is devoted to the proofs of our results on the longtime behavior. We start with the proof of Theorem 2.2.

First part. Since $y$ belongs to $L^{\infty}\left(0,+\infty ; V_{B}^{\sigma}\right)$ by the first conclusion of (3.17), we deduce that the $\omega$-limit $\omega$ given by ( $\mathbf{2 . 3 0}$ ) is nonempty. Thus, the first sentence of our result is established. Let us come to the second part.

Second part, first case. We first assume that $\lambda_{1}>0$. We pick an arbitrary element $y_{\omega} \in \omega$ and a sequence $\left\{t_{n}\right\}$ as in (2.30), and we prove that $y_{\omega}$ is a stationary solution in the sense of (2.32). To this end, we define the functions $y_{n}, \mu_{n}$, and $u_{n}$, on $(0,+\infty)$ by setting, for a.a. $t \in(0,+\infty)$,

$$
y_{n}(t):=y\left(t+t_{n}\right), \quad \mu_{n}(t):=\mu\left(t+t_{n}\right), \quad \text { and } \quad u_{n}(t):=u\left(t+t_{n}\right) .
$$

We notice that (2.20) and (2.31) imply that

$$
\begin{equation*}
\left\|u_{n}\right\|_{L^{\infty}(0,+\infty ; H)} \leq c, \quad \text { and } \quad u_{n}-u_{\infty} \rightarrow 0 \quad \text { strongly in } L^{2}(0,+\infty ; H) . \tag{4.1}
\end{equation*}
$$

Moreover, from (3.17) we clearly deduce that

$$
\begin{align*}
& \left\|y_{n}\right\|_{L^{\infty}\left(0,+\infty ; V_{B}^{\sigma}\right)} \leq c,  \tag{4.2}\\
& \partial_{t} y_{n} \rightarrow 0 \quad \text { strongly in } L^{2}\left(0,+\infty ; V_{A}^{-r}\right),  \tag{4.3}\\
& A^{r} \mu_{n} \rightarrow 0 \quad \text { strongly in } L^{2}(0,+\infty ; H), \tag{4.4}
\end{align*}
$$

whence also

$$
\begin{equation*}
\mu_{n} \rightarrow 0 \quad \text { strongly in } L^{2}\left(0,+\infty ; V_{A}^{r}\right), \tag{4.5}
\end{equation*}
$$

since $\lambda_{1}>0$. In addition, we have that

$$
\begin{equation*}
\partial_{t} y_{n} \rightarrow 0 \quad \text { strongly in } L^{2}(0,+\infty ; H) \quad \text { if } \tau>0 \tag{4.6}
\end{equation*}
$$

By weak-star compactness, we deduce from (4.2) that there exists some element $y_{\infty} \in$ $L^{\infty}\left(0,+\infty ; V_{B}^{\sigma}\right)$ such that

$$
\begin{equation*}
y_{n} \rightarrow y_{\infty} \quad \text { weakly star in } L^{\infty}\left(0,+\infty ; V_{B}^{\sigma}\right), \tag{4.7}
\end{equation*}
$$

at least for a (not relabeled) subsequence. Now, we fix an arbitrary time $T>0$ and look for the problem solved by $y_{\infty}$ on $(0, T)$. It is clear that $\left(y_{n}, \mu_{n}\right)$ satisfies the variational inequality

$$
\begin{align*}
& \left(\tau \partial_{t} y_{n}(t), y_{n}(t)-v\right)+\left(B^{\sigma} y_{n}(t), B^{\sigma}\left(y_{n}(t)-v\right)\right) \\
& \quad+\int_{\Omega} \widehat{\beta}\left(y_{n}(t)\right)+\left(\pi\left(y_{n}(t)\right)-u_{n}(t), y_{n}(t)-v\right) \leq\left(\mu_{n}(t), y_{n}(t)-v\right)+\int_{\Omega} \widehat{\beta}(v) \\
& \quad \text { for every } v \in V_{B}^{\sigma} \text { and for a.a. } t \in(0, T) \tag{4.8}
\end{align*}
$$

as well as its integrated version

$$
\begin{align*}
& \int_{0}^{T}\left(\tau \partial_{t} y_{n}(t), y_{n}(t)-v(t)\right) d t+\int_{0}^{T}\left(B^{\sigma} y_{n}(t), B^{\sigma}\left(y_{n}(t)-v(t)\right)\right) d t \\
& \quad+\int_{\Omega \times(0, T)} \widehat{\beta}\left(y_{n}\right)+\int_{0}^{T}\left(\pi\left(y_{n}(t)\right)-u_{n}(t), y_{n}(t)-v(t)\right) d t \\
& \leq \int_{0}^{T}\left(\mu_{n}(t), y_{n}(t)-v(t)\right) d t+\int_{\Omega \times(0, T)} \widehat{\beta}(v) \quad \text { for every } v \in L^{2}\left(0, T ; V_{B}^{\sigma}\right) . \tag{4.9}
\end{align*}
$$

Now, we want to let $n$ tend to infinity in (4.9). First, by (4.8) with $v=0$, we have that

$$
\begin{aligned}
& \left\|\widehat{\beta}\left(y_{n}(t)\right)\right\|_{L^{1}(\Omega)} \leq\left\|B^{\sigma} y_{n}(t)\right\|^{2}+\int_{\Omega} \widehat{\beta}\left(y_{n}(t)\right) \\
& \leq\left(\left\|\tau \partial_{t} y_{n}(t)\right\|+\left\|\pi\left(y_{n}(t)\right)\right\|+\left\|u_{n}(t)\right\|+\left\|\mu_{n}(t)\right\|\right)\left\|y_{n}(t)\right\| \quad \text { for a.a. } t \in(0, T)
\end{aligned}
$$

So, by accounting for the Lipschitz continuity of $\pi$, and owing to (4.1)-(4.5), we obtain that

$$
\begin{equation*}
\left\|\widehat{\beta}\left(y_{n}\right)\right\|_{L^{2}\left(0, T ; L^{1}(\Omega)\right)} \leq c_{T} \tag{4.10}
\end{equation*}
$$

On the other hand, by recalling (4.2), (4.3) and the compact embedding $V_{B}^{\sigma} \subset H$ that follows from (2.2), we can apply Simon (1986, Sect. 8, Cor. 4) and deduce that

$$
\begin{equation*}
y_{n} \rightarrow y_{\infty} \quad \text { strongly in } C^{0}([0, T] ; H) \tag{4.11}
\end{equation*}
$$

We infer that $\pi\left(y_{n}\right)$ converges to $\pi\left(y_{\infty}\right)$ in the same topology since $\pi$ is Lipschitz continuous. In order to deal with the nonlinearity $\widehat{\beta}$, we notice that we can assume that $y_{n} \rightarrow y_{\infty}$ a.e. in $\Omega \times(0, T)$ so that, by lower semicontinuity, we deduce the inequality

$$
\int_{\Omega \times(0, T)} \widehat{\beta}\left(y_{\infty}\right) \leq \liminf _{n \rightarrow \infty} \int_{\Omega \times(0, T)} \widehat{\beta}\left(y_{n}\right)
$$

where the last term is finite by $(\mathbf{4 . 1 0})$. As (4.7) also implies that

$$
\int_{0}^{T}\left\|B^{\sigma} y_{\infty}(t)\right\|^{2} d t \leq \liminf _{n \rightarrow \infty} \int_{0}^{T}\left\|B^{\sigma} y_{n}(t)\right\|^{2} d t
$$

and since the second statement in (4.1) yields that $u_{n} \rightarrow u_{\infty}$ strongly in $L^{2}(0, T ; H)$, from (4.9) and (4.7) it follows that $y_{\infty}$ satisfies the variational inequality

$$
\begin{align*}
& \int_{0}^{T}\left(B^{\sigma} y_{\infty}(t), B^{\sigma}\left(y_{\infty}(t)-v(t)\right)\right) d t+\int_{\Omega \times(0, T)} \widehat{\beta}\left(y_{\infty}\right) \\
& \quad+\int_{0}^{T}\left(\pi\left(y_{\infty}(t)\right)-u_{\infty}, y_{\infty}(t)-v(t)\right) d t \\
& \leq \int_{\Omega \times(0, T)} \widehat{\beta}(v) \quad \text { for every } v \in L^{2}\left(0, T ; V_{B}^{\sigma}\right) . \tag{4.12}
\end{align*}
$$

Equivalently, $y_{\infty}$ fulfills

$$
\begin{equation*}
\left(B^{\sigma} y_{\infty}(t), B^{\sigma}\left(y_{\infty}(t)-v\right)\right)+\int_{\Omega} \widehat{\beta}\left(y_{\infty}(t)\right)+\left(\pi\left(y_{\infty}(t)\right)-u_{\infty}, y_{\infty}(t)-v\right) \leq \int_{\Omega} \widehat{\beta}(v) \tag{4.13}
\end{equation*}
$$

for every $v \in V_{B}^{\sigma}$ and for a.a. $t \in(0, T)$.
At this point, we can easily conclude. In view of (4.3) and (4.7), we have that

$$
\partial_{t} y_{\infty}=0, \quad \text { whence } y_{\infty} \text { takes a constant value } \bar{y} \in V_{B}^{\sigma} \text { on }[0, T] .
$$

On the other hand, $y_{n}(0)$ converges to $y_{\infty}(0)$ in $H$ by (4.11). Thus, $y_{n}(0)$ converges to $\bar{y}$ in $H$. As $y_{n}(0)=y\left(t_{n}\right)$ converges weakly to $y_{\omega}$ in $V_{B}^{\sigma}$ by assumption, we conclude that $\bar{y}=y_{\omega}$, that is,

$$
\begin{equation*}
y_{\infty}(t)=y_{\omega} \quad \text { for every } t \in[0, T] . \tag{4.14}
\end{equation*}
$$

Therefore, (4.13) becomes (2.32).
Second part, second case. Assume now that $\lambda_{1}=0$. Coming back to the proof just concluded, we see that the assumption $\lambda_{1}>0$ has been used just to obtain (4.5), its consequence (4.10), and to make $\mu_{n}$ disappear in the limiting inequality (2.32). Therefore, the same argument essentially applies in the case $\lambda_{1}=0$ (with the modifications that are needed to prove (2.33) instead of (2.32)), provided we can derive a convergence property for $\mu_{n}$ (in place of (4.5), which should be false now) and (4.10). To this end, we recall assumption (2.22) and notice that it implies the existence of some $\delta>0$ such that $m_{0} \pm \delta$ belong to $D(\beta)$. Then, as $v$ in (4.8), we choose the convex combination $\frac{1}{2}\left(m_{0} \pm \delta\right)+\frac{1}{2} y_{n}(t)$ (thus, with values in $D(\widehat{\beta})$ ), which gives $y_{n}(t)-v=\frac{1}{2}\left(y_{n}(t)-m_{0} \mp \boldsymbol{\delta}\right)$. Thanks to the convexity of $\widehat{\beta}$, we obtain for a.a. $t \in(0, T)$

$$
\begin{aligned}
& \frac{\tau}{2}\left(\partial_{t} y_{n}(t), y_{n}(t)-m_{0} \mp \delta\right)+\frac{1}{2}\left(B^{\sigma} y_{n}(t), B^{\sigma}\left(y_{n}(t)-m_{0} \mp \delta\right)\right) \\
& \quad+\int_{\Omega} \widehat{\beta}\left(y_{n}(t)\right)+\frac{1}{2}\left(\pi\left(y_{n}(t)\right)-u_{n}(t), y_{n}(t)-m_{0} \mp \delta\right) \\
& \leq \frac{1}{2}\left(\mu_{n}(t), y_{n}(t)-m_{0} \mp \delta\right)+\int_{\Omega} \widehat{\beta}\left(\frac{1}{2}\left(m_{0} \pm \delta\right)+\frac{1}{2} y_{n}(t)\right) \\
& \leq \frac{1}{2}\left(\mu_{n}(t), y_{n}(t)-m_{0} \mp \delta\right)+\frac{1}{2} \int_{\Omega} \widehat{\beta}\left(y_{n}(t)\right)+\frac{1}{2} \int_{\Omega} \widehat{\beta}\left(m_{0} \pm \delta\right) .
\end{aligned}
$$

By multiplying by 2 and rearranging, we deduce that

$$
\begin{align*}
\pm & \delta\left(\mu_{n}(t), 1\right)+\left\|B^{\sigma} y_{n}(t)\right\|^{2}+\int_{\Omega} \widehat{\beta}\left(y_{n}(t)\right) \\
\leq & -\tau\left(\partial_{t} y_{n}(t), y_{n}(t)-m_{0} \mp \boldsymbol{\delta}\right)+\left(B^{\sigma} y_{n}(t), B^{\sigma}\left(m_{0} \pm \boldsymbol{\delta}\right)\right) \\
& -\left(\pi\left(y_{n}(t)\right)-u_{n}(t), y_{n}(t)-m_{0} \mp \boldsymbol{\delta}\right)+\int_{\Omega} \widehat{\beta}\left(m_{0} \pm \boldsymbol{\delta}\right) \\
& +\left(\mu_{n}(t), y_{n}(t)-m_{0}\right) . \tag{4.15}
\end{align*}
$$

Now, we recall the conservation property (2.29) and note that the Poincaré type inequality (2.15) is valid since $\lambda_{1}=0$. We thus have that for a.a. $t \in(0, T)$ it holds

$$
\begin{aligned}
& \left(\mu_{n}(t), y_{n}(t)-m_{0}\right)=\left(\mu_{n}(t)-\operatorname{mean} \mu_{n}(t), y_{n}(t)-m_{0}\right) \\
& \leq c\left\|A^{r}\left(\mu_{n}(t)-\operatorname{mean} \mu_{n}(t)\right)\right\|\left\|y_{n}(t)-m_{0}\right\|=c\left\|A^{r} \mu_{n}(t)\right\|\left\|y_{n}(t)-m_{0}\right\|
\end{aligned}
$$

so that we can use (4.4) in the right-hand side of (4.15). By also accounting for (4.1), (4.2) and (4.6), we deduce that the function

$$
t \mapsto \delta\left|\left(\mu_{n}(t), 1\right)\right|+\int_{\Omega} \widehat{\beta}\left(y_{n}(t)\right)
$$

is bounded in $L^{2}(0, T)$, uniformly with respect to $n$. In particular, (4.10) holds also in this case. Moreover, the mean value of $\mu_{n}$ is estimated in $L^{2}(0, T)$ so that the definition (2.13) of the norm in $V_{A}^{r}$ and (4.4) imply that $\mu_{n}$ is bounded in $L^{2}\left(0, T ; V_{A}^{r}\right)$. Therefore, we have, at least for a subsequence,

$$
\begin{equation*}
\mu_{n} \rightarrow \mu_{\infty} \quad \text { weakly in } L^{2}\left(0, T ; V_{A}^{r}\right) \tag{4.16}
\end{equation*}
$$

which is the desired convergence property to be established in place of (4.5). At this point, we repeat the argument used in the case $\lambda_{1}>0$ provided that we modify (4.13), since we have (4.16) instead of (4.5). In place of that variational inequality, we obtain the following one:

$$
\begin{aligned}
& \left(B^{\sigma} y_{\infty}(t), B^{\sigma}\left(y_{\infty}(t)-v\right)\right)+\int_{\Omega} \widehat{\beta}\left(y_{\infty}(t)\right)+\left(\pi\left(y_{\infty}(t)\right)-u_{\infty}, y_{\infty}(t)-v\right) \\
& \leq\left(\mu_{\infty}(t), y_{\infty}(t)-v\right)+\int_{\Omega} \widehat{\beta}(v)
\end{aligned}
$$

$$
\begin{equation*}
\text { for every } v \in V_{B}^{\sigma} \text { and for a.a. } t \in(0, T) \text {. } \tag{4.17}
\end{equation*}
$$

On the contrary, (4.14) holds true with the same proof also in the present case. Finally, (4.4) implies that $A^{r} \mu_{\infty}(t)=0$ for a.a. $t \in(0, T)$, i.e., that $\mu_{\infty}$ is space independent since $\lambda_{1}=0$ (cf. (2.11)). Therefore, (4.17) becomes

$$
\begin{align*}
& \left(B^{\sigma} y_{\omega}, B^{\sigma}\left(y_{\omega}-v\right)\right)+\int_{\Omega} \widehat{\beta}\left(y_{\omega}\right)+\left(\pi\left(y_{\omega}\right)-u_{\infty}, y_{\omega}-v\right) \\
& \leq\left(\mu_{\infty}(t), y_{\omega}-v\right)+\int_{\Omega} \widehat{\beta}(v) \\
& \quad \text { for every } v \in V_{B}^{\sigma} \text { and for a.a. } t \in(0, T) . \tag{4.18}
\end{align*}
$$

Hence, as (4.18) holds for arbitrary values of $T$, this implies the validity of (2.33) with a proper function $\mu_{\infty} \in L_{\text {loc }}^{\infty}([0,+\infty))$. Indeed, let us denote for $m=0,1, \ldots$ by $\mu_{\infty}^{m}$ the function $\mu_{\infty}$ which satisfies (4.18) with $T=m$, and we construct $\mu_{\infty}$ on $(0,+\infty)$ by setting $\mu_{\infty}(t):=\mu_{\infty}^{m}(t)$ for a.a. $t \in(m, m+1)$, for $m=0,1, \ldots$. Then, it turns out that
$\mu_{\infty} \in L_{l o c}^{\infty}([0,+\infty))$, and the inequality (2.33) clearly holds. This completes the proof of Theorem 2.2.
Proof of Proposition 2.4. We come back to the proof just concluded, keeping its notation. More precisely, we consider the part of the proof in the first case $i$ ) that also holds for the second one. For the first part of Proposition 2.4, it is sufficient to show that, for every $T \in(0,+\infty)$, the function $\mu_{\infty}$ is unique and constant on $(0, T)$. So, we fix an arbitrary $T>0$. By virtue of the results of Colli et al. (2018b) summarized just before the statement we are considering, we can replace the variational inequality (2.27) by the equation (2.37), so that (4.8) can be written in the strong form

$$
\begin{equation*}
\tau \partial_{t} y_{n}+B^{2 \sigma} y_{n}+\beta\left(y_{n}\right)+\pi\left(y_{n}\right)=\mu_{n}+u_{n} \quad \text { a.e. in } \Omega \times(0, T) . \tag{4.19}
\end{equation*}
$$

Now, we observe that our assumption (2.35) obviously implies that $y_{n}$ takes its values in $[a, b]$. From (4.11) and the Lipschitz continuity of $\beta+\pi$ in $[a, b]$, we thus infer that

$$
(\beta+\pi)\left(y_{n}\right) \rightarrow(\beta+\pi)\left(y_{\infty}\right) \quad \text { strongly in } C^{0}([0, T] ; H) .
$$

On the other hand, by comparison in (4.19), we see that $B^{2 \sigma} y_{n}$ is in fact bounded in $L^{2}(0, T ; H)$. Therefore, the limiting function $y_{\infty}$ belongs to $V_{B}^{2 \sigma}$ and satisfies the equation

$$
B^{2 \sigma} y_{\infty}+\beta\left(y_{\infty}\right)+\pi\left(y_{\infty}\right)=\mu_{\infty}+u_{\infty} \quad \text { a.e. in } \Omega \times(0, T) .
$$

But we already know that $y_{\infty}$ takes the constant value $y_{\omega}$. Therefore, $y_{\omega} \in V_{B}^{2 \sigma}$, and we have that

$$
B^{2 \sigma} y_{\omega}+\beta\left(y_{\omega}\right)+\pi\left(y_{\omega}\right)=\mu_{\infty}(t)+u_{\infty} \quad \text { for a.a. } t \in(0, T) .
$$

By comparison, we conclude that $\mu_{\infty}$ is unique and time independent, thus constant, and the above equation becomes (2.38). This completes the proof.

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${ }^{a}$ Università di Pavia
Dipartimento di Matematica "F. Casorati"
Research Associate at the IMATI - C.N.R. Pavia 27100 Pavia, Italy
b Humboldt-Universität zu Berlin
Department of Mathematics
Unter den Linden 6, 10099 Berlin, Germany
and Weierstrass Institute for Applied Analysis and Stochastics
Mohrenstrasse 39, 10117 Berlin, Germany

* To whom correspondence should be addressed | email: pierluigi.colli@unipv.it

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