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Guaranteed quality isotropic surface remeshing based on uniformization

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Abstract

Surface remeshing plays a significant role in computer graphics and visualization. Numerous surface remeshing methods have been developed to produce high quality meshes. Generally, the mesh quality is improved in terms of vertex sampling, regularity, triangle size and triangle shape. Many of such surface remeshing methods are based on Delaunay refinement. In particular, some surface remeshing methods generate high quality meshes by performing the planar Delaunay refinement on the conformal uniformization domain. However, most of these methods can only handle topological disks. Even though some methods can cope with high-genus surfaces, they require partitioning a high-genus surface into multiple simply connected segments, and remesh each segment in the parameterized domain.

In this work, we propose a novel surface remeshing method based on uniformization theorem using dynamic discrete Yamabe flow and Delaunay refinement, which is capable of handling surfaces with complicated topologies, without the need of partitioning. The proposed method has the following merits: Dimension deduction, it converts all 3*D* surface remeshing to 2*D* planar meshing; Theoretic rigor, the existence of the constant curvature measures and the lower bound of the corner angles of the generated meshes can be proven. Experimental results demonstrate the efficiency and efficacy of our proposed method.

Keywords: Surface remeshing, Conformal uniformization, Dynamic discrete Yamabe flow, Delaunay refinement

1. Introduction

The past decades have witnessed a remarkable growth of interest and research efforts in surface meshing and remeshing [4,17]. An important fact is that numerous geometric processing tasks are based on solving geometric PDEs on meshes, and the numerical stability as well as approximation errors are highly dependent on the mesh quality. Therefore, surface meshing plays a significant role in computer graphics and visualization.

Generally, surface remeshing refers to the process of improving the input mesh quality and then producing a refined mesh which satisfies a certain quality requirement. A large volume of research works have been studied to employ

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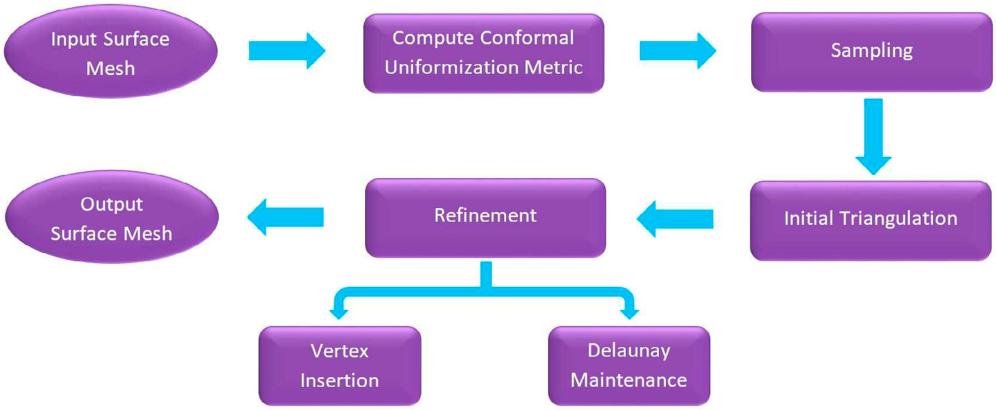

Fig. 1. The pipeline of our proposed surface remeshing method.

new methods for the improvement of mesh quality in terms of sampling, triangle size, element shape, etc. However, there still exist many challenging problems in surface remeshing such that it is highly desirable to develop novel techniques to obtain simpler and more computationally efficient algorithms, which can cope with surfaces with more complicated topologies and guarantee the convergence of curvature measures and so on.

Among various surface meshing methods, the Delaunay refinement based methods have attracted a great deal of attention. Some of these methods apply Delaunay refinement directly on the surfaces [3], and others first compute a conformal parameterization domain and then perform the Delaunay refinement on it [11]. Most of these proposed methods can only deal with topological disks, namely genus-zero surfaces with a single boundary, and few of them can handle genus-one or high-genus surfaces. In [11], for instance, the proposed surface meshing algorithm only handles genus-zero surfaces, although it is advantageous in guaranteeing the convergence of the curvature measures. Even though the presented harmonic map based high-quality surface remeshing method in [13] is capable of handling high-genus surfaces, it needs a series of complex operations including multiple cuts on high-genus surfaces into different genus-zero mesh partitions, remeshing the lines at the interfaces between different partitions, and the computation of a harmonic map for every partition as well as remeshing the partition in the parametric space. In contrast, our surface remeshing method proposed in this paper can handle surfaces with complex topologies without partitioning.

1.1. Our Approach

In this paper, we propose a novel surface remeshing method based on surface uniformization and Delauny refinement. The pipeline of our proposed method is shown in Figure 1. We illustrate the pipeline using a kitten surface model in Figure 2.

Given an input mesh with complicated topologies, our proposed method starts with computation of the conformal uniformization Riemannian metric using dynamic discrete Yamabe flow. In essence, dynamic discrete Yamabe flow deforms the Riemannian metric proportional to the curvature such that the curvature evolves like a non-linear diffusion-reaction process and eventually becomes constant everywhere. Moreover, the triangulation of the surface is updated to be Delaunay during the flow, and this guarantees the existence of the solution to the flow and the numerical stability. The proof of the existence is reported in our recent theoretic work [7].

After the conformal mapping is computed, the point sampling is performed on the 2D conformal uniformization domain. Then, initial triangulation of the sampled points is carried out, and planar Delaunay refinement is applied. Unlike the conventional methods which partition the surface into a number of segments, our proposed method converts all 3D surface remeshing to 2D planar meshing on 2D conformal uniformization domain without the need of partitioning of surface into several parts.

1.2. Contributions

The contributions in this paper can be summarized as follows:

- A novel surface remeshing method based on surface uniformization using dynamic discrete Yamabe flow and Delaunay refinement.

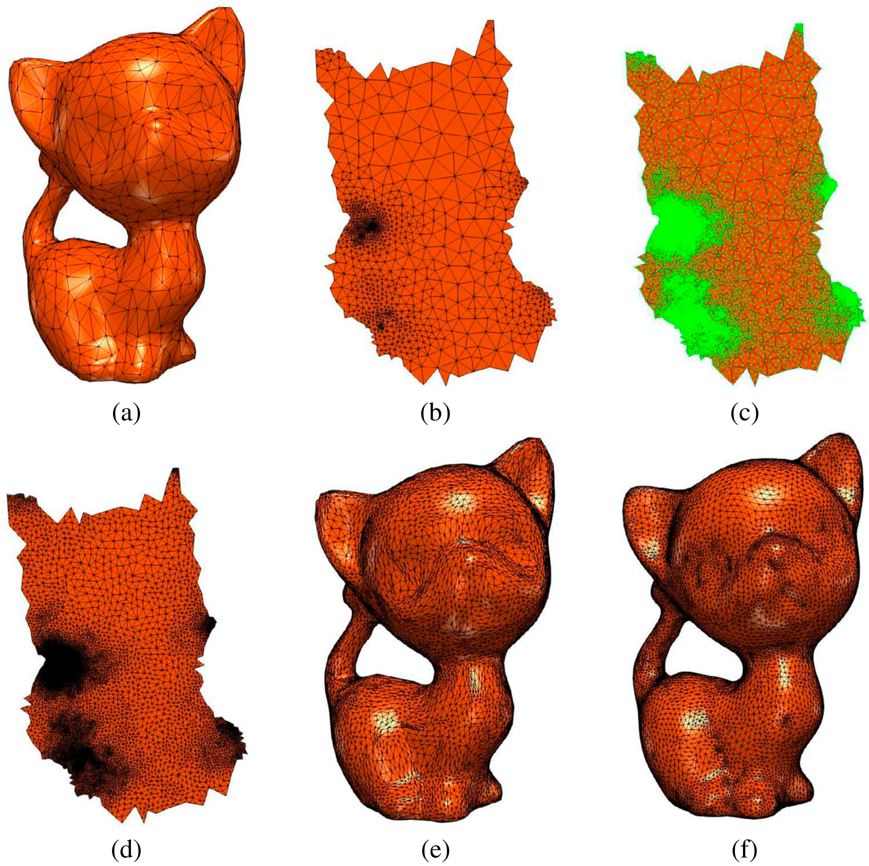

Fig. 2. Illustration of pipeline. (a) Input mesh. (b) Conformal mapping result on 2D domain. (c) Sampling on conformal mapping result with sampled points marked in green. (d) Triangulation of sampled points in (c). (e) The 3D surface which corresponds to (d). (f) Final surface remeshing result based on Delaunay refinement.

- The discrete Yamabe flow method is improved by updating the triangulation of the surface to be Delaunay during the flow, which guarantees the existence of the solution to the flow and the numerical stability.

The method has the following merits:

- General, the method can handle surfaces with complicated topologies;
- Holistic, the method computes the uniformization metric without surface segmentation;
- Dimension reducing, the method converts the 3D surface remeshing to a 2D problem;
- Robust, the existence of the uniformization metric, which is the solution to the dynamic Yamabe flow, is proven in [7]. Unlike other discrete Ricci flow methods, which are sensitive to the quality of the input meshes and may encounter degenerated cases during the flow, dynamic Yamabe flow is numerically robust to meshes with low qualities and never leads to degenerated triangles.

2. Previous Works

The literature of meshing and remeshing is vast, and in the following we briefly review only the most related works.

2.1. Conformal Parameterization

Surface parameterization [19] has received extensive research interest in mesh processing over the past decades. Among a variety of surface parameterization methods, conformal parameterization [14] has attracted increasing attention due to its nice properties including local shape preservation (angle preserving), naturally intrinsic to geometry and robustness to resolution change, thus rendering it suitable for many practical applications. In [9], a circle packing method was proposed to compute the conformal map, such that the canonical surface-based coordinate systems can be imposed on these flat maps by specifying anatomical or functional landmarks. In [8], a global conformal surface parameterization method was introduced based on the computation of holomorphic one-forms. Using this method, a conformal atlas can be constructed and used to build conformal geometry images which have very accurate reconstructed normals. In [10], a Ricci flow method was presented to conformally deform the Riemannian metric on a surface such that the curvature eventually becomes the user defined curvature. By optimizing the discrete Ricci energy using Newton's method, the Ricci energy reaches its minimum at the desired metric. In [12], a combinatorial Yamabe flow method was proposed on the space of all piecewise flat metrics associated to a triangulated surface. The flow can either develop removable singularities or converge exponentially fast to a constant combinatorial curvature metric. In [21], the discrete Yamabe flow algorithm was employed for computing arbitrary high-genus surface. A recent theoretic work [7] proved the uniformization theorem for polygonal surfaces, which ensures the existence of the solutions to the dynamic discrete Yamabe flow and the numerical stability of the flow. In this work, we employ the dynamic discrete Yamabe flow method to compute conformal surface parameterization.

2.2. Meshing and Remeshing

In the past years, numerous meshing or remeshing methods have been investigated. In [2], an efficient algorithm based on Delaunay triangulation was presented, which produces desirable quality triangles with angles between 30° and 120°. Moreover, the worst-case time of generating a triangulation is linear with the resulting number of triangles. In [3], a novel approach to producing high-quality triangle meshes for curved surfaces was proposed by circumcenter insertion and triangle splitting operations. This method guarantees all triangles to be well-shaped (no angle is smaller than 30 degree) and uniformly sized. In [17], a much simpler method of quality triangulation was achieved using successive Delaunay refinement. Also, all triangles are guaranteed to have angles no less than 20.7 degree. In [18], sharp features are handled.

An isotropic surface remeshing method was also devised in [1]. This method first samples a triangulated surface mesh with user-defined density, and then builds a weighted centroidal Voronoi tessellation (CVT) in a conformal parameter space using the resulting sampling, and finally constructs the mesh by lifting the corresponding constrained Delaunay triangulation. In [16], a surface remeshing method was developed to recover high quality meshes from complex surfaces, which is based on the parametrization of genus zero surfaces with harmonic maps. However, this method has limitations in handling surfaces with high genus and/or of large aspect ratio. To overcome these problems, [13] combined a multiscale version of the harmonic parametrization with a multilevel partitioning algorithm to obtain an automatic remeshing algorithm. In [6], a surface remeshing algorithm was presented. It constructs the principal curvature tensor using a local quadric surface fitting method. Based on the principal curvature and direction on the discrete surface, the curvature lines on the mesh are gained for surface remeshing.

To preserve the geometry at a high-order of accuracy, a remeshing method which can produce high-quality triangular meshes and untangle mildly folded triangles is proposed in [15]. In [4], an optimization-based surface remeshing method was introduced. The method can generate a curvature-adapted anisotropic surface mesh which preserves important geometric features and contains nicely shaped elements. To ensure the curvature convergence in conformal parameterization based Delaunay refinement, a novel method to triangulate the conformal uniformization domain of the surface is proposed in [11]. Although this method enables a high quality mesh to be generated with guaranteed curvature convergence, only genus zero surfaces can be handled.

Most of these methods have topological limitations, and lack the theoretic guarantees for the remeshing qualities and are vulnerable to low quality input meshes. In contrast, our proposed method can handle surfaces with complicated topologies, is robust to the input mesh quality and has a solid theoretic foundation.

3. Theoretical Foundation

In this section, the theoretical foundation of our framework is presented.

3.1. Uniformization Theorem

Suppose (M_1, g_1) and (M_2, g_2) are two smooth surfaces with Riemannian metrics. A diffeomorphism $\varphi : M_1 \to M_2$ is conformal (i.e., angle-preserving) if and only if

$$\phi^* g_2 = e^{2\lambda} g_1 \tag{1}$$

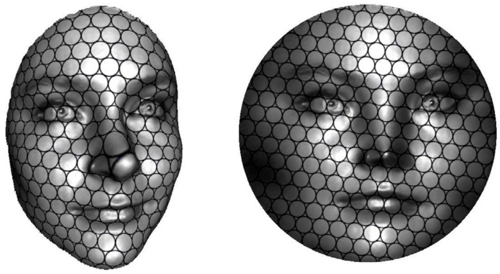

Fig. 3. Conformal mappings transform infinitesimal circles to infinitesimal circles and preserve the Delaunay triangulations.

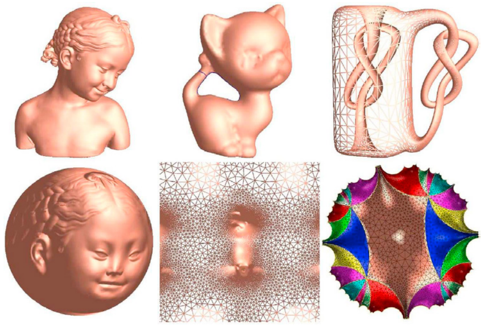

Fig. 4. Uniformization for closed surfaces. All closed surfaces can be conformally deformed to one of three canonical shapes depending on the topology: the unit sphere, the Euclidean plane or the hyperbolic plane.

where $\phi^* g_2$ is the pullback metric on M_1 induced by the mapping φ, and $\lambda : M_1 \to \mathbb{R}$ is a scalar function defined on M_1. The function $e^{2\lambda}$ is called the conformal factor, which denotes the area distortion induced by the conformal mapping.

Conformal mappings transform infinitesimal circles to infinitesimal circles, as shown in Figure 3. The human facial surface is conformally mapped onto the planar disk, a circle packing pattern is pulled back by the mapping onto the facial surface, and all the circles are well-preserved. Delaunay triangulation has the empty-circle property, therefore, **conformal mapping preserves Delaunay triangulations**.

Theorem 3.1 (Uniformization Theorem)**.** *Let* $(M, \mathbf{g})$ *be a compact two-dimensional surface with a Riemannian metric* $\mathbf{g}$*, then there is a metric* $\bar{\mathbf{g}}$ *conformal to* $\mathbf{g}$ *with constant Gaussian curvature everywhere; the constant is one of* $\{+1, 0, -1\}$.

The classical Riemann surface uniformization theorem states that every metric surface $(M, \mathbf{g})$ can be conformally deformed to one of three canonical surfaces: a sphere $\mathbb{S}^2$, a Euclidean plane $\mathbb{E}^2$ or a hyperbolic plane $\mathbb{H}^2$. That is, there exists a unique conformal factor function $\lambda : M \to \mathbb{R}$, such that the uniformization Riemannian metric $e^{2\lambda}\mathbf{g}$ is conformal to the metric g with constant Gaussian curvature, and the constant is one of $\{+1, 0, -1\}$ depending on the topology of the surface.

Figure 4 shows the uniformizations for the closed surfaces. The left-hand column shows that a genus zero surface can conformally deform to a unit sphere with $+1$ curvatures. The middle-column illustrates that the universal covering space of a genus one surface can be conformally mapped to a Euclidean plane with 0 curvature. The right-column demonstrates that the universal covering space of a high genus surface can be flattened to a hyperbolic plane with -1 curvature. Accordingly, we can perform the Delaunay refinement on the uniformization domain, and the triangulation is pulled back to the original surface by the conformal mapping. Due to the conformality of the mapping, the triangulation on the original surface is also Delaunay, and the minimal corner angle of the triangulation on the parameter domain is equal to that on the surface.

3.2. Dynamic Discrete Yamabe Flow

The uniformization metric of polygonal surfaces can be obtained using the discrete Ricci flow method [20]. In this paper, we focus on the dynamic Yamabe flow method for its robustness and rigorous foundation [7].

In the discrete setting, triangle meshes are employed to approximate smooth surfaces. A triangle mesh is denoted as $M = (V, E, F)$, where V, E and F represent vertex, edge and face sets, respectively.

A Riemannian metric on a mesh M is represented by a piecewise constant metric with cone singularities. A metric on a mesh with Euclidean geometry is a Euclidean metric with cone singularities. Similarly, a metric on a mesh with hyperbolic geometry is a hyperbolic metric with cone singularities. A discrete Riemannian metric is formulated as an edge length function $l : E \to \mathbb{R}^+$, which satisfies the triangle inequality on each face. The corner angles are determined by the cosine law given in Eqn. 11 or 12 accordingly.

The discrete Gaussian curvature is determined by the angle deficit. The Gaussian curvature for an interior vertex is defined as 2π minus the sum of the surrounding corner angles, while the Gaussian curvature for a boundary vertex is computed by π minus the sum of the surrounding corner angles. Formally speaking, given a triangle mesh M with a discrete Riemannian metric, the discrete Gaussian curvature for a vertex v_i is given as follows:

$$K(v_i) = \begin{cases} 2\pi - \Sigma_{f_{ijk}\in F}\, \theta_i^{jk}, & v_i \notin \partial M \\ \pi - \Sigma_{f_{ijk}\in F}\, \theta_i^{jk}, & v_i \in \partial M \end{cases} \tag{2}$$

where θ_i^{jk} denotes the corner angle adjacent to the vertex v_i in the face f_{ijk}, and ∂M represents the boundary of the mesh M.

According to the Gauss-Bonnet theorem, the total curvature is a topological invariant. The following formula holds on meshes:

$$\sum_{v_i\in V} K_i + \lambda \sum_{f_i\in F} A_i = 2\pi\chi(M), \tag{3}$$

where A_i represents the area of face f_i, λ denotes the constant curvature for the background geometry which is 0 for Euclidean geometry and -1 for hyperbolic geometry, and $\chi(M)$ is the Euler characteristic of mesh M.

Suppose e_{ij} is an edge which has end vertices v_i and v_j, and d_{ij} is the edge length of e_{ij} induced by the Euclidean metric. We define a discrete conformal factor u on vertices as $u : V \to \mathbb{R}$. Then the edge length l_{ij} in the discrete Euclidean Yamabe flow is defined as:

$$l_{ij} = e^{u_i} d_{ij} e^{u_j}. \tag{4}$$

The edge length in the discrete hyperbolic Yamabe flow is formulated as:

$$\sinh \frac{l_{ij}}{2} = e^{u_i} \sinh\left(\frac{d_{ij}}{2}\right) e^{u_j}. \tag{5}$$

Let K_i be the discrete Gaussian curvature defined on vertex v_i, and $\bar{K}_i$ be the target curvature. The discrete Euclidean (and Hyperbolic) Yamabe flow is defined as:

$$\frac{du_i}{dt} = \bar{K}_i - K_i. \tag{6}$$

During the flow, the triangulation is dynamically updated, such that the triangulation is Delaunay with respect to the current Riemannian metric. A discrete uniformization theorem which is proven in [7] demonstrates that, if the target curvature satisfies the Gauss-Bonnet condition and $\bar{K}_i < 2\pi$ holds for each vertex v_i, then the solution to the dynamic discrete Yamabe flow exists. Moreover, this solution is the unique optimal point of the following convex discrete Ricci energy:

$$E(\mathbf{u}) = \int_0^{\mathbf{u}} \sum_{i=1}^{n} (\bar{K}_i - K_i) du_i, \quad \mathbf{u} = (u_1, u_2, ..., u_n). \tag{7}$$

In practice, we employ Newton's method to optimize the convex discrete Ricci energy. The gradient of the Ricci energy is computed by the curvature difference, i.e.,

$$\nabla E(\mathbf{u}) = (\bar{K}_1 - K_1, \bar{K}_2 - K_2, ..., \bar{K}_n - K_n), \tag{8}$$

For Euclidean Yamabe flow, the Hessian matrix of the Ricci energy is given as follows:

$$\frac{\partial^2 E(\mathbf{u})}{\partial u_i \partial u_j} = \begin{cases} w_{ij}, & i \neq j \\ -\Sigma_k w_{ik}, & i = j \end{cases} \tag{9}$$

where w_{ij} is the weight of the edge e_{ij}. The edge weight w_{ij} is the classical cotangent edge weight:

$$w_{ij} = \cot\theta_k + \cot\theta_l, \tag{10}$$

where θ_k and θ_l are two corner angles against the edge e_{ij}. The Hessian matrix for hyperbolic Yambe flow can be defined similarly.

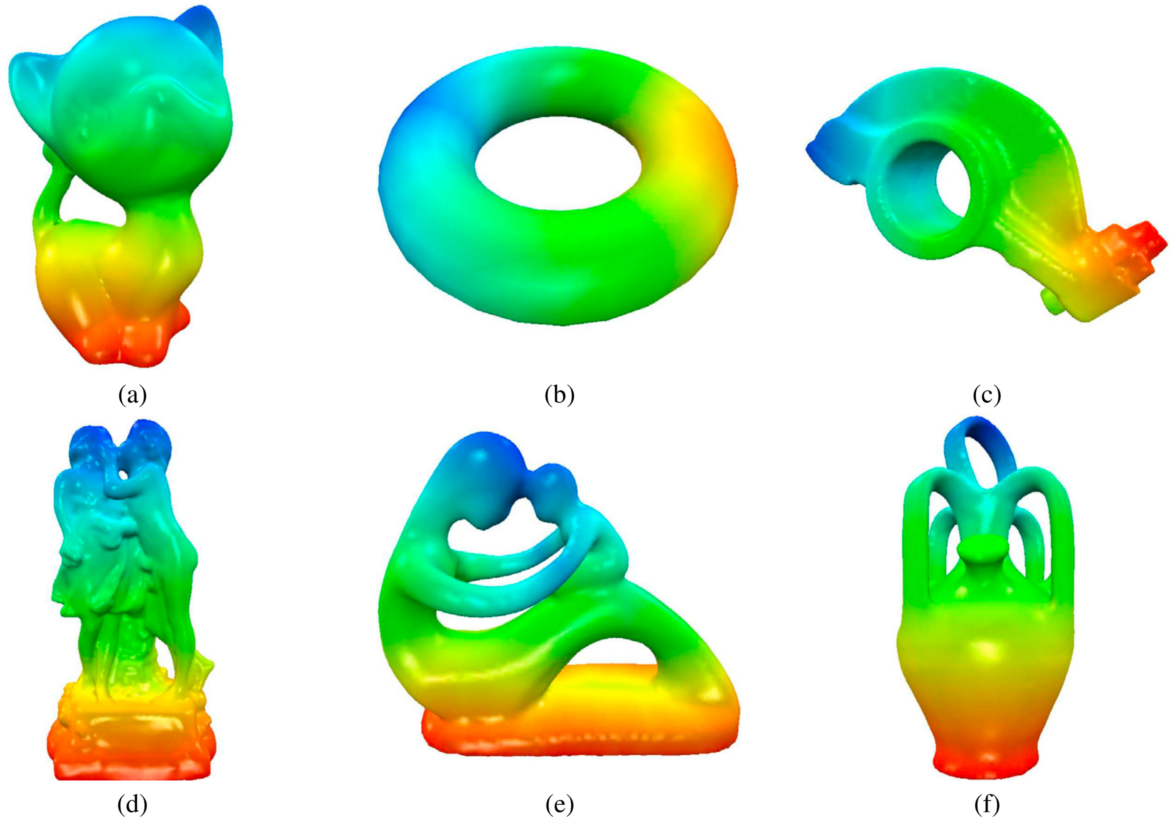

Fig. 5. Genus one and high genus surface models: (a) Kitten mesh. (b) Torus mesh. (c) Rocker arm mesh. (d) Kids mesh. (e) Fertility mesh. (f) Genus 5 mesh.

Note that in Eqn. 10, triangles with Euclidean or hyperbolic geometry (i.e., triangle in $\mathbb{E}^2$ or $\mathbb{H}^2$) satisfy different cosine laws:

$$\cos\theta_i = \frac{-l_i^2 + l_j^2 + l_k^2}{2l_j l_k}, \quad (in\ \mathbb{E}^2), \tag{11}$$

$$\cos\theta_i = \frac{-\cosh l_i + \cosh l_j \cosh l_k}{\sinh l_j \sinh l_k}, \quad (in\ \mathbb{H}^2). \tag{12}$$

3.3. Surface Remeshing based on Delaunay Refinement

Let $(M, \mathbf{g})$ be an initial surface. The proposed surface remeshing method based on Delaunay refinement starts with the computation of the conformal uniformization metric $\bar{\mathbf{g}}$. $(M, \bar{\mathbf{g}})$ is with constant curvature, and can be locally isometrically embedded onto the constant curvature space $\{\mathbb{S}^2, \mathbb{E}^2, \mathbb{H}^2\}$. Then the sampling and initial triangulation are carried out. Finally, Delaunay refinement is performed on the constant curvature space, until a certain quality requirement is satisfied.

The refinement is carried out based on inserting additional circumcenters of the bad triangles such that the bad triangles are gradually eliminated and replaced by better quality triangles while maintaining the Delaunay property using an edge-flipping operation.

Theorem 3.2. *Suppose $(M, \mathbf{g})$ is a closed Riemannian surface with genus one. For a given small enough constant ϵ, there exists a geodesic triangulation, such that the edge lengths are no greater than 2ϵ, and the minimal angle is no less than $30°$.*

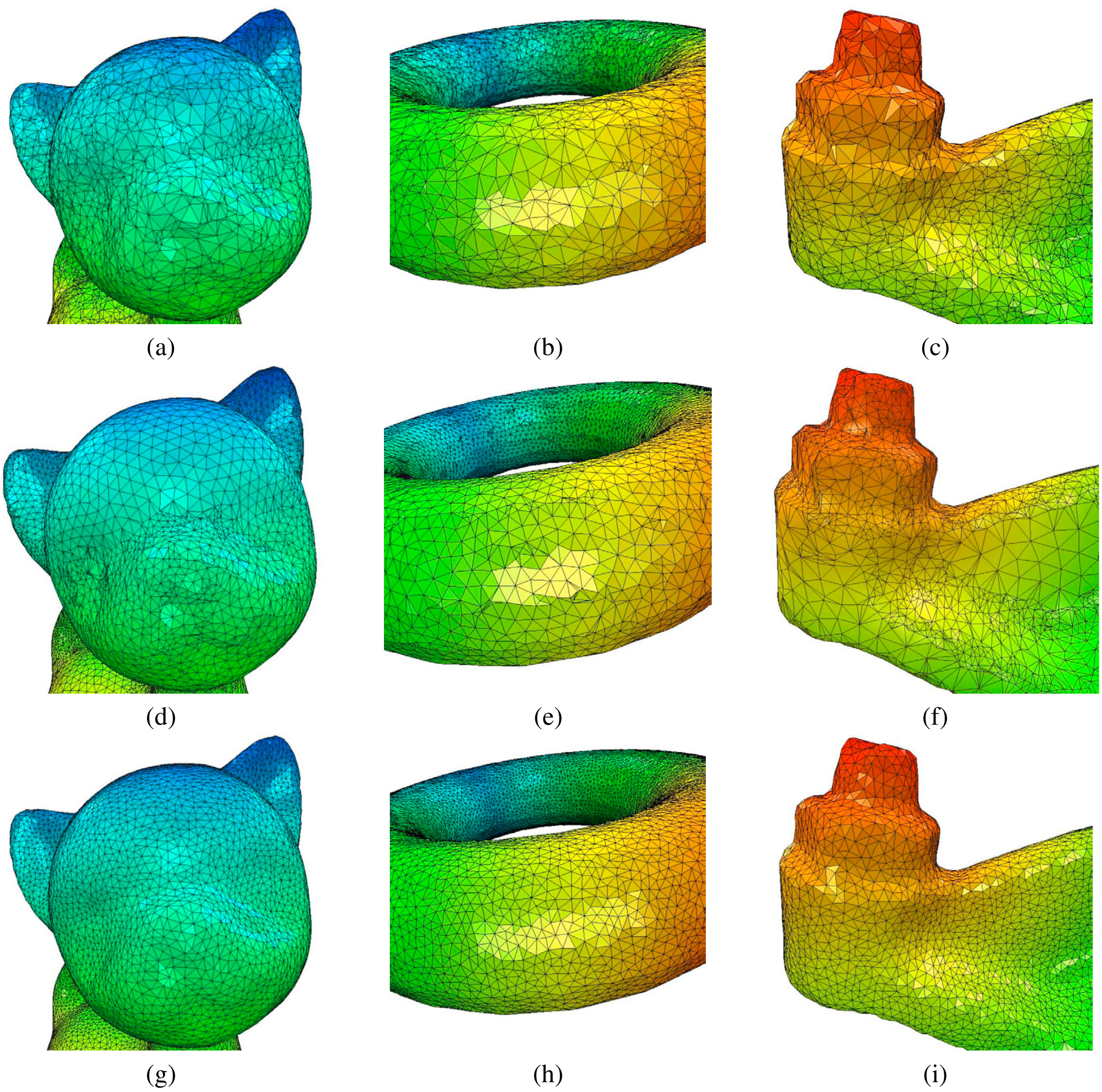

Fig. 6. Remeshing results on genus one models: (a-c) Initial meshes. (d-f) Remeshing results of CVT-based method. (g-i) Remeshing results of our method.

Proof. According to uniformization theorem, there exists a conformal factor function $\lambda : M \to \mathbb{R}$, such that $\bar{\mathbf{g}} = e^{2\lambda}\mathbf{g}$. Because M is compact, λ is bounded, and assume $C = \max e^{\lambda}$.

On the flat torus $(M, \bar{\mathbf{g}})$, we sample the surface, such that the shortest distance among the samples is $C^{-1}\epsilon$. Then we use Chow's Delaunay refinement algorithm: at each step, we pick a triangle with circum radius greater than $C^{-1}\epsilon$, and insert its circum center, maintain the Delaunay property by edge-flipping.

For each vertex in the triangulation, there is a disk centered at the vertex with radius $C^{-1}\epsilon/2$, and no further vertex can be inserted in this disk. Therefore, the total vertices must be finite, and the algorithm terminates after finite steps. In the final stage, all circum-circles are with radii no greater than $C^{-1}\epsilon$. On the other hand, the minimal edge length is always no less than $C^{-1}\epsilon$ during the whole process. Hence the minimal angle is no less than $30°$.

The triangulation on $(M, \bar{\mathbf{g}})$ is pulled back to $(M, \mathbf{g})$, then the edge length is no greater than ϵ, and the minimal corner angle is no less than $30°$. $\square$

Similarly, we can prove the following,

Theorem 3.3. *Suppose $(M, \mathbf{g})$ is a closed Riemannian surface with genus g, g equals to 0 (or greater than 1). For a given small enough constant ϵ, there exists a geodesic triangulation, such that the edge lengths are no greater than ϵ, and the minimal angle is no less than a positive constant $\theta > 0$.*

Proof. The proof is similar to the genus one case. When ϵ is small enough, a spherical or hyperbolic triangle with edge lengths no greater than 2ϵ can be approximated by a Euclidean triangle with the same edge lengths in the first order accurately, therefore the minimal corner angles are close to 30°. □

By using the Poincaré disk model, the hyperbolic Delaunay refinement algorithm can be performed by Euclidean Delaunay refinement, with one key difference: the hyperbolic circum-center doesn't coincide with the Euclidean circum-center, and at each step the hyperbolic circum-center should be inserted, instead of the Euclidean one. A similar principle is applied for spherical Delaunay refinement as well, where the sphere is conformally mapped onto the extended complex plane $\mathbb{C} \cup \{\infty\}$ using stereo-graphic projection. The spherical circum-center on the plane doesn't coincide with the Euclidean circum-center. At each step, the spherical circum-center should be inserted.

4. Computational Algorithms

In this section, we give the algorithmic implementation details for the dynamic discrete Yamabe flow method. We then present the proposed surface remeshing algorithm.

4.1. Dynamic Discrete Yamabe Flow Algorithm

The algorithm for discrete Yamabe flow is given in Alg. 1. The algorithm starts with determining the target curvature under the condition of Gauss-Bonnet in Eqn. 3 and initializing the conformal factor function. The algorithm then optimizes the discrete Ricci energy in Eqn. 7. To this end, either the gradient descent method or Newton's method can be utilized for the optimization. In this work, we employ Newton's method due to its quadratic convergence speed.

The energy optimization process iteratively computes the current edge lengths, the corner angles and the discrete Gaussian curvatures such that the gradient ∇E and Hessian matrix H of the Ricci energy are obtained for solving a linear system $\nabla E = Hx$. The solution x is then used for updating the new conformal factor in the next iteration. This process repeats until the maximal curvature error falls below a threshold. In each iteration, the triangulation is updated to be Delaunay by the edge-flipping operation.

Algorithm 1 Dynamic Discrete Yamabe Flow

Input: a closed triangle mesh M, the target curvature $\bar{K}$, step length δ, error threshould ε
Output: a discrete Riemannian metric realizing the target curvature

1. Set the target curvature value $\bar{K} : V \to \mathbb{R}$ for each vertex v_i and ensure that the total curvature satisfies the Gauss-Bonnet condition $\Sigma_{v_i \in V} \bar{K}(v_i) = 2\pi\chi(M)$.
2. Initialize the conformal factor function u_i by $u_i = 0$ for each vertex v_i.
3. Compute the current edge length l_{ij} for each edge e_{ij} using Eqn. 4 for the genus-one mesh or Eqn. 5 for the high-genus mesh, accordingly.
4. Update the triangulation to be Delaunay by edge flipping.
5. Compute the corner angles θ_i using cosine law in Eqn. 11 for the genus-one mesh or Eqn. 12 for the high-genus mesh, accordingly.
6. Compute the discrete Gaussian curvature $K(v_i)$ for each vertex v_i using Eqn. 2.
7. Compute the gradient of the Ricci energy using Eqn. 7.
8. Compute the Hessian matrix H of the Ricci energy using Eqn. 9.
9. Solve the linear system $\nabla E = Hx$.
10. Update the conformal factor $\mathbf{u} = \mathbf{u} - \delta x$, where δ is the step length.
11. Repeat steps 3 through 10 until the following condition is satisfied:
$$\max_{v_i \in V} |\bar{K}(v_i) - K(v_i)| < \varepsilon,$$
where ε denotes the threshold.

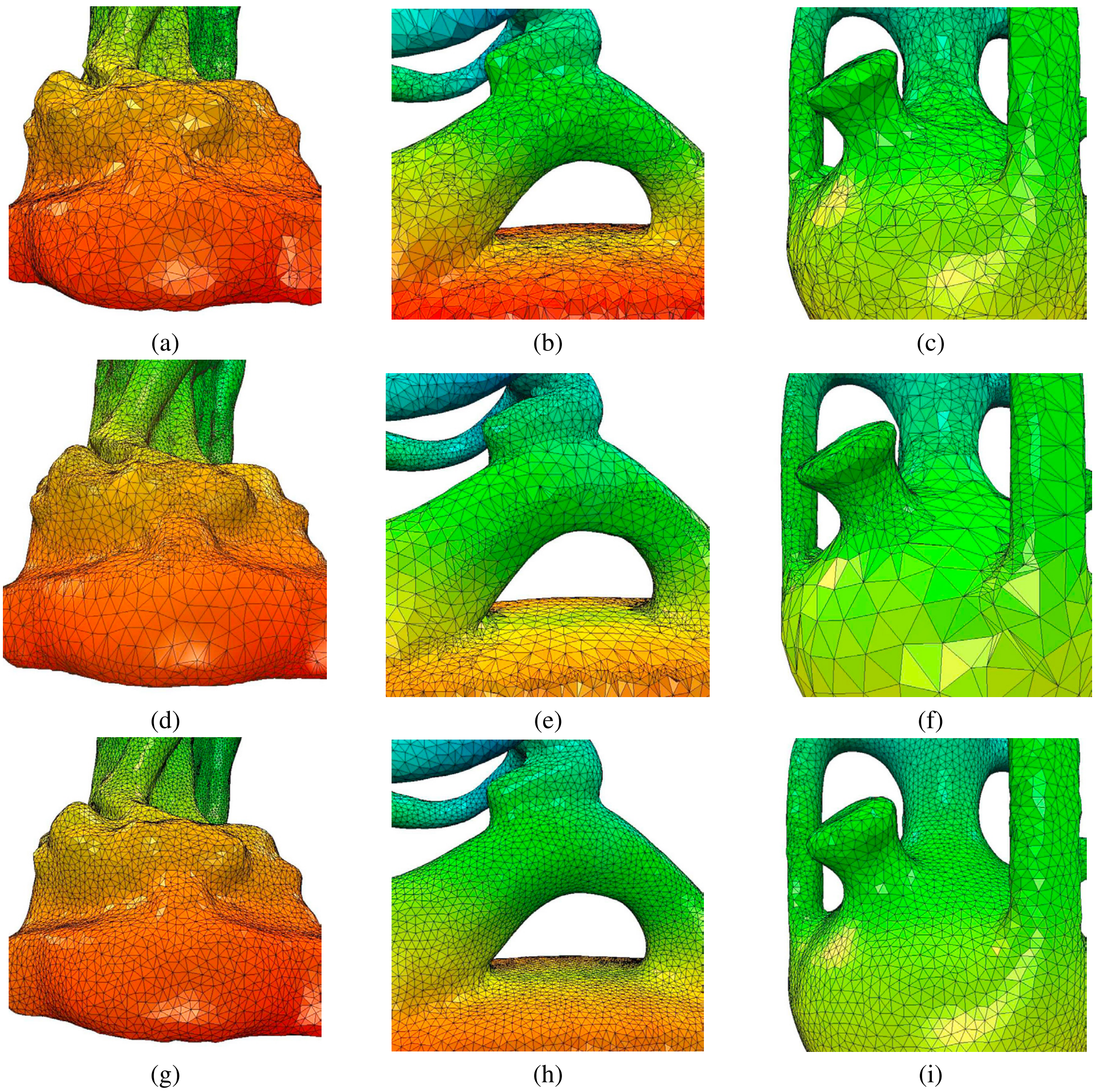

Fig. 7. Remeshing results on high-genus models: (a-c) Initial meshes. (d-f) Remeshing results of CVT-based method. (g-i) Remeshing results of our method.

4.2. *Surface Remeshing*

The surface remeshing algorithm based on conformal uniformization using dynamic discrete Yamabe flow and Delaunay refinement is presented in Alg. 2. The algorithm first computes the conformal uniformization metric using the dynamic discrete Yamabe flow method in Alg. 1. Then the sampling and an initial triangulation are performed. Finally, the Delaunay refinement procedure is applied. In this iterative procedure, a triangle with the worst quality is selected in each iteration for the vertex insertion operation. That is, the circumcenter of the worst-quality triangle is inserted and subsequent Delaunay maintenance is used for those affected edges. The algorithm repeats until the minimal quality threshold is satisfied.

5. Experimental Results

We implemented our proposed method using generic C++ on the Windows operating system, and all the experiments were conducted on a laptop computer with Intel Core i7 CPU, running at 2.60GHz, with 8GB of memory.

Algorithm 2 Surface Remeshing

Input: a closed triangle mesh M, meshing quality criteria

Output: a remeshing of M satisfying with the desired quality

1. Conformally deform the Riemannian metric on triangle mesh M to the uniformization metric with constant curvature using the dynamic discrete Yamabe flow in Alg. 1.
2. The sampling is computed and an initial triangulation is carried out.
3. Compute the triangle quality for each triangle $f_i \in F$.
4. Sort all the triangles according to the triangle quality and maintain the sorted triangles in a queue.
5. Select and pop the worst-quality triangle. Insert its circumcenter and maintain the Delaunay property.
6. Repeat steps 3 through 5 until a certain quality requirement is satisfied.

Table 1. Remeshing results in terms of the triangle quality.

Mesh	Initial θ_{min}	Initial Q_{ave}	CVT-based θ_{min}	CVT-based Q_{ave}	Our θ_{min}	Our Q_{ave}
Kitten	0.7718	0.7484	1.0795	0.8490	30.0048	0.9032
Torus	2.5373	0.7430	2.7988	0.8487	30.0034	0.9037
Rocker	4.6548	0.7449	1.3302	0.7431	30.0083	0.9049
Kids	1.0407	0.7462	0.9535	0.8129	30.0016	0.9039
Fertility	0.7165	0.7159	20.7838	0.9389	30.0110	0.9470
Genus 5	5.7908	0.7445	1.0715	0.7671	30.0020	0.9413

To demonstrate the effectiveness of our proposed surface remeshing method, we evaluated it on various surface models with complicated topologies in terms of the minimal angle and triangle quality , which are shown in Figure 5 . All the 3D surface models are represented as triangle meshes.

5.1. Improvement of Minimal Angles

In order to demonstrate the effectiveness of our method, we measured the minimal angle before and after the remeshing. The statistics of experimental results are reported in Table 1. The comparative results and analysis are given as follows.

Figures 6(a-c) and figures 7(a-c) show the initial mesh of genus-one models and high-genus models, respectively. According to the experimental results reported in Table 1, the minimal angles of the initial meshes are 0.7718°, 2.5373°, 4.6548°, 1.0407°, 0.7165°, and 5.7908° respectively. By contrast, the remeshing results of our method achieved minimal angle of 30.0048°, 30.0034°, 30.0083°, 30.0016°, 30.0110°, and 30.0020° respectively, which demonstrates our method improves the minimal angle.

5.2. Improvements of Triangle Qualities

We further applied our proposed method on a variety of surface models to evaluate our method based on the triangle quality measure. The triangle quality measure Q used in this work is formulated as follows:

$$Q = \frac{4\sqrt{3}A}{l_1^2 + l_2^2 + l_3^2}, \tag{13}$$

where A denotes the area of the triangle, and l_1, l_2 and l_3 represent the lengths of each edge in the triangle. The triangle quality has the value in the range [0, 1], where the highest quality is valued as 1 for equilateral triangles.

We report the statistics of the experimental results on triangle quality in Table 1, and present the comparative experimental results as follows.

We computed the triangle quality using Eqn. 13 for genus-one and high-genus meshes, including the triangle quality before and after the remeshing procedure. According to the experimental results, the remeshing results of our method have achieved better average triangle quality with the value of 0.9032, 0.9037, 0.9049, 0.9039, 0.9470, and 0.9470 respectively, which are greater than the average triangle quality of 0.7484, 0.7430, 0.7449, 0.7462, 0.7159 and 0.7445, respectively in the initial meshes.

5.3. Comparative Experiments

Centroidal Voronoi tessellation (CVT) has been extensively employed for surface remeshing in order to produce high quality triangulations. We compared our proposed method with a surface remeshing method based on CVT [5]. The comparative results are shown in Figure 6 and Figure 7. The results of comparative experiments in terms of the minimal angle are reported in Table 1. As shown in Table 1, all the resulting meshes produced by the CVT-based surface remeshing method [5] have minimal angles less than 30°, whereas our method can guarantee minimal angles greater than 30°.

We further compared the average triangle quality of the resulting meshes between our method and the CVT-based method. According to the results of comparative experiments, our uniformization based method outperformed the CVT-based method. The average triangle qualities obtained by the CVT-based method: 0.8490, 0.8487, 0.7431, 0.8129, 0.9389 and 0.7671 for the kitten, torus, rocker, kids, fertility and genus 5 meshes, respectively, are lower than those obtained by our method, 0.9032, 0.9037, 0.9049, 0.9039, 0.9470, and 0.9413 correspondingly.

6. Conclusion

In this work, a novel surface remeshing method based on surface uniformization and Delaunay refinement is proposed. The proposed method carries out Delaunay refinement on the conformal uniformization domain, which is obtained using dynamic discrete Yamabe flow.

The method is general to handle surfaces with complicated topologies, holistic without partitioning, simpler to convert 3D geometric processing to a 2D problem, robust to bad mesh qualities, and rigorous with solid theoretic foundation.

The algorithms for dynamic discrete Yamabe flow and Delaunay refinement based surface remeshing algorithm are presented with details. Experimental evaluations demonstrate the efficiency and efficacy of our proposed method. In the future, we will generalize our proposed method to surfaces with multiple boundaries.

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