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A one-dimensional symmetry result for solutions of an integral equation of convolution type

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Abstract

We consider an integral equation in the plane, in which the leading operator is of convolution type, and we prove that monotone (or stable) solutions are necessarily one-dimensiona.l

1 Introduction

In this paper, we consider solutions of an integral equation driven by the following nonlocal, linear operator of convolution type:

$$\mathcal{L}u(x) := \int_{\mathbb{R}^n} \big(u(x) - u(y)\big)\, k(x-y)\, dy. \tag{1}$$

Here we suppose[1] that k is an even, measurable kernel with normalization

$$\int_{\mathbb{R}^n} k(\zeta)\, d\zeta = 1$$

and such that

$$m_0 \chi_{B_{r_0}}(\zeta) \le k(\zeta) \le M_0 \chi_{B_{R_0}}(\zeta) \tag{2}$$

for any $\zeta \in \mathbb{R}^n$, for some fixed $M_0 \ge m_0 > 0$ and $R_0 \ge r_0 > 0$. Throughout the paper, B_r denotes the open Euclidean ball with radius $r > 0$ and centered at the origin.

We consider here solutions u of the semilinear equation

$$\mathcal{L}u(x) = f\big(u(x)\big). \tag{3}$$

In the past few years, there has been an intense activity in this type of equations, both for its mathematical interest and for its relation with biological models, see, among the others [17, 18, 20, 21]. In this case, the solution u is thought as the density of a biological species and the nonlinearity f is often a logistic map, which prescribes the birth and death rate of the population. In this framework, the nonlocal diffusion modeled by $\mathcal{L}$ is motivated by the long-range interactions between the individuals of the species.

The goal of this paper is to study the symmetry properties of solutions of (3) in the light of a famous conjecture of De Giorgi arising in elliptic partial differential equations, see [12]. The original problem consisted in the following question:

Conjecture 1. *Let u be a bounded solution of*

$$-\Delta u = u - u^3$$

in the whole of $\mathbb{R}^n$, with

$$\partial_{x_n} u(x) > 0 \text{ for any } x \in \mathbb{R}^n.$$

Then, u is necessarily one-dimensional, i.e. there exist $u_\star : \mathbb{R} \to \mathbb{R}$ and $\omega \in \mathbb{R}^n$ such that $u(x) = u_\star(\omega \cdot x)$, for any $x \in \mathbb{R}^n$, at least when $n \le 8$.

[1] For the sake of completeness, we point out that assumptions more general than (2) may be taken into account with the same methods as the ones used in this paper. For instance, one could follow assumptions (H1)–(H4) in [11] with $g \ge \alpha$ for some $\alpha > 0$. We focus on the simpler case of assumption (2) for simplicity.

The literature has presented several variations of Conjecture 1: in particular, a weak form of it has been investigated when the additional assumption

$$\lim_{x_n \to \pm\infty} u(x_1, \dots, x_n) = \pm 1 \tag{4}$$

is added to the hypotheses.

When the limit in (4) is uniform in the variables $(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$, the version of Conjecture 1 obtained in this way is due to Gibbons and is related to problems in cosmology.

In spite of the intense activity of the problem, Conjecture 1 is still open in its generality. Up to now, Conjecture 1 is known to have a positive answer in dimension 2 and 3 (see [2, 19] and also [1, 5]) and a negative answer in dimension 9 and higher (see [14]).

Also, the weak form of Conjecture 1 under the limit assumption in (4) was proved (up to the optimal dimension 8) in [23], and the version of Conjecture 1 under a uniform limit assumption in (4) holds true in any dimension (see [3, 6, 15]).

Since it is almost impossible to keep track in this short introduction of all the research developed on this important topic, we refer to [16] for further details and motivations.

The goal of this paper is to investigate whether results in the spirit of Conjecture 1 hold true when the Laplace operator is replaced by the nonlocal, integral operator in (1). We remark that symmetry results in nonlocal settings have been obtained in [7, 8, 9, 10, 13, 24], but all these works dealt with fractional operators with a regularizing effect. Namely, the integral kernel considered there is not integrable, therefore the solutions of the associated equation enjoy additional regularity and rigidity properties. Also, some of the problems considered in the previous works rely on an extension property of the operator that bring the problem into a local (though higher dimensional and either singular or degenerate) problem.

In this sense, as far as we know, this paper is the first one to take into account integrable kernels, for which the above regularization techniques do not hold and for which equivalent local problems are not available.

In this note, we prove the following one-dimensional result in dimension 2:

Theorem 2. *Let u be a solution of* (3) *in the whole of $\mathbb{R}^2$, with $\|u\|_{C^1(\mathbb{R}^2)} < +\infty$ and $f \in C^1(\mathbb{R})$. Assume that*

$$\partial_{x_2} u(x) > 0 \text{ for any } x \in \mathbb{R}^2. \tag{5}$$

Then, u is necessarily one-dimensional.

The proof of Theorem 2 relies on a technique introduced by [5] and refined in [2], which reduced the symmetry property to a Liouville type property for an associated equation (of course, differently from the classical case, we will have to deal with equations, and in fact inequalities, of integral type, in which the appropriate simplifications are more involved).

For the existence of one-dimensional solutions of (3) under quite general conditions, see Theorem 3.1(b) in [4].

The rest of the paper is devoted to the proof of Theorem 2.

2 Proof of Theorem 2

We observe that, for any $f \in L^\infty(\mathbb{R}^2)$ and $g \in L^1(\mathbb{R}^2)$,

$$\begin{aligned}
2\int_{\mathbb{R}^2} \mathcal{L}f(x)\,g(x)\,dx &= 2\int_{\mathbb{R}^2}\left[\int_{\mathbb{R}^2}\big(f(x)-f(y)\big)\,k(x-y)\,dy\right] g(x)\,dx \\
&= \int_{\mathbb{R}^2}\left[\int_{\mathbb{R}^2}\big(f(x)-f(y)\big)\,k(x-y)\,dy\right] g(x)\,dx \\
&\qquad + \int_{\mathbb{R}^2}\left[\int_{\mathbb{R}^2}\big(f(y)-f(x)\big)\,k(x-y)\,dx\right] g(y)\,dy \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\big(f(x)-f(y)\big)\big(g(x)-g(y)\big)\,k(x-y)\,dx\,dy.
\end{aligned} \tag{6}$$

Now we let $u_i := \partial_{x_i} u$, for $i \in \{1,2\}$. In light of (5), we can define

$$v(x) := \frac{u_1(x)}{u_2(x)}. \tag{7}$$

Also, fixed $R > 1$ (to be taken as large as we wish in the sequel), we consider a cut-off function $\tau := \tau_R \in C_0^\infty(B_{2R})$, such that $0 \le \tau \le 1$ in $\mathbb{R}^2$, $\tau = 1$ in B_R and

$$|\nabla \tau| \le C R^{-1}, \tag{8}$$

for some $C > 0$ independent of $R > 1$. Throughout the proof, C will denote a positive constant which may change from a line to another, but which is independent of $R > 1$.

By (3), we have that

$$\begin{aligned}
f'\big(u(x)\big)\,u_i(x) &= \partial_{x_i}\left(f\big(u(x)\big)\right) \\
&= \partial_{x_i}\big(\mathcal{L}u(x)\big) = \partial_{x_i}\left(\int_{\mathbb{R}^2}\big(u(x)-u(x-\zeta)\big)\,k(\zeta)\,d\zeta\right) \\
&= \int_{\mathbb{R}^2}\big(u_i(x)-u_i(x-\zeta)\big)\,k(\zeta)\,d\zeta = \int_{\mathbb{R}^2}\big(u_i(x)-u_i(y)\big)\,k(x-y)\,dy \\
&= \mathcal{L}u_i(x).
\end{aligned} \tag{9}$$

Accordingly,

$$\begin{aligned}
& f'(u)\,u_1u_2 = \big(\mathcal{L}u_1\big)\,u_2 \\
\text{and} \qquad & f'(u)\,u_1u_2 = \big(\mathcal{L}u_2\big)\,u_1.
\end{aligned}$$

By subtracting these two identities and using (7), we obtain

$$0 = \big(\mathcal{L}u_1\big)\,u_2 - \big(\mathcal{L}u_2\big)\,u_1 = \big(\mathcal{L}(vu_2)\big)\,u_2 - \big(\mathcal{L}u_2\big)\,(vu_2).$$

Now, we multiply the previous equality by $2\tau^2 v$ and we integrate over $\mathbb{R}^2$. Recalling (6), we conclude that

$$
\begin{aligned}
0 &= 2\int_{\mathbb{R}^2} \mathcal{L}(vu_2)(x)\,(\tau^2 vu_2)(x)\,dx - 2\int_{\mathbb{R}^2} \mathcal{L}u_2(x)\,(\tau^2 v^2 u_2)(x)\,dx \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \big(vu_2(x)-vu_2(y)\big)\,\big(\tau^2 vu_2(x)-\tau^2 vu_2(y)\big)\,k(x-y)\,dx\,dy \\
&\quad - \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \big(u_2(x)-u_2(y)\big)\,\big(\tau^2 v^2 u_2(x)-\tau^2 v^2 u_2(y)\big)\,k(x-y)\,dx\,dy \\
&=: I_1 - I_2.
\end{aligned}
\tag{10}
$$

By writing

$$
vu_2(x)-vu_2(y) = \big(u_2(x)-u_2(y)\big)\,v(x) + \big(v(x)-v(y)\big)\,u_2(y),
$$

we see that

$$
\begin{aligned}
I_1 &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \big(u_2(x)-u_2(y)\big)\,\big(\tau^2 vu_2(x)-\tau^2 vu_2(y)\big)\,v(x)\,k(x-y)\,dx\,dy \\
&\quad + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \big(v(x)-v(y)\big)\,\big(\tau^2 vu_2(x)-\tau^2 vu_2(y)\big)\,u_2(y)\,k(x-y)\,dx\,dy..
\end{aligned}
\tag{11}
$$

In the same way, if we write

$$
\tau^2 v^2 u_2(x)-\tau^2 v^2 u_2(y) = \big(\tau^2 vu_2(x)-\tau^2 vu_2(y)\big)\,v(x) + \big(v(x)-v(y)\big)\,\tau^2 vu_2(y),
$$

we get that

$$
\begin{aligned}
I_2 &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \big(u_2(x)-u_2(y)\big)\,\big(\tau^2 vu_2(x)-\tau^2 vu_2(y)\big)\,v(x)\,k(x-y)\,dx\,dy \\
&\quad + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \big(u_2(x)-u_2(y)\big)\,\big(v(x)-v(y)\big)\,\tau^2 vu_2(y)\,k(x-y)\,dx\,dy.
\end{aligned}
\tag{12}
$$

By (11) and (12), after a simplification we obtain that

$$
\begin{aligned}
I_1 - I_2 \;&=\; \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \big(v(x)-v(y)\big)\,\big(\tau^2 vu_2(x)-\tau^2 vu_2(y)\big)\,u_2(y)\,k(x-y)\,dx\,dy \\
&\qquad - \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \big(u_2(x)-u_2(y)\big)\,\big(v(x)-v(y)\big)\,\tau^2 vu_2(y)\,k(x-y)\,dx\,dy.
\end{aligned}
$$

Now we notice that

$$
\begin{aligned}
&\tau^2 vu_2(x)-\tau^2 vu_2(y) \\
= \;&\big(v(x)-v(y)\big)\,\tau^2(x)\,u_2(x) + \big(\tau^2(x)-\tau^2(y)\big)\,u_2(x)\,v(y) + \big(u_2(x)-u_2(y)\big)\,\tau^2(y)\,v(y),
\end{aligned}
$$

and so

$$
\begin{aligned}
I_1 - I_2 \;&=\; \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \big(v(x)-v(y)\big)^2\,\tau^2(x)\,u_2(x)\,u_2(y)\,k(x-y)\,dx\,dy \\
&\quad + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \big(v(x)-v(y)\big)\,\big(\tau^2(x)-\tau^2(y)\big)\,v(y)\,u_2(x)\,u_2(y)\,k(x-y)\,dx\,dy.
\end{aligned}
$$

Thus, using this and (10), and recalling (2), (5) and the support properties of τ, we deduce that

$$
\begin{aligned}
0 \le J_1 &:= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \big(v(x)-v(y)\big)^2\,\tau^2(x)\,u_2(x)\,u_2(y)\,k(x-y)\,dx\,dy \\
&\le \iint_{\mathcal{R}_R} \big|v(x)-v(y)\big|\,\big|\tau(x)-\tau(y)\big|\,\big|\tau(x)+\tau(y)\big|\,|v(y)|\,u_2(x)\,u_2(y)\,k(x-y)\,dx\,dy \\
&=: J_2,
\end{aligned}
\tag{13}
$$

where

$$
\begin{aligned}
\mathcal{R}_R &:= \{(x,y)\in\mathbb{R}^2\times\mathbb{R}^2 \text{ s.t. } |x-y|\le R_0\}\cap\mathcal{S}_R \\
\text{and }\ \mathcal{S}_R &:= \Big((B_{2R}\times B_{2R})\setminus(B_R\times B_R)\Big)\,\cup\,\Big(B_{2R}\times(\mathbb{R}^2\setminus B_{2R})\Big)\,\cup\,\Big((\mathbb{R}^2\setminus B_{2R})\times B_{2R}\Big).
\end{aligned}
$$

We use the symmetry in the (x,y) variables and the substitution $\zeta := x-y$ to see that the Lebesgue measure $|\mathcal{R}_R|$ of the set $\mathcal{R}_R$ is bounded by

$$
\begin{aligned}
|\mathcal{R}_R| &\le \Big|\{|x-y|\le R_0\}\cap\{|x|,|y|\le 2R\}\Big| + 2\,\Big|\{|x-y|\le R_0\}\cap\{|x|\le 2R\le|y|\}\Big| \\
&\le 3\int_{B_{2R}}\left[\int_{B_{R_0}} d\zeta\right] dx \\
&\le CR^2,
\end{aligned}
\tag{14}
$$

for some $C>0$, possibly depending on R_0, but independent of $R>1$.

Moreover, making use of the Cauchy-Schwarz Inequality, we see that

$$
\begin{aligned}
J_2^2 \le &\iint_{\mathcal{R}_R} \big(v(x)-v(y)\big)^2\,\big(\tau(x)+\tau(y)\big)^2\,u_2(x)\,u_2(y)\,k(x-y)\,dx\,dy \\
&\cdot\iint_{\mathcal{R}_R} \big(\tau(x)-\tau(y)\big)^2\,v^2(y)\,u_2(x)\,u_2(y)\,k(x-y)\,dx\,dy.
\end{aligned}
\tag{15}
$$

Now we claim that

$$
u_2(x)\le C\,u_2(y) \tag{16}
$$

for any $(x,y)\in\mathcal{R}_R$, for a suitable $C>0$, possibly depending on R_0 but independent of $R>1$ and $(x,y)\in\mathcal{R}_R$. For this, fix x and let $\Omega := B_{R_0}(x) = x+B_{R_0}$. Then we use the Harnack Inequality for integral equations (recall (2), (5) and (9), and see Corollary 1.7 in [11]), to obtain that

$$
u_2(x)\le \sup_{\Omega} u_2 \le C\inf_{\Omega} u_2 \le C\,u_2(y)
$$

for any $y\in B_{R_0}(x)$, where $C>0$ is independent of $R>1$ and $(x,y)\in\mathcal{R}_R$. This establishes (16).

From (7), (8) and (16), we obtain that, for any $(x,y)\in\mathcal{R}_R$,

$$
\big(\tau(x)-\tau(y)\big)^2\,v^2(y)\,u_2(x)\,u_2(y)\le CR^{-2}\,v^2(y)\,u_2^2(y) = CR^{-2}\,u_1^2(y)\le CR^{-2},
$$

for some $C > 0$ independent of $R > 1$. Hence, by (2) and (14),

$$\iint_{\mathcal{R}_R} \big(\tau(x) - \tau(y)\big)^2 v^2(y)\, u_2(x)\, u_2(y)\, k(x-y)\, dx\, dy \le C,$$

for some $C > 0$. Therefore, recalling (15),

$$J_2^2 \le C \iint_{\mathcal{R}_R} \big(v(x) - v(y)\big)^2 \big(\tau(x) + \tau(y)\big)^2\ u_2(x)\, u_2(y)\, k(x-y)\, dx\, dy. \tag{17}$$

Hence, since

$$\big(\tau(x) + \tau(y)\big)^2 = \tau^2(x) + \tau^2(y) + 2\tau(x)\,\tau(y) \le 2\tau^2(x) + 2\tau^2(y),$$

we can use the symmetric role played by x and y in (17) and obtain that

$$J_2^2 \le C \iint_{\mathcal{R}_R} \big(v(x) - v(y)\big)^2 \tau^2(x)\ u_2(x)\, u_2(y)\, k(x-y)\, dx\, dy,$$

up to renaming $C > 0$. So, we insert this information into (13) and we conclude that

$$\begin{aligned} &\left[\iint_{\mathbb{R}^2\times\mathbb{R}^2} \big(v(x) - v(y)\big)^2 \tau^2(x)\, u_2(x)\, u_2(y)\, k(x-y)\, dx\, dy\right]^2 = J_1^2 \\ &\qquad \le J_2^2 \le C \iint_{\mathcal{R}_R} \big(v(x) - v(y)\big)^2 \tau^2(x)\ u_2(x)\, u_2(y)\, k(x-y)\, dx\, dy, \end{aligned} \tag{18}$$

for some $C > 0$.

Since $\mathcal{R}_R \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ and u_2 and k are nonnegative, we can simplify the estimate in (18) by writing

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} \big(v(x) - v(y)\big)^2 \tau^2(x)\, u_2(x)\, u_2(y)\, k(x-y)\, dx\, dy \le C.$$

In particular, since $\tau = 1$ in B_R,

$$\iint_{B_R\times B_R} \big(v(x) - v(y)\big)^2 u_2(x)\, u_2(y)\, k(x-y)\, dx\, dy \le C.$$

Since C is independent of R, we can send $R \to +\infty$ in this estimate and obtain that the map

$$\mathbb{R}^2 \times \mathbb{R}^2 \ni (x, y) \mapsto \big(v(x) - v(y)\big)^2 u_2(x)\, u_2(y)\, k(x-y)$$

belongs to $L^1(\mathbb{R}^2 \times \mathbb{R}^2)$.

Using this and the fact that $\mathcal{R}_R$ approaches the empty set as $R \to +\infty$, we conclude from Lebesgue's dominated convergence theorem that

$$\lim_{R\to+\infty} \iint_{\mathcal{R}_R} \big(v(x) - v(y)\big)^2 u_2(x)\, u_2(y)\, k(x-y)\, dx\, dy = 0.$$

Therefore, going back to (18) and recalling the properties of $\tau = \tau_R$,

$$
\begin{aligned}
& \left[\iint_{\mathbb{R}^2\times\mathbb{R}^2} \big(v(x)-v(y)\big)^2\, u_2(x)\, u_2(y)\, k(x-y)\, dx\, dy\right]^2 \\
= \; & \lim_{R\to+\infty} \left[\iint_{\mathbb{R}^2\times\mathbb{R}^2} \big(v(x)-v(y)\big)^2\, \tau^2(x)\, u_2(x)\, u_2(y)\, k(x-y)\, dx\, dy\right]^2 \\
\le \; & \lim_{R\to+\infty} C \iint_{\mathcal{R}_R} \big(v(x)-v(y)\big)^2\, \tau^2(x)\; u_2(x)\, u_2(y)\, k(x-y)\, dx\, dy. \\
= \; & 0.
\end{aligned}
$$

This and (5) imply that $\big(v(x)-v(y)\big)^2 k(x-y) = 0$ for any $(x,y) \in \mathbb{R}^2\times\mathbb{R}^2$. Hence, recalling (2), we have that $v(x) = v(y)$ for any $x \in \mathbb{R}^2$ and any $y \in B_{r_0}(x)$.

As a consequence, the set $\{y \in \mathbb{R}^2 \text{ s.t. } v(y) = v(0)\}$ is open and closed in $\mathbb{R}^2$, and so, by connectedness, we obtain that v is constant, say $v(x) = a$ for some $a \in \mathbb{R}$. So we define $\omega := \frac{(a,1)}{\sqrt{a^2+1}}$ and we observe that

$$\nabla u(x) = u_2(x)\,(v(x),1) = u_2(x)\sqrt{a^2+1}\,\omega.$$

Thus, if $\omega\cdot y = 0$ then

$$u(x+y)-u(x) = \int_0^1 \nabla u(x+ty)\cdot y\, dt = \int_0^1 u_2(x+ty)\sqrt{a^2+1}\,\omega\cdot y\, dt = 0.$$

Therefore, if we set $u_\star(t) := u(t\omega)$ for any $t\in\mathbb{R}$, and we write any $x \in \mathbb{R}^2$ as

$$x = (\omega\cdot x)\,\omega + y_x$$

with $\omega\cdot y_x = 0$, we conclude that

$$u(x) = u\left((\omega\cdot x)\,\omega + y_x\right) = u\left((\omega\cdot x)\,\omega\right) = u_\star\left(\omega\cdot x\right).$$

This completes the proof of Theorem 2. □

For completeness, we observe that a more general version of Theorem 2 holds true, namely if we replace assumption (5) with a "stability assumption" in the sense of [2]: the precise statement goes as follows:

Theorem 3. *Let u be a solution of* (3) *in the whole of* $\mathbb{R}^2$, *with* $\|u\|_{C^1(\mathbb{R}^2)} < +\infty$ *and* $f \in C^1(\mathbb{R})$. *Assume that there exists* $\psi > 0$ *which solves*

$$\mathcal{L}\psi(x) = f'\big(u(x)\big)\,\psi(x) \text{ for any } x\in\mathbb{R}^2.$$

Then, u is necessarily one-dimensional.

Notice that, in this setting, Theorem 2 is a particular case of Theorem 3, choosing $\psi := u_2 = \partial_{x_2} u$ and recalling (9).

The proof of Theorem 3 is like the one of Theorem 2, with only a technical difference: instead of (7), one has to define, for $i \in \{1, 2\}$,

$$v(x) := \frac{u_i(x)}{\psi(x)}.$$

Then the proof of Theorem 2 goes through (replacing u_2 with ψ when necessary) and implies that v is constant, i.e. $u_i = a_i\psi$, for some $a_i \in \mathbb{R}$. This gives that $\nabla u(x) = \psi(x)\,(a_1, a_2)$, which in turn implies the one-dimensional symmetry of u.

Also, we think that it is an interesting open problem to investigate if symmetry results in the spirit of Theorems 2 and 3 hold true in higher dimension.

Additional note. Once our paper was completed, we heard that related results have been obtained simultaneously and independently in [22].

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