Weierstraß-Institut
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Is there an impact of small phase lags in the Kuramoto model?

Oleh Omel'chenko[1], Matthias Wolfrum[1]

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[1] Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: Oleh.Omelchenko@wias-berlin.de
Matthias.Wolfrum@wias-berlin.de

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Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

We discuss the influence of small phase lags on the synchronization transitions in the Kuramoto model for a large inhomogeneous population of globally coupled phase oscillators. Without a phase lag, all unimodal distributions of the natural frequencies give rise to a classical synchronization scenario, where above the onset of synchrony at the Kuramoto threshold there is an increasing synchrony for increasing coupling strength. We show that already for arbitrarily small phase lags there are certain unimodal distributions of natural frequencies such that for increasing coupling strength synchrony may decrease and even complete incoherence may regain stability. Moreover, our example allows a qualitative understanding of the mechanism for such non-universal synchronization transitions.

Systems of coupled oscillators play an important role in various fields. Collective rhythmical behavior can be found in biology, where examples range from cycles in the metabolism of cells, spiking of neurons, or behavior of whole organisms or populations, as well as in mechanical systems, electrochemistry, or economy. Global coupling through a common mean field, as in the classical Kuramoto model, is a particularly simple interaction structure and in this case the fundamental dynamical phenomenon is that in a population of oscillators with slightly varying natural frequencies a coupling above a certain strength can introduce a synchronous behavior. In the emerging partially synchronized state the oscillators with natural frequencies close to the maximum of the distribution are entrained, while those with more detuned natural frequencies still oscillate independently. For further increasing coupling strength more and more oscillators become synchronized. Recently, it has been shown that introducing a phase-lag parameter into the phase interaction function a counterintuitive scenario can be found where increasing coupling strength may lead to decreasing synchrony. We show here that this effect can be induced already by arbitrarily small phase-lags into the original Kuramoto system, where this effect is impossible.

1 Introduction

It has been pointed out by Kuramoto [1] that a general system of coupled limit cycle oscillators can be reduced to a system of phase oscillators assuming that they are only weakly coupled. Assuming also that the natural frequencies of the limit cycles are sufficiently large, a classical averaging procedure leads to a system of the form

$$\frac{d\theta_k}{dt} = \omega_k - \frac{K}{N}\sum_{j=1}^{N}\Gamma\left(\theta_k(t) - \theta_j(t)\right), \tag{1}$$

where the phases $\theta_k \in \mathbb{R} \bmod 2\pi$ rotate with their natural frequencies ω_k and interact through a coupling with strength K and a general phase interaction function Γ. We have chosen here a simple global coupling structure and assume that the natural frequencies are drawn at random following a given unimodal frequency distribution $g(\omega)$. Recall that a natural frequency distribution $g(\omega)$ is called *unimodal* if g is even, i.e. $g(-\omega) = g(\omega)$, and nonincreasing for $\omega \in [0, \infty)$.

It was the important achievement of Kuramoto to point out that choosing a sinusoidal coupling function opens the possibility for a deep analytical treatment of the resulting system [1]. Also more recent analytical approaches, as by Watanabe-Strogatz [2, 3] or Ott-Antonsen [4, 5], crucially rely on the fact that a general periodic coupling function Γ is replaced by its first Fourier modes only. At the other hand, it is known that many dynamical phenomena arising in globally coupled oscillator systems, such as clustering [6], heteroclinic cycles [7], chaos [8], or multiplicity of singular synchronous states [9, 10], can be only found in phase oscillators using higher harmonics in the interaction function, see also [11] for the recent overview.

At the other hand, Kuramoto and Sakaguchi already noticed that after modifying the original interaction function to

$$\Gamma(x) = \sin(x + \alpha) \tag{2}$$

by introducing the phase-lag α as an additional parameter, they were able to adapt the same analytical techniques [12] as in the case $\alpha = 0$. The variation of the phase-lag parameter α, which governs the attraction and repulsion between the oscillators, is essential for various interesting dynamical effects. In particular in spatially extended systems the emergence of self-organized patterns of coherence and incoherence, called chimera states [13, 14, 15], or macroscopic turbulence [16] can be observed only for a phase-lag α close below the value $\pi/2$, where the interaction switches into the repulsive regime.

However, for globally coupled systems and unimodal frequency distributions it was believed that the phase lag parameter does not introduce significant qualitative changes to the observed synchronization scenario, in which a single branch of partially synchronized solutions emerges at a critical coupling strength and continues with increasing synchrony above this value. Only recently [17, 18], it has been shown that one can find certain unimodal frequency distributions $g(\omega)$ giving rise to non-universal synchronization transitions. These transitions include decreasing synchronization with increasing coupling strength, incoherence regaining stability with increasing coupling strength and even coexistence of stable incoherence with a partially synchronized state. Up to now, in all examples these phenomena where observed for rather large values of the phase lag parameter, i.e. close to $\pi/2$. In the present paper, we will show that already for arbitrarily small α one can construct a unimodal frequency distribution, such that the corresponding synchronization transitions display non-universal features. It follows that the original Kuramoto model with $\alpha = 0$ is indeed singular in the sense that perturbing α to arbitrarily small positive values might already induce qualitative changes to the dynamics.

The paper is organized as follows. In Section 2 we introduce the framework of the continuum limit and recall the Ott-Antonsen reduction, which allows to derive a general form of the self-consistency equation for partially synchronized states. In Section 3 we proof our main result by explicitly constructing a specific family of unimodal frequency distributions and studying analytically the properties of its bifurcation curves. While this construction is based on the analytically tractable case of a linear superposition of two Lorentzian frequency distributions, we provide some numerical evidence that a similar scenario can be also found based on e.g. Gaussian distributions. Finally, we use these examples to give a qualitative explanation why in certain cases an increasing coupling strength may lead to a decrease of the synchrony.

2 Partially synchronized states in the continuum limit

Following the classical lines for the analytical treatment of system (1) with a large number N of oscillators [19], we pass to the limit $N \to \infty$ where we obtain the continuity equation

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \theta}(fv) = 0. \tag{3}$$

Using the Kuramoto-Sakaguchi phase interaction function (2), we can write the velocity as

$$v(\theta, \omega, t) := \omega + \frac{K}{2i}\left(e^{-i\alpha} r(t) e^{-i\theta} - e^{i\alpha}\overline{r}(t) e^{i\theta}\right) \tag{4}$$

where the continuum version of the global order parameter $r(t)$ is given by

$$r(t) = \int_{-\infty}^{\infty} d\omega \int_0^{2\pi} f(\omega, \theta, t) e^{i\theta} d\theta. \tag{5}$$

The continuity equation can be significantly simplified by restricting to the Ott-Antonsen manifold [4] of solutions of the form

$$f(\omega, \theta, t) = \frac{g(\omega)}{2\pi}\left(1 + \sum_{n=1}^{\infty}\left[\overline{z}^n(\omega, t) e^{in\theta} + z^n(\omega, t) e^{-in\theta}\right]\right), \tag{6}$$

where then the unknown complex function $z(\omega, t)$ has to satisfy the equation

$$\frac{dz}{dt} = i\omega z(\omega, t) + \frac{K}{2} e^{-i\alpha} \mathcal{G} z - \frac{K}{2} e^{i\alpha} z^2(\omega, t) \mathcal{G}\overline{z}. \tag{7}$$

Here, for any $\varphi \in C(\mathbb{R}; \mathbb{C})$ we denote by $\mathcal{G}\varphi$ the integral operator

$$\mathcal{G}\varphi := \int_{-\infty}^{\infty} g(\omega)\varphi(\omega) d\omega. \tag{8}$$

Note that only solutions with $|z| \leq 1$, which form an invariant set for equation (7), give rise to solutions of the continuity equation (3). For a more detailed exposition of this reduction, see also [18].

With respect to Eq. (7), we are interested in two types of solutions:

(i) the completely incoherent state $z(\omega, t) = 0$,

(iii) partially synchronized states $z(\omega, t) = a(\omega) e^{i\Omega t}$.

In [18] it has been shown that the amplitudes $a(\omega)$ and collective frequencies Ω of all partially synchronized states can be found from the self-consistency equation

$$\begin{aligned}\frac{1}{K}e^{i\alpha} &= \frac{i}{p}\int_{-\infty}^{\infty} g(\omega)h\left(\frac{\omega-\Omega}{p}\right)d\omega \\ &= i\int_{-\infty}^{\infty} g(\Omega+ps)h(s)ds =: H(p,\Omega)\end{aligned} \tag{9}$$

with

$$h(s) := \begin{cases} \left(1-\sqrt{1-s^{-2}}\right)s & \text{for} \quad |s|>1, \\ s-i\sqrt{1-s^2} & \text{for} \quad |s|\le 1. \end{cases} \tag{10}$$

More precisely, every pair $(p,\Omega) \in (0,\infty)\times\mathbb{R}$ determines a solution

$$z(\omega,t) = h\left(\frac{\omega-\Omega}{p}\right)e^{i\Omega t}$$

to equation (7) with $K = |H(p,\Omega)|^{-1}$ and $\alpha = \arg H(p,\Omega)$. Inserting this into formulas (5), (6) we obtain the corresponding global order parameter

$$|r| = p|H(p,\Omega)| = p/K. \tag{11}$$

Thus, using equations (9) and (11) we can describe the global structure of the synchronization transitions for different values of phase lag α, see Fig. 1 for the Lorentzian distribution

$$g(\omega) = L_\sigma(\omega) := \frac{1}{\pi}\frac{\sigma}{\omega^2+\sigma^2}. \tag{12}$$

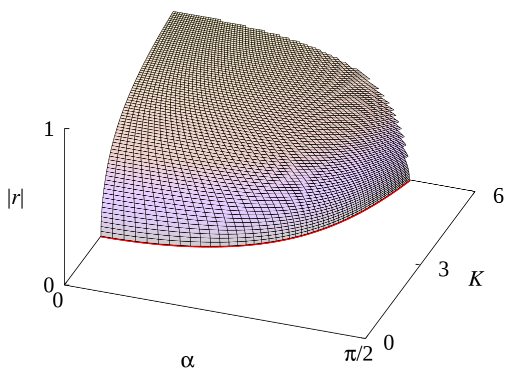

Figure 1: Order parameter $|r|$ of partially synchronized states in the Kuramoto-Sakaguchi model (1)–(2) with Lorentzian frequency distribution $L_1(\omega)$. Red line: stability boundary of the incoherent state.

In [18] it has been shown that the self-consistency equation (9) can be efficiently used to perform a bifurcation analysis for general frequency distributions $g(\omega)$. In particular, we obtain from the limit $(p \to +0)$ a bifurcation condition for the onset of synchronization, which coincides with the stability boundary of the completely incoherent state, given by the singular integral

$$\frac{1}{K}e^{i\alpha} = J(\Omega) := \frac{\pi}{2}g(\Omega) + \frac{i}{2}\lim_{\varepsilon\to+0}\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right)\frac{g(\omega+\Omega)}{\omega}d\omega. \tag{13}$$

3 Main result

In this section we give an explicit construction of a family of unimodal frequency distributions, which induce non-universal synchronization transitions for any small phase-lag α. Moreover, we show that for $\alpha = 0$ the order parameter increases monotonically along the branch of partially coherent states. To this end we start as in [17] with a linear superposition

$$g(\omega;\sigma,\tau) = \tau g_1(\omega) + (1-\tau)g_\sigma(\omega), \tag{14}$$

of two standard distributions where $\tau \in [0,1]$ governs mass balance between them and the width of the second distribution is scaled by $\sigma \in (0,1)$. Note that other versions of such superpositions can lead to symmetric but non-unimodal distributions, see [20] where multistability has been reported. A combination of two elementary distributions with different mean frequencies has been studied in [21]. A key assumption in all these cases is to choose the elementary distribution $g_\sigma(\omega)$ to be of Lorentzian type $L_\sigma(\omega)$, which allows for a explicit evaluation of the corresponding integrals in (13). However, as we will demonstrate below numerically, our construction leads to a similar scenario of nonuniversal synchronization transitions also for other choices of $g_\sigma(\omega)$.

We demonstrate now that for an appropriate choice of parameters $\sigma \to 0$ and $\tau \to 1$, we can observe nonuniversal synchronization transitions for arbitrarily small values of α.

Inserting the distribution (14) with the Lorentzian (12) into (13) we can perform a contour integration in the upper complex half-plane and obtain

$$J(\Omega) = \frac{1}{2}\left(\frac{\tau}{\Omega^2+1} + \frac{(1-\tau)\sigma}{\Omega^2+\sigma^2}\right) - \frac{i}{2}\left(\frac{\tau\Omega}{\Omega^2+1} + \frac{(1-\tau)\Omega}{\Omega^2+\sigma^2}\right). \tag{15}$$

Hence

$$\tan\alpha = F(\Omega) := \frac{\mathrm{Im}\ J(\Omega)}{\mathrm{Re}\ J(\Omega)} = -\Omega\frac{\Omega^2+A}{B\Omega^2+C},$$

where

$$\begin{aligned} A = A(\sigma,\tau) &= 1-\tau+\tau\sigma^2, \\ B = B(\sigma,\tau) &= \tau+\sigma-\sigma\tau, \\ C = C(\sigma,\tau) &= \sigma(1-\tau+\sigma\tau). \end{aligned}$$

It is easy to verify that $F(0) = 0$ and $F(\Omega) \to +\infty$ for $\Omega \to -\infty$. Therefore, varying Ω from $-\infty$ to 0 we obtain a parametric representation of the curve describing the onset of the synchronization in the half-plane of the parameters $\alpha \geq 0$ and K. Depending on the choice

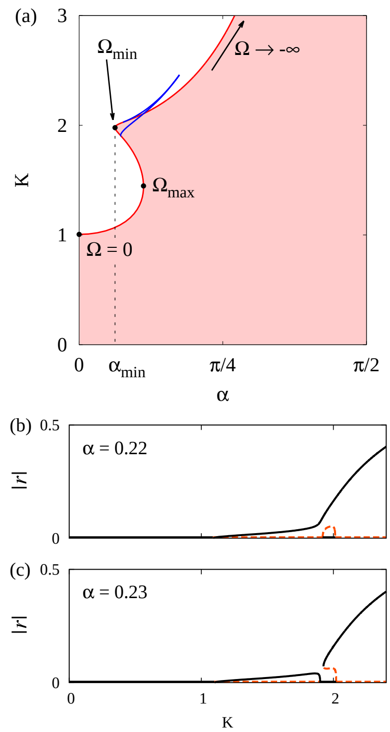

Figure 2: (a) Bifurcation diagram for the double Lorentzian distribution (12), (14) with $\sigma = 0.01$ and $\tau = 0.99$. Red line: stability boundary of the incoherent state. Blue line: fold of partially synchronized states. (b)–(c) Synchronization transitions for particular choices of α.

of the parameters (σ, τ) function $F(\Omega)$ may be monotonous as in Fig. 1, or non-monotonous as in Fig. 2(a). We are interested in the latter case, when the function $F(\Omega)$ has local extrema corresponding to folding points Ω_{min} and Ω_{max}. To find these extrema, we use the derivative

$$F'(\Omega) = \frac{B\Omega^4 + (3C - AB)\Omega^2 + AC}{(B\Omega^2 + C)^2} = 0. \tag{16}$$

Taking into account that $A, B, C \geq 0$ for all $(\sigma, \tau) \in [0, 1]^2$, we may encounter two situations. For (σ, τ) from the hatched region in Fig. 3 we have

$$A^2 B^2 + 9C^2 - 10ABC > 0 \tag{17}$$

and

$$AB - 3C > 0, \tag{18}$$

therefore Eq. (16) has two positive solutions

$$\Omega^2_{\text{min}} = \frac{AB - 3C + \sqrt{A^2 B^2 + 9C^2 - 10ABC}}{2B} \tag{19}$$

and

$$\Omega^2_{\text{max}} = \frac{AB - 3C - \sqrt{A^2 B^2 + 9C^2 - 10ABC}}{2B}.$$

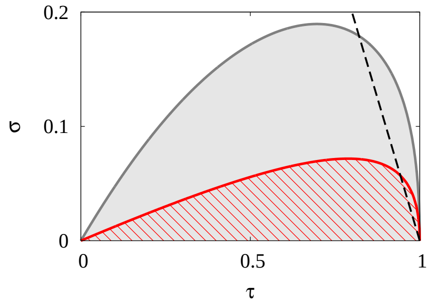

Figure 3: Parameter regions where inequalities (17) (hatched) and (18) (shaded) are satisfied; $\alpha_{\min} \to 0$ along the dashed line, given by (20).

For (σ, τ) outside of the hatched region the discriminant in formula (19) is negative and Eq. (16) has no solutions at all.

Let us analyze the behavior of

$$\alpha_{\min} = \arctan(F(\Omega_{\min}))$$

for $\sigma \to 0$ and $\tau \to 1$. For the sake of simplicity, we assume

$$\tau = 1 - \sigma \tag{20}$$

as shown by dashed line in Fig. 3. Then,

$$\begin{aligned} A &= \sigma + \sigma^2 - \sigma^3, \\ B &= 1 - \sigma + \sigma^2, \\ C &= 2\sigma^2 - \sigma^3. \end{aligned}$$

Inserting this into Eq. (19) we obtain

$$\Omega_{\min}^2 = \sigma + O(\sigma^2) \quad \text{for} \quad \sigma \to +0.$$

This yields

$$\alpha_{\min} = 2\sqrt{\sigma}(1 + O(\sigma)) \quad \text{for} \quad \sigma \to +0.$$

Therefore for arbitrary $\alpha_0 > 0$ there exists a pair (σ, τ) such that $\alpha_{\min} \le \alpha_0$. Hence, we have proven the following statement.

Proposition 1. *For every $|\alpha| \in (0, \pi/2)$ there exists a unimodal distribution of the form (14) such that the Kuramoto-Sakaguchi model (1)–(2) in the limit $N \to \infty$ exhibits a nonuniversal synchronization transition.*

Remark that the constructed non-universal synchronization transitions for small α require distributions with small width and contain regions where the order parameter $|r|$ has correspondingly small values, see Fig. 2(b)–(c). This implies that in a finite oscillator system (1)–(2) one needs to employ a correspondingly large number of oscillators N in order to resolve this scenario properly. The singular nature of our construction is also reflected by the fact that we have to avoid the case $\sigma = 0, \tau < 1$, where the distribution $g(\omega; \sigma, \tau)$ contains a delta-distribution and does no more satisfy the smoothness conditions that are necessary to justify a reasonable correspondence between the continuum limit (3), (4) and the finite oscillator system (1), see [22, 23]. At the other hand, inserting $\sigma = 0$ and $\tau = 1$ into (14), we obtain a single Lorentzian distribution $L_1(\omega)$. for which synchronization transitions are always monotonous independently of $\alpha \in [0, \pi/2)$, see Fig. 1.

Below we show that in our framework it is easy to conclude that for $\alpha = 0$ nonuniversal synchronization transitions are impossible.

Proposition 2. *For any unimodal distribution $g(\omega)$, the Kuramoto model (1)–(2) with $\alpha = 0$ exhibits only a monotonous synchronization transition.*

Proof: Relevant synchronization transitions can be obtained from the self-consistency equation (9) if we insert there $\alpha = \Omega = 0$. Taking into account that g is an even function we obtain

$$\frac{1}{K} = i \int_{-\infty}^{\infty} g(ps)h(s)ds = 2 \int_0^1 g(ps)\sqrt{1-s^2}ds =: I_1(p).$$

The corresponding global order parameter $|r|$ is given by (11) and reads

$$|r| = \frac{p}{K} = 2p \int_0^1 g(ps)\sqrt{1-s^2}ds =: I_2(p).$$

Varying p from 0 to ∞ we obtain a parametric representation of the synchronization transition in the form $(K, |r|) = (1/I_1(p), I_2(p))$. For a smooth distribution $g(\omega)$ one can easily verify

$$I_1'(p) = 2 \int_0^1 sg'(ps)\sqrt{1-s^2}ds \le 0$$

because $g(\omega)$ is nonincreasing for all $\omega \in [0, \infty)$. On the other hand,

$$\begin{aligned} I_2'(p) &= 2 \int_0^1 g(ps)\sqrt{1-s^2}ds + 2p \int_0^1 sg'(ps)\sqrt{1-s^2}ds \\ &= 2 \int_0^1 g(ps)\sqrt{1-s^2}ds + 2 \int_0^1 s\sqrt{1-s^2}dg(ps) \\ &= 2 \int_0^1 g(ps)\frac{s^2}{\sqrt{1-s^2}}ds \ge 0. \end{aligned}$$

This means that $d|r|/dK = -I_2'(p)I_1^2(p)/I_1'(p) \ge 0$.

Evaluating numerically the bifurcation condition for a superposition (14) of two Gaussian distributions

$$G_\sigma(\omega) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\omega^2/(2\sigma^2)}, \tag{21}$$

we obtain a qualitatively similar bifurcation scenario, see Fig. 4(a), with various types of nonuniversal synchronization transitions (see panels (b)–(e)). Note that we have chosen here the parameters $\tau = 0.9$ and $\sigma = 0.01$ more distant from the singular situation described above.

This shows that one can expect nonuniversal synchronization transitions close to $\alpha = 0$ in a similar way also for other frequency distributions, which do not allow for an analytical treatment as presented above.

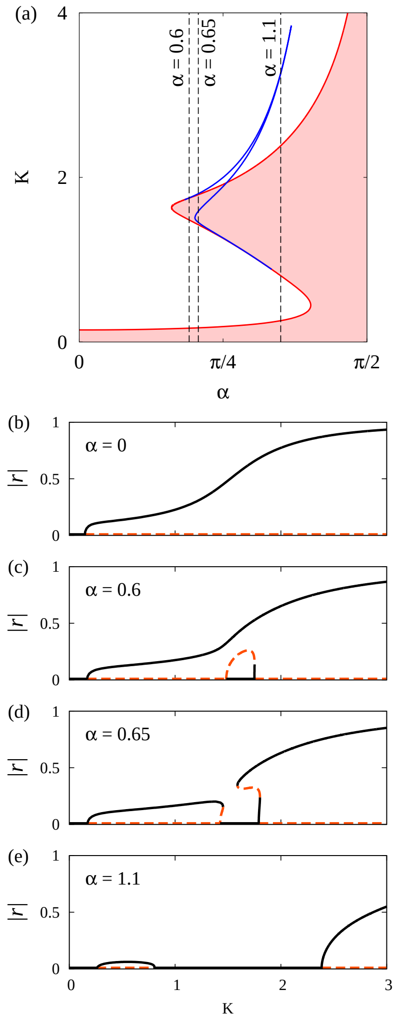

Figure 4: (a) Bifurcation diagram for the double Gaussian distribution (14), (21) with $\sigma = 0.01$ and $\tau = 0.9$. Red line: stability boundary of the incoherent state. Blue line: fold of partially synchronized states. (b)–(e) Synchronization transitions for particular choices of α.

4 Discussion: Interacting subpopulations

In all examples for non-universal synchronization transitions, it is an essential feature that the distribution of natural frequencies $g(\omega)$ is composed as a sum of two elementary distributions with significantly different widths. Based on this, we will now try to give a qualitative explanation of the observed phenomena by interpreting the globally coupled system (1)–(2), with distribution (14) as a system of two subpopulations $P_1,\ P_\sigma$, one of them distributed according to $g_1(\omega)$

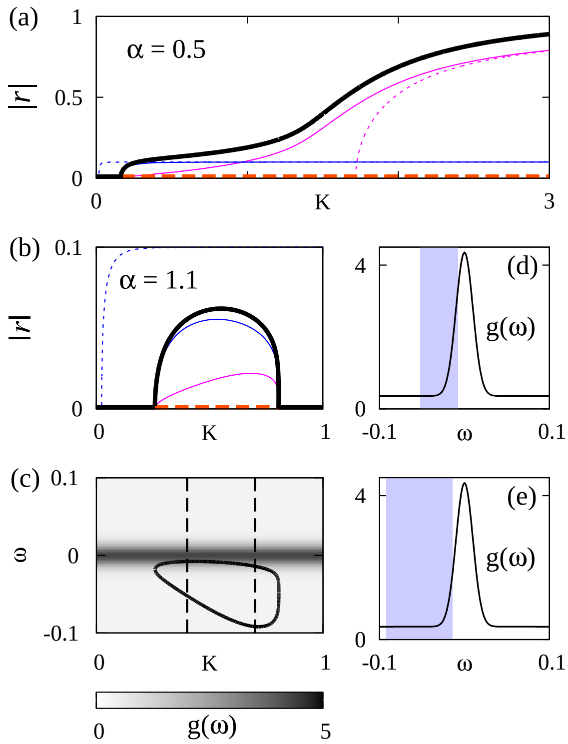

Figure 5: (a), (b): Synchronization transitions with global order parameter $|r|$ (bold) and subpopulation order parameters $|r_1|$, $|r_\sigma|$ (magenta, blue). Dotted lines: subpopulations without mutual coupling. (c): frequency distribution $g(\omega)$ (grey scale) and boundaries of the synchronization window (black curves) for varying K. (d), (e): frequency distribution $g(\omega)$ and synchronization windows for $K = 0.4$ and $K = 0.7$ (shaded intervals). Other parameters as in Fig. 4.

and the other one according to $g_\sigma(\omega)$. Respectively, the sizes of the subpopulations are

$$N_1 = \tau N \quad \text{and} \quad N_\sigma = (1-\tau)N$$

and the corresponding relative order parameters are given by

$$|r_1| = \tau \int_{-\infty}^{\infty} g_1(\omega)a(\omega)d\omega$$

and

$$|r_\sigma| = (1-\tau) \int_{-\infty}^{\infty} g_\sigma(\omega)a(\omega)d\omega.$$

We compare now the synchronization of the two populations with global coupling and without any mutual coupling. Fig. 5(a),(b) shows the behavior of the corresponding order parameters for the globally coupled case (solid) and the stand alone subpopulations (dotted). We use again a superposition (14) of two Gaussians with the same parameters as in Fig. 4. For $\alpha = 0.5$, i.e. in the range of the classical synchronization transition, the populations interact in an expectable way, see Fig. 4(a). Without the mutual coupling the oscillators from the narrow population P_σ synchronize at much smaller coupling than those of the wider population P_1. Under global coupling the synchronization starts also by entraining almost only oscillators from P_σ, however at a considerably higher threshold. The synchronization within P_1, supported by the already established synchrony within P_σ, sets in much earlier and it is higher than in the stand alone case. For $\alpha = 1.1$ we observe a quite different scenario, see Figure 5(b). Again we observe the onset of synchronization mainly within P_σ. But with the oscillators from P_1 being entrained little by little, they eventually start to suppress the synchrony within P_σ. This happens because due to the phase lag the increasing synchrony induces a shift of the synchronization window. Figure 5(c) shows how the overlap of the synchronization window (region between black curves) with the peak of the frequency distribution decreases. In this way, the entrained oscillators from P_1 suppress the synchrony within P_σ, on which their own synchrony relies, until the synchrony of the whole population breaks down. As shown in Fig. 4(c), there is also a bistability of the two regimes of cooperation and suppression possible.

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References

[1] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer, Berlin Heidelberg New York, 1984).

[2] S. Watanabe and S. H. Strogatz, Phys. Rev. Lett. **70**, 2391 (1993).

[3] S. Watanabe and S. H. Strogatz, Physica D **74**, 197 (1994).

[4] E. Ott and T. M. Antonsen, Chaos **18**, 037113 (2008).

[5] E. Ott and T. M. Antonsen, Chaos **19**, 023117 (2009).

[6] H. Daido, Physica D **91**, 24 (1996).

[7] P. Ashwin, O. Burylko, and Y. Maistrenko, Physica D **237**, 454 (2008).

[8] C. Bick, M. Timme, D. Paulikat, D. Rathlev, and P. Ashwin, Phys. Rev. Lett. **107**, 244101 (2011).

[9] M. Komarov and A. Pikovsky, Phys. Rev. Lett. **111**, 204101 (2013).

[10] M. Komarov and A. Pikovsky, Physica D **289**, 18 (2014).

[11] A. Pikovsky and M. Rosenblum, Chaos **25**, 097616 (2015).

[12] H. Sakaguchi and Y. Kuramoto, Prog. Theor. Phys. **76**, 576 (1986).

[13] Y. Kuramoto and D. Battogtokh, Nonlinear Phenom. Complex Syst. **5**, 380 (2002).

[14] D. M. Abrams and S. H. Strogatz, Phys. Rev. Lett. **93**, 174102 (2004).

[15] M. J. Panaggio and D. Abrams, Nonlinearity **28**, R67 (2015).

[16] M. Wolfrum, S. V. Gurevich, and O. E. Omel'chenko, Nonlinearity **29**, 257 (2016).

[17] O. E. Omel'chenko and M. Wolfrum, Phys. Rev. Lett. **109**, 164101 (2012).

[18] O. E. Omel'chenko and M. Wolfrum, Physica D **263**, 74 (2013).

[19] S. H. Strogatz, Physica D **143**, 1 (2000).

[20] D. Pazo and E. Montbrio, Phys. Rev. E **80**, 046215 (2009).

[21] C. R. Laing, Chaos **19**, 013113 (2009).

[22] H. Dietert, J. Math. Pures Appl. (to appear).

[23] B. Fernandez, D. Gerard-Varet, and G. Giacomin, Ann. Inst. H. Poincare Anal. Non Lineaire (to appear).