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für Angewandte Analysis und Stochastik
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A degenerating Cahn-Hilliard system coupled with complete damage processes

Christian Heinemann, Christiane Kraus

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Weierstraß-Institut
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: christian.heinemann@wias-berlin.de
christiane.kraus@wias-berlin.de

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Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

Complete damage in elastic solids appears when the material looses all its integrity due to high exposure. In the case of alloys, the situation is quite involved since spinodal decomposition and coarsening also occur at sufficiently low temperatures which may lead locally to high stress peaks. Experimental observations on solder alloys reveal void and crack growth especially at phase boundaries.

In this work, we investigate analytically a degenerating PDE system with a time-depending domain for phase separation and complete damage processes under time-varying Dirichlet boundary conditions. The evolution of the system is described by a degenerating parabolic differential equation of fourth order for the concentration, a doubly nonlinear differential inclusion for the damage process and a degenerating quasi-static balance equation for the displacement field. All these equations are strongly nonlinearly coupled.

Because of the doubly degenerating character and the doubly nonlinear differential inclusion we are confronted with introducing a suitable notion of weak solutions. We choose a notion of weak solutions which consists of weak formulations of the diffusion equation and the momentum balance, a one-sided variational inequality for the damage function and an energy estimate.

For the introduced degenerating system, we prove existence of weak solutions in an SBV-framework. The existence result is based on an approximation system, where the elliptic degeneracy of the displacement field and the parabolic degeneracy of the concentration are eliminated. In the framework of phase separation and damage, this means that the approximation system allows only for partial damage and a non-degenerating mobility tensor. For the approximation system, existence results are established. Then, a passage to the limit shows existence of weak solutions of the degenerating system.

1 Problem description

Phase separation and damage processes occur in many fields, including material sciences, biology and chemical reactions. In particular, for the manufacturing and lifetime prediction of micro-electronic devices it is of great importance to understand the mechanisms and the interplay between phase-separation and damage processes in solder alloys. As soon as elastic alloys are quenched sufficiently, spinodal decomposition leads to a fine-grained structure of different chemical mixtures on a short time-scale (see [DM01] for numerical simulations and experimental observations). The long-term evolution is determined by a chemical diffusion process which tends to minimize the bulk and the surface energy of the chemical substances. J.W. Cahn and J.E. Hilliard developed a phenomenological model for the kinetics of phase-separation in a thermodynamically consistent framework known as the Cahn-Hilliard equation [CH58], for which an extensive mathematical literature exists. An overview of modeling and some analytical aspects of the Cahn-Hilliard equation can be found in [Ell89]. The recent literature is mainly focused on coupled systems. For instance, physical observation and numerical simulations reveal that mechanical stresses influence the developing shapes of the chemical phases. A coupling between Cahn-Hilliard systems and elastic deformations have been analytically studied in [Gar00, BCD$^+$02, CMP00, Gar05a, Gar05b, BP05, PZ08]. For numerical results and simulations we refer [Wei01, Mer05, BB99, GRW01, BM10]. Phase separation of the chemical components may also lead to critical stresses at phase boundaries which result in cracks and formation of voids and are of particular interest to understand the aging process in solder materials, cf. [HCW91, USG07, GUaMM$^+$07, FK09].

A fully coupled system consisting of the Cahn-Hilliard equation, an elliptic equation for the displacement field and a differential inclusion for the damage variable has been recently investigated in [HK11, HK10].

However, in [HK11, HK10] and in the most mathematical damage literature [BS04, Gia05, MR06, MT10, KRZ11], it is usually assumed that damage cannot completely disintegrate the material (i.e. *incomplete damage*). Dropping this assumption gives rise to many mathematical challenges. Therefore, global-in-time existence results for complete damage models are rare. Modeling and existence of weak solutions for purely mechanical complete damage systems with quasi-static force balances are studied in [BMR09, Mie11, HK12] and with visco-elasticity in [MRZ10, RR12].

The main goal in the present work is to prove existence of weak solutions of a system coupling damage processes to an elastic Cahn-Hilliard system as in [HK11] but allowing for complete damage. The elasticity is considered to be linear and the system is assumed to be in quasi-static mechanical equilibrium since diffusion processes take place on a much slower time scale. A weak formulation of the degenerating system is given in Section 2 while the proof of existence is carried out in Section 3 and Section 4. In the first step of the proof, a degenerated limit of the corresponding incomplete damage system coupled with elastic Cahn-Hilliard equations is performed, see Section 3. Due to the additional coupling the passage to the limit becomes more involved than in [HK12] and a *conical Poincaré inequality* is used to control the chemical potential in a local sense.

However, as we will see, the limit functions may not form a weak solution since serious mathematical difficulties occur in the existence proof when not completely damaged material fragments become isolated from the Dirichlet boundary. The main idea in [HK12] has been to exclude these loosely parts from the considered evolutionary problem. The exclusions lead to jumps in the overall energy which have to be accounted for in the energy inequality. This issue is addressed in Section 4 by using methods from [HK12]. There, maximal local-in-time existence of weak solutions and a global existence result for approximate weak solutions, which are also introduced in the next section, are proven. In this context, the concept of maximal admissible subsets needs to be introduced to specify the domain of interest.

In the following, we fix a domain $\Omega \subseteq \mathbb{R}^n$, a maximal time $T > 0$ of interest and a constant p with $p > n$.

A relatively open subset G of $\overline{\Omega}$ is called *admissible* with respect to a part of the boundary $D \subseteq \partial\Omega$ if every path-connected component P_G of G satisfies $\mathcal{H}^{n-1}(P_G \cap D) > 0$, where $\mathcal{H}^{n-1}$ denotes the $(n-1)$ dimensional Hausdorff measure. The *maximal admissible subset* of G is denoted by $\mathfrak{A}_D(G)$. With the notion of maximal admissible subsets, we can formulate our evolutionary system with a time-depending domain in a smooth setting as follows:

Coupled PDE system with time-depending domain. *Find a relatively open subset* $F \subseteq \overline{\Omega_T} := \overline{\Omega} \times [0,T]$ *with the property* $F(t) = \mathfrak{A}_D\Big(\bigcap_{s<t} F(s)\Big)$ *and* $F(s) \subseteq F(t)$ *whenever* $s \geq t$ *(i.e.* F *is shrinking) and functions*

$$c \in \mathcal{C}^2(F;\mathbb{R}),\ u \in \mathcal{C}_x^2(F;\mathbb{R}^n),\ z \in \mathcal{C}^2(\overline{F};\mathbb{R}),\ \mu \in \mathcal{C}_x^2(F;\mathbb{R})$$

such that the PDE system

$$\begin{aligned}
&0 = \operatorname{div}(W_{,e}(c,\epsilon(u),z)),\\
&c_t = \operatorname{div}(m(z)\nabla\mu) && \textit{with } \mu = -\Delta c + \Psi_{,c}(c) + W_{,c}(c,\epsilon(u),z),\\
&z_t+\xi = \operatorname{div}(|\nabla z|^{p-2}\nabla z)+W_{,z}(c,\epsilon(u),z)+f'(z) && \textit{with } \xi \in \partial I_{(-\infty,0]}(z_t),\\
&0 < z
\end{aligned}$$

is satisfied pointwise in $\operatorname{int}(F)$ *with the initial-boundary conditions*

$$
\begin{aligned}
&c(0) = c^0,\, z(0) = z^0 && \textit{on } F(0),\\
&u(t) = b(t) && \textit{on } \Gamma_1(t) := F(t) \cap D,\\
&W_{,e}(c(t), \epsilon(u(t)), z(t)) \cdot \nu = 0 && \textit{on } \Gamma_2(t) := F(t) \cap (\partial\Omega \setminus D),\\
&z(t) = 0 && \textit{on } \Gamma_3(t) := \partial F(t) \setminus F(t),\\
&\nabla z(t) \cdot \nu = 0 && \textit{on } \Gamma_1(t) \cup \Gamma_2(t),\\
&\nabla c(t) \cdot \nu = 0 && \textit{on } \Gamma_1(t) \cup \Gamma_2(t),\\
&m(z(t))\nabla \mu(t) \cdot \nu = 0 && \textit{on } \Gamma_1(t) \cup \Gamma_2(t).
\end{aligned}
$$

The solution of the PDE system can physically be interpreted as follows: c denotes the chemical concentration ratio, $\epsilon(u)$ the linearized strain tensor of the deformation u, z the damage profile describing the degree of damage (i.e. $z = 1$ undamaged and $z = 0$ completely damaged material point) and μ the chemical potential. Moreover, W denotes the elastic energy density, Ψ the chemical energy density, f a damage dependent potential, m the mobility depending on the damage and b the time-depending Dirichlet boundary data for D.

We assume the following product structure for the elastic energy density:

$$W(c, e, z) = g(z)\varphi(c, e)$$

with a non-negative function $g \in \mathcal{C}^1([0, 1]; \mathbb{R}^+)$ such that the complete damage condition $g(0) = 0$ is fulfilled. The case $g(0) > 0$ is treated in [HK11] for constant mobility. The second function $\varphi \in \mathcal{C}^1(\mathbb{R} \times \mathbb{R}^{n\times n}_{\text{sym}}; \mathbb{R}^+)$ should have the following polynomial form

$$\varphi(c, e) = \varphi^1 e : e + \varphi^2(c) : e + \varphi^3(c), \tag{1}$$

for coefficients $\varphi^1 \in \mathcal{L}(\mathbb{R}^{n\times n}_{\text{sym}})$ with $\varphi^1 > 0$, $\varphi^2 \in \mathcal{C}^1(\mathbb{R}; \mathbb{R}^{n\times n}_{\text{sym}})$ and $\varphi^3 \in \mathcal{C}^1(\mathbb{R})$. Note that homogenous elastic energy densities of the type

$$W(c, e, z) = \frac{1}{2} z\mathbb{C}(e - e^\star(c)) : (e - e^\star(c))$$

with $e^\star$ eigenstrain and $\mathbb{C}$ stiffness tensor are covered in this approach. The mobility $m \in \mathcal{C}([0, 1]; \mathbb{R}^+)$, on the other hand, should satisfy the following condition for degeneracy, i.e.

$$m(z) = 0 \text{ if and only if } z = 0. \tag{2}$$

In the next section, we provide a weak formulation of the above system.

2 Weak solutions and approximate weak solutions

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded $\mathcal{C}^2$-domain and $D \subseteq \partial\Omega$ be the Dirichlet boundary with $\mathcal{H}^{n-1}(D) > 0$. The set $\{f > 0\}$ for a function $f \in W^{1,p}(\Omega)$ has to be read as $\{x \in \overline{\Omega} \,|\, f(x) > 0\}$ by employing the embedding $W^{1,p}(\Omega) \hookrightarrow \mathcal{C}(\overline{\Omega})$ (because of $p > n$).

The weak formulation of the PDE system presented here will be based on an energetic approach and uses the associated free energy $\mathcal{E}$. Let $G \subseteq \overline{\Omega}$ be a relatively open subset. Then, the free energy on G is given by

$$\mathcal{E}_G(c, e, z) := \int_G \frac{1}{p}|\nabla z|^p + \frac{1}{2}|\nabla c|^2 + \Psi(c) + W(c, e, z) + f(z)\,\mathrm{d}x$$

for $c \in H^1(G)$, $e \in L^2(G;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ and $z \in W^{1,p}(G)$. We will omit the subscript G in $\mathcal{E}_G$ and simply write $\mathcal{E}$.

In the following, a weak formulation of the system above combining the ideas in [HK11] and [HK12] is given.

Definition 2.1 (Weak solution of the coupled PDE system) *A quadruple (c, u, z, μ) is called a weak solution with the initial-boundary data (c^0, z^0, b) if*

(i) Trajectory spaces:

$$\begin{aligned} &c \in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))^\star), && u \in L^2_t H^1_{x,\mathrm{loc}}(F;\mathbb{R}^n),\\ &z \in L^\infty(0,T;W^{1,p}(\Omega)) \cap SBV^2(0,T;L^2(\Omega)), && \mu \in L^2_t H^1_{x,\mathrm{loc}}(F) \end{aligned}$$

with $e := \epsilon(u) \in L^2(F;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ where $F := \mathfrak{A}_D(\{z^- > 0\}) \subseteq \overline{\Omega_T}$ is a shrinking set.

(ii) Quasi-static mechanical equilibrium:

$$0 = \int_{F(t)} W_{,e}(c(t), e(t), z(t)) : \epsilon(\zeta)\,\mathrm{d}x \tag{3}$$

for a.e. $t \in (0,T)$ and for all $\zeta \in H^1_D(\Omega;\mathbb{R}^n)$. Furthermore, $u = b$ on $D_T \cap F$.

(iii) Diffusion:

$$\int_{\Omega_T} \partial_t\zeta(c - c^0)\,\mathrm{d}x\,\mathrm{d}t = \int_F m(z)\nabla\mu\cdot\nabla\zeta\,\mathrm{d}x\,\mathrm{d}t \tag{4}$$

for all $\zeta \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$ with $\zeta(T) = 0$ and

$$\int_F \mu\zeta\,\mathrm{d}x = \int_F \nabla c\cdot\nabla\zeta + \Psi_{,c}(c)\zeta + W_{,c}(c,e,z)\zeta\,\mathrm{d}x \tag{5}$$

for all $\zeta \in L^2(0,T;H^1(\Omega))$ with $\mathrm{supp}(\zeta) \subseteq F$.

(iv) Damage variational inequality:

$$\begin{aligned} &0 \le \int_{F(t)} |\nabla z(t)|^{p-2}\nabla z(t)\cdot\nabla\zeta + \big(W_{,z}(c(t), e(t), z(t)) + f'(z(t)) + \partial_t^{\mathrm{a}} z(t)\big)\zeta\,\mathrm{d}x \qquad (6)\\ &0 \le z(t),\\ &0 \ge \partial_t^{\mathrm{a}} z(t) \end{aligned}$$

for a.e. $t \in (0,T)$ and for all $\zeta \in W^{1,p}(\Omega)$ with $\zeta \le 0$. The initial value is given by $z^+(0) = z^0$.

(v) Damage jump condition:

$$z^+(t) = z^-(t)\mathbb{1}_{F(t)} \text{ in } \overline{\Omega} \tag{7}$$

for all $t \in [0,T]$.

(vi) Energy inequality:

$$\mathcal{E}(c(t), e(t), z(t)) + \int_0^t \int_{F(s)} |\partial_t^{\mathrm{a}} z|^2 + m(z)|\nabla\mu|^2 \,\mathrm{d}x\,\mathrm{d}s + \sum_{s \in J_z \cap (0,t]} \mathcal{J}_s$$
$$\leq \mathfrak{e}_0^+ + \int_0^t \int_{F(s)} W_{,e}(c, e, z) : \epsilon(\partial_t b) \,\mathrm{d}x\,\mathrm{d}s \tag{8}$$

for a.e. $t \in (0, T)$, where the jump part $\mathcal{J}_s$ satisfies $0 \leq \mathcal{J}_s$ and is given by

$$\mathcal{J}_s := \lim_{\tau \to s^-} \operatorname*{ess\,inf}_{\vartheta \in (\tau, s)} \mathcal{E}(c(\vartheta), e(\vartheta), z(\vartheta)) - \mathfrak{e}_s^+ \tag{9}$$

and the values $\mathfrak{e}_s^+ \in \mathbb{R}$ satisfy the upper energy estimate

$$0 \leq \mathfrak{e}_s^+ \leq \mathcal{E}(c(s), \epsilon(b(s) + \zeta), z^+(s)) \tag{10}$$

for all $\zeta \in H^1_{D \cap F(s)}(F(s); \mathbb{R}^n)$.

Remark 2.2 *(i) The vector-valued Banach space $SBV^2(0, T; L^2(\Omega))$ can be analogously defined as for real-valued SBV-functions on a time-interval (see [AFP00]).*

The space-time local Sobolev space $L_t^2 H^1_{x,\mathrm{loc}}(F; \mathbb{R}^N)$ for a shrinking set F, on the other hand, is given by

$$L_t^2 H^q_{x,\mathrm{loc}}(F; \mathbb{R}^N) := \Big\{ v : F \to \mathbb{R}^N \,\big|\, \forall t \in (0, T],\ \forall U \subset\subset F(t) \text{ open}:\ v|_{U \times (0,t)} \in L^2(0, t; H^q(U; \mathbb{R}^N)) \Big\}.$$

For both definitions, we refer to [HK12].

(ii) Under additional regularity assumptions, a weak solution reduces to the pointwise classical notion presented in Section 1 (cf. [HK12, Theorem 3.7]).

One aim of this paper is to prove maximal local-in-time existence of weak solutions according to Definition 2.1. In addition, following the approach in [HK12], existence of global solutions can be shown in an approximate sense. To be more precise, we use the notation

$$F \approx_\eta \mathfrak{A}_D(\{z^- > 0\})$$

for a measurable set $F \subseteq \overline{\Omega_T}$, a function $z \in SBV^2(0, T; L^2(\Omega))$ and a constant $\eta > 0$ if the conditions

$$\begin{aligned} &F(t) \supseteq \mathfrak{A}_D(\{z^-(t) > 0\}) \text{ for all } t \in [0, T], \\ &F(t) = \mathfrak{A}_D(\{z^-(t) > 0\}) \text{ for all } t \in [0, T] \setminus \bigcup_{t \in C_{z^\star}} [t, t + \eta), \\ &\mathcal{L}^n\big(F(t) \setminus \mathfrak{A}_D(\{z^-(t) > 0\})\big) < \eta \text{ for all } t \in \bigcup_{t \in C_{z^\star}} [t, t + \eta) \end{aligned}$$

are satisfied. Here, $C_{z^\star}$ denotes the set of cluster points from the right of the jump set $J_{z^\star}$ of the function $z^\star \in SBV^2(0, T; L^2(\Omega))$ given by $z^\star(t) := z(t)\mathbb{1}_{\mathfrak{A}_D(\{z^-(t)>0\})}$ ($\mathbb{1}_A : X \to \{0, 1\}$ is the characteristic function of a set $A \subseteq X$). Roughly speaking, $z^\star$ is the restricted damage profile of z which takes *all* material exclusions into account.

Definition 2.3 (Approximate weak solution) *A tuple (c, e, u, z, μ) and a shrinking set $F \subseteq \overline{\Omega_T}$ is called an approximate weak solution with fineness $\eta > 0$ and the initial-boundary data (c^0, z^0, b) if*

$$\begin{aligned} &c \in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))^\star), && u \in L^2_t H^1_{x,\mathrm{loc}}(\mathfrak{A}_D(F);\mathbb{R}^n),\\ &z \in L^\infty(0,T;W^{1,p}(\Omega)) \cap SBV^2(0,T;L^2(\Omega)), && \mu \in L^2_t H^1_{x,\mathrm{loc}}(F),\\ &e \in L^2(F;\mathbb{R}^{n\times n}_{\mathrm{sym}}) \end{aligned}$$

with $e = \epsilon(u)$ in $\mathfrak{A}_D(F)$, $F \approx_\eta \mathfrak{A}_D(\{z^- > 0\})$ and properties (ii)-(vi) of Definition 2.1 are satisfied.

The remaining part of this paper is devoted to establish local and global existence results.

3 Degenerate limit of the regularized system

In this section, we will start with a corresponding incomplete damage model coupled to an elastic Cahn-Hilliard system and perform a limit procedure. For each $\varepsilon > 0$, we define the regularized free energy $\mathcal{E}_\varepsilon$ as

$$\mathcal{E}_\varepsilon(c,e,z) := \int_\Omega \frac{1}{p}|\nabla z|^p + \frac{1}{2}|\nabla c|^2 + \Psi(c) + W^\varepsilon(c,e,z) + f(z)\,\mathrm{d}x$$

for functions $c \in H^1(\Omega)$, $e \in L^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ and $z \in W^{1,p}(\Omega)$. The regularized elastic energy density and mobility are given by

$$\begin{aligned} W^\varepsilon(c,e,z) &:= (g(z)+\varepsilon)\varphi(c,e),\\ m^\varepsilon(z) &:= m(z)+\varepsilon. \end{aligned}$$

From now on, we assume for φ, Ψ and g the following growth conditions:

$$|\varphi^2(c)|, |\varphi^2_{,c}(c)| \le C(1+|c|), \tag{11}$$

$$|\varphi^3(c)|, |\varphi^3_{,c}(c)| \le C(1+|c|^2), \tag{12}$$

$$|\Psi_{,c}(c)| \le C(1+|c|^{2^\star/2}), \tag{13}$$

$$\eta \le g'(z). \tag{14}$$

Here, $\eta, C > 0$ denote constants independently of c and z, and $2^\star$ denotes the Sobolev critical exponent. In the case $n = 2$, $\Psi_{,c}$ has to satisfy an r-growth condition for a fixed arbitrary $r > 0$ whereas we have no restrictions on $\Psi_{,c}$ in the one-dimensional case. The function f is assumed to be continuously differentiable.

A modification of the proof of Theorem 4.6 in [HK11] yields the following result.

Theorem 3.1 (ε-regularized coupled PDE problem) *Let $\varepsilon > 0$. For given initial-boundary data $c^0_\varepsilon \in H^1(\Omega)$, $z^0_\varepsilon \in W^{1,p}(\Omega)$ and $b_\varepsilon \in W^{1,1}(0,T;W^{1,\infty}(\Omega;\mathbb{R}^n))$ there exists a quadruple $q_\varepsilon = (c_\varepsilon, u_\varepsilon, z_\varepsilon, \mu_\varepsilon)$ such that*

(i) Trajectory spaces:

$$\begin{aligned} &c_\varepsilon \in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))^\star), && u_\varepsilon \in L^\infty(0,T;H^1(\Omega;\mathbb{R}^n)),\\ &z_\varepsilon \in L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega)), && \mu_\varepsilon \in L^2(0,T;H^1(\Omega)). \end{aligned}$$

(ii) Quasi-static mechanical equilibrium:

$$\int_\Omega W^\varepsilon_{,e}(c_\varepsilon(t), \epsilon(u_\varepsilon(t)), z_\varepsilon(t)) : \epsilon(\zeta)\,\mathrm{d}x = 0 \tag{15}$$

for a.e. $t \in (0,T)$ *and for all* $\zeta \in H^1_D(\Omega;\mathbb{R}^n)$*. Furthermore,* $u_\varepsilon = b_\varepsilon$ *on the boundary* D_T*.*

(iii) Diffusion:

$$\int_{\Omega_T} (c_\varepsilon - c^0_\varepsilon)\partial_t\zeta\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} m^\varepsilon(z_\varepsilon)\nabla\mu_\varepsilon \cdot \nabla\zeta\,\mathrm{d}x\,\mathrm{d}t \tag{16}$$

for all $\zeta \in L^2(0,T;H^1(\Omega))$ *with* $\partial_t\zeta \in L^2(\Omega_T)$ *and* $\zeta(T) = 0$ *and*

$$\int_\Omega \mu_\varepsilon(t)\zeta\,\mathrm{d}x = \int_\Omega \nabla c_\varepsilon(t)\cdot\nabla\zeta + \Psi_{,\mathrm{c}}(c_\varepsilon(t))\zeta + W^\varepsilon_{,\mathrm{c}}(c_\varepsilon(t), \epsilon(u_\varepsilon(t)), z_\varepsilon(t))\zeta\,\mathrm{d}x \tag{17}$$

for a.e. $t \in (0,T)$ *and for all* $\zeta \in H^1(\Omega)$*.*

(iv) Damage variational inequality:

$$0 \le \int_\Omega |\nabla z_\varepsilon(t)|^{p-2}\nabla z_\varepsilon(t)\cdot\nabla\zeta + \big(W^\varepsilon_{,z}(c_\varepsilon(t), \epsilon(u_\varepsilon(t)), z_\varepsilon(t)) + f'(z_\varepsilon(t)) + \partial_t z_\varepsilon(t) + r_\varepsilon(t)\big)\zeta\,\mathrm{d}x \tag{18}$$

$$0 \le z_\varepsilon(t),$$
$$0 \ge \partial_t z_\varepsilon(t)$$

for a.e. $t \in (0,T)$ *and for all* $\zeta \in W^{1,p}(\Omega)$ *with* $\zeta \le 0$ *where* $r_\varepsilon \in L^1(\Omega_T)$ *satisfies*

$$r_\varepsilon = -\chi_\varepsilon\big(W^\varepsilon_{,z}(c_\varepsilon, \epsilon(u_\varepsilon), z_\varepsilon) + f'(z)\big)^+ \tag{19}$$

with $\chi_\varepsilon \in L^\infty(\Omega)$ *fulfilling* $\chi_\varepsilon = 0$ *on* $\{z_\varepsilon > 0\}$ *and* $0 \le \chi_\varepsilon \le 1$ *on* $\{z_\varepsilon = 0\}$*. The initial value is given by* $z_\varepsilon(0) = z^0_\varepsilon$*.*

(v) Energy inequality:

$$\begin{aligned}\mathcal{E}_\varepsilon(c_\varepsilon(t), \epsilon(u_\varepsilon(t)), z_\varepsilon(t)) &+ \int_{\Omega_t} |\partial_t z_\varepsilon|^2 + m^\varepsilon(z_\varepsilon)|\nabla\mu_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}s \\ &\le \mathcal{E}_\varepsilon(c^0_\varepsilon, \epsilon(u^0_\varepsilon), z^0_\varepsilon) + \int_{\Omega_t} W_{,e}(c_\varepsilon, \epsilon(u_\varepsilon), z_\varepsilon) : \epsilon(\partial_t b_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s\end{aligned} \tag{20}$$

holds for a.e. $t \in (0,T)$ *where* u^0_ε *minimizes* $\mathcal{E}_\varepsilon(c^0, \epsilon(\cdot), z^0_\varepsilon)$ *in* $H^1(\Omega;\mathbb{R}^n)$ *with Dirichlet data* $b^0_\varepsilon := b_\varepsilon(0)$ *on* D*.*

Proof. The existence theorem presented in [HK11] can be adapted to our situation by considering the viscous semi-implicite time-discretized system (in a classical notation; we omit the ε-dependence in the notation for the discrete solution at the moment):

$$\begin{aligned}0 &= \mathrm{div}\big(W^\varepsilon_{,e}(c^m, \epsilon(u^m), z^m)\big) + \delta\,\mathrm{div}(|u^m|^2u^m),\\ \frac{c^m - c^{m-1}}{\tau} &= \mathrm{div}(m^\varepsilon(z^{m-1})\nabla\mu^m),\\ \mu^m &= -\Delta c^m + \Psi_{,c}(c^m) + W^\varepsilon_{,c}(c^m, \epsilon(u^m), z^m) + \delta\frac{c^m - c^{m-1}}{\tau},\end{aligned}$$

$$\frac{z^m - z^{m-1}}{\tau} + \xi + \zeta = \mathrm{div}(|\nabla z^m|^{p-2}\nabla z^m) + W^\varepsilon_{,z}(c^m, \epsilon(u^m), z^m) + f'(z^m),$$

with the sub-gradients $\xi \in \partial I_{(-\infty,0]}((z^m - z^{m-1})/\tau)$, $\zeta \in \partial I_{[0,\infty)}(z^m)$ and the discretization fineness $\tau = T/M$ for $M \in \mathbb{N}$. The discrete equations can be obtained recursively starting from (c^0, u^0, z^0) with $u^0 := \arg\min_{u \in H^1(\Omega;\mathbb{R}^n),\, u|_D = b^0|_D} \mathcal{E}_\varepsilon(c^0, u, z^0)$ by considering the Euler-Lagrange equations of the functional

$$\begin{aligned}\mathbb{E}^m(c,u,z) &:= \mathcal{E}_\varepsilon(c,u,z) + \int_\Omega \frac{\delta}{4}|\nabla u|^4 \,\mathrm{d}x \\ &\quad + \frac{\tau}{2}\left(\left\| \frac{z - z^{m-1}}{\tau} \right\|^2_{L^2(\Omega)} + \left\| \frac{c - c^{m-1}}{\tau} \right\|^2_{X(z^{m-1})} + \delta \left\| \frac{c - c^{m-1}}{\tau} \right\|^2_{L^2(\Omega)} \right)\end{aligned}$$

defined on the subspace of $W^{1,4}(\Omega;\mathbb{R}^n) \times H^1(\Omega) \times W^{1,p}(\Omega)$ given by the conditions $u|_D = b(m\tau)|_D$, $\int_\Omega c - c^0 \,\mathrm{d}x = 0$ and $0 \le z \le z^{m-1}$ a.e. in Ω. The scalar product $\langle \cdot, \cdot \rangle_{X(z^{m-1})}$ is given by

$$\langle u, v \rangle_{X(z^{m-1})} := \left\langle m^\varepsilon(z^{m-1})\nabla A^{-1}u, \nabla A^{-1}v \right\rangle_{L^2(\Omega)}$$

with the operator $A : V_0 \to \tilde{V}_0$, $Au := \left\langle m^\varepsilon(z^{m-1})\nabla u, \nabla \cdot \right\rangle_{L^2(\Omega)}$ and the spaces

$$\begin{aligned}V_0 &:= \left\{ \zeta \in H^1(\Omega) \;\Big|\; \int_\Omega \zeta \,\mathrm{d}x = 0 \right\}, \\ \tilde{V}_0 &:= \left\{ \zeta \in (H^1(\Omega))^* \;\Big|\; \langle \zeta, \mathbf{1} \rangle_{(H^1)^* \times H^1} = 0 \right\}.\end{aligned}$$

After passing the discretization fineness to 0, i.e. $\tau \to 0^+$, we obtain the corresponding equations and inequalities for (15)-(20). A further passage $\delta \to 0^+$ yields a weak solution as required. $\square$

We will need the Γ-limit of the *reduced* energy functional of $\mathcal{E}_\varepsilon$ in order to gain a suitable energy estimate in the limit $\varepsilon \to 0^+$. Define the reduced energy functionals $\mathfrak{E}_\varepsilon$ and $\mathfrak{F}_\varepsilon$ by

$$\mathfrak{E}_\varepsilon(c,\xi,z) := \begin{cases} \min\limits_{\zeta \in H^1_D(\Omega;\mathbb{R}^n)} \mathcal{E}_\varepsilon(c, \epsilon(\xi+\zeta), z) & \text{if } 0 \le z \le 1, \\ \infty & \text{else,} \end{cases}$$

$$\mathfrak{F}_\varepsilon(c,\xi,z) := \begin{cases} \min\limits_{\zeta \in H^1_D(\Omega;\mathbb{R}^n)} \mathcal{F}_\varepsilon(c, \epsilon(\xi+\zeta), z) & \text{if } 0 \le z \le 1, \\ \infty & \text{else} \end{cases}$$

with $\mathcal{F}_\varepsilon(c,e,z) := \int_\Omega W^\varepsilon(c,e,z)\,\mathrm{d}x$. The Γ-limits of $\mathfrak{E}_\varepsilon$ and $\mathfrak{F}_\varepsilon$ as $\varepsilon \to 0^+$ exist in the topological space $H^1_{\mathrm{w}}(\Omega) \times W^{1,\infty}(\Omega;\mathbb{R}^n) \times W^{1,p}_{\mathrm{w}}(\Omega)$ and are denoted by $\mathfrak{E}$ and $\mathfrak{F}$, respectively. Here, $H^1_{\mathrm{w}}(\Omega)$ denotes the space $H^1(\Omega)$ with its weak topology. The limit functional $\mathfrak{F}$ is needed as an auxiliary construction in the following because it already captures the essential properties of $\mathfrak{E}$. In the next section, we are going to prove some properties of the Γ-limit $\mathfrak{E}$ which are used in the global-in-time existence proof.

Let $(c^0_\varepsilon, b^0_\varepsilon, z^0_\varepsilon) \to (c^0, b^0, z^0)$ as $\varepsilon \to 0^+$ be a recovery sequence for $\mathfrak{E}_\varepsilon \xrightarrow{\Gamma} \mathfrak{E}$. In particular, $c^0_\varepsilon \rightharpoonup c^0$ in $H^1(\Omega)$, $b^0_\varepsilon \to b^0$ in $W^{1,\infty}(\Omega;\mathbb{R}^n)$ and $z^0_\varepsilon \rightharpoonup z^0$ in $W^{1,p}(\Omega)$. Furthermore, we set $b_\varepsilon := b - b^0 + b^0_\varepsilon$. For each $\varepsilon > 0$, we obtain a weak solution $(c_\varepsilon, u_\varepsilon, z_\varepsilon, \mu_\varepsilon)$ for $(c^0_\varepsilon, z^0_\varepsilon, b_\varepsilon)$ according to Theorem 3.1.

Applying Gronwall's lemma to the energy estimate (20) and following the argumentation in [HK12, Lemma 4.15] for the variables $\widehat{e}_\varepsilon$ and z_ε, we gain the following a-priori estimates:

- $\sup_{t\in[0,T]} \|c_\varepsilon(t)\|_{H^1(\Omega)} \le C$,
- $\|\widehat{e}_\varepsilon\|_{L^2(\Omega_T;\mathbb{R}^{n\times n})} \le C$ with $\widehat{e}_\varepsilon := e_\varepsilon \mathbb{1}_{\{z_\varepsilon>0\}}$,
- $\sup_{t\in[0,T]} \|z_\varepsilon(t)\|_{W^{1,p}(\Omega)} \le C$,
- $\|\partial_t z_\varepsilon\|_{L^2(\Omega_T)} \le C$,
- $\|W^\varepsilon(c_\varepsilon, e_\varepsilon, z_\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))} \le C$,
- $\|m^\varepsilon(z_\varepsilon)^{1/2}\nabla\mu_\varepsilon\|_{L^2(\Omega_T;\mathbb{R}^n)} \le C$,
- $\|\partial_t c_\varepsilon\|_{L^2(0,T;(H^1(\Omega))^\star)} \le \|m^\varepsilon(z_\varepsilon)\nabla\mu_\varepsilon\|_{L^2(\Omega_T;\mathbb{R}^n)} \le C$.

These estimates, Aubin-Lion type compactness theorems [Sim86], the variational inequality (18) and an approximation argument yield the following convergence properties (cf. [HK12]):

Lemma 3.2 *There exists functions*

(i) $c \in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))^\star)$,

(ii) $\widehat{e} \in L^2(\Omega_T;\mathbb{R}^{n\times n})$,

(iii) $z \in L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega))$, *$z$ is monotonically decreasing with respect to t, i.e.* $\partial_t z \le 0$,

and a subsequence (we omit the index) such that for $\varepsilon \to 0^+$

(a) $c_\varepsilon \rightharpoonup c$ *in* $H^1(0,T;(H^1(\Omega))^\star)$,
$c_\varepsilon \to c$ *in* $L^r(\Omega_T)$ *for all* $1 \le r < 2^\star$,
$c_\varepsilon(t) \rightharpoonup c(t)$ *in* $H^1(\Omega)$ *for all t,*
$c_\varepsilon \to c$ *a.e. in* Ω_T,

(b) $z_\varepsilon \rightharpoonup z$ *in* $H^1(0,T;L^2(\Omega))$,
$z_\varepsilon \to z$ *in* $L^p(0,T;W^{1,p}(\Omega))$,
$z_\varepsilon(t) \rightharpoonup z(t)$ *in* $W^{1,p}(\Omega)$ *for all t,*
$z_\varepsilon \to z$ *in* $\overline{\Omega_T}$,

(c) $b_\varepsilon \to b$ *in* $W^{1,1}(0,T;W^{1,\infty}(\Omega;\mathbb{R}^n))$,

(d) $\widehat{e}_\varepsilon \rightharpoonup \widehat{e}$ *in* $L^2(\Omega_T;\mathbb{R}^{n\times n})$,
$W^\varepsilon_{,e}(c_\varepsilon, e_\varepsilon, z_\varepsilon) \rightharpoonup W_{,e}(c,\widehat{e},z)$ *in* $L^2(\{z>0\};\mathbb{R}^{n\times n})$,
$W^\varepsilon_{,e}(c_\varepsilon, e_\varepsilon, z_\varepsilon) \to 0$ *in* $L^2(\{z=0\};\mathbb{R}^{n\times n})$,
$W^\varepsilon_{,c}(c_\varepsilon, e_\varepsilon, z_\varepsilon) \rightharpoonup W_{,c}(c,\widehat{e},z)$ *in* $L^2(\{z>0\};\mathbb{R}^{n\times n})$,
$W^\varepsilon_{,c}(c_\varepsilon, e_\varepsilon, z_\varepsilon) \to 0$ *in* $L^2(\{z=0\};\mathbb{R}^{n\times n})$.

To obtain a-priori estimates for the chemical potentials $\{\mu_\varepsilon\}$ in some local sense, we make use of the *conical* Poincaré inequality for star-shaped domains cited below.

Theorem 3.3 (Conical Poincaré inequality [BK98]) *Suppose that* $\Omega \subseteq \mathbb{R}^n$ *is a bounded and star-shaped domain,* $r \ge 0$ *and* $1 \le p < \infty$. *Then there exists a constant* $C = C(\Omega,p,r) > 0$ *such that*

$$\int_\Omega |w(x) - w_{\Omega,\delta^t}|^p \delta^r(x)\,\mathrm{d}x \le C \int_\Omega |\nabla w(x)|^p \delta^r(x)\,\mathrm{d}x$$

for all $w \in C^1(\Omega)$, *where the* δ^r*-weight* w_{Ω,δ^r} *is given by*

$$w_{\Omega,\delta^r} := \int_\Omega w(x)\delta^r(x)\,\mathrm{d}x, \quad \delta(x) := \mathrm{dist}(x,\partial\Omega).$$

By a density argument, the statement is, of course, also true for all $w \in W^{1,p}(\Omega)$ which will be used in this paper.

Lemma 3.4 (A-priori estimates for μ_ε)

(i) Interior estimate. For every $t \in [0,T]$ and for every open cube $Q \subset\subset \{z(t) > 0\} \cap \Omega$, there exists a $C > 0$ such that for all $0 < \varepsilon \ll 1$

$$\|\mu_\varepsilon\|_{L^2(0,t;H^1(Q))} \leq C. \tag{21}$$

(ii) Estimate at the boundary. For every $t \in [0,T]$ and every $x_0 \in \{z(t) > 0\} \cap \partial\Omega$, there exist a neighborhood U of x_0 and a $C > 0$ such that for all $0 < \varepsilon \ll 1$

$$\|\mu_\varepsilon\|_{L^2(0,t;H^1(U\cap\Omega))} \leq C. \tag{22}$$

Proof.

(i) We consider the smooth domain $\widetilde{Q} := B_\eta(Q) := \{x \in \mathbb{R}^n \,|\, \mathrm{dist}(x,Q) < \varepsilon\}$, where $\eta > 0$ is chosen so small such that $\widetilde{Q} \subset\subset \{z(t) > 0\} \cap \Omega$. Testing (17) with the $H^1(\Omega)$-function

$$\zeta(x) := \begin{cases} \delta(x) := \mathrm{dist}(x, \partial\widetilde{Q}) & \text{if } x \in \widetilde{Q}, \\ 0 & \text{else,} \end{cases}$$

and using the previous a-priori estimates yield boundedness of

$$\mu^\varepsilon_{\widetilde{Q},\delta} := \int_{\widetilde{Q}} \mu_\varepsilon(x,s)\delta(x)\,\mathrm{d}x \leq C \tag{23}$$

with respect to a.e. $s \in (0,T)$ and $\varepsilon \in (0,1)$.

There exists an $\eta > 0$ such that $z_\varepsilon(s) \geq \eta$ in $\widetilde{Q}$ for all $s \in [0,t]$ and for all $0 < \varepsilon \ll 1$ (see [HK12, Corollary 4.17]). Thus, by assumption (2), $m^\varepsilon(z_\varepsilon(s)) \geq \eta' > 0$ holds in $\widetilde{Q}$ for all $s \in [0,t]$ and all $0 < \varepsilon \ll 1$ for a common constant $\eta' > 0$. Consequently, we get by the a-priori estimate for $m^\varepsilon(z_\varepsilon)^{1/2}\nabla\mu_\varepsilon$

$$\|\nabla\mu_\varepsilon\|_{L^2(\widetilde{Q}\times[0,t])} \leq C \tag{24}$$

for all $\varepsilon \in (0,1)$. Applying Theorem 3.3 (we plug in $\Omega = \widetilde{Q}$, $r = p = 2$ and $w = \mu_\varepsilon(s)$ for $s \in [0,t]$), integrating from 0 to t and using boundedness properties (23) and (24), we obtain boundedness of $\|\mu_\varepsilon\delta\|_{L^2(\widetilde{Q}\times[0,t])}$ and thus boundedness of $\|\mu_\varepsilon\|_{L^2(Q\times[0,t])}$ with respect to $0 < \varepsilon \ll 1$. Together with (24), we get the claim (21).

(ii) By the properties of the domain, we can find a neighborhood $U \subseteq \mathbb{R}^n \setminus \{z(t) = 0\}$ of x_0 and a $\mathcal{C}^2$-diffeomorphism $\pi : (-1,1)^n \to U$ with the properties

- $\pi\big((-1,1)^{n-1} \times (-1,0)\big) \subseteq \Omega$,
- $\pi\big((-1,1)^{n-1} \times \{0\}\big) \subseteq \partial\Omega$,
- $\pi\big((-1,1)^{n-1} \times (0,1)\big) \subseteq \mathbb{R}^n \setminus \overline{\Omega}$.

Let $\vartheta : (-1,1)^n \to (-1,1)^n$ denote the reflection $x \mapsto (x_1,\ldots,x_{n-1},-x_n)$ and $\mathcal{T} := \pi \circ \vartheta \circ \pi^{-1}$. Furthermore, let $\widetilde{\mu}_\varepsilon \in L^2(0,t;H^1(U))$ be defined by

$$\widetilde{\mu}_\varepsilon(x,s) := \begin{cases} \mu_\varepsilon(x,s) & \text{if } x \in U \cap \Omega, \\ \mu_\varepsilon(\mathcal{T}(x),s) & \text{if } x \in U \setminus \overline{\Omega}. \end{cases}$$

Let $Q \subset\subset U$ be a non-empty open cube with $x_0 \in Q$. Then, integration by substitution with respect to the transformation $\mathcal{T}$ yields

$$\begin{aligned}\int_Q \widetilde{\mu}_\varepsilon(x,s)\delta(x)\,\mathrm{d}x = &\int_{Q\cap\Omega} \mu_\varepsilon(x,s)\delta(x)\,\mathrm{d}x \\ &+ \int_{\mathcal{T}(Q\setminus\Omega)} \mu_\varepsilon(x,s)\delta(\mathcal{T}(x))|\det(\nabla\mathcal{T}(x))|\,\mathrm{d}x \end{aligned} \tag{25}$$

with $\delta(x) := \mathrm{dist}(x, \partial Q)$. Here, we have used $\mathcal{T}\circ\mathcal{T} = \mathrm{Id}$. Testing (17) with

$$\zeta = \mathbb{1}_{Q\cap\Omega}\delta \in H^1(\Omega)$$

and with

$$\zeta = \mathbb{1}_{\mathcal{T}(Q\setminus\Omega)}\cdot(\delta\circ\mathcal{T})|\det(\nabla\mathcal{T})| \in H^1(\Omega)$$

and using the a-priori estimates, both terms on the right hand side of (25) are uniformly bounded with respect to $\varepsilon \in (0,1)$ and a.e. $s \in (0,T)$. Note that the property $|\det(\nabla\mathcal{T})| \in H^1(U)$ is based on the assumption that Ω has a $\mathcal{C}^2$-boundary. For $\nabla\widetilde{\mu}_\varepsilon$, we also get by integration via substitution:

$$\begin{aligned}&\int_0^t\int_Q |\nabla\widetilde{\mu}_\varepsilon(x,s)|^2\,\mathrm{d}x\,\mathrm{d}s \\ &\quad\leq \int_0^t\int_{Q\cap\Omega}|\nabla\mu_\varepsilon(x,s)|^2\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{Q\setminus\Omega}|\nabla\mu_\varepsilon(\mathcal{T}(x),s)|^2|\nabla\mathcal{T}(x)|^2\,\mathrm{d}x\,\mathrm{d}s \\ &\quad= \int_0^t\int_{Q\cap\Omega}|\nabla\mu_\varepsilon(x,s)|^2\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\mathcal{T}(Q\setminus\Omega)}|\nabla\mu_\varepsilon(x,s)|^2|\nabla\mathcal{T}(\mathcal{T}(x))|^2|\det(\nabla\mathcal{T}(x))|\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{26}$$

We know that $\nabla\mu_\varepsilon$ is bounded in $L^2((Q\cap\Omega)\times(0,t);\mathbb{R}^n)$ and in $L^2(\mathcal{T}(Q\setminus\Omega)\times(0,t);\mathbb{R}^n)$ with respect to $0<\varepsilon\ll 1$ by $Q\cap\Omega, \mathcal{T}(Q\setminus\Omega) \subset\subset \{z(t)>0\}$, by assumption (2) and by the a-priori estimates. Therefore, the left hand side of (26) is also bounded for all $0<\varepsilon\ll 1$. The Conical Poincaré inequality in Theorem 3.3 yields boundedness of $\widetilde{\mu}_\varepsilon\delta$ in $L^2(Q\times(0,t))$. Finally, we can find a neighborhood $V\subseteq Q$ of x_0 such that $\widetilde{\mu}_\varepsilon$ is bounded in $L^2(0,t;H^1(V))$.

□

Due to the a-priori estimates for $\{\mu_\varepsilon\}$ and $\{u_\varepsilon\}$, the limit functions μ and u can only be expected to be in some space-time local Sobolev space $L_t^2H_{x,\mathrm{loc}}^1$ (see Remark 2.2). In the sequel, it will be necessary to represent the maximal admissible subset of the not completely damaged area, i.e. $\mathfrak{A}_D(\{z>0\})$, as a union of Lipschitz domains which capture some parts of the Dirichlet boundary D. Following the argumentation in [HK12], we define the shrinking set $F := \{z>0\}$ and obtain the following result.

Lemma 3.5 (cf. [HK12, Lemma 4.18]) *There exists a function $u \in L_t^2H_{x,\mathrm{loc}}^1(\mathfrak{A}(F);\mathbb{R}^n)$ such that $\epsilon(u) = \widehat{e}$ a.e. in $\mathfrak{A}_\Gamma(F)$ and $u = b$ on the boundary $\Gamma_T\cap\mathfrak{A}_\Gamma(F)$.*

A related result can be shown for the sequence $\{\mu_\varepsilon\}$ by exploiting the estimates in Lemma 3.4. To proceed, we recall a basic definition from [HK12].

Definition 3.6 (cf. [HK12, Definition 4.1]) *Let $H \subseteq \overline{\Omega}$ be a relatively open subset. We call a countable family $\{U_k\}$ of open sets $U_k \subset\subset H$ a fine representation for H if for every $x \in H$ there exist an open set $U \subseteq \mathbb{R}^n$ with $x \in U$ and an $k \in \mathbb{N}$ such that $U \cap \Omega \subseteq U_k$.*

Lemma 3.7 *Let a sequence $\{t_m\} \subseteq [0,T]$ containing T be dense. There exists a fine representation $\{U_k^m\}_{k\in\mathbb{N}}$ for $F(t_m)$ for every $m \in \mathbb{N}$, a function $\mu \in L_t^2 H_{x,\mathrm{loc}}^1(F)$ and a subsequence of $\{\mu_\varepsilon\}$ (also denoted by $\{\mu_\varepsilon\}$) such that for all $k, m \in \mathbb{N}$*

$$\mu_\varepsilon \rightharpoonup \mu \text{ in } L^2(0, t_m; H^1(U_k^m)) \tag{27}$$

as $\varepsilon \to 0^+$.

Proof. A fine representation $\{U_k^m\}_{k\in\mathbb{N}}$ of $F(t_m)$ can be constructed by countably many open cubes $Q \subset\subset F(t_m) \cap \Omega$ and of finitely many open sets of the form $U \cap \Omega$ such that U satisfies (22) from Lemma 3.4 (ii). For each $k, m \in \mathbb{N}$, we have the estimate

$$\|\mu_\varepsilon\|_{L^2(0,t;H^1(U_k^m))} \leq C$$

for all $0 < \varepsilon \ll 1$ by Lemma 3.4. By successively choosing sub-sequences and by a diagonal argument, we obtain a $\mu \in L_t^2 H_{x,\mathrm{loc}}^1(F)$ such that (27) is satisfied (cf. proof of [HK12, Lemma 4.18]). ☐

The a-priori estimates and the convergence properties of $\{z_\varepsilon\}$ in Lemma 3.2 and of $\{\mu_\varepsilon\}$ in Lemma 3.7, respectively, yield the following corollary.

Corollary 3.8 *It holds for $\varepsilon \to 0^+$:*

$$\begin{aligned} m(z_\varepsilon)\nabla\mu_\varepsilon &\rightharpoonup m(z)\nabla\mu \text{ in } L^2(F;\mathbb{R}^n), \\ m(z_\varepsilon)\nabla\mu_\varepsilon &\to 0 \text{ in } L^2(\Omega_T \setminus F;\mathbb{R}^n). \end{aligned}$$

Now, we have all necessary convergence properties to perform the degenerate limit in (15)-(20). To proceed, we need the following auxiliary result.

Lemma 3.9 *Let $\{t_m\}$ and $\{U_k^m\}$ be as in Lemma 3.7. Then, for every compact subset $K \subseteq F$ there exist a finite set $I \subseteq \mathbb{N}$, values $m_k \in \mathbb{N}$, $k \in I$ and functions $\psi_k \in \mathcal{C}^\infty(\overline{\Omega_T})$, $k \in I$, such that*

(i) $K \cap \Omega_T \subseteq \bigcup_{k\in I} U_k^{m_k} \times (0, t_{m_k})$,

(ii) $\mathrm{supp}(\psi_k) \subseteq \overline{U_k^{m_k}} \times [0, t_{m_k}]$,

(iii) $\sum_{k\in I} \psi_k \equiv 1$ *on K.*

Proof. We extend the family of open sets $\{V_k^m\}$ given by $V_k^m := U_k^m \times (0, t_{m_k})$ in the following way. Define

$$\mathcal{P} := \Big\{ \{W_k^m\} \,\Big|\, W_k^m \subseteq \mathbb{R}^{n+1} \text{ is open with } W_k^m \cap \Omega_T = U_k^m \times (0, t_{m_k}) \Big\}.$$

We see that $\mathcal{P}$ is non-empty and every totally ordered subset of $\mathcal{P}$ has an upper bound with respect to the "$\leq$" ordering defined by

$$\{W_k^m\} \leq \{\widetilde{W}_k^m\} \Leftrightarrow W_k^m \subseteq \widetilde{W}_k^m \text{ for all } k, m \in \mathbb{N}.$$

By Zorn's lemma, we find a maximal element $\{\widetilde{V}_k^m\}$. It holds

$$F \subseteq \bigcup_{k,m\in\mathbb{N}} \widetilde{V}_k^m. \tag{28}$$

Assume that this condition fails. Then, because of $F \cap \Omega_T = \bigcup_{k,m\in\mathbb{N}} V_k^m$, there exists a $p = (x,t) \in F \cap \partial(\Omega_T)$ with $p \notin \bigcup_{k,m\in\mathbb{N}} \widetilde{V}_k^m$.

Let us consider the case $t < T$. Since $F \subseteq \overline{\Omega_T}$ is relatively open, we find an $m_0 \in \mathbb{N}$ with $x \in F(t_{m_0})$ and $t_{m_0} > t$. By the fine representation property of $\{U_k^{m_0}\}_{k\in\mathbb{N}}$ for $F(t_{m_0})$, we find an open set $U \subseteq \mathbb{R}^n$ with $x \in U$ and $k_0 \in \mathbb{N}$ such that $U \cap \Omega \subseteq U_{k_0}^{m_0}$.

The family $\{\widetilde{W}_k^m\}$ given by

$$\widetilde{W}_k^m := \begin{cases} \widetilde{V}_k^m \cup U \times (-\infty, t_{m_0}) & \text{if } k = k_0 \text{ and } m = m_0, \\ \widetilde{V}_k^m & \text{else,} \end{cases}$$

satisfies $\{\widetilde{W}_k^m\} \in \mathcal{P}$ and $p \in \bigcup_{k,m\in\mathbb{N}} \widetilde{W}_k^m$ which contradicts the maximality property of $\{\widetilde{V}_k^m\}$.

In the case $t = T$, we also find $k_0, m_0 \in \mathbb{N}$ and an open set $U \subseteq \mathbb{R}^n$ with $x \in U$ such that $U \cap \Omega \subseteq U_{k_0}^{m_0}$ and $t_{m_0} = T$. The family $\{\widetilde{W}_k^m\}$ given by

$$\widetilde{W}_k^m := \begin{cases} \widetilde{V}_k^m \cup U \times \mathbb{R} & \text{if } k = k_0 \text{ and } m = m_0, \\ \widetilde{V}_k^m & \text{else,} \end{cases}$$

also contradicts the maximality of $\{\widetilde{V}_k^m\}$. Therefore, (28) is proven.

Heine-Borel theorem yields

$$K \subseteq \bigcup_{k\in I} \widetilde{V}_k^{m_k}$$

for a finite set $I \subseteq \mathbb{N}$ and values $m_k \in \mathbb{N}$, $k \in I$. Together with a partition of unity argument, we get functions $\psi_k \in \mathcal{C}^\infty(\overline{\Omega_T})$ such that (i)-(iii) hold. ☐

The degenerate limit $\varepsilon \to 0^+$ can be performed as follows:

- We define the strain by $e := \widehat{e}|_F \in L^2(F;\mathbb{R}^{n\times n})$ and obtain for the remaining variables
  $$\begin{aligned} &c \in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))^\star), && u \in L_t^2 H_{x,\mathrm{loc}}^1(\mathfrak{A}_D(F);\mathbb{R}^n), \\ &z \in L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega)), && \mu \in L_t^2 H_{x,\mathrm{loc}}^1(F) \end{aligned}$$
  with $e = \epsilon(u)$ in $\mathfrak{A}(F)$.
- Passing to the limit $\varepsilon \to 0^+$ in (15), (18) and (20) imply properties (3), (6) and (8) as in [HK12].
- Using Lemma 3.2 (a) and Corollary 3.8, we can pass to $\varepsilon \to 0^+$ in (16) and obtain (4).

  Let $\zeta \in L^2(0,T;H^1(\Omega))$ with $\mathrm{supp}(\zeta) \subseteq F$ be a test-function. Furthermore, let $\{\psi_l\}$ be a partition of unity of the compact set $K := \mathrm{supp}(\zeta)$ according to Lemma 3.9. For each $l \in \mathbb{N}$, we obtain $\mathrm{supp}(\zeta\psi_l) \subseteq \overline{U_l^{m_l}} \times [0, t_{m_l}]$. Then, integrating (17) in time from 0 to t_{m_l}, testing the result with $\zeta\psi_l$ and passing to $\varepsilon \to 0^+$ by using Lemma 3.2 and Lemma 3.7 show
  $$\int_0^{t_m}\int_\Omega \mu\zeta\psi_l \,\mathrm{d}x\,\mathrm{d}s = \int_0^{t_m}\int_\Omega \nabla c\cdot\nabla(\zeta\psi_l) + \Psi_{,c}(c)\zeta\psi_l + W_{,c}(c,\widehat{e},z)\zeta\psi_l \,\mathrm{d}x\,\mathrm{d}s.$$
  Summing with respect to $l \in I$ and noticing $\sum_{l\in I}\psi_l \equiv 1$ on $\mathrm{supp}(\zeta)$ yield (5).

In conclusion, the limit procedure in this section yields functions (c, e, u, z, μ) with $e = \epsilon(u)$ in $\mathfrak{A}_D(F)$ and which satisfy properties (ii)-(vi) of Definition 2.1. In particular, the damage function z has no jumps with respect to time. We cannot ensure that $\{z > 0\}$ equals $\mathfrak{A}_D(\{z > 0\})$ and, moreover, if $F \backslash \mathfrak{A}_D(\{z > 0\}) \neq \emptyset$, it is not clear whether u can be extended to a function on F such that $e = \epsilon(u)$ also holds in F. This issue is addressed in the next section where such limit functions are concatenated in order to obtain global-in-time approximate weak solutions by Zorn's lemma.

4 Existence results

In this section, we are going to prove the main results in this paper. The proofs are based on [HK12].

Theorem 4.1 (Maximal local-in-time weak solutions)
Let $b \in W^{1,1}(0, T; W^{1,\infty}(\Omega; \mathbb{R}^n))$, $c^0 \in H^1(\Omega)$ and $z^0 \in W^{1,p}(\Omega)$ with $0 < \kappa \leq z^0 \leq 1$ in Ω be initial-boundary data. Then there exist a maximal value $\widehat{T} > 0$ with $\widehat{T} \leq T$ and functions c, u, z, μ defined on the time interval $[0, \widehat{T}]$ such that (c, u, z, μ) is a weak solution according to Definition 2.1. Therefore, if $\widehat{T} < T$, (c, u, z, μ) cannot be extended to a weak solution on $[0, \widehat{T} + \varepsilon]$.

Proof. Zorn's lemma can be applied to the set

$$\mathcal{P} := \big\{(\widehat{T}, c, u, z, \mu) \,|\, 0 < \widehat{T} \leq T \text{ and } (c, u, z, \mu) \text{ is a weak solution on } [0, \widehat{T}] \text{ according to Definition 2.1}\big\}$$

to find a maximal element with respect to the following partial ordering

$$(\widehat{T}_1, c_1, u_1, z_1, \mu_1) \leq (\widehat{T}_2, c_2, u_2, z_2, \mu_2) \quad \Leftrightarrow \quad \begin{aligned} &\widehat{T}_1 \leq \widehat{T}_2,\ c_2|_{[0,\widehat{T}_1]} = c_1,\ u_2|_{[0,\widehat{T}_1]} = u_1, \\ &z_2|_{[0,\widehat{T}_1]} = z_1,\ \mu_2|_{[0,\widehat{T}_1]} = \mu_1. \end{aligned} \tag{29}$$

Indeed, $\mathcal{P} \neq \emptyset$ by the result in Section 3. More precisely, since $z \in L^\infty(0, T; W^{1,p}(\Omega)) \cap H^1(0, T; L^2(\Omega))$ and since $0 < \kappa \leq z^0$, we find an $\varepsilon > 0$ such that $\{z(t) > 0\} = \mathfrak{A}_D(\{z(t) > 0\})$ for all $t \in [0, \varepsilon]$. For the proof that every totally ordered subset of $\mathcal{P}$ has an upper bound, we refer to [HK12]. $\square$

The proof of global-in-time existence of approximate weak solutions requires a concatenation property (see Lemma 4.4) which is, in turn, based on some deeper insights into the Γ-limit $\mathfrak{E}$. To this end, it is necessary to have more information about the recovery sequences for $\mathfrak{F}_\varepsilon \xrightarrow{\Gamma} \mathfrak{F}$.

We will introduce the following substitution method. Assume that $u \in H^1(\Omega; \mathbb{R}^n)$ minimizes $\mathcal{F}_\varepsilon(c, \epsilon(\cdot), z)$ with Dirichlet data ξ on D. Then, by expressing the elastic energy density W in terms of its derivative $W_{,e}$, i.e.

$$W = \frac{1}{2} W_{,e} : e + \frac{1}{2} z\varphi^2(c) : e + z\varphi^3(c),$$

and by testing the Euler-Lagrange equation for u with $\zeta = u - \widetilde{u}$ for a function $\widetilde{u} \in H^1(\Omega; \mathbb{R}^n)$ with $\widetilde{u} = \xi$ on D, the elastic energy term in $\mathcal{F}_\varepsilon$ can be rewritten as

$$\int_\Omega W_\varepsilon(c, \epsilon(u), z)\,\mathrm{d}x = \int_\Omega (g(z) + \varepsilon) \left(\varphi^1 \epsilon(u) : \epsilon(\widetilde{u}) + \frac{1}{2}\varphi^2(c) : (\epsilon(u) + \epsilon(\widetilde{u})) + \varphi^3(c) \right) \mathrm{d}x. \tag{30}$$

For convenience, in the following proof, the density $\widetilde{W}$ is defined as

$$\widetilde{W}_\varepsilon(c,e,e_1,z) := (g(z)+\varepsilon)\left(\varphi^1 e : e_1 + \frac{1}{2}\varphi^2(c) : (e+e_1) + \varphi^3(c)\right).$$

Lemma 4.2 *For every $c \in H^1(\Omega)$, $\xi \in W^{1,\infty}(\Omega)$ and $z \in W^{1,p}(\Omega)$ there exists a sequence $\delta_\varepsilon \to 0^+$ such that $(c,\xi,(z-\delta_\varepsilon)^+) \to (c,\xi,z)$ is a recovery sequence for $\mathfrak{F}_\varepsilon \xrightarrow{\Gamma} \mathfrak{F}$.*

Proof. We follow the idea of the proof in [HK12, Lemma 4.9]. But here we have an additional concentration variable which complicates the calculation. Let $(c_\varepsilon,\xi_\varepsilon,z_\varepsilon) \to (c,\xi,z)$ be a recovery sequence such that $(z-\delta_\varepsilon)^+ \le z_\varepsilon$ for some sequence $\delta_\varepsilon \to 0^+$. Consider

$$\mathfrak{F}_\varepsilon(c,\xi,(z-\delta_\varepsilon)^+) - \mathfrak{F}_\varepsilon(c_\varepsilon,\xi_\varepsilon,z_\varepsilon) = \underbrace{\mathfrak{F}_\varepsilon(c,\xi,(z-\delta_\varepsilon)^+) - \mathfrak{F}_\varepsilon(c,\xi,z_\varepsilon)}_{A_\varepsilon} + \underbrace{\mathfrak{F}_\varepsilon(c,\xi,z_\varepsilon) - \mathfrak{F}_\varepsilon(c_\varepsilon,\xi_\varepsilon,z_\varepsilon)}_{B_\varepsilon}.$$

Since $A_\varepsilon \le 0$, we focus on the second term of the right hand side. Let $u_\varepsilon, v_\varepsilon \in H^1_D(\Omega;\mathbb{R}^n)$ be given by

$$u_\varepsilon = \underset{\zeta\in H^1_D(\Omega;\mathbb{R}^n)}{\arg\min}\ \mathcal{F}_\varepsilon(c,\epsilon(\xi+\zeta),z_\varepsilon), \quad v_\varepsilon = \underset{\zeta\in H^1_D(\Omega;\mathbb{R}^n)}{\arg\min}\ \mathcal{F}_\varepsilon(c_\varepsilon,\epsilon(\xi_\varepsilon+\zeta),z_\varepsilon).$$

Using (30) for $(c,\xi+u_\varepsilon,z_\varepsilon)$ with test-function $\widetilde{u}=v_\varepsilon$ and (30) for $(c_\varepsilon,\xi_\varepsilon+v_\varepsilon,z_\varepsilon)$ with test-function $\widetilde{u}=u_\varepsilon$, we obtain a calculation as follows:

$$\begin{aligned}
B_\varepsilon &= \mathcal{F}_\varepsilon(c,\epsilon(\xi+u_\varepsilon),z_\varepsilon) - \mathcal{F}_\varepsilon(c_\varepsilon,\epsilon(\xi_\varepsilon+v_\varepsilon),z_\varepsilon,)\\
&= \int_\Omega \widetilde{W}(c,\epsilon(\xi+u_\varepsilon),\epsilon(\xi+v_\varepsilon),z_\varepsilon+\varepsilon) - \widetilde{W}(c_\varepsilon,\epsilon(\xi_\varepsilon+v_\varepsilon),\epsilon(\xi_\varepsilon+u_\varepsilon),z_\varepsilon+\varepsilon)\,\mathrm{d}x\\
&= \int_\Omega (g(z_\varepsilon)+\varepsilon)\Big(\varphi^1\epsilon(\xi+u_\varepsilon):\epsilon(\xi+v_\varepsilon) - \varphi^1\epsilon(\xi_\varepsilon+u_\varepsilon):\epsilon(\xi_\varepsilon+v_\varepsilon)\\
&\qquad + \frac{1}{2}\varphi^2(c):\epsilon(2\xi+u_\varepsilon+v_\varepsilon) - \frac{1}{2}\varphi^2(c_\varepsilon):\epsilon(2\xi+v_\varepsilon+u_\varepsilon) + \varphi^3(c) - \varphi^3(c_\varepsilon)\Big)\,\mathrm{d}x\\
&= \int_\Omega (g(z_\varepsilon)+\varepsilon)\Big(\varphi^1\epsilon(\xi):\epsilon(\xi) - \varphi^1\epsilon(\xi_\varepsilon):\epsilon(\xi_\varepsilon) + \varphi^1\epsilon(u_\varepsilon+v_\varepsilon):\epsilon(\xi-\xi_\varepsilon)\\
&\qquad + \varphi^2(c):\epsilon(\xi-\xi_\varepsilon) + \frac{1}{2}(\varphi^2(c)-\varphi^2(c_\varepsilon)):\epsilon(2\xi_\varepsilon+u_\varepsilon+v_\varepsilon) + \varphi^3(c)-\varphi^3(c_\varepsilon)\Big)\,\mathrm{d}x\\
&\le \int_\Omega (g(z_\varepsilon)+\varepsilon)\Big(\varphi^1\epsilon(\xi):\epsilon(\xi) - \varphi^1\epsilon(\xi_\varepsilon):\epsilon(\xi_\varepsilon) + \varphi^2(c):\epsilon(\xi-\xi_\varepsilon) + \varphi^3(c)-\varphi^3(c_\varepsilon)\Big)\,\mathrm{d}x\\
&\quad + \|(g(z_\varepsilon)+\varepsilon)\varphi^1\epsilon(u_\varepsilon+v_\varepsilon)\|_{L^2(\Omega)}\|\epsilon(\xi-\xi_\varepsilon)\|_{L^2(\Omega)}\\
&\quad + \frac{1}{2}\|\varphi^2(c)-\varphi^2(c_\varepsilon)\|_{L^2(\Omega)}\Big(\|(g(z_\varepsilon)+\varepsilon)\epsilon(\xi_\varepsilon+u_\varepsilon)\|_{L^2(\Omega)} + \|(g(z_\varepsilon)+\varepsilon)\epsilon(\xi_\varepsilon+v_\varepsilon)\|_{L^2(\Omega)}\Big)
\end{aligned}$$

Using the convergence properties $c_\varepsilon \rightharpoonup c$ in $H^1(\Omega)$, $\xi_\varepsilon \to \xi$ in $W^{1,\infty}(\Omega)$, $z_\varepsilon \rightharpoonup z$ in $W^{1,p}(\Omega)$ and the boundedness of $\mathcal{F}_\varepsilon(c,\epsilon(\xi+u_\varepsilon),z_\varepsilon)$ and $\mathcal{F}_\varepsilon(c_\varepsilon,\epsilon(\xi_\varepsilon+v_\varepsilon),z_\varepsilon)$ with respect to ε, we conclude $\limsup_{\varepsilon\to 0^+} B_\varepsilon \le 0$. The claim follows as in [HK12, Lemma 4.9]. $\square$

Remark 4.3 *The knowledge of such recovery sequences for $\mathfrak{F}_\varepsilon \xrightarrow{\Gamma} \mathfrak{F}$ gives also more information about $\mathfrak{E}$. In particular, we obtain an analogous result for $\mathfrak{E}$ as in Lemma 4.2 and, moreover, the following properties* *(cf. [HK12, Corollary 4.10, Lemma 4.11]):*

- $\mathfrak{E}(c,\xi,\mathbb{1}_F z)\leq\mathfrak{E}(c,\xi,z)$ $\quad$ $\forall c\in H^1(\Omega),\ \forall\xi\in W^{1,\infty}(\Omega;\mathbb{R}^n),\ \forall z\in W^{1,p}(\Omega)$ $\forall F\subseteq\Omega$ *open with* $\mathbb{1}_F z\in W^{1,p}(\Omega)$,
- $\mathfrak{E}(c,\xi,z)\leq\mathcal{E}(c,\epsilon(u),z)$ $\quad$ $\forall c\in H^1(\Omega),\ \forall\xi\in W^{1,\infty}(\Omega;\mathbb{R}^n),\ \forall z\in W^{1,p}(\Omega)$ *with* $0\leq z\leq 1, \forall u\in H^1_{\mathrm{loc}}(\{z>0\};\mathbb{R}^n)$ *with* $u=\xi$ *on* $D\cap\{z>0\}$.

Lemma 4.4 (cf. [HK12, Lemma 4.21]) *Let $t_1<t_2<t_3$ be real numbers and let $\eta>0$. Suppose that*

$$\widetilde{q}:=(\widetilde{c},\widetilde{e},\widetilde{u},\widetilde{z},\widetilde{\mu},\widetilde{F}) \text{ is an approximate weak solution on } [t_1,t_2],$$
$$\widehat{q}:=(\widehat{c},\widehat{e},\widehat{u},\widehat{z},\widehat{\mu},\widehat{F}) \text{ is an approximate weak solution on } [t_2,t_3]$$
$$\text{with } \widehat{\mathfrak{e}}_{t_2}^+=\mathfrak{E}(\widehat{c}(t_2),\widehat{b}(t_2),\widehat{z}^+(t_2)) \text{ (the value } \mathfrak{e}_{t_2}^+ \text{ for } \widehat{q} \text{ in Definition 2.1).}$$

Furthermore, suppose the compatibility condition $\widehat{c}(t_2)=\widetilde{c}(t_2)$ and $\widehat{z}^+(t_2)=\widetilde{z}^-(t_2)\mathbb{1}_{\mathfrak{A}_D(\{\widetilde{z}^-(t_2)>0\})}$ and the Dirichlet boundary data $b\in W^{1,1}(t_1,t_3;W^{1,\infty}(\Omega;\mathbb{R}^n))$.

Then, we obtain that $q:=(c,e,u,z,\mu,F)$ defined as $q|_{[t_1,t_2)}:=\widetilde{q}$ and $q|_{[t_2,t_3]}:=\widehat{q}$ is an approximate weak solution on $[t_1,t_3]$.

Proof. Because of the properties in Remark 4.3 we can prove the following crucial energy estimate at time point t_2:

$$\begin{aligned}
\lim_{s\to t_2^-}\operatorname*{ess\,inf}_{\tau\in(s,t_2)}\mathcal{E}(c(\tau),e(\tau),z(\tau)) &= \lim_{s\to t_2^-}\operatorname*{ess\,inf}_{\tau\in(s,t_2)}\mathcal{E}(c(\tau),e(\tau),z^-(\tau))\\
&\geq \lim_{s\to t_2^-}\operatorname*{ess\,inf}_{\tau\in(s,t_2)}\mathcal{E}(c(\tau),\epsilon(u(\tau)),z^-(\tau)\mathbb{1}_{\mathfrak{A}_D(\{z^-(\tau)>0\})})\\
&\geq \lim_{s\to t_2^-}\operatorname*{ess\,inf}_{\tau\in(s,t_2)}\mathfrak{E}(c(\tau),b(\tau),z^-(\tau)\mathbb{1}_{\mathfrak{A}_D(\{z^-(\tau)>0\})})\\
&\geq \mathfrak{E}(c(t_2),b(t_2),\chi)\\
&\geq \mathfrak{E}(c(t_2),b(t_2),z^+(t_2))
\end{aligned}$$

with $\chi:=z^-(t_2)\mathbb{1}_{\bigcap_{\tau\in(t_1,t_2)}\mathfrak{A}_D(\{z^-(\tau)>0\})}$. With this estimate, we can verify the claim by the same argumentation as for [HK12, Lemma 4.21]. $\square$

Theorem 4.5 (Global-in-time approximate weak solutions)
Let $b\in W^{1,1}(0,T;W^{1,\infty}(\Omega;\mathbb{R}^n))$, $c^0\in H^1(\Omega)$ and $z^0\in W^{1,p}(\Omega)$ with $0\leq z^0\leq 1$ in Ω and $\{z^0>0\}$ admissible with respect to D be initial-boundary data. Furthermore, let $\eta>0$. Then there exists an approximate weak solution (c,e,u,z,μ) with fineness $\eta>0$ according to Definition 2.3.

Proof. This result can also be proven by using Zorn's lemma on the set

$$\mathcal{P}:=\big\{(\widehat{T},c,e,u,z,\mu,F)\,|\,0<\widehat{T}\leq T \text{ and } (c,e,u,z,\mu,F) \text{ is an approximate weak solution on } [0,\widehat{T}] \text{ with fineness } \eta \text{ according to Definition 2.3}\big\}$$

with an ordering analogously to (29). The assumptions for Zorn's lemma can be proven as in Theorem 4.1 (see [HK12, Proof of Theorem 4.1]). To show that a maximal element from $\mathcal{P}$ is actually an approximate weak solution on the time-interval $[0,T]$, we need the concatenation property in Lemma 4.4. Indeed, if a maximal element $\widetilde{q}$ is only defined on an time-interval $[0,\widetilde{T}]$ with $\widetilde{T} < T$ we can apply the degenerated limit procedure in Section 3 to the initial values $c(\widetilde{T})$ and $z(\widetilde{T})$ to obtain a new limit function $\widehat{q}$. By exploiting Lemma 4.4, q is an approximate weak solution on the time-interval $[0,\widetilde{T}+\varepsilon]$ for a small $\varepsilon > 0$ which contradicts the maximality of $\widetilde{q}$. $\square$

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