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Existence of weak solutions for improved Nernst–Planck–Poisson models of compressible reacting electrolytes

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Abstract

We consider an improved Nernst–Planck–Poisson model for compressible electrolytes first proposed by Dreyer et al. in 2013. The model takes into account the elastic deformation of the medium. In particular, large pressure contributions near electrochemical interfaces induce an inherent coupling of mass and momentum transport. The model consists of convection–diffusion–reaction equations for the constituents of the mixture, of the Navier-Stokes equation for the barycentric velocity and the Poisson equation for the electrical potential. Cross–diffusion phenomena occur due to the principle of mass conservation. Moreover, the diffusion matrix (mobility matrix) has a zero eigenvalue, meaning that the system is degenerate parabolic. In this paper we establish the existence of a global–in–time weak solution for the full model, allowing for cross–diffusion and an arbitrary number of chemical reactions in the bulk and on the active boundary.

1 Introduction

Increasing the efficiency of energy storage systems nowadays requires a better understanding of their fundamental physical principles. Of particular interest are ion transport in electrolytes for instance in lithium-ion batteries. Classically this transport is modeled by the Nernst-Planck theory. But the classical Nernst-Planck theory has an important drawback: In the neighbourhood of interfaces, see [DGM13, DGL14], it is failing for various reasons:

First of all, the classical Nernst-Planck model neglects the high pressures induced by the Lorentz force which affects the charge transport. Secondly, it does not take into account the interaction between the solvent and the charged constituents.

A further drawback of the Nernst-Planck theory is the widely used assumption of local charge neutrality. This assumption completely fails in the vicinity of the boundaries where electric charges accumulate.

An improved model able to remedy these deficiencies was first proposed in the paper [DGM13]. In [DGL14, DGM15] this improved model was further extended to include (i) finite volume effects of the constituents, (ii) the viscosity of the mixture and (iii) chemical reactions in the bulk and on electrochemical interfaces. In the isothermal case, the new model consists of *universal balance equations* for mass and momentum and *general material–dependent constitutive equations* for the mass fluxes, the stress tensor and the reaction rates. These general constitutive equations use the driving forces of the system, which are derived from a single free energy function $\varrho\psi$. Here we choose a free energy function according to a *special constitutive model* for electrolytes proposed in [DGM15, LGD16]. Moreover, we use a generalization of the constitutive equations for the reaction rates as proposed in [DGM15].

In this paper we establish the existence of a global–in–time weak solution for the presented model, allowing for cross–diffusion, an arbitrary number of chemical reactions in the bulk and on an active boundary that represents a one–sided elecrochemical interface. Moreover we consider different specific volumes of the constituents.

Our method relies on the one hand on *a priori* estimates that result from the thermodynamically consistent modelling, and from the conservation of total mass. The estimates are partly a consequence of known results for the Poisson equation or the Navier-Stokes equations, but we can regard the estimates on the *chemical potentials* of the mixture constituents, in particular in the presence of chemical reactions, as original. The second supporting pillar of our method is compactness: Here we exploit the original idea of [Hop51] (rather than Aubin–Lions techniques and their generalisations), and the compactness properties of the Navier-Stokes operator established first in [Lio98] and extended in [FNP01]. In order to construct a thermodynamically consistent regularisation of the system and approximate solutions, we use standard techniques of convex analysis.

Since large parts of the modelling work in [DGM13] are original and not yet well known in the mathematical literature devoted to the analysis of mixtures, we are not able to quote a direct precursor for our analysis. In order to put the investigation into some context, let us mention [MPZ15] and [Zat15] where models of compressible mixtures, including the energy balance, but without the electric field, were studied. These models are not derived from the same thermodynamic principles that are used in our study: Particularly the constitutive equations for the pressure, for the diffusion fluxes and for the reaction terms, are different in [MPZ15] and in [DGM13]. The compactness question occurs like in our analysis but is solved assuming a special structure of the viscosity tensor, called Bresch–Desjardins condition. This allows to obtain estimates on the density gradient, a device which is not at our disposal here. A further difference between the two mixture models concerns cross–diffusion, which is described in [MPZ15] and [Zat15] by a special nonsymmetric choice of the mobility matrix, whereas we allow for general symmetric positive semidefinite matrices. Note that the mobility matrix must be symmetric at least in a binary mixture. Among recent less directly related investigations let us mention: In the context of general diffusion, [Bot11]; for models with simplified diffusion and pressure laws [FPT08], [BFPR16]; for the analysis of incompressible models of Nernst–Planck–Poisson type [BFS16], [JS13].

In Section 2 the model will be introduced following [DGM15]. The model is formulated for the *normal regime of the system*, i.e. it is assumed that the mass densities of the constituents do not vanish. For the mathematical analysis we will derive an equivalent formulation which exhibits more stability against possibly occurring extreme behavior, like the vanishing of species.

2 Improved Nernst–Planck–Poisson model

We consider a bounded domain $\Omega \subset \mathbb{R}^3$ representing an electrolytic mixture. The boundary of Ω possesses a disjoint decomposition $\partial\Omega = \Gamma \cup \Sigma$: The surface Γ represents an active *interface* between an electrode and the electrolyte, where chemical reactions and adsorption may occur. The other surface Σ represents an inert outer wall with no reactions and no adsorption.

The compressible mixture consists of $N \in \mathbb{N}$ species denoted by $\mathrm{A_1, \dots, A_N}$. A species $\mathrm{A_i}$ may be carrier of electric charge, z_i, and of molecular mass, m_i.

We assume that the system is isothermal, so that the absolute temperature denoted by θ is a positive constant. Under the isothermal assumption the thermodynamic state of the mixture at time $t \in [0, T]$ is described by the mass densities $\rho_1, \dots, \rho_N$ of the species, the barycentric velocity v of the mixture and the electric field E. As usual in electrochemistry, a quasi-static approximation of the electric field is considered, i.e. the magnetic field is constant and the electric field satisfies

$$E = -\nabla\phi. \tag{1}$$

The scalar function ϕ is called electrical potential.

The active boundary Γ can be viewed as a mixture of $N^\Gamma = N + N^{\text{ext}}$ constituents denoted by $\mathrm{A_1, \dots, A_{N^\Gamma}}$, where the additional N^{ext} constituents take into account the species of the adjacent exterior matter, i.e. electrode species. Thus we only consider surface chemical reactions with participating species that also exist in the adjacent bulk domains. The surface constituents have the surface mass densities $\rho_1^\Gamma, \dots,\ \rho_{N^\Gamma}^\Gamma$.

Moreover we consider $s \in \mathbb{N}$ chemical reactions in the bulk and $s^\Gamma \in \mathbb{N}$ surface reactions on the boundary Γ, respectively. The chemical reactions in the bulk and on the boundary have the general form

$$\mathrm{a_1^k A_1} + \cdots + \mathrm{a_N^k A_N} \underset{R_k^b}{\overset{R_k^f}{\rightleftharpoons}} \mathrm{b_1^k A_1} + \cdots + \mathrm{b_N^k A_N} \qquad \text{for } k \in \{1, \cdots, s\}, \tag{2}$$

$$\mathrm{a_{\Gamma,1}^k A_1} + \cdots + \mathrm{a_{\Gamma,N^\Gamma}^k A_{N^\Gamma}} \underset{R_k^{\Gamma,b}}{\overset{R_k^{\Gamma,f}}{\rightleftharpoons}} \mathrm{b_{\Gamma,1}^k A_1} + \cdots + \mathrm{b_{\Gamma,N^\Gamma}^k A_{N^\Gamma}} \qquad \text{for } k \in \{1, \cdots, s^\Gamma\}. \tag{3}$$

The constants a_α^i, b_α^i are positive intergers. We define the (mass related) stoechiometric coefficients of the kth bulk reactions as

$$\gamma^k \in \mathbb{R}^N, \quad \gamma_i^k := (\mathrm{a_i^k} - \mathrm{b_i^k})\, m_i \qquad \text{for } i = 1, \dots, N\,. \tag{4}$$

The inclusion of the molecular mass in the definition of the stoechiometric coefficients is not common, but it simplifies the notation. The forward reaction rate of the kth reaction is $R_k^f > 0$, and the backward reaction rate is rate $R_k^b > 0$. The net reaction rate of the kth reaction is defined as

$$R_k = R_k^f - R_k^b \qquad \text{for } k = 1, \dots, s\,. \tag{5}$$

The same definitions hold for the surface reactions on Γ. Here the stoechiometric coefficients are defined as

$$\gamma_\Gamma^k \in \mathbb{R}^{N^\Gamma}, \quad \gamma_{\Gamma,i}^k := (\mathrm{a_{\Gamma,i}^k} - \mathrm{b_{\Gamma,i}^k})\, m_i \qquad \text{for} \quad i = 1, \dots, N^\Gamma \tag{6}$$

and the surface reaction rates are

$$R_k^\Gamma = R_k^{\Gamma,f} - R_k^{\Gamma,b} \qquad \text{for} \quad k = 1, \dots, s^\Gamma\,. \tag{7}$$

Since charge and mass are conserved in every single reaction, we have

$$\sum_{i=1}^{N} \gamma_i^k = 0 \qquad \text{and} \qquad \sum_{i=1}^{N} \tfrac{z_i}{m_i}\, \gamma_i^k = 0 \qquad \text{for all} \quad k = 1, \ldots, s\,, \tag{8}$$

$$\sum_{i=1}^{N^\Gamma} \gamma_{\Gamma,i}^k = 0 \qquad \text{and} \qquad \sum_{i=1}^{N^\Gamma} \tfrac{z_i}{m_i}\, \gamma_{\Gamma,i}^k = 0 \qquad \text{for all} \quad k = 1, \ldots, s^\Gamma\,. \tag{9}$$

2.1 Balance equations in the bulk

In the isothermal case the evolution of the thermodynamic state is described by the equations of partial mass balances and momentum balance and the Poisson equation.

In $]0, T[\times\Omega$ the mixture obeys partial mass balances for $i = 1, \ldots, N$:

$$\frac{\partial \rho_i}{\partial t} + \operatorname{div}(\rho_i\, v + J^i) = r_i\,. \tag{10}$$

Here, v denotes the *barycentric velocity* of the mixture, and r_i is the mass production of the ith constituent due to chemical reactions. The quantities $J^1, \ldots, J^N$ are called the diffusion fluxes. We use upper indices in their case because they are vector fields of $\mathbb{R}^3$ and not scalars. The mass production of constituent $\mathrm{A_i}$ is related to the reaction rates by

$$r_i = \sum_{k=1}^{s} \gamma_i^k\, R_k \qquad \text{for} \quad i = 1, \ldots, N\,. \tag{11}$$

The total mass is defined as $\varrho = \sum_{i=1}^N \rho_i$ and has to satisfies the total mass balance equation

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho\, v) = 0\,. \tag{12}$$

Thus the conservation of total mass requires the additional constraints on the diffusion fluxes and mass productions

$$\sum_{i=1}^{N} J^i = 0 \qquad \text{and} \qquad \sum_{i=1}^{N} r_i = 0\,. \tag{13}$$

The side condition on the diffusion fluxes has to be guaranteed by an appropriate constitutive modeling. However, the constraint $(13)_2$ is already guaranteed by (11) and the conservation of mass in every chemical reaction (8).

The momentum balance has the form

$$\frac{\partial \varrho\, v}{\partial t} + \operatorname{div}(\varrho\, v \otimes v - \sigma) = \varrho\, b + n^F E\,, \tag{14}$$

Herein σ denotes the Cauchy stress tensor, $\varrho\, b$ is the force density due to gravitation, and the Lorentz force due to the electric field is given by $n^F\, E$. The quantity n^F represents the free charge density which may be written in terms of the species mass densities by

$$n^F = \sum_{i=1}^{N} \frac{z_i}{m_i}\,\rho_i \ . \tag{15}$$

Throughout the paper, we are going to neglect the gravitational force that plays no role in the analysis. In the electrostatic setting the balance equation for the electric field reduces to the Poisson equation for the electrical potential,

$$-\epsilon_0\,(1+\chi)\,\triangle\phi = n^F \,. \tag{16}$$

Here $\chi > 0$ is the constant susceptibility of the electrolyte.

2.2 Constitutive equations

The constitutive equations for the mass fluxes, the reaction rates and the stress tensor can be derived from a single free energy density $\varrho\psi$ of a general form

$$\varrho\psi = h(\theta,\ \rho_1,\ldots,\rho_N) \,. \tag{17}$$

The derivatives of the free energy function with respect to the mass densities are called *chemical potentials*,

$$\mu_i := \frac{\partial h}{\partial \rho_i}(\theta,\rho_1,\ldots,\rho_N) \,. \tag{18}$$

In the isothermal setting the balance equations and the free energy density yield a local *entropy production* ξ with three contributions due to diffusion, ξ_D, reaction, ξ_R, and viscosity, ξ_V, [MR59, BD15, DGM15],

$$\xi = \xi_D + \xi_R + \xi_V \geq 0 \,. \tag{19}$$

A constitutive model that relies on the free energy function (17) implies explicit expressions for the three entropy productions as binary products. From these expressions we may derive constitutive equations yielding three separate non-negative entropy productions. For more details regarding the derivation of the entropy production we refer to [MR59, dM63, BD15]. In [BD15] its is shown how cross–effects revealing the Onsager symmetry can be introduced.

Diffusion fluxes. The entropy production due to diffusion reads

$$\xi_D = -\sum_{i=1}^{N} J^i \cdot D^i \,, \tag{20}$$

where $D^1,\ldots,D^N$ are the thermodynamic driving forces for diffusion,

$$D^i := \nabla\left(\frac{\mu_i}{\theta}\right) - \frac{1}{\theta}\,\frac{z_i}{m_i}\,E \qquad \text{for} \quad i = 1,\ldots,N \,. \tag{21}$$

The simplest constitutive ansatz for the diffusion fluxes $J^1, \dots, J^N$ that implies $\xi_D \geq 0$ is given by

$$J^i = -\sum_{j=1}^{N} M_{i,j}\, D^j \qquad \text{for } i = 1, \dots, N\,, \tag{22}$$

where the *mobility matrix* $M \in \mathbb{R}^{N\times N}_{\mathsf{sym}}$ must be positive semidefinite. The matrix M may depend on ρ. Moreover, the side condition $\sum_{i=1}^{N} J^i = 0$ is complied if the mobility matrix satisfies

$$\sum_{i=1}^{N} M_{i,j} = 0 \ \text{ for } j = 1, \dots, N\,. \tag{23}$$

Exemplarily, following the paper [DGM13], one can construct M from an empirical mobility matrix $M_{\mathsf{emp}}(\rho)$ and a linear operator $\mathcal{P} :\ \mathbb{R}^N \to \mathbb{R}^{N-1} \times \{0\}$ via

$$M := \mathcal{P}^T\, M_{\mathsf{emp}}\, \mathcal{P}, \qquad M_{\mathsf{emp}} := \mathsf{diag}(d_1\, \rho_1, \dots,\, d_{N-1}\, \rho_{N-1},\, 1)\,, \tag{24}$$

where $d_1, \dots, d_{N-1} > 0$ are diffusion constants, and the lines of the matrix $\mathcal{P}$ are given by the differences $e^i - e^N$ of standard basis vectors for $i = 1, \dots, N$. In fact, any operator $\mathcal{P}$ that satisfies for $i = 1, \dots, N$ the condition $\sum_{j=1}^{N} \mathcal{P}_{i,j} = 0$ can be chosen in (24) in order to satisfy (23). Let us emphasize however that our analytical results do not rely on the particular structure (24) of the matrix M.

Reaction rates. The entropy production due to chemical reactions assumes the form

$$\xi_{\mathsf{R}} = -\sum_{k=1}^{s} R_k\, D^{\mathsf{R}}_k\,, \tag{25}$$

where the driving forces $D^{\mathsf{R}}_1, \dots, D^{\mathsf{R}}_s$ are given by

$$D^{\mathsf{R}}_k = \sum_{i=1}^{N} \gamma^k_i\, \mu_i \qquad \text{for } k = 1, \dots, s\,. \tag{26}$$

To achieve $\xi_{\mathsf{R}} \geq 0$, we assume that the vector of production rates are derived from a convex, non-negative potential

$$R = -\nabla_{D^{\mathsf{R}}} \Psi(D^{\mathsf{R}}), \qquad \text{with} \quad \Psi : \mathbb{R}^s \to \mathbb{R} \text{ convex} \quad \text{and} \quad \nabla_{D^{\mathsf{R}}} \Psi(0) = 0\,. \tag{27}$$

Note that this choice is more general as in [DGM15], where the following potential is employed,

$$\text{Dreyer et al.:} \qquad \Psi = -\sum_{k=1}^{s} \tfrac{1}{\beta_k A_k} e^{-\beta_k A_k D^{\mathsf{R}}_k} \Big(1 + \tfrac{\beta_k}{1-\beta_k} e^{A_k D^{\mathsf{R}}_k}\Big) + C\,, \tag{28}$$

with positive constants $A_1, \dots, A_s$ and constants $\beta_1, \dots \beta_s \in]0,\, 1[$, $C \in \mathbb{R}$ arbitrary. By this choice Dreyer et al. achieve an ansatz of Arrhenius typ, which is widely used in chemistry,

$$\text{Dreyer et al.:} \qquad R_k = e^{-\beta_k\, A_k\, D^{\mathsf{R}}_k}\, (1 - e^{A_k\, D^{\mathsf{R}}_k})\,. \tag{29}$$

Stress tensor. The entropy production due to viscosity is represented by

$$\xi_V = \tfrac{1}{2}(\sigma + p\,\mathsf{Id}) : D(v)\;, \tag{30}$$

where the driving force $D(v)$ is defined as $D(v) = (\partial_i v_j + \partial_j v_i)_{i,j=1,\dots,3}$, and Id denotes the identity matrix.

We split the Cauchy stress tensor into a viscous part $\mathbb{S}^{\mathsf{visc}}$ and the pressure p,

$$\sigma = -p\,\mathsf{Id} + \mathbb{S}^{\mathsf{visc}}\;. \tag{31}$$

Then the material *pressure* p can be calculated from the free energy function (17). The resulting representation is called Gibbs-Duhem equation and reads

$$p := -h + \sum_{i=1}^{N} \rho_i\,\mu_i\;. \tag{32}$$

The simplest constitutive choice for the viscous stress tensor $\mathbb{S}^{\mathsf{visc}}$ satisfying $\xi_V \geq 0$ describes a Newtonian fluid. It reads

$$\mathbb{S}^{\mathsf{visc}} = \eta\; D(v) + \lambda\;\operatorname{div} v\;\mathsf{Id}, \tag{33}$$

where $\eta > 0$ is the coefficient of shear viscosity, and the coefficient λ of bulk viscosity satisfies $\lambda + \frac{2}{3}\,\eta \geq 0$.

2.3 Choice of the free energy function

The constitutive model is derived from a free energy density of the general form (17). However, for the analysis of the model, we need in some extent to specify the choice of the free energy function. To this end the free energy density $\varrho\psi$ is additively split into three contributions,

$$h = \sum_{i=1}^{N} \rho_i\,h_i^{\mathsf{ref}} + h^{\mathsf{mech}} + h^{\mathsf{mix}}\,. \tag{34}$$

Here, the constants h_i^{ref} ($i = 1,\dots,N$) are related to the reference states of the pure constituents. The contribution h^{mech} is the mechanical part of the free energy that is neglected in the classical Nernst-Planck theory, and h^{mix} represents the mixing entropy.

In the presentation of [DGM13, DGL14], the contributions h^{mech} and h^{mix} are naturally given as functions of the *number densities* $n_1,\dots,n_N$ of the constituents. These are defined via $n_i := \rho_i/m_i$ ($i = 1,\dots,N$). Number fractions $y_i := n_i/(\sum_{j=1}^{N} n_j)$ for $i = 1,\dots,N$ are also involved.

The function h^{mech} is the free energy density associated with the isotropic elastic deformation of the mixture. The mechanical free energy takes into account the different *specific volumes* $V_1,\dots,V_N \in \mathbb{R}_+$ of the constituents. Assuming a constant bulk compression modulus $K > 0$ the mechanical free energy in [DGL14] is given by

$$h^{\mathsf{mech}} = (K - p_{\mathsf{ref}})\,(1 - n\cdot V) + K\,(n\cdot V)\,\ln(n\cdot V)\,.$$

Here p_{ref} is a constant reference value of the pressure, and $n \cdot V$ stands for $\sum_{i=1}^{N} n_i\, V_i$. Another typical choice in fluid mechanics is the Tait equation

$$h^{\text{mech}} = (K - p_{\text{ref}})\,(1 - n \cdot V) + \frac{K}{\alpha}\,((n \cdot V)^{\alpha} - n \cdot V)\,, \qquad \alpha > 1\,.$$

For the sake of generality, we express h^{mech} in the form

$$h^{\text{mech}} = K\,F(n \cdot V) + C\,n \cdot V \qquad \text{with} \quad F : \mathbb{R}_+ \to \mathbb{R} \text{ convex}\ . \tag{35}$$

Dreyer et al. use $F(x) := x\,\ln x + C_1$ for an simple mixture, whereas the Tait equation corresponds to $F(x) = c_\alpha\,x^\alpha + C_2$.

The free energy function h^{mix} results from the entropy of mixing and is given by

$$h^{\text{mix}} := \sum_{i=1}^{N} n_i\ k_B\ \theta\ \sum_{i=1}^{N} y_i\ \ln y_i\,, \tag{36}$$

where k_B is the Boltzmann constant.

2.4 The model for the boundary Γ

The active boundary Γ represents an interface between the electrolyte mixture and an *external material*. In the most important application the external material is an electrode which is likewise a mixture of $N^{\text{ext}} \in \mathbb{N}$ constituents. Here we have analogous quantities to those that occur in the electrolyte, namely the barycentric velocity, and diffusion fluxes and so on. To distinguish between the electrolyte and the external material we provide the external quantities the suffix $^{\text{ext}}$.

In this paper we assume for simplicity that on Γ we exclusively have constituents that also exist in the electrolyte and in the external material. Thus the interface Γ is a mixture of $N^\Gamma = N + N^{\text{ext}}$ constituents.

The equations of an interface representing a surface mixture are derivable in the context of surface thermodynamics and we refer the interested reader to [ABM75, DGM15, Guh14]. As in the bulk there are universal surface balance equation and material depending surface constitutive equations.

To simplify the surface equations we assume on $]0, T[\times\Gamma$

- Time variations of the surface mass densities and tangential transport are negligible in comparison to mass transfer across the surface and to chemical surface reactions. Then the surface balance equations become stationary.
- The interface is fixed in space, i.e. the interfacial normal speed is zero.
- There is no velocity slip and the normal barycentric velocity is equal to the interfacial normal speed, i.e. we have on $]0, T[\times\Gamma$

$$v = 0\,. \tag{37}$$

Surface mass balances and surface reaction rates. We assume that the interfacial unit normal ν points into the external material.

Under the above assumptions the surface mass balance equations on $]0,T[\times\Gamma$ then reduce to

$$0 = \begin{cases} r_i^\Gamma + J^i \cdot \nu \,, & \text{for } i = 1,\ldots,N \\ r_{N+i}^\Gamma - J^{\mathsf{ext},i} \cdot \nu & \text{for } i = 1,\ldots,N^{\mathsf{ext}} \end{cases} \tag{38}$$

Here we use the convention that the N first species on Γ are the electrolyte constituents, while the constituents with indices $N+1,\ldots,N+N^{\mathsf{ext}}$ are the external ones.

It remains to specify the surface mass production r^Γ due to surface reactions. As in the bulk, the production r^Γ is related to the surface reaction rates R^Γ by

$$r_i^\Gamma = \sum_{k=1}^{s^\Gamma} \gamma_{\Gamma,i}^k \, R_k^\Gamma \text{ for } i = 1,\ldots,N^\Gamma \;. \tag{39}$$

The interfacial entropy production ξ_{R}^Γ due to chemical reaction is, [DGM15],

$$\xi_{\mathsf{R}}^\Gamma = -\sum_{k=1}^{s^\Gamma} R_k^\Gamma \, D_k^{\Gamma,\mathsf{R}} \geq 0 \quad \text{with the driving force} \quad D_k^{\Gamma,\mathsf{R}} = \sum_{i=1}^{N^\Gamma} \gamma_{\Gamma,i}^k \, \mu_i^\Gamma \quad \text{for} \quad k = 1,\ldots,s^\Gamma \;. \tag{40}$$

The entropy production of the surface has the same structure as the corresponding entropy production in the bulk (25). Thus in order to satisfy the entropy inequality a similar ansatz to (29) may be used. We assume the existence of a potential Ψ^Γ so that

$$R^\Gamma = -\nabla_{D^{\Gamma,\mathsf{R}}} \Psi^\Gamma(D^{\Gamma,\mathsf{R}}) \qquad \text{with} \quad \Psi^\Gamma : \mathbb{R}^{s^\Gamma} \to \mathbb{R} \text{ convex} \quad \text{and} \quad \nabla_{D^{\Gamma,\mathsf{R}}} \Psi^\Gamma(0) = 0 \;. \tag{41}$$

Diffusion fluxes. Due to the above assumptions, the constitutive equations for the diffusion fluxes at $]0,T[\times\Gamma$ simplify to

$$J^i \cdot \nu = +\sum_{j=1}^{N} M_{i,j}^\Gamma \,(\mu_j - \mu_j^\Gamma) \qquad \text{for } i = 1,\ldots,N \,, \tag{42}$$

$$J^{\mathsf{ext},i} \cdot \nu = -\sum_{j=1}^{N^{\mathsf{ext}}} M_{i,j}^{\Gamma,\mathsf{ext}} \,(\mu_j^{\mathsf{ext}} - \mu_{N+j}^{\Gamma,\mathsf{ext}}) \qquad \text{for } i = 1,\ldots,N^{\mathsf{ext}} \,. \tag{43}$$

Here, $\mu_1^\Gamma,\ldots,\mu_N^\Gamma$ are the surface chemical potentials of the electrolytic species, whereas the external species induce the surface chemical potentials $\mu_{N+1}^\Gamma,\ldots,\mu_{N+N^{\mathsf{ext}}}^\Gamma$.

These equations describe the adsorption of a constituent from the bulk to the surface. The kinetics of this process is controlled by positive semidefinite matrices, viz.

$$M^\Gamma \in \mathbb{R}_{\mathsf{sym}}^{N\times N} \qquad \text{and} \qquad M^{\Gamma,\mathsf{ext}} \in \mathbb{R}_{\mathsf{sym}}^{N^{\mathsf{ext}}\times N^{\mathsf{ext}}} \,, \tag{44}$$

which satisfy the side condition

$$M^\Gamma \, 1^N = 0 \qquad \text{and} \qquad M^{\Gamma,\text{ext}} \, 1^{N^\text{ext}} = 0 \; . \tag{45}$$

In the general thermodynamic setting, the surface chemical potentials are derivatives of a surface free energy. Due to the assumption of stationary surface equations, and that the boundary is fixed, we are able to formulate all surface equations in terms of the surface chemical potentials. Thus, from a mathematical viewpoint the equation system (43) only serves to determine the surface chemical potentials μ^Γ and the fluxes.

Electrical potential. The boundary condition for the electrical potential can be derived from Maxwell's equations for surfaces, which are satisfied in the quasi-static stetting by a continuous electrical potential, [DGM15]. On $]0,T[\times\Gamma$ we have

$$\phi = \phi_0 \; , \tag{46}$$

where ϕ_0 is the electric potential at Γ. In this paper we assume that the surface potential ϕ_0 is a given function.

2.5 Summary model equations

Domain $\boldsymbol{\Omega}$. Summarising, the evolution of the state (ρ, v, φ) in $]0,T[\times\Omega$ is described by the PDE–system

$$\frac{\partial \rho_i}{\partial t} + \operatorname{div}(\rho_i \, v + J^i) = \sum_{k=1}^{s} \gamma_i^k \, R_k \text{ for } i = 1, \ldots, N \tag{47}$$

$$\frac{\partial \varrho \, v}{\partial t} + \operatorname{div}(\varrho \, v \otimes v - \mathbb{S}^\text{visc}(\nabla v)) + \nabla p = -n^F \, \nabla \phi \tag{48}$$

$$-\epsilon_0 \, (1+\chi) \, \triangle \phi = n^F \qquad . \tag{49}$$

Here n^F is given by the formula (15), the fluxes $J^1, \ldots, J^N$ obey (22), $R_1, \ldots, R_s$ obey (27), p obeys (32) and $\mathbb{S}^\text{visc}$ obeys (33).

Boundary $\boldsymbol{\Gamma}$. We have on $]0,T[\times\Gamma$ the boundary conditions

$$0 = r^\Gamma + \left(J - J^\text{ext}\right) \cdot \nu \; , \tag{50}$$

$$J \cdot \nu = +M^\Gamma \, (\mu - \mu^\Gamma) \qquad \text{for electrolyte constituents,} \tag{51}$$

$$J^\text{ext} \cdot \nu = -M^{\Gamma,\text{ext}} \, (\mu^\text{ext} - \mu^{\Gamma,\text{ext}}) \qquad \text{for external constituents,} \tag{52}$$

$$v = 0 \; , \tag{53}$$

$$\phi = \phi_0 \; , \tag{54}$$

where the external chemical potentials μ^ext, the external potential ϕ_0 and the kinetic matrices M^Γ and $M^{\Gamma,\text{ext}}$ are given. The reaction rates r^Γ obey(39) with R^Γ satisfying (41). Recall that the conditions (50) represent N^Γ equations and are a shorter form for (38).

Boundary Σ. We choose as simple as possible model on the surface $]0,T[\times\Sigma$: No mass flux,

$$(\rho_i\, v + J^i)\cdot\nu = 0 \quad \text{for } i = 1,\ldots,N\,; \tag{55}$$

complete adherence of the fluid,

$$v = 0 \quad \text{on }]0,T[\times\Sigma\,; \tag{56}$$

no surface charge,

$$\nabla\phi\cdot\nu = 0 \quad \text{on }]0,T[\times\Sigma\,. \tag{57}$$

Initial conditions. Initial conditions are prescribed for the variables $\rho_1,\ldots,\rho_N$. We denote them ρ_i^0, $i = 1,\ldots,N$. Moreover, an initial state v^0 is also given for the velocity vector.

2.6 Notation

To get rid of overstressed indexing, we simplify the notation by the convention that we use vectors for objects of the same type. For instance we write ρ for the vector of mass densities, n for the vector of number densities i.e.

$$\rho := (\rho_1,\rho_2,\ldots,\rho_N) \in \mathbb{R}^N, \quad n := (n_1,n_2,\ldots,n_N) \in \mathbb{R}^N\,. \tag{58}$$

Moreover we define the vector

$$\mathbb{1} := 1^N := (1,\,1,\ldots,1) \in \mathbb{R}^N \tag{59}$$

and the vectors of quotients of charge and mass, and of volume and mass

$$\frac{z}{m} := \big(\tfrac{z_1}{m_1},\,\tfrac{z_2}{m_2},\ldots,\tfrac{z_N}{m_N}\big) \in \mathbb{R}^N, \quad \frac{V}{m} := \big(\tfrac{V_1}{m_1},\tfrac{V_2}{m_2},\ldots,\tfrac{V_N}{m_N}\big) \in \mathbb{R}^N\,. \tag{60}$$

Using these conventions, we have a. o. the identities

$$\varrho = \mathbb{1}\cdot\rho, \quad n^F = \frac{z}{m}\cdot\rho, \quad n\cdot V = \rho\cdot\frac{V}{m} \ \text{etc.}$$

The diffusion fluxes $J^1,\ldots,J^N$ span a rectangular matrix $J = \{J^i_j\} \in \mathbb{R}^N\times\mathbb{R}^3$. The upper index corresponds to the lines of this matrix. Vectors of $\mathbb{R}^N$ are multiplicated from the left as for instance in $\mathbb{1}\cdot J = \sum_{i=1}^N J^i$ which is an identity in $\mathbb{R}^3$.

The vectors $\gamma^1,\ldots,\gamma^s$ span a rectangular matrix $\gamma = \{\gamma^k_i\} \in \mathbb{R}^s\times\mathbb{R}^N$. The upper index corresponds to the line of the matrix. Vectors of $\mathbb{R}^s$ are multiplicated from the left, as for instance in the identity $r = R\cdot\gamma = \sum_{k=1}^s R_k\,\gamma^k \in \mathbb{R}^N$.

Analogously the vectors $\gamma_\Gamma^1,\ldots,\gamma_\Gamma^{s^\Gamma}$ span a rectangular matrix $\gamma_\Gamma = \{\gamma^k_{\Gamma,i}\} \in \mathbb{R}^{s^\Gamma}\times\mathbb{R}^{N^\Gamma}$.

In order to describe the reactions, we will further use the abbreviations

$$\bar R:\ \mathbb{R}^s\to\mathbb{R}^s, \quad \bar R := -\nabla\Psi$$
$$\bar R^\Gamma:\ \mathbb{R}s^\Gamma\to\mathbb{R}^{s^\Gamma}, \quad \bar R^\Gamma := -\nabla\Psi^\Gamma\,.$$

3 Remarks on the concept of the solution

Consider the system of convection-diffusion-reaction equations (47) by given velocity and electric fields. A (weak) solution to this system is a pair of vector fields $(\rho, \, \mu) \, : \, [0,T] \times \Omega \to \mathbb{R}^N_+ \times \mathbb{R}^N$ such that (47) is valid (in the sense of distributions), and subject to the *algebraic relation* (cf. (18))

$$\mu = \nabla_\rho h(\rho) \, . \tag{61}$$

State–constraints If one thinks of applying functionalanalytic methods to the problem, it is natural seeking to eliminate the algebraic constraint and to resort to one set of variables, either μ or ρ. The choice of ρ as main variables is connected to the difficulty that there is no (known) maximum principle for systems with cross–diffusion. Since the mass densities must satisfy the physical condition $\rho_i \geq 0$ for $i = 1, \ldots, N$ the PDE system remains subject to an inequality constraint. Moreover, the $\rho-$variables are not natural to express the diffusion (22) and they would lead to uselessly complex structures at this level.

As to the $\mu-$variables, due to (61), they obey the constraint $\mu \in \text{Image}(\nabla h; \, \mathbb{R}^N_+)$. For a general function h, the range of ∇h applied to $\mathbb{R}^N_+$ might be a true subset of $\mathbb{R}^N$. Thus, we can state that in general, the PDE system is also subject to a constraint in μ. But *for the constitutive assumption* (34) *here under consideration*, we can show that $\nabla h \, : \, \mathbb{R}^N_+ \to \mathbb{R}^N$ is a bijection if the first derivative of the function F is surjective onto $\mathbb{R}$. Thus at least for relevant particular choices of h, the PDE system is *unconstrained* in μ, and the chemical potentials are a favourable set of variables for existence theory.

An 'hyperbolic' component As a next remark, it is important to note that the fluxes $J^1, \ldots, J^N$ and the functions $r_1, \ldots, r_N$ occurring in the system (47) in fact only depend on the projection of the vector μ on the subspace $\mathbb{1}^\perp := \{\xi \in \mathbb{R}^N \, : \, \xi \cdot \mathbb{1} = 0\}$ (see the side conditions (23) for the diffusion flux, and to the restriction (8) on the vectors $\gamma^1, \ldots, \gamma^s$). In fact, only $N-1$ coordinates of the vector μ in the plane $\mathbb{1}^\perp$ explicitely occur in the system. In particular, a control on the spatial gradient can be obtained only for the reduced vector.

Due to these remarks, a **change of variables** is necessary in order to define the solution. We keep as main variables:

(a) On the one hand, one coordinate of the vector field ρ, namely the total mass density $\rho \cdot \mathbb{1}$ that we shall denote ϱ throughout the paper. This is the 'hyperbolic' component subject to the continuity equation;

(b) On the other hand, $N-1$ coordinates of the vector of chemical potentials μ defined via a projection onto the linear space $\mathbb{1}^\perp \subset \mathbb{R}^N$.

The possibility of these choices relies on the following algebraic results that we only aforemention here.

Proposition 3.1. *Assume the free energy function h satisfies the Ansatz* (34)*,* (35)*,* (36)*, and that the function F occurring in* (35) *belongs to $C^2(\mathbb{R}_+) \cap C(\mathbb{R}_{0,+})$, is convex and possesses a surjective first derivative F'.*

Let $\xi^1, \ldots, \xi^N \in \mathbb{R}^N$ be a basis of $\mathbb{R}^N$ such that $\xi^N = \mathbb{1}$ and $\eta^1, \ldots, \eta^N \in \mathbb{R}^N$ the vectors such that $\xi^i \cdot \eta^j = \delta_i^j$ for $i, j = 1, \ldots, N$.

We define a 'projector'[1] $\Pi : \mathbb{R}^N \to \mathbb{R}^{N-1}$ *and an extension operator $\mathcal{E} : \mathbb{R}^{N-1} \to \mathbb{R}^N$ associated with the basis $\{\xi^i\}_{i=1\ldots,N}$ via*

$$\Pi\, X := (X \cdot \eta^1, \ldots, X \cdot \eta^{N-1}) \text{ for } X \in \mathbb{R}^N, \quad \mathcal{E}\, q := \sum_{k=1}^{N-1} q_k\, \xi^k \text{ for } q \in \mathbb{R}^{N-1}\,.$$

Then there are mappings $\mathscr{R} \in C^1(\mathbb{R}_+ \times \mathbb{R}^{N-1};\, \mathbb{R}^N_+)$ and $\mathscr{M} \in C^1(\mathbb{R}_+ \times \mathbb{R}^{N-1};\, \mathbb{R})$ such that the nonlinear algebraic equations (61) *are valid for $\mu \in \mathbb{R}^N$ and $\rho \in \mathbb{R}^N_+$ if and only if there are $\varrho \in \mathbb{R}_+$ and $q \in \mathbb{R}^{N-1}$ such that*

$$\rho = \mathscr{R}(\varrho,\, q), \quad \rho \cdot \mathbb{1} = \varrho \qquad \text{and} \qquad \Pi\mu = q, \quad \mu \cdot \eta^N = \mathscr{M}(\varrho,\, q)\,. \tag{62}$$

A proof of this elementary result is given in the section 5. In view of Proposition 3.1, we can *equivalently define* a solution to the system of equations (47) as a pair $(\varrho,\, q)$, with a function $\varrho :]0, T[\times\Omega \to \mathbb{R}_+$ and a vector field $q :]0, T[\times\Omega \to \mathbb{R}^{N-1}$ such that

$$\begin{cases} \partial_t \mathscr{R}_i(\varrho,\, q) + \operatorname{div}(\mathscr{R}_i(\varrho,\, q)\, v + J^i) = r_i \\ J^i = e^i \cdot M(\mathscr{R}(\varrho,\, q))\, (\nabla(\frac{\mathcal{E}\, q}{\theta}) + \frac{z}{m}\, \frac{1}{\theta}\, \nabla\phi) \qquad \text{for } i = 1, \ldots, N\,. \\ r_i = \sum_{k=1}^s\, \gamma_i^k\, \bar{R}_k(\gamma^1 \cdot \mathcal{E}q, \ldots, \gamma^s \cdot \mathcal{E}q) \end{cases} \tag{63}$$

Here we abbreviated $\bar{R} = -\nabla\Psi$. For instance one chooses the system $\{\xi^1, \ldots,\, \xi^{N-1}\} := \{e^1, \ldots, e^{N-1}\}$. In this case we see that $\eta^k = e^k - e^N$ for $k = 1, \ldots, N-1$ and $\eta^N = e^N$. Thus, $\Pi\mu$ is the vector $(\mu_1 - \mu_N, \ldots, \mu_{N-1} - \mu_N)$. For this reason, we propose to call *relative chemical potentials* the components of the new variable q.

Obviously, this approach in order to be possible also requires the reformulation of the system of boundary conditions (50), (51), (52). We mention a second algebraic result, which allows to eliminate the variables $\mu_1^\Gamma, \ldots, \mu_{N^\Gamma}^\Gamma$ so that only the vector μ and the data are involved in the boundary conditions.

In order to state this result, we first need to reinterpret via trivial extension the matrices M^Γ and $M^{\Gamma,\text{ext}}$ as positive semi-definite elements of $\mathbb{R}^{N^\Gamma \times N^\Gamma}_{\text{sym}}$. We introduce a linear space $\mathcal{V} \subset \mathbb{R}^{N^\Gamma}$ via

$$\mathcal{V} := \operatorname{span}\{\gamma_\Gamma^k\}_{k=1,\ldots,s^\Gamma} \oplus \operatorname{Image} M^{\Gamma,\text{ext}}$$

Exploiting standard results of linear algebra, we then find a representation $M^\Gamma = M^{\Gamma,1} + M^{\Gamma,2}$, with $M^{\Gamma,i} \in \mathbb{R}^{N^\Gamma \times N^\Gamma}_{\text{sym}}$ positive semi-definite for $i = 1, 2$, such that

$$M^{\Gamma,i}(0^N \times \mathbb{R}^{N^{\text{ext}}}) = 0, \quad M^{\Gamma,i}\, 1^{N^\Gamma} = 0 \text{ for } i = 1, 2$$
$$M^{\Gamma,1}(\mathcal{V}) = 0, \quad \operatorname{Image} M^{\Gamma,2} \subseteq \mathcal{V}\,.$$

[1] We should in fact speak *stricto sensu* of a reduction operator.

We define

(1) $d^\Gamma := \dim \mathcal{V}$;

(2) $\hat{s}^\Gamma := \dim(\mathrm{Image}\, M^{\Gamma,2})$;

(3) The *reduced boundary reaction vectors* $\hat{\gamma}^1, \ldots, \hat{\gamma}^{\hat{s}^\Gamma} \in \mathbb{1}^\perp \times 0^{N^{\text{ext}}}$ are the eigenvectors of $M^{\Gamma,2}$;

The following result is proved in the section 5.

Proposition 3.2. *Let $b^1, \ldots, b^{d^\Gamma}$ be a basis of $\mathcal{V}$ such that $b^k = \hat{\gamma}^k$ for $k = 1, \ldots, \hat{s}^\Gamma$.*

Assume that the function Ψ^Γ occuring in (41) *belongs to $C^2(\mathbb{R}^{s^\Gamma}; \mathbb{R}^{s^\Gamma})$, is strictly convex, coercive and has a global minimum at zero*[2].

There is a function $\hat{\Psi}^\Gamma \in C^1(\mathbb{R}^{\hat{s}^\Gamma} \times \mathbb{R}^{d^\Gamma})$, $(Y,\, w) \mapsto \hat{\Psi}^\Gamma(Y,\, w)$ such that

- *For all $w \in \mathbb{R}^{d^\Gamma}$, the function $Y \mapsto \hat{\Psi}^\Gamma(Y;\, w)$ is of class $C^2(\mathbb{R}^{\hat{s}^\Gamma})$, nonnegative, strictly convex and coercive on $\mathbb{R}^{\hat{s}^\Gamma}$ and it satisfies $\hat{\Psi}^\Gamma(0,\, w) = 0$;*
- *Defining $-\hat{R}^\Gamma := \nabla_Y \hat{\Psi}^\Gamma$, the boundary conditons* (50), (51), (52) *are valid if and only if for $i = 1, \ldots, N$,*

$$J^i \cdot \nu + \sum_{k=1}^{\hat{s}^\Gamma} \hat{R}_k^\Gamma(\hat{\gamma}^1 \cdot \mu, \ldots,\, \hat{\gamma}^{\hat{s}^\Gamma} \cdot \mu,\, w^0)\, \hat{\gamma}_i^k = -J_i^0$$

$$J_i^0 := -\sum_{k=1}^{\hat{s}^\Gamma} \lambda_k(M^{\Gamma,2})\, \hat{R}_k^\Gamma(0,\, w^0)\, \hat{\gamma}^k \,.$$

Here $\lambda_1(M^{\Gamma,2}), \ldots, \lambda_{\hat{s}^\Gamma}(M^{\Gamma,2})$ are the nontrivial eigenvalues of $M^{\Gamma,2}$, and the coefficients w^0 are choosen such that $M^{\Gamma,\text{ext}}\mu^{\text{ext}} = \sum_{k=1}^{d^\Gamma} w_k^0\, b^k$.

Owing to the Propositions 3.1 and 3.2, we define a solution to (47), (50), (51), (52) as a pair composed of the scalar $\varrho :]0, T[\times\Omega \to \mathbb{R}_+$ (total mass density) and of the vector field $q :]0, T[\times\Omega \to \mathbb{R}^{N-1}$ (relative chemical potentials). For the other occurences in (48) and (49) of the original variables ρ, μ, we use the following equivalences relying on (62)

$$p = -h(\rho) + \sum_{i=1}^N \rho_i\, \mu_i = K\,(-F + \mathrm{id}\, F')\left(\tfrac{V}{m} \cdot \mathscr{R}(\varrho,\, q)\right) =: P(\varrho,\, q)$$
$$n^F = \tfrac{z}{m} \cdot \rho = \tfrac{z}{m} \cdot \mathscr{R}(\varrho,\, q)\,.$$

[2]Since there is a free constant in the choice of the reaction potential, it is always possible to choose it nonnegative

The problem of vacuum In the context of weak solutions to the Navier-Stokes equations, the occurence of a set of positive measure where the total mass density ϱ vanishes cannot be exculded. Such a set is called a vacuum. For a mixture, a vacuum is additionaly characterised by the fact that the variables ρ and q are 'decoupled': Here we mean that the mapping $q \mapsto \mathscr{R}(\varrho = 0,\, q)$ is trivial on the entire $\mathbb{R}^{N-1}$. For the analysis of the model, an additional concrete difficulty is raised concerning the compacntness, since estimates for the time-derivatives are available only for the $\rho-$variables. Thus, a sequence of mass densities $\rho^n = \mathscr{R}(\varrho_n,\, q^n)$ $(n \in \mathbb{N})$ such that $\varrho_n \rightarrow 0$ can converge strongly while the corresponding q^n are exhibiting *oscillatory behaviour*.

The diffusion fluxes of the constituents $J^1, \ldots, J^N$ are linear expressions of the gradient of q, and therefore the vacuum-oscillations do not affect the concept of the solution at this level. However the reaction densities are in general nonlinear expressions in $q_1, \ldots, q_{N-1}$. For the concept of the solution, this means that the validity of the representation $r = \sum_{k=1}^s \bar{R}_k(\gamma^k \cdot \mathcal{E}q)\, \gamma^k$ is restricted to the set where ϱ is strictly positive. In a vacuum set, the reaction term r is the limit of a possibly oscillating sequence and is related to the variable q only via a dissipation inequality. An analogous situation occurs at the boundary $]0,T[\times\Gamma$ whenever it is in contact with a vacuum.

In order to include the possibility of this situation, we relax the concept of a solution to (47), (50), (51), (52). It now contains four entries: the scalar $\varrho :]0,T[\times\Omega \rightarrow \mathbb{R}_+$ (total mass density) and of the vector field $q :]0,T[\times\Omega \rightarrow \mathbb{R}^{N-1}$ (relative chemical potentials) like in the natural definition, but also the production factors in the bulk $R :]0,T[\times\Omega \rightarrow \mathbb{R}^s$ and on the interface $R^\Gamma :]0,T[\times\Gamma \rightarrow \mathbb{R}^{\hat{s}^\Gamma}$. We define the vacuum–free set via

$$Q^+(\varrho) := \{(t,x) \in]0,\, T[\times\Omega \,:\, \varrho(t,x) > 0\}\,. \tag{64}$$

For the representation of the bulk reactions, we require the following weaker condition

$$r = \sum_{k=1}^s \gamma^k\, R_k \text{ with } R = \bar{R}(\gamma^1 \cdot \mathcal{E}q, \ldots, \gamma^s \cdot \mathcal{E}q) \text{ in } Q^+(\varrho)\,. \tag{65}$$

We introduce a set $S^+(\varrho) \subseteq]0,T[\times\Gamma$ as the subset of all $(t,x) \in]0,T[\times\Gamma$ such that there is an open neighbourhood $U_{t,x}$ with the property

$$\lambda_4\Big(U_{t,x} \cap \{(s,y) \in]0,T[\times\Omega \,:\, \varrho(s,\, y) = 0\}\Big) = 0\,. \tag{66}$$

For the concept of the solution, we ask that

$$\hat{r} = \sum_{k=1}^{\hat{s}^\Gamma} \hat{\gamma}^k\, R^\Gamma_k \text{ with } R^\Gamma = \hat{R}^\Gamma(\hat{\gamma}^1 \cdot \mathcal{E}q, \ldots, \hat{\gamma}^{\hat{s}^\Gamma} \cdot \mathcal{E}q,\, w^0) \text{ in } S^+(\varrho)\,. \tag{67}$$

The weakening (65), (67) of the concept of the solution requires an equivalent representation of the entropy productions due to reactions in the bulk and on the interface.

The dissipation (entropy production) associated with the bulk reactions is given by the expression $\xi_{\mathsf{R}} := -\sum_{k=1}^s \bar{R}_k(D^{\mathsf{R}})\, D^{\mathsf{R}}_k$. Recall that $D^{\mathsf{R}}_k := \gamma^k \cdot \mu$ for $k = 1, \ldots, s$. On the boundary $]0,T[\times\Gamma$, the situation is slightly more complicated. Since the entropy production $\xi^\Gamma_{\mathsf{R}} = -\sum_{k=1}^{s^\Gamma} \bar{R}^\Gamma_k(\gamma^k_\Gamma \cdot \mu^\Gamma)\, \gamma^k_\Gamma \cdot \mu^\Gamma$, requires the introduction of interface chemical potentials, we instead use the reduced driving forces $\hat{D}^{\Gamma,\mathsf{R}}_k := \hat{\gamma}_k \cdot \mu$ for $k = 1, \ldots, \hat{s}^\Gamma$ (Proposition 3.2). A reduced entropy production, that we will show to be nonnegative, is given by

$$\hat{\xi}^\Gamma_{\mathsf{R}} = -\sum_{k=1}^{\hat{s}^\Gamma} \hat{R}^\Gamma_k(\hat{D}^{\Gamma,\mathsf{R}},\, w^0)\, \hat{D}^{\Gamma,\mathsf{R}}_k = -\hat{R}^\Gamma(\hat{D}^{\Gamma,\mathsf{R}},\, w^0) \cdot \hat{D}^{\Gamma,\mathsf{R}} \,.$$

For the validity of the following statement, we use the Theorem 26.5 of [Roc70].

Proposition 3.3. *Define $\bar{R} := -\nabla\Psi$ with a (strictly) convex potential $\Psi \in C^2(\mathbb{R}^s)$ with minimum at zero. Then*

$$\Psi(D) + \Psi^*(-\bar{R}(D)) = -\bar{R}(D) \cdot D \text{ for all } D \in \mathbb{R}^s \,. \tag{68}$$

Here, the convex conjugate function Ψ^* is itself a convex element of $C^2(\mathbb{R}^s)$.

Let $\Psi^\Gamma \in C^2(\mathbb{R}^{s^\Gamma})$ be a (strictly) convex potential. Define a reduced potential $\hat{\Psi}^\Gamma$ and $-\hat{R}^\Gamma = \partial\hat{\Psi}^\Gamma$ as in Proposition 3.2. Then,

$$\hat{\Psi}^\Gamma(D,\, w) + (\hat{\Psi}^\Gamma)^*(-\hat{R}^\Gamma(D,\, w),\, w) = -\,\hat{R}^\Gamma(D,\, w) \cdot D \quad \text{for all } (D,\, w) \in \mathbb{R}^{\hat{s}^\Gamma} \times \mathbb{R}^{d^\Gamma} \,. \tag{69}$$

Here, the convex conjugate function $(\hat{\Psi}^\Gamma)^*$ is taken in the first variable, and is itself convex in the first variable.

4 The mathematical results

Mathematical results can be obtained under suitable restrictions to the data of the problem. We at first formulate these assumptions.

Assumptions on the free energy function

Our estimates on the (relative) chemical potentials moreover require the special form $h = h^{\mathsf{ref}} + h^{\mathsf{mech}} + h^{\mathsf{mix}}$, where the mixing entropy obeys the precise representaion (36). We allow for a certain generality only at the level of the function h^{mech} which we assume of the form

$$h^{\mathsf{mech}} = K\, F(n \cdot V) + C\, n \cdot V \quad \text{for } s > 0 \,.$$

Here K is a constant, and we assume that F belongs to $C^2(\mathbb{R}_+) \cap C(\mathbb{R}_{0,+})$ and is a convex function.

We assume that there are $\frac{3}{2} < \alpha < +\infty$ and constants $0 < c_0,\, c_1$ such that

$$F(s) \geq c_0\, s^\alpha - c_1 \quad \text{for all } s > 0 \,. \tag{70}$$

In the neighbourhood of zero, we assume that $F(s)$ behaves like $s\,\ln s$: There are constants positive constants $k_0 < k_1$ and $s_0 > 0$ such that

$$\frac{k_0}{s} \leq F''(s) \leq \frac{k_1}{s} \quad \text{for all } s \in]0, s_0] \,. \tag{71}$$

In fact, in order to obtain an unconstrained PDE system, we crucialy need that

$$F' : \mathbb{R}_+ \to \mathbb{R} \text{ is surjective} \tag{72}$$

which is not satisfied for instance by the pure polynomial Ansatz according to Tait (see Section 5 below), but always follows from (71).

Assumptions on the mobility matrix

We assume that M is symmetric and positive semidefinite. Throughout the paper, we assume that M is mass conservative, that is

$$M\mathbb{1} = 0 \,. \tag{73}$$

Moreover we assume that the entries of $M = M(\rho)$ are linear–growth, continuous functions of the vector ρ of the partial mass densities.

Except for these few points, the exact structure of the mobility matrix is a delicate topic (in particular there are connections to the Maxwell–Stefan theory, see [BD15]). In this paper we restrict ourselves to the assumption that M has rank $N-1$ independently on ρ. In other words, denote $0 = \lambda_1(M) \leq \lambda_2(M) \leq \ldots \leq \lambda_N(M)$ the eigenvalues of the matrix M. We assume that there are positive constants $0 < \underline{\lambda} \leq \overline{\lambda}$ such that

$$\underline{\lambda} \leq \lambda_i(M(\rho)) \leq \overline{\lambda}\,(1 + |\rho|) \quad \text{for all } i = 2, 3, \ldots, N, \ \rho \in \mathbb{R}^N_+ \,. \tag{74}$$

Let us remark that due to this assumption, only regularisations of the original Ansatz of the paper [DGM13] are included in the analysis: In the formula (24) we can for example apply a cutoff from below to the entries of the empirical matrix M_{emp}. Note that (74) makes sense if we assume that the model applies to a mixture of N constituents which is not allowed to degenerate.

We will treat the case of degenerate mobilities in a further publication.

Assumptions on the reaction densities

We assume that the reaction rates are derived from a strictly convex, nonnegative potential[3] $\Psi \in C^2(\mathbb{R}^s)$. Moreover, Ψ satisfies

$$\Psi(0) = 0, \qquad \frac{\Psi(D)}{|D|} \to +\infty \text{ for } |D| \to \infty \,. \tag{75}$$

These assumptions are compatible with the choices (29).

Similarily, we require that the boundary reaction rates are derived from a strictly convex, nonnegative potential $\Psi^\Gamma \in C^2(\mathbb{R}^{s^\Gamma})$ such that

$$\Psi^\Gamma(0) = 0, \qquad \frac{\Psi^\Gamma(D)}{|D|} \to +\infty \text{ for } |D| \to \infty \,. \tag{76}$$

[3]It is always possible to achieve the nonnegativity because the modelling only requires that Ψ has a global minimum at zero

For simplicity we explicitely require at least linear growth of the reaction terms, that is,

$$\inf_{X\in\mathbb{R}^{s^\Gamma}} \lambda_{\min}(D^2\Psi^\Gamma(X)) > 0\,. \tag{77}$$

As to the adsorption coefficients M^Γ and $M^{\Gamma,\mathrm{ext}}$ occurring in the boundary conditions (42), (43) they play in the analysis a role similar to the reactions. We assume them to be symmetric and positive semidefinite matrices satisfying $M^\Gamma\, 1^N = 0$ and $M^{\Gamma,\mathrm{ext}}\, 1^{N^{\mathrm{ext}}} = 0$.

The reaction vectors: critical manifold

Denote $W \subseteq \mathbb{1}^\perp \subset \mathbb{R}^N$ the linear subspace given by

$$W := \operatorname{span}\left\{\gamma^1, \ldots, \gamma^s,\, \hat{\gamma}^1, \ldots, \hat{\gamma}^{\hat{s}^\Gamma}\right\}. \tag{78}$$

Recall that the reduced reaction vectors $\hat{\gamma}^1, \ldots, \hat{\gamma}^{\hat{s}^\Gamma}$ are associated with the matrix M^Γ and can be identified with elements from $\mathbb{1}^\perp$ (see Proposition 3.2). Call *selection* S of cardinality $|S| \leq N$ a subset $\{i_1, \ldots, i_{|S|}\}$ of $\{1, \ldots, N\}$ such that $i_1 \leq \ldots \leq i_{|S|}$. For every selection, we can introduce the corresponding projector $P_S : \mathbb{R}^N \to \mathbb{R}^N$ via $P_S(\xi)_i = \xi_i$ for $i \in S$, and $P_S(\xi)_i = 0$ otherwise. We can define a linear subspace $W_S \subset \mathbb{R}^N$ via

$$W_S := \operatorname{span}\left\{P_S(\gamma^1), \ldots, P_S(\gamma^s),\, P_S(\hat{\gamma}^1), \ldots, P_S(\hat{\gamma}^{\hat{s}^\Gamma})\right\}. \tag{79}$$

The selection S will be called *linearly independent* if $\dim(W_S) = |S|$ and *linearly dependent* otherwise.

For every selection S, we denote $S^\perp$ the selection $\{1, \ldots, N\} \setminus S$. It can easily be shown that the manifold

$$\mathcal{M}_{\text{crit}} := \mathbb{R}^N_+ \cap \bigcup_{S\subset\{1,\ldots,N\},\ S \text{ linearly dependent}} W_S \times P_{S^\perp}(\mathbb{R}^N) \tag{80}$$

is the finite union of submanifolds of dimension at most $N-1$. We say that the *initial compatibility condition* is satisfied if the intial vector of the total masses $\bar{\rho}_0 := \int_\Omega \rho^0\, dx \in \mathbb{R}^N_+$ satisfies $\bar{\rho}_0 \notin \mathcal{M}_{\text{crit}}$.

Assumptions on the domain Ω and the boundary Γ

The domain $\Omega \subset \mathbb{R}^3$ possesses a boundary of class $\mathcal{C}^{0,1}$. In connection with the optimal regullarity of the solution to the Poisson equation with mixed-boundary conditions, we need to introduce a further exponent $r(\Omega, \Gamma)$ as the largest number in the range $[2, +\infty[$ such that

$$\begin{aligned} -\triangle u = f \text{ in } [W^{1,\beta'}_\Gamma(\Omega)]^* \text{ implies } u \in W^{1,\beta}_\Gamma(\Omega) \\ \text{for all } f \in [W^{1,\beta'}_\Gamma(\Omega)]^* \text{ and all } \beta \in]r', r[\,. \end{aligned} \tag{81}$$

It is well kwnon that $r(\Omega, \Gamma) > 2$ in general (see [Grö89] a. o.), but there are numerous situations where, depending on the boundary of the domain and the structure of the surface Γ, the optimal exponent satisfies $r(\Omega, \Gamma) > 3$ (see [DKR15] for results and discussions on this topic). We require that

$$\alpha' := \frac{\alpha}{\alpha - 1} < r\,, \tag{82}$$

whith α from (70). This of course might be a restriction only if $\alpha < 2$.

Assumptions on the remaining boundary data

We consider only nondegenerate initial and boundary data. This means that

$$\begin{aligned} \rho^0 &\in L^\infty(\Omega;\, (\mathbb{R}_+)^N) \\ v^0 &\in L^\infty(\Omega;\, \mathbb{R}^3) \\ \phi_0 &\in L^\infty(0,T;\, W^{1,r}(\Omega)) \cap L^\infty(]0,T[\times\Omega) \\ \partial_t\phi_0 &\in W^{1,0}_2(]0,T[\times\Omega) \cap L^{\alpha'}(]0,T[\times\Omega) \\ \mu^{\text{ext}} &\in L^\infty(]0,T[\times\Gamma;\, \mathbb{R}^{N^{\text{ext}}}) \end{aligned} \tag{83}$$

Moreover we assume the compatibility condition $-\epsilon_0\,(1+\chi)\,\triangle\phi_0(0) = \frac{z}{m}\cdot\rho^0$ weakly.

Results For $t > 0$, we denote $Q_t :=]0,t[\times\Omega$ the space–time cylinder, and if $T > 0$ is the final time of the process, we abbreviate $Q := Q_T$. We denote $S_t :=]0,T[\times\Gamma$ and $S = S_T$.

Exploiting the preliminary considerations of the Section 3, a solution vector to the entire system (47), (48), (49) with boundary conditions (50), (51), (52), (53), (54) and initial conditions ($=:$ Problem (P)) is composed of the scalars $\varrho:\ Q \to \mathbb{R}_+$ (total mass density) and $\phi:\ Q \to \mathbb{R}$ (electrical potential) and of the vector fields $q:\ Q \to \mathbb{R}^{N-1}$ (relative chemical potentials), and $v\ :\ Q \to \mathbb{R}^3$ (barycentric velocity field). If we want to account for the possibility of vacuum, the productions factors are not everywhere functions of these components only. Thus we also introduce $R:\ Q \to \mathbb{R}^s$, $R^\Gamma:\ S \to \mathbb{R}^{\hat{s}^\Gamma}$ as variables.

In order to define the concept of a *weak solution* we introduce what one could call a *natural class* $\mathcal{B}$ because this class narurally arises from the global energy and mass conservation identities associated with the model. The class $\mathcal{B}$ reflects the regularity of the solution and essentially depends on several parameters

- The final time $T > 0$, the domain Ω and the partition $\Gamma\cup\Sigma$ of its boundary (see condition (81));
- The choice of the free energy function h and in particular the growth exponent of (70);
- The mobility matrix M, in particular the nummber $\operatorname{rk} M$;
- The choice of the potentials Ψ and Ψ^Γ for the reaction densities.

For the variables ϱ, ϕ and v we introduce the conditions

$$\varrho \in L^{\infty,\alpha}(Q_T;\, \mathbb{R}_{0,+}) \tag{84}$$

$$v \in W^{1,0}_{2,S}(Q_T;\, \mathbb{R}^3) \tag{85}$$

$$\sqrt{\varrho}\, v \in L^{\infty,2}(Q_T;\, \mathbb{R}^3) \tag{86}$$

$$\phi \in L^\infty(Q_T), \quad \nabla\phi \in L^{\infty,\beta}(Q_T;\, \mathbb{R}^3)\,, \tag{87}$$

with the exponents $\alpha > 3/2$ and $r(\Omega,\Gamma) > 2$ of the conditions (70) and (81), and with

$$\beta := \min\left\{r(\Omega,\,\Gamma),\, \frac{3\alpha}{(3-\alpha)^+}\right\}\,. \tag{88}$$

For the variable q, a control is achieved on the spatial gradient thanks to (74), but we obtain a very low regularity in time. In order to state this regularity, we introduce the function, $(\circ_N \ln) := \underbrace{\ln \circ \ldots \circ \ln}_{\times N}$ which is defined on the intervall $(\circ_N e)(1),\, +\infty[$. For Bochner measurable functions $u:\, [0,T] \to L^1(\Omega)$ we define a number

$$[u]_{L^w_{(\circ_N \ln)} L^1(Q)} := \sup_{k > (\circ_N e)(1)} (\circ_N \ln)(k)\, \lambda_1\big(\{t \in]0,T[:\, \|u(t)\|_{L^1(\Omega)} > k\}\big)\,. \tag{89}$$

We say that u belongs to the class $L^w_{(\circ_N \ln)} L^1(Q)$ if $[u]_{L^w_{(\circ_N \ln)} L^1(Q)} < +\infty$. For the variable q we consider the conditions

$$q \in L^w_{(\circ_N \ln)} L^1(Q;\, \mathbb{R}^{N-1}) \tag{90}$$

$$\nabla q \in L^2(Q;\, \mathbb{R}^{(N-1)\times 3})\,. \tag{91}$$

We recall that the reaction factor R is derived from a nonnegative, convex and coercive potential Ψ. The vectorial Orlicz classes $L_\Psi(Q_T;\, \mathbb{R}^s)$ and $L_{\Psi^*}(Q_T;\, \mathbb{R}^s)$ are then well known. We use the notation

$$[D]_{L_\Psi(Q_T;\mathbb{R}^s)} := \int_{Q_T} \Psi(D(t,\, x))\, dx\, dt\,.$$

Due to the preliminary considerations of Section 3, we know that the reduced reaction factor $\hat{R}$ is derived from a potential

$$\hat{D}^{\Gamma,\mathsf{R}} \mapsto \hat{\Psi}^\Gamma(\hat{D}^{\Gamma,\mathsf{R}},\, w^0) \quad \text{for } \hat{D}^{\Gamma,\mathsf{R}} \in \mathbb{R}^{\hat{s}^\Gamma}\,.$$

Here $w^0 \in L^\infty(S_T;\, \mathbb{R}^{d^\Gamma})$ depends linearly on the vector μ^{ext} of external chemical potentials. We can reinterpret $\hat{\Psi}^\Gamma \in L^\infty(S;\, C^2(\mathbb{R}^{\hat{s}^\Gamma}))$ as the mapping

$$(t,\, x,\, \hat{D}^{\Gamma,\mathsf{R}}) \mapsto \hat{\Psi}^\Gamma(\hat{D}^{\Gamma,\mathsf{R}},\, w^0(t,x))\,. \tag{92}$$

Then, we can introduce a vectorial Orlicz class $L_{\hat{\Psi}^\Gamma}(S;\, \mathbb{R}^{\hat{s}^\Gamma})$ as the set of all measurable $\hat{D}:\, S \to \mathbb{R}^{\hat{s}^\Gamma}$ such that

$$[\hat{D}]_{L_{\hat{\Psi}^\Gamma}(S;\mathbb{R}^{\hat{s}^\Gamma})} := \int_S \hat{\Psi}^\Gamma(t,\, x,\, \hat{D}(t,\, x))\, dS(x)\, dt < +\infty\,.$$

For the variable q we thus have the additional conditions

$$\begin{aligned} &(\gamma^1 \cdot \mathcal{E}q, \ldots, \gamma^s \cdot \mathcal{E}q) \in L_\Psi(Q_T;\, \mathbb{R}^s)\,,\\ &(\hat{\gamma}^1 \cdot \mathcal{E}q, \ldots, \hat{\gamma}^{\hat{s}^\Gamma} \cdot \mathcal{E}q) \in L_{\hat{\Psi}^\Gamma}([0,T]\times\Gamma;\, \mathbb{R}^{\hat{s}^\Gamma})\,. \end{aligned} \tag{93}$$

For the variables R and R^Γ we consider the conditions

$$-R \in L_{\Psi^*}(Q;\, \mathbb{R}^s), \quad -R^\Gamma \in L_{(\hat{\Psi}^\Gamma)^*}(S;\, \mathbb{R}^{\hat{s}^\Gamma})\,. \tag{94}$$

For a given vector $(\varrho,\, q,\, v,\, \phi,\, R,\, R^\Gamma)$ we introduce on the base of the Propositions 3.1, 3.2 the auxiliary variables

$$\rho = \mathscr{R}(\varrho,\, q) \tag{95a}$$

$$J = -M(\rho)\, D, \qquad D := \frac{\nabla \mathcal{E}\, q}{\theta} + \frac{1}{\theta}\, \frac{z}{m}\, \nabla\phi \tag{95b}$$

$$r = \sum_{k=1}^{s} \gamma^k\, R_k, \qquad D^{\mathsf{R}}_k := \gamma^k \cdot \mathcal{E}q \text{ for } k = 1, \ldots, s \tag{95c}$$

$$\hat{r} = \sum_{k=1}^{\hat{s}^\Gamma} \hat{\gamma}^k\, R^\Gamma_k, \qquad \hat{D}^{\Gamma,\mathsf{R}}_k := \hat{\gamma}^k \cdot \mathcal{E}q \text{ for } k = 1, \ldots, \hat{s}^\Gamma \tag{95d}$$

$$p = P(\varrho,\, q) \tag{95e}$$

$$n^F = \rho \cdot \frac{z}{m}\,. \tag{95f}$$

The natural class $\mathcal{B}$ also encodes an information concerning the conservation of global mass (integration of (10) over Ω). We additionally introduce the auxilliary variable

$$\bar{\rho} := \int_\Omega \rho = \int_\Omega \mathscr{R}(\varrho,\, q)\,, \tag{96}$$

and a nonnegative function $\Phi^* \in C([0,T]^2)$, $\Phi^*(t,\, t) = 0$ constructed from the functions Ψ, Ψ^Γ (and thus from R and R^Γ) via

$$\begin{aligned}\Phi^*(t_1,\, t_2) := & \sup_{i=1,\ldots,N;\, [R]_{L_{\Psi^*}(Q)} \leq C_0} \left| \int_{t_1}^{t_2} \int_\Omega R \cdot \gamma_i \right| \\ & + \sup_{i=1,\ldots,N;\, [\hat{R}]_{L_{(\hat{\Psi}^\Gamma)^*}(S)} \leq C_0} \left| \int_{t_1}^{t_2} \int_\Gamma \hat{R} \cdot \hat{\gamma}_i \right| + (t_2 - t_1)\,,\end{aligned} \tag{97}$$

for all $0 \leq t_1 \leq t_2 \leq T$. Here C_0 is an appropriate constant that we will choose later.

For a function $u \in C^1([0,T])$, we define a weighted modulus of uniform continuity via

$$[u]_{C_{\Phi^*}([0,T])} := \sup_{t_1,\, t_2 \in [0,T]} \frac{|u(t_1) - u(t_2)|}{\Phi^*(t_1,\, t_2)}\,. \tag{98}$$

We are finally in the position to introduce the solution class.

Definition 4.1. *Let* $(\varrho,\, q,\, v,\, \phi,\, R,\, R^\Gamma)$ *such that* ϱ *satisfies* (84)*,* v *satisfies* (85)*,* ϕ *satisfies* (87)*, and* q *satisfies* (90)*,* (91) *and* (93) *and* $R,\, R^\Gamma$ *satisfy* (94)*. We define a number*

$$\begin{aligned}&[(\varrho,\, q,\, v,\, \phi,\, R,\, R^\Gamma)]_{\mathcal{B}(T,\,\Omega,\,\alpha,\,\mathrm{rk}\,M,\,\Psi,\,\Psi^\Gamma)} := \\ &\quad \|\varrho\|_{L^{\infty,\alpha}(Q)} + \|v\|_{W^{1,0}_2(Q)} + \|\sqrt{\varrho}\, v\|_{L^{\infty,2}(Q_T)} + \|\phi\|_{L^\infty(Q)} + \|\nabla\phi\|_{L^{\infty,\beta}(Q)} \\ &\quad + [q]_{L^w_{(\circ_N \ln)} L^1(Q)} + \|\nabla q\|_{L^2(Q)} + [D^{\mathsf{R}}]_{L_\Psi(Q)} + [\hat{D}^{\Gamma,\mathsf{R}}]_{L_{\hat{\Psi}^\Gamma}(S)} \\ &\quad + \|J\|_{L^{2,\frac{2\alpha}{1+\alpha}}(Q)} + [-R]_{L_{\Psi^*}(Q)} + [-R^\Gamma]_{L_{(\hat{\Psi}^\Gamma)^*}(S)} + \|p\|_{L^{\min\{1+\frac{1}{\alpha},\, \frac{5}{3}-\frac{1}{\alpha}\}}(Q)} \\ &\quad + [\bar{\rho}]_{C_{\Phi^*}([0,T])}\,.\end{aligned} \tag{99}$$

We say that $(\varrho,\, q,\, v,\, \phi,\, R,\, R^\Gamma)$ *belongs to the class* $\mathcal{B}(T,\, \Omega,\, \alpha, \mathrm{rk}\, M,\, \Psi,\, \Psi^\Gamma)$ *if and only if* $[(\varrho,\, q,\, v,\, \phi,\, R,\, R^\Gamma)]_{\mathcal{B}(T,\, \Omega,\, \alpha,\, \mathrm{rk}\, M,\, \Psi,\, \Psi^\Gamma)}$ *is finite.*

An essential property of solutions is the mass and energy conservation.

Definition 4.2. *We say that* $(\varrho,\, q,\, v,\, \phi,\, R,\, R^\Gamma)$ *satisfies the* (global) energy (in)equality *with free energy function* h *and mobility matrix* M *if and only if the associated fields and variables* (95) *satisfy for almost all* $t \in]0, T[$

$$\begin{aligned}
&\int_\Omega \left\{\frac{1}{2}\,\varrho\, v^2 + \frac{1}{2}\,\epsilon_0\,(1+\chi)\,|\nabla\phi|^2 + h(\rho)\right\}(t) \\
&\quad + \int_{Q_t} \left\{\mathbb{S}(\nabla v)\,:\,\nabla v + \theta\, M\, D\cdot D + (\Psi(D^R) + \Psi^*(-R))\right\} \\
&\quad + \int_{S_t} \{\hat{\Psi}^\Gamma(\hat{D}^{\Gamma,R},\, w^0) + (\hat{\Psi}^\Gamma)^*(-R^\Gamma,\, w^0)\} \\
&\overset{(\leq)}{=} \int_\Omega \left\{\frac{1}{2}\,\varrho_0\,|v^0|^2 + \frac{1}{2}\,\epsilon_0\,(1+\chi)\,|\nabla\phi_0(0)|^2 + h(\rho^0)\right\} \\
&\qquad - \int_\Omega \left\{n^F\,\phi_0 - \epsilon_0\,(1+\chi)\,\nabla\phi\cdot\nabla\phi_0\right\}\Big|_0^t \\
&\quad + \int_{Q_t} \left\{n^F\,\phi_{0,t} - \epsilon_0\,(1+\chi)\,\nabla\phi\cdot\nabla\phi_{0,t}\right\} + \int_{S_t} ((\hat{r} + J^0)\cdot \frac{z}{m}\,\phi_0 + J^0\cdot \mathcal{E} q)\,. \qquad (100)
\end{aligned}$$

We say that $(\varrho,\, q,\, v,\, \phi,\, R,\, R^\Gamma)$ *satisfies the* global mass balance *if the vector field* $\bar{\rho}$ *(cf.* (96)*) satisfies*

$$\bar{\rho}(t) = \bar{\rho}^0 + \int_0^t \left\{\int_\Omega r + \int_\Gamma (\hat{r} + J^0)\right\}(s)\, ds \quad \textit{for all } t \in [0, T]\,. \qquad (101)$$

We now give the definition of a weak solution.

Definition 4.3. *We call* weak solution *to the Problem* (P) *a vector* $(\varrho,\, q,\, v,\, \phi,\, R,\, R^\Gamma) \in \mathcal{B}(T,\, \Omega,\, \alpha,\, N-1,\, \Psi,\, \Psi^\Gamma)$ *such that the energy inequality and the global mass identity of Definition 4.2 are valid and such that the quantities* ρ, J, r *and* $\hat{r}$, p *and* n^F *obeying the definitions* (95) *satisfy the relations*

$$- \int_Q \rho\cdot\psi_t - \int_Q (\rho_i\, v + J^i)\cdot\nabla\psi_i = \int_\Omega \rho^0\cdot\psi(0) + \int_Q r\cdot\psi + \int_{S_T} (\hat{r} + J^0)\cdot\psi \qquad (102)$$

$$\begin{aligned}
&- \int_Q \varrho\, v\cdot\eta_t - \int_Q \varrho\, v\otimes v\,:\,\nabla\eta - \int_Q p\,\mathrm{div}\,\eta + \int_Q \mathbb{S}(\nabla v)\,:\,\nabla\eta \qquad (103)\\
&\quad = \int_\Omega \varrho_0\, v^0\cdot\eta(0) - \int_Q n^F\,\nabla\phi\cdot\eta
\end{aligned}$$

$$\epsilon_0\,(1+\chi)\int_Q \nabla\phi\cdot\nabla\zeta = \int_Q n^F\,\zeta\,, \qquad \phi = \phi_0 \textit{ as traces on }]0, T[\times\Gamma\,. \qquad (104)$$

for all $\psi \in C^1_c([0,T[;\, C^1(\overline{\Omega};\, \mathbb{R}^N))$, $\eta \in C^1_c([0,T[;\, C^1_c(\Omega;\, \mathbb{R}^3))$ *and* $\zeta \in L^1(0,T;\, W^{1,2}_{\Gamma}(\Omega))$, *and the identities*

$$\begin{aligned} R &= \bar{R}(D^R) && \text{in } Q^+(\varrho) \\ R^\Gamma &= \hat{R}^\Gamma(\hat{D}^{\Gamma,R},\, w^0) && \text{in } S^+(\varrho)\,. \end{aligned} \tag{105}$$

The sets $Q^+(\varrho)$ *and* $S^+(\varrho)$ *are defined in* (64) *and* (66)*.*

The concept of weak solution is well defined owing to standard estimates (see also below). We state our main theorems.

Theorem 4.4. *[Gobal-in-time existence] Let* $\Omega \in C^{0,1}$*. Assume that the free energy function* h^{mech} *satisfies* (70) *and* (71) *and that the mobility matrix* M *satisfies* (73) *and* (74)*. Let* $\Psi \in C^2(\mathbb{R}^s)$ *and* $\Psi^\Gamma \in C^2(\mathbb{R}^{s^\Gamma})$ *be striclty convex and satisfy* (75), (76)*. Assume that the initial data* ρ^0 *and* v^0*, and the boundary data* μ^{ext}*,* ϕ_0 *are nondegenerate in the sense of* (83)*. Assume that one of the following conditions is valid:*

(1) $\alpha \geq 2$*;*

(2) $\frac{9}{5} \leq \alpha < 2$ *and* $r(\Omega,\, \Gamma) > \alpha'$*;*

(3) $\frac{3}{2} < \alpha < \frac{9}{5}$*,* $r(\Omega,\, \Gamma) > \alpha'$ *and the vectors* $m \in \mathbb{R}^N_+$ *and* $V \in \mathbb{R}^N_+$ *are parallel.*

Assume moreover that

(1) Either $s + \hat{s}^\Gamma = 0$*, that is, there are no bulk reactions and the adsorption coefficients and interface reaction vectors satisfy* $\dim(\mathrm{Image}(M^\Gamma) \cap \mathrm{span}\{\gamma^k_\Gamma\}_{k=1,\ldots,s^\Gamma}) = 0$*;*

(2) Or $s + \hat{s}^\Gamma \geq 1$ *and the vector* $\bar{\rho}^0$ *of the total initial masses has positive distance to the manifold* $\mathcal{M}_{crit}$ *of* (80)*.*

Then, for $T > 0$ *arbitrary, the problem* (P) *possesses a weak solution in the sense of Definition 4.3 in the class* $\mathcal{B}(T,\, \Omega,\, \alpha,\, N-1,\, \Psi,\, \Psi^\Gamma)$*. Moreover the following information on the complete vanishing of species is available:*

$$\lambda_1\big(\{t \in [0,T] \,:\, \inf_{i=1,\ldots,N} \bar{\rho}_i(t) = 0\}\big) = 0\,.$$

In the case that $s+\hat{s}^\Gamma = 0$ *we even obtain the additional time regularity* $\|q\|_{L^1(Q;\,\mathbb{R}^{N-1})} < +\infty$*.*

If one starts with total initial masses on the critical manifold, then it is possible that certain species completely vanish after finite time, and the solution then exists only up to this time. Afterwards, it might be necessary to restart the system with a smaller number of species.

Theorem 4.5. *[Local-in-time existence] Same assumptions as in Theorem 4.4, with* $\bar{\rho}_0 \in \mathcal{M}_{crit}$ *and* $s + \hat{s}^\Gamma \geq 1$*.*

Then, there are a time $0 < T_0$ *depending only on the data and a time* $T_0 \leq T^* \leq +\infty$ *such that there is a weak solution* $(\varrho,\, q,\, v,\, \phi,\, R,\, R^\Gamma) \in \mathcal{B}(t,\, \Omega,\, \alpha,\, N-1,\, \Psi,\, \Psi^\Gamma)$ *in the sense of Definition 4.3 to* (P_t) *for all* $t < T^*$*. Moreover the following alternative concerning* T^* *is valid:*

(1) Either $T^* = +\infty$*;*

(2) Or $\inf_{i=1,\ldots,N} \bar{\rho}_i(t) > 0$ *for all* $t \in [0, T^*[$ *and* $\lim_{t\to T^*} \inf_{i=1,\ldots,N} \bar{\rho}_i(t) = 0$*. Moreover* $\|q\|_{L^1(Q_t;\mathbb{R}^{N-1})} \to +\infty$ *as* $t \to T^*$*.*

Our plan is as follows. According to the preliminary Section 3, the algebraic properties of the equation (61) determines the analysis of the model. Our next Section 5 is therefore devoted to the proof of the Propositions 3.1 and 3.2. After that, we shall turn our attention to the PDEs. In the Section 6 we introduce thermodynamically consistent regularisations of the problem (P) for which it is easier to prove the solvability. For this larger class of problems, we then derive the energy and global mass balance identities (Section 7) and the resulting *a priori* estimates (Section 8). The Section 8 deals in particular with *a priori* estimates for the variable q, one of the most demanding part of the analysis. In order to pass to the limit in approximate problems with the numerous nonlinearities of the system, it is necessary to obtain compactness statements: This is the second pillar of the analysis, that we establish in the Sections 9 and 10. With all these tools at hand, we are able to complete the proof ot the main theorems in the Section 11 devoted to existence.

5 Algebraic properties

This section is devoted to the rigourous derivation of the statements announced in the Section 3. We at first enlight the choice of the variables in the bulk, and then prove the reduction of the boundary system on Γ.

5.1 The choice of variables in the bulk. General free energy

The algebraic relation between partial mass densities ρ and chemical potentials μ is given by

$$\mu_i = \partial_i h(\rho_1, \ldots, \rho_N) \text{ for } i = 1, \ldots, N\,. \tag{106}$$

In the isothermal case we can forget about the temperature-dependence, that is, $h \rightsquigarrow h(\rho)$. Using tools of convex analysis, we immediately obtain that the relation (106) is invertible if h is convex and smooth. In the remainder of the paper we always denote $\mathbb{R}^N_+ = (\mathbb{R}_+)^N = \{X \in \mathbb{R}^N \,:\, X_i > 0 \text{ for } i = 1, \ldots, N\}$, and $\mathbb{R}^N_{0,+} = (\mathbb{R}_{0,+})^N = \{X \in \mathbb{R}^N \,:\, X_i \geq 0 \text{ for } i = 1, \ldots, N\}$.

Lemma 5.1. *Let* $h \in C^2(\mathbb{R}^N_+) \cap C(\mathbb{R}^N_{0,+})$ *be convex. Let* $D^*_h \subseteq \mathbb{R}^N$ *be the set* $\mathrm{Image}(\nabla h;\, \mathbb{R}^N_+)$*, that is* $D^*_h = \{\mu \in \mathbb{R}^N \,:\, \exists\, \rho \in \mathbb{R}^N_+,\, \mu = \nabla h(\rho)\}$*. Then, the Legendre transform of* h*, denoted* h^**, is a well-defined function on* D^*_h*, convex and it satisfies* $h^* \in C^2(D^*_h)$*. Moreover the relation* (106) *is valid for* $\mu \in D^*_h$ *and* $\rho \in \mathbb{R}^N_+$ *if and only if* $\rho = \nabla h^*(\mu)$*.*

Proof. Since $h \in C(\mathbb{R}^N_{0,+})$, it is a closed proper convex function in the sense of [Roc70]. The claim follows from the Theorem 26.5 of this book. □

Next we investigate the possibility to introduce 'mixed' coordinates to describe the set of solutions to (106). Let $\xi^1,\ldots,\xi^N \in \mathbb{R}^N$ be a basis of $\mathbb{R}^N$ such that $\xi^N := \mathbb{1}$. Choose $\eta^1,\,\ldots,\eta^N \in \mathbb{R}^N$ such that $\xi^i\cdot\eta^j = \delta_i^j$, $i,j=1,\ldots,N$. We define a 'projector' $\Pi:\,\mathbb{R}^N\to\mathbb{R}^{N-1}$ and an extension operator $\mathcal{E}:\,\mathbb{R}^{N-1}\to\mathbb{R}^N$ associated with the basis $\{\xi^i\}_{i=1\ldots,N}$ via

$$\Pi X := (X\cdot\eta^1,\ldots,X\cdot\eta^{N-1}) \text{ for } X\in\mathbb{R}^N, \quad \mathcal{E}q := \sum_{k=1}^{N-1} q_k\,\xi^k \text{ for } q\in\mathbb{R}^{N-1}\,. \tag{107}$$

Corollary 5.2. *Assumptions of Lemma 5.1. Let $\xi^1,\ldots,\xi^N\in\mathbb{R}^N$ be a basis of $\mathbb{R}^N$ such that $\xi^N := \mathbb{1}$. Define a set $\mathscr{D}\subseteq\mathbb{R}_+\times\mathbb{R}^{N-1}$ via*

$$\mathscr{D} := \left\{(s,\,q)\in\mathbb{R}_+\times\mathbb{R}^{N-1}\,:\,\exists t\in\mathbb{R}\begin{cases}\mathcal{E}q+t\,\mathbb{1}\in D_h^*\\ \mathbb{1}\cdot\nabla h^*(\mathcal{E}q+t\,\mathbb{1})=s\end{cases}\right\}.$$

Then, $\mathscr{D}$ is open and there is a function $\mathscr{M}\in C^1(\mathscr{D})$, $(s,\,q)\mapsto\mathscr{M}(s,\,q)$ such that (106) *is valid for $\mu\in D_h^*$ and $\rho\in\mathbb{R}^N_+$ if and only if*

$$\begin{aligned}\mu &= \sum_{i=1}^{N-1}(\Pi\mu)_i\,\xi^i + +\mathscr{M}(\rho\cdot\mathbb{1},\,\Pi\mu)\,\mathbb{1}\\ &= (\mathcal{E}\circ\Pi)\mu + \mathscr{M}(\rho\cdot\mathbb{1},\,\Pi\mu)\,\mathbb{1}\,.\end{aligned}\tag{108}$$

The derivatives of $\mathscr{M}$ satisfy the idendities

$$\partial_s\mathscr{M}(\rho\cdot\mathbb{1},\,q) = \frac{1}{D^2h^*(\mu)\mathbb{1}\cdot\mathbb{1}},\quad \partial_{q_j}\mathscr{M}(\rho\cdot\mathbb{1},\,q) = -\,\frac{D^2h^*(\mu)\mathbb{1}\cdot\xi^j}{D^2h^*(\mu)\mathbb{1}\cdot\mathbb{1}}$$
$$j=1,\ldots,N-1\,. \tag{109}$$

Proof. Define an open set $\mathcal{U}\subset\mathbb{R}^{N-1}\times\mathbb{R}$ via

$$\mathcal{U} := \{(q,\,t)\in\mathbb{R}^{N-1}\times\mathbb{R}\,:\,\mathcal{E}q+t\,\mathbb{1}\in D_h^*\}\,.$$

We define a function $G:\,\mathcal{U}\times\mathbb{R}_+\to\mathbb{R}$ via

$$G(q,\,t,\,s) := \mathbb{1}\cdot\nabla h^*(\mathcal{E}q+t\,\mathbb{1}) - s\,. \tag{110}$$

We compute the partial derivatives of G and we use the representation (119) to obtain that

$$\partial_t G(q,\,t,\,s) = D^2h^*(\mathcal{E}q+t\,\mathbb{1})\,\mathbb{1}\cdot\mathbb{1}>0,\;\; \partial_{q_j}G(q,\,t,\,s) = D^2h^*(\mathcal{E}q+t\,\mathbb{1})\,\xi^j\cdot\mathbb{1}\,.$$

Consider now the solution manifold for $G=0$ in $\mathcal{U}\times\mathbb{R}_+$. Since $G_t>0$, we obtain from the implicit function theorem that there is $\mathscr{M}\in C^1(\mathscr{D})$

$$G(q,\,t,\,s)=0 \text{ if and only if } t=\mathscr{M}(s,\,q)\,.$$

In particular, $\partial_s\mathscr{M} = G_t^{-1}(q,\,t,\,s)$ and $\partial_q\mathscr{M} = -G_q/G_t$.

Assume now that (106) is valid for $\mu\in D_h^*$ and $\rho\in\mathbb{R}^N_+$. We express $\mu=\sum_{i=1}^{N-1}(\mu\cdot\eta^i)\,\xi^i + (\mu\cdot\eta^N)\,\mathbb{1}$. Then $G(\Pi\mu,\,\mu\cdot\eta^N,\,\rho\cdot\mathbb{1})=0$ so that $\mu\cdot\eta^N=\mathscr{M}(\rho\cdot\mathbb{1},\,\Pi\mu)$. □

Corollary 5.3. *Assumptions as in Corollary 5.2. Then there is a bijection* $\mathscr{R} : C^1(\mathscr{D}; \mathbb{R}^N_+)$ *such that* (106) *is valid for* $\mu \in D^*_h$ *and* $\rho \in \mathbb{R}^N_+$ *if and only if* $\rho_i = \mathscr{R}_i(\rho \cdot \mathbb{1}, \Pi\mu)$ *for* $i = 1, \ldots, N$.

Proof. For $(s, q) \in \mathscr{D}$, we define $\mathscr{R}(s, q) := (\nabla h^*)(\mathcal{E}\, q + \mathscr{M}(s, q)\, \mathbb{1})$. We can compute that

$$\partial_{q_j}\mathscr{R}_i(s, q) = D^2h^*e^i \cdot \xi^j - \frac{D^2h^*e^i \cdot \mathbb{1}\, D^2h^*\xi^j \cdot \mathbb{1}}{D^2h^*\mathbb{1} \cdot \mathbb{1}} \tag{111}$$

$$\partial_s\mathscr{R}_i(s, q) = \frac{D^2h^*e^i \cdot \mathbb{1}}{D^2h^*\mathbb{1} \cdot \mathbb{1}}\,. \tag{112}$$

In these formula, D^2h^* is evaluated at $\mu = \mathcal{E}q + \mathscr{M}(s,q)\,\mathbb{1}$. In order to prove that $\mathscr{R}$ is a bijection, it is sufficient to show that $d\mathscr{R}$ is invertible. Let $X = (r, q) \in \mathbb{R} \times \mathbb{R}^{N-1}$ arbitrary. Then $d\mathscr{R}\, X = 0$ means that for $i = 1, \ldots, N$ one has

$$e^i \cdot D^2h^* \left(\mathcal{E}q - \mathbb{1}\, \left(\frac{r + D^2h^*\mathbb{1} \cdot \mathcal{E}q}{D^2h^*\mathbb{1} \cdot \mathbb{1}}\right)\right) = 0\,.$$

Using the uniform invertibility of D^2h^*, we obtain that $\mathcal{E}q - \mathbb{1}\, \left(\frac{r+D^2h^*\mathbb{1}\cdot\mathcal{E}q}{D^2h^*\mathbb{1}\cdot\mathbb{1}}\right) = 0$. We can multiply this identity with $\eta^1, \ldots, \eta^{N-1}$, and since $\eta^j \cdot \mathbb{1} = 0$ for $j = 1, \ldots, N-1$, we obtain that $q_1, \ldots, q_{N-1} = 0$. Therefore also $r = 0$, and the claim follows. □

The pressure function The pressure function is given by the formula (32). We immediately see under (106) that $p = -h(\rho) + \rho \cdot \mu = h^*(\mu)$ where h^* is the convex conjugate of h. We define a function $P : \mathscr{D} \to \mathbb{R}$ via

$$P(s, q) := h^*(\mathcal{E}\, q + \mathscr{M}(s, q)\, \mathbb{1})\,. \tag{113}$$

Lemma 5.4. *Let* $(s, q) \in \mathscr{D}$. *Then* $P \in C^1(\mathscr{D})$ *satisfies*

$$P_s(s, q) = \frac{s}{D^2h^*\mathbb{1} \cdot \mathbb{1}}, \quad P_{q_j}(s, q) = \xi^j \cdot \nabla h^*(\mu) - s\, \frac{D^2h^*\mathbb{1} \cdot \xi^j}{D^2h^*\mathbb{1} \cdot \mathbb{1}}\,.$$

In these formula, D^2h^* *is evaluated at* $\mu = \mathcal{E}q + \mathscr{M}(s,q)\,\mathbb{1}$.

Proof. Define $\mu := \mathcal{E}\, q + \mathscr{M}(s, q)\, \mathbb{1}$ and $\rho = \nabla h^*(\mu)$. Then

$$P_s(s, q) = \mathbb{1} \cdot \nabla h^*(\mu)\, \mathscr{M}_s(s, q) = \rho \cdot \mathbb{1}\, \mathscr{M}_s(s, q)$$
$$P_{q_j}(s, q) = \xi^j \cdot \nabla h^*(\mu) + \mathbb{1} \cdot \nabla h^*(\mu)\mathscr{M}_{q_j}(s, q) = \rho \cdot \xi^j + \rho \cdot \mathbb{1}\, \mathscr{M}_{q_j}(s, q)$$

and the claim follows from the Lemma 5.2. □

5.2 The variables in the bulk. Special constitutive choice of the free energy

For special choices of the free energy, we can find more explicit formula than Lemma 5.1. Under the conditions (35) and (36), the relation (106) reads

$$\mu_i = c_i + K\,\frac{V_i}{m_i}\,F'(V\cdot n) + \frac{k_B\,\theta}{m_i}\,\ln y_i \quad i = 1,\ldots,N\,, \tag{114}$$

where $c_1,\ldots,c_N \in \mathbb{R}$ are certain constants depending on the reference states, $\theta > 0$ is the absolute temperature assumed constant and k_B is the Boltzmann constant.

Note that the free energy $h = h^{\text{ref}} + h^{\text{mech}} + h^{\text{mix}}$ satisfies the assumptions of Lemma 5.1 if we assume that the function $F \in C^2(\mathbb{R}_+)\cap C(\mathbb{R}_{0,+})$ is convex. At first we want to characterise the set D_h^* and we need a preliminary Lemma.

Lemma 5.5. *There is a function $f \in C^1(\mathbb{R}^N)$ such that if the identity (114) is valid for $\mu \in \mathbb{R}^N$ and $n \in \mathbb{R}^N_+$ then $F'(V\cdot n) = f(\mu)$. Moreover, the function f satisfies the following inequalities*

$$\tfrac{\underline{m}}{K\,\overline{V}}\,(\sup_i \mu_i - \sup_i c_i) \leq f(\mu) \leq \tfrac{\overline{m}}{K\,\underline{V}}\,(\sup_i \mu_i - \inf_i c_i) + \tfrac{k_B\,\theta}{K\,\underline{V}}\,\ln N \tag{115}$$

and $|\nabla f| \leq \overline{m}/(\underline{V}\,K)$. For a vector $V \in \mathbb{R}^N_+$ we here abbreviate $\underline{V} := \inf_{i=1,\ldots,N} V_i$ and $\overline{V} := \sup_{i=1,\ldots,N} V$.

Proof. Define a function $G:\,\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$, $(\mu,\,t)\mapsto G(\mu,\,t)$ via

$$G(\mu,\,t) := \sum_{i=1}^N \exp\left(\frac{m_i\,(\mu_i - c_i) - K\,V_i\,t}{k_B\,\theta}\right) - 1\,.$$

It is readily verified for all $\mu\in\mathbb{R}^N$ that $\lim_{t\to-\infty} G(\mu,\,t) = +\infty$, that $\lim_{t\to+\infty} G(\mu,\,t) = -1$, and that $G_t(\mu,\,t) < 0$. Thus, the solution manifold to $G(\mu,\,t) = 0$ is a curve $\{(\mu,\,f(\mu))\,:\,\mu\in\mathbb{R}^N\}$ where $\partial_i f(\mu) = -G_t^{-1}(\mu,\,f(\mu))\,G_{\mu_i}(\mu,\,f(\mu)$. Easy computations show that

$$\partial_i f(\mu) = \frac{m_i}{K}\,\frac{\exp\left(\frac{m_i\,(\mu_i-c_i)-K\,V_i\,f(\mu)}{k_B\,\theta}\right)}{\sum_{j=1}^N V_j\,\exp\left(\frac{m_j\,(\mu_j-c_j)-K\,V_j\,f(\mu)}{k_B\,\theta}\right)}\,. \tag{116}$$

In particular $|\nabla f| \leq \overline{m}\,\underline{V}\,K^{-1}$. Moreover, if $G(\mu,\,t) = 0$, then setting

$$y_i = \exp\left(\frac{m_i\,(\mu_i - c_i) - K\,V_i\,F'(t)}{k_B\,\theta}\right)\,, \tag{117}$$

we see that $\mu_i = c_i + K\,\frac{V_i}{m_i}\,t + \frac{k_B\,\theta}{m_i}\,\ln y_i$ for $i = 1,\ldots,N$. Since $y \in]0,1[^N$ and $y\cdot\mathbb{1} = 1$, it easily follows that

$$\tfrac{\underline{m}}{K\,\overline{V}}\,(\sup_i \mu_i - \sup_i c_i) \leq t \leq \tfrac{\overline{m}}{K\,\underline{V}}\,(\sup_i \mu_i - \inf_i c_i) + \tfrac{k_B\,\theta}{K\,\underline{V}}\,\ln N$$

proving (115). □

We are now ready to prove an inversion formula for the relation (114).

Corollary 5.6. *Assume that the function $F \in C^2(\mathbb{R}_+) \cap C(\mathbb{R}_{0,+})$ is convex. Define $D_h^* := \mathrm{Image}(\nabla h;\, \mathbb{R}^N_+)$. Then $D_h^* = \{\mu \in \mathbb{R}^N \,:\, f(\mu) \in \mathrm{Image}(F',\, \mathbb{R}_+)\}$. If $\mu \in D_h^*$, then*

$$\begin{aligned}\partial_i h^*(\mu) &= m_i\, ([F']^{-1} \circ f)(\mu)\, \frac{\exp\left(\frac{m_i\,(\mu_i - c_i) - K\, V_i\, f(\mu)}{k_B\,\theta}\right)}{\sum_{j=1}^N V_j\, \exp\left(\frac{m_j\,(\mu_j - c_j) - K\, V_j\, f(\mu)}{k_B\,\theta}\right)} \\ &= \partial_i (F^* \circ f)(\mu)\,.\end{aligned} \tag{118}$$

with $F^* =$ Legendre transform of F.

Proof. If $\mu \in D_h^*$, then there is $\rho \in \mathbb{R}^N_+$ such that $\mu = \nabla h(\rho)$. Thus, (114) is valid, and Lemma 5.5 shows that $F'(\frac{V}{m} \cdot \rho) = f(\mu)$. Thus, $f(\mu) \in \mathrm{Image}(F',\, \mathbb{R}_+)$ and this yields

$$D_h^* \subseteq \{\mu \in \mathbb{R}^N \,:\, f(\mu) \in \mathrm{Image}(F',\, \mathbb{R}_+)\}\,.$$

In order to prove the reverse inclusion, consider $\mu \in \mathbb{R}^N$ such that $f(\mu) \in \mathrm{Image}(F',\, \mathbb{R}_+)$. Denote

$$g(\mu) := [F']^{-1} \circ f(\mu)\,, \;\; \rho_i := m_i\, g(\mu)\, \frac{\exp\left(\frac{m_i\,(\mu_i - c_i) - K\, V_i\, f(\mu)}{k_B\,\theta}\right)}{\sum_{j=1}^N V_j\, \exp\left(\frac{m_j\,(\mu_j - c_j) - K\, V_j\, f(\mu)}{k_B\,\theta}\right)}$$

We easily show that $\nabla h(\rho) = \mu$. Using (116) we see that

$$\partial_i h^*(\mu) = K\, g(\mu)\, \partial_i f(\mu) = \partial_i (F^* \circ f)(\mu)\,.$$

□

Lemma 5.7. *Assumptions of Corollary 5.6. Then $\nabla h^* \in C^1(D_h^*)$. The representation*

$$\begin{aligned}&D^2 h^*_{i,j}(\nabla h(\rho)) = \\ &\frac{m_i\, \rho_j\, \delta_i^j}{k_B\, \theta} + \frac{\rho_i\, \rho_j}{n \cdot V}\, \left(\frac{1}{K\, n \cdot V\, F''(n \cdot V)} + \frac{V^2 \cdot n}{k_B\, \theta\, n \cdot V} - \frac{V_i + V_j}{k_B\, \theta}\right)\end{aligned} \tag{119}$$

is valid with $V^2 \cdot n := \sum_{i=1}^N V_i^2\, n_i$. There holds

$$|D^2 h^*(\nabla h(\rho))| \leq C_1\, \rho \cdot \mathbb{1} \tag{120}$$

$$D^2 h^*(\nabla h(\rho))\, \mathbb{1} \cdot \mathbb{1} \geq C_0\, \frac{1}{K\, F''(\rho \cdot \mathbb{1})}\,. \tag{121}$$

Proof. By direct computation starting from (118) we obtain (119). This entails

$$\begin{aligned}|D^2 h^*_{i,j}(\nabla h(\rho))| &\leq \rho_i\, \left(\frac{m_i}{k_B\, \theta} + \frac{m_j}{\underline{V}}\, \left(\frac{1}{K\, n \cdot V\, F''(n \cdot V)} + \frac{\overline{V}^2}{k_B\, \theta\, \underline{V}} + 2\, \frac{\overline{V}}{k_B\, \theta}\right)\right) \\ &\leq C\, \rho_i\, \left(1 + \frac{1}{K\, n \cdot V\, F''(n \cdot V)}\right)\,.\end{aligned} \tag{122}$$

The function $s\,F''(s)$ is asymptotically equivalent to $s\,s^{-1} = const$ near zero (cf. (71)) and to $s\,s^{\alpha-2} = s^{\alpha-1}$ for s large. Thus, there is a constant $c_0 > 0$ such that $\inf_{s\in\mathbb{R}_+} s\,F''(s) \geq c_0$, and (120) follows. Further

$$\begin{aligned} D^2 h^* \mathbb{1} \cdot e^i = & \frac{\rho_i\,\rho\cdot\mathbb{1}}{K\,F''(n\cdot V)\,(n\cdot V)^2} \\ & + \frac{\rho_i}{k_B\,\theta}\left(m_i + \frac{\rho\cdot\mathbb{1}\,V^2\cdot n}{(n\cdot V)^2} - \frac{V_i\,\rho\cdot\mathbb{1}}{n\cdot V} - \frac{\rho\cdot V}{n\cdot V}\right)\,. \end{aligned} \tag{123}$$

Thus

$$\begin{aligned} & \sum_{i,j=1}^N D^2 h^*_{j,i} \\ & = \frac{(\rho\cdot\mathbb{1})^2}{K\,(V\cdot n)^2\,F''(V\cdot n)} + \frac{1}{k_B\,\theta}\left(m\cdot\rho + \frac{(\rho\cdot\mathbb{1})^2\,V^2\cdot n}{(V\cdot n)^2} - 2\,\frac{\rho\cdot V\,\rho\cdot\mathbb{1}}{n\cdot V}\right) \\ & = \frac{(\rho\cdot\mathbb{1})^2}{K\,(V\cdot n)^2\,F''(V\cdot n)} + \frac{1}{k_B\,\theta}\left(\sqrt{m\cdot\rho} - \frac{\rho\cdot\mathbb{1}\,\sqrt{V^2\cdot n}}{V\cdot n}\right)^2 \\ & \quad + \frac{2}{k_B\,\theta}\,\frac{\rho\cdot\mathbb{1}}{n\cdot V}\,(\sqrt{m\cdot\rho}\,\sqrt{V^2\cdot n} - V\cdot\rho)\,. \end{aligned} \tag{124}$$

The estimate (121) is a straightforward consequence of (124) and of the Cauchy-Schwarz inequality, since we can express $\sum_{i=1}^N V_i\,\rho_i = \sum_{i=1}^N (V_i\,\sqrt{n_i})\,(m_i\,\sqrt{n_i})$. We further use in (121) that $F''(n\cdot V) \geq F''(c\,\varrho) \geq \tilde{c}\,F''(\varrho)$ (cf. (71)). □

As corollaries of Lemma 5.7, note that the functions $\mathscr{M} \in C^1(\mathscr{D})$ of Lemma 5.2 and $P \in C^1(\mathscr{D})$ satisfy for all $(s,\,q) \in \mathscr{D}$ the following inequalities (cp. (109), Lemma 5.4):

$$\begin{aligned} \frac{1}{C_1\,s} \leq \partial_s\mathscr{M}(s,\,q) & \leq \frac{K\,F''(s)}{C_0} \\ |\partial_q\mathscr{M}(s,\,q)| & \leq \frac{C_1}{C_0}\,K\,s\,F''(s) \end{aligned} \tag{125}$$

$$\begin{aligned} \frac{1}{C_1} \leq P_s(q,\,s) & \leq \frac{K\,s\,F''(s)}{C_0} \\ |P_{q_j}(s,\,q)| & \leq C\,s\,(1 + K\,s\,F''(s))\,. \end{aligned} \tag{126}$$

Remark 5.8. *For the applicability of our approximation methods we are restricted to the case that $D^*_h = \mathbb{R}^N$. In view of the Corollary 5.6 this is basically the case if F' is surjective. In this case, $\mathscr{D} = \mathbb{R}_+ \times \mathbb{R}^{N-1}$ and there is no state-constraint on μ.*

Remark 5.9. *In the case that the polynomial growth of the function F is less than $9/5$, we rely in the analysis of the PDE system on second derivatives and on the* convexity *of the function $s \mapsto P(s,\,q)$ at fixed q. We are able to establish this property only in the very special case that P is a function of the total mass density. We note the following trivial observation. Define P as in the Lemma 5.4. Assume that the vector $V \in \mathbb{R}^N_+$ and $m \in \mathbb{R}^N_+$ are parallel. Then P depends only on the first variable.*

5.3 The boundary reduction

The second step is to show that the boundary conditions (50), (51), (52) can be equivalently expressed by means of only a $(N-1)-$dimensional reduction of the vector μ from the bulk. The idea is to solve the algebraic equations

$$r^{\Gamma} - (M^{\Gamma} + M^{\Gamma,\text{ext}})\,\mu^{\Gamma} = -M^{\Gamma}\,\mu - M^{\Gamma,\text{ext}}\mu^{\text{ext}}\,, \tag{127}$$

which result from (50), (51). We show that these equation allow to eliminate the occurrences of the surface potentials μ^{Γ}.

Note that (127) makes sense if we reinterpret via trivial extension the matrices M^{Γ} and $M^{\Gamma,\text{ext}}$ as positive semidefinite elements of $\mathbb{R}^{N^{\Gamma}\times N^{\Gamma}}_{\text{sym}}$. The vectors μ and μ^{ext} are trivially extended as well according to the scheme $\mu \rightsquigarrow (\mu,\,0) \in \mathbb{R}^N \times 0^{N^{\text{ext}}}$ and $\mu^{\text{ext}} \rightsquigarrow (0,\,\mu^{\text{ext}}) \in 0^N \times \mathbb{R}^{N^{\text{ext}}}$. For the sake of simplicity we do not introduce explicitely these operators by means of additional symbols.

For the solution to (127), we define a linear subspace of $\mathbb{R}^{N^{\Gamma}}$ via

$$\mathcal{V} := \operatorname{span}\{\gamma^1_{\Gamma}, \ldots, \gamma^{s^{\Gamma}}_{\Gamma}\} \oplus \operatorname{Image} M^{\Gamma,\text{ext}}\,. \tag{128}$$

Now we can orthogonaly decompose $\operatorname{Image} M^{\Gamma} = (\operatorname{Image} M^{\Gamma} \cap \mathcal{V}) \oplus \mathcal{V}_1$, where $\mathcal{V}_1$ is the orthogonal complement of $\mathcal{V}$ in $\operatorname{Image} M^{\Gamma}$. There is an associated decompositon $M^{\Gamma} = M^{\Gamma,1} + M^{\Gamma,2}$ with positive semidefinite $M^{\Gamma,i} \in \mathbb{R}^{N^{\Gamma}\times N^{\Gamma}}_{\text{sym}}$ for $i = 1, 2$ satisfying

$$M^{\Gamma,i}((\mathbb{1}^{\perp} \times \{0\}) \oplus (\{0\} \times \mathbb{R}^{N^{\text{ext}}})) = 0 \text{ for } i = 1, 2, \quad \operatorname{Image} M^{\Gamma,1} = \mathcal{V}_1\,,$$

and $\operatorname{Image} M^{\Gamma,2} \subseteq \mathcal{V}$. Then, it is obvious that (127) is equivalent to

$$r^{\Gamma} - (M^{\Gamma,2} + M^{\Gamma,\text{ext}})\,\mu^{\Gamma} = -M^{\Gamma,\text{ext}}\,\mu^{\text{ext}} - M^{\Gamma,2}\,\mu \tag{129}$$

$$P_{\mathcal{V}_1}(\mu^{\Gamma} - \mu) = 0\,. \tag{130}$$

We now focus on the conditions (129). In order to solve these equations, we introduce

- The numbers $d^{\Gamma} = \dim \mathcal{V}$ and $\hat{s}^{\Gamma} := \dim(\operatorname{Image} M^{\Gamma,2}) \le d^{\Gamma}$;
- The eigenvalules $\lambda_1, \ldots, \lambda_{\hat{s}^{\Gamma}}$ and the orthonormal eigenvectors $b^1, \ldots, b^{\hat{s}^{\Gamma}}$ of $M^{\Gamma,2}$;
- We choose further vectors $b_{\hat{s}^{\Gamma}+1}, \ldots, b_{d^{\Gamma}} \in \mathbb{R}^{N^{\Gamma}}$ such that $\{b_1, \ldots, b_{d^{\Gamma}}\}$ is a basis of $\mathcal{V}$.

We introduce the abbreviation $d^{\text{ext}} = \operatorname{rk} M^{\text{ext}}$. Since the matrix $M^{\Gamma,\text{ext}}$ occurring in (129) is symmetric and positive semidefinite, there are orthonormal vectors $e^1, \ldots, e^{d^{\text{ext}}} \in \mathbb{R}^{N^{\Gamma}}$ such that

$$M^{\Gamma,\text{ext}}_{i,j} = \sum_{k=1}^{d^{\text{ext}}} \lambda^{\text{ext}}_k\, e^k_i\, e^k_j\,, \quad i, j = 1, \ldots, N^{\Gamma} \tag{131}$$

where $\lambda_1^{\text{ext}}, \ldots, \lambda_{d^{\text{ext}}}^{\text{ext}}$ are the nonzero eigenvalues of $M^{\Gamma,\text{ext}}$. Recalling now that $\{b^1, \ldots, b^{d^\Gamma}\}$ is a basis of $\mathcal{V}$, there are coefficients $\{A_{j,\ell}\}_{j=1\ldots,s^\Gamma,\ \ell=1,\ldots,d^\Gamma}$ and $\{\tilde{A}_{j,\ell}\}_{j=1\ldots,d^{\text{ext}},\ \ell=1,\ldots,d^\Gamma}$ such that

$$\gamma_\Gamma^j = \sum_{\ell=1}^{d^\Gamma} A_{j,\ell}\, b^\ell, \quad e^j = \sum_{\ell=1}^{d^{\text{ext}}} \tilde{A}_{j,\ell}\, b^\ell\,. \tag{132}$$

Employing these notations and properties

$$\begin{aligned} r^\Gamma - (M^{\Gamma,2} + M^{\Gamma,\text{ext}})\,\mu^\Gamma = &\sum_{k=1}^{d^\Gamma} b^k \left(\sum_{j=1}^{s^\Gamma} A_{j,k}\, R^{\Gamma,j} - \sum_{j=1}^{d^{\text{ext}}} \tilde{A}_{j,k}\, \lambda_j^{\text{ext}}\, e^j \cdot \mu^\Gamma\right) \\ &- \sum_{k=1}^{\hat{s}^\Gamma} b^k\, \lambda_k\, b^k \cdot \mu^\Gamma\,. \end{aligned}$$

Moreover there is a representation

$$-M^{\Gamma,\text{ext}}\,\mu^{\text{ext}} = \sum_{k=1}^{d^\Gamma} \left(\sum_{j=1}^{d^{\text{ext}}} \tilde{A}_{j,k}\, \lambda_j^{\text{ext}}\, e^j \cdot \mu^{\text{ext}}\right) b^k =: \sum_{k=1}^{d^\Gamma} w_k\, b^k\,. \tag{133}$$

Due to the two latter relations, (129) is equivalent to

$$\begin{cases} \sum_{j=1}^{s^\Gamma} A_{j,k}\, R^{\Gamma,j} - \sum_{j=1}^{d^{\text{ext}}} \tilde{A}_{j,k}\, \lambda_j^{\text{ext}}\, e^j \cdot \mu^\Gamma - \lambda_k\, b^k \cdot \mu^\Gamma = w_k - \lambda_k\, b^k \cdot \mu \\ \quad \text{for } k = 1, \ldots, \hat{s}^\Gamma \\ \sum_{j=1}^{s^\Gamma} A_{j,k}\, R^{\Gamma,j} - \sum_{j=1}^{d^{\text{ext}}} \tilde{A}_{j,k}\, \lambda_j^{\text{ext}}\, e^j \cdot \mu^\Gamma = w_k \\ \quad \text{for } k = \hat{s}^\Gamma + 1, \ldots, d^\Gamma \end{cases}\,. \tag{134}$$

Choose Ψ^Γ from (41). We introduce auxiliary potentials $\tilde{\Psi}^1,\ \tilde{\Psi}^2 \in C^2(\mathbb{R}^{d^\Gamma})$ via

$$\tilde{\Psi}^1(X) := \Psi^\Gamma(A\,X) + \frac{1}{2}\sum_{k=1}^{d^{\text{ext}}} \lambda_k^{\text{ext}} (\tilde{A}_k \cdot X)^2 \tag{135}$$

$$\tilde{\Psi}^2(X) := \tilde{\Psi}^1(X) + \frac{1}{2}\sum_{i=1}^{\hat{s}^\Gamma} \lambda_i\, X_i^2\,. \tag{136}$$

At $X = (b^1 \cdot \mu^\Gamma, \ldots, b^{d^\Gamma} \cdot \mu^\Gamma) \in \mathbb{R}^{d^\Gamma}$ and $Y := (b^1 \cdot \mu, \ldots, b^{\hat{s}^\Gamma} \cdot \mu) \in \mathbb{R}^{\hat{s}^\Gamma}$ the identities (130) are valid if and only if

$$\begin{cases} -\nabla_{X_k} \tilde{\Psi}^1(X) - \lambda_k\, X_k = w_k - \lambda_k\, Y_k & \text{for } k = 1, \ldots, \hat{s}^\Gamma \\ -\nabla_{X_k} \tilde{\Psi}^1(X) = w_k & \text{for } k = \hat{s}^\Gamma + 1, \ldots, d^\Gamma \end{cases} \tag{137}$$

or using the second potential also

$$-\nabla_X \tilde{\Psi}^2(X) = w - \overline{\mathcal{D}}\,Y\,. \tag{138}$$

Here $\overline{D} \in \mathbb{R}^{d^\Gamma \times \hat{s}^\Gamma}$ is the block-structured matrix

$$\overline{\mathcal{D}} = \begin{pmatrix} \mathcal{D} \\ 0 \end{pmatrix}, \quad \mathcal{D} = \mathrm{diag}(\lambda_1, \ldots, \lambda_{\hat{s}^\Gamma}) \in \mathbb{R}^{\hat{s}^\Gamma \times \hat{s}^\Gamma} . \tag{139}$$

The following auxiliary statement is then obvious.

Lemma 5.10. *The solution to the equation* (134) *at the point* $X = (b^1 \cdot \mu^\Gamma, \ldots, b^{d^\Gamma} \cdot \mu^\Gamma) \in \mathbb{R}^{d^\Gamma}$ *and* $Y := (b^1 \cdot \mu, \ldots, b^{\hat{s}^\Gamma} \cdot \mu) \in \mathbb{R}^{\hat{s}^\Gamma}$ *is given by*

$$X = \nabla(\tilde{\Psi}^2)^*(\overline{\mathcal{D}}\, Y - w) .$$

Here $(\tilde{\Psi}^2)^* \in C^2(\mathbb{R}^{d^\Gamma})$ *is the convex conjugate to* $\tilde{\Psi}^2$ *and* $\overline{\mathcal{D}} \in \mathbb{R}^{d^\Gamma \times \hat{s}^\Gamma}$ *is defined in* (139).

The Lemma 5.10 yields a representation of the vector $(b^1 \cdot \mu^\Gamma, \ldots, b^{d^\Gamma} \cdot \mu^\Gamma)$ as a function of $(b^1 \cdot \mu, \ldots, b^{\hat{s}^\Gamma} \cdot \mu)$. Recall also (130) to see that $\mu^\Gamma - \mu = 0$ on $\mathcal{V}_1$. Thus, the flux J_ν in (50) given by the expression $J_\nu = M^\Gamma\,(\mu - \mu^\Gamma)$ possesses at $Y = (b^1 \cdot \mu, \ldots, b^{\hat{s}^\Gamma} \cdot \mu) \in \mathbb{R}^{\hat{s}^\Gamma}$ the equivalent representation

$$\begin{aligned} J_\nu &= M^\Gamma\,(\mu - \mu^\Gamma) = M^{\Gamma,2}\,(\mu - \mu^\Gamma) \\ &= \sum_{i=1}^{\hat{s}^\Gamma} \lambda_i\,(b^i \cdot \mu - b^i \cdot \mu^\Gamma)\, b_i = \sum_{i=1}^{\hat{s}^\Gamma} \lambda_i\,(Y_i - \partial_i(\tilde{\Psi}^2)^*(\overline{\mathcal{D}}\, Y - w))\, b^i . \end{aligned}$$

We introduce a potential $\hat{\Psi}^\Gamma \in C^2(\mathbb{R}^{\hat{s}^\Gamma} \times \mathbb{R}^{d^\Gamma})$ via

$$\begin{aligned} \hat{\Psi}^\Gamma(Y,\, w) :=& \frac{1}{2}\,\mathcal{D}Y \cdot Y - (\tilde{\Psi}^2)^*(\overline{\mathcal{D}}\, Y - w) \\ &+ (\tilde{\Psi}^2)^*(-w) + \overline{\mathcal{D}}\, Y \cdot \nabla(\tilde{\Psi}^2)^*(-w) . \end{aligned} \tag{140}$$

Then, at the point $Y = (b^1 \cdot \mu, \ldots, b^{\hat{s}^\Gamma} \cdot \mu)$ we obtain the equivalence

$$J_\nu = \sum_{i=1}^{\hat{s}^\Gamma} \lambda_i\,(\partial_{Y_i}\hat{\Psi}^\Gamma(Y,\, w) - \partial_{Y_i}\hat{\Psi}^\Gamma(0,\, w))\, b^i . \tag{141}$$

We reinterpret the identity (141) by defining

- A modified reaction rate vector field $\hat{R}^\Gamma \in C^1(\mathbb{R}^{\hat{s}^\Gamma} \times \mathbb{R}^{d^\Gamma})$

$$\hat{R}^\Gamma(Y,\, w) := -\nabla_Y \hat{\Psi}^\Gamma(Y,\, w) , \tag{142}$$

- Modified reaction vectors

$$\hat{\gamma}^k := b^k \text{ for } k = 1, \ldots, \hat{s}^\Gamma , \tag{143}$$

- Modified reaction driving forces

$$\hat{D}_k^{\Gamma,\mathsf{R}} := \hat{\gamma}^k \cdot \mu \text{ for } k = 1, \ldots, \hat{s}^\Gamma , \tag{144}$$

- An outer flux $J^0 = J^0(\mu^{\text{ext}})$ taking values in $\operatorname{Image} M^{\Gamma,2} = \operatorname{span}\{\hat\gamma^1, \dots, \hat\gamma^{\hat s^\Gamma}\}$ via

$$J^0 = \sum_{i=1}^{\hat s^\Gamma} \lambda_i \, \partial_{Y_i} \hat\Psi^\Gamma(0,\, w)\, \hat\gamma^i \,. \tag{145}$$

Lemma 5.11. *We define*

(a) A reduced number of boundary reactions $\hat s^\Gamma := \dim(\operatorname{Image} M^\Gamma \cap \mathcal{V})$*;*

(b) Modified reactions vectors $\{\hat\gamma^1, \dots, \hat\gamma^{\hat s^\Gamma}\}$ as the eigenvectors of the matrix $M^{\Gamma,2}$ (cf. (143)*);*

Using the potential $\hat\Psi^\Gamma$ from (140)*, we define*

$$\hat r := \sum_{k=1}^{\hat s^\Gamma} \hat R_k^\Gamma(\hat D^{\Gamma,R},\, w)\, \hat\gamma^k = - \sum_{k=1}^{\hat s^\Gamma} \partial_Y \hat\Psi_k^\Gamma(\hat D^{\Gamma,R},\, w)\, \hat\gamma^k \,.$$

We moreover define (cf. (133)*,* (145)*)*

$$w_k^0 := \sum_{j=1}^{d^{ext}} \tilde A_{j,k}\, \lambda_j^{ext}\, e^j \cdot \mu^{ext} \text{ for } k = \dots, d^\Gamma$$

$$J^0 = \sum_{i=1}^{\hat s^\Gamma} \lambda_i \, \partial_{Y_i} \hat\Psi^\Gamma(0,\, w^0)\, \hat\gamma^i \,.$$

Then the conditions (50)*,* (51)*,* (52) *are satisfied if and only if $J_\nu = -\hat r - J^0$.*

It remains to investigate the properties of the potential $\hat\Psi^\Gamma$ in order to show that $\hat r$ has the desired structure of a reaction term.

Proposition 5.12. *Assume that $\Psi^\Gamma \in C^2(\mathbb{R}^{s^\Gamma})$ is a strictly convex, nonnegative and coercive potential. Assume that M^Γ and $M^{\Gamma,ext}$ are positive semidefinite elements of $M_{sym}^{N^\Gamma \times N^\Gamma}$. Let $\hat s^\Gamma := \dim(\operatorname{Image} M^\Gamma \cap \mathcal{V})$ be the* reduced number of boundary reactions*. We define the reduced potential $\hat\Psi^\Gamma$ as in* (140)*.*

Then, $\hat\Psi^\Gamma \in C^1(\mathbb{R}^{\hat s^\Gamma} \times \mathbb{R}^{d^\Gamma})$ is nonnegative, and the function $Y \mapsto \hat\Psi^\Gamma(Y,\, w)$ is of class $C^2(\mathbb{R}^{\hat s^\Gamma})$, strictly convex and coercive for all $w \in \mathbb{R}^{d^\Gamma}$.

Proof. Due to the representation (140), we directly obtain that $\hat\Psi^\Gamma$ is of class C^1 and even of class C^2 in the first variable. The second derivative $D^2_{Y,Y}\hat\Psi^\Gamma$ is given by

$$D^2_{Y,Y}\hat\Psi^\Gamma = \mathcal{D} - \overline{\mathcal{D}}^T\, D^2_{X,X}(\tilde\Psi^2)^*(\overline{\mathcal{D}}\, Y - w))\, \overline{\mathcal{D}} \,.$$

Due to convex conjugation, $D^2(\tilde\Psi^2)^*(\mathcal{D}\, Y - w) = [D^2\tilde\Psi^2(X)]^{-1}$ at $X = \nabla(\tilde\Psi^2)^*(\overline{\mathcal{D}}\, Y - w)$. The definition of $\tilde\Psi^2$ induces $D^2\tilde\Psi^2(X) = D^2\tilde\Psi^1(X) + \widetilde{\mathcal{D}}$. Here we denote $\widetilde{\mathcal{D}} \in \mathbb{R}^{d^\Gamma \times d^\Gamma}$ the matrix $\operatorname{diag}(\lambda_1, \dots, \lambda_{\hat s^\Gamma}, 0, \dots, 0)$.

Therefore $D^2_{Y,Y}\hat{\Psi}^\Gamma(Y,\,w) = \mathcal{D} - \overline{\mathcal{D}}^T\,[D^2\tilde{\Psi}^1(X) + \widetilde{\mathcal{D}}]^{-1}\,\overline{\mathcal{D}}$.

By definition (recall also the definitions (132) of the matrices A and $\tilde{A}$), for $\eta \in \mathbb{R}^{d^\Gamma}$ arbitrary

$$
\begin{aligned}
D^2\tilde{\Psi}^1(X)\eta\cdot\eta &= D^2\Psi^\Gamma(A\,X)\,A\eta\cdot A\eta + \frac{1}{2}\sum_{i=1}^{d^{\text{ext}}}\lambda_i^{\text{ext}}\,(\tilde{A}\eta)_i^2\\
&\geq \inf\{\lambda_{\min}(D^2\Psi^\Gamma),\,\lambda_1^{\text{ext}},\ldots,\lambda_{d^{\text{ext}}}^{\text{ext}}\}\,(|A\eta|^2 + |\tilde{A}\eta|^2) \geq c_0\,|\eta|^2\,,
\end{aligned}
$$

where we make use of the assumption (77). From the latter estimate, we obtain via elementary arguments that $\lambda_{\min}(D^2_{Y,Y}\hat{\Psi}^\Gamma) \geq \frac{c_0\,\lambda_{\min}(\mathcal{D})}{c_0+\lambda_{\max}(\mathcal{D})}$. This proves the claims. ☐

6 Approximate solutions. Regularisation strategy

For the existence theory we shall embedd the problem (P) into a larger class of approximate, regularised problems that are easier to solve. These approximations (in the spirit of 'viscosity solutions') are constructed in such a way that the integrability of the entire vector of chemical potentials μ as main variable can be expected.

6.1 The regularisation strategy

The regularisation strategy, though not mass conservative, will be chosen *thermodynamically consistent*, since it consists in two essential steps:

(1) A positive definite regularisation of the mobility matrix M;

(2) A convex regularisation of the free energy function h.

The method involves three levels associated with positive parameter, say σ, δ and τ. The first level, associated with the diffusion parameter σ, consisits in modifying the mobility matrix M in order that it becomes elliptic via $M \rightsquigarrow M + \sigma\,I$. We denote $M_\sigma(\rho)$ the corresponding diffusion matrix, that is, we set

$$
M_\sigma(\rho) = M(\rho) + \sigma\,I\,. \tag{146}
$$

This regularisation will allow a control on $\nabla\mu$.

The $\delta-$ and $\tau-$ regularisations are associated with the free energy function h. The $\delta-$regularisation consists in increasing the growth of the (mechamical) free energy modifying the function F that occurs in the definition of h^{mech} via $F(n\cdot V) \rightsquigarrow F(n\cdot V) + \delta\,(n\cdot V)^\alpha,\ \alpha > 3$. If the original growth exponent of F is larger than 3, this step can be omitted. We denote h_δ the corresponding free energy function, that is

$$
h_\delta(\rho) := h(\rho) + \delta\,\left(\rho\cdot\tfrac{V}{m}\right)^\alpha\,. \tag{147}
$$

The $\tau-$regularisation is a stabilisation for the vector of chemical potentials. It consists in modifiying the function h^* (or $(h_\delta)^*$) via

$$(h_\delta)^*(X) \rightsquigarrow (h_\delta)^*(X) + \tau \sum_{i=1}^N \omega(X_i) \ \ (= (h_\delta)^*(X) + \tau\, \omega(X) \cdot \mathbb{1}) \text{ for } X \in \mathbb{R}^N \, .$$

Here $\omega \in C^2(\mathbb{R})$ is a convex and increasing function for which we impose the growth conditions

$$c_0\,(\sqrt{|s^-|} + |s^+|^{\alpha'}) \leq \omega'(s)\,s - \omega(s) \leq c_1\,(\sqrt{|s^-|} + |s^+|^{\alpha}) \tag{148}$$

$$\omega'(s) \leq c_2\,(1 + \omega'(s)\,s - \omega(s))^{1/\alpha} \, . \tag{149}$$

For example, we could choose the function

$$\omega(s) := \begin{cases} -2\,\sqrt{|s|} & \text{for } s \leq -1 \\ \frac{1}{4}\,s^2 + \frac{3}{2}\,s - \frac{3}{4} & \text{for } -1 < s < 1 \\ \frac{1}{2\alpha'\,(\alpha'-1)}\,s^{\alpha'} + (2 - \frac{1}{2(\alpha'-1)})\,s + \frac{1}{2\alpha(\alpha'-1)} - 1 & \text{otherwise} \, . \end{cases} \tag{150}$$

which satisfies these assumptions. The choice of the regularisation ω is by no means unique, the constants in (150) are determined from simple interpolation conditions. Essential for our purposes is in fact only the sublinear growth for $s \to -\infty$ that garanties convexity.

Combining with the $\delta-$regularisation, we define on $\mathbb{R}^N$ a function

$$h^*_{\delta,\tau}(X) := (h_\delta)^*(X) + \tau \sum_{i=1}^N \omega(X_i) \, , \tag{151}$$

which is twice differentiable and convex. Making use of the convexity we easily show that the mapping $\nabla h^*_{\tau,\delta}\, :\, \mathbb{R}^N \to \mathbb{R}^N_+$ is bijective. Interpreting (151) as Legendre transform, we introduce a regularised free energy function via

$$h_{\tau,\delta} := \text{ convex conjugate of the function } h^*_{\tau,\delta} = (h^*_{\tau,\delta})^* \, , \tag{152}$$

which is a twice differentiable convex function on $\mathbb{R}^N_+$. The main motivation for this construction is that the new free energy function has improved coercivity properties over the variables ρ *and* μ as exposed in the following statement.

Lemma 6.1. *Let the original free energy function h satisfy*

$$c_0\,|\rho|^{\alpha_0} - c_1 \leq h(\rho) \leq C_0\,|\rho|^{\alpha_0} + C_1, \text{ for all } \rho \in \mathbb{R}^N_+ \, . \tag{153}$$

with constants $3/2 < \alpha_0 < +\infty$ and $0 < c_0,\, c_1,\, C_0,\, C_1 < +\infty$. Let $\alpha > 3$ be the regularisation exponent of (147)*, and ω a function satisfying* (148)*. Define*

$$\Phi_\omega(X) := \sum_{i=1}^N \omega'(X_i)\,X_i - \omega(X_i) \text{ for } X \in \mathbb{R}^N \, . \tag{154}$$

Then there are $\tilde{c}_0,\, \tilde{c}_1 > 0$, and $\tau_0(\alpha,\, \alpha_0) > 0$ such that if $\tau \leq \tau_0$

$$h_{\tau,\delta}(\rho) \geq \tilde{c}_0\,(|\rho|^{\alpha_0} + \delta\,|\rho|^{\alpha} + \tau\,\Phi_\omega(\mu)) - \tilde{c}_1 \tag{155}$$

for all $\rho \in \mathbb{R}^N_+$ and $\mu \in \mathbb{R}^N$ connected by the identity $\rho = \nabla h^*_{\tau,\delta}(\mu)$.

Proof. The definition (152) implies that

$$h_{\tau,\delta}(\nabla h^*_{\tau,\delta}(X)) = h_\delta(\nabla(h_\delta)^*(X)) + \tau\, \Phi_\omega(X)\,.$$

By assumption, ρ and μ are related via

$$\rho := \nabla h^*_{\tau,\delta}(\mu) = \nabla(h_\delta)^*(\mu) + \tau\, \omega'(\mu)\,, \tag{156}$$

and we obtain for the regularised free enery the identity

$$\begin{aligned} h_{\tau,\delta}(\rho) &= h_\delta(\nabla(h_\delta)^*(\mu)) + \tau\, \Phi_\omega(\mu) \\ &= h_\delta(\rho - \tau\, \omega'(\mu)) + \tau \sum_{i=1}^N (\mu_i\, \omega'(\mu_i) - \omega(\mu_i))\,. \end{aligned} \tag{157}$$

Using the properties of $h_\delta(Y) = h(Y) + \delta\,(Y \cdot \frac{V}{m})^\alpha$, we obtain that

$$h_{\tau,\delta}(\rho) \geq h(\rho - \tau\, \omega'(\mu)) + \delta\,((\rho - \tau\, \omega'(\mu)) \cdot \tfrac{V}{m})^\alpha + \tau \sum_{i=1}^N (\mu_i\, \omega'(\mu_i) - \omega(\mu_i))\,.$$

On the other hand, the condition (148) ensures that $\omega'(\mu_i) \leq c\,(1 + \omega'(\mu_i)\,\mu_i - \omega(\mu_i))^{1/\alpha}$. For $\alpha > 1$, denote $\underline{c}(\alpha),\ \bar{c}(\alpha)$ two constants such that $|a-b|^\alpha \geq \underline{c}(\alpha)\, a^\alpha - \bar{c}(\alpha)\, b^\alpha$ for all $a,\, b > 0$. If follows that

$$\begin{aligned} h_{\tau,\delta}(\rho) &\geq h(\rho - \tau\, \omega'(\mu)) + c_2\, \delta\, |\rho - \tau\, \omega'(\mu))|^\alpha + \tau \sum_{i=1}^N (\mu_i\, \omega'(\mu_i) - \omega(\mu_i)) \\ &\geq h(\rho - \tau\, \omega'(\mu)) + \min\{c_2\, \delta,\, \underline{c}(\alpha)\}\, |\rho|^\alpha \\ &\qquad + \tau \sum_{i=1}^N (\mu_i\, \omega'(\mu_i) - \omega(\mu_i)) - c_2\, \delta\, \tau^\alpha\, \bar{c}(\alpha)\, |\omega'(\mu)|^\alpha \\ &= h(\rho - \tau\, \omega'(\mu)) + \min\{c_2\, \delta,\, \underline{c}(\alpha)\}\, |\rho|^\alpha \\ &\qquad + (1 - c_2\, \delta\, \bar{c}(\alpha)\, \tau^{\alpha-1})\, \tau \sum_{i=1}^N (\mu_i\, \omega'(\mu_i) - \omega(\mu_i)) - C\,. \end{aligned}$$

If we asssume that $C\, \delta\, \tau^{\alpha-1} \leq 1/4$,

$$h_{\tau,\delta}(\rho) \geq h(\rho - \tau\, \omega'(\mu)) + \min\{c_2\, \delta,\, \underline{c}(\alpha)\}\, |\rho|^\alpha + \frac{3}{4}\, \tau \sum_{i=1}^N (\mu_i\, \omega'(\mu_i) - \omega(\mu_i)) - C\,.$$

Making use of the growth of the free energy h and analogous arguments, the claim follows. □

We also note that the pressure in the system is naturallly defined via $p = -h(\rho) + \sum_{i=1}^N \rho_i\, \mu_i = h^*(\mu)$. In the context of the approximmation scheme for $\delta > 0$ and $\tau > 0$, we obtain natural definitions for the variables p and n^F via

$$\rho = \nabla h^*_{\tau,\delta}(\mu) = \nabla(h_\delta)^*(\mu) + \tau\, \Phi'_\omega(\mu), \quad p = h^*_{\tau,\delta}(\mu) = h^*_\delta(\mu) + \tau \sum_{i=1}^N \Phi_\omega(\mu_i)$$

6.2 Approximation scheme

For the existence proof we shall embedd the problem (P) into a larger class of (approximate) problems $(P_{\tau,\sigma,\delta})$ charcterised by an elliptic diffusion matrix M_σ and a regularised free energy $h_{\tau,\delta}$. Since in this approach it is possible to control the entire vector μ, a solution vector consists of the entries μ, v and ϕ.

In order to define the concept of solution, we introduce also in this case a natural class $\mathcal{B}$ for the approximate solutions. If $\delta,\,\sigma > 0$ and $\tau \geq 0$, we say that $(\mu,\,v,\,\phi)$ belongs to $\mathcal{B}(T,\,\Omega,\,\alpha,\,N,\,\Psi,\,\Psi^\Gamma)$ if and only if

$$
\begin{aligned}
&(\varrho,\,q,\,v,\,\phi,\,R,\,R^\Gamma) \in \mathcal{B}(T,\,\Omega,\,\alpha,\,N-1,\,\Psi,\,\Psi^\Gamma)\\
&\qquad \text{with } \varrho := \nabla h^*_{\tau,\delta}(\mu)\cdot \mathbb{1} \text{ and } q := \Pi\,\mu,\\
&\qquad R_k = \bar{R}_k(D^{\mathsf{R}}), \quad D^{\mathsf{R}}_k := \gamma^k\cdot\mu \text{ for } k = 1,\ldots,s,\\
&\qquad R^\Gamma_k = \hat{R}^\Gamma_k(\hat{D}^{\Gamma,\mathsf{R}},\,w), \quad \hat{D}^{\Gamma,\mathsf{R}}_k := \hat{\gamma}^k\cdot\mu, \quad k = 1,\ldots,\hat{s}^\Gamma
\end{aligned}
\tag{158}
$$

$$
\mu \in \begin{cases} W^{1,0}_2(Q;\,\mathbb{R}^N) & \text{if } \tau > 0 \text{ and } \sigma > 0\\ L^w_{(\circ_N \ln)}L^1(Q;\,\mathbb{R}^N),\ \nabla\mu \in L^2(Q;\,\mathbb{R}^{N\times 3}) & \text{for } \tau = 0 \text{ and } \sigma > 0 \end{cases}
\tag{159}
$$

We say that $(\mu,\,v,\,\phi)$ satisfies the approximate energy (in)equality if and only if the corresponding vector $(\varrho,\,q,\,v,\,\phi,\,R,\,R^\Gamma)$ satisfies the energy (in)equality of Definition 4.2, with free energy function $h_{\tau,\delta}$ and mobility matrix M_σ. For $\delta > 0$, $\sigma > 0$ and $\tau \geq 0$ we call weak solution to the problem $(P_{\tau,\,\sigma,\,\delta})$ a vector $(\mu,\,v,\,\phi) \in \mathcal{B}$ subject to the energy inequality and such that the quantities

$$
\begin{aligned}
\rho &= \nabla h^*_{\tau,\delta}(\mu)\\
J &= -M_\sigma(\rho)\,D, \quad D := \frac{\nabla\mu}{\theta} + \frac{1}{\theta}\,\frac{z}{m}\,\nabla\phi\\
r &= \sum_{k=1}^{s} \hat{\gamma}^k\,\bar{R}_k(D^{\mathsf{R}}), \quad D^{\mathsf{R}} = (\gamma^1\cdot\mu,\ldots,\gamma^s\cdot\mu)\\
\hat{r} &= \sum_{k=1}^{\hat{s}^\Gamma} \hat{\gamma}^k\,\hat{R}^\Gamma_k(\hat{D}^{\Gamma,\mathsf{R}},\,w^0), \quad \hat{D}^{\Gamma,\mathsf{R}} = (\hat{\gamma}^1\cdot\mu,\ldots,\hat{\gamma}^{\hat{s}^\Gamma}\cdot\mu)\\
p &= h^*_{\tau,\delta}(\mu)\\
n^F &= \rho\cdot\frac{z}{m}
\end{aligned}
\tag{160}
$$

satisfy the identities

$$-\int_Q \rho\cdot\psi_t - \int_Q (\rho_i\, v + J^i)\cdot\nabla\psi_i = \int_\Omega \rho^0\cdot\psi(0) + \int_Q r\cdot\psi + \int_{S_T}(\hat r + J^0)\cdot\psi \quad (161)$$

$$-\int_Q \varrho\, v\cdot\eta_t - \int_Q \varrho\, v\otimes v\, :\, \nabla\eta - \int_Q p\,\operatorname{div}\eta + \int_Q \mathbb{S}(\nabla v)\, :\, \nabla\eta \quad (162)$$

$$= \int_\Omega \varrho_0\, v^0\cdot\eta(0) - \int_Q n^F\,\nabla\phi\cdot\eta - \int_Q (\sum_{i=1}^N J^i\cdot\nabla)\eta\cdot v$$

$$\epsilon_0\,(1+\chi)\int_Q \nabla\phi\cdot\nabla\zeta = \int_Q n^F\,\zeta, \quad \phi = \phi_0 \text{ as traces on }]0,T[\times\Gamma\,. \quad (163)$$

for all $\psi \in C^1_c([0,T[;\, C^1(\overline{\Omega};\, \mathbb{R}^N))$, $\eta \in C^1_c([0,T[;\, C^1_c(\Omega;\mathbb{R}^3))$ and $\zeta \in L^1(0,T;\, W^{1,2}_\Gamma(\Omega))$.

7 Derivation of the global energy and mass balance identities

In this section we *motivate* the definition 4.2 by stating an *energy identity* naturally associated with the problem (P) (or its thermodynamically consistent approximations $(P_{\tau,\sigma,\delta})$).

Proposition 7.1. *Assume that there are vector fields* $\mu \in C^{0,1}([0,T]\times\Omega;\, \mathbb{R}^N)$*,* $v \in C^{0,1}([0,T]\times\Omega;\, \mathbb{R}^3)$ *and* $\phi \in L^\infty([0,T];\, C^{0,1}(\Omega))$ *that satisfy together with their associate variables* ρ*,* J*,* r*,* $\hat r$*,* ϱ*,* p*,* n^F *defined in* (160) *the relations*

$$\int_\Omega \partial_t\rho\cdot\psi - \int_\Omega (\rho_i\, v + J^i)\cdot\nabla\psi^i = \int_\Omega r\cdot\psi + \int_\Gamma (\hat r + J^0)\cdot\psi \quad (164)$$

$$\int_\Omega \varrho\,\partial_t v\cdot\eta + \int_\Omega \varrho\,(v\cdot\nabla)v\cdot\eta + \int_\Omega \mathbb{S}(\nabla v)\, :\, \nabla\eta - \int_\Omega p\,\operatorname{div}\eta$$

$$= -\int_\Omega (\sum_{i=1}^N J^i\cdot\nabla)v\cdot\eta - \int_\Omega n^F\,\nabla\phi\cdot\eta \quad (165)$$

$$\epsilon_0\,(1+\chi)\int_\Omega \nabla\phi\cdot\nabla\zeta = \int_\Omega n^F\,\zeta\,, \quad (166)$$

for all $\psi \in W^{1,1}(\Omega;\, \mathbb{R}^N)$*, all* $\eta \in W^{1,1}_0(\Omega;\, \mathbb{R}^3)$ *and for all* $\zeta \in W^{1,1}_\Gamma(\Omega)$ *together with conditions*

$$\begin{aligned} \mu(0) = \mu^0 \in C^{0,1}(\Omega;\, \mathbb{R}^N), \quad v(0) = v^0 \in C^{0,1}(\Omega;\, \mathbb{R}^3) \textit{ in } \Omega \\ \phi = \phi_0 \in C^{0,1}([0,T]\times\Omega)) \textit{ on }]0,T[\times\Gamma,\ v = 0 \textit{ on } [0,T]\times\partial\Omega\,. \end{aligned} \quad (167)$$

We define $\rho^0 = \nabla h^*_{\tau,\delta}(\mu^0)$. Then, for all $t \in]0,T[$, the following identity is valid:

$$\begin{aligned}
&\int_\Omega \left\{\frac{1}{2}\,\varrho\, v^2 + \frac{1}{2}\,\epsilon_0\,(1+\chi)\,|\nabla\phi|^2 + h_{\tau,\delta}(\rho)\right\}(t)\\
&+ \int_{Q_t} \{\mathbb{S}(\nabla v) \,:\, \nabla v - \theta\, J\cdot D - r\cdot\mu\} - \int_{S_t} \hat{r}\cdot\mu\\
&= \int_\Omega \left\{\frac{1}{2}\,\varrho_0\,|v^0|^2 + \frac{1}{2}\,\epsilon_0\,(1+\chi)\,|\nabla\phi_0(0)|^2 + h_{\tau,\delta}(\rho^0)\right\}\\
&\quad - \int_\Omega \left\{n^F\,\phi_0 + \epsilon_0\,(1+\chi)\,\nabla\phi\cdot\nabla\phi_0\right\}\Big|_0^t\\
&\quad + \int_{Q_t} \left\{n^F\,\phi_{0,t} - \epsilon_0\,(1+\chi)\,\nabla\phi\cdot\nabla\phi_{0,t}\right\} + \int_{S_t} \{J^0\cdot\mu + (\hat{r}+J^0)\cdot\tfrac{z}{m}\,\phi_0\}\,.
\end{aligned}$$

Proof. We choose $\psi = \mu(t)$ in (164). Because of Lemma 5.4, we observe that $\sum_{i=1}^N \rho_i\,\nabla\mu_i = \nabla h^*_{\tau,\delta}(\mu) = \nabla p$. Moreover, the definition of ρ yields $\mu = \nabla h_{\tau,\delta}(\rho)$ and therefore $\partial_t\rho\cdot\mu = \partial_t h_{\tau,\delta}(\rho)$. Thus, we obtain that

$$\partial_t \int_\Omega h_{\tau,\delta}(\rho) - \int_\Omega \left(v\cdot\nabla p + \sum_{i=1}^N J^i\cdot\nabla\mu_i\right) = \int_\Omega r\cdot\mu + \int_\Gamma (\hat{r}+J^0)\cdot\mu\,. \tag{168}$$

We next choose $\psi = \frac{z}{m}\,\phi$ in (164). We recall that $r\cdot\frac{z}{m} = 0$, and obtain that

$$\int_\Omega \partial_t n^F\,\phi - \int_\Omega \left(n^F\,v\cdot\nabla\phi + \sum_{i=1}^N J^i\,\frac{z_i}{m_i}\cdot\nabla\phi\right) = \int_\Gamma (\hat{r}+J^0)\cdot\tfrac{z}{m}\,\phi_0\,. \tag{169}$$

Next we differentiate in time (166), and choose $\zeta = \phi(t) - \phi_0(t)$. We obtain that

$$\int_\Omega n^F_t\,\phi = \int_\Omega n^F_t\,\phi_0 + \frac{\epsilon_0\,(1+\chi)}{2}\,\partial_t\int_\Omega |\nabla\phi|^2 - \epsilon_0\,(1+\chi)\,\int_\Omega \nabla\phi_t\cdot\nabla\phi_0\,. \tag{170}$$

Thus, (169) and (170) yield

$$\begin{aligned}
\frac{\epsilon_0\,(1+\chi)}{2}\,\partial_t\int_\Omega |\nabla\phi|^2 - \int_\Omega \left(n^F\,v\cdot\nabla\phi + \sum_{i=1}^N J^i\,\frac{z_i}{m_i}\cdot\nabla\phi\right) &= \int_\Gamma (\hat{r}+J^0)\cdot\tfrac{z}{m}\,\phi_0\\
&\quad + \epsilon_0\,(1+\chi)\,\int_\Omega \nabla\phi_t\cdot\nabla\phi_0 - \int_\Omega n^F_t\,\phi_0\,.
\end{aligned} \tag{171}$$

If we now add (171) to (168), we obtain that

$$\begin{aligned}
&\partial_t\int_\Omega \{h_{\tau,\delta}(\rho) + \frac{\epsilon_0\,(1+\chi)}{2}\,|\nabla\phi|^2\} - \int_\Omega v\cdot(\nabla p + n^F\,\nabla\phi)\\
&- \int_\Omega \sum_{i=1}^N J^i\cdot(\nabla\mu_i + \tfrac{z_i}{m_i}\cdot\nabla\phi) - \int_\Omega r\cdot\mu - \int_\Gamma \hat{r}\cdot\mu\\
&= \int_\Gamma (J^0\cdot\mu + (J^0+\hat{r})\cdot\tfrac{z}{m}\,\phi_0) + \epsilon_0\,(1+\chi)\,\int_\Omega \nabla\phi_t\cdot\nabla\phi_0 - \int_\Omega n^F_t\,\phi_0\,.
\end{aligned} \tag{172}$$

Next we choose $\eta = v(t)$ in (165), and we obtain that

$$\frac{1}{2}\int_\Omega \varrho\,\partial_t v^2 + \int_\Omega \varrho\,(v\cdot\nabla)v^2 + \int_\Omega \mathbb{S}(\nabla v)\,:\,\nabla v$$
$$+\int_\Omega v\cdot(\nabla p + n^F\,\nabla\phi) = -\frac{1}{2}\int_\Omega \sum_{i=1}^N J^i\cdot\nabla v^2\,. \tag{173}$$

If we choose $\psi = v^2\,\mathbb{1}$ in (164) and observe that $r\cdot\mathbb{1} = 0 = \hat{r}\cdot\mathbb{1}$ by definition, we see that

$$\int_\Omega \partial_t\varrho\, v^2 - \int_\Omega (\varrho\, v + \sum_{i=1}^N J^i)\cdot\nabla v^2 = 0\,. \tag{174}$$

Due to (174), we have

$$\int_\Omega \varrho\,\partial_t v^2 + \int_\Omega \varrho\, v\cdot\nabla v^2 + \int_\Omega \sum_{i=1}^N J^i\cdot\nabla v^2 = \partial_t\int_\Omega \varrho\, v^2\,, \tag{175}$$

and thus (173) yields

$$\frac{1}{2}\,\partial_t\int_\Omega \varrho\, v^2 + \int_\Omega \mathbb{S}(\nabla v)\,:\,\nabla v + \int_\Omega v\cdot(\nabla p + n^F\,\nabla\phi) = 0\,. \tag{176}$$

We add (176) to (172) and obtain that

$$\partial_t\int_\Omega\{\frac{1}{2}\,\varrho\, v^2 + h_{\tau,\delta}(\rho) + \frac{\epsilon_0\,(1+\chi)}{2}\,|\nabla\phi|^2\} + \int_\Omega \mathbb{S}(\nabla v)\,:\,\nabla v$$
$$-\int_\Omega \theta\, J\cdot D - \int_\Omega r\cdot\mu - \int_\Gamma \hat{r}\cdot\mu$$
$$= \int_\Gamma (J^0\cdot\mu + (J^0+\hat{r})\cdot\tfrac{z}{m}\,\phi_0) + \epsilon_0\,(1+\chi)\int_\Omega \nabla\phi_t\cdot\nabla\phi_0 - \int_\Omega n^F_t\,\phi_0\,. \tag{177}$$

We integrate over time and are done. □

The proof of the global mass conservation identities is comparatively simpler. It suffices to insert $\psi = e^i$ for $i = 1,\ldots,N$ into (263).

Proposition 7.2. *Assumptions of Proposition 7.1. Then for all* $t\in[0,T]$

$$\bar{\rho}(t) = \bar{\rho}^0 + \int_0^t \left\{\int_\Omega r + \int_\Gamma (\hat{r} + J^0)\right\}(s)\,ds\,. \tag{178}$$

8 A priori estimates

In this section we derive *a priori* estimates on solutions to the problem (P) that result from the energy identity. In order to include in our considerations both approximationn scheme and limit problem, we here consider generic free energy functions satisfying the following growth assumption: there are $c_1 > 0$, $c_2 \geq 0$ and $C_i \geq 0$, $i = 1,2,3$ and $\tau \geq 0$ such that for all $\rho \in \mathbb{R}^N_+$

$$c_1\,|\rho|^\alpha + \tau\,\Phi_\omega[\nabla h(\rho)] - c_2 \leq h(\rho) \leq C_1\,|\rho|^\alpha + C_2\,\tau\,\Phi_\omega[\nabla h(\rho)] + C_3\,. \tag{179}$$

Moreover we consider mobility matrices $M_\sigma = M(\rho) + \sigma\, I$, $\sigma \geq 0$, such that M satisfies (73) and (74).

We commence with a few standard estimates. In the second subsection we prove the most demanding estimate on the chemical potentials.

8.1 Standard estimates

Proposition 8.1. *Let* $(\varrho,\, q,\, v,\, \phi,\, R,\, R^\Gamma)$ *satisfy the energy inequality of the Definition 4.2 with free energy function* h *satisfying* (179) *and mobility matrix* M *satisfying* (74). *Then, there is a number* $C_0 > 0$ *depending only on* Ω, *on the constants* c_i, C_i *in the conditions* (179), *and on the quantity*

$$\begin{aligned}\mathcal{B}_0 := \|\rho^0\|_{L^\alpha(\Omega)} + \tau\,\|\Phi_\omega(\mu^0)\|_{L^1(\Omega)} + \|\sqrt{\varrho_0}\,v^0\|_{L^2(\Omega)} + \|\phi_0\|_{L^\infty(Q)}\\ + \|\phi_0\|_{L^\infty(0,T;\,W^{1,2}(\Omega))} + \|\phi_{0,t}\|_{W^{1,0}_2(Q)} + \|\phi_{0,t}\|_{L^{\alpha'}(Q)} + \|\mu^{ext}\|_{L^\infty(S)}\,,\end{aligned} \tag{180}$$

such that

$$\begin{aligned}
&\|\rho\|_{L^{\infty,\alpha}(Q)} + \tau\,\|\Phi_\omega(\mu)\|_{L^{\infty,1}(Q)} + \|\sqrt{\varrho}\,v\|_{L^{\infty,2}(Q)} + \|\nabla\phi\|_{L^{\infty,2}(Q)} \leq C_0\\
&\|v\|_{W^{1,0}_2(Q)} + \|\nabla q\|_{L^2(Q)} \leq C_0\\
&\|D^R\|_{L_\Psi(Q)} + \|\hat{D}^{\Gamma,R}\|_{L_{\hat{\Psi}^\Gamma}(S)} \leq C_0\\
&\sum_{i=1}^N \|J^i\|_{L^{2,\frac{2\alpha}{1+\alpha}}(Q)} + [-R]_{L_{\Psi^*}(Q)} + [-R^\Gamma]_{L_{(\hat{\Psi}^\Gamma)^*}(S)} \leq C_0\\
&\sqrt{\sigma}\,\|\nabla\mu\|_{L^2(Q)} + \min\{\sigma,\,\tau^2\}\,\|\mu\|_{L^{2,3}(Q)} \leq C_0\\
&\|\mathbb{1}\cdot J\|_{L^2(Q)} \leq C_0\,\sqrt{\sigma}, \quad \|\tau\,\omega'(\mu)\|_{L^{\infty,\alpha}(Q)} \leq C_0\,\tau^{1/\alpha'}\,.
\end{aligned}$$

Here the quantities ρ, J, *etc. obey the definitions* (95), (160).

Proof. Due to the assumption (179) we have for the left-hand of the energy identity (100)

$$\int_\Omega h(\rho)(t) \geq c_1 \int_\Omega |\rho(t)|^\alpha + \tau \int_\Omega \Phi_\omega(\mu(t)) - c_2\,|\Omega|\,.$$

Moreover, for general velocity fields $v \in W^{1,2}(\Omega;\,\mathbb{R}^3)$

$$\int_\Omega \mathbb{S}(\nabla v)\,:\,\nabla v = \int_\Omega \frac{\eta}{4}\,|D(v) - \frac{2}{3}\,\operatorname{div} v\,\mathsf{Id}|^2 + \int_\Omega (\lambda + \frac{2}{3}\,\eta)\,(\operatorname{div} v)^2\,.$$

In the case that $v = 0$ on $\partial\Omega$ one even has

$$\int_\Omega \mathbb{S}(\nabla v)\,:\,\nabla v = \int_\Omega (\eta\,|\nabla v|^2 + (\lambda + \eta)\,(\operatorname{div} v)^2)\,.$$

For the right hand of (100), we first observe that

$$\left|\int_\Omega n^F(t)\,\phi_0(t)\right| \leq \left|\tfrac{z}{m}\right| \int_\Omega |\rho|\,|\phi_0(t)| \leq \frac{c_1}{2}\int_\Omega |\rho|^\alpha + c\int_\Omega |\phi_0|^{\alpha'}$$

$$\left|\epsilon_0\,(1+\chi)\int_\Omega \nabla\phi\cdot\nabla\phi_0\right| \leq \frac{\epsilon_0\,(1+\chi)}{4}\int_\Omega |\nabla\phi(t)|^2 + c\int_\Omega |\nabla\phi_0|^2\,.$$

Moreover by standard considerations

$$\begin{aligned}
&\left|\int_{Q_t}\{n^F\,\phi_{0,t} - \epsilon_0\,(1+\chi)\,\nabla\phi\cdot\nabla\phi_{0,t}\}\right| \\
&\leq \int_0^t \{\|n^F(s)\|_{L^\alpha(\Omega)}\,\|\phi_{0,t}(s)\|_{L^{\alpha'}(\Omega)} + \epsilon_0\,(1+\chi)\,\|\nabla\phi(s)\|_{L^2(\Omega)}\,\|\nabla\phi_{0,t}(s)\|_{L^2(\Omega)} \\
&\leq \int_0^t \{\|n^F(s)\|^\alpha_{L^\alpha(\Omega)} + \epsilon_0\,(1+\chi)\,\|\nabla\phi(s)\|^2_{L^2(\Omega)}\} \\
&\quad + C\int_0^t \{\|\phi_{0,t}(s)\|^{\alpha'}_{L^{\alpha'}(\Omega)} + \|\nabla\phi_{0,t}(s)\|^2_{L^2(\Omega)}\}\,.
\end{aligned}$$

We further note by the Young inequality that

$$\begin{aligned}
-\int_{S_t} R^\Gamma_k\,\hat\gamma^k\cdot\frac{z}{m}\,\phi_0 \leq &\int_{S_t}(\hat\Psi^\Gamma)^*(t,x,\,-\frac{1}{4}\,R^\Gamma) \\
&+\int_{S_t}\hat\Psi^\Gamma(t,x,\,4\,\phi_0\,(\hat\gamma^1\cdot\tfrac{z}{m},\ldots,\,\hat\gamma^{\hat s^\Gamma}\cdot\tfrac{z}{m}))\,.
\end{aligned}$$

Thus, using convexity and that $(\hat\Psi^\Gamma)^*(t,x,\,-\frac{1}{4}\,R^\Gamma) = (\hat\Psi^\Gamma)^*(t,x,\,\frac{1}{4}\,(-R^\Gamma)+\frac{3}{4}\,0)$

$$\begin{aligned}
-\int_{S_t} R^\Gamma_k\,\hat\gamma^k\cdot\frac{z}{m}\,\phi_0 \leq &\frac{1}{4}\int_{S_t}(\hat\Psi^\Gamma)^*(t,x,\,-R^\Gamma) \\
&+\int_{S_t}\hat\Psi^\Gamma(t,x,\,\frac{3}{4}\,\phi_0\,(\hat\gamma^1\cdot\tfrac{z}{m},\ldots,\,\hat\gamma^{\hat s^\Gamma}\cdot\tfrac{z}{m})) \\
&=\frac{1}{4}\int_{S_t}(\hat\Psi^\Gamma)^*(t,x,\,-R^\Gamma) + C_0(\|\phi_0\|_{L^\infty([0,T]\times\Gamma)})\,.
\end{aligned}$$

Next we use the fact that J^0 possesses a representation $J^0 = \sum_{k=1}^{\hat s^\Gamma}\jmath_k\,\hat\gamma^k$, and therefore

$$\begin{aligned}
\int_{S_t} J^0\cdot\mu &\leq \int_{S_t}\hat\Psi^\Gamma(t,x,\,\frac{1}{4}\,\hat D^{\Gamma,\mathsf{R}}) + \int_{S_t}(\hat\Psi^\Gamma)^*(t,x,\,4\,\jmath) \\
&\leq \frac{1}{4}\int_{S_t}\hat\Psi^\Gamma(t,x,\,\hat D^{\Gamma,\mathsf{R}}) + C_0(\|\mu^{\mathsf{ext}}\|_{L^\infty(S)})\,.
\end{aligned}$$

Recall the identities

$$\Psi(D^{\mathsf{R}}) + (\Psi)^*(-\bar R(D^{\mathsf{R}})) = -\sum_{k=1}^s \bar R_k(D^{\mathsf{R}})\,\gamma^k\cdot\mu$$

$$\hat\Psi^\Gamma(t,x,\,\hat D^{\Gamma,\mathsf{R}}) + (\hat\Psi^\Gamma)^*(t,x,\,-\hat R^\Gamma(\hat D^{\Gamma,\mathsf{R}},\,w^0)) = -\sum_{k=1}^{\hat s^\Gamma}\hat R^\Gamma_k(\hat D^{\Gamma,\mathsf{R}},\,w^0)\,\hat\gamma^k\cdot\mu\,.$$

Thus, for all $t \in]0,T[$, the dissipation inequality implies that

$$\int_\Omega \left\{\frac{1}{2}\,\varrho\, v^2 + \frac{\epsilon_0\,(1+\chi)}{4}\,|\nabla\phi|^2 + \frac{c_1}{2}\,|\rho|^\alpha + \tau\,\Phi_\omega(\mu)\right\}(t)$$
$$+ \int_{Q_t}\left\{(\eta\,|\nabla v|^2 + (\lambda+\eta)\,(\operatorname{div} v)^2) - \theta\,\sum_{i=1}^N J^i\cdot D^i + (\Psi(D^{\mathsf{R}}) + (\Psi)^*(-R))\right\}$$
$$+ \frac{1}{2}\int_{S_t}\{\hat{\Psi}^\Gamma(t,x,\,\hat{D}^{\Gamma,\mathsf{R}}) + (\hat{\Psi}^\Gamma)^*(t,x,\,-R^\Gamma)\}$$
$$\leq C_0 + C\int_0^t\{\|\rho\|^\alpha_{L^\alpha(\Omega)} + \epsilon_0\,(1+\chi)\,\|\nabla\phi\|^2_{L^2(\Omega)}\}$$

Owing to the thermodynamical consistency, we (at least) obtain that $\sum_{i=1}^N J^i\cdot D^i \leq 0$. Moreover, recall that $\lambda + \frac{2}{3}\,\eta \geq 0$. Exploiting the Gronwall Lemma, we thus obtain bounds for the quantities $\|\sqrt{\varrho}\,v\|_{L^{\infty,2}(Q)}$, $\|\nabla\phi\|_{L^{\infty,2}(Q)}$ and $\|\rho\|_{L^{\infty,\alpha}(Q)}$ and $\tau\,\|\Phi_\omega(\mu)\|_{L^{\infty,1}(Q)}$. It next follows that

$$\int_\Omega \left\{\frac{1}{2}\,\varrho\, v^2 + \frac{1}{4}\,\epsilon_0\,(1+\chi)\,|\nabla\phi|^2 + \frac{c_1}{2}\,|\rho|^\alpha + \tau\,\Phi_\omega(\mu)\right\}(t)$$
$$+ \int_{Q_t}\left\{(\eta\,|\nabla v|^2 + (\lambda+\eta)\,(\operatorname{div} v)^2) - \theta\,\sum_{i=1}^N J^i\cdot D^i + (\Psi(D^{\mathsf{R}}) + (\Psi)^*(-R))\right\}$$
$$+ \frac{1}{2}\int_{S_t}\{\hat{\Psi}^\Gamma(t,x,\,\hat{D}^{\Gamma,\mathsf{R}}) + (\hat{\Psi}^\Gamma)^*(t,x,\,-R^\Gamma)\} \leq C_0(T)\,.$$

In turn this implies a bound for $\|\operatorname{div} v\|_{L^2(Q)}$, and for $\|\nabla v\|_{L^2(Q)}$. Moreover the production factors R and R^Γ are bounded in Orlicz classes

$$[-R]_{L_{(\Psi)^*}(Q;\mathbb{R}^s)} + [-R^\Gamma]_{L_{(\hat{\Psi}^\Gamma)^*}(S_T;\mathbb{R}^{\hat{s}^\Gamma})} \leq C_0\,. \tag{181}$$

whereas the reaction driving forces satisfy

$$[D^{\mathsf{R}}]_{L_\Psi(Q;\mathbb{R}^s)} + [\hat{D}^{\Gamma,\mathsf{R}}]_{L_{\hat{\Psi}^\Gamma}(S_T;\mathbb{R}^{\hat{s}^\Gamma})} \leq C_0\,. \tag{182}$$

It remains to exploit the dissipation due to diffusion and the driving forces $D^1,\ldots,D^N$. At first we note that $-\theta\,\sum_{i=1}^N J^i\cdot D^i = \theta\,\sum_{i,j} M_{i,j}\,D^i\cdot D^j$. For $i=1,\ldots,N$ the Cauchy-Schwarz inequality and the growth condition (74) on M (or M_σ) imply that

$$|J^i| = |\sum_{j=1}^N M_{i,j}\,D^j| \leq (MD\cdot D)^{1/2}\,(Me^i\cdot e^i)^{1/2}$$
$$\leq (\sqrt{\sigma}+\sqrt{\lambda})\,(1+|\rho|)^{1/2}\,(MD\cdot D)^{1/2}\,.$$

Therefore, we obtain for the diffusion fluxes that

$$\|J^i(t)\|_{L^{\frac{2\alpha}{1+\alpha}}(\Omega)} \leq c\,\|MD\cdot D(t)\|^{1/2}_{L^1(\Omega)}\,(1+\|\rho(t)\|^{1/2}_{L^\alpha(\Omega)}) \leq C_0\,\|MD\cdot D(t)\|^{1/2}_{L^1(\Omega)}\,.$$

It follows that $\|J^i\|_{L^{2,\frac{2\alpha}{1+\alpha}}(Q)} \leq c\,\left(\int_Q MD \cdot D\right)^{1/2} \leq C_0$.

We finally want to obtain estimates on the gradients of the (relative) chemical potentials. Here we use the assumption (74) which yields that

$$-\theta \sum_{i=1}^N J^i \cdot D^i = \theta \sum_{i,j=1}^N M_{i,j}\, D^i \cdot D^j \geq \theta\, \underline{\lambda}\, |P_{\mathbb{1}^\perp}\, D|^2\,.$$

Here $P_{\mathbb{1}^\perp}$ the orthogonal projection on the space $\mathbb{1}^\perp$. We now split the driving force $D^i = \theta^{-1}\,(\nabla\mu_i + \frac{z_i}{m_i}\,\nabla\phi)$, and we obtain that

$$-\theta \sum_{i=1}^N J^i \cdot D^i \geq \frac{\lambda}{2\,\theta}\,|P_{\mathbb{1}^\perp}\nabla\mu|^2 - \frac{3\,\underline{\lambda}}{\theta}\,\left|\frac{z}{m}\right|^2\,|\nabla\phi|^2\,.$$

We use the identity $P_{\mathbb{1}^\perp}\mu = \sum_{i=1}^{N-1} q_i\, P_{\mathbb{1}^\perp}\xi^i$. Due to the choice of $\xi^1, \ldots, \xi^{N-1}$, the vectors $P_{\mathbb{1}^\perp}\xi^1, \ldots, P_{\mathbb{1}^\perp}\xi^{N-1}$ are a basis of $\mathbb{1}^\perp$. Thus, there is a constant depending only on the choice of the projector Π such that $|P_{\mathbb{1}^\perp}\nabla\mu|^2 \geq c_\Pi\,|\nabla q|^2$. This entails

$$|\nabla q|^2 \leq c\,(-\theta^2 \sum_{i=1}^N J^i \cdot D^i + |\nabla\phi|^2)\,, \tag{183}$$

proving that $\|\nabla q\|_{L^2(Q)} \leq C_0$. Since $M_\sigma D \cdot D \geq \sigma\, D^2$

$$C_0 \geq -\theta^2 \sum_{i=1}^N \int_Q J^i \cdot D^i \geq \frac{\sigma}{2}\int_Q |\nabla\mu|^2 - 3\,\sigma\,\left|\tfrac{z}{m}\right|\,\|\nabla\phi\|^2_{L^2(Q)}\,,$$

which yields the bound for $\sqrt{\sigma}\,\|\nabla\mu\|_{L^2(Q)}$. Finally

$$\|\mathbb{1}\cdot J\|_{L^2(Q)} = \sigma\,\|\mathbb{1}\cdot D\|_{L^2(Q)} \leq c\,\sqrt{\sigma}\,(\sqrt{\sigma}\,\|\nabla\mu\|_{L^2(Q)} + \sqrt{\sigma}\,\|\nabla\phi\|_{L^2(Q)})\,.$$

Due to the conditions (148), we verify that $|\omega'|^\alpha \leq (1+\Phi_\omega)$ and this directly yields

$$\|\tau\,\omega'(\mu)\|_{L^{\infty,\alpha}(Q)} \leq \tau^{1/\alpha'}\,\|\tau\,\Phi_\omega(\mu)\|_{L^{\infty,1}(Q)} \leq \tau^{1/\alpha'}\,C_0\,.$$

At last we can verify using the growth property of Φ_ω that the function $w = \sqrt{1+|\mu|}$ possesses a distributional gradient in $L^2(Q)$ and is bounded in $L^{\infty,1}(Q)$ via

$$\begin{aligned}
&\|\nabla w\|_{L^2(Q)} \leq \frac{1}{2}\,\|\nabla\mu\|_{L^2(Q)} \leq C_0\,\sigma^{-1/2},\\
&\|w\|_{L^{\infty,1}(Q)} \leq |\Omega| + \|\sqrt{|\mu|}\|_{L^{\infty,1}(Q)} \leq |\Omega| + \|\Phi_\omega(\mu)\|_{L^{\infty,1}(Q)} \leq C_0\,\tau^{-1}\,.
\end{aligned}$$

Thus, $\|w\|_{L^{2,6}(Q)} \leq C_{\sigma,\tau}$. $\square$

In many cases it is possible to increase the regularity of the electrical potential.

Lemma 8.2. *Assumptions of Proposition 8.1. Assume moreover that for almost all $t \in]0,T[$, the electrical potential $\phi \in L^\infty(0,T;\, W^{1,2}(\Omega))$ satisfies*

$$-\epsilon_0\,(1+\chi)\,\triangle\phi(t) = n^F(t) \textit{ in } [W^{1,2}_\Gamma(\Omega)]^*, \quad \phi(t) = \phi_0(t) \textit{ as traces on } \Gamma\,,$$

with $\phi_0 \in L^\infty(Q) \cap L^\infty(0,T;\, W^{1,\beta}(\Omega))$, $\beta = \min\{r(\Omega,\,\Gamma),\, \frac{3\alpha}{(3-\alpha)^+}\}$. Then

$$\begin{aligned} \|\phi\|_{L^\infty(Q)} &\leq \|\phi_0\|_{L^\infty(Q)} + c\,\|\rho\|_{L^{\infty,\alpha}(Q)} \\ \|\phi\|_{L^\infty(0,T;\,W^{1,\beta}(\Omega))} &\leq c\,(\|\phi_0\|_{L^\infty(0,T;\,W^{1,\beta}(\Omega))} + \|\rho\|_{L^{\infty,\alpha}(Q)})\,. \end{aligned} \tag{184}$$

Moreover, if $\beta \geq \alpha'$ we obtain that

$$\|n^F\,\nabla\phi\|_{L^{\infty,\frac{\beta\alpha}{\beta+\alpha}}(Q)} \leq \|n^F\|_{L^{\infty,\alpha}(Q)}\,\|\nabla\phi\|_{L^{\infty,\beta}(Q)}\,. \tag{185}$$

Proof. We only need to recall that $\alpha > 3/2$ and the definition of the exponent $r(\Omega,\Gamma) \geq 2$ (see (81)). The estimates (184) are standard consequences of second order elliptic theory, whereas (185) follows from the Hölder inequality. □

Next we can derive the uniform continuity estimate that results from the mass balance equations.

Proposition 8.3. *Assumptions of Proposition 8.1. If $\bar\rho$ satisfies the global mass balance identity of Definition 4.2, then $[\bar\rho]_{C_{\Phi^*}([0,T])} \leq C_0$.*

Proof. Let $0 \leq t_1 < t_2 \leq T$. Note that by assumption $\bar\rho(t_2)-\bar\rho(t_1) = \int_{t_1}^{t_2}\{\int_\Omega r + \int_\Gamma(\hat r + J^0)\}$. We note that

$$\left|\int_{t_1}^{t_2}\int_\Omega r_i\right| = \left|\int_{t_1}^{t_2}\int_\Omega R\cdot\gamma_i\right| \leq \sup_{i=1,\ldots,N,\,[R]_{L_{\Psi^*}}\leq C_0}\left|\int_{t_1}^{t_2}\int_\Omega R\cdot\gamma_i\right|\,.$$

We argue similarily with the other right-hand side terms, and recalling the definition (97), we obtain that

$$|\bar\rho(t_2) - \bar\rho(t_1)| \leq \bar C_0\,\Phi^*(t_1,\,t_2)\,. \tag{186}$$

□

In the course of the proofs, we shall also need bounds of more technical nature obtained via Hölder and Sobolev inequalities. We denote α the growth exponent of the function h at infinity and $\beta := \min\{r(\Omega,\,\Gamma),\, \frac{3\alpha}{(3-\alpha)^+}\}$ the optimal regularity of the electric field.

Lemma 8.4. *We assume that the bounds in the Propositions 8.1, 8.2 are valid. Then*

$$\begin{aligned} &\|\varrho\,v\|_{L^{2,\frac{6\alpha}{6+\alpha}}(Q)} \leq c\,\|\varrho\|_{L^{\infty,\alpha}(Q)}\,\|v\|_{W^{1,0}_2(Q)} \leq C_0 \\ &\|\varrho\,v\|_{L^{\infty,\frac{2\alpha}{1+\alpha}}(Q)} \leq \|\sqrt{\varrho}\,v\|_{L^{\infty,2}(Q)}\,\|\varrho\|^{1/2}_{L^{\infty,\alpha}(Q)} \leq C_0 \\ &\|\varrho\,v^2\|_{L^{1,\frac{3\alpha}{3+\alpha}}(Q)} \leq c\,\|\varrho\|_{L^{\infty,\alpha}(Q)}\,\|v\|^2_{W^{1,0}_2(Q)} \leq C_0 \\ &\|\varrho\,v^2\|_{L^{\frac{5\alpha-3}{3\alpha}}(Q)} \leq c\,\|\varrho\,v^2\|^{(2\alpha-3)/(3\alpha)}_{L^{\infty,1}(Q)}\,\|\varrho\,v^2\|^{(3+\alpha)/(3\alpha)}_{L^{1,\frac{3\alpha}{3+\alpha}}(Q)} \leq C_0 \\ &\left\|\sum_{i=1}^N J^i\,v\right\|_{L^{2,3/2}(Q)} \leq \left\|\sum_{i=1}^N J^i\right\|_{L^2(Q)}\,\|v\|_{L^{2,6}(Q)} \leq C_0\,\sqrt{\sigma}\,. \end{aligned}$$

Proof. We prove exemplarily only the last statement.

Employing Hölder's inequality $\| \sum_{i=1}^N J^i_\sigma(t)\, v(t)\|_{L^{3/2}(\Omega)} \leq \| \sum_{i=1}^N J^i_\sigma(t)\|_{L^2(\Omega)} \, \|v(t)\|_{L^6(\Omega)}$. $\square$

Further we shall need an improved bound on the pressure. This is also fairly standard, and therefore we give the proof in the appendix.

Lemma 8.5. *Assume that the relations* (161) *and* (162) *are valid.*

- *If* $\alpha > 3$, *then* $\|p\|_{L^{1+1/\alpha}(Q)} \leq C_0$;
- *If* $3/2 < \alpha \leq 3$, $r(\Omega, \Gamma) > \alpha'$ *and* $\mathbb{1} \cdot J \equiv 0$, *then* $\|p\|_{L^{1+\frac{2}{3}-\frac{1}{\alpha}}(Q)} \leq C_0$.

The only piece of information still missing in order to obtain a bound in the natural class is the estimate on the vector q. This is the object of the next section.

8.2 The $L^w_{(\circ_N \ln)}\, L^1(Q)$norm of the relative chemical potentials

In this section we show that a combination between the estimates on the reactions (cf. (181), (182)) and the control on the gradient of the relative potentials $(q_1, \ldots, q_{N-1}) = \Pi\mu$ (cf. (183)) allows a control in time on the L^1-norm of these functions in the sense of the natural class $\mathcal{B}$.

A first essential ingredient of the proof is the global mass balance identity which implies for the vector of total masses $\bar{\rho} := \int_\Omega \rho$ that

$$\bar{\rho}(t) \in \{\bar{\rho}^0\} \oplus \mathrm{span}\{\gamma^1, \ldots, \gamma^s, \, \hat{\gamma}^1, \ldots, \hat{\gamma}^{\hat{s}^\Gamma}\} =: \{\bar{\rho}^0\} \oplus W \quad \text{for all } t \in]0, T[\,. \tag{187}$$

The estimate is not obvious we have to devote an entire section to its proof. We start from a pari $(\varrho, \, q) \in L^{\infty,\alpha}(Q) \times L^w_{(\circ_N \ln)} L^1(Q; \, \mathbb{R}^{N-1})$. We define $\rho := \mathscr{R}(\varrho, \, q)$, and $\mu := \mathcal{E}q$ if q is finite. The estimate method relies on a precise structure of the free energy

$$\partial_i h = c_i + K \, \frac{V_i}{m_i} \, F'(\rho \cdot \tfrac{V}{m}) + k_B \, \theta \, \frac{1}{m_i} \, \ln y_i \, . \tag{188}$$

Note that at every point where $\varrho > 0$, we can resort to the representation

$$\begin{aligned}\mu_i - \mu_k &= \mathcal{E}q \cdot (e^i - e^k) = (\mathcal{E}q + \mathscr{M}(\varrho, \, q) \, \mathbb{1}) \cdot (e^i - e^k) \\ &= c_i - c_k + K \left(\frac{V_i}{m_i} - \frac{V_k}{m_k} \right) F'(\rho \cdot \tfrac{V}{m}) + k_B \, \theta \left(\frac{1}{m_i} \, \ln y_i - \frac{1}{m_k} \, \ln y_k \right) .\end{aligned} \tag{189}$$

We commence stating an obvious estimate that results from the energy identity.

Lemma 8.6. *Define* $W := \mathrm{span}\{\gamma^1, \ldots, \gamma^s, \, \hat{\gamma}^1, \ldots, \hat{\gamma}^{\hat{s}^\Gamma}\}$, *and* $P_W \, : \, \mathbb{R}^N \to W$ *the orthogonal projection on the subspace* W. *There is* C *depending only on* Ω *such that*

$$\|P_W \mu\|_{L^1(Q)} + \|P_W \mu\|_{L^1(]0,T[\times\Gamma)} \leq C \, (1 + \|\nabla q\|_{L^1(Q)} + [D^{R}]_{L_\Psi(Q)} + [\hat{D}^{\Gamma,R}]_{L_{\hat{\Psi}^\Gamma}(S)}) \, .$$

Proof. Consider at first a vector $\gamma^k \in \mathbb{R}^N$, $k \in \{1, \ldots, s\}$ associated with the bulk reactions (see (4))). Obviously $\int_Q |\mu \cdot \gamma^k| \leq \int_Q |D^{\mathsf{R}}| \leq C_0$. By assumption, $\gamma^k \cdot \mathbb{1} = 0$ for all k. This means that there is a constant $c_{W,\Pi}$ depending on W and the choice of the projector Π such that $|\nabla(\gamma^k \cdot \mu)| \leq c_{W,\Pi} \, |\nabla \Pi\mu|$. We also obtain (trace theorem) that

$$\int_{]0,T[\times\Gamma} |\mu \cdot \gamma^k| \leq C \, \|\mu \cdot \gamma^k\|_{W^{1,0}_1(Q)} \, .$$

Analogously for $k \in \{1, \ldots, \hat{s}^\Gamma\}$, we obtain that $\int_{]0,T[\times\Gamma} |\mu \cdot \hat{\gamma}^k| \leq \int_{]0,T[\times\Gamma} |\hat{D}^{\Gamma,\mathsf{R}}| \leq C_0$. We use that $\int_Q |\mu \cdot \hat{\gamma}^k| \leq C \, (\|\nabla(\mu \cdot \hat{\gamma}^k)\|_{L^1(Q)} + \|\mu \cdot \hat{\gamma}^k\|_{L^1(]0,T[\times\Gamma)})$, and the claim follows. $\square$

In general we cannot expect to control the entire vector $\|\Pi\mu\|_{L^1(Q)}$ using Lemma 8.6: In fact W is always a true subset of$\mathbb{1}^\perp$ due to the charge conservation condition $\gamma^k \cdot \frac{z}{m} = 0$. In order to pursue, we need the auxiliary functions

$$d_0(t) := \|\nabla q(t)\|_{L^1(\Omega)}, \quad r_0(t) := \|P_W \mu(t)\|_{L^1(\Omega)} \, . \tag{190}$$

In order to achieve notational simplicity troughout the section, we denote for $A \subseteq \Omega$ measurable its three-dim. Lebesgue measure via $|A| := \lambda_3(A)$.

Lemma 8.7. *Let $i, \, k \in \{1, \ldots, N\}$, $i \neq k$ and $\epsilon, \, \delta > 0$ be arbitrary numbers. For $t \in]0, T[$ define $d_0(t)$ like in* (190)*. Then, there is a disjoint splitting $]0, T[= I_1(i, k, \epsilon, \, \delta) \cup I_2(i, k, \epsilon, \, \delta)$ of the interval $]0, T[$ such that*

$$\begin{cases} \int_\Omega |\mu_i(t) - \mu_k(t)| \leq C^*(\delta) \, (d_0(t) + \epsilon^{-1}) & \text{for } t \in I_1 \\ \begin{cases} |\{x \in \Omega \, : \, \mu_i(t,x) - \mu_k(t,x) < \epsilon^{-1}\}| < \delta \\ \text{or} \\ |\{x \in \Omega \, : \, \mu_i(t,x) - \mu_k(t,x) > -\epsilon^{-1}\}| < \delta \end{cases} & \text{for } t \in I_2 \end{cases}$$

Here, C^* is a continuous, nonincreasing function on $[0, 1]$ depending only on Ω.

Proof. We show that for all $\delta > 0$, there is $c = c(\delta)$ depending only on Ω such that for all $u \in W^{1,1}(\Omega)$

$$\|u\|_{L^1(\Omega)} \leq c(\delta) \, \left(\|\nabla u\|_{L^1(\Omega)} + \max\left\{\int_A |u^+|, \, \int_B |u^-|\right\}\right)$$
$$\text{for all } A, \, B \subset \Omega \text{ such that } \min\{|A|, \, |B|\} \geq \delta \, . \tag{191}$$

Otherwise, there is $\delta_0 > 0$ such that for all $j \in \mathbb{N}$, one finds $u_j \in W^{1,1}(\Omega)$ and $A_j, \, B_j \subset \Omega$, $|A_j|, \, |B_j| \geq \delta_0$ and

$$\|u_j\|_{L^1(\Omega)} \geq j \, \left(\|\nabla u_j\|_{L^1(\Omega)} + \max\left\{\int_{A_j} |u_j^+|, \, \int_{B_j} |u_j^-|\right\}\right) \, .$$

Consider $\bar{u}_j := u_j / \|u_j\|_{L^1(\Omega)}$. Then, $\|\bar{u}_j\|_{W^{1,1}(\Omega)} \leq \|\nabla \bar{u}_j\|_{L^1(\Omega)} + 1 \leq j^{-1} + 1$. Consequently, there is a subsequence (no new labels) and a limiting element $\bar{u} \in L^1(\Omega)$ such that

$\bar{u}_j \to \bar{u}$ strongly in $L^1(\Omega)$. But since $\nabla \bar{u}_j \to 0$ strongly in $L^1(\Omega)$, we easily show that $\bar{u}$ is a constant. Now, we see that also $\bar{u}^+ |A_j| + |\bar{u}^-| |B_j| \to 0$, and obviously $\bar{u} \equiv 0$. Thus $1 = \|\bar{u}_j\|_{L^1(\Omega)} \to 0$, a contradiction.

For $u \in L^1(\Omega)$, we apply (191) with the choices

$$A := \{x \in \Omega \,:\, u(x) < \epsilon^{-1}\},\; B := \{x \in \Omega \,:\, u(x) > -\epsilon^{-1}\}\,.$$

It follows that either $\min\{|A|,\, |B|\} < \delta$ or that

$$\begin{aligned}\|u\|_{L^1(\Omega)} &\leq c(\delta)\,\left(\|\nabla u\|_{L^1(\Omega)} + \max\left\{\int_A |u^+|,\, \int_B |u^-|\right\}\right)\\ &\leq c(\delta)\,(\|\nabla u\|_{L^1(\Omega)} + \frac{1}{\epsilon}\,\max\{|A|,\, |B|\})\,.\end{aligned}$$

We apply the latter inequality to $u = (\mu_i - \mu_k)(t)$, $i,\, k \in \{1, \ldots, N\}$. Note that independently of the choice of Π, there is c_Π such that $\|\nabla(\mu_i(t) - \mu_k(t))\|_{L^1(\Omega)} \leq c_\Pi\, d_0(t)$. The claim follows. □

Under the assumptions (189), (188) we can also translate the result in the following way.

Lemma 8.8. *Recall the definition of the number densities* $n_i := \frac{1}{m_i}\,\rho_i$. *Let* $i,\, k \in \{1, \ldots, N\}$, $i \neq k$ *arbitrary. For all* $\delta,\, \epsilon \geq 0$ *there is a disjoint splitting* $]0, T[= J_1(i, k,\, \epsilon, \delta) \cup J_2(i, k,\, \epsilon, \delta)$ *of the interval* $]0, T[$ *such that*

$$\begin{cases} \int_\Omega |\mu_i(t) - \mu_k(t)| \leq C^*(\delta)\,(d_0(t) + \epsilon^{-\alpha}) & \text{for } t \in J_1\\ \begin{cases} |\{x \in \Omega \,:\, n_i(t, x) \geq \epsilon\}| \leq C_0\,\epsilon^\alpha + \delta \\ \text{or} \\ |\{x \in \Omega \,:\, n_k(t, x) \geq \epsilon\}| < C_0\,\epsilon^\alpha + \delta \end{cases} & \text{for } t \in J_2\,. \end{cases}$$

Proof. We consider $t \in I_2(i, k,\, \gamma,\, \delta)$ for $\gamma,\, \delta > 0$ with I_2 defined by Lemma 8.7. We for example assume that $|\{x \in \Omega \,:\, \mu_i(t, x) - \mu_k(t, x) < \gamma^{-1}\}| < \delta$. The other case of the alternative is completely similar.

We abbreviate $A = A(\gamma) := \{x \in \Omega \,:\, \mu_i(t, x) - \mu_k(t, x) > \gamma^{-1}\}$. For constants $0 < M_1 < M_2 < +\infty$, we introduce also the sets $B_{M_2} := \{x \in \Omega \,:\, a_0 \leq n(t, x) \cdot V \leq M_2\}$ and $B_{M_1} := \{x \in \Omega \,:\, M_1 \leq n(t, x) \cdot V \leq a_0\}$. Here, $a_0 \in \mathbb{R}_+$ denotes the number such that $F'(a_0) = 0$. Moreover we put $C_{M_1} := \{x \in \Omega \,:\, n(t, x) \cdot V \leq M_1\}$ and $C_{M_2} := \{x \in \Omega \,:\, n(t, x) \cdot V \geq M_2\}$. On the set C_{M_1}, we directly obtain that $n_k \leq n \cdot \mathbb{1} \leq \underline{V}^{-1}\, M_1$. Thus we note that

$$C_{M_1} \subseteq \{x \,:\, n_k(t, x) \leq \underline{V}^{-1}\, M_1\}\,. \tag{192}$$

Due to the estimate $\|n \cdot \mathbb{1}\|_{L^{\infty,\alpha}(Q)} \leq C_0$, one moveover has

$$|C_{M_2}| \leq C_0\, M_2^{-\alpha}\,. \tag{193}$$

Due to (189), we see that in the set $A \cap (B_{M_1} \cup B_{M_2})$ we have

$$\gamma^{-1} \leq \mu_i - \mu_k = c_i - c_k + K\,F'(n\cdot V)\,(\tfrac{V_i}{m_i} - \tfrac{V_k}{m_k}) + \ln\left(\frac{y_i^{1/m_i}}{y_k^{1/m_k}}\right)^{k_B\,\theta} ,$$

which in turn implies for $x \in A \cap B_{M_2}$ that

$$\begin{aligned}\left(\frac{y_k^{1/m_k}}{y_i^{1/m_i}}\right)^{k_B\,\theta} &\leq e^{-\gamma^{-1}}\, e^{c_i - c_k}\, e^{K\,F'(M_2)\,(\frac{V_i}{m_i} - \frac{V_k}{m_k})^+} \\ &\leq C_1\, e^{b\,K\,F'(M_2)}\, e^{-\gamma^{-1}} ,\end{aligned}$$

where $b := \sup \frac{V}{m} - \inf \frac{V}{m}$. On the other hand, for $x \in B_{M_1} \cap A$ we obtain that

$$\begin{aligned}\left(\frac{y_k^{1/m_k}}{y_i^{1/m_i}}\right)^{k_B\,\theta} &\leq e^{-\gamma^{-1}}\, e^{c_i - c_k}\, e^{K\,F'(M_1)\,(\frac{V_i}{m_i} - \frac{V_k}{m_k})^-} \\ &\leq C_1\, e^{b\,K\,|F'(M_1)|}\, e^{-\gamma^{-1}} ,\end{aligned}$$

Thus, for $x \in A \cap B_{M_i}$, $i = 1, 2$, we obtain that

$$\begin{aligned} n_k &\leq \tilde{C}_1\,(n\cdot \mathbb{1})\,[e^{b\,K\,|F'(M_i)|}\, e^{-\gamma^{-1}}]^{\frac{m_k}{k_B\theta}} \\ &\leq \tilde{C}_1\, M_2\,[e^{b\,K\,|F'(M_i)|}\, e^{-\gamma^{-1}}]^{\frac{m_k}{k_B\,\theta}} . \end{aligned} \tag{194}$$

For $\epsilon > 0$, consider the set $U = U_{\epsilon,k,t} := \{x \in \Omega \,:\, n_k(t,x) \geq \epsilon\}$. Observe that

$$U \subset [U \cap C_{M_1}] \cup [U \cap C_{M_2}] \cup [U \cap A \cap B_{M_1}] \cup [U \cap A \cap B_{M_2}] \cup [U \setminus A] .$$

For $M_1 = \underline{V}\,\epsilon$, we obviously obtain from (192) that $U \cap C_{M_1} = \emptyset$. Choosing $M_2 := \epsilon^{-1}$ and $\gamma = \gamma(\epsilon)$ so small that

$$\tilde{C}_1\,\epsilon^{-1}\,[e^{b\,K\,\max\{|F'(\underline{V}\,\epsilon)|,\,|F'(\epsilon^{-1})|\}}\, e^{-\gamma^{-1}}]^{\frac{m_k}{k_B\,\theta}} < \epsilon , \tag{195}$$

we obtain from (194) that $U \cap A \cap B_{M_i} = \emptyset$ for $i = 1, 2$, and therefore that $U \subset [U \setminus A] \cup [U \setminus C_{M_2}]$. Since $|\Omega \setminus A| < \delta$ by assumption and $|C_{M_2}| \leq C_0\,\epsilon^{\alpha}$ owing to (193), the latter yields

$$|U| \leq C_0\,\epsilon^{\alpha} + \delta . \tag{196}$$

Thus, we have shown that for all $\epsilon > 0$, if $t \in I_2(i, k,\, \gamma(\epsilon),\, \delta)$ then (196) holds. Define $J_2(i, k,\, \epsilon,\, \delta)$ via

$$\begin{aligned} &J_2 := \{t \in [0,T] \,: \\ &\inf\{|\{x \,:\, n_i(t,x) \geq \epsilon\}|,\, |\{x \,:\, n_k(t,x) \geq \epsilon\}|\} \leq C_0\,\epsilon^{\alpha} + \delta\} . \end{aligned}$$

Then, $t \notin J_2$ implies $t \notin I_2(i, k,\, \gamma(\epsilon),\, \delta)$, and applying the dichotomy of Lemma 8.7, we obtain that $t \in I_1(i, k,\, \gamma(\epsilon),\, \delta)$. This means that $J_1 :=]0, T[\setminus J_2 \subset I_1$ and it follows that

$$\int_{\Omega} |\mu_i(t) - \mu_k(t)| \leq C^*(\delta)\,(d_0(t) + \gamma^{-1}(\epsilon)) \text{ for all } t \in J_1 .$$

We can find $\gamma(\epsilon)$ according to (195) by setting

$$\frac{1}{\gamma(\epsilon)} = \ln \frac{C}{\epsilon^{2\, k_B\, \theta/\overline{m}}\, e^{-b\, K\, \max\{|F'(\underline{V}\, \epsilon)|,\, |F'(\epsilon^{-1})|\}}}\,,$$

with a constant C that depends only on m and V and $b := \sup \frac{V}{m} - \inf \frac{V}{m}$. Thus

$$\begin{aligned}\frac{1}{\gamma(\epsilon)} &\leq C\,(1 + |\ln\epsilon| + K\, \max\{|F'(\underline{V}\, \epsilon)|,\, |F'(\epsilon^{-1})|\})\\ &\leq \tilde{C}\,(1 + |\ln\epsilon| + \epsilon^{-\alpha}) \,\leq \bar{C}\,\epsilon^{-\alpha}\,.\end{aligned}$$

The claim follows. □

The Lemma 8.8 shows that the set J_2 of all times such that the $L^1(\Omega)$ norm of $\mu_i - \mu_k$ might be 'large' must be contained in the set where the number density of at least one specie is 'vanishing' almost entirely. For this set it is possible to obtain estimates using the property (187) for the vector of total masses $\bar{\rho} := \int_\Omega \rho$. In a certain special case, we can show it directly.

Lemma 8.9. *Assume* (187)*, and that $s + \hat{s}^\Gamma = 0$ (no reactions and $M^{\Gamma,2} = 0$). Then, there is a constant C_0 depending on $\mathcal{B}_0$ (cf.* (180)*) and on $\inf_{i=1,\ldots,N} \int_\Omega \rho_i^0$ such that $\|\mu_i - \mu_k\|_{L^1(Q)} \leq C_0$ for all $i,\, k \in \{1,\ldots,N\}$.*

Proof. If $s + \hat{s}^\Gamma = 0$, then $\hat{r} = 0 = J^0$ and the total mass of each constituent is conserved, that is $\int_\Omega \rho_i(t) = \int_\Omega \rho_i^0$ for $i = 1,\ldots,N$ and all $t \in]0,T[$. We thus easily show for all $a_0 < |\Omega|^{-1} \inf_{i=1,\ldots,N} \bar{\rho}_i^0$ and all $t \in]0,T[$ that

$$|\{x \in \Omega \,:\, \rho_i(t) \geq a_0\}| \geq \left[\frac{1}{\|\varrho\|_{L^{\infty,\alpha}(Q)}}\, \left(\inf_{i=1,\ldots,N} \bar{\rho}_i^0 - a_0\,|\Omega|\right)\right]^{\alpha'}\,.$$

In particular, for $a_0 := (2|\Omega|)^{-1} \inf_{i=1,\ldots,N} \bar{\rho}_i^0$, we obtain that

$$|\{x \in \Omega \,:\, \rho_i(t) \geq a_0\}| \geq \left[\frac{1}{2\,\|\varrho\|_{L^{\infty,\alpha}(Q)}\,|\Omega|}\, \inf_{i=1,\ldots,N} \bar{\rho}_i^0\right]^{\alpha'} =: b_0\,. \tag{197}$$

Thus, recalling the dichotomy of the Lemma 8.8, the interval $J_2(i,k,\,\epsilon,\delta)$ is empty for all parameters $\epsilon,\,\delta$ satisfying the conditions

$$\epsilon < a_0, \quad C_0\,\epsilon^\alpha + \delta < b_0\,.$$

Choose for instance $\delta_0 = b_0/2$ and $\epsilon_0 = \min\{a_0/2,\, (b_0/(2C_0))^{1/\alpha}\}$, and we obtain for all $t \in]0,T[$ that $\int_\Omega |\mu_i(t) - \mu_k(t)| \leq C^*(\delta_0)\,(d_0(t) + \epsilon_0^{-\alpha})$. This proves the claim. □

For a system with reactions, we have an equivalent property in the case that the time intervall is sufficiently short

Lemma 8.10. *Assume that* (187) *is valid. Define*

$$T^* := \inf\{t \in [0,T] \,:\, \min_{i=1,\ldots,N} \bar{\rho}_i(t) = 0\}\,. \tag{198}$$

Then, there is a time $T_0 > 0$ *depending on* $\mathcal{B}_0$ *(cf.* (180)*) and on* $\inf_{i=1,\ldots,N} \bar{\rho}_i^0$ *such that* $T^* \geq T_0$*, and* $\|\mu_i - \mu_k\|_{L^1(Q_t)} \leq C_{0,t}$ *for all* $i,\, k \in \{1,\ldots,N\}$ *and* $t < T^*$*.*

Proof. We recall (186), and we see that

$$|\bar{\rho}(t) - \bar{\rho}^0| \leq \tilde{C}_0\, \Phi^*(t,0) \text{ for all } t \in [0,T]\,. \tag{199}$$

Thus, if T_0 is such that $\inf_{i=1,\ldots,N} \bar{\rho}_i^0 - \tilde{C}_0\, \Phi^*(T_0,0) \geq c_0 > 0$, we obtain that $\inf_{i=1,\ldots,N} \bar{\rho}_i(t) > 0$ for all $t \in [0,T^*]$ and we can conclude as in Lemma 8.9. □

Our next purpose is to prove an equivalent of Lemma 8.9 in the case that $s + \hat{s}^\Gamma > 0$ and this globally in time. Our idea relies on the concept of a *species selection*. A selection S of cardinality $|S| \in \{1,\ldots,N\}$ is defined via

$$S := \{i_1,\ldots,\, i_{|S|}\} \text{ with } i_j \in \{1,\ldots,N\} \text{ for } j = 1,\ldots,|S| \text{ and } i_1 < \ldots < i_{|S|}\,.$$

For a selection $S = \{i_1,\ldots,\, i_{|S|}\}$, we dentote $P_S \,:\, \mathbb{R}^N \to \mathbb{R}^N$ the projection $(P_S\xi)_i = \xi_i$ for $i \in S$ and $(P_S\xi)_i = 0$ otherwise. The orthogonal selection is defined via $S^\perp = \{1,\ldots,N\} \setminus S$. We call a selection S *linearly independent* (with respect to the vectors $\gamma^1,\ldots,\gamma^s$ and $\hat{\gamma}^1,\ldots,\hat{\gamma}^{\hat{s}^\Gamma}$) if the condition

$$W_S := \operatorname{span}\{P_S(\gamma^1),\,\ldots,P_S(\gamma^s),\; P_S(\hat{\gamma}^1),\ldots P_S(\hat{\gamma}^{\hat{s}^\Gamma})\} = P_S(\mathbb{R}^N)$$

is satisfied, and linearly dependent if $\dim(W_S) \leq |S| - 1$. We recall at this point also the definition (80) of the manifold $\mathcal{M}_{\text{crit}}$.

Lemma 8.11. *Assume that the vector* $\bar{\rho}^0 := \int_\Omega \rho^0 \in \mathbb{R}^N_+$ *of total initial masses does not belong to* $\mathcal{M}_{\text{crit}}$*. Then, there are* $a_0,\, b_0 > 0$ *depending on* $\operatorname{dist}(\bar{\rho}^0,\, \mathcal{M}_{\text{crit}})$ *and on* $\mathcal{B}_0$ *such that for all* linearly dependent *selections* S*, and all* $a < a_0$ *and* $b < b_0$

$$\{t \in]0,T[\,:\, |\{x \in \Omega \,:\, n_i(t,x) \geq a\}| < b \;\textit{ for all } i \in S\} = \emptyset\,. \tag{200}$$

Proof. Due to (187), we obtain that $P_S(\bar{\rho}(t)) \in \{P_S(\bar{\rho}^0)\} \oplus W_S$. If S is a linearly dependent selection, then by the definition of the critical manifold

$$|P_S(\bar{\rho}(t))| \geq \operatorname{dist}(\bar{\rho}^0,\, \mathcal{M}_{\text{crit}})\,.$$

Thus, for all $t \in]0,T[$, there is at least one indice $i_1 = i_1(t) \in S$ such that $\bar{\rho}_{i_1}(t) \geq \frac{\operatorname{dist}(\bar{\rho}^0,\, \mathcal{M}_{\text{crit}})}{|S|^{1/2}}$. As in the proof of (197), we see that

$$|\{x \in \Omega \,:\, \rho_{i_1}(t,x) \geq a_0\}| \geq \left[\frac{1}{\|\varrho\|_{L^{\infty,\alpha}(Q)}} \left(\frac{\operatorname{dist}(\bar{\rho}^0,\, \mathcal{M}_{\text{crit}})}{|S|^{1/2}} - a_0\,|\Omega|\right)\right]^{\alpha'}\,.$$

Thus, in particular for $a_0 := \operatorname{dist}(\bar{\rho}^0,\, \mathcal{M}_{\text{crit}})/(2\,N^{1/2}\,|\Omega|)$,

$$|\{x \in \Omega \,:\, \rho_{i_1}(t,x) \geq a_0\}| \geq \left[\frac{\operatorname{dist}(\bar{\rho}^0,\, \mathcal{M}_{\text{crit}})}{2\,\|\varrho\|_{L^{\infty,\alpha}(Q)}\,|\Omega|}\right]^{\alpha'} =: b_0\,.$$

The claim follows. □

Thus, few restrictions on the vector of the initial global masses (note: the critical manifold as at most dimension $N-1$) are sufficient to garanty that the indices of a linearly dependent selection cannot correspond to global masses that vanish at the same time. In the remaining, most technical part of the estimate, we are going to show that this is also the case for linearly independent selections. We aforemention that owing to the conservation of the total mass $\int_\Omega \varrho$, we can introduce

$$\bar{a}_0 := \frac{1}{2|\Omega|} \int_\Omega \varrho_0, \quad \bar{b}_0 = \left(\frac{|\Omega|}{2\|\varrho\|_{L^{\infty,\alpha}(Q)}} \int_\Omega \varrho_0\right)^{\alpha'} \tag{201}$$

and show that the set $A_0(t) := \{x \in \Omega \, : \, \varrho(t,x) \geq \bar{a}_0\}$ satisfies $|A_0(t)| \geq \bar{b}_0$ for all $t \in]0,T[$. Note that in the set $|A_0(t)|$ it is always possible to introduce the entire vector of chemical potentials $\mu = \mathcal{E}q + \mathscr{M}(\varrho, \, q)\, \mathbb{1}$, and the identity $\mu = \nabla h(\rho)$ is valid at $\rho = \mathscr{R}(\varrho, \, q)$. We commence with two auxiliary statements.

Lemma 8.12. *Define $\bar{a}_0, \, \bar{b}_0 > 0$ like in* (201)*. Then there is for $t \in [0,T]$ arbitrary a set $E = E(t) \subseteq A_0(t)$, $|E| \geq 4^{-1}\,\bar{b}_0$ such that for all $i \in \{1,\ldots,N\}$ and all $0 \leq \epsilon < \min\{\frac{\bar{a}_0}{N\bar{m}}, \, \left(\frac{\bar{b}_0}{4NC_0}\right)^{1/\alpha}\}$ and $\delta < \bar{b}_0/(2N^2)$, the following alternative is valid:*

1. *Either $|\{x \in \Omega \, : \, n_i(t,x) \geq \epsilon\}| \leq C_0\, \epsilon^\alpha + \delta$;*
2. *Or $\int_E |\mu_i(t)| \leq c\,(1 + C^*(\delta)\,(d_0(t) + \epsilon^{-\alpha}))$.*

where c depends only on the parameters appearing in the definition of the chemical potentials, on $\bar{a}_0$ and $\bar{b}_0$, and C^* is as in Lemma 8.7.

Proof. Let $t \in]0,T[$. We define $\mathscr{I} = \mathscr{I}(t,\epsilon,\delta) \subseteq \{1,\ldots,N\}$ as the set of indices i such that the condition $|\{x \, : \, n_i(t,x) \geq \epsilon\}| \leq C_0\, \epsilon^\alpha + \delta$ is satisfied. For $i \in \mathscr{I}$, the first member (1) always applies. Note that $\mathscr{I}(t,\epsilon,\delta) \subseteq \mathscr{I}(t,\epsilon_0,\delta_0)$ if $\epsilon \leq \epsilon_0$ and $\delta \leq \delta_0$.

Consider $i \notin \mathscr{I}$ fixed. Then, we obtain due to Lemma 8.8 for all $j \in \{1,\ldots,N\}$ the alternative

1. Either $\int_\Omega |\mu_i(t) - \mu_j(t)| \leq C^*(\delta)\,(d_0(t) + \epsilon^{-\alpha})$;
2. Or $|\{x \in \Omega \, : \, n_j(t,x) \geq \epsilon\}| \leq C_0\, \epsilon^\alpha + \delta$.

We call $\mathscr{J} = \mathscr{J}(t, \, i, \, \epsilon, \delta) \subseteq \{1,\ldots,N\}$ the subset such that the first condition in this new alternative is satisfied. Thus

$$\int_\Omega |\mu_i(t) - \mu_j(t)| \leq C^*(\delta)\,(d_0(t) + \epsilon^{-\alpha}) \text{ for all } j \in \mathscr{J} \, . \tag{202}$$

Due to the nonincreasing character of the latter estimate in $\epsilon, \, \delta$, we note that

$$\mathscr{J}(t, \, i, \, \epsilon_0, \, \delta_0) \subseteq \mathscr{J}(t, \, i, \, \epsilon, \, \delta) \text{ if } \epsilon \leq \epsilon_0 \text{ and } \delta \leq \delta_0 \, . \tag{203}$$

We can decompose Ω into disjoint sets $B_1, \ldots, B_N$ such that

$$n_j(t, x) = \sup_{\ell=1,\ldots,N} n_\ell(t, x) \text{ for } x \in B_j\,.$$

Call $A_0(t) := \{x \in \Omega \,:\, \varrho(t,x) \geq \bar{a}_0\}$ like in (201). Note that on $B_j \cap A_0(t)$, the estimates

$$\mu_j^+ \leq c_j + \frac{KV_j}{m_j}\,[F']^+ \leq c\,(1 + (n\cdot \mathbb{1})^{\alpha-1})$$
$$\mu_j^- \geq c_j + \frac{KV_j}{m_j}\,[F'(\bar{a}_0\, \inf \tfrac{V}{m})]^- - \frac{k_B\,\theta}{m_j}\,\ln\frac{1}{N}$$

ensure that $|\mu_j| \leq \sup c\,(1 + (n\cdot \mathbb{1})^{\alpha-1} + |[F'(\bar{a}_0\, \inf \frac{V}{m})]^-|$. Therefore

$$\int_{B_j\cap A_0} |\mu_j| \leq C_0 + \bar{C}(\bar{a}_0)\,|B_j \cap A_0|\,. \tag{204}$$

For $i \neq j$ arbitrary, we observe that

$$\int_{B_j\cap A_0} |\mu_i| \leq \int_{B_j\cap A_0} |\mu_j| + \int_{B_j\cap A_0} |\mu_i - \mu_j| \text{ for } j = 1,\ldots,N\,.$$

If $j \in \mathscr{J}(t,\, i,\, \epsilon, \delta)$, the inequality (204) joined to the fact (202) implies that

$$\begin{aligned}\int_{B_j\cap A_0} |\mu_i| &\leq C_0 + \bar{C}(\bar{a}_0)\,|B_j \cap A_0| + \int_\Omega |\mu_i - \mu_j|\\ &\leq C(\bar{a}_0) + C^*(\delta)\,(d_0(t) + \epsilon^{-\alpha})\,,\end{aligned} \tag{205}$$

We define $E(t,i,\epsilon,\delta) := \bigcup_{j\in\mathscr{J}(t,i,\epsilon,\delta)} B_j \cap A_0$, and we obtain as a consequence of (205) that

$$\int_{E(t,i,\epsilon,\delta)} |\mu_i| \leq N\,(C(\bar{a}_0) + C^*(\delta)\,(d_0(t) + \epsilon^{-\alpha}))\,.$$

Further, the property (203) yields $E(t,i,\epsilon_0,\delta_0) \subset E(t,i,\epsilon,\delta)$.

Consider now $j \in \{1,\ldots,N\} \setminus \mathscr{J}$. Then by definition, $|\{x \,:\, n_j(t,x) \geq \epsilon\}| \leq C_0\,\epsilon^\alpha + \delta$. Recall also that $\sup_{k=1,\ldots,N} n_k \geq \bar{a}_0/(N\overline{m})$ on A_0. Therefore, the set inclusion $B_j \cap A_0 \subset \{x \,:\, n_j \geq \bar{a}_0/(N\overline{m})\}$ is valid. Thus, if $j \notin \mathscr{J}$ and if $\epsilon \leq \bar{a}_0/(N\overline{m})$ then $|B_j \cap A_0| \leq C_0\,\epsilon^\alpha + \delta$. Thus, for all $\epsilon \leq \bar{a}_0/(N\overline{m})$

$$|A_0 \setminus E(t,i,\epsilon,\delta)| \leq \sum_{j\notin\mathscr{J}} |A_0 \cap B_j| \leq N\,(C_0\,\epsilon^\alpha + \delta)\,,$$

We now introduce a set E independent of i, ϵ and δ via

$$E = E(t) = \bigcap_{i\in\{1,\ldots,N\}\setminus\mathscr{J}} E(t,\, i,\, \epsilon_0,\, \delta_0)\,,$$
$$\epsilon_0 := \min\left\{\frac{\bar{a}_0}{N\overline{m}},\, \left(\frac{\bar{b}_0}{4NC_0}\right)^{1/\alpha}\right\}\,, \qquad \delta_0 := \frac{\bar{b}_0}{2N^2}\,.$$

Then

$$|E| \geq |A_0| - \sum_{i\in\{1,\ldots,N\}\setminus\mathscr{I}} |A_0 \setminus E(t,i,\epsilon_0,\, \delta_0)|$$
$$\geq |A_0| - N\, C_0\, \epsilon_0^\alpha - N^2\, \delta_0 \geq \frac{\bar{b}_0}{4}\,.$$

Since $E \subseteq E(t,i,\epsilon_0,\, \delta_0) \subseteq E(t,i,\epsilon,\, \delta)$ for all ϵ and δ

$$\int_E |\mu_i| \leq \int_{E(t,i,\epsilon,\delta)} |\mu_i| \leq N\,(C(\bar{a}_0) + C^*(\delta)\,(d_0(t) + \epsilon^{-\alpha}))\,.$$

□

We need a second auxiliary statement.

Lemma 8.13. *Let $i \in \{1,\ldots,N\}$. Then there are $C_1 > 0$ and $\delta_1 > 0$ depending only on the parameters appearing in the definition of the chemical potentials such that for all $0 < \delta < \delta_1$, there is $N_{i,\delta} \subset \Omega$, such that $|N_{i,\delta}| \leq C_0\,[\ln\frac{1}{\delta}]^{-\alpha'}$ and such that*

$$\{x \in \Omega\,:\, n_i(t,x) \leq \delta\} \subset \{x \in \Omega\,:\, |\mu_i(t,x)| \geq -C_1\, \ln\delta\} \cup N_{i,\delta}\,.$$

Proof. Consider $x \in \Omega$ such that $n_i(x) \leq \delta$. We distiguish two cases.

First case $n(x)\cdot\mathbb{1} \geq 1$.

We introduce for $i = 1,\ldots,N$ the function G_i such that $\mu_i = G_i + (k_B\,\theta/m_i)\,\ln n_i$, that is,

$$G_i := c_i + \frac{KV_i}{m_i}\,F'(n\cdot V) - \frac{k_B\,\theta}{V_i}\,\ln(n\cdot\mathbb{1})\,.$$

We easily show that for $n\cdot\mathbb{1} \geq 1$ that $G_i \leq \bar{c}\,(1 + |n\cdot\mathbb{1}|^{\alpha-1})$. If $G_i \leq -\frac{k_B\,\theta}{2m_i}\,\ln\delta$, then $\mu_i \leq \frac{k_B\,\theta}{2m_i}\,\ln\delta < 0$, and therefore $|\mu_i| \geq -\frac{k_B\,\theta}{2m_i}\,\ln\delta$. On the other hand, if $G_i(x) \geq -\frac{k_B\,\theta}{2m_i}\,\ln\delta$, then

$$-\frac{k_B\,\theta}{2m_i}\,\ln\delta \leq \bar{c}\,(1 + (n\cdot\mathbb{1}))^{\alpha-1})\,.$$

Thus, if δ is such that $-\frac{k_B\,\theta}{2m_i}\,\ln\delta > 4\,\bar{c}$, we see that x belongs to the set $N_\delta := \{x\,:\, (n\cdot\mathbb{1}))^{\alpha-1} \geq -\frac{k_B\,\theta}{4m_i}\,\ln\delta\}$. The estimate

$$|N_\delta| \leq \|n\cdot\mathbb{1}\|^{\alpha-1}_{L^{\infty,\alpha}(Q)}\,[\ln\frac{1}{\delta^{\frac{k_B\,\theta}{4m_i}}}]^{-\alpha'}\,,$$

is valid.

Second case $n(x)\cdot\mathbb{1} < 1$.

If $n(x)\cdot\mathbb{1} \leq \sqrt{\delta}$, then $n(x)\cdot V \leq \overline{V}\,\sqrt{\delta}$, and we can use the logarithmic growth of F' to see that that $\mu_i \leq c_i + K\,V_i\,m_i^{-1}\,\ln(\overline{V}\,\sqrt{\delta})$.

If otherwise $n(x)\cdot\mathbb{1} > \sqrt{\delta}$, then $y_i(x) \leq \sqrt{\delta}$, and $\mu_i \leq c_i + K\,V_i\,m_i^{-1}\,F'(\overline{V}) + k_B\,\theta\,m_i^{-1}\,\ln\sqrt{\delta}$.

Thus in both cases $\mu_i \leq C_i\,(1 + \ln\delta)$, and the claim follows. □

Next we are going to use the structure of the reaction terms. Recall that we denote $P_W(\mu)$ the projection of the vector μ and that the inequality of Lemma 8.6 ensures a control of the quantity $\|P_W(\mu)\|_{L^1(Q)}$. We denote $r_0(t) := \|P_W(\mu(t))\|_{L^1(\Omega)}$.

Lemma 8.14. *There are*

1 *Numbers* $\epsilon_0,\, \delta_0 > 0$ *depending only on* $\bar{a}_0,\, \bar{b}_0$ *(cf.* (201)*);*

2 *A continuous, nonnegative and nondecreasing function* h^* *defined on* $[0,\, \delta_0]$*, such that* $h^*(0) = 0$ *and* $h^*(\delta) \geq \delta$ *for all* δ*;*

with the following property: If S *is a linearly independent selection and* $t \in]0, T[$*,* $\epsilon \in]0,\, \epsilon_0]$ *and* $\delta \in]0,\, \delta_0]$ *are such that*

$$\begin{cases} |\{x \in \Omega \,:\, n_i(t,x) \geq \epsilon\}| \leq C_0\, \epsilon^\alpha + \delta & \textit{for all } i \in S \\ r_0(t) \leq \frac{C_1\, \bar{b}_0}{16}\, |\ln \epsilon|, \quad d_0(t) \leq |\ln \epsilon| \end{cases}$$

then, there is a selection $S' \supset S$*,* $|S'| = |S| + 1$ *such that*

$$|\{x \in \Omega \,:\, n_i(t,x) \geq |\ln \epsilon|^{-1/\alpha}\}| \leq C_0\, (|\ln \epsilon|^{-1} + h^*(\delta)) \qquad \textit{for all } i \in S' \,.$$

Proof. Let S be linearly independent. There are for $i \in S$ coefficients $\lambda_1, \ldots, \lambda_s$ and $\hat{\lambda}_1, \ldots, \hat{\lambda}_{\hat{s}^\Gamma}$ in $\mathbb{R}$ depending only on the vectors γ and $\hat{\gamma}$ such that the $i-$th standard basis vector of $\mathbb{R}^N$ has the representation

$$\begin{aligned} e^i &= \sum_{\ell=1}^{s} \lambda_\ell\, P_S(\gamma^\ell) + \sum_{\ell=1}^{\hat{s}^\Gamma} \hat{\lambda}_\ell\, P_S(\hat{\gamma}^\ell) \\ &= \sum_{\ell=1}^{s} \lambda_\ell\, \gamma^\ell + \sum_{\ell=1}^{\hat{s}^\Gamma} \hat{\lambda}_\ell\, \hat{\gamma}^\ell + \underbrace{\sum_{\ell=1}^{s} \lambda_\ell\, (I - P_S)(\gamma^\ell) + \sum_{\ell=1}^{\hat{s}^\Gamma} \hat{\lambda}_\ell\, (I - P_S)(\hat{\gamma}^\ell)}_{\in \text{Image}\, P_{S^\perp}} \\ &\in W \oplus \text{Image}\, P_{S^\perp} \,. \end{aligned}$$

Thus, for all $i \in S$ there is a constant $C_{i,S}$ such that

$$|\mu_i| - C_{i,S}\, |P_{S^\perp}(\mu)| \leq |P_W(\mu)| \,. \tag{206}$$

Introducing $\bar{C} := \sup_{S \text{ linearly independent},\, i \in S} C_{i,S}$ it follows that

$$|P_{S^\perp}(\mu)| \geq \frac{1}{\bar{C}}\, (|\mu_i| - |P_W(\mu)|) \,. \tag{207}$$

Assume now that $|\{x \,:\, n_i(t,x) \geq \epsilon\}| \leq C_0\, \epsilon^\alpha + \delta$ for all $i \in S$. Owing to Lemma 8.13, we then see for all $i \in S$ that

$$\begin{aligned} &|\{x \in \Omega \,:\, |\mu_i(t,x)| \geq -C_1\, \ln \epsilon\}| \\ &\quad \geq |\{x \in \Omega \,:\, n_i(t,x) \leq \epsilon\}| - C_0\, [\ln \frac{1}{\epsilon}]^{-\alpha'} \\ &\quad \geq |\Omega| - \delta - C_0\, \epsilon^\alpha - C_0\, [\ln \frac{1}{\epsilon}]^{-\alpha'} \,. \end{aligned}$$

Define E as in the Lemma 8.12. Then, we can ensure that the intersection $F := E \cap \{x \in \Omega \, : \, |\mu_i(t,x)| \geq -C_1 \, \ln \epsilon\}$ satisfies

$$
\begin{aligned}
|F| &\geq |E| - \delta - C_0\, \epsilon^\alpha - C_0\, [\ln \frac{1}{\epsilon}]^{-\alpha'} \\
&\geq \frac{\bar{b}_0}{4} - \delta - C_0\, \epsilon^\alpha - C_0\, [\ln \frac{1}{\epsilon}]^{-\alpha'}\, .
\end{aligned}
$$

If we now choose

$$
\delta \leq \frac{\bar{b}_0}{8}, \quad \epsilon \leq \min\left\{ \left(\frac{\bar{b}_0}{8C_0}\right)^{1/\alpha}, \left[e^{-\left(\frac{8}{\bar{b}_0}\right)^{1/\alpha'}} \right] \right\} ,
$$

we obtain that $|F| \geq \bar{b}_0/8$. We integrate (207) over F. This yields

$$
\begin{aligned}
\int_F |P_S^\perp(\mu)| &\geq \frac{1}{\bar{C}} \left(|F|\, [C_1\, |\ln \epsilon|] - \int_F |P_W(\mu)| \right) \\
&\geq \frac{1}{\bar{C}} \left(\frac{\bar{b}_0}{8}\, C_1\, |\ln \epsilon| - r_0(t) \right) .
\end{aligned}
$$

Thus, there must exist $j_0 = j_0(t) \in S^\perp$ such that

$$
\int_E |\mu_{j_0}| \geq \int_F |\mu_{j_0}| \geq \frac{1}{N\, \bar{C}}\, (\frac{\bar{b}_0}{8}\, C_1\, |\ln \epsilon| - r_0(t))
$$

Therefore, we obtain again from the alternative of Lemma 8.12 that for all τ, η such that

$$
\begin{cases}
0 \leq \eta < \min\left\{ \frac{\bar{a}_0}{N\bar{m}}, \left(\frac{\bar{b}_0}{4NC_0}\right)^{1/\alpha} \right\}, & \tau < \frac{\bar{b}_0}{2N^2} \\
c\, (1 + C^*(\tau)\, (d_0(t) + \eta^{-\alpha})) < \frac{1}{N\, \bar{C}}\, (\frac{\bar{b}_0}{8}\, C_1\, |\ln \epsilon| - r_0(t))
\end{cases}
\tag{208}
$$

it must follow that $|\{x \, : \, n_{j_0}(t,x) \geq \eta\}| \leq C_0\, \eta^\alpha + \tau$.

We next compute good choices of τ and η as functions of ϵ and δ. Define two functions

$$
\tau(\delta) := \min\{y \geq 0 \, : \, C^*(y) \leq \delta\}, \quad \eta(\epsilon) := |\ln \epsilon|^{-1/\alpha} . \tag{209}
$$

Then

$$
c\, (1 + C^*(\tau(\delta))\, (d_0(t) + \eta(\epsilon)^{-\alpha})) \leq c\, (1 + \delta\, d_0(t) + \delta\, |\ln \epsilon|) .
$$

Thus, if t is such that $d_0(t) \leq |\ln \epsilon|$ and $r_0(t) \leq \frac{\bar{b}_0}{16}\, C_1\, |\ln \epsilon|$, (209) imply that

$$
\begin{aligned}
c\, (1 + C^*(\tau(\delta))\, (d_0(t) + \eta(\epsilon)^{-\alpha})) &\leq c\, (1 + 2\, \delta\, |\ln \epsilon|) \\
\frac{\bar{b}_0}{8}\, C_1\, |\ln \epsilon| - r_0(t) &\geq \frac{\bar{b}_0}{16}\, C_1\, |\ln \epsilon| .
\end{aligned}
$$

The second of the conditions (208) is satisfied for all $2\, c\, \bar{C}\, \delta < \frac{\bar{b}_0}{N\, 16}\, C_1$ and all $\epsilon \leq e^{-\frac{64\, c\, \bar{C}}{b_0\, C_1}}$.

Define finally $h^*(\delta) := \max\{\delta, \, \tau(\delta)\}$. Set $S' = S \cup \{j_0\}$ and the claim follows. $\square$

We are now going to iterate the result of Lemma 8.14. Consider the auxiliary function

$$g^*(\epsilon) := \left(\frac{1}{(\circ_{N-1}\ln)(\epsilon^{-1})}\right)^{1/\alpha} . \tag{210}$$

Corollary 8.15. *There are*

1 *Numbers $\bar\epsilon_0,\ \bar\delta_0 > 0$ depending only on $\bar a_0,\ \bar b_0$;*

2 *A continuous, nonnegative and nondecreasing function $\bar h^*$ defined on $[0,\ \bar\delta_0]$, $\bar h^*(0) = 0$;*

with the following property: If S is any linearly independent selection and $t \in]0, T[$, $\epsilon \in]0,\ \bar\epsilon_0]$ and $\delta \in]0,\ \bar\delta_0]$ are such that

$$\begin{cases} |\{x \in \Omega\ :\ n_i(t,x) \geq \epsilon\}| \leq C_0\,\epsilon^\alpha + \delta & \text{for all } i \in S \\ r_0(t) \leq \frac{C_1\,\bar b_0}{16}\,|\ln g^*(\epsilon)|, \quad d_0(t) \leq |\ln g^*(\epsilon)| \end{cases}$$

then, there is a linearly dependent selection $S' \supset S$ such that

$$|\{x \in \Omega\ :\ n_i(t,x) \geq g^*(\epsilon)\}| \leq C_0\,((g^*(\epsilon))^\alpha + \bar h^*(\delta)) \quad \text{for all } i \in S' .$$

Proof. Taking the numbers $\delta_0,\ \epsilon_0$ and the function h^* from Lemma 8.14, we denote $\bar h^* := \circ_{N-1} h^* := \underbrace{h^* \circ \ldots \circ h^*}_{\times N-1}$, and we choose

$$\bar\epsilon_0 := (\circ_{N-1} e)\left(-\frac{1}{\epsilon_0^\alpha}\right) \quad \bar\delta_0 := \sup\{y \geq 0\ :\ \bar h^*(y) \leq \delta_0\} . \tag{211}$$

If S is any linearly independent selection and $t \in]0, T[$, $\epsilon \in]0,\ \epsilon_0]$ and $\delta \in]0,\ \delta_0]$ are such that

$$\begin{cases} |\{x \in \Omega\ :\ n_i(t,x) \geq \epsilon\}| \leq C_0\,\epsilon^\alpha + \delta & \text{for all } i \in S \\ r_0(t) \leq \frac{C_1\,\bar b_0}{16}\,|\ln \epsilon|, \quad d_0(t) \leq |\ln \epsilon| \end{cases}$$

then, due to Lemma 8.14, there is a selection $S' \supset S$, $|S'| = |S| + 1$ such that

$$|\{x \in \Omega\ :\ n_i(t,x) \geq \epsilon_1\}| \leq C_0\,((\epsilon_1)^\alpha + \delta_1) \quad \text{for all } i \in S' ,$$

with $\epsilon_1 = |\ln \epsilon|^{-1/\alpha}$ and $\delta_1 = h^*(\delta)$. Our assumptions (see (211)) are suited in such a way that $\epsilon_1 < \epsilon_0$ and $\delta_1 \leq \delta_0$, and that moreover

$$r_0(t) \leq \frac{C_1\,\bar b_0}{16}\,|\ln \epsilon_1|, \quad d_0(t) \leq |\ln \epsilon_1| .$$

Thus, we can apply to S' Lemma 8.14 again, and we obtain the existence of a selection $S'' \supset S$, $|S''| = |S| + 2$ such that

$$|\{x \in \Omega\ :\ n_i(t,x) \geq \epsilon_2\}| \leq C_0\,((\epsilon_2)^\alpha + \delta_2) \quad \text{for all } i \in S'' ,$$

with $\epsilon_2 = |\ln \epsilon_1|^{-1/\alpha} = \alpha^{1/\alpha}\,(\ln|\ln \epsilon|)^{-1/\alpha}$ and $\delta_2 = h^*(\delta_1) = (h^* \circ h^*)(\delta)$. After at most $N - 1$ steps, we attain the (linearly dependent) selection of cardinality N. □

Now, we can combine these estimates with the Lemma 8.11.

Corollary 8.16. *Assumptions of Lemma 8.11. Then, for all* $0 < \epsilon < \bar{\epsilon}_0$, $\delta \leq \bar{\delta}_0$ *and* $i \in \{1, \ldots, N\}$

$$\lambda_1(\{t \in]0, T[: \, |\{x \in \Omega \, : \, n_i(t, x) \geq \epsilon\}| \leq C_0 \, \epsilon^\alpha + \delta\}) \leq C_0 \, \frac{1}{(\circ_N \ln)(\epsilon^{-1})} \, .$$

Proof. Consider $I_M := \{t \in]0, T[: \; d_0(t) \geq |\ln g^*(\epsilon)| \, , r_0(t) \geq \frac{C_1 \, \bar{b}_0}{16} \, |\ln g^*(\epsilon)|\}$. Then, since both $d_0(t)$ and $r_0(t)$ are bounded in $L^1(0, T)$, we can show that $|I_M| \leq C_0 \, |\ln g^*(\epsilon)|^{-1}$. Let $t \in]0, T[\setminus I_M$, and $i \in \{1, \ldots, N\}$ be such that $|\{x \, : \, n_i(t, \, x) \geq \epsilon\}| < C_0 \, \epsilon^\alpha + \delta$. Then, due to Corollary 8.15, there is a linearly dependent selection $\tilde{S} \supset \{i\}$ such that

$$|\{x \, : \, n_i(t, x) \geq g^*(\epsilon)\}| < C_0 \, ((g^*(\epsilon))^\alpha + \bar{h}^*(\delta)) \text{ for all } j \in \tilde{S} \, .$$

Applying the Lemma 8.11, we obtain that if $g^*(\epsilon) \leq a_0$ and $C_0 \, ((g^*(\epsilon))^\alpha + \bar{h}^*(\delta)) \leq b_0$, it must follow

$$\{t \in]0, T[\setminus I_M \, : \, |\{x \, : \, n_i(t, \, x) \geq \epsilon\}| < C_0 \, \epsilon^\alpha + \delta\} = \emptyset \, .$$

Thus, for all $i = 1, \ldots, N$

$$\lambda_1(\{t \in]0, T[: \, |\{x \, : \, n_i(t, x) \geq \epsilon\}| < C_0 \, \epsilon^\alpha + \delta\} \leq \lambda_1(I_M) \, .$$

$\square$

We can conclude to the main estimate.

Proposition 8.17. *Assume that the free energy function* h *possesses the explicit form* (188)*. Assume that the vector of total masses* $\bar{\rho}$ *is subject to* (187)*. Then*

(1) Define T^* *as in* (198)*. Then* $T^* \geq T_0 > 0$ *with* T_0 *depending only on the data, and for all* $t \in [0, \, T^*[$, $\|\Pi\mu\|_{L^1(Q_t; \mathbb{R}^{N-1})} \leq C_0(t)$*;*

(2) If $s + \hat{s}^\Gamma = 0$*, there is a constant* C_0 *such that* $\|\Pi\mu\|_{L^1(Q; \mathbb{R}^{N-1})} \leq C_0$*;*

(3) Assume that the vector of total initial masses $\bar{\rho}^0$ *does not belong to the manifold* $\mathcal{M}_{crit}$ *(cp.* (80)*). Then, there is a constant* C_0 *such that* $[\Pi\mu]_{L^w_{(\circ_N \ln)} L^1(Q; \mathbb{R}^{N-1})} \leq C_0$*.*

Here C_0 *depends only on* $\|\nabla\Pi\mu\|_{L^1(Q)}$ *on* $\|P_W\mu\|_{L^1(Q \cup S_T)}$*, on* $\inf_{i=1,\ldots N} \bar{\rho}^0_i$*, and in the last case also on* $\operatorname{dist}(\bar{\rho}^0, \, \mathcal{M}_{crit})$*.*

Proof. See the Lemmas 8.9 and 8.10 for the two first cases. Otherwise, we apply the Lemma 8.8 together with the consideration of Corollary 8.16. We see that the set $J_2(i, k, \, \epsilon, \, \delta)$ satisfies for $\delta < \bar{\delta}_0$ and $\epsilon < \bar{\epsilon}_0$ the inequality $\lambda_1(J_2(i, k, \, \epsilon, \, \delta)) \leq C_0 \, \frac{1}{|\ln g^*(\epsilon)|}$. Thus also for $J_2(\epsilon, \, \delta) :=$

$\bigcup_{i,k=1}^N J_2(i,k,\,\epsilon,\,\delta)$, we have $\lambda_1(J_2(\epsilon,\,\delta)) \leq C_0\,\frac{1}{|\ln g^*(\epsilon)|}$. For $t \in]0,T[\setminus J_2$, the inequality $\int_\Omega |\Pi\mu(t)| \geq C^*(\delta)\,(d_0(t)+\epsilon^{-\alpha})$ is valid. Observe that

$$\left\{t \in]0,T[:\ \int_\Omega |\Pi\mu(t)| \geq 2\,C^*\left(\frac{\bar\delta_0}{2}\right)\,\frac{1}{\epsilon^\alpha}\right\} \subset J_2(\epsilon,\,\frac{\bar\delta_0}{2}) \cup \left\{t \in]0,T[:\ d_0(t) \geq \frac{1}{\epsilon^\alpha}\right\}\,.$$

Therefore

$$\lambda_1\left(\left\{t \in]0,T[:\ \int_\Omega |\Pi\mu(t)| \geq 2\,C^*\left(\frac{\bar\delta_0}{2}\right)\,\frac{1}{\epsilon^\alpha}\right\}\right) \leq C_0\,\frac{1}{|\ln g^*(\epsilon)|} + C_0\,\epsilon^\alpha\,.$$

The claim follows. $\square$

8.3 Special estimates for $\sigma > 0$ and $\tau > 0$

In the case $\sigma > 0$, the dissipation inequality provides $\sqrt{\sigma}\,\|\nabla\mu\|_{L^2(Q)} \leq C_0$ as an additional information. Thus, a gradient bound for *all* coordinates of the vector μ. In this case, we can apply the reasoning a simplified version of the preceding subsection to the chemical potentials instead of the differences.

Lemma 8.18. *Let $\sigma > 0$ be fixed. Then*

(1) Define $T^* := \inf\{t \in [0,T]\ :\ \min_{i=1,\ldots,N} \bar\rho_i(t) = 0\}$. Then $T^* \geq T_0 > 0$ with T_0 depending only on the data, and for all $t \in [0,\,T^*[$, $\|\mu\|_{L^1(Q_t;\,\mathbb{R}^N)} \leq C_{0,\sigma}(t)$;

(2) If $s + \hat{s}^\Gamma = 0$, then $\|\mu\|_{L^1(Q;\,\mathbb{R}^N)} \leq C_{0,\sigma}$;

(3) If the vector of total initial masses $\bar\rho^0$ does not belong to $\mathcal{M}_{crit}$, then $[\mu]_{L^w_{(\circ_N \ln)}L^1(Q;\,\mathbb{R}^N)} \leq C_{0,\sigma}$.

Here $C_{0,\sigma}$ depends on σ, on $\|\nabla\Pi\mu\|_{L^1(Q)}$ on $\|P_W\mu\|_{L^1(Q\cup S_T)}$, on $\inf_{i=1,\ldots N}\bar\rho_i^0$, and in the second case also on $\operatorname{dist}(\bar\rho^0,\,\mathcal{M}_{crit})$.

Proof. For all $t \in]0,T[$, the global mass conservation implies that $\int_\Omega \varrho(t) = \int_\Omega \varrho^0 \cdot \mathbb{1}$, and this implies by means of a well-known argument that there is are $+\infty > a_1 > a_0 > 0$ and $b_0 > 0$ such that the set $A_0(t) := \{x \in \Omega\ :\ a_1 \geq \varrho(t)\cdot\mathbb{1} \geq a_0\}$ satisfies $|A_0(t)| \geq b_0$. Consider now the disjoint decomposition of Ω into sets $B_1(t),\ldots,B_N(t)$ where $B_j(t) := \{x \in \Omega\ :\ \mu_j(t,x) = \sup_{i=1,\ldots,N}\mu_i(t,x)\}$. We can show that

$$c\,(1+[F'(a_1)]^+) \geq \mu_j(t) \geq -c\,(1+[F'(a_0)]^-) \text{ on } A_0(t)\cap B_j(t)\,. \tag{212}$$

Thus, for $i \in \{1,\ldots,N\}$ arbitrary

$$\begin{aligned}\int_{A_0(t)\cap B_j(t)} |\mu_i(t)| &\leq \int_{A_0(t)\cap B_j(t)} |\mu_j(t)| + \int_{A_0(t)\cap B_j(t)} |\mu_i(t)-\mu_j(t)|\\ &\leq c(a_0,\,a_1) + \int_{A_0(t)\cap B_j(t)} |\mu_i(t)-\mu_j(t)|\,.\end{aligned}$$

Summing up over $j = 1, \ldots, N$, the latter yields

$$\int_{A_0(t)} |\mu_i(t)| \leq N\, c(a_0,\, a_1) + \sum_{j=1}^N \int_\Omega |\mu_i(t) - \mu_j(t)| \,.$$

Now we apply the inequality (191) in the proof of Lemma 8.7 with $A := A_0(t) =: B$ and $\delta := b_0$, and we obtain that

$$\begin{aligned} \|\mu(t)\|_{L^1(\Omega)} &\leq c(b_0) \, \left(\|\nabla \mu(t)\|_{L^1(\Omega)} + \int_{A_0(t)} |\mu(t)|\right) \\ &\leq c(b_0) \, \left(\|\nabla \mu(t)\|_{L^1(\Omega)} + N\, c(a_0,\, a_1) + \sum_{j=1}^N \int_\Omega |\mu_i(t) - \mu_j(t)|\right) \\ &\leq C_0 \, \left(\|\nabla \mu(t)\|_{L^1(\Omega)} + \|\Pi \mu(t)\|_{L^1(\Omega;\, \mathbb{R}^{N-1})} + 1\right) \,. \end{aligned}$$

We apply the Proposition 8.17 to control $\|\Pi\mu(t)\|_{L^1(\Omega;\, \mathbb{R}^{N-1})}$, and the fact that $\|\nabla \mu\|_{L^1(Q)} \leq C_{0,\sigma}$, and the claim follows. $\square$

We recall that we can always express $\rho = \nabla h^*(\mu)$ with the mapping of Lemma 5.7. If $\sigma > 0$, we thus obtain that

$$\nabla \rho = (\nabla \mu \cdot D^2 h^*(\mu)) \,, \tag{213}$$

and using the relation (120), this shows that

$$|\nabla \rho| \leq C_1\, \varrho\, |\nabla \mu| \,. \tag{214}$$

If $\alpha \geq 2$ (we in fact always assume $\alpha > 3$ if $\sigma > 0$), it follows that

$$\sqrt{\sigma}\, \|\nabla \rho\|_{L^{2, \frac{2\alpha}{2+\alpha}}(Q)} \leq \tilde{C}\, \|\varrho\|_{L^{\infty,\alpha}(Q)}\, \|\sqrt{\sigma}\, \nabla \mu\|_{L^2(Q)} \leq C_0 \,. \tag{215}$$

We will need the following statement.

Lemma 8.19. *Assume* $\sigma > 0$. *Then* $\|\ln \varrho\|_{W^{1,0}_2(Q)} \leq C_0\, \sigma^{-1/2}$.

Proof. Let $1 > \gamma > 0$. Using (213), (214), we obtain that

$$|\nabla \ln(\varrho + \gamma)| \leq C_1\, \frac{\varrho}{\varrho + \gamma}\, |\nabla \mu| \leq C_1\, |\nabla \mu| \,.$$

Thus, $\sqrt{\sigma}\, \|\nabla \ln(\varrho + \gamma)\|_{L^2(Q)} \leq C$. Let $\epsilon > 0$. For $t \in]0, T[$, we can always show that

$$|\{x \in \Omega \,:\, \ln(\varrho(t) + \gamma) \leq \epsilon^{-1}\}| \geq |\Omega| - C_0\, e^{-\frac{1}{\epsilon}} \,.$$

Thus, applying (191) (see the proof of Lemma 8.7), we see that there is a decomposition $]0, T[= I_1 \cup I_2$ such that

$$\begin{cases} \int_\Omega |\ln(\varrho(t) + \gamma)| \leq C^*(\delta)\, \left((\|\nabla \ln(\varrho(t) + \gamma)\|_{L^1(\Omega)} + \epsilon^{-1}\right) & \text{for } t \in I_1 \\ |\{x \in \Omega \,:\, \ln(\varrho(t) + \gamma) \geq -\epsilon^{-1}\}| \leq \delta & \text{for } t \in I_2 \,. \end{cases}$$

In particular, choosing $\gamma < 2^{-1}\, e^{-1/\epsilon}$,

$$\begin{cases} \int_\Omega |\ln(\varrho(t)+\gamma)| \leq C^*(\delta)\, (\|\nabla \ln(\varrho(t)+\gamma)\|_{L^1(\Omega)} + \epsilon^{-1}) & \text{for } t \in I_1 \\ |\{x \in \Omega \,:\, \varrho(t) \geq 2^{-1}\, e^{-1/\epsilon}\}| \leq \delta & \text{for } t \in I_2\,. \end{cases}$$

Due to the global mass conservation, we find parameter $\epsilon_0 > 0,\ \delta_0 > 0$ depending only on the data such that $I_2 \equiv \emptyset$ for all $\delta \leq \delta_0$ and $\epsilon \leq \epsilon_0$. Thus

$$\int_\Omega |\ln(\varrho(t)+\gamma)| \leq C^*(\delta_0)\, (\|\nabla \ln(\varrho(t)+\gamma)\|_{L^1(\Omega)} + \epsilon_0^{-1}) \text{ for } t \in]0,T[\,.$$

It follows that $\int_Q |\ln(\varrho(t)+\gamma)| \leq C^*(\delta_0)\, (C_0\, \sigma^{-1/2} + \epsilon_0^{-1})$, and letting γ tend to zero, the claim follows. □

We now resume the estimates obained in the two last sections for the chemical potentials.

Proposition 8.20. *Let $(\varrho,\, q,\, v,\, \phi,\, R,\, R^\Gamma)$ satisfy the dissipation inequality. Assume that $\mu := \mathcal{E} q + \mathscr{M}(\varrho,\, q)\, \mathbb{1}$ is a measurable mapping from $[0,T]$ into $L^1(\Omega;\, \mathbb{R}^N)$ and that the vector of total masses $\bar{\rho} := \int_\Omega \rho = \int_\Omega \mathscr{R}(\varrho,\, q)$ belongs to $C_{\Phi^*}([0,T])$ and satisfies $\bar{\rho}(t) \in \{\bar{\rho}^0\} \oplus W$ for all $t \in]0,T[$. Then*

(1) Define $T^* := \inf\{t \in [0,T] \,:\, \min_{i=1,\ldots,N} \bar{\rho}_i(t) = 0\}$. Then $T^* \geq T_0 > 0$ with T_0 depending only on the data. For all $t \in [0,\, T^*[$, we have $\|q\|_{L^1(Q_t;\mathbb{R}^{N-1})} \leq C_0(t)$;

(2) If $s + \hat{s}^\Gamma = 0$, then $\|q\|_{L^1(Q;\mathbb{R}^{N-1})} \leq C_0$.

(3) If $\operatorname{dist}(\bar{\rho}^0,\, \mathcal{M}_{crit}) > 0$, then $[q]_{L^w_{(\circ_N \ln)} L^1(Q;\mathbb{R}^{N-1})} \leq C_0$.

If $\sigma > 0$, there is $C_{0,\sigma}$ such that

(1) For all $t \in [0,\, T^*[$, we have $\|\mu\|_{L^1(Q_t;\mathbb{R}^N)} \leq C_{0,\sigma}(t)$;

(2) If $s + \hat{s}^\Gamma = 0$, then $\|\mu\|_{L^1(Q;\mathbb{R}^N)} \leq C_{0,\sigma}$.

(3) If $\operatorname{dist}(\bar{\rho}^0,\, \mathcal{M}_{crit}) > 0$ then $[\mu]_{L^w_{(\circ_N \ln)} L^1(Q;\mathbb{R}^N)} \leq C_{0,\sigma}$.

Remark 8.21. *Recall also that if $\tau > 0$, then independently on additional conditions, there is $C_{0,\sigma,\tau} > 0$ depending on $\mathcal{B}_0$ and $\sigma,\, \tau$ such that $\|\mu\|_{L^1(Q)} \leq C_{0,\sigma,\tau}$ (Proposition 8.1).*

9 Compactness I

Our aim in this section is to derive a general compactness tool in order to pass to the limit with approximate solutions to the problem (P). Since we do not want to specify with which of the approximation parameters δ, σ or τ we pass to the limit, we will considers families indexed by a generic parameter $\epsilon > 0$.

In order to obtain the compactness we shall need the informations on distributional times derivative contained in the system (102), (103). For technical reasons it is convenient to express these informations in an older (though elementary) fashion (see [Hop51], Lemma 5.1 for the inspiring precursor of all Aubin–Lions–type techniques). For the sake of brevity, we introduce an auxiliary vector $\mathcal{A}$ associated with $(\varrho,\, q,\, v,\, \phi,\, R,\, R^{\Gamma})$ and the auxiliary quantities (95) via

$$\mathcal{A} := (J,\, \varrho\, v,\, r,\, \hat{r},\, \nabla v,\, \varrho\, v\otimes v,\, v\otimes(\mathbb{1}\cdot J),\, p,\, n^F\,\nabla\phi) \in [L^1(Q)]^a\,, \tag{216}$$

where $a > 1$ is the number of scalar compnents of the vector $\mathcal{A}$. A functional $\mathcal{F}$ belongs to $C([0,T]\,;\mathscr{L}(L^1(Q),\, [C^1_c(\Omega;\,\mathbb{R}^k)]^*))$ if $\mathcal{F}$ maps $[0,T]\times L^1(Q)$ into $[C^1_c(\Omega;\,\mathbb{R}^k)]^*$ and satisfies moreover the conditions

$$\begin{aligned} & t\mapsto \mathcal{F}(t,\, y) \text{ is absolutely continuous for all } y\in L^1(Q)\\ & \sup_{\|y\|_{L^1(Q)}\leq R,\, |t_1-t_2|\leq\delta} \|\mathcal{F}(t_1,\, y)-\mathcal{F}(t_2,\, y)\|_{[C^1_c(\Omega;\,\mathbb{R}^k)]^*}\to 0 \text{ for } \delta\to 0 \end{aligned} \tag{217}$$

We consider a 'solution family' $\{(\varrho_\epsilon,\, q^\epsilon,\, v^\epsilon,\, \phi_\epsilon,\, R^\epsilon,\, R^{\Gamma,\epsilon})\}_{\epsilon>0}$ which might for example correspond to free energy functions $\{h_\epsilon\}_{\epsilon>0}$ and mobility matrices $\{M_\epsilon\}_{\epsilon>0}$. We assume that the conditions

$$\begin{aligned} h_\epsilon(\rho) &\geq c_0\,|\rho|^\alpha - c_1, \quad \text{ for all } \rho\in\mathbb{R}^N_+\\ M_\epsilon\xi\cdot\xi &\geq \underline{\lambda}\,|P_{\mathbb{1}^\perp}\xi|^2 \text{ for all } \xi\in\mathbb{R}^N\,. \end{aligned} \tag{218}$$

are satisfied uniformly in ϵ.

At first we need to extract weakly convergent subsequences.

Lemma 9.1. *Consider a family $\{(\varrho_\epsilon,\, q^\epsilon,\, v^\epsilon,\, \phi_\epsilon,\, R^\epsilon,\, R^{\Gamma,\epsilon})\}_{\epsilon>0}$ which satisfies a uniform bound in the class $\mathcal{B}(T,\,\Omega,\,\alpha,\,N-1,\,\Psi,\,\Psi^\Gamma)$. Define auxiliary quantities ρ^ϵ, J_ϵ, r^ϵ, $\hat{r}^\epsilon$, p_ϵ, n^F_ϵ and $\mathcal{A}^\epsilon$ in the fashion of* (95)*,* (216)*.*

Assume that there is a mapping $\mathcal{F}\in C([0,T]\,;\mathscr{L}([L^1(Q)]^a,\, [C^1_c(\Omega;\,\mathbb{R}^{N+3})]^*))$ such that for almost all $t\in]0,T[$

$$\begin{pmatrix} \int_\Omega \rho^\epsilon(t)\cdot\psi\\ \int_\Omega \varrho_\epsilon(t)\, v^\epsilon(t)\cdot\eta \end{pmatrix} = \mathcal{F}(t,\,\mathcal{A}^\epsilon)(\psi,\,\eta)\,,\ \forall\ (\psi,\,\eta)\in C^1_c(\Omega;\,\mathbb{R}^N)\times C^1_c(\Omega;\,\mathbb{R}^3)\,. \tag{219}$$

Assume that for almost all $t\in]0,T[$, $\phi_\epsilon(t)$ satisfies in the weak sense

$$-\epsilon_0\,(1+\chi)\,\triangle\phi_\epsilon(t) = \frac{z}{m}\cdot\rho^\epsilon(t),\quad -\nu\cdot\nabla\phi_\epsilon(t) = 0 \textit{ on } \Sigma,\quad \phi_\epsilon(t) = \phi_0(t) \textit{ on } \Gamma\,. \tag{220}$$

Then, there are

$$\begin{aligned} &\rho\in L^{\infty,\alpha}(Q;\,\mathbb{R}^N_+),\quad J\in L^{2,\frac{2\alpha}{1+\alpha}}(Q;\,\mathbb{R}^{N\times 3}),\quad -R\in L_\Psi(Q;\,\mathbb{R}^s),\quad -R^\Gamma\in L_{\hat{\Psi}^\Gamma}(S;\,\mathbb{R}^{\hat{s}^\Gamma})\\ &v\in W^{1,0}_2(Q),\quad p\in L^{\infty,1}(Q)\cap L^{\min\{1+\frac{1}{\alpha},\,\frac{5}{3}-\frac{1}{\alpha}\}}(Q)\\ &\phi\in L^\infty(Q)\cap L^\infty(0,T;\,W^{1,\beta}(\Omega)) \end{aligned}$$

and a subsequence $\{\epsilon_n\}_{n\in\mathbb{N}}$ *such that as* $n \to \infty$*:*

$$
\begin{aligned}
&\rho^{\epsilon_n} \to \rho \text{ weakly in } L^{\alpha}(Q;\, \mathbb{R}^N)\\
&\rho^{\epsilon_n}(t) \to \rho(t) \text{ weakly in } L^{\alpha}(\Omega;\, \mathbb{R}^N) \text{ for almost all } t \in [0,T]\\
&\bar{\rho}^{\epsilon_n} \to \bar{\rho} \text{ strongly in } C([0,T];\, \mathbb{R}^N)\\
&J_{\epsilon_n} \to J \text{ weakly in } L^{2,\frac{2\alpha}{1+\alpha}}(Q;\, \mathbb{R}^{N\times 3})\\
&R^{\epsilon_n} \to R \text{ weakly in } L^1(Q;\, \mathbb{R}^s),\ R^{\Gamma,\epsilon_n} \to R^{\Gamma} \text{ weakly in } L^1(S;\, \mathbb{R}^{\hat{s}^{\Gamma}})\\
&v^{\epsilon_n} \to v \text{ weakly in } W^{1,0}_2(Q;\ \mathbb{R}^3)\\
&p_{\epsilon_n} \to p \text{ weakly in } L^{1+\min\{\frac{1}{\alpha},\, \frac{2}{3}-\frac{1}{\alpha}\}}(Q)\\
&\phi_{\epsilon_n} \to \phi \text{ strongly in } W^{1,0}_2(Q)\\
&\frac{z}{m}\cdot \rho^{\epsilon_n}\, \nabla\phi_{\epsilon_n} \to \frac{z}{m}\cdot \rho\, \nabla\phi \text{ weakly in } L^1(Q;\, \mathbb{R}^3)\\
&\varrho_{\epsilon_n}\, v^{\epsilon_n} \to \varrho\, v \text{ weakly in } L^{2,\frac{6\alpha}{6+\alpha}}(Q;\, \mathbb{R}^3)\\
&(\varrho_{\epsilon_n}\, v^{\epsilon_n})(t) \to \varrho(t)\, v(t) \text{ weakly in } L^{\frac{2\alpha}{1+\alpha}}(\Omega;\, \mathbb{R}^3) \text{ for almost all } t \in [0,T]\\
&\varrho_{\epsilon_n}\, v^{\epsilon_n} \otimes v^{\epsilon_n} \to \varrho\, v \otimes v \text{ weakly in } L^{\frac{5\alpha-3}{3\,\alpha}}(Q;\, \mathbb{R}^{3\times 3})\,.
\end{aligned}
$$

Proof. At first, using the bounds in the natural class $\mathcal{B}$ we extract a subsequence such that

$$
\begin{aligned}
&\rho^{\epsilon_n} \to \rho \text{ weakly in } L^{\alpha}(Q;\, \mathbb{R}^N), \quad \bar{\rho}^{\epsilon_n} \to \bar{\rho} \text{ strongly in } C([0,T];\, \mathbb{R}^N)\\
&J_{\epsilon_n} \to J \text{ weakly in } L^{2,\frac{2\alpha}{1+\alpha}}(Q;\, \mathbb{R}^{N\times 3})\\
&R^{\epsilon_n} \to R \text{ weakly in } L^1(Q;\, \mathbb{R}^s), \quad R^{\Gamma,\epsilon_n} \to R^{\Gamma} \text{ weakly in } L^1(S;\, \mathbb{R}^{\hat{s}^{\Gamma}})\\
&v^{\epsilon_n} \to v \text{ weakly in } W^{1,0}_2(Q;\, \mathbb{R}^3)\\
&p_{\epsilon_n} \to p \text{ weakly in } L^{1+\min\{1/\alpha,\, 2/3-1/\alpha\}}(Q)\\
&\varrho_{\epsilon_n}\, v^{\epsilon_n} \to \xi \text{ weakly in } L^{2,\frac{6\alpha}{6+\alpha}}(Q;\, \mathbb{R}^3)\\
&\varrho_{\epsilon_n}\, v^{\epsilon_n} \otimes v^{\epsilon_n} \to \tilde{\xi} \text{ weakly in } L^{\frac{5\alpha-3}{3\,\alpha}}(Q;\, \mathbb{R}^{3\times 3})\\
&\phi_{\epsilon_n} \to \phi \text{ weakly } W^{1,0}_2(Q)\\
&n^F_{\epsilon_n}\, \nabla\phi_{\epsilon_n} \to k_L \text{ weakly in } L^1(Q;\, \mathbb{R}^3)\\
&v^{\epsilon_n} \otimes (\mathbb{1}\cdot J_{\epsilon_n}) \to \hat{\xi} \text{ weakly in } L^1(Q;\, \mathbb{R}^{3\times 3})\,.
\end{aligned}
$$

It is then easily seen that the corresponding quantity $\mathcal{A}^{\epsilon_n}$ defined via (216) satisfies

$$\mathcal{A}^{\epsilon_n} \to \mathcal{A} := (J,\, \xi,\, r,\, \hat{r},\, \nabla v,\, \tilde{\xi},\, \hat{\xi},\, p,\, k_L) \text{ weakly in } [L^1(Q)]^a\,.$$

We now make use of the identity (219). Due to the fact that the mapping $\mathcal{F}$ is linear in the second argument, we obtain for almost all $t \in]0,T[$ that

$$\lim_{n\to\infty} \begin{pmatrix} \int_{\Omega} \rho^{\epsilon_n}(t)\cdot\psi \\ \int_{\Omega} \varrho_{\epsilon_n}(t)\, v^{\epsilon_n}(t)\cdot\eta \end{pmatrix} = \mathcal{F}(t,\, \mathcal{A})(\psi,\, \eta) \quad \forall\, (\psi,\, \eta) C^1_c(\Omega;\, \mathbb{R}^N)\times C^1_c(\Omega;\, \mathbb{R}^3)\,. \tag{221}$$

Thus, for almost all $t \in [0,T]$, we realise that the entire sequence $\{\rho^{\epsilon_n}(t)\}$ converges as distributions. Since it is uniformly bounded in $L^{\alpha}(\Omega)$, we obtain that $\{\rho^{\epsilon_n}(t)\}$ weakly converges in $L^{\alpha}(\Omega)$. The limit must be identical with $\rho(t)$ for almost all $t \in [0,T]$. Thus, using the Remark 9.3 hereafter, $\rho^{\epsilon_n} \to \rho$ strongly in $[W^{1,0}_2(Q)]^*$, and this allows to show that $\varrho^{\epsilon_n}\, v^{\epsilon_n} \to \varrho\, v$ as distributions in Q. Clearly $\xi = \varrho\, v$.

Next we define $\phi(t) \in W^{1,2}(\Omega)$ to be the unique weak solution to the problem

$$-\epsilon_0\,(1+\chi)\,\triangle\phi(t) = \frac{z}{m}\cdot\rho(t), \quad -\nu\cdot\nabla\phi(t) = 0 \text{ on } \Sigma, \quad \phi(t) = \phi_0(t) \text{ on } \Gamma\,.$$

We can verify due to the Remark 9.3 that for almost all $t \in]0,T[$ the convergence $\phi_{\epsilon_n}(t) \to \phi(t)$ strongly in $W^{1,2}(\Omega)$ is valid.

Thus, we also obtain that $\frac{z}{m}\cdot\rho^{\epsilon_n}\,\nabla\phi_{\epsilon_n} \to \frac{z}{m}\cdot\rho\,\nabla\phi$ weakly in $L^1(Q)$. It follows that $k_L = n^F\,\nabla\phi$.

The relation (221) implies that $\varrho_{\epsilon_n}(t)\,v^{\epsilon_n}(t)$ converges as distributions to $\varrho(t)\,v(t)$ for almost all $t \in]0,T[$, and therefore also weakly in $L^{2\alpha/(1+\alpha)}(\Omega)$. Since $2\alpha/(1+\alpha) > 6/5$, it also follows $\varrho_{\epsilon_n}\,v^{\epsilon_n} \to \varrho\,v$ strongly in $[W^{1,0}_2(Q)]^*$. This in turn allows to show that $\varrho_{\epsilon_n}\,v^{\epsilon_n}\otimes v^{\epsilon_n} \to \varrho\,v\otimes v$ as distributions, that means $\tilde{\xi} = \varrho\,v\otimes v$. □

Remark 9.2. *The condition* (219) *is naturally motivated by the structure of the weak formulation.*

Remark 9.3.
- *Let $1 \leq p \leq +\infty$. Let $\mathcal{K} : L^p(\Omega) \to W^{1,p}(\Omega)$ be a linear, bounded, compact operator. Assume that $\{u_n\}_{n\in\mathbb{N}} \subset L^p(Q)$ is a sequence such that $u_n(t) \to u(t)$ weakly in $L^p(\Omega)$ for almost all $t \in]0,T[$. Then $\mathcal{K}(u_n(t)) \to \mathcal{K}(u(t))$ strongly in $W^{1,p}(\Omega)$ for almost all $t \in]0,T[$.*
- *If $v_n \to v$ weakly in $W^{1,0}_2(Q)$ and $u_n(t) \to u(t)$ strongly in $[W^{1,2}(\Omega)]^*$ for almost all $t \in]0,T[$, then $u_n\,v_n \to u\,v$ weakly in $L^1(Q)$.*

We next can obtain the strong convergence of the velocity field. This result is in principle known (see [Lio98], page 9). For the convenience of the reader, we give a proof in the Appendix.

Corollary 9.4. *Assumptions of Lemma 9.1. Then, there is a subsequence such that $\varrho_{\epsilon_n}\,(v^{\epsilon_n} - v)$ converges to zero strongly in $L^1(Q)$ and pointwise almost everywhere in Q.*

We now can prove the conditional compactness of the family $\{\rho^{\epsilon}\}_{\epsilon>0}$. We will need the following auxiliary statements.

Lemma 9.5. *Consider the mapping $\mathscr{R} \in C(\mathbb{R}_{0,+}\times\mathbb{R}^{N-1};\, \mathbb{R}^N_+)$ (cf. (5.3)). For $x \in \mathbb{R}_+\times\mathbb{R}^{N-1}$, we denote $x = (x_1,\, \bar{x})$. Let $K \subset L^1(\Omega;\,\mathbb{R}^N)$ be a weakly sequentially compact set, and $K^* \subset L^1(\Omega)$ a sequentially compact set. Let $\phi^1,\, \phi^2, \ldots \in C^{\infty}(\overline{\Omega})$ be a countable, dense subset of $C(\overline{\Omega};\,\mathbb{R}^N)$.*

For all $\delta > 0$, there are $C(\delta) > 0$ and $m(\delta) \in \mathbb{N}$ such that

$$\begin{aligned}&\|\mathscr{R}(w^1) - \mathscr{R}(w^2)\|_{L^1(\Omega)}\\ &\leq \delta\left(1 + \sum_{i=1,2}\|\bar{w}^i\|_{W^{1,1}(\Omega)}\right) + C(\delta)\sum_{i=1}^{m}\left|\int_{\Omega}(\mathscr{R}(w^1) - \mathscr{R}(w^2))\cdot\phi^i\right|\end{aligned}$$

for all $w^1,\ w^2 \in L^1(\Omega;\ \mathbb{R}_+ \times \mathbb{R}^{N-1})$ *such that*

$$\mathscr{R}(w^i) \in K, \quad w_1^i \in K^*, \quad \bar{w}_i \in W^{1,1}(\Omega;\ \mathbb{R}^{N-1}) \quad \textit{for}\ i = 1, 2\,.$$

Proof. Clearly, it is sufficient to prove the claim for all $w^1,\ w^2 \in L^1(\Omega;\ \mathbb{R}_+ \times \mathbb{R}^{N-1})$ such that $\mathscr{R}(w^i) \in K$, $w_1^i \in K^*$ and $\bar{w}^i \in W^{1,1}(\Omega;\ \mathbb{R}^{N-1})$ for $i = 1, 2$ and such that

$$\|\mathscr{R}(w^1) - \mathscr{R}(w^2)\|_{L^1(\Omega)} \geq \delta\,.$$

If this is not true, there is $\delta_0 > 0$ such that for all $n \in \mathbb{N}$ and $i = 1, 2$, we can find $w^{i,n} \in L^1(\Omega;\ \mathbb{R}_+ \times \mathbb{R}^{N-1})$ such that $\mathscr{R}(w^{i,n}) \in K$, $w_1^{i,n} \in K^*$, $\bar{w}^{i,n} \in W^{1,1}(\Omega;\ \mathbb{R}^{N-1})$ $(i = 1, 2)$ satisfying moreover the properties

$$\begin{aligned} &\|\mathscr{R}(w^{1,n}) - \mathscr{R}(w^{2,n})\|_{L^1(\Omega)} \\ &\geq \delta_0 \sum_{i=1,2} \|\bar{w}^{i,n}\|_{W^{1,1}(\Omega)} + n \sum_{i=1}^{n} \left| \int_\Omega (\mathscr{R}(w^{1,n}) - \mathscr{R}(w^{2,n})) \cdot \phi^i \right| \qquad (222) \\ &\|\mathscr{R}(w^{1,n}) - \mathscr{R}(w^{2,n})\|_{L^1(\Omega)} \geq \delta_0\,. \qquad (223) \end{aligned}$$

Since we assume that $\mathscr{R}(w^{i,n}) \in K$ for $i = 1, 2$ and since K is a bounded set of $L^1(\Omega)$, we obtain first that $\|\bar{w}^{i,n}\|_{W^{1,1}(\Omega)} \leq C$ for all $n \in \mathbb{N}$. Thus we can extract a subsequence that we not relabel such that for almost all $x \in \Omega$ there exists $\bar{w}^i(x) := \lim_{n\to\infty} \bar{w}^{i,n}(x)$.

Moreover as $w_1^{i,n} \in K^*$, we can extract a subsequence such that $w_1^{i,n} \to w_1^i$ strongly in $L^1(\Omega)$ and almost everywhere in Ω. Consequently, we obtain for a subsequence and for $i = 1, 2$ that

$$w^{i,n} \to w^i := (w_1^i,\ \bar{w}^i) \text{ strongly in } L^1(\Omega;\ \mathbb{R}_+ \times \mathbb{R}^{N-1}) \text{ and a. e. in } \Omega\,.$$

Now using that $\mathscr{R}(w^{i,n}) \in K$, we can pass to a subsequence again to see that $\mathscr{R}(w^{i,n}) \to u^i$ weakly in $L^1(\Omega;\ \mathbb{R}^N)$ for $i = 1, 2$. Obviously the continuity of $\mathscr{R}$ and the pointwise convergence yield $u^i = \mathscr{R}(w^i)$. We next use the second implication of (222), that is,

$$\sum_{i=1}^{n} \left| \int_\Omega (\mathscr{R}(w^{1,n}) - \mathscr{R}(w^{2,n})) \cdot \phi^i \right| \leq c\, n^{-1}\,,$$

so that we easily conclude that $\mathscr{R}(w^1) = \mathscr{R}(w^2)$ almost everywhere in Ω. It remains to observe that $\mathscr{R}(w^{1,n}) - \mathscr{R}(w^{2,n}) \to 0$ in $L^1(\Omega)$ to show that the condition (223) is violated. □

In order to apply the Lemma 9.5 in the context of parabolic problems, we introduce the following way of speaking:

Remark 9.6. *We say that a family of vector-valued functions* $\{u_\epsilon\}_{\epsilon\in[0,1]} \subset C([0,T];\ L^1(\Omega))$ *is compact in* $L^1(\Omega)$ uniformly in time *if and only if the family* $\bigcup_{\epsilon\in[0,1]} \bigcup_{t\in[0,T]} \{u_\epsilon(t)\}$ *is sequentially compact in* $L^1(\Omega)$.

We now state and prove our main compactness tool.

Corollary 9.7. *For $n \in \mathbb{N}$, let $w^n : [0,T] \to L^1(\Omega;\, \mathbb{R}_+ \times \mathbb{R}^{N-1})$ be continuous. Assume that w_1^n is compact in $L^1(\Omega)$ uniformly in time (sense of Remark 9.6). Moreover assume that there is C_1 independent on n such that*

$$[\bar{w}^n]_{L^w_{(\circ_N \ln)} L^1(Q)} + \|\nabla \bar{w}^n\|_{L^1(Q)} \leq C_1$$

Suppose that $\|\mathscr{R}(w^n)\|_{L^{\infty,\alpha}(Q)} \leq C_1$, and that the sequence $\{\mathscr{R}(w^n(t))\}_{n\in\mathbb{N}}$ converges as distributions in Ω for almost all t.

Then, there is a subsequence (no new labels) for which $\rho(t,\, x) := \lim_{n\to\infty} \mathscr{R}(w^n(t,x))$ exists for almost all $(t,x) \in Q$, and $\mathscr{R}(w^n(t,x)) \to \rho$ strongly in $L^1(Q;\, \mathbb{R}^N)$.

Proof. For $n \in \mathbb{N}$, the assumptions imply that $\mathscr{R}(w^n(t)) \in L^\alpha(\Omega;\, \mathbb{R}^N)$ for all $t \in [0,T]$. We define $K \subset L^1(\Omega;\, \mathbb{R}^N)$ via $K := \bigcup_{n\in\mathbb{N}} \bigcup_{t\in[0,T]} \{\mathscr{R}(w^n(t))\}$ By assumption K is bounded in $L^\alpha(\Omega)$ and thus also weakly sequentially compact in $L^1(\Omega)$.

By assumption again, the set $K^* := \bigcup_{n\in\mathbb{N}} \bigcup_{t\in[0,T]} \{w_1^n(t)\}$ is compact in $L^1(\Omega)$.

For $\delta > 0$, we apply the inequality of Lemma 9.5 with the following choices: $w^1 = w^n(t)$, $w^2 := w^{n+p}(t)$. We obtain for $t \in [0,T]$ that

$$\begin{aligned}
&\|\mathscr{R}(w^n(t)) - \mathscr{R}(w^{n+p}(t))\|_{L^1(\Omega)} \\
&\quad \leq \delta \left(1 + \|\bar{w}^n(t)\|_{W^{1,1}(\Omega)} + \|\bar{w}^{n+p}(t)\|_{W^{1,1}(\Omega)}\right) \\
&\qquad + C(\delta,\, K^*) \sum_{i=1}^m \left| \int_\Omega (\mathscr{R}(w^n(t)) - \mathscr{R}(w^{n+p}(t))) \cdot \phi^i \right| .
\end{aligned} \tag{224}$$

Fix $\ell \in \mathbb{N}$ and define $I_{\ell,n} \subset [0,T]$ via $I_{\ell,n} := \{t \in]0,T[:\ \|\bar{w}^n(t)\|_{L^1(\Omega)} \geq \ell\}$. Note that by assumption

$$\lambda_1(I_{\ell,n}) \leq \frac{[\bar{w}^n]_{L^w_{(\circ_N \ln)} L^1(Q)}}{(\circ_N \ln)(\ell)} \leq \frac{C_1}{(\circ_N \ln)(\ell)} .$$

Thus $\lambda_1([0,T] \setminus I_{\ell,n}) \geq T - \frac{C_1}{(\circ_N \ln)(\ell)}$.

We integrate the relation (224) over the set $J := [0,T] \setminus (I_{\ell,n} \cup I_{\ell,n+p})$ and this yields

$$\begin{aligned}
&\|\mathscr{R}(w^n) - \mathscr{R}(w^{n+p})\|_{L^1(J\times\Omega)} \\
&\quad \leq \delta \, (T + 2 \sup_n \|\bar{w}^n\|_{W^{1,1}(J\times\Omega)}) \\
&\qquad + C(\delta) \sum_{i=1}^m \int_J \left| \int_\Omega (\mathscr{R}(w^n(t)) - \mathscr{R}(w^{n+p}(t))) \cdot \phi^i \right| \\
&\quad \leq \delta \, (T + C_\ell) + C(\delta) \sum_{i=1}^m \int_0^T \left| \int_\Omega (\mathscr{R}(w^n(t)) - \mathscr{R}(w^{n+p}(t))) \cdot \phi^i \right|
\end{aligned}$$

Due to boundedness of $\{\mathscr{R}(w^n)\}$ in $L^{\infty,1}(Q)$, and the fact that $\lambda_1([0,T] \setminus J) \leq 2\, \frac{C_1}{(\circ_N \ln)(\ell)}$, we obtain that

$$
\begin{aligned}
&\|\mathscr{R}(w^n)-\mathscr{R}(w^{n+p})\|_{L^1(Q)}\\
&\quad\le 4\,\sup_n \|\mathscr{R}(w^n)\|_{L^{\infty,1}(Q)}\,\frac{C_1}{(\circ_N \ln)(\ell)}+\delta\,(T+C_\ell)\\
&\qquad + C(\delta)\,\sum_{i=1}^m \int_0^T \left|\int_\Omega (\mathscr{R}(w^n(t))-\mathscr{R}(w^{n+p}(t)))\cdot\phi^i\right| .
\end{aligned}
$$

The vector fields $\mathscr{R}(w^n)$ weakly converges in $L^1(\Omega;\,\mathbb{R}^N)$ for almost all t to some element $\rho\in L^{\infty,\alpha}(Q;\,\mathbb{R}^N)$. Invoking the triangle inequality,

$$
\int_0^T \left|\int_\Omega (\mathscr{R}(w^n(t))-\mathscr{R}(w^{n+p}(t)))\cdot\phi^i\right| \le 2\,\sup_{k\ge n}\int_0^T \left|\int_\Omega (\mathscr{R}(w^k(t))-\rho(t))\cdot\phi^i\right| .
$$

Thus

$$
\begin{aligned}
&\sup_{p\ge 0}\|\mathscr{R}(w^n)-\mathscr{R}(w^{n+p})\|_{L^1(Q)}\\
&\quad\le 4\,\sup_n \|\mathscr{R}(w^n)\|_{L^{\infty,1}(Q)}\,\frac{C_1}{(\circ_N \ln)(\ell)}+\delta\,(T+C_\ell)\\
&\qquad + 2\,C(\delta)\,\sum_{i=1}^m \sup_{k\ge n}\int_0^T \left|\int_\Omega (\mathscr{R}(w^k(t))-\rho(t))\cdot\phi^i\right| .
\end{aligned}
$$

Invoking the Fatou Lemma and the bounds in $L^{\infty,\alpha}$, we have

$$
\begin{aligned}
\limsup_{n\to\infty}\sup_{k\ge n}\int_0^T \left|\int_\Omega (\mathscr{R}(w^k(t))-\rho(t))\cdot\phi^i\right| &= \limsup_{n\to\infty}\int_0^T \left|\int_\Omega (\mathscr{R}(w^n(t))-\rho(t))\cdot\phi^i\right|\\
&\le \int_0^T \limsup_{n\to\infty}\left|\int_\Omega (\mathscr{R}(w^n(t))-\rho(t))\cdot\phi^i\right| .
\end{aligned}
$$

The vector fields $\mathscr{R}(w^n(t))$ weakly converges in $L^1(\Omega)$ for almost all t.

Therefore $\limsup_{n\to\infty}\left|\int_\Omega (\mathscr{R}(w^n(t))-\rho(t))\cdot\phi^i\right|=0$ for almost all t. Thus

$$
\limsup_{n\to\infty}\sup_{k\ge n}\int_0^T \left|\int_\Omega (\mathscr{R}(w^k(t))-\rho(t))\cdot\phi^i\right| = 0\,.
$$

It next follows that

$$
\begin{aligned}
&\limsup_{n\to\infty}\sup_{p\ge 0}\|\mathscr{R}(w^n)-\mathscr{R}(w^{n+p})\|_{L^1(Q)}\\
&\quad\le 4\,\sup_n \|\mathscr{R}(w^n)\|_{L^{\infty,1}(Q)}\,\frac{C_1}{(\circ_N \ln)(\ell)}+\delta\,(T+C_\ell)\,,
\end{aligned}
$$

and since ℓ and δ are arbitrary, $\limsup_{n\to\infty}\sup_{p\ge 0}\|\mathscr{R}(w^n)-\mathscr{R}(w^{n+p})\|_{L^1(Q)}=0$. This means that $\{\mathscr{R}(w^n)\}$ is a Cauchy sequence in $L^1(Q)$. In particular, we can extract a subsequence such that $\lim_{n\to+\infty}\mathscr{R}(w^n)$ exists almost everywhere in Q. $\square$

Corollary 9.8. *Assumptions of Lemma 9.1. Assume moreover that the family of the total mass densities $\{\varrho_\epsilon\}_{\epsilon\geq 0}$ is compact in $L^1(\Omega)$ uniformly with respect to time (sense of Remark 9.6). Then*

$$\rho^{\epsilon_n} \to \rho \text{ strongly in } L^1(Q;\, \mathbb{R}^N)$$
$$\exists q(t,\,x) := \lim_{n\to\infty} q^{\epsilon_n}(t,\,x) \text{ for almost every } (t,\,x) \text{ such that } \varrho(t,\,x) > 0\,.$$

Consequently, the identity $\rho = \mathscr{R}(\varrho,\, q)$ is valid at almost every point of the set $\{(t,\,x)\ :\ \varrho(t,\,x) > 0\}$.

Proof. We at first obtain the convergence properties of Lemma 9.1 for a sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$. We define $w^n = (\varrho_{\epsilon_n},\, q^{\epsilon_n})$, and verify easily that all requirements of the Corollary 9.7 are satisfied. We apply the Corollary 9.7, and we first obtain that $\rho^{\epsilon_n} = \mathscr{R}(\varrho_{\epsilon_n},\, q^{\epsilon_n})$ converges strongly in $L^1(Q)$ and pointwise almost everywhere.

Next we use the formula (111) and the inequalities of Lemma 5.7 to see that for a certain $\lambda \in [0,1]$

$$\begin{aligned}|\mathscr{R}(\varrho_{\epsilon_n},\, q^{\epsilon_n}) - \mathscr{R}(\varrho,\, q^{\epsilon_n})| &= \mathscr{R}_s(\lambda\,\varrho_{\epsilon_n} + (1-\lambda)\,\varrho,\, q^{\epsilon_n})|\,|\varrho_{\epsilon_n} - \varrho|\\ &\leq C\,\max\{\varrho_{\epsilon_n},\,\varrho\}^{\frac{\alpha-1}{2}}\,|\varrho_{\epsilon_n} - \varrho|\,.\end{aligned}$$

The latter implies that

$$\begin{aligned}\|\mathscr{R}(\varrho_{\epsilon_n},\, q^{\epsilon_n}) - \mathscr{R}(\varrho,\, q^{\epsilon_n})\|_{L^1(Q)} \leq& C\,(\sup_n \|\varrho_{\epsilon_n}\|_{L^\alpha(Q)} + \|\varrho\|_{L^\alpha(Q)})\,\|\varrho_{\epsilon_n} - \varrho\|_{L^{\frac{2\alpha}{1+\alpha}}(Q)}\\ &\to 0\,.\end{aligned}$$

Thus passing to a subsequence we obtain also that $\mathscr{R}(\varrho,\, q^{\epsilon_n})$ converges almost everywhere in Q. Next, we use the fact that for all $s > 0$, the mapping $\mathscr{R}$ is a bijection between $[s,\, +\infty[\times\mathbb{R}^{N-1}$ and $\{\rho \in \mathbb{R}^N_+\ :\ \rho\cdot\mathbb{1} \geq s\}$. Thus, from the existence of $\lim_{n\to\infty}\mathscr{R}(\varrho(t,x),\, q^{\epsilon_n}(t,x))$, we first obtain that

$$\liminf_{n\to\infty} q^{\epsilon_n}(t,x) = \limsup_{n\to\infty} q^{\epsilon_n}(t,x) \text{ for almost all} (t,x) \text{ such that } \varrho(t,x) > 0\,. \tag{225}$$

Next we use the estimates available on q^{ϵ_n} to see for $k,\,\ell > 0$ that

$$\begin{aligned}&\text{meas}\{(t,x)\in Q\ :\ |q^{\epsilon_n}| \geq k\} = \int_0^T |\{x\in\Omega\ :\ |q^{\epsilon_n}(t)| \geq k\}|\\ &= \int_{\{t\,:\,\|q^{\epsilon_n}(t)\|_{L^1(\Omega)}\leq\ell\}} |\{x\in\Omega\ :\ |q^{\epsilon_n}(t)| \geq k\}|\\ &\quad + \int_{\{t\,:\,\|q^{\epsilon_n}(t)\|_{L^1(\Omega)}>\ell\}} |\{x\in\Omega\ :\ |q^{\epsilon_n}(t)| \geq k\}|\\ &\leq \frac{1}{k}\int_{\{t\,:\,\|q^{\epsilon_n}(t)\|_{L^1(\Omega)}\leq\ell\}} \|q^{\epsilon_n}(t)\|_{L^1(\Omega)} + \lambda_1(\{t\ :\ \|q^{\epsilon_n}(t)\|_{L^1(\Omega)} > \ell\})\\ &\leq T\,\frac{\ell}{k} + [q^{\epsilon_n}]_{L^w_{(\circ_N \ln)}L^1(Q)}\,\frac{1}{(\circ_N\ln)(\ell)} \leq C\,\left(\frac{\ell}{k} + \frac{1}{(\circ_N\ln)(\ell)}\right)\,.\end{aligned}$$

Combining the latter observation and (225) we see that that there is a set $\mathcal{N}(k,\,\ell)$ with $\operatorname{meas}\mathcal{N}(k,\ell) \leq C\,(\frac{\ell}{k} + \frac{1}{(\circ_N \ln)(\ell)})$ such that

$$\lim_{n\to\infty} q^{\epsilon_n}(t,x) \text{ exists in } \mathbb{R}^{N-1} \text{ for all } (t,x) \in \{(t,x)\,:\,\varrho(t,x) > 0\} \setminus \mathcal{N}(k,\ell)\,.$$

Choosing appropriate sequences of numbers $\ell,\,k$, the measure of $\mathcal{N}(k,\ell)$ can be made arbitrarily small.

We then use that $\rho^{\epsilon_n} = \mathscr{R}(\varrho_{\epsilon_n},\,q^{\epsilon_n})$ to see that ρ^{ϵ_n} also converges almost everywhere in Q to $\mathscr{R}(\varrho,\,q)$, and the claim follows. □

In order to pass to the limit in the boundary reaction terms, we also discuss the strong convergence of the relative chemical potentials on the boundary Γ.

Lemma 9.9. *Assumptions of Corollary 9.8. Then*

$$\exists\, q(t,\,x) := \lim_{n\to\infty} q^{\epsilon_n}(t,\,x) \text{ for almost every } (t,\,x) \in S^+(\varrho)\,.$$

Proof. By definition, the surface $S^+(\varrho)$ is relatively open and possesses an open neighbourhood U in Q such that $|U \cap \{(t,x)\,:\,\varrho(t,\,x) = 0\}| = 0$. Thus, for $(t_0,\,x^0) \in S^+(\varrho)$ arbitrary, there is $R > 0$ such that the cube $Q_R(t_0,x^0)$ with radius $R > 0$ and centered at $(t_0,\,x^0)$ is contained in U. For all $\epsilon > 0$, there is a constant $c = c(\Omega,\,\epsilon)$ such that

$$\|u\|_{L^1(\Gamma_R(x^0))} \leq \epsilon\,\|\nabla u\|_{L^1(\Omega_R(x^0))} + c(\epsilon,\,\Omega)\,\|u\|_{L^1(\Omega_R(x^0))} \text{ for all } u \in W^{1,1}(\Omega)\,.$$

Here Γ_R and Ω_R denote the intersectoin of Γ and Ω with $Q_R(x^0)$, the three-dimensional cube with radius R centered at x^0. With the help of this inequality, we obtain for almost all $t \in]t_0 - R,\,t_0 + R[$ that

$$\begin{aligned}\|q^{\epsilon_n}(t) - q(t)\|_{L^1(\Gamma\cap Q_R(x^0))} \leq & \epsilon\,(\|\nabla q^{\epsilon_n}(t)\|_{L^1(\Omega)} + \|\nabla q(t)\|_{L^1(\Omega)}) \\ & + c(\epsilon,\,\Omega)\,\|q^{\epsilon_n}(t) - q(t)\|_{L^1(\Omega\cap Q_R(x^0))}\,.\end{aligned}$$

Choosing $I \subset]t_0 - R,\,t_0 + R[$ of arbitrary small measure so that the norms $\|q^{\epsilon_n}(t) - q(t)\|_{L^1(\Omega)}$ are uniformly bounded for $t \in]t_0 - R,\,t_0 + R[\setminus I$, we obtain that

$$\int_I \|q^{\epsilon_n}(t) - q(t)\|_{L^1(\Gamma_R(x^0))}\,dt \leq C_0\,\epsilon + c(\epsilon,\,\Omega)\,\int_I \|q^{\epsilon_n}(t) - q(t)\|_{L^1(\Omega_R(x^0))}\,dt$$

Now as $I \times (\Omega \cap Q_R(x^0))$ is a subset of U, we obtain with the help of Corollary 9.8 that $\int_I \|q^{\epsilon_n}(t) - q(t)\|_{L^1(\Omega_R(x^0))}\,dt \to 0$ for $n \to \infty$, and this yields $\limsup_{n\to\infty} \int_I \|q^{\epsilon_n}(t) - q(t)\|_{L^1(\Gamma_R(x^0))}\,dt = 0$. The claim follows. □

It remains to enlight the global convergence property of the variables $\{q^{\epsilon_n}\}$ inclusively of the set where vacuum possibly occurs.

Lemma 9.10. *Assumptions of Corollary 9.8. Then, there are open sets $K_1 \subseteq K_2 \subseteq \ldots [0,T]$, $|K_m| \to T$ for $m \to \infty$ and a subsequence such that*

$$\begin{aligned} q^{\epsilon_n} &\to q \text{ weakly in } L^2(K_m \times [\Omega \cup \Gamma];\, \mathbb{R}^{N-1}) \text{ for all } m \in \mathbb{N}\\ \nabla q^{\epsilon_n} &\to \nabla q \text{ weakly in } L^2(Q;\, \mathbb{R}^{(N-1)\times 3})\\ D^{R,\epsilon_n} &\to (\gamma^1 \cdot \mathcal{E}q, \ldots, \gamma^s \cdot \mathcal{E}q) \text{ weakly in } L^1(Q;\, \mathbb{R}^s)\\ \hat{D}^{\Gamma,R,\epsilon_n} &\to (\hat{\gamma}^1 \cdot \mathcal{E}q, \ldots, \hat{\gamma}^{\hat{s}^\Gamma} \cdot \mathcal{E}q) \text{ weakly in } L^1(S;\, \mathbb{R}^{\hat{s}^\Gamma})\,. \end{aligned}$$

Proof. Let $m \in \mathbb{N}$, and consider the sets $I_{n,m} \subseteq [0,T]$ defined via

$$I_{n,m} := \big\{t \in [0,T] \,:\, \exists i \le N,\, |\{x \,:\, \rho_i^{\epsilon_n}(t,\,x) \ge m^{-1}\}| \le m^{-1}\big\} \tag{226}$$

For $t \in I_{n,m}$, we find $i_0 \in \{1, \ldots, N\}$ such that

$$|\{x \,:\, \rho_{i_0}^{\epsilon_n}(t,\,x) \ge m^{-1}\}| \le \frac{1}{m}\,. \tag{227}$$

Recall the definition of $\bar{a}_0 > 0$ and $\bar{b}_0 > 0$ in (201). For $M > 0$ set $A(t) := \{x \,:\, \bar{a}_0 \le \varrho^{\epsilon_n}(t,\,x) \le M\}$ satisfies

$$\begin{aligned} |A(t)| &\ge |\{x \,:\, \bar{a}_0 \le \varrho^{\epsilon_n}(t,\,x)\}| - |\{x \,:\, M \le \varrho^{\epsilon_n}(t,\,x)\}|\\ &\ge \bar{b}_0 - C_0\, M^{-\alpha}\,. \end{aligned}$$

Thus, there is M_0 depending only on the data such that $|A(t)| \ge \bar{b}_0/2$. Defining next $B(t) := A(t) \cap \{x \,:\, \rho_{i_0}^{\epsilon_n}(t,\,x) < m^{-1}\}$, we easily show that if $m \ge 4/\bar{b}_0$ then $|B(t)| \ge \bar{b}_0/2 - 1/m \ge \bar{b}_0/4$. Pursuing this reasoning we obtain for $x \in B(t)$ that

$$\mu_{i_0}(t,x) - \sup_{j=1,\ldots,N} \mu_j(t,x) \le \frac{k_B\,\theta}{m_{i_0}}\, \ln\frac{1}{m} + C(\bar{a}_0,\, M_0)\,.$$

It follows that $|q(t,x)| \ge c\,|\mu_{i_0}(t,x) - \sup_{j=1,\ldots,N} \mu_j(t,x)|$ on $B(t)$, and therefore

$$\|q(t)\|_{L^1(\Omega;\,\mathbb{R}^{N-1})} \ge \int_{B(t)} |q(t,x)| \ge \frac{\bar{b}_0}{4}\,\left(\frac{k_B\,\theta}{m_{i_0}}\, \ln m - C(\bar{a}_0,\, M_0)\right)\,.$$

Thus, for all $m \ge m_0(\text{data})$, we achieve that

$$I_{n,m} \subset \{t \,:\, \|q(t)\|_{L^1(\Omega;\,\mathbb{R}^{N-1})} \ge c\, \ln m\}\,.$$

Due to the estimate in the class $L^w_{(\circ_N \ln)} L^1(Q)$, we obtain that

$$\lambda_1(I_{n,m}) \le \bar{c}\,\frac{1}{(\circ_{N+1}\ln)(m)}\,. \tag{228}$$

Recall that in the natural class, the regularity $\bar{\rho} \in C_{\Phi^*}([0,T])$ is available. Consider for $k \in \mathbb{N}$ the compact sets $J_k := \{t \in [0,T] \,:\, \inf_{i=1,\ldots,N} \bar{\rho}_i(t) \le k^{-1}\}$, and define

$$J_{n,k} := \{t \in [0,T] \,:\, \inf_{i=1,\ldots,N} \bar{\rho}_i^{\epsilon_n}(t) \le k^{-1}\}\,.$$

Due to the *uniform convergence* of $\bar{\rho}^{\epsilon_n}$, there is $n_0 = n_0(k)$ such that

$$J_{n,2\,k} \subseteq J_k \subseteq J_{n,k/2} \text{ for all } n \geq n_0\,.$$

Observe further that

$$\begin{aligned} t \in J_{n,k} &\Longrightarrow \inf_{i=1,\ldots,N} |(\{x\,:\,\rho_i^{\epsilon_n}(t,\,x) \geq k^{-1/2}\}| \leq k^{-1/2} \\ &\Longrightarrow t \in I_{n,\sqrt{k}}\,. \end{aligned} \tag{229}$$

Moreover $t \in I_{n,m}$ implies for a i_0 that

$$\begin{aligned} \int_\Omega \rho_{i_0}^{\epsilon_n}(t) &= \int_{\{\rho_{i_0}^{\epsilon_n}(t)<1/m\}} \rho_{i_0}^{\epsilon_n}(t) + \int_{\{\rho_{i_0}^{\epsilon_n}(t)\geq 1/m\}} \rho_{i_0}^{\epsilon_n}(t) \\ &\leq m^{-1}\,|\Omega| + \|\rho_{i_0}^{\epsilon_n}\|_{L^{\infty,\alpha}(\Omega)}\, m^{-1/\alpha'}\,. \end{aligned}$$

Thus

$$t \in I_{n,m} \Longrightarrow t \in J_{n,C\,m^{1/\alpha'}}\,. \tag{230}$$

Define now $K_m :=]0,T[\setminus J_{2^{-1}\,C\,m^{1/\alpha'}}$ open. Owing to (230), $K_m \subset [0,T] \setminus I_{n,m}$ for all $n \geq n_0(m)$. Using the alternative of Lemma 8.8

$$\|q^{\epsilon_n}(t)\|_{L^2(\Omega;\,\mathbb{R}^{N-1})} \leq C_m\,(\|\nabla q^{\epsilon_n}(t)\|_{L^2(\Omega)} + 1) \tag{231}$$

$$\|q^{\epsilon_n}\|_{L^2(K_m\times\Omega;\,\mathbb{R}^{N-1})} \leq C_{0,m}\,. \tag{232}$$

Moreover, by the definition of K_m, the inclusion (229) and (228), we see that

$$\lambda_1([0,T] \setminus K_m) \leq \lambda_1(I_{n,2^{-1}\,(C\,m^{1/\alpha'})^{1/2}}) \leq C\,\frac{1}{(\circ_{N+1}\ln)(m)}\,. \tag{233}$$

Making use of (232) can now extract a diagonal subsequence such that $q^{\epsilon_n} \rightarrow q$ weakly in $L^2(K_m \times [\Omega \cup \Gamma];\,\mathbb{R}^{N-1})$ for all $m \in \mathbb{N}$. Let next $\zeta \in C_c(K_m)$ arbitrary. From (231) we deduce that

$$\begin{aligned} \int_0^T \zeta(t)\,\|q(t)\|_{L^2(\Omega;\,\mathbb{R}^{N-1})}\,dt &\leq \liminf_{n\to+\infty} \int_0^T \zeta(t)\,\|q^{\epsilon_n}(t)\|_{L^2(\Omega;\,\mathbb{R}^{N-1})}\,dt \\ &\leq C_m \int_0^T \zeta(t)\,(A(t)+1)\,dt \end{aligned}$$

Here, $A \in L^2(0,T)$ is a weak limit in $L^2(0,T)$ of the sequence $\{\|\nabla q^{\epsilon_n}(t)\|_{L^2(\Omega)}\}$. Thus, we obtain the majoration $\|q(t)\|_{L^2(\Omega;\,\mathbb{R}^{N-1})} \leq C_m\,(A(t)+1)$ for almost all $t \in K_m$. From this we deduce that

$$\|q(t)\|_{L^2(\Omega;\,\mathbb{R}^{N-1})} \leq C_m \text{ or } t \in [0,T] \setminus K_m \cup \{t\,:\,A(t) > m\}\,.$$

Clearly, together with (233), this yields a bound for q in $L^w_{(\circ_N\ln)}L^1(Q)$.

We then easily see that q possesses for almost all t weak partial derivatives, and that $\nabla q^{\epsilon_n} \rightarrow \nabla q$ weakly in $L^2(Q;\,\mathbb{R}^{(N-1)\times 3})$. $\square$

Finally we can identitfy the remaining limits.

Corollary 9.11. *Assumptions of Corollary 9.8. Let J, p, r and $\hat{r}$ denote the weak limit of J^{ϵ_n}, p_{ϵ_n}, r^{ϵ_n} and $\hat{r}^{\epsilon_n}$ constructed in the Lemma 9.1. Then, for almost all $t \in]0, T[$*

$$\begin{aligned}
&J = M(\rho)\,(\nabla\mathcal{E}\,q + \frac{z}{m}\,\nabla\phi)\\
&p = P(\varrho,\, q)\\
&r = \sum_{k=1}^{s} \gamma^k\,\bar{R}_k(D^R) \text{ with } D^R_k = \gamma^k \cdot \mathcal{E}q \text{ in } Q^+(\varrho)\\
&\hat{r} = \sum_{k=1}^{s^\Gamma} \hat{\gamma}^k\,\hat{R}^\Gamma_k(\hat{D}^{\Gamma,R},\, w^0) \text{ with } \hat{D}^{\Gamma,R}_k = \hat{\gamma}^k \cdot \mathcal{E}q \text{ on } S^+(\varrho)\,.
\end{aligned}$$

Proof. Exploiting the convergence properties stated in the Corollary 9.8 and the Lemma 9.1, 9.10 we see that

$$J_{\epsilon_n} = M(\rho^{\epsilon_n})\,(\nabla\mathcal{E}q^{\epsilon_n} + \frac{z}{m}\,\nabla\phi_{\epsilon_n}) \to M(\rho)\,(\nabla\mathcal{E}\,q + \frac{z}{m}\,\nabla\phi) \text{ weakly in } L^{2,\frac{2\alpha}{1+\alpha}}(Q)\,.$$

Moreover, $P(\varrho_{\epsilon_n},\, q^{\epsilon_n}) \to P(\varrho,\, q)$ pointwise in $Q^+(\varrho)$, while $|P(\varrho_{\epsilon_n},\, q^{\epsilon_n})| \leq c\,\varrho^\alpha_{\epsilon_n} \to 0$ pointwise in $Q \setminus Q^+(\varrho)$. The other claims are proved similarly. □

We now resume the results of the section formulating our main (conditional) compactness statement.

Proposition 9.12. *Consider a family $\{(\varrho_\epsilon,\, q^\epsilon,\, v^\epsilon,\, \phi_\epsilon,\, R^\epsilon,\, R^{\Gamma,\epsilon})\}_{\epsilon>0}$ which satisfies a uniform bound in the class $\mathcal{B}(T,\, \Omega,\, \alpha,\, N-1,\, \Psi,\, \Psi^\Gamma)$. Assume the condition* (219) *on the time derivatives. Assume that the family ϱ_ϵ is compact in $L^1(\Omega)$ uniformly in time (sense of Remark 9.6).*

Then, there is a limiting element $(\varrho,\, q,\, v,\, \phi,\, R,\, R^\Gamma) \in \mathcal{B}$ and subsequence $\{\epsilon_n\}_{n\in\mathbb{N}}$ such that

$$\begin{aligned}
&\rho^{\epsilon_n} \to \rho \text{ strongly in } L^1(Q;\, \mathbb{R}^N)\\
&J^{\epsilon_n} \to J \text{ weakly in } L^{2,\frac{2\alpha}{1+\alpha}}(Q;\, \mathbb{R}^{N\times 3})\\
&R^{\epsilon_n} \to R \text{ weakly in } L^1(Q;\, \mathbb{R}^s), \quad R^{\Gamma,\epsilon_n} \to R^\Gamma \text{ weakly in } L^1(S;\, \mathbb{R}^{\hat{s}^\Gamma})\\
&v^{\epsilon_n} \to v \text{ weakly in } W^{1,0}_2(Q;\, \mathbb{R}^3)\\
&\phi_{\epsilon_n} \to \phi \text{ strongly in } W^{1,0}_2(Q)\\
&n^F_{\epsilon_n}\,\nabla\phi_{\epsilon_n} \to n^F\,\nabla\phi \text{ weakly in } L^1(Q;\, \mathbb{R}^3)\\
&\varrho_{\epsilon_n}\,v^{\epsilon_n} \to \varrho\,v \text{ strongly in } L^1(Q;\, \mathbb{R}^3)\\
&\varrho_{\epsilon_n}\,v^{\epsilon_n} \otimes v^{\epsilon_n} \to \varrho\,v \otimes v \text{ weakly in } L^{\frac{5\alpha-3}{3\,\alpha}}(Q;\, \mathbb{R}^{3\times 3})\,.
\end{aligned}$$

Here the quantities ρ, J, r, $\hat{r}$, p, n^F obey the natural definitions (95).

We finally note an important consequence of Proposition 9.12.

Corollary 9.13. *Assumptions of Proposition 9.12. Suppose that* $\{(\varrho_\epsilon,\, q^\epsilon,\, v^\epsilon,\, \phi_\epsilon,\, R^\epsilon,\, R^{\Gamma,\epsilon})\}_{\epsilon>0}$ *satisfies the enerqy inequality with mobility matrix* $M_\epsilon \geq M$*, and a free energy function* h^ϵ *having the property*

$$\rho^\epsilon \to \rho \in \mathbb{R}^N_{0,+} \Longrightarrow \liminf_{\epsilon\to 0} h^\epsilon(\rho^\epsilon) \geq h(\rho)\,.$$

Then the limiting element $(\varrho,\, q,\, v,\, \phi,\, R,\, R^\Gamma)$ *constructed in Proposition 9.12 satisfies the enerqy inequality with free energy function* h *and mobility matrix* M*.*

Proof. We first prove that

$$\liminf_{\epsilon\to 0} \int_{Q_t} M(\rho^\epsilon)\, D^\epsilon \cdot D^\epsilon \geq \int_{Q_t} M(\rho)\, D\cdot D\,. \tag{234}$$

Since $M(\rho^\epsilon)$ is a positive semidefinite matrix, we can introduce its square-root $M^{\frac{1}{2}}(\rho^\epsilon)$, and we easily realise that

$$M^{\frac{1}{2}}(\rho^\epsilon) \to M^{\frac{1}{2}}(\rho^\epsilon) \text{ strongly in } L^r(Q) \text{ for all } 1 \leq r < 2\,\alpha\,.$$

On the other hand, the driving forces $D^\epsilon = \nabla \mathcal{E} q^\epsilon + \frac{z}{m}\,\nabla\phi_\epsilon$ converge weakly to $D = \nabla\mathcal{E}q + \frac{z}{m}\,\nabla\phi$ in $L^2(Q)$ owing to the Proposition 9.1. Thus, since $\alpha > 1$, it follows that

$$M^{\frac{1}{2}}(\rho^\epsilon)\, D^\epsilon \to M^{\frac{1}{2}}(\rho)\, D \text{ weakly in } L^1(Q)\,. \tag{235}$$

Since the dissipation inequality implies that $M^{\frac{1}{2}}(\rho^\epsilon)\, D^\epsilon$ is uniformly bounded even in $L^2(Q)$, we can show that the weak convergence (235) takes place even in $L^2(Q)$. Thus, due to the lower semicontinuity of the L^2-norm, we obtain (234).

In order to prove the lower semicontinuity of the reaction terms, we use the lower semicontinuity (convexity) of the functions Ψ and Ψ^*, to obtain that

$$\liminf_{\epsilon\to 0} \int_Q (\Psi(D^{\mathsf{R},\epsilon}) + \Psi^*(-R^\epsilon)) \geq \int_Q (\Psi(D^{\mathsf{R}}) + \Psi^*(-R))\,.$$

Analogoulsy, on the boundary

$$\begin{aligned}\liminf_{\epsilon\to 0} &\int_S (\hat{\Psi}^\Gamma(\hat{D}^{\Gamma,\mathsf{R},\epsilon},\, w^0) + (\hat{\Psi}^\Gamma)^*(-R^{\Gamma,\epsilon},\, w^0))\\ &\geq \int_S (\hat{\Psi}^\Gamma(\hat{D}^{\Gamma,\mathsf{R}},\, w^0) + (\hat{\Psi}^\Gamma)^*(-R^\Gamma,\, w^0))\,.\end{aligned}$$

□

10 The structure of the Navier-Stokes operator

In the section 9 we showed that boundedness in the energy class together with the existence of weak time derivatives implies the compactness of the solution vector if the condition $\varrho(t) \in K^*$ is satisfied, where K^* is a compact of $L^1(\Omega)$. Using an extension of the method of Lions for the compressible Navier-Stokes operator, we are going to show that this condition is satisfied for the approximation schemes of interest to us. We commence formulating our main statement.

Proposition 10.1. *Consider a family $\{(\varrho_\epsilon,\, q^\epsilon,\, v^\epsilon,\, \phi_\epsilon,\, R^\epsilon,\, R^{\Gamma,\epsilon})\}_{\epsilon>0} \subset \mathcal{B}$ which is uniformly bounded in the natural class $\mathcal{B}(T,\, \Omega,\, \alpha,\, N-1,\, \Psi,\, \Psi^\Gamma)$ and satisfies the assumptions of Lemma 9.1. Let $\{\bar{J}^\epsilon\}_{\epsilon>0} \subset L^2(Q;\, \mathbb{R}^3)$ be a family of perturbations such that $\bar{J}^\epsilon \to 0$ strongly in $L^2(Q)$ as $\epsilon \to 0$ and such that*

$$\begin{cases} \limsup_{\epsilon\to 0} \|(\bar{J}^\epsilon \cdot \nabla \ln \varrho_\epsilon)^+\|_{L^1(Q)} = 0 & \text{if } \alpha > 3 \\ \bar{J}^\epsilon \equiv 0 & \text{if } \frac{3}{2} < \alpha \leq 3\,. \end{cases} \tag{236}$$

Suppose that the identities

$$- \int_Q \varrho_\epsilon\, \psi_t - \int_Q (\varrho_\epsilon\, v^\epsilon + \bar{J}^\epsilon) \cdot \nabla \psi = \int_\Omega \varrho_0\, \psi(0) \tag{237}$$

$$\begin{aligned} - \int_Q \varrho_\epsilon\, v^\epsilon \cdot \eta_t - \int_Q \varrho_\epsilon\, v^\epsilon \otimes v^\epsilon \,:\, \nabla\eta - \int_Q p_\epsilon \operatorname{div} \eta + \int_Q \mathbb{S}(\nabla v^\epsilon) \cdot \nabla \eta \\ = \int_\Omega \varrho_0\, v^0 \cdot \eta(0) + \int_Q (\bar{J}^\epsilon \cdot \nabla)\eta \cdot v^\epsilon - \int_Q n^F_\epsilon\, \nabla \phi_\epsilon \cdot \eta\,. \end{aligned} \tag{238}$$

are valid for all $\psi \in C^1_c([0, T[;\, C^1(\overline{\Omega}))$ and all $\eta \in C^1_c([0, T[;\, C^1_c(\Omega;\, \mathbb{R}^3))$. Assume that either $\alpha \geq 9/5$, or that the function P of Lemma 5.4 is convex in the first variable and that $\frac{3}{2} < \alpha < 9/5$.

Then for every sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$, the sequence $\{\varrho_{\epsilon_n}\}_{n\in\mathbb{N}}$ is compact in $L^1(\Omega)$ uniformly with respect to time (sense of Remark 9.6).

Remark 10.2. *Under the assumptions of the Proposition 10.1, we apply the Lemmas 9.1, 9.4 and we find a subsequence such that*

$\rho^{\epsilon_n} \to \rho$ *weakly in* $L^\alpha(Q;\, \mathbb{R}^N)$

$\rho^{\epsilon_n}(t) \to \rho(t)$ *weakly in* $L^\alpha(\Omega;\, \mathbb{R}^N)$ *for almost all* $t \in [0, T]$

$v^{\epsilon_n} \to v$ *weakly in* $W^{1,0}_2(Q;\, \mathbb{R}^3)$

$p_{\epsilon_n} \to p$ *weakly in* $L^{1+\min\{\frac{1}{\alpha},\, \frac{2}{3}-\frac{1}{\alpha}\}}(Q)$

$\phi_{\epsilon_n} \to \phi$ *strongly in* $W^{1,0}_2(Q)$

$\dfrac{z}{m} \cdot \rho^{\epsilon_n}\, \nabla \phi_{\epsilon_n} \to \dfrac{z}{m} \cdot \rho\, \nabla\phi$ *weakly in* $L^1(Q)$

$\varrho_{\epsilon_n}\, v^{\epsilon_n} \to \varrho\, v$ *weakly in* $L^{2, \frac{6\alpha}{6+\alpha}}(Q; \mathbb{R}^3)$

$(\varrho_{\epsilon_n}\, v^{\epsilon_n})(t) \to \varrho(t)\, v(t)$ *weakly in* $L^{\frac{2\alpha}{1+\alpha}}(\Omega;\, \mathbb{R}^3)$ *for almost all* $t \in [0, T]$

$\varrho_{\epsilon_n}\, v^{\epsilon_n} \otimes v^{\epsilon_n} \to \varrho\, v \otimes v$ *weakly in* $L^{\frac{5\alpha-3}{3\,\alpha}}(Q;\, \mathbb{R}^{3\times 3})$

$\varrho_{\epsilon_n}\, (v^{\epsilon_n} - v) \to 0$ *strongly in* $L^1(Q; \mathbb{R}^3)\,.$

and these weak limiting elements satisfy

$$- \int_Q \varrho \, \psi_t - \int_Q \varrho \, v \cdot \nabla \psi = \int_\Omega \varrho_0 \, \psi(0) \tag{239}$$

$$\begin{aligned} & - \int_Q \varrho \, v \cdot \eta_t - \int_Q \varrho \, v \otimes v \, : \, \nabla \eta - \int_Q p \, \operatorname{div} \eta + \int_Q \mathbb{S}(\nabla v) \cdot \nabla \eta \\ & \quad = \int_\Omega \varrho_0 \, v^0 \cdot \eta(0) - \int_Q n^F \, \nabla \phi \cdot \eta \end{aligned} \tag{240}$$

for all $\psi \in C^1_c([0,T[; \, C^1(\overline{\Omega}))$ *and for all* $\eta \in C^1_c([0,T[; \, C^1_c(\Omega; \, \mathbb{R}^3))$.

The section is devoted to the proof of Proposition 10.1. There is a branching in the proof: We consider separately the cases $\alpha > 3$ and $3/2 < \alpha \leq 3$.

10.1 The case $\alpha > 3$

We are going to establish after Lions convergence properties associated with the effective viscous flux $p_{\epsilon_n} - \eta' \, \operatorname{div} v^{\epsilon_n}$. Here we abbreviate $\eta' := \lambda + 2\,\eta > 0$.

Lemma 10.3. *Let* p, v *and* ϱ *denotes the weak limits according to the Remark 10.2. Then*

$$(p_{\epsilon_n} - \eta' \, \operatorname{div} v^{\epsilon_n}) \, \varrho_{\epsilon_n} \to (p - \eta' \, \operatorname{div} v) \, \varrho \text{ as distributions in } Q \, .$$

Proof. For convenience we give a proof in the Appendix. □

We next use an important property of our regularisation.

Lemma 10.4. *Let* ϱ_ϵ *satisfy* (237)*. Then* $\varrho^\epsilon \in C([0,T]; \, L^1(\Omega))$*, and for all* $t \in [0,T]$

$$\int_\Omega \varrho_\epsilon(t) \, \ln \varrho_\epsilon(t) - \int_\Omega \varrho_0 \, \ln \varrho_0 + \int_0^t \int_\Omega \varrho_\epsilon \, \operatorname{div} v^\epsilon \leq C \, \|(\bar{J}^\epsilon \cdot \nabla \ln \varrho_\epsilon)^+\|_{L^1(Q)} \, . \tag{241}$$

Denote ϱ *the weak limit of* $\{\varrho_{\epsilon_n}\}$*. Then* $\varrho_\epsilon \in C([0,T]; \, L^1(\Omega))$ *and for all* $t \in [0,T]$

$$\int_\Omega \varrho(t) \, \ln \varrho(t) - \int_\Omega \varrho_0 \, \ln \varrho_0 + \int_0^t \int_\Omega \varrho \, \operatorname{div} v = 0 \, . \tag{242}$$

Proof. Owing to the Lemma 8.19, we can rely for $\epsilon > 0$ on the fact that $\ln \varrho_\epsilon \in W^{1,0}_2(Q)$ (cf. Lemma 8.19). Using well known time smoothing techniques, of which we spare the details here, we can multiply the equation (237) with the function $1 + \ln \varrho_\epsilon$. If follows for almost all $t \in]0,T[$ that

$$\int_\Omega \varrho_\epsilon(t) \, \ln \varrho_\epsilon(t) - \varrho_0 \, \ln \varrho_0 + \int_0^t \int_\Omega \varrho_\epsilon \, \operatorname{div} v^\epsilon - \int_0^t \int_\Omega \bar{J}^\epsilon \cdot \nabla \ln \varrho_\epsilon = 0 \, .$$

The first claim (241) follows.

The second claim (242) follows from the fact that ϱ is a renormalised solution to (239). This was shown in [Lio98] (for instance on page 14, see also [Lio96], Lemma 2.3) and [FNP01], section 3.5.

In order to state (241), (242) for all $t \in [0,T]$, we need $\varrho^{\epsilon},\ \varrho \in C([0,T];\ L^1(\Omega))$. This was proved in [Lio98], page 23. We provide a proof for convenience in the appendix. $\square$

In order to prove the compactness of the mass density we need a last observation in the following Lemma. Comparable ideas are to find for instance in [Lio98], section 8.5.

Lemma 10.5. *If $p,\, v$ and ϱ denotes the weak limits according to the Remark 10.2. Then, for all $\zeta \in C^1(\overline{Q})$ such that $\zeta \geq 0$ in Q, there holds $\liminf_{n\to\infty} \int_Q p_{\epsilon_n}\, \varrho_{\epsilon_n}\, \zeta \geq \int_Q p\, \varrho\, \zeta + c_0 \liminf_{n\to\infty} \int_Q (\varrho_{\epsilon_n} - \varrho)^2\, \zeta$.*

Proof. We note that $p_{\epsilon_n} = P(\varrho_{\epsilon_n},\, q^{\epsilon_n})$ with the function P of Lemma 5.4. Moreover, due to Lemma 5.4 $\partial_s P \geq c_0$. For arbitrary nonnegative $u \in C^1(\overline{Q})$, we therefore obtain that $(P(\varrho_{\epsilon_n},\, q^{\epsilon_n}) - P(u,\, q^{\epsilon_n}))\,(\varrho_{\epsilon_n} - u) \geq c_0\,(\varrho_{\epsilon_n} - u)^2$. This entails

$$\liminf_{n\to\infty} \int_Q p_{\epsilon_n}\, \varrho_{\epsilon_n}\, \zeta - \int_Q p\, u\, \zeta \geq \liminf_{n\to\infty} \int_\Omega P(u,\, q^{\epsilon_n})\,(\varrho_{\epsilon_n} - u)\, \zeta \\ + c_0 \liminf_{n\to\infty} \int_Q (\varrho_{\epsilon_n} - u)^2\, \zeta\,.$$

We note that $\nabla P(u,\, q^{\epsilon_n}) = P_s(u,\, q^{\epsilon_n})\, \nabla u + \sum_{j=1}^{N-1} P_{q_j}(u,\, q^{\epsilon_n})\, \nabla q_j^{\epsilon_n}$, the Lemma 5.4 implies that

$$|\nabla P(u,\, q^{\epsilon_n})| \leq c\, \{|u|^{\alpha-1}\, |\nabla u| + |u|^{\alpha}\, |\nabla q^{\epsilon_n}|\}\,.$$

It follows that $\|\nabla P(u,\, q^{\epsilon_n})\|_{L^2(Q)} \leq C_u\, C_0$. Since moreover $|P(u,\, q^{\epsilon_n})| \leq C\, |u|^{\alpha}$, there is $a = a_u \in L^\infty(Q) \cap W^{1,0}_2(Q)$ and a subsequence such that

$$P(u,\, q^{\epsilon_n}) \to a \text{ weakly in } W^{1,0}_2(Q)\,.$$

We easily show that $\int_Q P(u,\, q^{\epsilon_n})\,(\varrho_{\epsilon_n} - u)\, \zeta \to \int_Q a\,(\varrho - u)\, \zeta$. Note that the inequality $P(u,\, q^{\epsilon_n}) \leq c\, |u|^{\alpha}$ implies that $|a| \leq c\, |u|^{\alpha}$. We obtain that

$$\liminf_{n\to\infty} \int_Q p_{\epsilon_n}\, \varrho_{\epsilon_n}\, \zeta - \int_Q p\, u\, \zeta \geq \int_Q a\,(\varrho - u)\, \zeta + c_0\, \liminf_{n\to\infty} \int_Q (\varrho_{\epsilon_n} - u)^2\, \zeta\,.$$

It suffices now to approximate ϱ in $L^\alpha(Q)$ with functions u of $C^1(\overline{Q})$ to obtain the claim. $\square$

Lemma 10.6. *Assumptions of the Proposition 10.1 for $\alpha > 3$. Then for every sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$ such that the convergence properties* (10.2) *are valid:*

1. *$\varrho_{\epsilon_n}(t) \to \varrho(t)$ strongly in $L^1(\Omega)$ for almost all $t \in]0,T[$.*
2. *The family $\bigcup_{t\in[0,T]} \bigcup_{n\in\mathbb{N}} \{\varrho_{\epsilon_n}(t)\}$ is sequentially compact in $L^1(\Omega)$.*

Proof. We consider an arbitrary sequence of times $\{t_n\}_{n\in\mathbb{N}} \subset]0,T[$ such that $t_n \to t^*$ for $n \to \infty$. We choose for $j \in \mathbb{N}$ a nonnegative function $f_j \in C^1(\mathbb{R})$ with the following properties

$$f_j(s) \begin{cases} = 0 & \text{for } s \leq j^{-1} \\ \in [0,1] & \text{for } s \in [j^{-1},\, 2j^{-1}]\,, \quad |f_j'(s)| \leq c\, j\,. \\ = 1 & \text{for } s \geq 2\, j^{-1} \end{cases}$$

We define $\zeta_{j,n} \in C^1_c(Q)$ via

$$\zeta_{j,n}(t,\, x) := f_j(t_n - t)\, f_j(\mathrm{dist}(x,\, \partial\Omega))\,.$$

Note that $\zeta_{j,n} \to \zeta_j$ uniformly in Q for $n \to \infty$ with $\zeta_j := f_j(t^*-t)\, f_j(\mathrm{dist}(x,\, \partial\Omega))$. Moreover $|\nabla_4 \zeta_{j,n}| \leq c\, j$ and

$$\|\zeta_{j,n} - \chi_{[0,t_n]}\, \chi_\Omega\|_{L^{2,\frac{2\alpha}{\alpha-2}}(Q)} \leq c\, \left(\frac{1}{j}\right)^{\frac{1}{2}+\frac{\alpha-2}{2\alpha}}\,. \tag{243}$$

We then express

$$\int_Q p_{\epsilon_n}\, \varrho_{\epsilon_n}\, \zeta_{j,n} = \int_Q (p_{\epsilon_n} - \eta'\, \mathrm{div}\, v^{\epsilon_n})\, \varrho_{\epsilon_n}\, \zeta_{j,n}$$
$$+ \eta' \int_Q \mathrm{div}\, v^{\epsilon_n}\, \varrho_{\epsilon_n}\, (\zeta_{j,n} - \chi_{[0,t_n]}\, \chi_\Omega) + \eta' \int_{Q_{t_n}} \mathrm{div}\, v^{\epsilon_n}\, \varrho_{\epsilon_n}\,.$$

Thus, because of the identity (241)

$$\eta' \int_\Omega (\varrho_{\epsilon_n}(t_n)\, \ln \varrho_{\epsilon_n}(t_n) - \varrho_0\, \ln \varrho_0) + \int_Q p_{\epsilon_n}\, \varrho_{\epsilon_n}\, \zeta_{j,n}$$
$$= \int_Q (p_{\epsilon_n} - \eta'\, \mathrm{div}\, v^{\epsilon_n})\, \varrho_{\epsilon_n}\, \zeta_{j,n} + \eta' \int_Q \mathrm{div}\, v^{\epsilon_n}\, \varrho_{\epsilon_n}\, (\zeta_{j,n} - \chi_{[0,t_n]}\, \chi_\Omega)\,.$$

Moreover, owing to (242),

$$\eta' \int_\Omega (\varrho(t^*)\, \ln \varrho(t^*) - \varrho_0\, \ln \varrho_0) + \int_Q p\, \varrho\, \zeta_{j,n}$$
$$= \int_Q (p - \eta'\, \mathrm{div}\, v)\, \varrho\, \zeta_{j,n} + \eta' \int_Q \mathrm{div}\, v\, \varrho\, (\zeta_{j,n} - \chi_{[0,t^*]}\, \chi_\Omega)\,.$$

Thus, subtracting the two latter identities

$$\eta' \int_\Omega (\varrho_{\epsilon_n}(t_n)\, \ln \varrho_{\epsilon_n}(t_n) - \varrho(t^*)\, \ln \varrho(t^*)) + \int_Q (p_{\epsilon_n}\, \varrho_{\epsilon_n} - p\, \varrho)\, \zeta_{j,n}$$
$$= \int_Q ((p_{\epsilon_n} - \eta'\, \mathrm{div}\, v^{\epsilon_n})\, \varrho_{\epsilon_n} - (p - \eta'\, \mathrm{div}\, v)\, \varrho)\, \zeta_{j,n}$$
$$+ \eta' \int_Q \mathrm{div}\, v^{\epsilon_n}\, \varrho_{\epsilon_n}\, (\zeta_{j,n} - \chi_{[0,t_n]}\, \chi_\Omega) - \eta' \int_Q \mathrm{div}\, v\, \varrho\, (\zeta_{j,n} - \chi_{[0,t^*]}\, \chi_\Omega)\,.$$

Due to (243)

$$\left|\int_Q \operatorname{div} v^{\epsilon_n}\, \varrho_{\epsilon_n}\, (\zeta_{j,n} - \chi_{[0,t_n]}\, \chi\Omega)\right| \le \|\operatorname{div} v^{\epsilon_n}\, \varrho_{\epsilon_n}\|_{L^{2,\frac{2\alpha}{2+\alpha}}(Q)}\, \|\zeta_{j,n} - \chi_{[0,t_n]}\, \chi\Omega\|_{L^{2,\frac{2\alpha}{2-\alpha}}(Q)}$$
$$\le C_0\, j^{-1}\,.$$

Moreover, we easily show that $\|\zeta_{j,n} - \chi_{[0,t^*]}\, \chi\Omega\|_{L^{2,\frac{2\alpha}{\alpha-2}}(Q)} \le c\, j^{-1} + |t_n - t^*|^{1/2}$, and therefore

$$\left|\int_Q \operatorname{div} v\, \varrho\, (\zeta_{j,n} - \chi_{[0,t^*]}\, \chi\Omega)\right| \le C_0\, (j^{-1} + |t_n - t^*|^{1/2})\,.$$

Since $\zeta_{n,j} \to \zeta_j$ uniformly in Q, the Lemma 10.3 implies that

$$\lim_{n\to\infty} \int_Q ((p_{\epsilon_n} - \eta'\, \operatorname{div} v^{\epsilon_n})\, \varrho_{\epsilon_n} - (p - \eta'\, \operatorname{div} v)\, \varrho)\, \zeta_{j,n}$$
$$= \lim_{n\to\infty} \int_Q ((p_{\epsilon_n} - \eta'\, \operatorname{div} v^{\epsilon_n})\, \varrho_{\epsilon_n} - (p - \eta'\, \operatorname{div} v)\, \varrho)\, \zeta_j = 0\,.$$

Further

$$\lim_{n\to\infty} \int_Q (p_{\epsilon_n}\, \varrho_{\epsilon_n} - p\, \varrho)\, \zeta_{j,n} = \lim_{n\to\infty} \int_Q (p_{\epsilon_n}\, \varrho_{\epsilon_n} - p\, \varrho)\, \zeta_j \ge c_0\, \lim_{n\to\infty} \int_Q (\varrho_{\epsilon_n} - \varrho)^2\, \zeta_j$$
$$= c_0\, \lim_{n\to\infty} \int_Q (\varrho_{\epsilon_n} - \varrho)^2\, (\zeta_j - \chi_{[0,t^*]}\, \chi\Omega) + c_0\, \lim_{n\to\infty} \int_{Q_{t^*}} (\varrho_{\epsilon_n} - \varrho)^2$$
$$\ge c_0\, \lim_{n\to\infty} \int_{Q_{t^*}} (\varrho_{\epsilon_n} - \varrho)^2 - \|\varrho_{\epsilon_n} - \varrho\|^2_{L^{\infty,\alpha}(Q)}\, \left(\frac{1}{j}\right)^{1+\frac{\alpha-2}{\alpha}}$$

It follows that there is $r > 0$ such that

$$\eta'\, \liminf_{n\to\infty} \int_\Omega (\varrho_{\epsilon_n}(t_n)\, \ln \varrho_{\epsilon_n}(t_n) - \varrho(t^*)\, \ln \varrho(t^*))$$
$$+ c_0\, \liminf_{n\to\infty} \int_{Q_t} (\varrho_{\epsilon_n} - \varrho)^2 \le C_0\, j^{-r}\,. \tag{244}$$

Since $\varrho_{\epsilon_n} \in C([0,T];\, \mathcal{D}^*(\Omega))$, we show easily that $\varrho_{\epsilon_n}(t_n) \to \varrho(t^*)$ as distributions in Ω, and this added to (244) yields

$$\varrho_{\epsilon_n}(t_n) \to \varrho(t^*) \text{ strongly in } L^1(\Omega)\,. \tag{245}$$

We now deduce both claims of the Lemma.

In order to estabilsh (1), we choose $t_n = t \in [0,T]$ fixed. Then, due to (245), we see that $\varrho_{\epsilon_n}(t) \to \varrho(t)$ strongly in $L^1(\Omega)$. The claim (1) follows

In order to prove (2), we observe that $\varrho_{\epsilon_n} \in C([0,T];\, L^1(\Omega))$ for all $n \in \mathbb{N}$ and that also $\varrho \in C([0,T];\, L^1(\Omega))$. This was observed in Lemma 10.4. Consider any sequence in the set $\bigcup_{t\in[0,T]} \bigcup_{n\in\mathbb{N}} \{\varrho_{\epsilon_n}(t)\}$. Such a sequence is of the form $\{\varrho_{\epsilon_{n_k}}(t_k)\}_{k\in\mathbb{N}}$. We can always extract a subsequence such that $t_k \to t^* \in [0,T]$, an applying the result (245), it follows that $\varrho_{\epsilon_{n_k}}(t_k) \to \varrho(t^*)$ strongly in $L^1(\Omega)$. Thus, the set $\bigcup_{t\in[0,T]} \bigcup_{n\in\mathbb{N}} \{\varrho_{\epsilon_n}(t)\}$ is sequentially compact in $L^1(\Omega)$. $\square$

10.2 The case $3/2 < \alpha \le 3$

Since we cannot rely on the condition $\alpha > 3$, additional technical problems occur. Nevertheless the passage to the limit can be carried over using an extension of the method of Lions ($\alpha \ge 9/5$, [Lio98], Chapter 5) and Feireisl, Novotný and Petzeltová ($3/2 < \alpha < 9/5$, [FNP01]) . Here we have to assume that the approximate solutions satisfy global mass conservation exactly (the perturbation $\bar{J}^\epsilon$ in (239), (240) vanishes). In particular

$$-\int_Q \varrho_\epsilon \, \psi_t - \int_Q \varrho_\epsilon \, v^\epsilon \cdot \nabla\psi = \int_\Omega \varrho_0 \, \psi(0) \text{ for all } \psi \in C^1_c([0,T[;\, C^1(\overline{\Omega})) \tag{246}$$

The Lemma 10.3 and the further reasoning have to be modified. Here we will stick to the approach of Feireisl, Novotný and Petzeltová in [FNP01]. One introduces for $k \in \mathbb{N}$ the cutoff function

$$T_k(\varrho_\epsilon) := \min\{\varrho_\epsilon,\, k\}\,.$$

It is possible to extract a subsequence (which might be a different one for all values of k), and to find $a_k \in L^\infty(Q)$ such that

$$T_k(\varrho_\epsilon) \to a_k \text{ weakly in } L^p(Q) \text{ for all } 1 < p < \infty\,.$$

Exploiting the a priori bounds it follows that

$$\|T_k(\varrho_\epsilon)(t) - \varrho_\epsilon(t)\|_{L^1(\Omega)} \le (|\{x \,:\, \varrho_\epsilon(t,x) \ge k\}|^{1/\alpha'}\, \|\varrho_\epsilon\|_{L^\alpha(\Omega)} \le C_0\, \left(\frac{1}{k}\right)^{\frac{\alpha}{\alpha'}}\,,$$

so that $\|a_k(t) - \varrho(t)\|_{L^1(\Omega)} \le C_0\, \left(\frac{1}{k}\right)^{\frac{\alpha}{\alpha'}}$. Thus, a_k is an approximation of ϱ. Now, the arguments of [FNP01], Lemma 4.4 allow to prove that the limit ϱ is also a renormalised solution to (239), and to obtain the following statement.

Lemma 10.7. *Let ϱ_ϵ satisfy* (246)*. Define*

$$L_k(\varrho) := \begin{cases} \varrho\, \ln\varrho & \text{if } \rho \le k \\ \varrho\, \ln k + \varrho - k & \text{otherwise} \end{cases}$$

Then, for all $\epsilon > 0$, the function ϱ_ϵ belongs to $C([0,T];\, L^1(\Omega))$ and for all $t \in [0,T]$

$$\int_\Omega L_k(\varrho_\epsilon)(t) - \int_\Omega L_k(\varrho_0) + \int_t^0 \int_\Omega T_k(\varrho_\epsilon)\, \operatorname{div} v^\epsilon = 0\,. \tag{247}$$

Denote ϱ the weak limit of $\{\varrho_{\epsilon_n}\}$. Then $\varrho \in C([0,T];\, L^1(\Omega))$ and for all $t \in [0,T]$

$$\int_\Omega L_k(\varrho)(t) - \int_\Omega L_k(\varrho_0) + \int_0^t \int_\Omega T_k(\varrho)\, \operatorname{div} v = 0\,. \tag{248}$$

Proof. We can reproduce the proof [FNP01], Lemma 4.4 (see also the section 4.6) one to one. □

With the same method as in the Lemma 10.3, one moreover proves

Lemma 10.8. *Let p, v and ϱ denotes the weak limits according to 10.2. Then, for one subsequence possibly depending on k*

$$(p_{\epsilon_n} - \eta' \operatorname{div} v^{\epsilon_n})\, T_k(\varrho_{\epsilon_n}) \to (p - \eta' \operatorname{div} v)\, a_k \text{ weakly in } L^1(Q)\,.$$

Proof. For conveninence, the reader can find a proof in the appendix. □

We next can establish the essential property of Lemma 10.5 also if $\alpha \leq 3$.

Lemma 10.9. *For all $t \in [0, T]$ there holds:*

$$\limsup_{n\to\infty} \int_{Q_t} (p_{\epsilon_n}\, T_k(\varrho_{\epsilon_n}) - p\, a_k) \geq c_0 \limsup_{n\to\infty} \int_{Q_t} (T_k(\varrho_{\epsilon_n}) - a_k)^2\,.$$

If P is moreover a convex function of ϱ (see Lemma 5.9), then

$$\limsup_{n\to\infty} \int_{Q_t} p_{\epsilon_n}\, T_k(\varrho_{\epsilon_n}) \geq \int_{Q_t} p\, T_k(\varrho) + c_0 \limsup_{n\to\infty} \int_{Q_t} (T_k(\varrho_{\epsilon_n}) - T_k(\varrho))^2\,.$$

Proof. We note that we can represent $p_{\epsilon_n} = P(\varrho_{\epsilon_n},\, q^{\epsilon_n})$ with the function P of Lemma 5.4. Recall that $\partial_s P \geq c_0$. For arbitrary nonnegative $u \in C^1(\overline{Q})$, we have

$$(P(\varrho_{\epsilon_n},\, q^{\epsilon_n}) - P(u,\, q^{\epsilon_n}))\, (T_k(\varrho_{\epsilon_n}) - T_k(u)) \geq c_0\, (T_k(\varrho_{\epsilon_n}) - T_k(u))^2\,.$$

As in the proof of the Lemma 10.5, we use that the functions $P(u,\, q^{\epsilon_n})$ have a bounded gradient in $L^{2,2\alpha/(1+\alpha)}(Q)$ for fixed u. Exploiting the weak convergence $p_{\epsilon_n} \rightharpoonup p$ and $T_k(\varrho_{\epsilon_n}) \rightharpoonup a_k$, we can show that

$$\begin{aligned}\limsup_{n\to\infty} \int_{Q_t} p_{\epsilon_n}\, T_k(\varrho_{\epsilon_n}) - \int_{Q_t} p\, T_k(u) \geq & \int_{Q_t} \beta(u)\, (a_k - T_k(u)) \\ & + c_0 \limsup_{n\to\infty} \int_{Q_t} (T_k(\varrho_{\epsilon_n}) - T_k(u))^2\,.\end{aligned}$$

Here, $\beta(u)$ denote a weak limit of $P(u,\, q^{\epsilon_n})$. Since $a_k \leq k$ almost everywhere in Q, it is possible to represent $a_k = T_k(a_k)$. Therefore, we can approximate a_k with functions $T_k(u)$, $u \in C^1(\overline{Q})$, and it follows that

$$\limsup_{n\to`\infty} \int_{Q_t} p_{\epsilon_n}\, T_k(\varrho_{\epsilon_n}) - \int_{Q_t} p\, a_k \geq c_0 \limsup_{n\to\infty} \int_{Q_t} (T_k(\varrho_{\epsilon_n}) - a_k)^2\,.$$

If P is a *convex* function depending only on ϱ, then we follow [FNP01], Lemma 4.3. □

At last we prove the equivalent of Lemma 10.6.

Lemma 10.10. *1 $\varrho_{\epsilon_n}(t) \to \varrho(t)$ strongly in $L^1(\Omega)$ for almost all $t \in]0, T[$.*

2 The family $\bigcup_{t\in[0,T]} \bigcup_{n\in\mathbb{N}} \{\varrho_{\epsilon_n}(t)\}$ is sequentially compact in $L^1(\Omega)$.

Proof. We have

$$\int_{Q_{t_n}} p_{\epsilon_n}\, T_k(\varrho_{\epsilon_n}) = \int_{Q_{t_n}} (p_{\epsilon_n} - \eta' \operatorname{div} v^{\epsilon_n})\, T_k(\varrho_{\epsilon_n}) + \eta' \int_{Q_{t_n}} \operatorname{div} v^{\epsilon_n}\, T_k(\varrho_{\epsilon_n})$$
$$= \int_{Q_{t_n}} (p_{\epsilon_n} - \eta' \operatorname{div} v^{\epsilon_n})\, T_k(\varrho_{\epsilon_n}) - \eta' \int_\Omega (L_k(\varrho_{\epsilon_n}(t_n)) - L_k(\varrho_0))$$

Invoking the Lemma 10.9

$$\limsup_{n\to\infty} \int_{Q_t} p_{\epsilon_n}\, T_k(\varrho_{\epsilon_n}) = \int_{Q_t} (p - \eta' \operatorname{div} v)\, a_k - \eta' \liminf_{n\to\infty} \int_\Omega (L_k(\varrho_{\epsilon_n}(t_n)) - L_k(\varrho_0))$$
$$= \int_{Q_t} p\, a_k - \eta' \int_{Q_t} \operatorname{div} v\, (a_k - T_k(\varrho)) - \eta' \int_{Q_t} \operatorname{div} v\, T_k(\varrho)$$
$$- \eta' \liminf_{n\to\infty} \int_\Omega (L_k(\varrho_{\epsilon_n}(t_n)) - L_k(\varrho_0))$$
$$= \int_{Q_t} p\, a_k - \eta' \int_{Q_t} \operatorname{div} v\, (a_k - T_k(\varrho)) + \eta' \left(\int_\Omega L_k(\varrho(t)) - \liminf_{n\to\infty} \int_\Omega L_k(\varrho_{\epsilon_n}(t_n)) \right) .$$

Now we distinguish two cases. In general, we obtain the inequality

$$c_0 \limsup_{n\to\infty} \int_{Q_t} (T_k(\varrho_{\epsilon_n}) - a_k)^2 + \eta' \liminf_{n\to\infty} \int_\Omega (L_k(\varrho_{\epsilon_n})(t_n) - L_k(\varrho)(t))$$
$$\leq -\eta' \int_{Q_t} \operatorname{div} v\, (a_k - T_k(\varrho)) .$$

In the case where the function P is convex in the first argument, there is the stronger statement

$$c_0 \limsup_{n\to\infty} \int_{Q_t} (T_k(\varrho_{\epsilon_n}) - T_k(\varrho))^2 + \eta' \liminf_{n\to\infty} \int_\Omega (L_k(\varrho_{\epsilon_n})(t_n) - L_k(\varrho)(t))$$
$$\leq -\eta' \int_{Q_t} \operatorname{div} v\, (a_k - T_k(\varrho)) .$$

Thus using that both terms on the left-hand are nonnegative

$$c_0 \limsup_{n\to\infty} \int_{Q_t} (T_k(\varrho_{\epsilon_n}) - T_k(\varrho))^2 \leq -\eta' \int_{Q_t} \operatorname{div} v\, (a_k - T_k(\varrho))$$
$$= -\eta' \lim_{n\to\infty} \int_{Q_t} \operatorname{div} v\, (T_k(\varrho_{\epsilon_n}) - T_k(\varrho))$$
$$\leq |\eta'|\, \|\operatorname{div} v\|_{L^2(Q)} \limsup_{n\to\infty} \|T_k(\varrho_{\epsilon_n}) - T_k(\varrho)\|_{L^2(Q_t)} .$$

This shows that $c_0\, \|a_k - T_k(\varrho)\|_{L^2(Q_t)} \leq |\eta'|\, \|\operatorname{div} v\|_{L^2(Q)}$.

Thus, in both cases, we will find that $a_k - T_k(\varrho)$ converges strongly to zero in $L^2(Q)$ as $k \to \infty$, and it follows that

$$
\begin{aligned}
&c_0 \limsup_{k\to\infty} \limsup_{n\to\infty} \int_{Q_t} (T_k(\varrho_{\epsilon_n}) - a_k)^2 \\
&\qquad + \liminf_{k\to\infty}\ \liminf_{n\to\infty} \int_\Omega (L_k(\varrho_\epsilon)(t_n) - \varrho(t)\ \ln \varrho(t)) \\
&= c_0 \limsup_{k\to\infty} \limsup_{n\to\infty} \int_{Q_t} (T_k(\varrho_{\epsilon_n}) - a_k)^2 \\
&\qquad + \liminf_{k\to\infty}\ \liminf_{n\to\infty} \int_\Omega (L_k(\varrho_{\epsilon_n})(t_n) - L_k(\varrho)(t)) \le 0\,.
\end{aligned} \tag{249}
$$

Now we introduce for $k > 2$ and $n \in \mathbb{N}$ the variables $u_{k,\epsilon}$ such that

$$
u_{k,\epsilon_n}\ \ln u_{k,\epsilon_n} = L_k(\varrho_{\epsilon_n})\,.
$$

Denoting ψ the inverse of the function $t \mapsto t\ \ln t$ in the range $[2, +\infty]$, we have

$$
\varrho_{\epsilon_n} - u_{k,\epsilon_n} = \begin{cases} 0 & \text{if } \varrho_{\epsilon_n} \le k \\ \varrho_{\epsilon_n} - \psi(\varrho_{\epsilon_n}\ \ln k + \varrho_{\epsilon_n} - k) & \text{otherwise} \end{cases}
$$

Thus

$$
\|u_{k,\epsilon_n}(t) - \varrho_{\epsilon_n}(t)\|_{L^1(\Omega)} \le\ (\|\varrho_{\epsilon_n}(t)\|_{L^\alpha(\Omega)} + \|\varrho(t)\|_{L^\alpha(\Omega)})\, k^{-\alpha}\,. \tag{250}
$$

We use this to show that

$$
u_{k,\epsilon_n}(t_n) = u_{k,\epsilon_n}(t_n) - \varrho_{\epsilon_n}(t_n) + \varrho_{\epsilon_n}(t_n) \to \varrho(t) \text{ as distributions for } k, n \to \infty \tag{251}
$$

It follows that $\liminf_{k,\, n\to\infty} \int_\Omega u_{k,\epsilon_n}(t_n)\ \ln u_{k,\epsilon_n}(t) \ge \int_\Omega \varrho(t)\ \ln \varrho(t)$. Using the definition of u_{k,ϵ_n} and (249), we condluce that the equality sign is valid, showing that $u_{k,\epsilon}(t) \to \varrho(t)$ strongly in $L^1(\Omega)$, and thus due to (250) also that

$$
\varrho_{\epsilon_n}(t_n) \to \varrho(t) \text{ strongly in} L^1(\Omega)\,. \tag{252}
$$

The claims (1), (2) follow using the same argument as in Lemma 10.6. □

11 Existence of solutions

Weak solutions to (P) are defined in the spirit of viscosity solutions by passing to the limit $\sigma \to 0$ and then $\delta \to 0$ in the approximation scheme $(P_{\sigma,\delta}) = (P_{\tau=0,\sigma,\delta})$.

Proposition 11.1. *Assumptions of the Theorems 4.4, 4.5. For $\sigma > 0$ and $\delta > 0$ assume that there is $(\mu^{\sigma,\delta},\, v^{\sigma,\delta},\, \phi_{\sigma,\delta}) \in \mathcal{B}(T,\, \Omega,\, \alpha,\, N,\, \Psi,\, \Psi^\Gamma)$ subject to the energy inequality and the global mass conservation identity (Definition 4.2) that weakly solves $(P_{\sigma,\delta})$.*

Then, (P) possesses a weak solution (as stated in the theorems 4.4, 4.5).

Proof. We first show the claim under the assumptions of the Theorem 4.4 (Global existence).

The validity of the mass conservation identity implies that the vector of total mass densities $\bar{\rho}^{\sigma,\delta} \in C_{\Phi^*}([0,T];\, \mathbb{R}^N)$ satisfies

$$\bar{\rho}^{\sigma,\delta}(t) \in \{\bar{\rho}^0\} \oplus W \text{ for all } t \in [0,T]\,.$$

We apply the Propositions 8.1, 8.2, 8.3 and 8.20 and obtain that

$$\begin{aligned}
[(\varrho_{\sigma,\delta},\, q^{\sigma,\delta},\, v^{\sigma,\delta},\, \phi_{\sigma,\delta},\, R^{\sigma,\delta},\, R^{\Gamma,\sigma,\delta})]_{\mathcal{B}(T,\,\Omega,\,\alpha_\delta,\,N-1,\,\Psi,\,\Psi^\Gamma)} \leq C_{0,\delta}\\
[(\varrho_{\sigma,\delta},\, q^{\sigma,\delta},\, v^{\sigma,\delta},\, \phi_{\sigma,\delta},\, R^{\sigma,\delta},\, R^{\Gamma,\sigma,\delta})]_{\mathcal{B}(T,\,\Omega,\,\alpha,\,N-1,\,\Psi,\,\Psi^\Gamma)} \leq C_0\,.
\end{aligned} \tag{253}$$

Here we distinguish the regularisation exponent $\alpha_\delta > 3$ and the original growht exponent $3/2 < \alpha < +\infty$ of the free energy function.

Moreover, time integration in (161) and (162) means that there is a mapping $\mathcal{F}$ in the class $C([0,T]\,;\mathscr{L}([L^1(Q)]^a,\,[C^1_c(\Omega;\,\mathbb{R}^{N+3})]^*))$ such that for almost all $t \in]0,T[$

$$\begin{pmatrix} \int_\Omega \rho^{\sigma,\delta}(t)\cdot\psi \\ \int_\Omega \varrho_{\sigma,\delta}(t)\, v^{\sigma,\delta}(t)\cdot\eta \end{pmatrix} = \mathcal{F}(t,\,\mathcal{A}^{\sigma,\delta})(\psi,\,\eta)$$

$$\text{for all } (\psi,\,\eta) \in C^1_c(\Omega;\,\mathbb{R}^N)\times C^1_c(\Omega;\,\mathbb{R}^3)\,. \tag{254}$$

Fix $\delta > 0$. Then, by construction (see (147)), we can rely on the growth condition $\alpha_\delta > 3$. In order to apply the proposition 10.1, we need to verify the condition

$$\|((\mathbb{1}\cdot J^\sigma)\cdot\nabla\ln\varrho_{\sigma,\delta})^+\|_{L^2(Q)} \to 0 \text{ for } \sigma\to 0\,. \tag{255}$$

In order to show (255), note that $\varrho_\sigma = \sum_{i=1}^N \nabla h^*_i(\mu^\sigma)$ with the mapping of Lemma 5.7 (here we can forget for a while about the δ indices). We obtain that

$$\begin{aligned}
\nabla\ln\varrho_\sigma &= \varrho_\sigma^{-1}\sum_{i,j=1}^N D^2h^*_{i,j}(\mu^\sigma)\,\nabla\mu^\sigma_j\\
&= \frac{D^2h^*\mathbb{1}\cdot\mathbb{1}}{\varrho_\sigma}\,\nabla(\mu^\sigma\cdot\mathbb{1}) + \sum_{l=1}^{N-1}\frac{D^2h^*\mathbb{1}\cdot\xi^l}{\varrho_\sigma}\,\nabla(\mu^\sigma\cdot\xi^l)\,,
\end{aligned}$$

where $\xi^1,\,\ldots,\,\xi^{N-1}$ are chosen as to form an orthonormal basis of $\mathbb{1}^\perp$. Thus, introducing for $k=1,\ldots,N$ the driving forces $D_k := \nabla\mu^\sigma_k + \frac{z_k}{m_k}\,\nabla\phi_\sigma$, we obtain that

$$\nabla\ln\varrho_\sigma = \frac{D^2h^*\mathbb{1}\cdot\mathbb{1}}{\varrho_\sigma}\,(\mathbb{1}\cdot D) + \sum_{\ell=1}^{N-1}\frac{D^2h^*\mathbb{1}\cdot\xi^\ell}{\varrho_\sigma}\,(\xi^\ell\cdot D) - \frac{D^2h^*\mathbb{1}\cdot\frac{z}{m}}{\varrho_\sigma}\,\nabla\phi_\sigma\,.$$

Recall that $-\sum_{i=1}^N J^{i,\sigma} = \sigma\,(\mathbb{1}\cdot D)$. Thus

$$\begin{aligned}
-\sum_{i=1}^N J^{i,\sigma}\cdot\nabla\ln\varrho_\sigma &= \sigma\,\frac{D^2h^*\mathbb{1}\cdot\mathbb{1}}{\varrho_\sigma}\,(\mathbb{1}\cdot D)^2 - \sum_{\ell=1}^{N-1}\frac{D^2h^*\mathbb{1}\cdot\xi^\ell}{\varrho_\sigma}\,(\sum_{i=1}^N J^{i,\sigma})\cdot(\xi^\ell\cdot D)\\
&\qquad - \frac{D^2h^*\mathbb{1}\cdot\frac{z}{m}}{\varrho_\sigma}\,(\sum_{i=1}^N J^{i,\sigma}\cdot\nabla\phi_\sigma)\\
&\geq -\sum_{\ell=1}^{N-1}\frac{D^2h^*\mathbb{1}\cdot\xi^\ell}{\varrho_\sigma}\,(\sum_{i=1}^N J^{i,\sigma})\cdot(\xi^\ell\cdot D) - \frac{D^2h^*\mathbb{1}\cdot\frac{z}{m}}{\varrho_\sigma}\,(\sum_{i=1}^N J^{i,\sigma}\cdot\nabla\phi_\sigma)\,.
\end{aligned} \tag{256}$$

Recall that $|\xi \cdot D| \leq c\,|\Pi\, D| \leq c\,\sqrt{MD \cdot D}$. Thus

$$\|(\sum_{i=1}^N J^{i,\sigma}) \cdot (\xi^\ell \cdot D)\|_{L^1(Q)} \leq \|\sum_{i=1}^N J^{i,\sigma}\|_{L^2(Q)}\, \|\xi \cdot D\|_{L^2(Q)} \leq C_0\,\sqrt{\sigma}$$
$$\|(\sum_{i=1}^N J^{i,\sigma}) \cdot \nabla\phi_\sigma\|_{L^1(Q)} \leq \|\sum_{i=1}^N J^{i,\sigma}\|_{L^2(Q)}\, \|\nabla\phi_\sigma\|_{L^2(Q)} \leq C_0\,\sqrt{\sigma}\,.$$

Using (120) we moreover see that $\frac{|D^2h^*|}{\varrho_\sigma} \leq C_1$. Thus, (256) implies that

$$\|((\mathbb{1}\cdot J^\sigma)\cdot\nabla\ln\varrho_\sigma)^+\|_{L^1(Q)} \leq C_0\,\tilde{C}_1\,\sqrt{\sigma}\,.$$

This establishes (255), and the Proposition 10.1 applied with $\bar{J}^\sigma := \mathbb{1}\cdot J^\sigma$ now garanties that the familty $\{\varrho_{\sigma,\delta}\}_{\sigma>0}$ is compact in $L^1(\Omega)$ uniformly with respect to time (see the Remark 9.6). It remains to apply the Propositon 9.12 in order to obtain the convergence to a weak solution $(\varrho_\delta,\, q^\delta,\, v^\delta,\, \phi_\delta,\, R^\delta,\, R^{\Gamma,\delta}) \in \mathcal{B}(T,\,\Omega,\,\alpha,\,N-1,\,\Psi,\,\Psi^\Gamma)$ to $(P_{\sigma=0,\delta})$.

For the passage to the limit $\delta \to 0$ the reasoning is the same. We have a uniform bound for $[(\varrho_\delta,\, q^\delta,\, v^\delta,\, \phi_\delta,\, R^\delta,\, R^{\Gamma,\delta})]_{\mathcal{B}(T,\,\Omega,\,\alpha,\,N-1,\,\Psi,\,\Psi^\Gamma}$. Since there is no perturbation $\bar{J}^\delta$ in the mass conservation equation, the Proposition 10.1 garanties at once the uniform in time compactness in $L^1(\Omega)$ of $\{\varrho_\delta\}_{\delta>0}$, and the Propositon 9.12 garanties the convergence to a weak solution to (P).

It remains to discuss the case of Theorem 4.5 (Local-in-time existence). We first note that due to the Proposition 8.1, 8.3, we have a bound

$$\|\rho^{\sigma,\delta}\|_{L^{\infty,\alpha}(Q)} + [\bar{\rho}^{\sigma,\delta}]_{C_{\Phi^*}([0,T])} \leq C_0\,. \tag{257}$$

We can extract subsequences such that $\rho^{\sigma,\delta}$ converges weakly in $L^\alpha(Q)$, and $\bar{\rho}^{\sigma,\delta}$ uniformly on $[0,T]$.

We define a time $T^*_{\sigma,\delta}$ via

$$T^*_{\sigma,\delta} = \inf\{t \in [0,T[\,:\, \inf_{i=1,\ldots,N} \bar{\rho}_i^{\sigma,\delta}(t) = 0\}\,.$$

We know that $T^*_{\sigma,\delta} \geq T_0 > 0$ where T_0 is fixed by the data. At first we can extract a subsequence such that $T^*_{\sigma,\delta} \to T^*$. Due to (257), we see that $0 = \inf\bar{\rho}^{\sigma,\delta}(T^*_{\sigma,\delta}) \to \inf\bar{\rho}(T^*)$.

Consider now $T' \in [0,\, T^*[$ arbitrary. Then, for all $\sigma \leq \sigma_0(T^*-T')$, and $\delta \leq \delta_0(T^*-T')$, we can apply the Propositions 8.1, 8.2, 8.3 and 8.20 and obtain the esimate (253) with T replaced by T'. We then finish the proof as for Theorem 4.4 with T replaced by T'. The claim follows. □

Due to Proposition 11.1 it is sufficient to prove the solvability of the problem $(P_{\sigma,\delta})$ in order to complete the proof of the existence Theorems. We are going to carry out this last step by means of a Galerkin approximation described hereafter.

Construction of approximate solutions for $(P_{\delta,\sigma})$ We choose

(1) A countable linearly independent system $\eta^1,\, \eta^2, \ldots \in W^{1,\infty}_0(\Omega;\, \mathbb{R}^3)$ dense in $W^{1,2}_0(\Omega;\, \mathbb{R}^3)$ in order to approximate the variable v;

(2) A countable linearly independent system $\zeta^1,\, \zeta^2, \ldots \in W^{1,\infty}_\Gamma(\Omega)$ dense in $W^{1,2}_\Gamma(\Omega)$ in order to approximate the variable ϕ;

In order to approximmate the variables μ we need a countable system $\psi^1,\, \psi^2, \ldots$ of the space $W^{1,\infty}(\Omega;\, \mathbb{R}^N)$ dense in $W^{1,2}(\Omega;\, \mathbb{R}^N)$. For technical reasons, we have to require additional properties of this set. For $n \in \mathbb{N}$, and $i, j \in \{1, \ldots, n\}$ such that $i \leq j$, we introduce the functions $\tilde{\eta}^{i,j} = \eta^i \cdot \eta^j$ whith $\eta^1, \ldots, \eta^n$ from (1). By means of an obvious renumbering, we denote these funtions $\tilde{\eta}^s$ for $s = 1, \ldots, n\,(n+1)/2$. For all $n \in \mathbb{N}$, we assume that there is $p = p(n) > n$ such that the following additional conditions are valid

$$\begin{cases} \mathbb{1} \in \operatorname{span}\{\psi^1, \ldots, \psi^p\} & \\ \tilde{\eta}^s\, \mathbb{1} \in \operatorname{span}\{\psi^1, \ldots, \psi^p\} & \text{for all } s = 1, \ldots, n\,(n+1)/2 \\ \phi_0\, \frac{z}{m},\, \zeta^s\, \frac{z}{m} \in \operatorname{span}\{\psi^1, \ldots, \psi^p\} & \text{for all } s = 1, \ldots, n \end{cases} \tag{258}$$

Obvious corollaries of this property are

$$\begin{cases} v \in \operatorname{span}\{\eta^1, \ldots, \eta^n\} \Longrightarrow |v|^2 \in \operatorname{span}\{\psi^1, \ldots, \psi^{p(n)}\} \\ \tilde{\phi} \in \operatorname{span}\{\zeta^1, \ldots, \zeta^n\} \Longrightarrow (\tilde{\phi} + \phi_0)\, \frac{z}{m} \in \operatorname{span}\{\psi^1, \ldots, \psi^{p(n)}\} \end{cases} \tag{259}$$

For $n \in \mathbb{N}$, we are looking for approximmate solutions

$$\begin{gathered} \mu^n \in C^1([0,T];\, W^{1,\infty}(\Omega;\, \mathbb{R}^N)) \quad v^n \in C^1([0,T];\, W^{1,\infty}_0(\Omega;\, \mathbb{R}^3)) \\ \phi_n \in C^1([0,T];\, W^{1,\infty}(\Omega)) \end{gathered} \tag{260}$$

following the Ansatz

$$\mu^n = \sum_{\ell=1}^{p(n)} a_\ell(t)\, \psi^\ell(x), \quad v^n = \sum_{\ell=1}^{n} b_\ell(t)\, \eta^\ell(x), \quad \phi_n = \phi_0 + \sum_{\ell=1}^{n} c_\ell(t)\, \zeta^\ell(x)\,. \tag{261}$$

where the vector fieds $a = a^{(n)} \in C^1([0,T];\, \mathbb{R}^p)$, $b = b^{(n)} \in C^1([0,T];\, \mathbb{R}^n)$ and $c = c^{(n)} \in C^1([0,T];\, \mathbb{R}^n)$ are to determine.

Our approximation scheme is $(P_{\tau,\sigma,\delta})$ as described in the section 6. We project this scheme on the Galerkin space and choose $\tau = \tau_n = \frac{1}{n}$. In order to state approximate equations, we need for $i = 1, \ldots, N$ the free energy functions $h_{\tau_n,\delta}$ (cp. (152)). In this point we introduce the abbreviation

$$\mathscr{R}^*(\mu) := \nabla h^*_{\tau_n,\delta} = \nabla (h_\delta)^*(\mu^n) + \tau_n\, \omega'(\mu^n)\,. \tag{262}$$

In order to approximate the equations (47), we consider for $s \in \{1, \ldots, p(n)\}$

$$\int_\Omega \partial_t \mathscr{R}^*(\mu^n) \cdot \psi^s = \int_\Omega ((\mathscr{R}^*(\mu^n)\, v^n + J^n) \cdot \nabla\psi^s + r(\mu^n) \cdot \psi^s) \\ + \int_\Gamma (\hat{r}(\mu^n) + J^0) \cdot \psi^s \,. \tag{263}$$

Introduce a Matrix-valued mapping $\mu \mapsto A^1(\mu) = \{a_{i,j}(\mu)\}_{i,j=1,\ldots,p(n)}$ via

$$a_{i,j}(\mu) := \int_\Omega \mathscr{R}^*_{\ell,\mu_s}(\mu)\, \psi^j_\ell\, \psi^i_s = \int_\Omega (D^2_{l,s} h^*_\delta(\mu) + \tau_n\, \omega''(\mu)\, \delta_{s,\ell})\, \psi^j_\ell\, \psi^i_s \,. \tag{264}$$

Owing to the convexity of h^*_δ and of the function ω, we see that $A^1(\mu)$ is symmetric and positive semidefinite. Due to the Ansatz (261) for μ^n, we can now express (263) in the equivalent form

$$A^1(\mu^n(t))\, a'(t) = F^1(a(t),\, b(t),\, c(t))$$
$$F^1_s := \int_\Omega (\mathscr{R}^*(\mu^n)\, v^n + J^n) \cdot \nabla\psi^s + \int_\Omega r(\mu^n) \cdot \psi^s + \int_\Gamma (\hat{r}(\mu^n) + J^0) \cdot \psi^s \,.$$

In order to approximate the equations (48), we consider for $s \in \{1, \ldots, n\}$

$$\int_\Omega \mathscr{R}^*(\mu^n) \cdot \mathbb{1}\, \partial_t v^n \cdot \eta^s = - \int_\Omega \mathscr{R}^*(\mu^n) \cdot \mathbb{1}\, (v^n \cdot \nabla) v^n \cdot \eta^s + \int_\Omega h^*_{\tau_n,\delta}(\mu^n)\, \operatorname{div} \eta^s \\ - \int_\Omega \mathbb{S}(\nabla v^n) \cdot \nabla \eta^s - \int_\Omega (\sum_{i=1}^N J^{n,i} \cdot \nabla) v^n \cdot \eta^s - \int_\Omega \frac{z}{m} \cdot \mathscr{R}^*(\mu^n)\, \nabla\phi_n \cdot \eta^s \,. \tag{265}$$

Introduce a matrix-valued mapping $\mu \mapsto A^2(\mu)$ via $A^2(\mu) = \{a^{(2)}_{i,j}(\mu)\}_{i,j=1,\ldots,n}$

$$a^{(2)}_{i,j}(\mu) := \int_\Omega \mathscr{R}^*(\mu) \cdot \mathbb{1}\, \eta^i \cdot \eta^j = \int_\Omega (\nabla h^*_\delta(\mu) + \tau_n\, \omega'(\mu^n)) \cdot \mathbb{1}\, \eta^i \cdot \eta^j \;. \tag{266}$$

Owing to the nonnegativity of ∇h^*_δ and of ω', we see that $A^2(\mu)$ is symmetric and positive semidefinite. Due to the Ansatz (261) for v^n and μ^n, we can express (265) in the equivalent form

$$A^2(\mu^n(t))\, b'(t) = F^2(a(t),\, b(t),\, c(t))$$
$$F^2_s := - \int_\Omega \mathscr{R}^*(\mu^n) \cdot \mathbb{1}\, (v^n \cdot \nabla) v^n \cdot \eta^s + \int_\Omega h^*_{\tau_n,\delta}(\mu^n)\, \operatorname{div} \eta^s \\ - \int_\Omega \mathbb{S}(\nabla v^n) \cdot \nabla\eta^s + \int_\Omega (\sum_{i=1}^N J^{n,i} \cdot \nabla) \eta^s \cdot v^n - \int_\Omega \frac{z}{m} \cdot \mathscr{R}^*(\mu^n)\, \nabla\phi^n \cdot \eta^s \,.$$

In order to determine ϕ_n, we use the ansatz $\phi_n = \tilde{\phi}_n + \phi_0$ and we consider the projection onto $\operatorname{span}\{\zeta^1, \ldots, \zeta^n\}^*$ of the Poisson equation, that is

$$\epsilon_0\, (1 + \chi) \int_\Omega \nabla\tilde{\phi}_n \cdot \nabla\zeta^i = -\epsilon_0\, (1+\chi) \int_\Omega \nabla\phi_0 \cdot \nabla\zeta^i + \int_\Omega \frac{z}{m} \cdot \mathscr{R}^*(\mu^n)\, \zeta^i \,. \tag{267}$$

We make use of the Ansatz (261) for ϕ_n, and we see that the vector $c_1, \ldots c_n$ can be determined via for a linear system $A\,c = f$ where

$$A_{i,j} := \epsilon_0\,(1+\chi) \int_\Omega \nabla\zeta^i \cdot \nabla\zeta^j \text{ for } i,j = 1,\ldots,n$$
$$f_i := -\epsilon_0\,(1+\chi) \int_\Omega \nabla\phi_0 \cdot \nabla\zeta^i + \int_\Omega \frac{z}{m} \cdot \mathscr{R}^*(\mu^n)\,\zeta^i \text{ for } i = 1,\ldots,n\,. \tag{268}$$

Since the matrix A is by assumption invertible, we obtain that $c = A^{-1} f =: \tilde{f}(a)$.

Overall, the Galerkin approximation (263), (265), (267) has the form

$$\begin{pmatrix} A^1(a(t)) & 0 \\ 0 & A^2(a(t)) \end{pmatrix} \begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} F^1(a(t),\, b(t),\, \tilde{f}(a(t))) \\ F^2(a(t),\, b(t),\, \tilde{f}(a(t))) \end{pmatrix} \tag{269}$$

We consider the initial conditions

$$a(0) = a^{0,n} \in \mathbb{R}^p, \quad b(0) = b^{0,n} \in \mathbb{R}^n\,. \tag{270}$$

Here we require for the reason of consistency that

$$\begin{aligned} \mu^{0,n} &:= \sum_{\ell=1}^{p(n)} a_\ell^{0,n}\,\psi^\ell \to \mu^0 := \nabla_\rho h(\rho^0) \text{ in } L^1(\Omega;\,\mathbb{R}^N) \\ v^{0,n} &= \sum_{\ell=1}^{n} b_\ell^{0,n}\,\eta^\ell \to v^0 \text{ in } L^1(\Omega;\,\mathbb{R}^3) \text{ as } n \to \infty\,. \end{aligned} \tag{271}$$

We moreover assume that

$$\|\mu^{0,n}\|_{L^\infty(\Omega)} \leq C_0\,, \tag{272}$$

which by definition also yields for $i = 1,\ldots,N$

$$\rho_i^{0,n} := \mathscr{R}_i^*(\mu^{0,n}) \geq c_0 > 0 \text{ everywhere in } \Omega\,. \tag{273}$$

At first we can obtain local existence for the problem (269), (270).

Proposition 11.2. *There is $\epsilon = \epsilon(n,\, a^{0,n},\, b^{0,n})$ such that the problem* (269), (270) *possesses a solution in $C^1([0,\,\epsilon];\,\mathbb{R}^p \times \mathbb{R}^n)$.*

Proof. Recall (273). Consider the matrix $A^1(\mu^0)$ (cf. (264))

$$A^1_{i,j}(\mu^0) = \int_\Omega D^2_{\ell,s} h^*_{\tau_n,\delta}(\mu^0)\,\psi^j_\ell\,\psi^i_s\,.$$

Owing to the strict convexity of $h^*_{\tau_n,\delta}$ on compact sets, $A^1(\mu^0)$ is positive definite and therefore invertible, and $\|[A^1(\mu^0)]^{-1}\| \leq C(a^0,\,n)$. The matrix $A^2(\mu^0)$ (cf. (266)) is uniformly invertible because $\nabla h^*_{\tau_n,\delta}$ is stricly positive on compact sets, and $\|[A^2(\mu^0)]^{-1}\| \leq C(a^0,\,n)$.

The system matrix A in (269) satisfies $\det A = \det A^1\,\det A^2$. Thus, A is invertible at $a^0,\,b^0$, and standard perturbation arguments yield the claim. $\square$

Next we want establish a continuation property for the solution, and we need *a priori* estimates.

Proposition 11.3. *Assume that the approximmate system* (269)*,* (270) *possesses a solution* $(a,\, b) \in C^1([0, T^*[;\, \mathbb{R}^p \times \mathbb{R}^n)$ *for a* $T^* > 0$*. Then,* μ^n*,* v^n *and* ϕ_n *satisfy the dissipation inequality with free energy* $h_{\tau_n,\delta}$ *and mobility matrix* M_σ*.*

Proof. We apply the ideas of Proposition 7.1. We can multiply (263) with μ^n. Due to the additional property (258) and to (259) on the system $\{\psi^1, \ldots, \psi^p\}$, we can also multiply (263) with $\frac{z}{m}\,\phi_n$.

Second, we multiply (265) with v^n. Due again to the additional property (258) and to (259) we can also choose $|v^n|^2\,\mathbb{1}$ as a test function in (263) to obtain that the perturbations vanish. The claim follows. $\square$

Next we verify a continuation criterion.

Proposition 11.4. *Assumptions of Proposition 11.3. Then* $\|\mu^n\|_{L^\infty([0,T^*]\times\Omega)} + \|v^n\|_{L^\infty([0,T^*]\times\Omega)} + \|\phi_n\|_{L^\infty([0,T^*]\times\Omega)} \leq C(n)$*.*

Proof. The bound $\|\mathscr{R}^*(\mu^n)\|_{L^{\infty,\alpha}(Q_{T^*})} \leq C_0$ also implies that $\sup_{(t,x)\in Q_{T^*}} |\mathscr{R}^*(\mu^n)| \leq C(n)$. The reason is that the subset $M := \mathscr{R}^*(\mathrm{span}\{\psi^1, \ldots, \psi^{p(n)}\})$ ($\subset L^1(\Omega;\, \mathbb{R}^N)$) is parameterised by a finite dimensional linear space. Thus, there exists a constant c_M such that $\|u\|_{L^1(\Omega)} \geq c_M\, \|u\|_{L^\infty(\Omega)}$ for all $u \in M$.

We want to obtain a L^∞ bound for μ^n. By construction we have for $t \in]0, T^*[$ arbitrary

$$c\,\tau_n \sum_{i=1}^N \int_\Omega \sqrt{|\mu_i^n(t)|} \leq \tau_n \int_\Omega \Phi_\omega(\mu^n) \leq C_0\,.$$

Now we prove: There is $c = c(n)$ such that $|x|_{L^\infty}^{1/2} \leq c\,\||x\cdot\psi|^{1/2}\|_{L^1(\Omega)}$ for all $x \in \mathbb{R}^p$. Otherwise there is for each $j \in \mathbb{N}$ a $x^j \in \mathbb{R}^p$ such that $|x^j|_\infty^{1/2} \geq j\,\||x^j\cdot\psi|^{1/2}\|_{L^1(\Omega)}$. Thus $\||\bar{x}^j\cdot\psi|^{1/2}\|_{L^1(\Omega)} \leq j^{-1}$ with $\bar{x}^j = x^j/|x^j|_{L^\infty}$. For a subsequence, $\bar{x}^j \to \bar{x}$ in $\mathbb{R}^p$, $|\bar{x}|_\infty = 1$. But since $\||\bar{x}\cdot\psi|^{1/2}\|_{L^1(\Omega)} = 0$, we obtain that $\bar{x}\cdot\psi = 0$ in Ω, and due to the choice of the system $\{\psi^1, \ldots, \psi^p\}$, it follows that $\bar{x} = 0$, a contradiction.

It follows that

$$\|\mu^n(t)\|_{L^\infty(\Omega)}^{1/2} \leq c(n)\,|a(t)|_\infty^{1/2} \leq \tilde{c}(n)\,\||\mu^n(t)|^{1/2}\|_{L^1(\Omega)} \leq \frac{C(n)}{\tau_n}\,C_0\,.$$

Thus $\|\mu^n\|_{L^\infty([0,T^*]\times\Omega)} \leq C(n)$.

It follows from the properties of $\mathscr{R}^*$ that $\inf_{i=1,\ldots,N}\inf_{[0,T^*]\times\Omega}\mathscr{R}_i^*(\mu^n) \geq c(n) > 0$. From the bound $\int_\Omega \mathscr{R}^*(\mu^n(t))\cdot\mathbb{1}\,|v^n(t)|^2 \leq C_0$, we obtain that $\|v^n\|_{L^\infty([0,T^*]\times\Omega)} \leq C_0\,c(n)^{-1}$. Analogously, $\int_\Omega |\nabla\phi_n(t)|^2 \leq C_0$ implies that $\|\nabla\phi_n\|_{L^\infty([0,T^*]\times\Omega)} \leq C(n)$, and since $\phi_n = \phi_0$ on $[0, T^*]\times\Gamma$, the claim follows. $\square$

As a Corollary of these estimates, we obtain the global solvability of the approximate system.

Corollary 11.5. *Let $T > 0$. Then, the approximmate system* (269), (270) *possesses a solution* $(a,\, b) \in C^1([0,T];\, \mathbb{R}^p \times \mathbb{R}^n)$.

Proof. Owing to the Proposition 11.2, there is $T^* > 0$ such that (269), (270) possesses a solution $(a,\, b) \in C^1([0,\, T^*];\, \mathbb{R}^p \times \mathbb{R}^n)$. Since $\|\mu^n\|_{L^\infty([0,T^*]\times\Omega)} \leq C(n)$, it follows from the properties of $\mathscr{R}^*$ that $\inf_{i=1,\ldots,N} \inf_{[0,T^*]\times\Omega} \mathscr{R}^*_i(\mu^n) \geq c(n)$.

The matrix $A^1(\mu^n(t))$ is invertible for all $t \in [0,\, T^*]$, and $\|[A^1(\mu^n(t))]^{-1}\| \leq C(n)$. The matrix $A^2(\mu^n(t))$ (cf. (266)) is uniformly invertible, and $\|[A^2(\mu^n(t))]^{-1}\| \leq C(n)$ on $[0, T^*]$. Due to the Proposition 11.4, the functions $\mu^n(T^*)$, $v^n(T^*)$ and $\phi_n(T^*)$ belong to $L^\infty(\Omega)$ and their norm in this space is bounded independently on t.

Thus, the problem (269), with initial data $(a(T^*), b(T^*))$ possesses solution in an intervall $[T^*,\, T^* + \epsilon(n)]$, and the claim follows reiterating this argument. □

Proposition 11.6. *Let $n \in \mathbb{N}$ and $T > 0$. The Galerkin approximation* (263), (265), (267), *possesses a solution with the regularity* (11) *such that the dissipation inequality is valid with free energy function $h_{\tau_n,\delta}$ and mobility matrix M_σ.*

Uniform estimates We define

$$\rho^n := \mathscr{R}^*(\mu^n) = \nabla(h_\delta)^*(\mu^n) + \tau_n\, \omega'(\mu^n), \quad p_n := h^*_{\tau_n,\delta}(\mu^n)\,.$$

The approximate vector of total masses $\bar{\rho}^n \in C^1([0,T];\, \mathbb{R}^N)$ defined via $\bar{\rho}^n(t) = \int_\Omega \rho^n(t)$ satisfies $\bar{\rho}^n(0) = \bar{\rho^0} + \tau_n \int_\Omega \omega'(\mu^0)$. Therefore, for $c_0 := |\int_\Omega \omega'(\mu^0)|$ we have

$$\bar{\rho}^n(t) \in B_{c_0\,\tau_n}(\bar{\rho}^0) \oplus W = B_{c_0\,\tau_n}(\bar{\rho}^0) \oplus \mathrm{span}\{\gamma^1, \ldots, \gamma^s,\, \hat{\gamma}^1, \ldots, \hat{\gamma}^{\hat{s}^\Gamma}\}\,. \tag{274}$$

We obtain for $c_0\,\tau_n \leq \frac{1}{2}\, \mathrm{dist}(\bar{\rho}^0,\, \mathcal{M}_{\text{crit}})$ that the conclusion of Proposition 8.17 is valid. Applying the conclusion of the Propositions 8.1, 8.2, 8.3 and 8.20 to $\varrho_n := \rho^n \cdot \mathbb{1}$ and $q^n := \Pi\mu^n$ we obtain that $[(\mu^n,\, v^n,\, \phi_n)]_{\mathcal{B}(T,\,\Omega,\,\alpha,\,N,\,\Psi,\,\Psi^\Gamma)} \leq C_0$.

Passage to the limit $n \to \infty$ Due to the condition (258), we can multiply the equations (263) with $\psi = v^n \cdot \eta^s\, \mathbb{1}$, $s \in \{1, \ldots, n\}$ arbitrary. We obtain that

$$\int_\Omega \partial_t\varrho_n\, v^n \cdot \eta^s - \int_\Omega \varrho_n\, v^n \cdot \nabla(v^n \cdot \eta^s) = \int_\Omega (\mathbb{1} \cdot J^n) \cdot \nabla(v^n \cdot \eta^s)\,.$$

Thus, it follows that

$$\begin{aligned} \int_\Omega \partial_t(\varrho_n\, v^n) \cdot \eta^s - \int_\Omega \varrho_n\, \partial_t v^n \cdot \eta^s - \int_\Omega \varrho_n\, (v^n \cdot \nabla)v^n \cdot \eta^s \\ - \int_\Omega \varrho_n\, (v^n \otimes v^n) \,:\, \nabla\eta^s = \int_\Omega (\mathbb{1} \cdot J^n) \cdot \nabla(v^n \cdot \eta^s)\,. \end{aligned}$$

Rearranging terms

$$\begin{aligned} \int_\Omega \partial_t(\varrho_n\, v^n) \cdot \eta^s - \int_\Omega \varrho_n\, (v^n \otimes v^n) \,:\, \nabla\eta^s - \int_\Omega (\mathbb{1} \cdot J^n) \cdot \nabla(v^n \cdot \eta^s) \\ = \int_\Omega \varrho_n\, (\partial_t v^n + (v^n \cdot \nabla)v^n) \cdot \eta^s\,. \end{aligned}$$

Using the latter identity and (265), we obtain that

$$\int_\Omega \partial_t(\varrho_n\, v^n)\cdot \eta^s - \int_\Omega \varrho_n\,(v^n\otimes v^n)\cdot\nabla\eta^s = \int_\Omega p_n\,\operatorname{div}\eta^s - \int_\Omega \mathbb{S}(\nabla v^n)\cdot\nabla\eta^s$$
$$+ \int_\Omega (\sum_{i=1}^N J^{n,i}\cdot\nabla)\eta^s\cdot v^n - \int_\Omega \frac{z}{m}\cdot\rho^n)\,\nabla\phi_n\cdot\eta^s\,. \tag{275}$$

Due to the identities (263) and (275) we obtain for all $t\in[0,T]$ the representation

$$\begin{pmatrix} \int_\Omega \rho^n(t)\cdot\psi \\ \int_\Omega \varrho_n(t)\,v^n(t)\cdot\eta \end{pmatrix} = \mathcal{F}(t,\,\mathcal{A}^n)(\psi,\,\eta)$$
$$\text{for all } (\psi,\,\eta)\in \operatorname{span}\{\psi^1,\ldots,\,\psi^{p(n)}\}\times\operatorname{span}\{\eta^1,\ldots,\eta^n\}\,. \tag{276}$$

Here $\mathcal{F}\in C([0,T]\,;\mathscr{L}(L^1(Q),\,[C^1_c(\Omega;\,\mathbb{R}^{N+3})]^*))$ is the functional naturally defined by the right-hands of (263) and (265), that is

$$\mathcal{F}_1(t,\,\mathcal{A}^n)(\psi) := \int_\Omega \rho^0\cdot\psi + \int_0^t\int_\Omega(\rho_i^n\,v^n + J^{n,i})\cdot\nabla\psi_i$$
$$+\int_0^t\int_\Omega r(\mu^n)\cdot\psi + \int_0^t\int_\Gamma(\hat r(\mu^n)+J^0)\cdot\psi\,,$$
$$\mathcal{F}_2(t,\,\mathcal{A}^n)(\eta) = \int_0^t\int_\Omega \varrho_n\,(v^n\otimes v^n)\cdot\eta + \int_0^t\int_\Omega p_n\,\operatorname{div}\eta^s - \int_0^t\int_\Omega \mathbb{S}(\nabla v^n)\cdot\nabla\eta^s$$
$$+\int_0^t\int_\Omega(\sum_{i=1}^N J^{n,i}\cdot\nabla)\eta^s\cdot v^n - \int_0^t\int_\Omega\frac{z}{m}\cdot\rho^n\,\nabla\phi_n\cdot\eta^s\,.$$

As the systems $\operatorname{span}\{\psi^1,\ldots,\,\psi^{p(n)}\}$ and $\operatorname{span}\{\eta^1,\ldots,\eta^n\}$ are dense in C^1 for $n\to\infty$, we easily show that there is a subsequence such that $\rho^n(t)$ and $\varrho_n(t)\,v^n(t)$ converge as distributions for all $t\in]0,T[$. Thus, the conclusions of Lemma 9.1 are valid and we can produce a limit element $(\mu,\,v,\,\phi)\in\mathcal{B}(T,\,\Omega,\,\alpha,\,N,\,\Psi,\,\Psi^\Gamma)$.

In order to obtain the strong convergence of the sequence, we use the estimate of Lemma 8.18 valid for the class $\mathcal{B}(T,\,\Omega,\,\alpha,\,N,\,\Psi,\,\Psi^\Gamma)$ and fixed $\sigma>0$, and a variant of Corollary 9.7. Abbreviate $\mathscr{R}=\nabla(h_\delta)^*\in C(\mathbb{R}^N;\,\mathbb{R}^N_+)$. We show that for all $\epsilon>0$, there are $C_\epsilon>0$ and $m_\epsilon\in\mathbb{N}$ such that for all $w^1,\,w^2\in W^{1,1}(\Omega;\,\mathbb{R}^N)$

$$\|\mathscr{R}(w^1)-\mathscr{R}(w^2)\|_{L^1(\Omega)} \leq \epsilon\,(1+\sup_{i=1,2}\|w^i\|_{W^{1,1}(\Omega;\,\mathbb{R}^N)})$$
$$+ C_\epsilon\sum_{j=1}^{m_\epsilon}\left|\int_\Omega(\mathscr{R}(w^1)-\mathscr{R}(w^2))\cdot\phi^j\right|\,.$$

Here $\phi^1,\,\phi^2,\ldots$ is a dense subset of $C_c(\Omega;\,\mathbb{R}^N)$. Then, we choose $w^1=\mu^n(t)$ and $w^2=\mu^{n+p}(t)$, and integrate over the interval $J=[0,T]\setminus(I_{\ell,n}\cup I_{\ell,n+p})$ where $I_{\ell,n}=\{t\,:\,\|\mu^n(t)\|_{L^1(\Omega)}\geq\ell\}$. Arguing as in the Corollary 9.7, we obtain after few steps the inequality

$$\|\mathscr{R}(\mu^n)-\mathscr{R}(\mu^{n+p})\|_{L^1([0,T]\times\Omega)} \leq \frac{C}{(\circ_N\ln)(\ell)} + \epsilon\,(T+C_\ell)$$
$$+ C_\epsilon\sum_{j=1}^{m_\epsilon}\int_0^T\left|\int_\Omega(\mathscr{R}(\mu^n)-\mathscr{R}(\mu^{n+p}))\cdot\phi^j\right|\,.$$

Due to Proposition 8.1, $|\mathscr{R}(\mu^n) - \rho^n| = |\tau_n\, \omega'(\mu^n)| \le C_0\, \tau_n^{1/\alpha'}$. Thus, in view of Lemma 9.1, we ensure that $\mathscr{R}(\mu^n(t)) \to \rho(t)$ as distributions for almost all t. This yields

$$\limsup_{n\to\infty} \|\mathscr{R}(\mu^n) - \mathscr{R}(\mu^{n+p})\|_{L^1([0,T]\times\Omega)} \le \frac{C}{(\circ_N \ln)(\ell)} + \epsilon\,(T + C_\ell)\,.$$

We conclude as in the proof of Corollary 9.7 that $\{\mathscr{R}(\mu^n)\}$ converges strongly in $L^1(Q;\, \mathbb{R}^N)$. Then, owing to the uniform bound $[\mu^n]_{L^w_{(\circ_N \ln)} L^1(Q)} \le C_0$, we obtain that $\mu := \lim_{n\to\infty} \mu^n$ exists almost everywhere in Q, and we easily identify $(\mu,\, v,\, \phi) \in \mathcal{B}(T,\, \Omega,\, \alpha,\, N,\, \Psi,\, \Psi^\Gamma)$ as a weak solution to the problem $(P_{\sigma,\delta})$.

A Proofs of some auxiliary statements

Proof of the Lemma 8.5

Proof. The proof relies on the availibility of a solution operator to the problem

$$\operatorname{div} X = f \text{ in } \Omega, \quad X = 0 \text{ on } \partial\Omega\,, \tag{277}$$

for all f having mean value zero over Ω, so that for all $1 < q < +\infty$ the estimates

$$\|X\|_{W^{1,q}(\Omega)} \le c_q\, \|f\|_{L^q(\Omega)}, \quad \|X\|_{L^q(\Omega)} \le c_q\, \|f\|_{[W^{1,q'}_0(\Omega)]^*} \tag{278}$$

are valid. For details about the solution operator, see among others [FNP01], section 3.1.

We begin with the case $\alpha > 3$. Then, for all $\eta \in C^1_c([0,T[;\, C^1_c(\Omega;\, \mathbb{R}^3))$ the function p satisfies

$$\begin{aligned}\int_Q p\, \operatorname{div}\eta = & -\int_Q \varrho\, v\cdot \eta_t - \int_Q \varrho\, v\otimes v\, :\, \nabla\eta + \int_Q \mathbb{S}(\nabla v)\, :\, \nabla \eta\\ & - \int_Q (\sum_{i=1}^N J^i\cdot\nabla)\eta\cdot v - \int_\Omega \varrho_0\, v^0\cdot\eta(0) + \int_Q n^F\, \nabla\phi\cdot\eta\,.\end{aligned}$$

We make use of the estimates

$$\begin{aligned}\left|\int_Q \varrho\, v\cdot\eta_t\right| & \le \|\varrho\, v\|_{L^{2,\,\frac{6\alpha}{6+\alpha}}(Q)}\, \|\eta_t\|_{L^{2,\,\frac{6\alpha}{5\alpha-6}}(Q)}\\ \left|\int_Q \varrho\, v\otimes v\, :\, \nabla\eta\right| & \le \|\varrho\, v^2\|_{L^{1,\,\frac{3\alpha}{3+\alpha}}(Q)}\, \|\nabla\eta\|_{L^{\infty,\,\frac{3\alpha}{2\alpha-3}}(Q)}\\ \left|\int_Q \mathbb{S}(\nabla v)\, :\, \nabla\eta\right| & \le c\, \|\nabla v\|_{L^2(Q)}\, \|\nabla\eta\|_{L^2(Q)}\\ \left|\int_Q (\sum_{i=1}^N J^i_\sigma\cdot\nabla)\eta\cdot v\right| & \le \left\|\sum_{i=1}^N J^i_\sigma\, v\right\|_{L^{2,3/2}(Q)} \|\nabla\eta\|_{L^{\infty,3}(Q)}\\ \left|\int_Q n^F\, \nabla\phi\cdot\eta\right| & \le \|n^F\, \nabla\phi\|_{L^{\infty,1}(Q)}\, \|\eta\|_{L^{1,\infty}(Q)}\\ & \le c\, \|n^F\, \nabla\phi\|_{L^{\infty,1}(Q)} \|\eta\|_{L^\infty(0,T;\, W^{1,\alpha}(\Omega))}\,.\end{aligned} \tag{279}$$

Let $t \in]0,T[$ and consider according to (277) a solution to the problem

$$\operatorname{div} X = \varrho(t) - \bar{\varrho}(t) \text{ in } \Omega, \; X = 0 \text{ on } \partial\Omega$$

Since $\bar{\varrho}(t) = \|\varrho_0\|_{L^1(\Omega)}$ for all t as a consequence of (161), we obtain the estimate

$$\|X\|_{W^{1,\alpha}(\Omega)} \leq c\,(\|\varrho(t)\|_{L^\alpha(\Omega)} + \|\varrho_0\|_{L^1(\Omega)})\,.$$

Further the identity (161) also implies that

$$-\int_Q \varrho\,\psi_t = \int_Q \varrho\, v \cdot \nabla\psi + \int_Q \sum_{i=1}^N J^i \cdot \nabla\psi = 0 \text{ for all } \psi \in C^1_c(0,T;\, C^1(\overline{\Omega}))\,,$$

and since we assume $\alpha > 3$, this yields

$$\begin{aligned}\|\varrho_t\|_{L^2(0,T;\,[W^{1,2}(\Omega)]^*)} &\leq \|\varrho\, v\|_{L^2(Q)} + \|\sum_{i=1}^N J^i\|_{L^2(Q)}\\ &\leq c\,\|\varrho\, v\|_{L^{2,\frac{6\alpha}{6+\alpha}}(Q)} + \|\sum_{i=1}^N J^i\|_{L^2(Q)} \leq C_0\,.\end{aligned}$$

Thus we also obtain from the properties (278) that

$$\|X_t\|_{L^2(Q)} \leq c\,\|\varrho_t\|_{L^2(0,T;\,[W^{1,2}(\Omega)]^*)} \leq C_0\,.$$

Owing to the inqualities $6\alpha/(5\alpha-6) < 2$ and $3\alpha/(2\alpha-3) < \alpha$, we see that $|\int_Q p\,\operatorname{div} X| \leq C_0$. Thus $\int_Q p\,\varrho \leq C_0$, and since $\varrho \geq p^{1/\alpha}$ the claim follows.

If $\alpha \leq 3$, then we assume that $\mathbb{1}\cdot J = 0$, then p satisfies for all $\eta \in C^1_c([0,T[;\, C^1_c(\Omega;\mathbb{R}^3))$

$$\begin{aligned}\int_Q p\,\operatorname{div}\eta =& -\int_Q \varrho\, v\cdot\eta_t - \int_Q \varrho\, v\otimes v \,:\, \nabla\eta + \int_Q \mathbb{S}(\nabla v)\,:\,\nabla\eta\\ &-\int_\Omega \varrho_0\, v^0\cdot\eta(0) + \int_Q n^F\,\nabla\phi\cdot\eta\,.\end{aligned}$$

We apply the estimates (279) for the right-hand except for the last one. Note now that $3\alpha/(2\alpha-3) \geq 3$, and therefore $\beta \geq \min\{3,\, r(\Omega,\,\Gamma)\} > \alpha'$ by assumption. It follows that $\frac{\beta\alpha}{\beta+\alpha} > 1$, and therefore

$$\left|\int_Q n^F\,\nabla\phi\cdot\eta\right| \leq \|n^F\,\nabla\phi\|_{L^{\frac{\beta\alpha}{\beta+\alpha}}(Q)}\,\|\eta\|_{L^{\frac{\beta\alpha}{\beta\alpha-\beta-\alpha}}(Q)} \leq C_0\,\|\eta\|_{L^\infty(0,T;\,W^{1,\frac{3\alpha}{2\alpha-3}}(\Omega))}\,.$$

It can be shown using (161) that ϱ is a solution to the continuity equation in the sense of *renormalised solutions* (see [Lio98] or [FNP01]) and that it satisfies for all $s > 0$ and $\psi \in C^1_c(0,T;\, C^1(\overline{\Omega}))$

$$-\int_Q \varrho^s\,\psi_t = \int_Q \varrho^s\, v\cdot\nabla\psi + (1-s)\int_Q \rho^s\,\operatorname{div} v\,\psi\,.$$

Defining $r := 2\alpha/(2s+\alpha)$

$$\|\varrho^s(t)\,\operatorname{div} v(t)\|_{L^r(\Omega)} \leq \|\operatorname{div} v(t)\|_{L^2(\Omega)}\,\|\varrho(t)\|^s_{L^\alpha(\Omega)} \leq C_0\,\|\operatorname{div} v(t)\|_{L^2(\Omega)}\,.$$

Thus, $\|\varrho^s\,\operatorname{div} v\|_{L^{2,r}(Q)} \leq C_0$. Moreover, defining $\tilde{r} = 6\alpha/(6s+\alpha)$

$$\|\varrho(t)^s\,v(t)\|_{L^{\tilde{r}}(\Omega)} \leq \|\varrho(t)\|^s_{L^\alpha(\Omega)}\,\|v(t)\|_{L^6(\Omega)} \leq C_0\,\|v(t)\|_{L^6(\Omega)}\,,$$

and this shows that $\|\varrho^s\,v\|_{L^{2,\tilde{r}}(Q)} \leq C_0$, $\tilde{r} = 6\alpha/(6s+\alpha)$. Using the Sobolev inequality

$$\left|\int_Q \varrho^s\,\psi_t\right| \leq C_0\,(\|\nabla\psi\|_{L^{2,\tilde{r}'}(Q)} + \|\psi\|_{L^{2,r'}(Q)}) \leq C_0\,\|\psi\|_{L^2(0,T;\,W^{1,\frac{6\alpha}{5\alpha-6s}}(\Omega))}\,.$$

For the choice $s = \frac{2}{3}\alpha - 1$, it follows that $\|(\varrho^s)'\|_{L^2(0,T;\,[W^{1,\frac{6\alpha}{6+\alpha}}(\Omega)]^*)} \leq C_0$. Now we consider a solution to the problem

$$\operatorname{div} X = \varrho^s(t) - \bar{\varrho}^s(t) \text{ in } \Omega,\; X = 0 \text{ on } \partial\Omega$$

We obtain that $\|X\|_{L^\infty(0,T;\,W^{1,\frac{3\alpha}{2\alpha-3}}(\Omega))} \leq C_0$ and that $\|X_t\|_{L^{2,\frac{6\alpha}{7\alpha-6}}(Q)} \leq C_0$. We see again that $\int_Q p\,\operatorname{div} X$ is finite, and the claim follows. □

Proof of the Corollary 9.4 At first, we prove the Corollary 9.4. Here we need the following auxiliary statement.

Lemma A.1. *Let $\phi_1,\,\phi_2,\ldots \in C^\infty(\overline{\Omega})$ be a countable, dense subset of $C(\overline{\Omega})$. Let $K \subset L^1(\Omega)$ be a weakly sequentially compact set. We denote $L^1_+(\Omega)$ the cone of nonnegative functions in $L^1(\Omega)$. Then, for every $\delta > 0$, there are a constant $C = C(\delta) \in \mathbb{R}_+$ and $m(\delta) \in \mathbb{N}$ such that*

$$\|\lambda\,u\|_{L^1(\Omega)} \leq \delta\,\|\nabla u\|_{L^1(\Omega)} + C\,\sum_{i=1}^m \left|\int_\Omega \lambda\,u\,\phi_i\right|$$

for all $u \in W^{1,1}(\Omega)$ and $\lambda \in L^1_+(\Omega)$ such that $\lambda\,u \in K$ and $\|\lambda\,u\|_{L^1(\Omega)} \geq \delta$.

Proof. If the claim is not true, then one can find $\delta_0 > 0$ and for each $n \in \mathbb{N}$ functions $u_n \in W^{1,1}(\Omega)$ and $\lambda_n \in L^1_+(\Omega)$ such that $u_n\,\lambda_n \in K$, $\|\lambda_n\,u_n\|_{L^1(\Omega)} \geq \delta_0$ and such that

$$\|\lambda_n\,u_n\|_{L^1(\Omega)} \geq \delta_0\,\|\nabla u_n\|_{L^1(\Omega)} + n\,\sum_{i=1}^n \left|\int_\Omega \lambda_n\,u_n\,\phi_i\right| \tag{280}$$

Since $\lambda_n\,u_n \in K$, there is a subsequence and $\beta^* \in L^1(\Omega)$ such that $\lambda_n\,u_n \to \beta^*$ weakly in $L^1(\Omega)$ and $\sup_{n\in\mathbb{N}} \|\lambda_n\,u_n\|_{L^1(\Omega)} < +\infty$. Thus, we easily obtain that

$$\sum_{i=1}^{n_0} \left|\int_\Omega \beta^*\,\phi_i\right| = 0 \text{ for all } n_0 \in \mathbb{N}\,,$$

and using the properties of the system $\{\phi_i\}$, this yields $\beta^* = 0$. On the other hand, the sequence ∇u_n is bounded in $L^1(\Omega)$ owing to (280). Passing to a subsequence, we obtain that $u(x) := \lim_{n\to\infty} u_n(x) \in \overline{\mathbb{R}}$ exists for almost all $x \in \Omega$. In particular, the characteristic functions $\chi_n^+ := \chi_{\{x\in\Omega\,:\,u_n(x)\geq 0\}}$ and $\chi_n^- := \chi_{\{x\in\Omega\,:\,u_n(x)\leq 0\}}$ converge pointwise almost everywhere in Ω to $\chi^+ := \chi_{\{x\in\Omega\,:\,u(x)\geq 0\}}$ and $\chi^- := \chi_{\{x\in\Omega\,:\,u(x)\leq 0\}}$. We can then use for instance the Proposition 1. in [GMS98], Section 1.2.4, to see that

$$\lambda_n\, u_n\, \chi_n^{\pm} \to \beta^*\, \chi^{\pm} = 0 \text{ weakly in } L^1(\Omega)\,.$$

Together with the fact that $\|\lambda_n\, u_n\|_{L^1(\Omega)} = \int_\Omega \lambda_n\, u_n\, \{\chi_n^+ - \chi_n^-\}$, this proves that $\delta_0 \leq \|\lambda_n\, u_n\|_{L^1(\Omega)} \to 0$, a clear contradiction. $\square$

Proof of Corollary 9.4. For $j \in \mathbb{N}$, define $I_j := \{t \in]0,T[\,:\, \|v(t)\|_{L^{2\alpha'}(\Omega)} \leq j\}$.

Since $\alpha > 3/2$, one always can show that $2\alpha' < 6$, and thus that $\|v\|_{L^{2,2\alpha'}(Q)} \leq C_0$. Thus, $\lambda_1(]0,T[\setminus I_j) \leq c\, j^{-2}$.

We now define a set $K = K(j) \subset L^1(\Omega)$ via

$$K := \bigcup_{t\in I_j} \bigcup_{\epsilon>0} \{\varrho_\epsilon(t)\,(v^\epsilon(t) - v(t))\}\,.$$

Note that for $t \in I_j$, one has $\|\varrho^\epsilon(t)\, v(t)\|_{L^{\frac{2\alpha}{1+\alpha}}(\Omega)} \leq \|\varrho^\epsilon(t)\|_{L^\alpha(\Omega)}\, \|v(t)\|_{L^{2\alpha'}(\Omega)} \leq C_0\, j$. Thus the set K is bounded in $L^{\frac{2\alpha}{1+\alpha}}(\Omega)$ and it is weakly sequentially compact in $L^1(\Omega)$.

Let $\delta > 0$ arbitrary. Consider $\lambda := \varrho_\epsilon(t)$ and $u := v^\epsilon(t) - v(t)$. According to the Lemma A.1, if $\|\lambda\, u\|_{L^1(\Omega)} \geq \delta$, then

$$\begin{aligned}\|\varrho_\epsilon(t)\,(v^\epsilon(t) - v(t))\|_{L^1(\Omega)} \leq & \delta\, \|\nabla(v^\epsilon(t) - v(t))\|_{L^1(\Omega)} \\ & + C(\delta) \sum_{j=1}^{m(\delta)} \left|\int_\Omega \varrho_\epsilon(t)\,(v^\epsilon(t) - v(t))\,\phi_i\right|\,,\end{aligned}$$

Thus it also follows that

$$\begin{aligned}\|\varrho_\epsilon(t)\,(v^\epsilon(t) - v(t))\|_{L^1(\Omega)} \leq & \delta\, \max\{1,\, \|\nabla(v^\epsilon(t) - v(t))\|_{L^1(\Omega)}\}) \\ & + C(\delta) \sum_{i=1}^{m(\delta)} \left|\int_\Omega \varrho_\epsilon(t)\,(v^\epsilon(t) - v(t))\,\phi_i\right|\,,\end{aligned}$$

where $C(\delta)$ depends only on the set K. We integrate over $I_j \subset]0,T[$ the latter inequality and we obtain that

$$\begin{aligned}& \|\varrho_\epsilon\,(v^\epsilon - v)\|_{L^1(I_j\times\Omega)} \leq \delta\,(T + \|\nabla(v^\epsilon - v)\|_{L^1(Q)}) \\ & \quad + C(\delta) \sum_{i=1}^{m(\delta)} \int_{I_j} \left|\int_\Omega \varrho_\epsilon(t)\,(v^\epsilon(t) - v(t))\,\phi_i\right|\, dt \\ & \quad \leq (T + C_0)\,\delta + C(\delta) \sum_{i=1}^{m(\delta)} \int_{I_j} \left|\int_\Omega \varrho_\epsilon(t)\,(v^\epsilon(t) - v(t))\,\phi_i\right|\, dt\end{aligned}$$

We now consider the subsequence $\{\epsilon_n\}_{n\in\mathbb{N}}$ of Lemma 9.1. Using the distributional convergence $\varrho_{\epsilon_n}(t)\, v^{\epsilon_n}(t) \to \varrho(t)\, v(t)$ and $\varrho_{\epsilon_n}(t) \to \varrho(t)$ for almost all $t \in]0,T[$, we see that the function $g_{\epsilon_n}(t) := \int_\Omega \varrho_{\epsilon_n}(t)\,(v^{\epsilon_n}(t)-v(t))\,\phi_i$ converges pointwise to zero for almost all $t \in]0,T[$, and $|g_{\epsilon_n}(t)| \leq C$ for $t \in I_j$. Thus

$$\limsup_{n\to\infty} \|\varrho_{\epsilon_n}\,(v^{\epsilon_n}-v)\|_{L^1(I_j\times\Omega)} \leq (T+C_0)\,\delta\,.$$

On the other hand, the sequence $\varrho_{\epsilon_n}\,(v^{\epsilon_n}-v)$ is uniformly bounded in $L^{2,\frac{6\alpha}{6+\alpha}}(Q)$, and therefore

$$\begin{aligned}\|\varrho_{\epsilon_n}\,(v^{\epsilon_n}-v)\|_{L^1(Q)} &\leq \|\varrho_{\epsilon_n}\,(v^{\epsilon_n}-v)\|_{L^1(I_j\times\Omega)} + \|\varrho_{\epsilon_n}\,(v^{\epsilon_n}-v)\|_{L^1(]0,T[\setminus I_j\times\Omega)}\\ &\leq \|\varrho_{\epsilon_n}\,(v^{\epsilon_n}-v)\|_{L^1(I_j\times\Omega)} + C_0\,j^{-s}, \quad s:=\min\{2,\,\frac{6\alpha}{6+\alpha}\}\,.\end{aligned}$$

Now, $\limsup_{n\to\infty}\|\varrho_{\epsilon_n}\,(v^{\epsilon_n}-v)\|_{L^1(Q)} \leq (T+C_0)\,\delta + C_0\,j^{-s}$ and the claim follows. □

Proof of Lemma 10.3 Next we want to provide the proof of Lemma 10.3. We commence with the fundamental technical observations due to Lions ([Lio98], page 17–21) about the compensated compactness of the acceleration terms (see also [FNP01], section 3.4).

Lemma A.2. *Assumptions of Proposition 10.1. Let $a > \frac{6\alpha}{5\alpha-6}$, and $b > \max\{2, \frac{3\alpha}{2\alpha-3}\}$. Consider $\{X^\epsilon\}_{\epsilon>0},\ X \subset L^2(Q;\,\mathbb{R}^3)$ such that for $\epsilon\to 0$*

$$X^\epsilon \to X \text{ strongly in } L^2(Q;\,\mathbb{R}^3), \quad \partial_t X^\epsilon \to \partial_t X \text{ weakly in } L^{2,a}(Q;\mathbb{R}^3)$$
$$\nabla X^\epsilon \to \nabla X \text{ weakly in } L^b(Q;\mathbb{R}^9)\,.$$

Then

$$\varrho_\epsilon\, v^\epsilon\cdot\partial_t X^\epsilon + \varrho_\epsilon\, v^\epsilon\otimes v^\epsilon \,:\, \nabla X^\epsilon \to \varrho\, v\cdot\partial_t X + \varrho\, v\otimes v \,:\, \nabla X \text{ weakly in } L^1(Q)\,.$$

Proof. The assumptions imply that $a' < \frac{6\alpha}{6+\alpha}$. Using $L^{p,q}$ interpolation, there are $r_1 > 2$ and $r_2 > a'$ and a certain interpolation exponent $\lambda \in]0,1[$ so that

$$\|\varrho_\epsilon\, v^\epsilon\|_{L^{r_1,r_2}(Q)} \leq \|\varrho_\epsilon\, v^\epsilon\|^{\lambda}_{L^{\infty,\frac{2\alpha}{1+\alpha}}(Q)}\,\|\varrho_\epsilon\, v^\epsilon\|^{1-\lambda}_{L^{2,\frac{6\alpha}{6+\alpha}}(Q)} \leq C_0\,.$$

Similarily since $b' < \frac{3\alpha}{3+\alpha}$, there are $p_1 > 1$ and $p_2 > b'$ and an interpolation exponent $\lambda \in]0,1[$ such that

$$\|\varrho_\epsilon\,|v^\epsilon|^2\|_{L^{p_1,p_2}(Q)} \leq \|\varrho_\epsilon\,|v^\epsilon|^2\|^{\lambda}_{L^{\infty,1}(Q)}\,\|\varrho_\epsilon\,|v^\epsilon|^2\|^{1-\lambda}_{L^{1,\frac{3\alpha}{3+\alpha}}(Q)} \leq C_0\,.$$

As a consequence of Hölder's inequality, one then finds a $z > 1$ such that

$$\|\varrho_\epsilon\, v^\epsilon\,\partial_t X^\epsilon\|_{L^z(Q)} + \|\varrho_\epsilon\, v^\epsilon\otimes v^\epsilon \,:\, \nabla X^\epsilon\|_{L^z(Q)} \leq C\,. \tag{281}$$

Since the exact value of z is not necessary to our purpose we avoid to burden the proof by calculating this exponent. We also note that

$$\begin{aligned}\|\varrho_\epsilon\,\partial_t X^\epsilon\|_{L^{2,q_1}(Q)} &\leq \|\varrho_\epsilon\|_{L^{\infty,\alpha}(Q)}\,\|\partial_t X^\epsilon\|_{L^{2,a}(Q)} \leq C\\ \|\varrho_\epsilon\, v^\epsilon \,:\, \nabla X^\epsilon\|_{L^{2,q_2}(Q)} &\leq \|\varrho_\epsilon\, v^\epsilon\|_{L^{2,\frac{6\alpha}{6+\alpha}}(Q)}\,\|\nabla X^\epsilon\|_{L^{\infty,b}(Q)} \leq C\\ q_1 = \frac{a\,\alpha}{a+\alpha} > \frac{6}{5}\,,&\quad q_2 = \frac{6\alpha\,b}{(6+\alpha)\,b+6\,\alpha} > \frac{6}{5}\,.\end{aligned} \tag{282}$$

After these preliminary observations, we now want to establish the convergence property. We define vector fields $w^\epsilon,\, w : Q \to \mathbb{R}^4$ via $w^\epsilon := (1, v^\epsilon)$ and analogously $w := (1,\, v)$. The first step is to show for $i = 1, 2, 3$ that the sequence $A_{i,\epsilon} := \varrho_\epsilon\, w^\epsilon \cdot \nabla_4 X_i^\epsilon$ converges weakly in $L^1(Q)$ to $A_i := \varrho\, w \cdot \nabla_4 X_i$. Here $\nabla_4 := (\partial_t,\, \nabla)$ is the total differential. Owing to (282), note that $\|A_\epsilon\|_{L^{2,y}(Q)} \leq C$, $y := \min\{\frac{a\alpha}{a+\alpha},\, \frac{6\alpha\, b}{(6+\alpha)\, b + 6\,\alpha}\} > 6/5$. Consider now arbitrary $\eta \in C^1_c(Q;\, \mathbb{R}^3)$. Then, due to the Gauss integration by parts theorem

$$\int_Q (\varrho_\epsilon\, w^\epsilon \cdot \nabla_4) X^\epsilon \cdot \eta = - \int_Q (\varrho_\epsilon\, w^\epsilon \cdot \nabla_4)\eta \cdot X^\epsilon + \int_Q \varrho_\epsilon\, w^\epsilon \cdot \nabla_4(\eta\, X^\epsilon)\,. \tag{283}$$

At this point we recall the continuity equation (237). We consider $\eta \in C^1_c(Q)$ arbitrary, and we define $\psi := \eta \cdot X^\epsilon$. We easily verify that $\psi_t \in L^{1,\alpha'}(Q)$ and that $\nabla\psi \in L^{2,\max\{2,\, \frac{6\alpha}{5\alpha-6}\}}(Q)$. Thus, by density we can use this test function in (237), and therefore

$$- \int_Q \varrho_\epsilon\, w^\epsilon\, \cdot \nabla_4(\eta\, X^\epsilon) = \int_Q \bar{J}^\epsilon \cdot \nabla(\eta \cdot X^\epsilon)\,.$$

Owing to the property $\bar{J}^\epsilon \to 0$ strongly in $L^2(Q)$ as $\epsilon \to 0$ this yields

$$\left| \int_Q \varrho_\epsilon\, w^\epsilon \cdot \nabla_4(\eta\, X^\epsilon) \right| \leq c_\eta\, \|\bar{J}^\epsilon\|_{L^2(Q)}\, (\|\nabla X^\epsilon\|_{L^2(Q)} + \|X^\epsilon\|_{L^2(Q)}) \;\to 0 \text{ as } \epsilon \to 0\,.$$

Owing to (283) and the strong convergence of X^ϵ in $L^2(Q)$, it follows that

$$\int_Q (\varrho_\epsilon\, w^\epsilon \cdot \nabla_4) X^\epsilon \cdot \eta \to - \int_Q (\varrho\, w \cdot \nabla_4)\eta \cdot X\,.$$

Using that $\operatorname{div}_4(\varrho\, w) = 0$ in the sense of (239), this yields that $\varrho_\epsilon\, w^\epsilon \cdot \nabla_4 X^\epsilon \to \varrho\, w \cdot \nabla_4 X$ as distributions. Therefore,

$$A_{i,\epsilon} \to A_i \text{ weakly in } L^{2,y}(Q) \text{ for } i = 1, 2, 3\,. \tag{284}$$

For arbitrary $i, j \in \{1, 2, 3\}$ and $\zeta \in C_c(Q)$, we now express

$$\begin{aligned}
\int_Q v_i^\epsilon\, A_j^\epsilon\, \zeta = & \int_Q (v_i^\epsilon - T_n(v_i^\epsilon))\, A_j^\epsilon\, \zeta \\
& + \int_Q \varrho_\epsilon\, (T_n(v_i^\epsilon) - T_n(v_i))\, w^\epsilon \cdot \nabla_4 X_j^\epsilon\, \zeta + \int_Q T_n(v_i)\, (A_j^\epsilon - A_j)\, \zeta \\
& + \int_Q (T_n(v_i) - v_i)\, A_j\, \zeta + \int_Q v_i\, A_j\, \zeta\,.
\end{aligned} \tag{285}$$

For $n \in \mathbb{N}$ and $s \in \mathbb{R}$, we denoted $T_n(s) := \operatorname{sign}(s) \min\{|s|,\, n\}$. Denote $B_{\epsilon,n} := \bigcup_{i=1,2,3}\{(t, x) \,:\, |v_i^\epsilon(t, x)| \geq n\}$. Then, $\operatorname{meas}(B_{\epsilon,n}) \leq 3\, \|v^\epsilon\|_{L^2}\, n^{-2}$. Then, epxloiting (281), we obtain that

$$\begin{aligned}
\left| \int_Q (v_i^\epsilon - T_n(v_i^\epsilon))\, A_j^\epsilon\, \zeta \right| = \left| \int_{B_{\epsilon,n}} (v_i^\epsilon - n)\, A_j^\epsilon\, \zeta \right| & \leq 2\, \|\zeta\|_{L^\infty(Q)} \int_{B_{\epsilon,n}} |v^\epsilon|\, |A_\epsilon| \\
& \leq 2\, C\, \|\zeta\|_{L^\infty(Q)}\, \operatorname{meas}(B_{\epsilon,n})^{1/z'} \leq 2\, C\, \|\zeta\|_{L^\infty(Q)}\, n^{-2/z'}\,.
\end{aligned}$$

On the other hand, owing to Remark 10.2, we have $\varrho_\epsilon \, |T_n(v^\epsilon) - T_n(v)| \leq \varrho_\epsilon \, |v^\epsilon - v| \to 0$ almost everywhere in Q, and therefore $\varrho_\epsilon \, (T_n(v^\epsilon) - T_n(v)) \to 0$ strongly in $L^{\alpha-\delta}(Q)$ for all $\delta > 0$. We verify easily that $\|w^\epsilon \cdot \nabla_4 X^\epsilon\|_{L^{2,\min\{a,\,\frac{6b}{6+b}\}}(Q)} \leq C$, and thus there is $\delta_0 > 0$ such that $\|w^\epsilon \cdot \nabla_4 X^\epsilon\|_{L^{\alpha'+\delta_0}(Q)} \leq C$. Thus

$$\left|\int_Q \varrho_\epsilon \, (T_n(v_i^\epsilon) - T_n(v_i)) \, w^\epsilon \cdot \nabla_4 X_j^\epsilon \, \zeta\right|$$
$$\leq \|\zeta\|_{L^\infty(Q)} \, \|w^\epsilon \cdot \nabla_4 X^\epsilon\|_{L^{\alpha'+\epsilon_0}(Q)} \, \|\varrho_\epsilon \, (T_n(v^\epsilon) - T_n(v))\|_{L^{\alpha-\epsilon}(Q)} \to 0 \text{ as } \epsilon \to 0 \, .$$

Due to the weak convergence (284)

$$\left|\int_Q T_n(v_i) \, (A_j^\epsilon - A_j) \, \zeta\right| \to 0 \text{ as } \epsilon \to 0 \, ,$$

and for $B_n := \bigcup_{i=1,2,3}\{(t,x) \, : \, |v_i(t,x)| \geq n\}$,

$$\left|\int_Q (v_i - T_n(v_i)) \, A_j \, \zeta\right| = \left|\int_{B_n} (v_i - n) \, A_j \, \zeta\right| \leq 2\, C \, \|\zeta\|_{L^\infty(Q)} \, n^{-2/z'} \, .$$

It follows from (285) that $\limsup_{\epsilon\to 0} \left|\int_Q (v_i^\epsilon \, A_j^\epsilon - v_i \, A_j) \, \zeta\right| \leq C \, \|\zeta\|_{L^\infty(Q)} \, n^{-2/z'}$, which yields the claim. □

Proof of the Lemma 10.3. Let $\zeta \in C^1_c(0,T)$. Consider for $t \in]0,T[$ the weak solution $\psi_\epsilon \in W^{1,2}(\Omega)$ to the auxiliary Problem

$$-\triangle \psi_\epsilon = \varrho_\epsilon(t) \, \zeta(t) \text{ in } \Omega, \quad \psi = 0 \text{ on } \partial\Omega \, .$$

Then the estimate $\|\psi_\epsilon\|_{L^\infty(0,T;\, W^{2,\alpha}_{\text{loc}}(\Omega))} \leq c_\zeta \, \|\varrho_\epsilon\|_{L^\infty(0,T;\, L^\alpha(\Omega))}$ is valid. Moreover, $u = \psi_t$ is a weak solution to

$$-\triangle u = (\varrho_\epsilon(t) \, \zeta(t))_t = \zeta(t) \, (-\operatorname{div}(\varrho_\epsilon \, v^\epsilon + \bar{J}^\epsilon) + \varrho_\epsilon(t) \, \zeta'(t), \quad u = 0 \text{ on } \partial\Omega \, .$$

This yields for $6\alpha/(6+\alpha) \geq 2$ that

$$\|\psi_t\|_{L^2(0,T;\, W^{1,2}(\Omega))} \leq c_\zeta \, (\|\varrho_\epsilon \, v^\epsilon\|_{L^2(Q)} + \|\bar{J}^\epsilon\|_{L^2(Q)}) \leq c_\zeta \, C_0 \, .$$

Let $\eta \in C^1_c(\Omega)$ arbitrary, and consider the field $X^\epsilon := \eta \, \nabla \psi_\epsilon$. Then

$$\|\partial_t X^\epsilon\|_{L^2(0,T;\, W^{1,2}(\Omega))} \leq C_0, \quad \|\nabla X^\epsilon\|_{L^{\infty,\alpha}(Q)} \leq c_\eta \, \|\psi_\epsilon\|_{L^\infty(0,T;\, W^{2,\alpha}(\operatorname{supp}\eta))} \leq C_0 \, .$$

Define $\psi \in W^{1,2}(\Omega)$ to be the weak solution to the auxiliary Problem $-\triangle\psi = \varrho_\epsilon(t) \, \zeta(t)$ in $[W^{1,2}_0(\Omega)]^*$. Then it is readily proved (use the Remark 9.3) for $X = \nabla\psi \, \eta$ that

$$X^\epsilon \to X \text{ strongly in } L^2(Q), \quad \partial_t X^\epsilon \to \partial_t X \text{ weakly in } L^2(Q) \, ,$$
$$\nabla X^\epsilon \to \nabla X \text{ weakly in } L^\alpha(Q) \, .$$

Since $2 > 6\alpha/(5\alpha - 6)$ and $\alpha > \frac{3\alpha}{2\alpha-3}$ (this is exactly the case for $\alpha > 3$), we can show that the assumptions of Lemma A.2 are satisfied. Thus

$$\varrho_\epsilon\, v^\epsilon \cdot \partial_t X^\epsilon + \varrho_\epsilon\, v^\epsilon \otimes v^\epsilon \,:\, \nabla X^\epsilon \to \varrho\, v \cdot \partial_t X + \varrho\, v \otimes v \,:\, \nabla X \text{ weakly in } L^1(Q)\,.$$

Moreover, calling here η_0 the coefficient of shear viscosity

$$\begin{aligned}
\int_Q \mathbb{S}(\nabla v^\epsilon) \,:\, \nabla X^\epsilon &= \eta_0 \int_Q D(\nabla v^\epsilon) \,:\, \nabla X^\epsilon + \lambda \int_Q \operatorname{div} v^\epsilon \, \operatorname{div} X^\epsilon \\
&= -\eta_0 \int_Q v^\epsilon \cdot \triangle X^\epsilon - \eta_0 \int_Q v^\epsilon \cdot \nabla(\operatorname{div} X^\epsilon) + \lambda \int_Q \operatorname{div} v^\epsilon \, \operatorname{div} X^\epsilon \\
&= \eta_0 \int_Q v^\epsilon \cdot \operatorname{curl}\operatorname{curl} X^\epsilon - 2\,\eta_0 \int_Q v^\epsilon \cdot \nabla(\operatorname{div} X^\epsilon) + \lambda \int_Q \operatorname{div} v^\epsilon \, \operatorname{div} X^\epsilon \\
&= \eta_0 \int_Q \operatorname{curl} v^\epsilon \cdot \operatorname{curl} X^\epsilon + (\lambda + 2\,\eta_0) \int_Q \operatorname{div} v^\epsilon \, \operatorname{div} X^\epsilon\,.
\end{aligned}$$

Thus

$$\begin{aligned}
\int_Q \mathbb{S}(\nabla v^\epsilon) \,:\, \nabla X^\epsilon =& (\lambda + 2\,\eta_0) \int_Q \operatorname{div} v^\epsilon \, \varrho_\epsilon\, \zeta \\
&+ \int_Q \{(\lambda + 2\,\eta_0)\, \operatorname{div} v^\epsilon\, \nabla\psi_\epsilon \cdot \nabla\eta - \eta_0\, \operatorname{curl} v^\epsilon \cdot (\nabla\psi_\epsilon \times \nabla\eta)\}\,.
\end{aligned}$$

We also note that

$$\int_Q p_\epsilon\, \operatorname{div} X^\epsilon = \int_Q p_\epsilon\, \varrho_\epsilon\, \eta\, \zeta + \int_Q p_\epsilon\, \nabla\psi_\epsilon \cdot \nabla\eta\,.$$

Moreover

$$\left| \int_Q (\bar{J}^\epsilon \cdot \nabla) X^\epsilon \cdot v^\epsilon \right| \le \|\bar{J}^\epsilon\|_{L^2(Q)}\, \|\nabla X^\epsilon\|_{L^{\infty,3}(Q)}\, \|v^\epsilon\|_{L^{2,6}(Q)} \to 0\,.$$

Multiplying the Navier-Stokes equation with X^ϵ and the limiting equation with X, we then easily obtain that

$$\int_Q \zeta\, \eta\, (p_\epsilon - (\lambda + 2\,\eta_0)\, \operatorname{div} v^\epsilon)\,, \varrho_\epsilon \to \int_Q \zeta\, \eta\, (p - (\lambda + 2\,\eta_0)\, \operatorname{div} v)\, \varrho\,.$$

□

Proof of the Lemma 10.8

Proof. One uses the Lemma A.2 with a vector field $X^{\epsilon_n} = \nabla\psi_{\epsilon_n}\, \eta$, where ψ_{ϵ_n} solves

$$-\triangle\psi_{\epsilon_n} = T_k(\varrho_{\epsilon_n}(t))\, \zeta(t) \text{ in } \Omega, \quad \psi_{\epsilon_n} = 0 \text{ on } \partial\Omega\,.$$

Then, one obtains for all $1 < p < \infty$ the bound $\|\psi_{\epsilon_n}\|_{L^\infty(0,T;\, W^{2,p}_{\text{loc}}(\Omega))} \leq C_p$. Moreover, since ϱ_{ϵ_n} are assumed renormalized solutions to (246), we obtain for k fixed that

$$\begin{aligned} & -\int_Q T_k(\varrho_{\epsilon_n})\, \psi_t - \int_Q T_k(\varrho_{\epsilon_n})\, v^{\epsilon_n} \cdot \nabla \psi \\ & \quad = -\int_Q (T_k'(\varrho_{\epsilon_n})\, \varrho_{\epsilon_n} - T_k(\varrho_{\epsilon_n}))\, \operatorname{div} v^{\epsilon_n}\, \psi \;\; \text{for all } \psi \in C^1_c(Q)\,. \end{aligned}$$

This proves the bounds

$$\begin{aligned} \|\partial_t T_k(\varrho_{\epsilon_n})\|_{L^2(0,T;\,[W^{1,6/5}_0(\Omega)]^*)} & \leq c\, k\, (\|v^{\epsilon_n}\|_{L^{2,6}(Q)} + \|\operatorname{div} v^{\epsilon_n}\|_{L^2(Q)}) \\ \|(\nabla \psi_{\epsilon_n})'\|_{L^{2,6}(Q)} & \leq c\, k\, (\|v^{\epsilon_n}\|_{L^{2,6}(Q)} + \|\operatorname{div} v^{\epsilon_n}\|_{L^2(Q)} + \|\zeta'\|_{L^\infty(Q)})\,. \end{aligned}$$

We obtain from the Lemma A.2 that

$$\varrho_{\epsilon_n}\, v^{\epsilon_n} \cdot \partial_t X^{\epsilon_n} + \varrho_{\epsilon_n}\, v^{\epsilon_n} \otimes v^{\epsilon_n} \, : \, \nabla X^{\epsilon_n} \to \varrho\, v \cdot \partial_t X + \varrho\, v \otimes v \, : \, \nabla X \text{ weakly in } L^1(Q)\,.$$

But then we can conclude exactly as in the Lemma 10.3 that $(p_{\epsilon_n} - \eta' \operatorname{div} v^{\epsilon_n})\, T_k(\varrho_{\epsilon_n}) \to (p - \eta' \operatorname{div} v)\, a_k$ as distributions.

Finally we note that the sequence $\{p_{\epsilon_n} - \eta' \operatorname{div} v^{\epsilon_n}\}$ is uniformly bounded in $L^{1+\frac{2}{3}-\frac{1}{\alpha}}(Q)$ and the claim follows. $\square$

At last we justify the continuity statements in the Lemma 10.4 and 10.7.

Proof. We use that the solution to (237), (239) is renormalised. We are going to prove the claim for a solution to (239). The arguments for solutions to (237) case are similar. For $k \in \mathbb{N}$, denote

$$f_k(s) := \min\{k,\, \max\{s,\, 1/k\}\} \text{ for } s \geq 0\,.$$

If ϱ satisfies (239), we then obtain for the function $F(s) := s \ln s$ and almost all $0 < t_1 < t_2 < T$ that

$$\begin{aligned} \int_\Omega (F \circ f_k)(\varrho(t_2)) - \int_\Omega (F \circ f_k)(\varrho(t_1)) + \int_{t_1}^{t_2} \int_\Omega b_k \operatorname{div} v = 0 \qquad (286) \\ b_k = \chi_k\, \varrho + (1 - \chi_k)\, (F \circ f_k)(\varrho), \quad \chi_k = \chi_{\{\varrho \in [k^{-1},\, k]\}}\,. \end{aligned}$$

Thus,

$$\left| \int_\Omega (F \circ f_k)(\varrho(t_2)) - \int_\Omega (F \circ f_k)(\varrho(t_1)) \right| \leq c_k\, \|\operatorname{div} v\|_{L^2(Q)}\, \sqrt{t_2 - t_1}\,.$$

Next we use that for postitive real numbers $F(x_2) - F(x_1) = F'(x_1)\,(x_2 - x_1) + \frac{1}{2}\int_0^1 F'(\lambda\, x_1 + (1-\lambda)\, x_2)\, d\lambda\, (x_2 - x_1)^2$, which yields

$$\frac{|x_2 - x_1|^2}{1 + \max\{x_1, x_2\}} \leq F(x_2) - F(x_1) - F'(x_1)\,(x_2 - x_1)\,.$$

From this inequality, we easily deduce for all $u,\, v \in L^\infty(\Omega;\, [k^{-1},\, k])$ that

$$\int_\Omega |u-v| \le \left(\int_\Omega \frac{(u-v)^2}{\max\{u,\, v\}}\right)^{1/2} \left(\int_\Omega (u+v)\right)^{1/2}$$
$$\le (\|u\|_{L^{\infty,1}(\Omega)} + \|v\|_{L^{\infty,1}(\Omega)})^{1/2} \left(\left|\int_\Omega (F(v) - F(u) - F'(u)\,(v-u))\right|\right)^{1/2} .$$

Thus, for $u = f_k(\varrho(t_1))$ and $v = f_k(\varrho(t_2))$

$$\begin{aligned}
&\|f_k(\varrho(t_2)) - f_k(\varrho(t_1))\|^2_{L^1(\Omega)} \\
&\le 2\,\|\varrho\|_{L^{\infty,1}(Q)} \left|\int_\Omega \{(F\circ f_k)(\varrho(t_2)) - (F\circ f_k)(\varrho(t_1)) - (1+\ln f_k(\varrho(t_1)))\,(\varrho(t_2)-\varrho(t_1))\}\right| \\
&\le C_0 \left(c_k\,\|\operatorname{div} v\|_{L^2(Q)}\sqrt{t_2-t_1} + \left|\int_\Omega (1+\ln f_k(\varrho(t_1)))\,(\varrho(t_2)-\varrho(t_1))\right|\right)
\end{aligned}$$

Further, we note that

$$\|f_k(\varrho(t_2)) - f_k(\varrho(t_1))\|_{L^1(\Omega)} \ge \|\varrho(t_2)-\varrho(t_1)\|_{L^1(\Omega)} - c_0\,\left(\frac{1}{k}\right)^{\frac{1}{\alpha'}} .$$

Thus

$$\|\varrho(t_2)-\varrho(t_1)\|^2_{L^1(\Omega)} \le C_0\,\left(c_k\sqrt{t_2-t_1} + \left|\int_\Omega (1+\ln f_k(\varrho(t_1)))\,(\varrho(t_2)-\varrho(t_1))\right|\right) + k^{-\frac{1}{\alpha'}} .$$

It remains to observe that $\varrho \in C([0,T];\, \mathcal{D}^*(\Omega))$. Thus, $\varrho(t_1) \to \varrho(t_2)$ as distributions for $t_2 \to t_1$. On the other hand $\{\varrho(t_2)\}$ is a bounded family in $L^\alpha(\Omega)$. We obtain that

$$\limsup_{t_2\to t_1} \|\varrho(t_2)-\varrho(t_1)\|_{L^1(\Omega)} \le k^{-\frac{1}{\alpha'}} ,$$

proving that $\varrho \in C([0,T];\, L^1(\Omega))$. □

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