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Unsaturated deformable porous media flow with phase transition

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Abstract

In the present paper, a continuum model is introduced for fluid flow in a deformable porous medium, where the fluid may undergo phase transitions. Typically, such problems arise in modeling liquid-solid phase transformations in groundwater flows. The system of equations is derived here from the conservation principles for mass, momentum, and energy and from the Clausius-Duhem inequality for entropy. It couples the evolution of the displacement in the matrix material, of the capillary pressure, of the absolute temperature, and of the phase fraction. Mathematical results are proved under the additional hypothesis that inertia effects and shear stresses can be neglected. For the resulting highly nonlinear system of two PDEs, one ODE and one ordinary differential inclusion with natural initial and boundary conditions, existence of global in time solutions is proved by means of cut-off techniques and suitable Moser-type estimates.

Introduction

A model for fluid flow in partially saturated porous media with thermomechanical interaction was proposed and analyzed in [2, 5, 6]. Here, we extend the model by including the effects of freezing and melting of the fluid in the pores. Typical examples, in which such situations arise, are related to groundwater flows and to the freezing-melting cycles of water sucked into the pores of concrete. Notice that the latter process forms one of the main reasons for the degradation of concrete in buildings, bridges, and roads. However, many of the governing effects in concrete like the multi-component microstructure, the breaking of pores, chemical reactions, the hysteresis of the saturation-pressure curves, and the occurrence of shear stresses, are still neglected in our model.

While often in continuum models for three-component and multi-component porous media the intention is to describe the propagation of sound waves in these media (e.g., [1, 28]), we investigate in the present paper – instead of using partial balance equations for each component – a continuum model combining the principles of conservation of mass and momentum with the first and the second principles of thermodynamics. In, e. g., [1], flow in porous media is described in Eulerian coordinates in order to incorporate, for example, the effects of fast convection. Here, instead, we assume that slow diffusion is dominant, and choose the Lagrangian description as in [2, 5, 28]. The resulting system of coupled ODEs and PDEs then appears to be a nonlinear extension of the linear model in [26], referred to as a simplified Biot system, to the case when also the occurrence of temperature changes and phase transitions is taken into account. In addition to the model studied in [5], we include here the effects of freezing and melting. The idea is the following. The pores in the matrix material contain a mixture of H_2O and gas, and H_2O itself is a mixture of the liquid (water) and the solid phase (ice). That is, in addition to the other physical quantities like capillary pressure, displacement, and absolute temperature, we need to consider the evolution of a phase parameter χ representing the relative proportion of water in the H_2O part and its influence on pressure changes due to the different mass densities of water and ice. Unlike in [2, 5, 6, 27], we do not consider hysteresis in the model. We believe that the mathematical results can be extended to the case of capillary hysteresis as in [2, 5, 6]. In our model without shear stresses, elastoplastic hysteresis effects as in [2, 5, 27] cannot occur.

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As it will be detailed in Section 1, we assume that the deformations are small, so that $\operatorname{div} u$ is the relative local volume change, where u represents the displacement vector. Moreover, we assume that the volume of the matrix material does not change during the process, and thus the volume and mass balance equations with Darcy's law for the water flux lead to a nonlinear degenerate parabolic equation for the capillary pressure, see (1.7). In the equation of motion, we take into account the pressure components due to phase transition and temperature changes, and we further simplify the system in order to make it mathematically tractable by assuming that the process is quasistatic and the shear stresses are negligible. The problem of existence of solutions for the coupled system without this assumption is open and, in our opinion, very challenging. Finally, we use the balance of internal energy and the entropy inequality to derive the dynamics for absolute temperature and phases; they turn out to be, respectively, a parabolic equation for the temperature with highly nonlinear right-hand side (quadratic in the derivatives) and an ordinary differential inclusion for the phase parameter χ.

Finally, let us note that – in order to model the freezing and melting phenomena in the pores – we have borrowed here some ideas from our earlier publications on freezing and melting in containers filled with water with rigid, elastic, or elastoplastic boundaries (cf. [11, 14, 15, 16, 17]). It was shown there how important it is to account for the difference in specific volumes of water and of ice.

There is an abundant classical mathematical literature on phase transition processes, see, e.g., the monographs [4], [7], [29], and the references therein. It seems, however, that only few publications take into account that the mass densities and specific volumes of the phases differ. In [8], the authors proposed to interpret a phase transition process in terms of a balance equation for macroscopic motions, and to include the possibility of voids. Well-posedness of an initial-boundary value problem associated with the resulting PDE system was proved there, and the case of two different densities ϱ_1 and ϱ_2 for the two substances undergoing phase transitions was pursued in [9].

Let us also mention the papers [22, 23, 24, 25] dealing with macroscopic stresses in phase transitions models, where the different properties of the viscous (liquid) and elastic (solid) phases were taken into account and the coexisting viscous and elastic properties of the system were given a distinguished role, under the working assumption that they indeed influence the phase transition process. The model studied there includes inertia, viscous, and shear viscosity effects (depending on the phases). This is reflected in the analytical expressions of the associated PDEs for the strain u and the phase parameter χ: the χ-dependence, e. g., in the stress-strain relation, leads to the possible degeneracy of the elliptic operator therein. Finally, we can quote in this framework the model analyzed in [18] and [19], which pertains to nonlinear thermoviscoplasticity: in the spatially one-dimensional case, the authors prove the global well-posedness of a PDE system that both incorporates hysteresis effects and models phase change but, however, does not display a degenerating character.

Another coupled system for temperature, displacement, and phase parameter has been derived in order to model the full thermomechanical behavior of shape memory alloys. A long list of references for further developments can be found in the monographs [7] and [29].

The paper is organized as follows: in the next Section 1, we derive the model in full generality from the basic principles of continuum thermodynamics. In Section 2, we state the mathematical problem, the main assumptions on the data, and the main Theorem 2.2, the proof of which is split into Sections 3, 4, and 5. The steps of the proof are as follows: we first cut off some of the pressure and temperature dependent terms in the system in Section 3 by means of a cut-off parameter R and solve the related problem employing a special Galerkin approximation scheme. Then, in Section 4, we first prove the positivity of the temperature by means of a maximum principle technique, and then we perform the – independent of R – estimates on the system. They mainly consist of: the energy estimate, the so-called Dafermos estimate (with negative small powers of the temperature), Moser-type and then higher-order estimates for the capillary pressure and for the temperature. This allows us in Section 5 to pass to the limit in the cut-off system as $R \to \infty$, which will conclude the proof of the existence result.

1 The model

We consider a connected domain $\Omega \subset \mathbb{R}^3$ filled by a deformable matrix material with pores containing a mixture of H_2O and gas, where we assume that H_2O may appear in one of the two phases: water or ice. We also assume that the volume of the solid matrix remains constant during the process, and let $c_s \in (0, 1)$ be the relative proportion of solid in the total reference volume. We denote, for $x \in \Omega$ and time $t \in [0, T]$,

$W(x, t) \in [0, 1]$ ... relative proportion of H_2O in the total pore volume;

$A(x, t) \in [0, 1]$ ... relative proportion of gas in the total pore volume;

$\chi(x, t) \in [0, 1]$ ... relative proportion of water in the H_2O part;

$\xi(x, t)$ ... mass flux vector;

$p(x, t)$ ... capillary pressure;

$u(x, t)$ ... displacement vector;

$\sigma(x, t)$ ... stress tensor;

$\theta(x, t)$ ... absolute temperature.

Then χW represents the relative proportion of water in the total pore volume, and $(1 - \chi)W$ represents the relative proportion of ice in the total pore volume.

We assume that the deformations are small, so that $\operatorname{div} u$ is the relative local volume change. By hypothesis, the volume of the matrix material does not change, so that the volume balance reads

$$W(x, t) + A(x, t) + c_s = 1 + \operatorname{div} u(x, t). \tag{1.1}$$

For A, we assume the functional relation

$$A = 1 - c_s - \varphi(p)\,, \tag{1.2}$$

where φ is an increasing function that satisfies $\varphi(-\infty) = \varphi^\flat \in (0, 1)$ and $\varphi(\infty) = 1 - c_s$, $\varphi^\flat + c_s < 1$. This means that the porous medium cannot be made completely dry by thermomechanical processes alone. Combining (1.1) with (1.2), we obtain that

$$W = \varphi(p) + \operatorname{div} u\,. \tag{1.3}$$

1.1 Mass balance

Consider an arbitrary control volume $V \subset \Omega$. The water content in V is given by the integral $\int_V \rho_L \chi W \,\mathrm{d}x$, where ρ_L is the water mass density, and the ice content is $\int_V \rho_S(1-\chi)W \,\mathrm{d}x$, where ρ_S is the ice mass density. The mass conservation principle then reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho_L \chi W \,\mathrm{d}x + \int_{\partial V} \xi \cdot n \,\mathrm{d}s(x) = -\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho_S(1-\chi)W \,\mathrm{d}x\,, \tag{1.4}$$

where n the unit outward normal vector to ∂V. In differential form, we obtain

$$\rho_L(\chi W)_t + \operatorname{div}\xi = -\rho_S((1-\chi)W)_t\,. \tag{1.5}$$

The right-hand side of (1.5) is the positive or negative liquid water source due to the solidification or melting of the ice. We assume the water flux in the form of the Darcy law

$$\xi = -\mu(p)\nabla p, \tag{1.6}$$

with a proportionality factor $\mu(p) > 0$. This, (1.3), and (1.5), yield the equation

$$\big((\chi{+}\rho^*(1{-}\chi))(\varphi(p) + \operatorname{div} u)\big)_t - \frac{1}{\rho_L}\operatorname{div}(\mu(p)\nabla p) = 0, \tag{1.7}$$

with $\rho^* = \rho_S/\rho_L \in (0,1)$.

1.2 Equation of motion

The equation of motion is considered in the form

$$\rho_M u_{tt} - \operatorname{div}\sigma = g\,, \tag{1.8}$$

where ρ_M is the mass density of the matrix material, σ is the stress tensor, and g is a volume force acting on the body (e. g., gravity). For σ, we prescribe the constitutive equation

$$\sigma = B\varepsilon_t + A\varepsilon + \big((\chi{+}\rho^*(1{-}\chi))(\lambda \operatorname{div} u - p) - \beta(\theta - \theta_c)\big)\delta\,, \tag{1.9}$$

where $\varepsilon = \nabla_s u := \frac{1}{2}(\nabla u + \nabla u^T)$ is the small strain tensor, δ is the Kronecker tensor, B is a symmetric positive definite viscosity tensor, A is the symmetric positive definite elasticity tensor of the matrix material, $\lambda > 0$ is the bulk elasticity modulus of water, $\theta > 0$ is the absolute temperature, $\theta_c > 0$ is a fixed referential temperature, and $\beta \in \mathbb{R}$ is the relative solid-liquid thermal expansion coefficient. The term $(\chi{+}\rho^*(1{-}\chi))(\lambda \operatorname{div} u - p)$ accounts for the pressure component due to the phase transition.

1.3 Energy and entropy balance

We have to derive formulas for the densities of internal energy U and entropy S such that the energy balance balance equation and the Clausius–Duhem inequality hold for all processes. Let q be the heat flux vector, and let $V \subset \Omega$ be again an arbitrary control volume. The total internal energy in V is $\int_V U \,\mathrm{d}x$, and the total mechanical power $Q(V)$ supplied to V equals

$$Q(V) = \int_V \sigma : \varepsilon_t \,\mathrm{d}x - \int_{\partial V} \frac{1}{\rho_L} p\,\xi \cdot n \,\mathrm{d}s(x)\,,$$

where ξ is the fluid mass flux (1.6). We thus have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V U\,\mathrm{d}x + \int_{\partial V} q\cdot n\,\mathrm{d}s(x) = \int_V \sigma : \varepsilon_t\,\mathrm{d}x - \int_{\partial V}\frac{1}{\rho_L}p\,\xi\cdot n\,\mathrm{d}s(x)\,. \tag{1.10}$$

Again, by the Gauss formula, we obtain the energy balance equation in differential form, namely

$$U_t + \operatorname{div} q = \sigma : \varepsilon_t - \frac{1}{\rho_L}\operatorname{div}(p\xi)\,. \tag{1.11}$$

The internal energy and entropy densities U and S, as well as the heat flux vector q, have to be chosen in order to satisfy, for all processes, the Clausius–Duhem inequality

$$S_t + \operatorname{div}\left(\frac{q}{\theta}\right) \ge 0, \tag{1.12}$$

or, taking into account the energy balance (1.11),

$$U_t - \theta S_t + \frac{q\cdot\nabla\theta}{\theta} \le \sigma : \varepsilon_t - \frac{1}{\rho_L}\operatorname{div}(p\xi)\,. \tag{1.13}$$

We consider $\varepsilon, \chi, p, \theta$ as state variables and U, S as state functions, independent of $\nabla\theta$. Hence, as a consequence of (1.13), two inequalities have to hold separately for all processes, namely

$$q\cdot\nabla\theta \le 0\,, \quad U_t - \theta S_t \le \sigma : \varepsilon_t - \frac{1}{\rho_L}\operatorname{div}(p\xi)\,. \tag{1.14}$$

For simplicity, we assume Fourier's law for the heat flux,

$$q = -\kappa(\theta)\nabla\theta, \tag{1.15}$$

with the heat conductivity coefficient $\kappa = \kappa(\theta) > 0$. We further introduce the free energy F by the formula $F = U - \theta S$, so that, in terms of F, the second inequality in (1.14) takes the form

$$F_t + \theta_t S \le \sigma : \varepsilon_t + \frac{1}{\rho_L}\operatorname{div}(p\mu(p)\nabla p)\,. \tag{1.16}$$

We claim that the right choice of F for (1.16) to hold is given by

$$\begin{aligned} F &= \frac{1}{2}A\varepsilon : \varepsilon + (\chi + \rho^*(1-\chi))\left(V(p) + \frac{\lambda}{2}(\operatorname{div} u)^2\right) \\ &\quad + L\chi\left(1 - \frac{\theta}{\theta_c}\right) - \beta(\theta - \theta_c)\operatorname{div} u + F_0(\theta) + I(\chi), \end{aligned} \tag{1.17}$$

$$S = -\frac{\partial F}{\partial\theta} = \frac{L}{\theta_c}\chi + \beta\operatorname{div} u - F_0'(\theta), \tag{1.18}$$

where

$$V(p) = p\varphi(p) - \Phi(p)\,, \quad \Phi(p) = \int_0^p \varphi(\tau)\,\mathrm{d}\tau\,, \tag{1.19}$$

$F_0(\theta)$ is a purely caloric component of F, $L > 0$ is the latent heat, and I is the indicator function of the interval $[0,1]$. It remains to check that if we choose the phase dynamics equation in the form

$$\gamma(\theta)\chi_t + \partial I(\chi) \ni (1-\rho^*)\left(\Phi(p) + p\operatorname{div} u - \frac{\lambda}{2}(\operatorname{div} u)^2\right) + L\left(\frac{\theta}{\theta_c} - 1\right) \tag{1.20}$$

with a coefficient $\gamma(\theta) > 0$, then (1.16) holds for all processes. Indeed, by (1.17)–(1.18) and (1.7) we have that

$$
\begin{aligned}
F_t + \theta_t S &= A\varepsilon : \varepsilon_t + (\chi{+}\rho^*(1{-}\chi))(V'(p)p_t + \lambda \operatorname{div} u \operatorname{div} u_t) \\
&\quad + (1-\rho^*)\chi_t \left(V(p) + \frac{\lambda}{2}(\operatorname{div} u)^2 \right) + L\chi_t \left(1 - \frac{\theta}{\theta_c} \right) - \beta(\theta-\theta_c) \operatorname{div} u_t \,, \\
\sigma : \varepsilon_t &= B\varepsilon_t : \varepsilon_t + A\varepsilon : \varepsilon_t + (\chi{+}\rho^*(1{-}\chi))(\lambda \operatorname{div} u \operatorname{div} u_t - p \operatorname{div} u_t) \\
&\quad - \beta(\theta-\theta_c) \operatorname{div} u_t \,, \\
\frac{1}{\rho_L} \operatorname{div}(p\mu(p)\nabla p) &= \frac{1}{\rho_L}\mu(p)|\nabla p|^2 + p(\chi{+}\rho^*(1{-}\chi))(\varphi'(p)p_t + \operatorname{div} u_t) \\
&\quad + p(1-\rho^*)\chi_t(\varphi(p) + \operatorname{div} u) \,.
\end{aligned}
$$

Hence (note that $V(p) - p\varphi(p) = -\Phi(p)$),

$$
\begin{aligned}
F_t + \theta_t S - \sigma : \varepsilon_t - \frac{1}{\rho_L} \operatorname{div}(p\mu(p)\nabla p) &= -B\varepsilon_t : \varepsilon_t - \frac{1}{\rho_L}\mu(p)|\nabla p|^2 \\
&\quad + \chi_t \left(L\left(1 - \frac{\theta}{\theta_c}\right) + (1-\rho^*)\left(\frac{\lambda}{2}(\operatorname{div} u)^2 - \Phi(p) - p \operatorname{div} u \right) \right) \\
&= -B\varepsilon_t : \varepsilon_t - \frac{1}{\rho_L}\mu(p)|\nabla p|^2 - \gamma(\theta)\chi_t^2 \le 0, \qquad (1.21)
\end{aligned}
$$

by virtue of (1.20), so that (1.16) holds.

Now observe that

$$
\begin{aligned}
U &= F + \theta S \\
&= \frac{1}{2} A\varepsilon : \varepsilon + (\chi{+}\rho^*(1{-}\chi)) \left(V(p) + \frac{\lambda}{2}(\operatorname{div} u)^2 \right) \\
&\quad + L\chi + \beta\theta_c \operatorname{div} u + F_0(\theta) - \theta F_0'(\theta) + I(\chi) \,. \qquad (1.22)
\end{aligned}
$$

The derivative of the purely caloric component $F_0(\theta) - \theta F_0'(\theta)$ is the specific heat capacity $c(\theta) = -\theta F''(\theta)$. Assuming that $c(\theta) = c_0$ is a positive constant, we obtain that $F_0(\theta) = -c_0\theta\log(\theta/\theta_c)$ up to a linear function, and

$$
U = \frac{1}{2} A\varepsilon : \varepsilon + (\chi{+}\rho^*(1{-}\chi)) \left(V(p) + \frac{\lambda}{2}(\operatorname{div} u)^2 \right) + L\chi + \beta\theta_c \operatorname{div} u + c_0\theta + I(\chi) \,. \qquad (1.23)
$$

We now rewrite Eq. (1.11) in a more suitable form, using (1.21). We have

$$
\begin{aligned}
0 &= U_t + \operatorname{div} q - \sigma : \varepsilon_t - \frac{1}{\rho_L} \operatorname{div}(p\mu(p)\nabla p) \\
&= (F + \theta S)_t + \operatorname{div} q - \sigma : \varepsilon_t - \frac{1}{\rho_L} \operatorname{div}(p\mu(p)\nabla p) \\
&= -B\varepsilon_t : \varepsilon_t - \frac{1}{\rho_L}\mu(p)|\nabla p|^2 - \gamma(\theta)\chi_t^2 + \theta S_t + \operatorname{div} q \,, \qquad (1.24)
\end{aligned}
$$

which yields the identity

$$
c_0\theta_t - \operatorname{div}(\kappa(\theta)\nabla\theta) = B\varepsilon_t : \varepsilon_t + \frac{1}{\rho_L}\mu(p)|\nabla p|^2 + \gamma(\theta)\chi_t^2 - \frac{L}{\theta_c}\theta\chi_t - \beta\theta \operatorname{div} u_t \,. \qquad (1.25)
$$

2 The mathematical problem

We consider the system

$$\big((\chi+\rho^*(1-\chi))(\varphi(p)+\operatorname{div} u)\big)_t = \frac{1}{\rho_L}\operatorname{div}(\mu(p)\nabla p)\,, \tag{2.1}$$

$$\rho_M u_{tt} = \operatorname{div}\sigma + g\,, \tag{2.2}$$

$$\sigma = B\nabla_s u_t + A\nabla_s u + ((\chi+\rho^*(1-\chi))(\lambda\operatorname{div} u - p) - \beta(\theta-\theta_c))\delta\,, \tag{2.3}$$

$$\gamma(\theta)\chi_t + \partial I(\chi) \ni (1-\rho^*)\left(\Phi(p) + p\operatorname{div} u - \frac{\lambda}{2}(\operatorname{div} u)^2\right) + L\left(\frac{\theta}{\theta_c} - 1\right), \tag{2.4}$$

$$c_0\theta_t - \operatorname{div}(\kappa(\theta)\nabla\theta) = B\nabla_s u_t : \nabla_s u_t + \frac{1}{\rho_L}\mu(p)|\nabla p|^2 + \gamma(\theta)\chi_t^2 - \frac{L}{\theta_c}\theta\chi_t - \beta\theta\operatorname{div} u_t\,, \tag{2.5}$$

for the unknown functions p, u, χ, θ, coupled with the boundary conditions

$$u = 0\,, \tag{2.6}$$

$$\xi\cdot n = \alpha(x)(p-p^*)\,, \tag{2.7}$$

$$q\cdot n = \omega(x)(\theta-\theta^*)\,, \tag{2.8}$$

on $\partial\Omega$, where p^* is a given outer pressure, θ^* is a given outer temperature, $\alpha(x)\ge 0$ is the permeability of the boundary, and $\omega(x)\ge 0$ is the heat conductivity of the boundary.

We can also simplify the problem by assuming that water is incompressible. This corresponds to the choice $\lambda = 0$, whence the system becomes

$$\big((\chi+\rho^*(1-\chi))(\varphi(p)+\operatorname{div} u)\big)_t = \frac{1}{\rho_L}\operatorname{div}(\mu(p)\nabla p)\,, \tag{2.9}$$

$$\rho_M u_{tt} = \operatorname{div}\sigma + g\,, \tag{2.10}$$

$$\sigma = B\nabla_s u_t + A\nabla_s u - (p(\chi+\rho^*(1-\chi)) - \beta(\theta-\theta_c))\delta\,, \tag{2.11}$$

$$\gamma(\theta)\chi_t + \partial I(\chi) \ni (1-\rho^*)(\Phi(p) + p\operatorname{div} u) + L\left(\frac{\theta}{\theta_c} - 1\right), \tag{2.12}$$

$$c_0\theta_t - \operatorname{div}(\kappa(\theta)\nabla\theta) = B\nabla_s u_t : \nabla_s u_t + \frac{1}{\rho_L}\mu(p)|\nabla p|^2 + \gamma(\theta)\chi_t^2 - \frac{L}{\theta_c}\theta\chi_t - \beta\theta\operatorname{div} u_t\,. \tag{2.13}$$

We further simplify the system by assuming that the process is quasistatic and that the shear stresses are negligible. Then (2.10)–(2.11) can be reduced to

$$0 = \operatorname{div}\sigma + g\,, \tag{2.14}$$

$$\sigma = (\nu\operatorname{div} u_t + \lambda_M\operatorname{div} u - p(\chi+\rho^*(1-\chi)) - \beta(\theta-\theta_c))\delta\,. \tag{2.15}$$

Assuming that the force g admits a potential G, that is, $g = \nabla G$, this yields

$$\nu\operatorname{div} u_t + \lambda_M\operatorname{div} u - p(\chi+\rho^*(1-\chi)) - \beta(\theta-\theta_c) = -G + H(t)\,, \tag{2.16}$$

where $H(t)$ is an "integration constant", ν is the bulk viscosity coefficient, and λ_M is the bulk elasticity modulus of the matrix material. In view of the boundary condition (2.6), we have that

$$H(t) \;=\; -\frac{1}{|\Omega|}\int_\Omega (p(\chi{+}\rho^*(1{-}\chi)) + \beta(\theta-\theta_c) - G)(x,t)\,\mathrm{d}x\,. \tag{2.17}$$

With the new unknown function $w = \operatorname{div} u$, which represents the *relative volume change*, the system (2.9)–(2.13) then becomes

$$\big((\chi{+}\rho^*(1{-}\chi))(\varphi(p)+w)\big)_t \;=\; \frac{1}{\rho_L}\operatorname{div}(\mu(p)\nabla p)\,, \tag{2.18}$$

$$\nu w_t + \lambda_M w \;=\; p(\chi{+}\rho^*(1{-}\chi)) + \beta(\theta-\theta_c) - G + H(t)\,, \tag{2.19}$$

$$\gamma(\theta)\chi_t + \partial I(\chi) \;\ni\; (1-\rho^*)\,(\Phi(p)+pw) + L\left(\frac{\theta}{\theta_c}-1\right), \tag{2.20}$$

$$c_0\theta_t - \operatorname{div}(\kappa(\theta)\nabla\theta) \;=\; \nu w_t^2 + \frac{1}{\rho_L}\mu(p)|\nabla p|^2 + \gamma(\theta)\chi_t^2 - \frac{L}{\theta_c}\theta\chi_t - \beta\theta w_t\,. \tag{2.21}$$

We prescribe the initial conditions

$$p(x,0) \;=\; p^0(x)\,, \tag{2.22}$$

$$w(x,0) \;=\; w^0(x)\,, \tag{2.23}$$

$$\chi(x,0) \;=\; \chi^0(x)\,, \tag{2.24}$$

$$\theta(x,0) \;=\; \theta^0(x)\,. \tag{2.25}$$

The weak formulation of Problem (2.18)–(2.21) reads as follows:

$$\int_\Omega\left(((\chi{+}\rho^*(1{-}\chi))(\varphi(p)+w))_t\eta + \frac{1}{\rho_L}\mu(p)\nabla p\cdot\nabla\eta\right)\mathrm{d}x \;=\; \int_{\partial\Omega}\alpha(x)(p^*-p)\eta\,\mathrm{d}s(x)\,, \tag{2.26}$$

$$\nu w_t + \lambda_M w - p(\chi{+}\rho^*(1{-}\chi)) - \beta(\theta-\theta_c) \;=\; -G + H(t) \quad \text{a. e.}, \tag{2.27}$$

$$\gamma(\theta)\chi_t + \partial I(\chi) - (1-\rho^*)(\Phi(p)+pw) \;\ni\; L\left(\frac{\theta}{\theta_c}-1\right) \quad \text{a. e.}, \tag{2.28}$$

$$\int_\Omega\left(c_0\theta_t - \gamma(\theta)\chi_t^2 + \frac{L}{\theta_c}\theta\chi_t - \nu w_t^2 + \beta\theta w_t\right)\zeta\,\mathrm{d}x + \int_\Omega\left(-\frac{1}{\rho_L}\mu(p)|\nabla p|^2\zeta + \kappa(\theta)\nabla\theta\cdot\nabla\zeta\right)\mathrm{d}x \;=\; \int_{\partial\Omega}\omega(x)(\theta^*-\theta)\zeta\,\mathrm{d}s(x)\,, \tag{2.29}$$

almost everywhere in $(0,T)$ and for all test functions $\eta \in W^{1,2}(\Omega)$ and $\zeta \in W^{1,q^*}(\Omega)$, with some $q^* > 1$ that will be specified below in Theorem 2.2.

Hypothesis 2.1. We fix a time interval $[0,T]$ and assume that the data of Problem (2.26)–(2.29) have the following properties:

(i) $\gamma : [0,\infty) \to [0,\infty)$ is continuous; $\exists 0 < c_\gamma < C_\gamma : c_\gamma(1+\theta) \le \gamma(\theta) \le C_\gamma(1+\theta)$ for all $\theta \ge 0$;

(ii) $\kappa : [0,\infty) \to [0,\infty)$ is continuous; $\exists 0 < c_\kappa < C_\kappa$, $0 < a < 1$, $a < \hat a < \frac{16}{5} + \frac{6}{5}a$: $c_\kappa(1+\theta^{1+a}) \le \kappa(\theta) \le C_\kappa(1+\theta^{1+\hat a})$ for all $\theta \ge 0$;

(iii) $\theta^0 \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$, $\theta^* \in L^\infty(\partial\Omega\times(0,T))$, $\theta^*_t \in L^2(\partial\Omega\times(0,T))$, $\exists\bar\theta > 0 : \theta^0(x) \ge \bar\theta$, $\theta^*(x,t) \ge \bar\theta$;

(iv) $\exists\, 0 < \hat\delta \le \delta < 1/4$, $\exists\, 0 < c_\varphi < C_\varphi$ such that for all $p \in \mathbb{R}$ we have that $c_\varphi \max\{1,|p|\}^{-1-\delta} \le \varphi'(p) \le C_\varphi \max\{1,|p|\}^{-1-\hat\delta}$;

(v) $\exists 0 < c_\mu < C_\mu : c_\mu \le \mu(p) \le C_\mu$ for all $p \in \mathbb{R}$;

(vi) $p^0 \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$, $p^* \in L^\infty(\partial\Omega\times(0,T)) \cap L^2(0,T;W^{1,2}(\partial\Omega))$, $p^*_t \in L^2(\partial\Omega \times (0,T))$;

(vii) $w^0, \chi^0 \in L^\infty(\Omega)$, $\chi^0(x) \in [0,1]$ a. e., $\int_\Omega w^0(x)\,\mathrm{d}x = 0$;

(viii) $G \in L^\infty(\Omega\times(0,T))$, $G_t \in L^2(\Omega\times(0,T))$;

(ix) $\Omega \subset \mathbb{R}^3$ is a bounded connected set of class $C^{1,1}$, $\alpha : \partial\Omega \to [0,\infty)$ is Lipschitz continuous, $\omega \in L^\infty(\partial\Omega)$, $\omega(x) \ge 0$ a. e., $\int_{\partial\Omega} \alpha(x)\,\mathrm{d}s(x) > 0$, $\int_{\partial\Omega} \omega(x)\,\mathrm{d}s(x) > 0$.

It is worth noting that it follows from (2.19) and (vii), using the definition of the functions G and H, that

$$\int_\Omega w(x,t)\,\mathrm{d}x \;=\; \int_\Omega w^0(x)\,\mathrm{d}x \;=\; 0 \quad \forall\, t \in [0,T]. \tag{2.30}$$

The main result of this paper is the following existence result.

Theorem 2.2. *Let Hypothesis 2.1 hold true. Then there exists a solution (p,w,χ,θ) to the system* (2.22)–(2.29), (2.17)*, with the regularity*

$$p \in L^\infty(\Omega\times(0,T)), \quad p_t, \nabla\theta \in L^2(\Omega\times(0,T)), \quad \nabla p \in L^\infty(0,T;L^2(\Omega)), \tag{2.31}$$

$$\theta, w_t \in L^{\bar p}(\Omega\times(0,T)), \quad w, \chi_t \in L^\infty(0,T;L^{\bar p}(\Omega)) \ \textit{for}\ \bar p < 8+a, \tag{2.32}$$

$$\theta_t \in L^2(0,T;W^{-1,q^*}(\Omega)) \ \textit{with}\ q^* > 1 \ \textit{given by}\ (4.82). \tag{2.33}$$

The proof of Theorem 2.2 will be divided into several steps which each constitutes a new section in this paper.

3 Cut-off system

The strategy for solving Problem (2.26)–(2.29) and proving Theorem 2.2 is the following: we choose a parameter $R > 0$ and first solve a cut-off system with the intention to let R tend to infinity. More precisely, for $R > 0$ and $z \in \mathbb{R}$ we denote by

$$Q_R(z) = \max\{-R, \min\{z,R\}\}$$

the projection onto $[-R,R]$, and set

$$P_R(z) = z - Q_R(z).$$

We further denote

$$\varphi_R(p) = \varphi(p) + P_R(p)\,, \quad \Phi_R(p) = \int_0^p \varphi_R(\tau)\,\mathrm{d}\tau\,, \quad V_R(p) = p\varphi_R(p) - \Phi_R(p)\,, \tag{3.1}$$

and

$$\gamma_R(p,\theta) = \big(1 + (p^2 - R^2)^+\big)\, \gamma(Q_R(\theta^+)), \tag{3.2}$$

for $p, \theta \in \mathbb{R}$, and replace (2.26)–(2.29) by the cut-off system

$$\int_\Omega \left(((\chi{+}\rho^*(1{-}\chi))(\varphi_R(p) + w))_t \eta + \frac{1}{\rho_L}\mu(p)\nabla p \cdot \nabla \eta \right) \mathrm{d}x \ = \ \int_{\partial\Omega} \alpha(x)(p^* - p)\eta \, \mathrm{d}s(x), \tag{3.3}$$

$$\nu w_t + \lambda_M w - p(\chi{+}\rho^*(1{-}\chi)) - \beta(Q_R(\theta^+) - \theta_c) \ = \ -G + H_R(t) \quad \text{a. e.}, \tag{3.4}$$

$$\gamma_R(p,\theta)\chi_t + \partial I(\chi) - (1-\rho^*)(\Phi_R(p) + pw) \ \ni \ L\left(\frac{Q_R(\theta^+)}{\theta_c} - 1\right) \quad \text{a. e.}, \tag{3.5}$$

$$\begin{aligned} &\int_\Omega \big(c_0\theta_t\zeta + \kappa(Q_R(\theta^+))\nabla\theta\cdot\nabla\zeta\big)\, \mathrm{d}x - \int_\Omega \left(\frac{1}{\rho_L}\mu(p)Q_R(|\nabla p|^2) + \gamma_R(p,\theta)\chi_t^2 + \nu w_t^2\right)\zeta \, \mathrm{d}x \\ &\quad + \int_\Omega Q_R(\theta^+)\left(\frac{L}{\theta_c}\chi_t + \beta w_t\right)\zeta \, \mathrm{d}x \ = \ \int_{\partial\Omega} \omega(x)(\theta^* - \theta)\zeta \, \mathrm{d}s(x), \end{aligned} \tag{3.6}$$

for all test functions $\eta, \zeta \in W^{1,2}(\Omega)$, with

$$H_R(t) = -\frac{1}{|\Omega|}\int_\Omega (p(\chi{+}\rho^*(1{-}\chi)) + \beta(Q_R(\theta^+) - \theta_c) - G)\, \mathrm{d}x\,. \tag{3.7}$$

For the system (3.3)–(3.7), we prove the following result.

Proposition 3.1. *Let Hypothesis 2.1 hold and let $R > 0$ be given. Then there exists a solution (p, w, χ, θ) to (3.3)–(3.7), (2.22)–(2.25) with the regularity $p, w, \chi, \theta, w_t \in L^q(\Omega; C[0,T])$ for $1 \le q < 3$, $p_t, \theta_t \in L^2(\Omega \times (0,T))$, and $\nabla p, \nabla\theta, \chi_t \in L^\infty(0,T; L^2(\Omega))$.*

Proof of Proposition 3.1. Let

$$M(p) := \int_0^p \mu(\tau)\,\mathrm{d}\tau, \quad K_R(\theta) := \int_0^\theta \kappa(Q_R(\tau^+))\,\mathrm{d}\tau,$$

and set $v = M(p)$, $z = K_R(\theta)$. Then (3.3)–(3.6) is transformed into the system

$$\int_\Omega \left(((\chi{+}\rho^*(1{-}\chi))(\varphi_R(p) + w))_t \eta + \frac{1}{\rho_L}\nabla v \cdot \nabla \eta \right) \mathrm{d}x \ = \ \int_{\partial\Omega} \alpha(x)(p^* - p)\eta \, \mathrm{d}s(x), \tag{3.8}$$

$$\nu w_t + \lambda_M w - p(\chi{+}\rho^*(1{-}\chi)) - \beta(Q_R(\theta^+) - \theta_c) \ = \ -G + H_R(t) \quad \text{a. e.}, \tag{3.9}$$

$$\gamma_R(p,\theta)\chi_t + \partial I(\chi) - (1-\rho^*)(\Phi_R(p) + pw) \ \ni \ L\left(\frac{Q_R(\theta^+)}{\theta_c} - 1\right) \quad \text{a. e.}, \tag{3.10}$$

$$\begin{aligned} &\int_\Omega (c_0\theta_t\zeta + \nabla z\cdot\nabla\zeta)\, \mathrm{d}x - \int_\Omega \left(\frac{1}{\rho_L}\mu(p)Q_R(|\nabla p|^2) + \gamma_R(p,\theta)\chi_t^2 + \nu w_t^2\right)\zeta \, \mathrm{d}x \\ &\quad + \int_\Omega Q_R(\theta^+)\left(\frac{L}{\theta_c}\chi_t + \beta w_t\right)\zeta \, \mathrm{d}x \ = \ \int_{\partial\Omega} \omega(x)(\theta^* - \theta)\zeta \, \mathrm{d}s(x), \end{aligned} \tag{3.11}$$

which we solve by Galerkin approximations. To this end, let $\{e_k; k = 0, 1, \dots\}$ denote the complete orthonormal system of eigenfunctions of the problem

$$-\Delta e_k = \lambda_k e_k \ \text{ in } \Omega\,, \quad \nabla e_k \cdot n = 0 \ \text{ on } \partial\Omega\,. \tag{3.12}$$

We approximate v and z by the finite sums

$$v^{(n)}(x,t) = \sum_{k=0}^{n} v_k(t) e_k(x)\,, \quad z^{(n)}(x,t) = \sum_{k=0}^{n} z_k(t) e_k(x)\,, \tag{3.13}$$

where $v_k, z_k, w^{(n)}, \chi^{(n)}$ satisfy the system

$$\begin{aligned}
&\int_\Omega \left(((\chi^{(n)}+\rho^*(1-\chi^{(n)}))(\varphi_R(p^{(n)}) + w^{(n)}))_t e_k + \frac{1}{\rho_L} \nabla v^{(n)} \cdot \nabla e_k \right) \mathrm{d}x \\
&\quad = \int_{\partial\Omega} \alpha(x)(p^* - p^{(n)}) e_k \,\mathrm{d}s(x), \quad k = 0,1,\dots,n,
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
&\nu w_t^{(n)} + \lambda_M w^{(n)} - p^{(n)}(\chi^{(n)}+\rho^*(1-\chi^{(n)})) - \beta(Q_R((\theta^{(n)})^+) - \theta_c) \\
&\quad = -G + H_R^{(n)}(t) \quad \text{a. e.},
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
&\gamma_R(p^{(n)}, \theta^{(n)}) \chi_t^{(n)} + \partial I(\chi^{(n)}) - (1-\rho^*)(\Phi_R(p^{(n)}) + p^{(n)} w^{(n)}) \\
&\quad \ni L \left(\frac{Q_R((\theta^{(n)})^+)}{\theta_c} - 1 \right) \quad \text{a. e.},
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
&\int_\Omega \left(c_0 \theta_t^{(n)} e_k + \nabla z^{(n)} \cdot \nabla e_k \right) + Q_R((\theta^{(n)})^+) \left(\frac{L}{\theta_c} \chi_t^{(n)} + \beta w_t^{(n)} \right) e_k \,\mathrm{d}x \\
&\qquad - \int_\Omega \left(\frac{1}{\rho_L} \mu(p) Q_R(|\nabla p^{(n)}|^2) + \gamma_R(p^{(n)}, \theta^{(n)}) (\chi_t^{(n)})^2 + \nu (w_t^{(n)})^2 \right) \zeta \,\mathrm{d}x \\
&\quad = \int_{\partial\Omega} \omega(x)(\theta^* - \theta^{(n)}) e_k \,\mathrm{d}s(x),
\end{aligned} \tag{3.17}$$

with $p^{(n)} := M^{-1}(v^{(n)})$, $\theta^{(n)} := K_R^{-1}(z^{(n)})$, and

$$H_R^{(n)}(t) := -\frac{1}{|\Omega|} \int_\Omega (p^{(n)}(\chi^{(n)} + \rho^*(1-\chi^{(n)})) + \beta(Q_R((\theta^{(n)})^+) - \theta_c) - G) \,\mathrm{d}x\,, \tag{3.18}$$

and with the initial conditions

$$v_k(0) = \int_\Omega M(p^0(x)) e_k(x) \,\mathrm{d}x\,, \tag{3.19}$$

$$z_k(0) = \int_\Omega K_R(\theta^0(x)) e_k(x) \,\mathrm{d}x\,, \tag{3.20}$$

$$w^{(n)}(x,0) = w^0(x)\,, \tag{3.21}$$

$$\chi^{(n)}(x,0) = \chi^0(x)\,. \tag{3.22}$$

This is an easy ODE system that admits a unique solution on some interval $[0,T_n) \subset [0,T]$. Moreover, the solution $w^{(n)}$ of (3.14) enjoys the explicit representation

$$\begin{aligned}
w^{(n)}(x,t) \;=\; &\mathrm{e}^{-(\lambda_M/\nu)t} w^0(x) + \frac{1}{\nu} \int_0^t \mathrm{e}^{(\lambda_M/\nu)(t'-t)} (-G + H_R^{(n)})(x,t') \,\mathrm{d}t' \\
&+ \frac{1}{\nu} \int_0^t \mathrm{e}^{(\lambda_M/\nu)(t'-t)} \left(p^{(n)}(\chi^{(n)}+\rho^*(1-\chi^{(n)})) + \beta(Q_R((\theta^{(n)})^+) - \theta_c) \right) (x,t') \,\mathrm{d}t'.
\end{aligned} \tag{3.23}$$

Also (3.16) is of a standard form, namely,

$$\chi_t^{(n)} + \partial I(\chi^{(n)}) \ni F^{(n)}, \tag{3.24}$$

with

$$F^{(n)} = (1-\rho^*)\frac{\Phi_R(p^{(n)}) + p^{(n)}w^{(n)}}{\gamma_R(p^{(n)},\theta^{(n)})} + \frac{L(Q_R((\theta^{(n)})^+) - \theta_c)}{\theta_c\gamma_R(p^{(n)},\theta^{(n)})}, \tag{3.25}$$

or, equivalently,

$$\chi^{(n)} \in [0,1], \quad (F^{(n)} - \chi_t^{(n)})(\chi^{(n)} - \tilde\chi) \ge 0 \text{ a.e. } \forall \tilde\chi \in [0,1]. \tag{3.26}$$

By virtue of (3.23)–(3.25), we have for all $(x,t) \in \Omega \times (0,T_n)$ the inequalities

$$\left.\begin{array}{rcl} |w^{(n)}(x,t)| + |\chi_t^{(n)}(x,t)| & \le & C_R\Big(1 + \int_0^t |p^{(n)}(x,t')|\,\mathrm{d}t' + \int_0^t\int_\Omega |p^{(n)}(x',t')|\,\mathrm{d}x'\,\mathrm{d}t'\Big) \\ |w_t^{(n)}(x,t)| & \le & C_R\Big(1 + |p^{(n)}(x,t)| + \int_0^t |p^{(n)}(x,t')|\,\mathrm{d}t' \\ & & + \int_\Omega |p^{(n)}(x',t)|\,\mathrm{d}x' + \int_0^t\int_\Omega |p^{(n)}(x',t')|\,\mathrm{d}x'\,\mathrm{d}t'\Big) \end{array}\right\} \tag{3.27}$$

where, here and in the following, $C_R > 0$ denote constants which possibly depend on R and on the data, but not on n.

We now derive some a priori estimates for the solutions to the Galerkin system. To begin with, we first test (3.14) by $v_k(t)$ and sum over $k = 1,\dots,n$ to obtain the identity (note that $v^{(n)} = M(p^{(n)})$, by definition)

$$\begin{aligned} &(1-\rho^*)\int_\Omega \chi_t^{(n)}(\varphi_R(p^{(n)}) + w^{(n)})M(p^{(n)})\,\mathrm{d}x + \int_\Omega (\chi^{(n)}+\rho^*(1-\chi^{(n)}))\varphi_R(p^{(n)})_t M(p^{(n)})\,\mathrm{d}x \\ &\quad + \int_\Omega (\chi^{(n)}+\rho^*(1-\chi^{(n)}))w_t^{(n)} M(p^{(n)})\,\mathrm{d}x + \frac{1}{\rho_L}\int_\Omega |\nabla v^{(n)}|^2\,\mathrm{d}x \\ &\quad + \int_{\partial\Omega} \alpha(x)(p^{(n)} - p^*)M(p^{(n)})\,\mathrm{d}s(x) = 0. \end{aligned} \tag{3.28}$$

We rewrite the first term of (3.28), using the identity

$$\begin{aligned} \int_\Omega \chi_t^{(n)}(\varphi_R(p^{(n)}) + w^{(n)})M(p^{(n)})\,\mathrm{d}x &= \int_\Omega \chi_t^{(n)}(\Phi_R(p^{(n)}) + p^{(n)}w^{(n)})\frac{M(p^{(n)})}{p^{(n)}}\,\mathrm{d}x \\ &\quad + \int_\Omega \chi_t^{(n)} V_R(p^{(n)})\frac{M(p^{(n)})}{p^{(n)}}\,\mathrm{d}x. \end{aligned} \tag{3.29}$$

From (3.16) it follows that for a. e. $(x,t) \in \Omega \times (0,T)$ we have, by Young's inequality,

$$\begin{aligned} &(1-\rho^*)\chi_t^{(n)}(\Phi_R(p^{(n)}) + p^{(n)}w^{(n)})\frac{M(p^{(n)})}{p^{(n)}} \\ &= \frac{M(p^{(n)})}{p^{(n)}}\left(\gamma_R(p^{(n)},\theta^{(n)})\big|\chi_t^{(n)}\big|^2 + \frac{L}{\theta_c}(\theta_c - Q_R((\theta^{(n)})^+))\chi_t^{(n)}\right) \\ &\ge \frac{c_\mu}{2}\gamma_R(p^{(n)},\theta^{(n)})\big|\chi_t^{(n)}\big|^2 - C_R\,. \end{aligned} \tag{3.30}$$

The second term in (3.28) can be rewritten as

$$\begin{aligned} \int_\Omega (\chi^{(n)}+\rho^*(1-\chi^{(n)}))\varphi_R(p^{(n)})_t M(p^{(n)})\,\mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\chi^{(n)}+\rho^*(1-\chi^{(n)}))V_{R,M}(p^{(n)})\,\mathrm{d}x \\ &\quad - (1-\rho^*)\int_\Omega \chi_t^{(n)} V_{R,M}(p^{(n)})\,\mathrm{d}x, \end{aligned} \tag{3.31}$$

where we denote

$$V_{R,M}(p) = \int_0^p \varphi_R'(\tau) M(\tau)\,\mathrm{d}\tau\,. \tag{3.32}$$

We see, in particular, that there exist constants $0 < c_{R,\mu} < C_{R,\mu}$ such that

$$c_{R,\mu} p^2 \le V_{R,M}(p) \le C_{R,\mu} p^2$$

for all $p \in \mathbb{R}$. Combining (3.29)–(3.32) with (3.28), we obtain that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\chi^{(n)}{+}\rho^*(1{-}\chi^{(n)}))V_{R,M}(p^{(n)})\,\mathrm{d}x + \frac{c_\mu}{2}\int_\Omega \gamma_R(p^{(n)},\theta^{(n)})\big|\chi_t^{(n)}\big|^2\,\mathrm{d}x + \frac{1}{\rho_L}\int_\Omega |\nabla v^{(n)}|^2\,\mathrm{d}x\\
&\qquad + \int_{\partial\Omega} \alpha(x)(p^{(n)} - p^*)M(p^{(n)})\,\mathrm{d}s(x)\\
&\quad \le\; C_R - \int_\Omega (\chi^{(n)}{+}\rho^*(1{-}\chi^{(n)}))w_t^{(n)} M(p^{(n)})\,\mathrm{d}x\\
&\qquad + (1-\rho^*)\int_\Omega \chi_t^{(n)}\left(V_{R,M}(p^{(n)}) - V_R(p^{(n)})\frac{M(p^{(n)})}{p^{(n)}}\right)\,\mathrm{d}x.
\end{aligned} \tag{3.33}$$

By (3.27) and Hölder's inequality, we have that

$$\left|\int_\Omega (\chi^{(n)}{+}\rho^*(1{-}\chi^{(n)}))w_t^{(n)} M(p^{(n)})(x,t)\,\mathrm{d}x\right| \le C_R\int_\Omega\left(|p^{(n)}|^2(x,t) + \int_0^t |p^{(n)}|^2(x,t')\,\mathrm{d}t'\right)\,\mathrm{d}x, \tag{3.34}$$

and, similarly,

$$\begin{aligned}
&\left|\int_\Omega \chi_t^{(n)}\left(V_{R,M}(p^{(n)}) - V_R(p^{(n)})\frac{M(p^{(n)})}{p^{(n)}}\right)\,\mathrm{d}x\right| \le C_R\int_\Omega |\chi_t^{(n)}||p^{(n)}|^2\,\mathrm{d}x\\
&\quad\le\; \frac{c_\mu}{4}\int_\Omega \gamma_R(p^{(n)},\theta^{(n)})\big|\chi_t^{(n)}\big|^2\,\mathrm{d}x + C_R\int_\Omega \frac{|p^{(n)}|^4}{\gamma_R(p^{(n)},\theta^{(n)})}\,\mathrm{d}x\\
&\quad\le\; \frac{c_\mu}{4}\int_\Omega \gamma_R(p^{(n)},\theta^{(n)})\big|\chi_t^{(n)}\big|^2\,\mathrm{d}x + C_R\int_\Omega \left(1+|p^{(n)}|^2\right)\,\mathrm{d}x.
\end{aligned} \tag{3.35}$$

Let $[0,T_n)$ denote the maximal interval of existence of our solution. Using (3.33)–(3.35), and Gronwall's lemma, we thus can infer that

$$\sup_{t\in(0,T_n)}\operatorname{ess}\int_\Omega |p^{(n)}|^2(x,t)\,\mathrm{d}x + \int_0^{T_n}\int_\Omega |\nabla p^{(n)}|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^{T_n}\int_{\partial\Omega}\alpha(x)|p^{(n)}|^2\,\mathrm{d}s(x)\,\mathrm{d}t \le C_R\,. \tag{3.36}$$

In particular, the Galerkin solution exists globally, and for every $n \in \mathbb{N}$ we have $T_n = T$.

In what follows, we denote by $|\cdot|_p$ the norm in $L^p(\Omega)$, by $\|\cdot\|_p$ the norm in $L^p(\Omega\times(0,T))$, by $\|\cdot\|_{\partial\Omega,p}$ the norm in $L^p(\partial\Omega\times(0,T))$, and by $\|\cdot\|_{W^{\ell,p}(\Omega)}$ the norm in $W^{\ell,p}(\Omega)$ for $\ell\in\mathbb{N}$ and $1 \le p \le \infty$.

Let us recall the Gagliardo-Nirenberg inequality

$$|u|_q \le C\left(|u|_s + |u|_s^{1-\rho}|\nabla u|_p^\rho\right), \tag{3.37}$$

with

$$\rho = \frac{\frac{1}{s} - \frac{1}{q}}{\frac{1}{s} + \frac{1}{N} - \frac{1}{p}}\,,$$

which is valid for all $1 \le s < q$, $1/q > (1/p) - (1/N)$, every bounded open set $\Omega \subset \mathbb{R}^N$ with Lipschitzian boundary, and every function $u \in W^{1,p}(\Omega)$. For $t \in (0,T)$, $N = 3$, $s = p = 2$, and $q = 4$, we have, in particular,

$$|p^{(n)}(t)|_4 \le C\left(|p^{(n)}(t)|_2 + |p^{(n)}(t)|_2^{1/4}|\nabla p^{(n)}(t)|_2^{3/4}\right).$$

Hence, by (3.36),

$$\int_0^T \left(\int_\Omega |p^{(n)}(x,t)|^4 \,\mathrm{d}x\right)^{2/3} \mathrm{d}t \le C_R\,, \tag{3.38}$$

independently of n.

Next, we test (3.14) by $\dot v_k(t)$ and sum over $k = 0,1,\dots n$ to obtain the identity

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2\rho_L}\int_\Omega |\nabla v^{(n)}|^2 \,\mathrm{d}x + \int_{\partial\Omega} \alpha(x)(\hat M(v^{(n)}) - p^* v^{(n)})\,\mathrm{d}s(x)\right) \\
&\qquad + \int_\Omega \left(((\chi^{(n)}{+}\rho^*(1{-}\chi^{(n)}))(\varphi_R(p^{(n)}) + w^{(n)}))_t v_t^{(n)}\right)\mathrm{d}x \\
&= \ -\int_{\partial\Omega} \alpha(x) p_t^* v^{(n)}\,\mathrm{d}s(x),
\end{aligned} \tag{3.39}$$

where $\hat M' = M^{-1}$. We have the pointwise lower bound

$$\varphi_R(p^{(n)})_t v_t^{(n)} \ge C_R |v_t^{(n)}|^2,$$

and thus, by (3.39) and Hölder's inequality, we have for all $t \in [0,T]$ that

$$\begin{aligned}
&\int_0^t\int_\Omega |v_t^{(n)}|^2 \,\mathrm{d}x\,\mathrm{d}t' + \left(\int_\Omega |\nabla v^{(n)}|^2(x,t)\,\mathrm{d}x + \int_{\partial\Omega}\alpha(x)|v^{(n)}|^2(x,t)\,\mathrm{d}s(x)\right) \\
&\quad \le \ C_R\left(1 + \int_0^t\int_\Omega (|w_t^{(n)}|^2 + |\chi_t^{(n)}|^2|w^{(n)}|^2)\,\mathrm{d}x\,\mathrm{d}t' + \int_0^t\int_{\partial\Omega}\alpha(x)|v^{(n)}|^2\,\mathrm{d}s(x)\,\mathrm{d}t'\right)
\end{aligned} \tag{3.40}$$

A bound for $\|w_t^{(n)}\|_2^2$ follows from (3.27) and (3.36). Moreover, owing to (3.27), we have for $t \in (0,T)$ that

$$\int_\Omega |\chi_t^{(n)}|^2|w^{(n)}|^2(x,t)\,\mathrm{d}x \le C_R\left(1 + \int_\Omega\left(\int_0^t |p^{(n)}(x,t')|\,\mathrm{d}t'\right)^4 \mathrm{d}x\right).$$

We use the Minkowski inequality in the form

$$\left(\int_\Omega\left(\int_0^t |p^{(n)}(x,t')|\,\mathrm{d}t'\right)^4 \mathrm{d}x\right)^{1/4} \le \int_0^t \left(\int_\Omega |p^{(n)}(x,t')|^4\,\mathrm{d}x\right)^{1/4}\mathrm{d}t'$$

to check that

$$\int_\Omega |\chi_t^{(n)}|^2|w^{(n)}|^2(x,t)\,\mathrm{d}x \le C_R\left(1 + \left(\int_0^t\left(\int_\Omega |p^{(n)}(x,t')|^4\,\mathrm{d}x\right)^{1/4}\mathrm{d}t'\right)^4\right) \le C_R,$$

by virtue of (3.38). Then (3.40) and the Gronwall argument imply that

$$\|v^{(n)}(t)\|^2_{W^{1,2}(\Omega)} + \int_0^t |v_t^{(n)}(t')|_2^2\,\mathrm{d}t' \le C_R, \tag{3.41}$$

whence also

$$\|p^{(n)}(t)\|^2_{W^{1,2}(\Omega)} + \int_0^t |p_t^{(n)}(t')|_2^2 \,\mathrm{d}t' \le C_R \tag{3.42}$$

for $t \in (0,T)$.

We continue by testing (3.17) by $\dot z_k(t)$ and summing over $k = 0, 1, \dots n$. Note that, thanks to (3.24)–(3.25), (3.27), and (3.36), we have that

$$\gamma_R(p^{(n)}, \theta^{(n)})(\chi_t^{(n)}(x,t))^2 + \nu(w_t^{(n)}(x,t))^2 \le C_R \left(1 + |p^{(n)}(x,t)| + \int_0^t |p^{(n)}(x,t')|\,\mathrm{d}t'\right)^3$$

for a. e. $(x,t) \in \Omega \times (0,T)$. This yields the inequality

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_\Omega |\nabla z^{(n)}|^2 \,\mathrm{d}x + \int_{\partial\Omega} \omega(x)(\hat K_R(v^{(n)}) - \theta^* v^{(n)})\,\mathrm{d}s(x)\right) + \frac{1}{2}\int_\Omega c_0 \theta_t^{(n)} z_t^{(n)} \,\mathrm{d}x \\ \le \int_{\partial\Omega} \omega(x)|\theta_t^*||z^{(n)}|\,\mathrm{d}s(x) + C_R \int_\Omega \left(1 + |p^{(n)}(x,t)| + \int_0^t |p^{(n)}(x,t')|\,\mathrm{d}t'\right)^6 \mathrm{d}x, \end{aligned}$$

where $\hat K_R' = K_R^{-1}$. Using (3.42) and the Sobolev embedding theorem, we obtain, as before, that

$$\|\theta^{(n)}(t)\|^2_{W^{1,2}(\Omega)} + \int_0^t |\theta_t^{(n)}(t')|_2^2 \,\mathrm{d}t' \le C_R \tag{3.43}$$

for $t \in (0,T)$. Hence, there exist a subsequence of $\{(p^{(n)}, \theta^{(n)}) : n \in \mathbb{N}\}$, which is again indexed by n, and functions p, θ, such that

$$\begin{aligned} p_t^{(n)} \to p_t,\ \theta_t^{(n)} \to \theta_t, &\qquad \text{weakly in } L^2(\Omega \times (0,T)), \\ \nabla p^{(n)} \to \nabla p,\ \nabla\theta^{(n)} \to \nabla\theta, &\qquad \text{weakly-star in } L^\infty(0,T; L^2(\Omega)), \\ p^{(n)} \to p,\ \theta^{(n)} \to \theta, &\qquad \text{strongly in } L^q(\Omega; C[0,T]) \text{ for } 1 \le q < 3, \end{aligned}$$

where we used the compact embedding $W^{1,2}(\Omega \times (0,T)) \hookrightarrow\hookrightarrow L^q(\Omega; C[0,T])$ for $1 \le q < 3$, see [3].

We now check that the sequences $\{w^{(n)}\}, \{\chi^{(n)}\}, \{w_t^{(n)}\}, \{\chi_t^{(n)}\}$ converge strongly in appropriate function spaces and that the limit functions satisfy the system (3.3)–(3.7). Passing again to a subsequence if necessary, we may fix a set $\Omega' \subset \Omega$ with meas$(\Omega \setminus \Omega') = 0$ such that

$$\lim_{n\to\infty} \sup_{t\in[0,T]} |p^{(n)}(x,t) - p(x,t)| = 0\,, \quad \lim_{n\to\infty} \sup_{t\in[0,T]} |\theta^{(n)}(x,t) - \theta(x,t)| = 0, \quad \forall\, x \in \Omega', \tag{3.44}$$

and such that the functions $t \mapsto p(x,t)$ and $t \mapsto \theta(x,t)$ belong to $C[0,T]$ for all $x \in \Omega'$. In particular, we can define the real numbers

$$\widetilde p(x) := \sup_{t\in[0,T]} |p(x,t)|\,, \quad \widetilde\theta(x) := \sup_{t\in[0,T]} |\theta(x,t)|, \ \text{ for } x \in \Omega'\,.$$

Let $x \in \Omega'$ be arbitrarily fixed now. Then there is some $n_0(x) \in \mathbb{N}$ such that for $n > n_0(x)$ we have $|p^{(n)}(x,t)| \le 2\widetilde p(x)$ and $|\theta^{(n)}(x,t)| \le 2\widetilde\theta(x)$, for all $t \in [0,T]$ and $x \in \Omega'$.

For $n, m \in \mathbb{N}$, $n, m > n_0(x)$, we have by (3.23) for $t \in [0,T]$ and $x \in \Omega'$ that

$$|w^{(n)}(x,t) - w^{(m)}(x,t)| \le C_R(1+\widetilde p(x)) \int_0^t (|\chi^{(n)} - \chi^{(m)}| + |p^{(n)} - p^{(m)}| + |\theta^{(n)} - \theta^{(m)}|)(x,t')\,\mathrm{d}t'. \tag{3.45}$$

Hence, with the notation of (3.25),

$$\int_0^t |F^{(n)}(x,t') - F^{(m)}(x,t')|\,\mathrm{d}t'$$
$$\leq \ C_R(1+\widetilde{p}(x))^2 \int_0^t (|\chi^{(n)} - \chi^{(m)}| + |p^{(n)} - p^{(m)}| + |\theta^{(n)} - \theta^{(m)}|)(x,t')\,\mathrm{d}t'. \tag{3.46}$$

The well-known L^1-Lipschitz continuity result for variational inequalities (see, e. g., [10, Theorem 1.12]) tells us that

$$\int_0^t |\chi_t^{(n)} - \chi_t^{(m)}|(x,t')\,\mathrm{d}t' \leq 2\int_0^t |F^{(n)}(x,t') - F^{(m)}(x,t')|\,\mathrm{d}t'. \tag{3.47}$$

Since $\{p^{(n)}(x,\cdot)\}$ and $\{\theta^{(n)}(x,\cdot)\}$ converge uniformly for each $x \in \Omega'$, we may apply the Gronwall argument to conclude that $\{\chi^{(n)}(x,\cdot)\}$, $\{w^{(n)}(x,\cdot)\}$, $\{w_t^{(n)}(x,\cdot)\}$ are Cauchy sequences in $C[0,T]$ and that $\{\chi_t^{(n)}(x,\cdot)\}$ is a Cauchy sequence in $W^{1,1}(0,T)$, for every $x \in \Omega'$. Hence, there exist functions $\chi, w : \Omega' \times (0,T)$ such that, as $n \to \infty$,

$$\sup_{t\in[0,T]} |w^{(n)}(x,t) - w(x,t)| \to 0\,,\ \sup_{t\in[0,T]} |\chi^{(n)}(x,t) - \chi(x,t)| \to 0\,,\ \sup_{t\in[0,T]} |w_t^{(n)}(x,t) - w_t(x,t)| \to 0 \tag{3.48}$$

and

$$\int_0^T |\chi_t^{(n)}(x,t) - \chi_t(x,t)|\,\mathrm{d}t \to 0, \tag{3.49}$$

for all $x \in \Omega'$. Since $|\chi^{(n)}|, |w^{(n)}|, |\chi_t^{(n)}|, |w_t^{(n)}|$ admit a pointwise upper bound (3.27) in terms of convergent sequences in $L^q(\Omega; C[0,T])$ for $1 \leq q < 3$, we can use the Lebesgue Dominated Convergence Theorem to conclude that

$$w^{(n)} \to w\,, \quad w_t^{(n)} \to w_t\,, \quad \chi^{(n)} \to \chi, \quad \text{strongly in } L^q(\Omega; C[0,T])\,, \tag{3.50}$$

and

$$\chi_t^{(n)} \to \chi_t \quad \text{strongly in } L^1(\Omega \times (0,T)). \tag{3.51}$$

Moreover, from Hölder's inequality we obtain that

$$\int_0^T\int_\Omega |\chi_t^{(n)} - \chi_t|^2\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega |\chi_t^{(n)} - \chi_t|^{1/3}|\chi_t^{(n)} - \chi_t|^{5/3}\,\mathrm{d}x\,\mathrm{d}t$$
$$\leq \left(\int_0^T\int_\Omega |\chi_t^{(n)} - \chi_t|\,\mathrm{d}x\,\mathrm{d}t\right)^{1/3} \left(\int_0^T\int_\Omega |\chi_t^{(n)} - \chi_t|^{5/2}\,\mathrm{d}x\,\mathrm{d}t\right)^{2/3} \to 0,$$

by virtue of (3.51) and (3.27). We can therefore pass to the limit in (3.14)–(3.18), where (3.16) is interpreted as the variational inequality (3.26), and check that its limit is the desired solution to (3.3)–(3.7). ∎

4 Estimates independent of R

In this section, we derive estimates for the solutions to (3.3)–(3.6) which are independent of the cut-off parameter R. In the entire section, we denote by C positive constants which may depend on the data of the problem, but not on R.

4.1 Positivity of temperature

For every nonnegative test function $\zeta \in W^{1,2}(\Omega)$ we have, by virtue of (3.6), and using the fact that $\gamma_R(p,\theta) \ge c_\gamma > 0$,

$$\int_\Omega \left(c_0\theta_t\zeta + \kappa(Q_R(\theta^+))\nabla\theta\cdot\nabla\zeta\right)\,\mathrm{d}x + \int_{\partial\Omega}\omega(x)(\theta-\theta^*)\zeta\,\mathrm{d}s(x) \ge -C\int_\Omega (Q_R(\theta^+))^2\zeta\,\mathrm{d}x \tag{4.1}$$

with a constant C depending only on the constants $L, \theta_c, \beta, \nu, c_\gamma$. Let ψ be the solution of the equation

$$c_0\dot\psi(t) + C\psi^2(t) = 0\,, \quad \psi(0) = \bar\theta\,. \tag{4.2}$$

Then

$$\psi(t) = \frac{\bar\theta c_0}{c_0 + \bar\theta C t}\,, \tag{4.3}$$

and we have

$$\begin{aligned}\int_\Omega \left(c_0(\psi-\theta)_t\zeta + \kappa(Q_R(\theta^+))\nabla(\psi-\theta)\cdot\nabla\zeta\right)\,\mathrm{d}x &- \int_{\partial\Omega}\omega(x)(\theta-\theta^*)\zeta\,\mathrm{d}s(x)\\ \le\ C\int_\Omega ((Q_R(\theta^+))^2 - \psi^2)\zeta\,\mathrm{d}x\end{aligned} \tag{4.4}$$

for every nonnegative test function $\zeta \in W^{1,2}(\Omega)$. In particular, for $\zeta(x,t) = (\psi(t)-\theta(x,t))^+$, we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{c_0}{2}\int_\Omega((\psi-\theta)^+)^2\,\mathrm{d}x + \int_{\partial\Omega}\omega(x)(\theta^*-\theta)(\psi-\theta)^+\,\mathrm{d}s(x) \le C\int_\Omega((Q_R(\theta^+))^2-\psi^2)(\psi-\theta)^+\,\mathrm{d}x\,. \tag{4.5}$$

From Hypothesis 2.1 (iii), we obtain for all values of x and t that

$$(\theta^*-\theta)(\psi-\theta)^+ \ge 0\,,\quad ((Q_R(\theta^+))^2-\psi^2)(\psi-\theta)^+ = (Q_R(\theta^+)-\psi)(Q_R(\theta^+)+\psi)(\psi-\theta)^+ \le 0\,,$$

and from

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{c_0}{2}\int_\Omega((\psi-\theta)^+)^2\,\mathrm{d}x \le 0\,,\quad (\psi-\theta)^+(x,0) = 0, \tag{4.6}$$

we conclude that, independently of $R > 0$,

$$\theta(x,t) \ge \psi(t) \ge \frac{\bar\theta c_0}{c_0 + \bar\theta C T} > 0 \quad \text{for all } x \text{ and } t. \tag{4.7}$$

4.2 Energy estimate

We test (3.3) by $\eta = p$, (3.6) by $\zeta = 1$, and sum up. With the notation (3.1), we use the identities

$$\begin{aligned}\int_\Omega((\chi+\rho^*(1-\chi))\varphi_R(p))_t p\,\mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(\chi+\rho^*(1-\chi))V_R(p)\,\mathrm{d}x + \int_\Omega(1-\rho^*)\Phi_R(p)\chi_t\,\mathrm{d}x\,, &(4.8)\\ \int_\Omega((\chi+\rho^*(1-\chi))w)_t p\,\mathrm{d}x &= \int_\Omega(\chi+\rho^*(1-\chi))w_t p\,\mathrm{d}x + \int_\Omega(1-\rho^*)wp\chi_t\,\mathrm{d}x\,, &(4.9)\\ \chi_t\left(\gamma_R(p,\theta)\chi_t - L\frac{Q_R(\theta)}{\theta_c}\right) &= -L\chi_t + (1-\rho^*)(\Phi_R(p)+pw)\chi_t, &(4.10)\end{aligned}$$

which follow from (3.5), and we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(c_0\theta + L\chi + (\chi{+}\rho^*(1{-}\chi))V_R(p) + \frac{\lambda_M}{2}w^2 + \beta\theta_c w\right)\,\mathrm{d}x \\ + \int_{\partial\Omega} (\omega(x)(\theta-\theta^*) + \alpha(x)(p-p^*)p)\,\mathrm{d}s(x) \le \int_\Omega w_t(H_R(t)-G)\,\mathrm{d}x. \tag{4.11}$$

Note that $V_R'(p) = p\varphi_R'(p)$, $V_R(0)=0$, so that $V_R(p)>0$ for all $p\neq 0$. Furthermore,

$$\int_\Omega w_t H_R(t)\,\mathrm{d}x = H_R(t)\int_\Omega w_t\,\mathrm{d}x = 0,$$

so that (4.11) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(c_0\theta + L\chi + (\chi{+}\rho^*(1{-}\chi))V_R(p) + \frac{\lambda_M}{2}w^2 + (\beta\theta_c+G)w\right)\,\mathrm{d}x \\ + \int_{\partial\Omega} (\omega(x)(\theta-\theta^*) + \alpha(x)(p-p^*)p)\,\mathrm{d}s(x) \le \int_\Omega G_t w\,\mathrm{d}x. \tag{4.12}$$

By Gronwall's argument and Hypothesis 2.1, we thus have

$$\sup_{t\in(0,T)}\mathrm{ess}\int_\Omega(\theta + V_R(p) + w^2)\,\mathrm{d}x + \int_0^T\int_{\partial\Omega}(\omega(x)\theta + \alpha(x)p^2)\,\mathrm{d}s(x)\,\mathrm{d}t \le C\,. \tag{4.13}$$

4.3 The Dafermos estimate

We denote $\hat\theta = Q_R(\theta) = Q_R(\theta^+)$ and rewrite (3.6) in the form

$$\int_\Omega\left(c_0\theta_t\zeta + \kappa(\hat\theta)\nabla\theta\cdot\nabla\zeta\right)\,\mathrm{d}x \;-\int_\Omega\left(\frac{1}{\rho_L}\mu(p)Q_R(|\nabla p|^2) + \gamma_R(p,\theta)\chi_t^2 + \nu w_t^2\right)\zeta\,\mathrm{d}x \\ + \int_\Omega \hat\theta\left(\frac{L}{\theta_c}\chi_t + \beta w_t\right)\zeta\,\mathrm{d}x \;=\; \int_{\partial\Omega}\omega(x)(\theta^*-\theta)\zeta\,\mathrm{d}s(x), \tag{4.14}$$

for every $\zeta\in W^{1,2}(\Omega)$. We test (4.14) by $\zeta = -\hat\theta^{-a}$. This yields the identity

$$\int_\Omega \frac{a\kappa(\hat\theta)}{\hat\theta^{1+a}}|\nabla\hat\theta|^2\,\mathrm{d}x + \int_\Omega\hat\theta^{-a}\left(\frac{1}{\rho_L}\mu(p)Q_R(|\nabla p|^2) + \gamma_R(p,\theta)\chi_t^2 + \nu w_t^2\right)\,\mathrm{d}x \\ = \int_\Omega \hat\theta^{1-a}\left(\frac{L}{\theta_c}\chi_t + \beta w_t\right)\,\mathrm{d}x + \int_{\partial\Omega}\omega(x)(\theta-\theta^*)\hat\theta^{-a}\,\mathrm{d}s(x) + \frac{c_0}{1-a}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\hat\theta^{1-a}\,\mathrm{d}x. \tag{4.15}$$

By Hypothesis 2.1 (ii), we have $\frac{\kappa(\hat\theta)}{\hat\theta^{1+a}} \ge c_\kappa$. Furthermore, Hölder's and Young's inequalities give the estimate

$$\int_\Omega\hat\theta^{1-a}\left(|\chi_t| + |w_t|\right)\,\mathrm{d}x \le \frac{C}{\tau}\int_\Omega\hat\theta^{2-a}\,\mathrm{d}x + \tau\int_\Omega\hat\theta^{-a}\left(\chi_t^2 + w_t^2\right)\,\mathrm{d}x \tag{4.16}$$

for every $\tau>0$. This and (4.13) yield the estimate

$$\int_0^T\int_\Omega|\nabla\hat\theta(t)|^2\,\mathrm{d}x\,\mathrm{d}t \le C\left(1 + \int_0^T\int_\Omega\hat\theta^{2-a}\,\mathrm{d}x\,\mathrm{d}t\right)\,. \tag{4.17}$$

From the Gagliardo-Nirenberg inequality (3.37) with $s = 1$, $p = 2$, and $N = 3$, we obtain that

$$|\hat{\theta}(t)|_q \leq C\left(1 + |\nabla\hat{\theta}(t)|_2^{\rho}\right), \tag{4.18}$$

with $\rho = (6/5(1 - (1/q))$, where we used (4.13) once more. In particular, for every $q \leq 8/3$, we have by (4.17) and (4.18) that

$$\int_0^T |\hat{\theta}(t)|_q^{5q/3(q-1)}\,\mathrm{d}t \leq C\left(1 + \int_0^T |\nabla\hat{\theta}(t)|_2^2\,\mathrm{d}t\right) \leq C\left(1 + \int_0^T |\hat{\theta}|_{2-a}^{2-a}\,\mathrm{d}x\,\mathrm{d}t\right). \tag{4.19}$$

Moreover, using (4.20) first for $q = 2 - a$ and then for $q = 8/3$, we obtain that

$$\int_0^T |\hat{\theta}(t)|_{8/3}^{8/3}\,\mathrm{d}t + \int_0^T |\nabla\hat{\theta}(t)|_2^2\,\mathrm{d}t \leq C, \tag{4.20}$$

independently of R.

4.4 *Estimates for the capillary pressure*

We choose an even function $b : \mathbb{R} \to (0, \infty)$ such that the functions b and $p \mapsto pb(p)$ are Lipschitz continuous and such that $b'(p) \geq 0$ for $p > 0$. Then, owing to (3.42), $\eta = pb(p)$ is an admissible test function in (3.3), and the term under the time derivative has the form

$$\begin{aligned}\int_\Omega ((\chi+\rho^*(1-\chi))(\varphi_R(p) + w))_t pb(p)\,\mathrm{d}x &= \int_\Omega (\chi+\rho^*(1-\chi))\varphi_R(p)_t pb(p)\,\mathrm{d}x \\ &+ \int_\Omega (\chi+\rho^*(1-\chi))w_t pb(p)\,\mathrm{d}x + (1-\rho^*)\int_\Omega \chi_t\left(\varphi_R(p) + w\right)pb(p)\,\mathrm{d}x.\end{aligned} \tag{4.21}$$

For $p \in \mathbb{R}$, we put

$$\begin{aligned} V_b(p) &:= \int_0^p \varphi'(\tau)\,\tau\,b(\tau)\,\mathrm{d}\tau, \quad \hat{P}_{R,b}(p) := \int_0^p P_R'(\tau)\,\tau\,b(\tau)\,\mathrm{d}\tau, \\ \Psi_{R,b}(p) &:= \varphi_R(p)pb(p) - \hat{P}_{R,b}(p) - V_b(p) - \Phi_R(p)b(p) = \int_0^p V_R(\tau)\,\tau\,b'(\tau)\,\mathrm{d}\tau\,. \end{aligned}$$

Then $V_b(p) > 0$ and $\Psi_{R,b}(p) \geq 0$ for all $p \neq 0$, and (4.21) can be rewritten as

$$\begin{aligned}\int_\Omega ((\chi+\rho^*(1-\chi))(\varphi_R(p) + w))_t pb(p)\,\mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\chi+\rho^*(1-\chi))(\hat{P}_{R,b}(p) + V_b(p))\,\mathrm{d}x \\ &+ \int_\Omega (\chi+\rho^*(1-\chi))w_t pb(p)\,\mathrm{d}x + (1-\rho^*)\int_\Omega \chi_t\left((\Phi_R(p) + wp)b(p) + \Psi_{R,b}(p)\right)\,\mathrm{d}x\end{aligned} \tag{4.22}$$

Owing to (4.10), we have, with the notation from Subsection 4.3, that

$$(1-\rho^*)\chi_t(\Phi_R(p) + wp) = \gamma_R(p,\theta)\chi_t^2 + \frac{L}{\theta_c}(\theta_c - \hat{\theta})\chi_t \geq \frac{1}{2}\gamma_R(p,\theta)\chi_t^2 - C(1+\hat{\theta})$$

with a constant $C > 0$ independent of R. Similarly,

$$\left|\int_\Omega \chi_t\Psi_{R,b}(p)\,\mathrm{d}x\right| \leq \frac{1}{4}\int_\Omega \gamma_R(p,\theta)\chi_t^2 b(p)\,\mathrm{d}x + C\int_\Omega \frac{\Psi_{R,b}^2(p)}{b(p)\gamma_R(p,\theta)}\,\mathrm{d}x.$$

We have, by definition, that $\Psi_{R,b}(p) \le V_R(p)b(p)$, hence

$$\frac{\Psi^2_{R,b}(p)}{b(p)\gamma_R(p,\theta)} \le C\frac{V^2_R(p)b(p)}{\gamma_R(p,\theta)} \le Cb(p)\frac{(V(p)+\frac{1}{2}(p^2-R^2)^+)^2}{1+(p^2-R^2)^+} \le Cp^2b(p)$$

independently of R. We conclude that

$$\begin{aligned}\int_\Omega((\chi+\rho^*(1-\chi))(\varphi_R(p)+w))_t pb(p)\,\mathrm{d}x &\ge \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(\chi+\rho^*(1-\chi))(\hat{P}_{R,b}(p)+V_b(p))\,\mathrm{d}x\\ &+\frac{1}{4}\int_\Omega\gamma_R(p,\theta)\chi_t^2 b(p)\,\mathrm{d}x - C\int_\Omega\left(1+|w_t|+|p|+\hat{\theta}\right)|p|b(p)\,\mathrm{d}x.\end{aligned} \tag{4.23}$$

From (3.3), with $\eta = pb(p)$, we thus obtain, in particular, that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(\chi+\rho^*(1-\chi))(\hat{P}_{R,b}(p)+V_b(p))\,\mathrm{d}x + \int_\Omega\mu(p)(pb'(p)+b(p))|\nabla p|^2\,\mathrm{d}x\\ +\int_{\partial\Omega}\alpha(x)(p-p^*)pb(p)\,\mathrm{d}s(x) \le C\int_\Omega\left(1+|w_t|+|p|+\hat{\theta}\right)|p|b(p)\,\mathrm{d}x\,,\end{aligned} \tag{4.24}$$

with a constant $C > 0$ which is independent of both b and R. To estimate the right-hand side of (4.24), we first notice that $|H(t)| \le C(1+\int_\Omega|p|\,\mathrm{d}x)$, and from (3.4) and Hypothesis 2.1 (viii) we obtain the pointwise bounds

$$|w(x,t)| \le C\left(1+\int_0^t(|p(x,t')|+\hat{\theta}(x,t'))\,\mathrm{d}t' + \int_0^t\int_\Omega|p(x',t')|\,\mathrm{d}x'\,\mathrm{d}t'\right), \tag{4.25}$$

$$|w_t(x,t)| \le |w(x,t)| + C\left(1+|p(x,t)|+\hat{\theta}(x,t)+\int_\Omega|p(x',t)|\,\mathrm{d}x'\right). \tag{4.26}$$

In particular, for $b(p)\equiv 1$ we have $\Psi_{R,b}(p) = 0$ and $V_b = V$, and it follows from (4.24)–(4.25) that

$$\begin{aligned}\int_\Omega((\chi+\rho^*(1-\chi))V(p))(x,t)\,\mathrm{d}x + \int_0^t\int_\Omega\mu(p)|\nabla p|^2(x,t')\,\mathrm{d}x\,\mathrm{d}t' + \int_0^t\int_{\partial\Omega}\alpha(x)|p|^2(x,t')\,\mathrm{d}s(x)\,\mathrm{d}t'\\ \le C\left(1+\int_0^t\int_\Omega\left(\hat{\theta}|p|+|p|^2\right)\,\mathrm{d}x\,\mathrm{d}t'\right).\end{aligned} \tag{4.27}$$

We have, by Hypothesis 2.1 (iv), that $V(p) \ge c_\varphi(|p|^{1-\delta}-\delta)/(1-\delta)$. The energy estimate (4.13) then yields that

$$\int_\Omega|p|^{1-\delta}(x,t)\,\mathrm{d}x \le C\,. \tag{4.28}$$

Moreover, by (4.20), $\hat{\theta}$ is bounded in $L^{8/3}(\Omega\times(0,T))$. We thus obtain from (4.27) that

$$\begin{aligned}\int_\Omega((\chi+\rho^*(1-\chi))V(p))(x,t)\,\mathrm{d}x + \int_0^t\int_\Omega\mu(p)|\nabla p|^2\,\mathrm{d}x\,\mathrm{d}t' + \int_0^t\int_{\partial\Omega}\alpha(x)|p|^2\,\mathrm{d}s(x)\,\mathrm{d}t'\\ \le C\left(1+\int_0^t\int_\Omega|p|^2\,\mathrm{d}x\,\mathrm{d}t'\right).\end{aligned} \tag{4.29}$$

Furthermore, by Hypothesis 2.1 (ix), Ω is connected, and $\int_{\partial\Omega}\alpha(x)\,\mathrm{d}s(x) > 0$. This implies that there exists a constant $C_\Omega > 0$, which depends only on Ω, such that, a. e. in $(0,T)$,

$$C_\Omega\,\|p\|^2_{W^{1,2}(\Omega)} \le \int_{\partial\Omega}\alpha(x)|p|^2\,\mathrm{d}s(x) + \int_\Omega\mu(p)|\nabla p|^2\,\mathrm{d}x\,. \tag{4.30}$$

Moreover, we infer from Hölder's inequality that

$$\int_\Omega |p|^2 \,\mathrm{d}x = \int_\Omega |p|^{(1-\delta)/2}|p|^{(3+\delta)/2}\,\mathrm{d}x \le \left(\int_\Omega |p|^{1-\delta}\,\mathrm{d}x\right)^{1/2}\left(\int_\Omega |p|^{3+\delta}\,\mathrm{d}x\right)^{1/2}. \tag{4.31}$$

Hence, by (4.28),

$$\begin{aligned}\int_0^t\int_\Omega |p|^2\,\mathrm{d}x\,\mathrm{d}t' &\le C\int_0^t \left(\int_\Omega |p|^{3+\delta}\,\mathrm{d}x\right)^{1/2}\mathrm{d}t' \le C\left(\int_0^t\left(\int_\Omega |p|^{3+\delta}\,\mathrm{d}x\right)^{2/(3+\delta)}\mathrm{d}t'\right)^{(3+\delta)/4}\\ &\le C\left(\int_0^t |p(t')|_{3+\delta}^2\,\mathrm{d}t'\right)^{(3+\delta)/4}.\end{aligned} \tag{4.32}$$

Since $\delta < 1$, we have the embedding inequality

$$|p(t)|_{3+\delta}^2 \le C\,\|p(t)\|_{W^{1,2}(\Omega)}^2,$$

so that from (4.32) it follows that

$$\int_0^t\int_\Omega |p|^2\,\mathrm{d}x\,\mathrm{d}t' \;\le\; C\left(\int_0^t \|p(t')\|_{W^{1,2}(\Omega)}^2\,\mathrm{d}t'\right)^{(3+\delta)/4}. \tag{4.33}$$

Employing Young's inequality, we therefore conclude from (4.27) and (4.30) that

$$\|p\|_{L^2(0,T;W^{1,2}(\Omega))} \le C. \tag{4.34}$$

Moreover, for $(x,t) \in \Omega\times(0,T)$, $q \ge 1$ and $s > 1$, we have

$$|p(x,t)|^q = |p(x,t)|^{(1-\delta)/s}|p(x,t)|^{q-(1-\delta)/s},$$

whence, by Hölder's inequality with $s' = s/(s-1)$,

$$|p(t)|_q^q = \left(\int_\Omega |p(x,t)|^{1-\delta}\,\mathrm{d}x\right)^{1/s}\left(\int_\Omega |p(x,t)|^{(q-(1-\delta)/s)s'}\,\mathrm{d}x\right)^{1/s'}.$$

We thus obtain from (4.28) and (4.34) that

$$\int_0^T |p(t)|_q^q\,\mathrm{d}t \le C, \tag{4.35}$$

provided that $(q-(1-\delta)/s)s' \le 6$ and $s' \ge 3$. In other words,

$$q \le \frac{1-\delta}{s} + \frac{6}{s'} = \frac{5+\delta}{s'} + 1 - \delta \le \frac{5+\delta}{3} + 1 - \delta,$$

and the maximal admissible value for q in (4.35) is given by

$$q = \frac{8-2\delta}{3}. \tag{4.36}$$

Let now the function b in (4.24) be arbitrary. For $p \in \mathbb{R}$, we put $\hat b(p) := \int_0^p \tau\, b(\tau)\,\mathrm{d}\tau$. Then $\hat b$ is convex, and we have the inequality

$$\hat b(p) - \hat b(p^*) \le (p-p^*)\hat b'(p) = (p-p^*)pb(p). \tag{4.37}$$

From (4.24), (4.20), (4.25), (4.35), (4.36), and (4.37) it follows that there exists a function $h \in L^q(\Omega \times (0,T))$ such that

$$\|h\|_q \leq C,$$

with a constant $C > 0$ independent of b and R, as well as

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\chi + \rho^*(1-\chi))(\hat{P}_{R,b}(p) + V_b(p))\,\mathrm{d}x + c_\mu \int_\Omega b(p)|\nabla p|^2\,\mathrm{d}x + \int_{\partial\Omega} \alpha(x)\hat{b}(p)\,\mathrm{d}s(x) \\ &\quad\leq \int_{\partial\Omega} \alpha(x)\hat{b}(p^*)\,\mathrm{d}s(x) + \int_\Omega h|p|b(p)\,\mathrm{d}x. \end{aligned} \tag{4.38}$$

Integration of (4.38) in time, using the fact that $\chi + \rho^*(1-\chi) \geq \rho^* > 0$, yields the estimate

$$\begin{aligned} &\int_\Omega V_b(p)(x,t)\,\mathrm{d}x + \int_0^T \int_\Omega b(p)|\nabla p|^2\,\mathrm{d}x\,\mathrm{d}t' + \int_0^T \int_{\partial\Omega} \alpha(x)\hat{b}(p)\,\mathrm{d}s(x)\,\mathrm{d}t' \\ &\leq C\left(\int_\Omega V_b(p)(x,0)\,\mathrm{d}x + \int_0^T \int_{\partial\Omega} \alpha(x)\hat{b}(p^*)\,\mathrm{d}s(x)\,\mathrm{d}t' + \left(\int_0^T \int_\Omega (|p|b(p))^{q'}\,\mathrm{d}x\,\mathrm{d}t'\right)^{1/q'}\right), \end{aligned} \tag{4.39}$$

for all $t \in [0,T]$, with $q' = \frac{q}{q-1} = \frac{8-2\delta}{5-2\delta}$.

Now let $k > 0$ be given, and let $\{b_n\}_{n\in\mathbb{N}}$ be a sequence of even, smooth, bounded functions which are nondecreasing in $(0,\infty)$ and such that $b_n(p) \nearrow |p|^{2k}$ locally uniformly in $\mathbb{R}$. Then $(|p|b_n(p))^{q'} \nearrow |p|^{(1+2k)q'}$ locally uniformly. From (4.35) we know that $p \in L^q(\Omega \times (0,T))$, where q is given by (4.36). Hence, the integral on the right-hand side of (4.39) is meaningful if $(1+2k)q' \leq q$, that is, if $3k \leq 1-\delta$. In particular, thanks to Hypothesis 2.1 (iv), $k = \delta$ is an admissible choice.

We continue by induction. To this end, assume that

$$\int_0^T \int_\Omega |p|^{(1+2k)q'}\,\mathrm{d}x\,\mathrm{d}t =: J_k < \infty \tag{4.40}$$

holds true for some $k \geq \delta$. Using the denotations

$$V_{b_n}(p) := \int_0^p \varphi'(\tau)\,\tau\,b_n(\tau)\,\mathrm{d}\tau, \quad \hat{b}_n(p) := \int_0^p \tau\,b_n(\tau)\,\mathrm{d}\tau \quad \text{for } n \in \mathbb{N},$$

we can estimate the terms occurring on the right-hand side of (4.39) for $n \in \mathbb{N}$ as follows:

$$\begin{aligned} \int_\Omega V_{b_n}(p)(x,0)\,\mathrm{d}x &\leq C|p^0|_\infty^{2k+1-\hat{\delta}}, \\ \int_0^T \int_{\partial\Omega} \alpha(x)\hat{b}_n(p^*)\,\mathrm{d}s(x)\,\mathrm{d}t' &\leq C\|p^*\|_{\partial\Omega,\infty}^{2k+2}. \end{aligned}$$

Put $E = \max\{1, |p^0|_\infty, \|p^*\|_{\partial\Omega,\infty}\}$. Then (4.39) can for $n \in \mathbb{N}$ be rewritten as

$$\int_\Omega V_{b_n}(p)(x,t)\,\mathrm{d}x + \int_0^T\int_\Omega b_n(p)|\nabla p|^2\,\mathrm{d}x\,\mathrm{d}t' + \int_0^T\int_{\partial\Omega} \alpha(x)\hat{b}_n(p)\,\mathrm{d}s(x)\,\mathrm{d}t' \leq C\max\{E^{2k+2}, J_k^{1/q'}\}, \tag{4.41}$$

independently of k, R, and n. By virtue of Fatou's lemma, we can take the limit as $n \to \infty$ to obtain that (4.41) holds true for $b(p) = |p|^{2k}$. Using the estimate

$$V_b(p) \geq \frac{c_\varphi}{2k+1-\delta}\left(|p|^{2k+1-\delta} - \frac{1+\delta}{2k+2}\right), \tag{4.42}$$

we thus have shown that

$$\frac{1}{2k+1}\int_\Omega |p|^{2k+1-\delta}(x,t)\,\mathrm{d}x + \int_0^T\int_\Omega |p|^{2k}|\nabla p|^2\,\mathrm{d}x\,\mathrm{d}t' + \frac{1}{2k+2}\int_0^T\int_{\partial\Omega}\alpha(x)|p|^{2k+2}\,\mathrm{d}s(x)\,\mathrm{d}t'$$
$$\leq\ C\max\{E^{2k+2}, J_k^{1/q'}\}. \tag{4.43}$$

4.5 Moser iterations

We first recall a technical lemma proved in [12, Lemma 3.1].

Lemma 4.1. *Let $\Omega\subset\mathbb{R}^N$ be a bounded Lipschitzian domain, $N\geq 2$. Moreover, let $q_0=(N+2)/2$, $q_0'=(N+2)/N$, and suppose that the real numbers s,r satisfy the inequalities*

$$\frac{1}{2}\leq s\leq r\leq \frac{N+2s}{N+2}\leq 1\,. \tag{4.44}$$

Furthermore, assume that a function $v\in L^2(0,T;W^{1,2}(\Omega))$ satisfies for a. e. $t\in(0,T)$ the inequality

$$|v(t)|_{2s}^{2s}+\int_0^T |v(t')|^2_{W^{1,2}(\Omega)}\,\mathrm{d}t'\leq A\max\{B,\|v\|_{2rq'}\}^{2r}\,, \tag{4.45}$$

for some $q'<q_0'$, $A\geq 1$, and $B\geq 1$. Then there exists a constant $C\geq 1$, which is independent of the choice of v, B, and A, such that

$$\|v\|_{2rq_0'}\leq CA^{1/(2r)}\max\{B,\|v\|_{2rq'}\}\,. \tag{4.46}$$

We now apply Lemma 4.1 to the inequality (4.43) with q given by (4.36). Put $v_k:=p|p|^k$. Then (4.43) can be rewritten, using Hölder's inequality, as

$$|v_k(t)|_{2s}^{2s}+\int_0^T |v_k(t')|^2_{W^{1,2}(\Omega)}\,\mathrm{d}t'\leq (k+1)^2A\max\left\{E^{k+1},\|v_k\|_{2rq'}\right\}^{2r} \tag{4.47}$$

with

$$2s=\frac{2k+1-\delta}{k+1}\,,\quad 2r=\frac{2k+1}{k+1},\quad q'=\frac{q}{q-1}\,,$$

and with a constant $A\geq 1$ depending only on the initial and boundary data. We see that the hypothesis (4.44) of Lemma 4.1 is fulfilled whenever $k\geq\delta$. The assertion of Lemma 4.1 then ensures that

$$\|v_k\|_{2rq_0'}\leq C((k+1)^2A)^{1/(2r)}\max\left\{E^{k+1},\|v_k\|_{2rq'}\right\}, \tag{4.48}$$

which entails that

$$\max\left\{E,\|p\|_{(2k+1)q_0'}\right\}\leq C^{1/(k+1)}((k+1)^2A)^{1/(2k+1)}\max\left\{E,\|p\|_{(2k+1)q'}\right\}. \tag{4.49}$$

By induction, we check that the choice $b(p)=|p|^{2k}$ is justified for every $k\geq\delta$. Moreover, we set $\widetilde{\nu}:=(q_0'/q')-1>0$ and define the sequence $\{k_j\}_{j\geq 0}$ by the formula

$$2k_j+1=(2\delta+1)(1+\widetilde{\nu})^j. \tag{4.50}$$

Set $D_j:=\max\left\{E,\|p\|_{(2k_j+1)q_0'}\right\}$. Then (4.49) takes the form

$$D_j\leq C^{1/(k_j+1)}((k_j+1)^2A)^{1/(2k_j+1)}D_{j-1}\ \text{ for }\ j\in\mathbb{N}, \tag{4.51}$$

and therefore,

$$\log D_j - \log D_{j-1} \le \frac{1}{k_j+1} \log C + \frac{1}{2k_j+1} \log((k_j+1)^2 A). \tag{4.52}$$

We have $k_0 = \delta$ and $D_0 \le C$, by (4.35)–(4.36) and the condition $\delta < 1/4$ in Hypothesis 2.1 (iv). The series on the right-hand side of (4.52) is convergent, and we thus have

$$D_j \le D_0 \prod_{j=1}^{\infty} C^{1/(k_j+1)^2} ((k_j+1)^2 A)^{1/(2k_j+1)} \le C^*$$

with a constant C^* independent of j, which enables us to conclude that

$$\|p\|_\infty \le C^*. \tag{4.53}$$

4.6 Higher order estimates for the capillary pressure

We aim at taking the limit as $R \nearrow \infty$ in (3.3)–(3.7). Hence, we can restrict ourselves to parameter values $R > C^*$ with C^* from (4.53) and rewrite (3.3)–(3.7) in the form

$$\begin{aligned}
\int_\Omega \left(((\chi{+}\rho^*(1{-}\chi))(\varphi(p)+w))_t \eta + \frac{1}{\rho_L} \mu(p) \nabla p \cdot \nabla \eta \right) \mathrm{d}x \;&=\; \int_{\partial\Omega} \alpha(x)(p^*-p)\eta \, \mathrm{d}s(x) && (4.54)\\
\nu w_t + \lambda_M w - p(\chi+\rho^*(1{-}\chi)) - \beta(\hat\theta - \theta_c) \;&=\; -G + H_R(t) \quad a.\,e., && (4.55)\\
\gamma(\hat\theta)\chi_t + \partial I(\chi) - (1-\rho^*)(\Phi(p)+pw) \;&\ni\; L\left(\frac{\hat\theta}{\theta_c} - 1\right) \quad a.\,e., && (4.56)\\
\int_\Omega \left(c_0 \theta_t \zeta + \kappa(\hat\theta) \nabla\theta \cdot \nabla\zeta \right) \mathrm{d}x + \int_{\partial\Omega} \omega(x)(\theta - \theta^*)\zeta \, \mathrm{d}s(x) \;&=\; \frac{1}{\rho_L} \int_\Omega \mu(p) Q_R(|\nabla p|^2) \zeta \, \mathrm{d}x\\
+ \int_\Omega \Big(\chi_t \big((1{-}\rho^*)(\Phi(p)+pw) - L \big) + w_t \big((\chi{+}\rho^*(1{-}\chi))p{-}\lambda_M w{-}\beta\theta_c{-}G{+}H_R(t) \big) \Big) \zeta \, \mathrm{d}x && (4.57)
\end{aligned}$$

for every test functions $\eta, \zeta \in W^{1,2}(\Omega)$, with $\hat\theta = Q_R(\theta)$ and

$$H_R(t) = -\frac{1}{|\Omega|} \int_\Omega (p(\chi{+}\rho^*(1{-}\chi)) + \beta(\hat\theta - \theta_c) - G)(x,t) \, \mathrm{d}x \,. \tag{4.58}$$

We test (4.54) by $\eta = \mu(p)p_t$, which is an admissible choice by (3.42). Then

$$\begin{aligned}
&\int_\Omega (\chi{+}\rho^*(1{-}\chi))\varphi'(p)\mu(p)|p_t|^2 \, \mathrm{d}x + \frac{1}{2\rho_L} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \mu^2(p) |\nabla p|^2 \, \mathrm{d}x + \int_{\partial\Omega} \alpha(x)(p-p^*)\mu(p)p_t \, \mathrm{d}s(x)\\
&= \int_\Omega ((1-\rho^*)\chi_t w + (\chi{+}\rho^*(1{-}\chi))w_t)\, \mu(p) p_t \, \mathrm{d}x. && (4.59)
\end{aligned}$$

Note that by Hypothesis 2.1 (iv) and (4.53), we have

$$\varphi'(p) \ge \frac{c_\varphi}{\max\{1, C^*\}^{1+\delta}}.$$

We set

$$\hat\mu(p) = \int_0^p \tau\mu(\tau) \, \mathrm{d}\tau, \quad M(p) = \int_0^p \mu(\tau) \, \mathrm{d}\tau, \tag{4.60}$$

and integrate (4.59) in time to obtain the estimate

$$
\begin{aligned}
&\int_0^t \int_\Omega |p_t|^2 \,\mathrm{d}x\,\mathrm{d}t' + \int_\Omega |\nabla p|^2(x,t)\,\mathrm{d}x + \int_{\partial\Omega} \alpha(x)\hat{\mu}(p)(x,t)\,\mathrm{d}s(x) \\
&\quad\le\; C\Bigg(1+\int_{\partial\Omega} \alpha(x)M(p)|p^*|(x,t)\,\mathrm{d}s(x) + \int_0^t \int_{\partial\Omega} \alpha(x)M(p)|p_t^*|(x,t')\,\mathrm{d}s(x)\,\mathrm{d}t' \\
&\qquad + \int_0^t \int_\Omega (|\chi_t w| + |w_t|)|p_t|\,\mathrm{d}x\,\mathrm{d}t'\Bigg) \\
&\quad\le\; C\left(1+\int_0^t \int_\Omega (|\chi_t w| + |w_t|)|p_t|\,\mathrm{d}x\,\mathrm{d}t'\right)
\end{aligned}
\tag{4.61}
$$

for all $t \in [0,T]$, whence we infer that

$$
\begin{aligned}
&\int_0^t \int_\Omega |p_t|^2 \,\mathrm{d}x\,\mathrm{d}t' + \int_\Omega |\nabla p|^2(x,t)\,\mathrm{d}x + \int_{\partial\Omega} \alpha(x)\hat{\mu}(p)(x,t)\,\mathrm{d}s(x) \\
&\quad\le\; C\left(1+\int_0^t \int_\Omega (|\chi_t w|^2 + |w_t|^2)\,\mathrm{d}x\,\mathrm{d}t'\right).
\end{aligned}
\tag{4.62}
$$

By virtue of (4.25)–(4.26), (4.56), and (4.53), we have the pointwise bounds

$$
|w(x,t)| \;\le\; C\left(1+\int_0^t \hat{\theta}(x,t')\,\mathrm{d}t'\right), \tag{4.63}
$$

$$
|\chi_t(x,t)| \;\le\; C(1+|w(x,t)|) \;\le\; C\left(1+\int_0^t \hat{\theta}(x,t')\,\mathrm{d}t'\right), \tag{4.64}
$$

$$
|w_t(x,t)| \;\le\; C\left(1+\hat{\theta}(x,t)+\int_0^t \hat{\theta}(x,t')\,\mathrm{d}t'\right). \tag{4.65}
$$

By (4.20) and the Sobolev embedding theorem, we know that $\hat{\theta}$ is bounded in $L^{8/3}(\Omega \times (0,T)) \cap L^2(0,T;L^6(\Omega))$. Let us recall again the Minkowski inequality

$$
\left(\int_\Omega \left(\int_0^t \hat{\theta}(x,t')\,\mathrm{d}t'\right)^6 \mathrm{d}x\right)^{1/6} \le \int_0^t \left(\int_\Omega \hat{\theta}^6(x,t')\,\mathrm{d}x\right)^{1/6} \mathrm{d}t',
$$

which implies that

$$
\int_\Omega \left(|w(x,t)|^6 + |\chi_t(x,t)|^6\right)\,\mathrm{d}x \;\le\; C \quad \text{for a.e. } t \in (0,T), \tag{4.66}
$$

$$
\|w_t\|_{8/3} \;\le\; C. \tag{4.67}
$$

Hence, the right-hand side of (4.62) is bounded independently of R, and we have for all $t \in [0,T]$ that

$$
\int_0^t \int_\Omega |p_t|^2 \,\mathrm{d}x\,\mathrm{d}t' + \int_\Omega |\nabla p|^2(x,t)\,\mathrm{d}x + \int_{\partial\Omega} \alpha(x)\hat{\mu}(p)(x,t)\,\mathrm{d}s(x) \le C\,. \tag{4.68}
$$

Now let $M(p)$ be as in (4.60). By (4.66)–(4.68), and by comparison in (4.54), the term $\Delta M(p)$ is bounded in $L^2(\Omega\times(0,T))$, independently of R. In terms of the new variable $\tilde{p} = M(p)$, the boundary condition (2.7) is nonlinear, and the $W^{2,2}$-regularity of $M(p)$ follows from considerations similar to those used in the proof of [13, Theorem 4.1], inspired by [21]. We thus may employ the Gagliardo-Nirenberg inequality (4.69) in the form

$$|\nabla M(p)(t)|_q \le C\left(|\nabla M(p)(t)|_2 + |\nabla M(p)(t)|_2^{1-\rho}|\Delta M(p)(t)|_2^{\rho}\right) \tag{4.69}$$

with $\rho = 3(\frac{1}{2} - \frac{1}{q})$. Together with (4.68), we conclude that

$$\int_0^T |\nabla p(t)|_q^s \,\mathrm{d}t \le C \quad \text{for } q \in (2,6] \text{ and } \frac{1}{q} + \frac{2}{3s} = \frac{1}{2}. \tag{4.70}$$

In particular, for $s = 4$ and $s = q$ we obtain, respectively,

$$\int_0^T |\nabla p(t)|_3^4 \,\mathrm{d}t \le C\,, \quad \|\nabla p\|_{10/3} \le C. \tag{4.71}$$

4.7 Higher order estimates for the temperature

The previous estimates (4.66)–(4.67) and (4.71) entail that (4.57) has the form

$$\int_\Omega \left(c_0\theta_t\zeta + \kappa(\hat\theta)\nabla\theta\cdot\nabla\zeta\right)\,\mathrm{d}x + \int_{\partial\Omega}\omega(x)(\theta-\theta^*)\zeta\,\mathrm{d}s(x) = \int_\Omega \tilde F\zeta\,\mathrm{d}x \tag{4.72}$$

for every $\zeta \in W^{1,2}(\Omega)$, with a function $\tilde F$ such that

$$\|\tilde F\|_{5/3} \le C\,, \quad \int_0^T |\tilde F(t)|_{3/2}^2 \,\mathrm{d}t \le C\,, \tag{4.73}$$

independently of R.

Assume now that for some $p_0 \ge 8/3$ we have

$$\|\hat\theta\|_{p_0} \le C. \tag{4.74}$$

We know that this is true for $p_0 = 8/3$ by virtue of (4.20). Set $r_0 = 2p_0/5$. Then we may put $\zeta = \hat\theta^{r_0}$ in (4.72) and obtain, using Hypothesis 2.1 (ii), that

$$\frac{1}{r_0+1}\int_\Omega \hat\theta^{r_0+1}(x,t)\,\mathrm{d}x + r_0\int_0^t\int_\Omega \hat\theta^{r_0+a}|\nabla\hat\theta|^2\,\mathrm{d}x\,\mathrm{d}t' \le C\,. \tag{4.75}$$

We now denote

$$v = \hat\theta^p\,, \quad p = 1 + \frac{r_0+a}{2}\,, \quad s = \frac{r_0+1}{p}\,,$$

and rewrite (4.75) as

$$\int_\Omega |v|^s(x,t)\,\mathrm{d}x + \int_0^t\int_\Omega |\nabla v|^2\,\mathrm{d}x\,\mathrm{d}t' \le C(r_0+1)\,. \tag{4.76}$$

By the Gagliardo-Nirenberg inequality (3.37), we have $\|v\|_q \le C(r_0+1)$ for $q = 2 + \frac{2s}{3}$. Hence,

$$\|\hat\theta\|_{p_1} \le C(r_0+1) \quad \text{for } p_1 = pq = \frac{2p_0}{3} + \frac{8}{3} + a\,. \tag{4.77}$$

We now proceed by induction according to the recipe $p_{j+1} = \frac{2p_j}{3} + \frac{8}{3} + a\,,\ r_j = \frac{2p_j}{5}$. We have $\lim_{j\to\infty} p_j = 8+3a$. After finitely many steps, we may stop the algorithm and put $\bar{p} := p_j < 8+3a$ with

$$\|\hat\theta\|_{\bar p} + \sup\text{ess}\ |\hat\theta(t)|_{\bar r+1} \le C\,, \quad \bar r = \frac{2\bar p}{5} > \hat a, \tag{4.78}$$

with the constant $\hat a$ introduced in Hypothesis 2.1 (ii). By Proposition 3.1, we may test (4.72) by θ, which yields

$$\int_\Omega \theta^2(x,t)\,\mathrm{d}x + \int_0^t\int_\Omega \kappa(\hat\theta)|\nabla\theta|^2\,\mathrm{d}x\,\mathrm{d}t' + \int_0^t\int_{\partial\Omega}\omega(x)\theta^2\,\mathrm{d}s(x)\,\mathrm{d}t' \le C\|\theta\|_{5/2}\,. \tag{4.79}$$

Using the Gagliardo-Nirenberg inequality again, for instance, we conclude that

$$\int_\Omega \theta^2(x,t)\,\mathrm{d}x + \int_0^t\int_\Omega \kappa(\hat\theta)|\nabla\theta|^2\,\mathrm{d}x\,\mathrm{d}t' + \int_0^t\int_{\partial\Omega}\omega(x)\theta^2\,\mathrm{d}s(x)\,\mathrm{d}t' \le C\,. \tag{4.80}$$

This enables us to derive an upper bound for the integral $\int_\Omega \kappa(\hat\theta)\nabla\theta\cdot\nabla\zeta\,\mathrm{d}x$, which we need for getting an estimate for θ_t from the equation (4.80). We have, by Hölder's inequality and Hypothesis 2.1 (ii), that

$$\begin{aligned}\int_\Omega |\kappa(\hat\theta)\nabla\theta\cdot\nabla\zeta|\,\mathrm{d}x &= \int_\Omega |\kappa^{1/2}(\hat\theta)\nabla\theta\cdot\kappa^{1/2}(\hat\theta)\nabla\zeta|\,\mathrm{d}x \\ &\le C\left(\int_\Omega \kappa(\hat\theta)|\nabla\theta|^2\,\mathrm{d}x\right)^{1/2}\left(\int_\Omega \hat\theta^{1+\hat a}|\nabla\zeta|^2\,\mathrm{d}x\right)^{1/2}.\end{aligned} \tag{4.81}$$

We now choose $\hat q > 1$ such that $(1+\hat a)\hat q = 1+\bar r$, where $\bar r$ is defined in (4.78). Choosing now

$$q^* = \frac{2\hat q}{\hat q - 1}, \tag{4.82}$$

we obtain from Hölder's inequality that

$$\int_\Omega \hat\theta^{1+\hat a}|\nabla\zeta|^2\,\mathrm{d}x \le \left(\int_\Omega \hat\theta^{1+\bar r}\,\mathrm{d}x\right)^{1/\hat q}\left(\int_\Omega |\nabla\zeta|^{q^*}\,\mathrm{d}x\right)^{2/q^*} \le C\left(\int_\Omega |\nabla\zeta|^{q^*}\,\mathrm{d}x\right)^{2/q^*}, \tag{4.83}$$

by virtue of (4.78). Eq. (4.81) then yields the bound

$$\int_\Omega |\kappa(\hat\theta)\nabla\theta\cdot\nabla\zeta|\,\mathrm{d}x \le C\left(\int_\Omega \kappa(\hat\theta)|\nabla\theta|^2\,\mathrm{d}x\right)^{1/2}\left(\int_\Omega |\nabla\zeta|^{q^*}\,\mathrm{d}x\right)^{1/q^*}. \tag{4.84}$$

Hence, by (4.80),

$$\int_0^T\int_\Omega |\kappa(\hat\theta)\nabla\theta\cdot\nabla\zeta|\,\mathrm{d}x\,\mathrm{d}t \le C\|\zeta\|_{L^2(0,T;W^{1,q^*}(\Omega))}\,. \tag{4.85}$$

From (4.73) it follows that testing with $\zeta \in L^2(0,T;W^{1,q^*}(\Omega))$ is admissible. We thus obtain from (4.72) that

$$\int_0^T\int_\Omega \theta_t\zeta\,\mathrm{d}x\,\mathrm{d}t \le C\|\zeta\|_{L^2(0,T;W^{1,q^*}(\Omega))}\,. \tag{4.86}$$

5 Proof of Theorem 2.2

Let $R_i \nearrow \infty$ be a sequence such that $R_1 > C^*$ with C^* as in (4.53), and let $(p, w, \chi, \theta) = (p^{(i)}, w^{(i)}, \chi^{(i)}, \theta^{(i)})$ be solutions of (4.54)–(4.58) corresponding to $R = R_i$, with $\hat{\theta} = \hat{\theta}^{(i)} = Q_{R_i}(\theta^{(i)})$ and test functions $\eta, \zeta \in W^{1,2}(\Omega)$. Our aim is to check that at least a subsequence converges as $i \to \infty$ to a solution of (2.26)–(2.29), (2.17), with test functions $\eta \in W^{1,2}(\Omega)$, $\zeta \in W^{1,q^*}(\Omega)$ with q^* as in Theorem 2.2.

First, for the capillary pressure $p = p^{(i)}$ we have the estimates (4.53), (4.68), (4.71), which imply that, passing to a subsequence if necessary,

$$\begin{aligned} p^{(i)} \to p \qquad & \text{strongly in } L^r(\Omega \times (0,T)) \ \text{ for every } \ r \ge 1\,, \\ p_t^{(i)} \to p_t \qquad & \text{weakly in } L^2(\Omega \times (0,T))\,, \\ \nabla p^{(i)} \to \nabla p \qquad & \text{strongly in } L^r(\Omega \times (0,T)) \ \text{ for every } \ 1 \le r < \frac{10}{3}\,. \end{aligned}$$

We easily show that

$$Q_{R_i}\big(|\nabla p^{(i)}|^2\big) \to |\nabla p|^2 \ \text{ strongly in } L^r(\Omega \times (0,T)) \ \text{ for every } \ 1 \le r < \frac{5}{3}\,. \tag{5.1}$$

Indeed, let $\Omega_T^{(i)} \subset \Omega \times (0,T)$ be the set of all $(x,t) \in \Omega \times (0,T)$ such that $|\nabla p^{(i)}(x,t)|^2 > R_i$. By (4.71), we have

$$C \ge \int_0^T \int_\Omega |\nabla p^{(i)}(x,t)|^{10/3} \,\mathrm{d}x\,\mathrm{d}t \ge \iint_{\Omega_T^{(i)}} |\nabla p^{(i)}(x,t)|^{10/3} \,\mathrm{d}x\,\mathrm{d}t \ge |\Omega_T^{(i)}| R_i^{5/3}\,,$$

hence $|\Omega_T^{(i)}| \le C R_i^{-5/3}$. For $r < \frac{5}{3}$, we use Hölder's inequality to get the estimate

$$\begin{aligned} \int_0^T \int_\Omega \big|Q_{R_i}(|\nabla p^{(i)}|^2) - |\nabla p^{(i)}|^2\big|^r \,\mathrm{d}x\,\mathrm{d}t &= \iint_{\Omega_T^{(i)}} \big|R_i - |\nabla p^{(i)}|^2\big|^r \,\mathrm{d}x\,\mathrm{d}t \le \iint_{\Omega_T^{(i)}} |\nabla p^{(i)}|^{2r} \,\mathrm{d}x\,\mathrm{d}t \\ &\le \left(\iint_{\Omega_T^{(i)}} |\nabla p^{(i)}|^{10/3} \,\mathrm{d}x\,\mathrm{d}t\right)^{3r/5} |\Omega_T^{(i)}|^{1-(3r/5)}\,, \end{aligned}$$

and (5.1) follows.

For the temperature $\theta = \theta^{(i)}$, we proceed in a similar way. From the compactness result in [20, Theorem 5.1], it follows that, for a subsequence,

$$\theta^{(i)} \to \theta \ \text{ strongly in } L^2(\Omega \times (0,T)).$$

Furthermore, by (4.78), $\hat{\theta}^{(i)}$ are uniformly bounded in $L^r(\Omega \times (0,T))$ for every $r < 8 + 3a$. A similar argument as above yields that

$$\hat{\theta}^{(i)} \to \theta \ \text{ strongly in } L^r(\Omega \times (0,T)) \ \text{ for every } \ 1 \le r < 8 + 3a\,.$$

Indeed, by (4.80) and (4.86),

$$\begin{aligned} \theta_t^{(i)} \to \theta_t \qquad & \text{weakly in } L^2(0,T; W^{-1,q^*}(\Omega))\,, \\ \nabla \theta^{(i)} \to \nabla \theta \qquad & \text{weakly in } L^2(\Omega \times (0,T))\,. \end{aligned}$$

The strong convergences of $w^{(i)} \to w$, $w_t^{(i)} \to w_t$, $\chi^{(i)} \to \chi$, $\chi_t^{(i)} \to \chi_t$ are handled using the estimates (4.63)–(4.65) similarly as in the proof of Proposition 3.1 at the end of Section 3. This enables us to pass to the limit as $R \nearrow \infty$ in the system (4.54)–(4.58) and thus to complete the proof of Theorem 2.2.

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