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Higher-order discontinuous Galerkin time stepping and local projection stabilization techniques for the transient Stokes problem

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Abstract

We introduce and analyze discontinuous Galerkin time discretizations coupled with continuous finite element methods based on equal-order interpolation in space for velocity and pressure in transient Stokes problems. Spatial stability of the pressure is ensured by adding a stabilization term based on local projection. We present error estimates for the semi-discrete problem after discretization in space only and for the fully discrete problem. The fully discrete pressure shows an instability in the limit of small time step length. Numerical tests are presented which confirm our theoretical results including the pressure instability.

1. Introduction

The use of equal-order interpolation for velocity and pressure in incompressible flow problems does not satisfy the inf-sup stability condition and may produce oscillations in the pressure. In order to overcome this difficulty, several stabilization methods have been proposed in the literature. The streamline-upwind Petrov–Galerkin (SUPG) [1–3] and the pressure stabilized Petrov–Galerkin (PSPG) [4] methods are popular tools for the approximation of incompressible flow problems using equal-order interpolation. The common point in the stabilized methods is the addition of extra terms to the discrete formulation in order to enhance the stability of the numerical scheme. The SUPG method combined with the PSPG method has been used to cope the instability of dominating advection due to high Reynolds numbers and the violation of the discrete inf-sup condition of the Navier–Stokes equations. Concerning steady incompressible flow problems, this class of residual based stabilization techniques is still very popular due to its robustness. However, a fundamental drawback of the SUPG is that several terms need to be added to the variational form to ensure the strong consistency of the method. Furthermore, the strong coupling between velocity and pressure in the stabilization terms makes the analysis difficult. In order to relax the strong consistency in SUPG and PSPG type methods, several stabilization techniques have been developed. In particular, we mention edge stabilization methods [5, 6], local projection stabilization (LPS) methods as two-level approach [7, 8] or as one-level enrichment method [9], and variational multiscale method [10].

In the discretization of time-dependent problems, one often uses the methods of lines. In this approach, the problem is discretized in space first whereas the time remains continuous. This methodology leads to a large system of ordinary differential equations which can be solved by a suitable ODE solver. Numerical studies of the SUPG method in space combined with implicit/explicit transient algorithms can be found in [11]. The extension to transient Stokes problems of different stabilization methods including Galerkin/least squares (GLS) methods in small time step limit are studied in [12–14]. In [13], it has been shown for the small time step limit that even the first order backward difference methods perturbs the stability of the numerical scheme. This behavior is caused by the finite difference operator included in the stabilization terms of SUPG to guarantee consistency. Furthermore, the appearing non-symmetric term is difficult to handle. Nevertheless, by using a different analysis, optimal error estimates for time-dependent convection-diffusion-reaction problems with time-independent coefficients on uniform grids were proven for the standard choice of the stabilization parameter independent of the time step length, see [15].

The case of symmetric stabilization methods was investigated in [16] and the PSPG stabilization method was considered in [17] for the transient Stokes problem. It was shown that the small time step instability can be circumvented if the initial data is chosen as Ritz projection onto a space of discretely divergence-free functions. Furthermore, the convergence of velocities and pressure can be obtained without any coupling of mesh width and time step length. To prevent oscillation in the pressure approximation in the case where the discrete initial condition is chosen as some interpolant of the continuous initial data, mesh width and time step length have to fulfill a coupling condition similar to that in [13].

We consider discontinuous Galerkin (dG) methods to discretize the problem in time. Discontinuous Galerkin methods were first introduced for neutron transport problems in [18] and then analyzed in [19]. The theoretical analysis of dG methods for scalar hyperbolic equations can be found in [20] and for space-time dG methods applied to convection-diffusion-reaction problems in [21]. The dG finite element techniques were developed for the numerical solution of elliptic problems [22] and compressible and incompressible flow problems, see e.g. [23, 24] and the references therein. The dG time discretization was introduced and analyzed in [25] for the solution of ordinary differential equations, see also [26]. Space-time discontinuous Galerkin finite element methods have been applied to solve transient advection-diffusion problems [27] and flow problems [28]. The combination of LPS in space and dG in time for transient convection-diffusion-reaction equations has been studied in [29].

In this paper, we consider the stabilized finite element method for the transient Stokes problem which is based on a one-level local projection stabilization method. Our main focus is the higher order time discretization using discontinuous Galerkin methods. We derive error estimates for the semi-discrete problem in space. For the fully discrete scheme, we prove the stability of the method and error estimates.

The remainder of the paper is organized as follows. Section 2 introduces the model problem under consideration and some basic notation. The stabilized finite element semi-discretization in space is presented in Section 3. Furthermore, an optimal error estimate of velocity and pressure for the semi-discrete problem will be given. In Section 4, we address the full discretization of the problem by considering discontinuous Galerkin methods in time. We derive the unconditional stability of velocity. However, the L^2-norm bound of the pressure shows an instability for small time step length. Furthermore, we present the error analysis of the fully discrete problem. Finally, numerical results illustrating the theoretical predictions are reported in Section 5. Some conclusion will be given in Section 6.

2. Model problem and basic notation

Let Ω be a Lipschitz domain in $\mathbb{R}^d$ $(d = 2, 3)$ with polyhedral boundary $\partial\Omega$ and $T > 0$. We consider the following time-dependent Stokes problem:

Find $u : \Omega \times (0, T) \to \mathbb{R}$ and $p : \Omega \times (0, T) \to \mathbb{R}$ such that

$$\begin{aligned} u' - \Delta u + \nabla p &= f && \text{in } \Omega \times (0, T), \\ \nabla \cdot u &= 0 && \text{in } \Omega \times (0, T), \\ u &= 0 && \text{on } \partial\Omega \times (0, T), \\ u(\cdot, 0) &= u_0 && \text{in } \Omega, \end{aligned} \tag{1}$$

where u is the velocity field, p the pressure, f the external force, and u_0 the initial velocity. For the sake of simplicity, homogeneous Dirichlet boundary conditions will be considered.

Throughout this paper, standard notation and conventions will be used. For a measurable set $G \subset \mathbb{R}^d$, the inner product in $L^2(G)$ will be denoted by $(\cdot,\cdot)_G$. The norm and semi-norm in $W^{m,p}(G)$ are given by $\|\cdot\|_{m,p,G}$ and $|\cdot|_{m,p,G}$, respectively. In the case $p=2$, we write $H^m(G)$, $\|\cdot\|_{m,G}$, and $|\cdot|_{m,G}$ instead of $W^{m,2}(G)$, $\|\cdot\|_{m,2,G}$, and $|\cdot|_{m,2,G}$. If $G=\Omega$, the index G in inner products, norms, and semi-norms will be omitted. The subspace of functions from $H^1(\Omega)$ having zero boundary trace is denoted by $H_0^1(\Omega)$. The above definition apply component-wise to vector-valued and tensor-valued cases. Furthermore, let $L_0^2(\Omega)$ denote the subspace of functions from $L^2(\Omega)$ with vanishing integral mean.

We will denote by C a generic constant which is always independent of the mesh size defined in Section 3. We will write shortly $\alpha \sim \beta$ if there exist two positive constants C_1 and C_2 being independent of the mesh size such that $\alpha \le C_1\beta$ and $\beta \le C_2\alpha$ hold true.

Let X be a Banach space with norm $\|\cdot\|_X$ and $I := [0,T]$. We consider the Bochner spaces

$$
\begin{aligned}
C\big(I;X\big) &:= \big\{v : I \to X, \quad v \text{ continuous}\big\},\\
C^s\big(I;X\big) &:= \big\{v : I \to X, \quad v \text{ is } s \text{ times continuously differentiable}\big\},\\
L^2(I;X) &:= \left\{v : I \to X, \quad \int_0^T \|v(t)\|_X^2 \, dt < \infty\right\},\\
H^m(I;X) &:= \left\{v \in L^2(I;X) : \frac{\partial^j v}{\partial t^j} \in L^2(I;X),\ 1 \le j \le m\right\},
\end{aligned}
$$

where the derivatives $\partial^j v/\partial t^j$, $j=1,\ldots,m$, are understood in the sense of distributions on I. We use in the following the short notation $Y(X) := Y(I;X)$. The norms in the above defined spaces are given by

$$
\|v\|_{C(X)} := \sup_{t\in I} \|v(t)\|_X, \qquad \|v\|_{C^s(X)} := \max_{j=0,\ldots,s} \sup_{t\in I} \|v^{(j)}(t)\|_X,
$$

$$
\|v\|_{L^2(X)} := \left(\int_0^T \|v(t)\|_X^2 \, dt\right)^{1/2}, \qquad \|v\|_{H^m(X)} := \left(\sum_{j=0}^m \left\|\frac{\partial^j v}{\partial t^j}\right\|_{L^2(X)}^2\right)^{1/2},
$$

where we denote by v', v'', and $v^{(k)}$ the first, second, and kth order time derivative of v, respectively.

Let $V := H_0^1(\Omega)^d$, $Q := L_0^2(\Omega)$, and $W := \big\{v \in L^2(V) : v' \in L^2(V')\big\}$ where $V' = H^{-1}(\Omega)^d$ denotes the dual space of V. Note that $v \in W$ ensures the continuity of the mapping $v : [0,T] \to L^2(\Omega)$.

We assume in the remainder of this paper that $f \in L^2(L^2)$. Hence, a variational form of (1) reads as follows:

Find $u \in W$, $p \in L^2(Q)$ such that $u(0) = u_0$ and for almost all $t \in (0,T)$

$$
(u'(t),v) + (\nabla u(t),\nabla v) - (p(t),\nabla\cdot v) + (q,\nabla\cdot u(t)) = (f(t),v) \quad \forall (v,q) \in V \times Q. \tag{2}
$$

Note that the initial condition $u(0) = u_0$ is well defined since u belongs to W.

3. Space discretization

For finite element discretizations of (2), let $\{\mathcal{T}_h\}$ denote a family of shape regular triangulations of Ω into d-simplices, quadrilaterals, or hexahedra. The diameter of $K \in \mathcal{T}_h$ will be denoted by h_K and the mesh size h is defined by $h := \max_{K\in\mathcal{T}_h} h_K$. Let Y_h be a space of continuous, piecewise polynomial functions of order r over $\mathcal{T}_h$.

We consider in this paper the case of equal-order interpolation. Thus, we define $V_h := Y_h^d \cap V$ and $Q_h := Y_h \cap Q$. Now, the standard Galerkin discretization of (2) reads:

Find $u_h \in H^1(V_h)$ and $p_h \in L^2(Q_h)$ such that $u_h(0) = u_{0,h}$ and for almost every $t \in (0,T)$

$$
\big(u_h'(t),v_h\big) + A\big((u_h(t),p_h(t));(v_h,q_h)\big) = \big(f(t),v_h\big) \qquad \forall (v_h,q_h) \in V_h \times Q_h \tag{3}
$$

where $u_{0,h} \in V_h$ is a suitable approximation of u_0 which will be specified later. The bilinear form A is given by

$$A\big((v,q);(w,r)\big) = (\nabla v, \nabla w) - (q, \nabla \cdot w) + (r, \nabla \cdot v).$$

In general, the pairs (V_h, Q_h) do not satisfy the discrete Babuška–Brezzi condition

$$\exists \beta_0 > 0 : \inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{(q_h, \nabla \cdot v_h)}{\|q_h\|_0 \, \|v_h\|_1} \geq \beta_0 \qquad \forall h. \tag{4}$$

Hence, inaccurate pressure approximations have to be expected. In order to overcome this instability, we will add to (3) a stabilizing term based on local projection. We concentrate on the one-level local projection stabilization method where approximation space and projection space are defined on the same mesh. Let $D(K)$, $K \in \mathcal{T}_h$, denote finite dimensional function spaces with the associated local L^2-projections $\pi_K : L^2(K) \to D(K)$ and the fluctuation operators $\kappa_K : L^2(K) \to L^2(K)$ with $\kappa_K \varphi := \varphi - \pi_K \varphi$. The stabilization term S_h is defined by

$$S_h(p_h, q_h) = \sum_{K \in \mathcal{T}_h} \mu_K (\kappa_K \nabla p_h, \kappa_K \nabla q_h)_K \tag{5}$$

where μ_K, $K \in \mathcal{T}_h$, are user chosen non-negative constants and the fluctuation operators κ_K are applied component-wise. The local projection stabilization term S_h gives additional control on the fluctuation of the pressure gradient.

The stabilized semi-discrete scheme reads:

Find $(u_h, p_h) \in H^1(V_h) \times L^2(Q_h)$ with $u_h(0) = u_{0,h}$ such that for almost every $t \in (0,T)$

$$\big(u_h'(t), v_h\big) + A_h\big((u_h(t), p_h(t)); (v_h, q_h)\big) = \big(f(t), v_h\big) \qquad \forall (v_h, q_h) \in V_h \times Q_h \tag{6}$$

where the bilinear form A_h is defined by

$$A_h\big((v,q);(w,r)\big) := A\big((v,q);(w,r)\big) + S_h(q,r). \tag{7}$$

Let us introduce the mesh-dependent semi-norm on the product space $V \times Q$

$$|||(v,q)||| := \left(|v|_1^2 + \sum_{K \in \mathcal{T}_h} \mu_K \|\kappa_K \nabla q\|_{0,K}^2 \right)^{1/2}. \tag{8}$$

Note that $|||\cdot|||$ is just a semi-norm on $V \times Q$ since there are non-zero pressure functions with vanishing pressure part inside $|||\cdot|||$. An immediate consequence of definition (7) of the bilinear form A_h is given in the following lemma.

Lemma 1. *The stabilized bilinear form A_h fulfills*

$$A_h\big((v_h, q_h); (v_h, q_h)\big) = |||(v_h, q_h)|||^2 \qquad \forall (v_h, q_h) \in V_h \times Q_h, \tag{9}$$

i.e., A_h is coercive on $V_h \times Q_h$ with respect to $|||\cdot|||$.

Stability and convergence properties of the local projection stabilization method (6) are based on the following assumptions, see [8]:

Assumption 1. *Let the fluctuation operator satisfy the following approximation property*

$$\|\kappa_K q\|_{0,K} \leq C h_K^l |q|_{l,K} \qquad \forall K \in \mathcal{T}_h,\ \forall q \in H^l(K),\ 0 \leq l \leq r. \tag{10}$$

Assumption 2. *There exists an interpolation operator $j_h : H^1(\Omega) \to Y_h$ with $j_h v \in H_0^1(\Omega)$ for all $v \in H_0^1(\Omega)$ that satisfies on all $K \in \mathcal{T}_h$ the orthogonality property*

$$(w - j_h w, q_h)_K = 0 \qquad \forall q_h \in D(K),\ \forall w \in H^1(\Omega), \tag{11}$$

and the approximation property

$$\|w - j_h w\|_{0,K} + h_K |w - j_h w|_{1,K} \le C h_K^l \|w\|_{l,\omega(K)} \qquad \forall w \in H^l\big(\omega(K)\big),\ 1 \le l \le r+1, \tag{12}$$

where $\omega(K)$ denotes a certain local neighborhood of $K \in \mathcal{T}_h$ which appears in the definition of interpolation operators for non-smooth functions, see [30] for details.

The following modified inf-sup condition shows the stability of the discrete pressure.

Lemma 2. *Suppose Assumptions 1, 2, and $h_K^2/\mu_K \le C$ for all $K \in \mathcal{T}_h$. Then, there exist positive constants β and C independent of h such that*

$$\beta \|q\|_0 \le \sup_{v_h \in V_h} \frac{(q, \nabla \cdot v_h)}{|v_h|_1} + C S_h(q,q)^{1/2} \qquad \forall q \in Q \cap H^1(\Omega) \tag{13}$$

holds true.

Proof. The continuous inf-sup condition ensures that there exists for any $q \in Q \cap H^1(\Omega)$ an element $v_q \in V$ satisfying

$$(\nabla \cdot v_q, q) = \|q\|_0^2, \qquad \|v_q\|_1 \le \beta_0 \|q\|_0. \tag{14}$$

Hence, we have

$$\|q\|_0^2 = (q, \nabla \cdot v_q) = \big(q, \nabla \cdot (j_h v_q)\big) + \big(q, \nabla \cdot (v_q - j_h v_q)\big). \tag{15}$$

Integrating the second term on the right-hand side of (15) by parts and using the orthogonality condition (11) and the approximation property (12), one gets

$$\begin{aligned}
\Big|\big(q, \nabla \cdot (v_q - j_h v_q)\big)\Big| &= \Big|\big(\nabla q, v_q - j_h v_q\big)\Big| \\
&= \Bigg|\sum_{K \in \mathcal{T}_h} \Big(\kappa_K \nabla q, v_q - j_h v_q\Big)_K\Bigg| \\
&\le \left(\sum_{K \in \mathcal{T}_h} h_K^2 \|\kappa_K \nabla q\|_{0,K}^2\right)^{1/2} \left(\sum_{K \in \mathcal{T}_h} h_K^{-2} \|v_q - j_h v_q\|_{0,K}^2\right)^{1/2} \\
&\le C \left(\sum_{K \in \mathcal{T}_h} h_K^2 \|\kappa_K \nabla q\|_{0,K}^2\right)^{1/2} \|v_q\|_1 \\
&\le C S_h(q,q)^{1/2}\, \|q\|_0
\end{aligned}$$

where we have used the shape-regularity of $\mathcal{T}_h$, the assumption on the choice of the parameters μ_K, $K \in \mathcal{T}_h$, and the second condition of (14). Combining the above inequality with (15) leads to

$$\|q\|_0^2 \le (q, \nabla \cdot (j_h v_q)) + C S_h(q,q)^{1/2}\, \|q\|_0. \tag{16}$$

We conclude the proof by dividing (16) by $\|q\|_0$ and using

$$\|j_h v_q\|_1 \le C \|v_q\|_1 \le C \|q\|_0$$

where the first inequality follows from (12). $\square$

In contrast to residual-based stabilization methods, see [31] for several examples, we do not have strong consistency which is also known as Galerkin orthogonality. Consequently, we have to investigate the consistency error.

Lemma 3. *Let $(u,p) \in W \times L^2(Q)$ and $(u_h,p_h) \in H^1(V_h) \times L^2(Q_h)$ be the solutions of (2) and (6), respectively. Then, the relation*

$$\big(u'(t) - u_h'(t), v_h\big) + A\big((u(t)-u_h(t), p(t)-p_h(t)); (v_h,q_h)\big) = S_h\big(p_h(t), q_h\big) \qquad \forall (v_h,q_h) \in V_h \times Q_h \quad (17)$$

holds true for almost all $t \in (0,T)$.

Proof. The statement follows by subtracting (6) from (2). □

3.1. Velocity and pressure estimates

This section studies the error analysis of the semi-discrete problem (6). In particular, we prove convergence of velocity and pressure. The next theorem states the main result of this section.

Theorem 4. *Suppose Assumptions 1, 2, and $\mu_K \sim h_K^2$ for all $K \in \mathcal{T}_h$. Let $(u,p) \in W \times L^2(Q)$ be the solution of the continuous problem (2) and $(u_h,p_h) \in H^1(V_h) \times L^2(Q_h)$ be the solution of the stabilized semi-discrete problem (6) with $u_{0,h} = j_h u_0$. Furthermore, we assume that $u \in H^1\big(H^{r+1}(\Omega)^d\big)$ and $p \in L^2\big(H^r(\Omega)\big)$. Then, there exists a positive constant C independent of h such that*

$$\int_0^T |||(u-u_h, p-p_h)|||^2 \le C h^{2r} \int_0^T \Big[h^2 \|u'\|_{r+1}^2 + \|u\|_{r+1}^2 + \|p\|_r^2\Big], \tag{18}$$

and

$$\begin{aligned}\|u(T) - u_h(T)\|_0^2 &+ \int_0^T |||(u-u_h,p-p_h)|||^2 \\ &\le C h^{2r} \int_0^T \Big[h^2 \|u'\|_{r+1}^2 + \|u\|_{r+1}^2 + \|p\|_r^2\Big] + C h^{2(r+1)} \|u(T)\|_{r+1}^2\end{aligned} \tag{19}$$

hold true.

Proof. The proof starts by decomposing the error into an interpolation error and the difference of the interpolant and the solution of the semi-discrete problem (6)

$$u - u_h = (u - j_h u) + (j_h u - u_h)\,, \quad p - p_h = (p - j_h p) + (j_h p - p_h)\,.$$

The interpolation error can be estimated using (12). We denote $\xi_h := j_h u - u_h$, $\vartheta_h := j_h p - p_h$, and skip indicating their time-dependence for brevity. A straightforward calculation yields the following error equation

$$(\xi_h', \xi_h) + A_h\big((\xi_h,\vartheta_h);(\xi_h,\vartheta_h)\big) = \big((j_h u)' - u', \xi_h\big) + A\big((j_h u - u, j_h p - p); (\xi_h,\vartheta_h)\big) + S_h(j_h p, \vartheta_h)$$

where (2) and (6) have been applied. Using $(\xi_h', \xi_h) = \frac{1}{2}\frac{d}{dt}(\xi_h,\xi_h)$, the coercivity (9) of the bilinear form A_h with respect to $|||\cdot|||$, and the fact that the spatial interpolation j_h commutes with the time derivative, one obtains

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\xi_h\|_0^2 &+ |||(\xi_h,\vartheta_h)|||^2 \\ &= (j_h u' - u', \xi_h) + \big(\nabla(j_h u - u), \nabla \xi_h\big) - (j_h p - p, \nabla\cdot\xi_h) + \big(\vartheta_h, \nabla\cdot(j_h u - u)\big) + S_h(j_h p, \vartheta_h).\end{aligned} \tag{20}$$

The arising terms on the right-hand side of (20) will be bounded by the norms of u and p from the continuous problem (2). Applying the Cauchy–Schwarz inequality, the approximation properties (12), and Young's inequality yields

$$\begin{aligned}\big|(j_hu'-u',\xi_h)\big| &\le \|j_hu'-u'\|_0\|\xi_h\|_0 \le C\,h^{r+1}\|u'\|_{r+1}\|\xi_h\|_0 \le Ch^{2r+2}\|u'\|_{r+1}^2+\frac{1}{10}|||(\xi_h,\vartheta_h)|||^2,\\ \big|\,(\nabla(j_hu-u),\nabla\xi_h)\,\big| &\le |j_hu-u|_1\,|\xi_h|_1 \le Ch^r\|u\|_{r+1}\,|||(\xi_h,\vartheta_h)||| \le Ch^{2r}\|u\|_{r+1}^2+\frac{1}{10}|||(\xi_h,\vartheta_h)|||^2,\\ \big|\,(j_hp-p,\nabla\cdot\xi_h)\,\big| &\le \|j_hp-p\|_0|\xi_h|_1 \le C\,h^r\|p\|_r|||(\xi_h,\vartheta_h)||| \le Ch^{2r}\|p\|_r^2+\frac{1}{10}|||(\xi_h,\vartheta_h)|||^2,\end{aligned}$$

where the Friedrichs' inequality was used for the first estimate.

For estimating the fourth term on the right-hand side of (20), the orthogonality property (11) will be used. Integration by parts, the approximation properties (12), and the parameter choice $\mu_K\sim h_K^2$ give

$$\begin{aligned}\big|(\vartheta_h,\nabla\cdot(j_hu-u))\big| = \big|(\nabla\vartheta_h,j_hu-u)\big| &= \Big|\sum_{K\in\mathcal{T}_h}(\kappa_K\nabla\vartheta_h,j_hu-u)_K\Big|\\ &\le \left(\sum_{K\in\mathcal{T}_h}\mu_K\|\kappa_K\nabla\vartheta_h\|_{0,K}^2\right)^{1/2}\left(\sum_{K\in\mathcal{T}_h}\mu_K^{-1}\|j_hu-u\|_{0,K}^2\right)^{1/2}\\ &\le Ch^r\|u\|_{r+1}|||(\xi_h,\vartheta_h)||| \le Ch^{2r}\|u\|_{r+1}^2+\frac{1}{10}|||(\xi_h,\vartheta_h)|||^2.\end{aligned}$$

The stabilizing term in (20) is estimated by using the Cauchy–Schwarz inequality, the L^2-stability and the approximation properties (10) of the fluctuation operators κ_K, the approximation properties of j_h, and $\mu_K\sim h_K^2$. We get

$$\begin{aligned}S_h(j_hp,\vartheta_h) &= S_h(j_hp-p,\vartheta_h)+S_h(p,\vartheta_h)\\ &\le \Big(S_h(j_hp-p,j_hp-p)^{1/2}+S_h(p,p)^{1/2}\Big)S_h(\vartheta_h,\vartheta_h)^{1/2}\\ &\le \left\{\left(\sum_{K\in\mathcal{T}_h}\mu_K\|\kappa_K\nabla(j_hp-p)\|_{0,K}^2\right)^{1/2}+\left(\sum_{K\in\mathcal{T}_h}\mu_K\|\kappa_K\nabla p\|_{0,K}^2\right)^{1/2}\right\}|||(\xi_h,\vartheta_h)|||\\ &\le Ch^r\|p\|_r|||(\xi_h,\vartheta_h)||| \le Ch^{2r}\|p\|_r^2+\frac{1}{10}|||(\xi_h,\vartheta_h)|||^2.\end{aligned}$$

Inserting all the estimates into (20) and absorbing the $|||(\xi_h,\vartheta_h)|||$-contributions on the left-hand side give after an integration over $(0,T)$ the estimate

$$\|\xi_h(T)\|_0^2+\int_0^T|||(\xi_h,\vartheta_h)|||^2 \le \|\xi_h(0)\|_0^2+Ch^{2r}\int_0^T\Big[h^2\|u'\|_{r+1}^2+\|u\|_{r+1}^2+\|p\|_r^2\Big].$$

Using $\xi_h(0)=0$, the estimates (18) and (19) follow from the triangle inequality and the interpolation error estimates. The last term in (19) arises from estimating $\|u(T)-j_hu(T)\|_0^2$. □

The next lemma gives an estimate for the $L^2(L^2)$-norm of the time derivative of the velocity which will be used later to derive an error bound for the pressure error.

Lemma 5. *Suppose Assumptions 1, 2, and $\mu_K\sim h_K^2$ for all $K\in\mathcal{T}_h$. Let $(u,p)\in W\times L^2(Q)$ be the solution of the continuous problem (2) and $(u_h,p_h)\in H^1(V_h)\times L^2(Q_h)$ be the solution of the stabilized semi-discrete problem (6) with $u_{0,h}=j_hu_0$. Let $u\in H^2\big(H^{r+1}(\Omega)^d\big)$ and $p\in H^1\big(H^r(\Omega)\big)$. Then, there exists a positive constant C independent of h such that*

$$\int_0^T\|u'-u_h'\|_0^2 \le \|j_hu'(0)-u_h'(0^+)\|_0^2+Ch^{2r}\int_0^T\Big[h^2\|u''\|_{r+1}^2+\|u'\|_{r+1}^2+\|p'\|_r^2\Big].$$

Proof. Differentiating (2) and (6) with respect to time and subtracting one from the other gives the following error equation

$$(\xi_h'', v_h) + A_h\big((\xi_h', \vartheta_h'); (v_h, q_h)\big) = (j_h u'' - u'', v_h) + A\big((j_h u' - u', j_h p' - p'); (v_h, q_h)\big) + S_h(j_h p', q_h)$$

with $\xi_h := j_h u - u_h$ and $\vartheta_h := j_h p - p_h$. Setting $v_h = \xi_h'$ and $q_h = \vartheta_h'$, one gets

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\xi_h'\|_0^2 + |\xi_h'|_1^2 + S_h(\vartheta_h', \vartheta_h') &= (j_h u'' - u'', \xi_h') + \big(\nabla(j_h u' - u'), \nabla \xi_h'\big) - \big(j_h p' - p', \nabla \cdot \xi_h'\big) \\ &\quad + \big(\vartheta_h', \nabla \cdot (j_h u' - u')\big) + S_h(j_h p', \vartheta_h').\end{aligned}$$

Using the Cauchy–Schwarz inequality, the interpolation error estimates, and Young's inequality, the first three term can be bounded. The last term is estimated similar to the term $S_h(j_h p, \vartheta_h)$ in the proof of Thm. 4. In order to get an estimate for the fourth term, we integrate by parts and exploit the orthogonality (11). Altogether, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\xi_h'\|_0^2 + \frac{1}{2}|\xi_h'|_1^2 + \frac{1}{2}S_h(\vartheta_h', \vartheta_h') \\ &\quad \le C\Bigg[\|j_h u'' - u''\|_0^2 + \|j_h u' - u'\|_1^2 + \|j_h p' - p'\|_0^2 \\ &\qquad\qquad + S_h(j_h p' - p', j_h p' - p') + S_h(p', p') + \sum_{K \in \mathcal{T}_h} \mu_K^{-1}\|j_h u' - u'\|_{0,K}^2\Bigg] \\ &\quad \le C h^{2r}\Big[h^2\|u''\|_{r+1}^2 + \|u'\|_{r+1}^2 + \|p'\|_r^2\Big].\end{aligned}$$

The final estimate follows from the triangle inequality, Friedrichs' inequality, interpolation error estimates, and integrating over 0 to T. Note that the homogeneous Dirichlet boundary conditions also ensure $u' = 0$ on $\partial\Omega \times (0, T)$. □

Now, we are able to provide a bound for the pressure error in the $L^2(L^2)$-norm.

Theorem 6. *Suppose Assumptions 1, 2, and $\mu_K \sim h_K^2$ for all $K \in \mathcal{T}_h$. Let $(u, p) \in W \times L^2(Q)$ be the solution of the continuous problem (2) and $(u_h, p_h) \in H^1(V_h) \times L^2(Q_h)$ be the solution of the stabilized semi-discrete problem (6) with $u_{0,h} = j_h u_0$. Let $u \in H^2\big(H^{r+1}(\Omega)^d\big)$ and $p \in H^1\big(H^r(\Omega)\big)$. Then, there exists a positive constant C independent of h such that*

$$\int_0^T \|p - p_h\|_0^2 \le \|j_h u'(0) - u_h'(0^+)\|_0^2 + C h^{2r}\int_0^T \Big[\|u\|_{r+1}^2 + \|u'\|_{r+1}^2 + h^2\|u''\|_{r+1}^2 + \|p\|_r^2 + \|p'\|_r^2\Big].$$

holds true.

Proof. Setting $q_h = 0$ in (17), one gets

$$(p - p_h, \nabla \cdot v_h) = (u' - u_h', v_h) + (\nabla(u - u_h), \nabla v_h).$$

The Cauchy–Schwarz and the Friedrichs' inequalities give

$$(p - p_h, \nabla \cdot v_h) \le \|u' - u_h'\|_0\|v_h\|_0 + |u - u_h|_1|v_h|_1 \le \Big[C_{\mathrm{F}}\|u' - u_h'\|_0 + |u - u_h|_1\Big]|v_h|_1.$$

Hence, by using the inf-sup condition (13), one obtains

$$\|p - p_h\|_0^2 \le C\left[\|u' - u_h'\|_0^2 + |u - u_h|_1^2 + S_h(p - p_h, p - p_h)\right].$$

The statement follows by integrating over 0 to T and the application of Theorem 4 and Lemma 5. □

4. Time discretization by discontinuous Galerkin method

We discretize in this section the semi-discrete problem (6) in time by using discontinuous Galerkin (dG) methods to obtain the fully discrete LPS/dG formulation of (2). To this end, we consider a partition $0 = t_0 < t_1 < \cdots < t_N = T$ of the time interval $I := [0,T]$ and set $I_n := (t_{n-1}, t_n]$, $\tau_n := t_n - t_{n-1}$, $n = 1, \ldots, N$, and

$$\tau := \max_{1\le n\le N} \tau_n, \qquad \tau_{\min} := \min_{1\le n\le N} \tau_n. \tag{21}$$

For a given non-negative integer k, we define the fully discrete time-discontinuous velocity and pressure spaces as follows:

$$\begin{aligned} X_k &:= \Big\{ v \in L^2(I, V_h) : v|_{I_n} \in \mathbb{P}_k(I_n, V_h),\ n = 1, \ldots, N \Big\}, \\ Y_k &:= \Big\{ q \in L^2(I, Q_h) : q|_{I_n} \in \mathbb{P}_k(I_n, Q_h),\ n = 1, \ldots, N \Big\}, \end{aligned}$$

where

$$\mathbb{P}_k(I_n, W_h) := \left\{ w : I_n \to W_h : w(t) = \sum_{j=0}^{k} W^j t^j,\ W^j \in W_h,\ j = 0, \ldots, k \right\}$$

denotes the W_h-valued polynomials of degree less than or equal to k in time. For a piecewise smooth function w, we define the left-sided values w_n^-, the right-sided values w_n^+, and the jumps $[w]_n$ as

$$w_n^- := \lim_{t\to t_n - 0} w(t), \qquad w_n^+ := \lim_{t\to t_n + 0} w(t), \qquad [w]_n = w_n^+ - w_n^-.$$

The discontinuous Galerkin method applied to (6) leads to the following fully discrete problem

Find $(u_{h,\tau}, p_{h,\tau}) \in X_k \times Y_k$ such that

$$\begin{aligned} \sum_{n=1}^{N} \int_{I_n} (u'_{h,\tau}, v_{h,\tau}) + A_h\left((u_{h,\tau}, p_{h,\tau}); (v_{h,\tau}, q_{h,\tau})\right) dt + \sum_{n=1}^{N-1} \left([u_{h,\tau}]_n, v_n^+\right) + (u_0^+, v_0^+) \\ = (u_{0,h}, v_0^+) + \int_0^T (f, v_{h,\tau}) dt \end{aligned} \tag{22}$$

for all $v_{h,\tau} \in X_k$ and all $q_{h,\tau} \in Y_k$.

In order to evaluate the time integrals in (22) numerically, the right-sided Gauß–Radau quadrature with $(k+1)$ points will be applied. Let $-1 < \hat{t}_1 < \cdots < \hat{t}_{k+1} = 1$ and $\widehat{\omega}_j$, $j = 1, \ldots, k+1$, denote the points and weights of this quadrature formula on the reference time interval $[-1, 1]$. We define on I_n, $n = 1, \ldots, N$, the transformed quadrature formula Q_n by

$$Q_n[\phi] := \frac{\tau_n}{2} \sum_{j=1}^{k+1} \widehat{\omega}_j \phi(t_{n,j}), \qquad t_{n,j} := T_n(\hat{t}_j),$$

where

$$T_n : [-1, 1] \to \overline{I}_n, \quad \hat{t} \mapsto t_{n-1} + \frac{\tau_n}{2}(\hat{t} + 1) \tag{23}$$

is an affine mapping, see [32]. Note that Q_n integrates polynomials of degree less than or equal to $2k$ exactly. Furthermore, we set

$$Q[\phi] := \sum_{n=1}^{N} Q_n[\phi]$$

as abbreviation.

Let us introduce the bilinear forms B and B_h as

$$B\big((v,q);(w,r)\big) := Q\left[(v',w) + A\big((v,q);(w,r)\big)\right] + \sum_{n=1}^{N-1}\big([v]_n, w_n^+\big) + (v_0^+, w_0^+)$$

and

$$B_h\big((v,q);(w,r)\big) := Q\left[(v',w) + A_h\big((v,q);(w,r)\big)\right] + \sum_{n=1}^{N-1}\big([v]_n, w_n^+\big) + (v_0^+, w_0^+).$$

If the solution (u,p) of (2) belongs to $C^1(V)\times C(Q)$, we have

$$B\big((u,p);(v,q)\big) = (u_0, v_0^+) + Q\left[(f,v)\right] \qquad \forall (v,q)\in L^2(V)\times L^2(Q). \tag{24}$$

The numerically integrated, fully discrete problem reads:

Find $(u_{h,\tau}, p_{h,\tau}) \in X_k \times Y_k$ such that

$$B_h\big((u_{h,\tau},p_{h,\tau});(v_{h,\tau},q_{h,\tau})\big) = (j_h u_0, v_0^+) + Q\left[(f,v_{h,\tau})\right] \qquad \forall (v_{h,\tau},q_{h,\tau})\in X_k\times Y_k. \tag{25}$$

Note the $j_h u_0 \in V_h$ acts as discrete initial condition.

Lemma 7. *The bilinear form B_h associated with the LPS/dG method satisfies*

$$B_h\big((v_{h,\tau},q_{h,\tau});(v_{h,\tau},q_{h,\tau})\big) = \|(v_{h,\tau},q_{h,\tau})\|_{\mathrm{S}}^2 \qquad \forall (v_{h,\tau},q_{h,\tau})\in X_k\times Y_k \tag{26}$$

with

$$\|(v,q)\|_{\mathrm{S}} := \left(\int_0^T |||(v,q)|||^2 + \frac{1}{2}\|v_0^+\|_0^2 + \frac{1}{2}\sum_{n=1}^{N-1}\|[v]_n\|_0^2 + \frac{1}{2}\|v_N^-\|_0^2\right)^{1/2} \tag{27}$$

as corresponding semi-norm.

Proof. Since

$$Q_n\left[(v_{h,\tau}',w_{h,\tau}) + A_h\big((v_{h,\tau},q_{h,\tau});(w_{h,\tau},r_{h,\tau})\big)\right] = \int_{I_n}\left[(v_{h,\tau}',w_{h,\tau}) + A_h\big((v_{h,\tau},q_{h,\tau});(w_{h,\tau},r_{h,\tau})\big)\right]$$

for $n=1,\dots,N$ and $w_{h,\tau}\in X_k$, $r_{h,\tau}\in Y_k$, we have

$$\begin{aligned} B_h\big((v_{h,\tau},p_{h,\tau});(w_{h,\tau},q_{h,\tau})\big) = \int_0^T &\left[(v_{h,\tau}',w_{h,\tau}) + A_h\big((v_{h,\tau},p_{h,\tau});(w_{h,\tau},q_{h,\tau})\big)\right] \\ &+ \sum_{n=1}^{N-1}\big([v_{h,\tau}]_n, w_{h,\tau}(t_n^+)\big) + \big(v_{h,\tau}(t_0^+), w_{h,\tau}(t_0^+)\big). \end{aligned} \tag{28}$$

The integration by parts

$$\int_{I_n}(v',w) = (v_n^-, w_n^-) - (v_{n-1}^+, w_{n-1}^+) - \int_{I_n}(v,w')$$

leads to the representation

$$\begin{aligned} B_h\big((v_{h,\tau},q_{h,\tau});(w_{h,\tau},r_{h,\tau})\big) = \int_0^T &\left[-(v_{h,\tau},w_{h,\tau}') + A_h\big((v_{h,\tau},q_{h,\tau});(w_{h,\tau},r_{h,\tau})\big)\right] \\ &- \sum_{n=1}^{N-1}\big(v_{h,\tau}(t_n^-), [w_{h,\tau}]_n\big) + \big(v_{h,\tau}(t_N^-), w_{h,\tau}(t_N^-)\big). \end{aligned} \tag{29}$$

The statement of this Lemma follows by adding the representations (28) and (29) of B_h, setting $(w_{h,\tau}, r_{h,\tau}) = (v_{h,\tau}, q_{h,\tau})$, and using the coercivity (9) of the bilinear form A_h with respect to the spatial semi-norm $|||\cdot|||$. $\square$

The consistency error is estimated by the following lemma.

Lemma 8. *Let $(u_{h,\tau}, p_{h,\tau}) \in X_k \times Y_k$ be the solution of the fully discrete problem* (25) *and $(u, p) \in C^1(V) \times C(Q)$ be the solution of the continuous problem* (2)*. Then, we have*

$$B\big((u - u_{h,\tau}, p - p_{h,\tau}); (v_{h,\tau}, q_{h,\tau})\big) = (u_0 - u_{0,h}, v_0^+) + Q\left[S_h(p_{h,\tau}, q_{h,\tau})\right] \tag{30}$$

for all $(v_{h,\tau}, q_{h,\tau}) \in X_k \times Y_k$.

Proof. The statement follows by subtracting (24) from (25) and using the definitions of A and A_h. □

4.1. Representation of the fully discrete problem

Since the test functions are allowed to be discontinuous at the discrete time points t_n, $n = 1, \ldots, N-1$, we can choose test functions $v_{h,\tau}$ and $q_{h,\tau}$ which are zero on $I \setminus I_n$. Hence, the solution of the LPS/dG(k)-method can be determined by successively solving one local problem on each time interval I_n. The fully discrete I_n-problem associated to (25) reads as follows

Given u_n^- with $u_0^- = j_h u_0$, find $u_{h,\tau}\big|_{I_n} \in \mathbb{P}_k(I_n, V_h)$ and $p_{h,\tau}\big|_{I_n} \in \mathbb{P}_k(I_n, Q_h)$ such that

$$Q_n\left[(u'_{h,\tau}, v_{h,\tau}) + A_h\big((u_{h,\tau}, p_{h,\tau}); (v_{h,\tau}, q_{h,\tau})\big)\right] + \big([u_{h,\tau}]_{n-1}, v_{n-1}^+\big) = Q_n\left[(f, v_{h,\tau})\right] \tag{31}$$

for all $v_{h,\tau} \in \mathbb{P}_k(I_n, V_h)$ and all $q_{h,\tau} \in \mathbb{P}_k(I_n, Q_h)$.

In order to get an algebraic formulation of (31), let $\hat{\phi}_1, \ldots, \hat{\phi}_{k+1}$ denote the Lagrange basis functions with respect to the Gauß–Radau points $\hat{t}_1, \ldots, \hat{t}_{k+1}$ on $[-1, 1]$. Following [32], we define

$$\phi_{n,j}(t) := \hat{\phi}_j\big(T_n^{-1}(t)\big) \tag{32}$$

on I_n, $n = 1, \ldots, N$, with T_n from (23). Since $u_{h,\tau}$ and $p_{h,\tau}$ restricted to the interval I_n are V_h-valued and Q_h-valued polynomials of degree less than or equal to k, they can be represented as

$$u_{h,\tau}\big|_{I_n}(t) := \sum_{j=1}^{k+1} U_{n,h}^j \phi_{n,j}(t), \qquad p_{h,\tau}\big|_{I_n}(t) := \sum_{j=1}^{k+1} P_{n,h}^j \phi_{n,j}(t) \tag{33}$$

with $(U_{n,h}^j, P_{n,h}^j) \in V_h \times Q_h$, $j = 1, \ldots, k+1$. The choice of the ansatz basis guarantees

$$u_{h,\tau}(t_{n,j}) = U_{n,h}^j, \qquad p_{h,\tau}(t_{n,j}) = P_{n,h}^j, \qquad j = 1, \ldots, k+1,$$

with $t_{n,j} = T_n(\hat{t}_j)$, $j = 1, \ldots, k+1$. Taking into consideration that the Gauß–Radau formula Q_n is exact for polynomials up to degree $2k$, the particular choices

$$q_{h,\tau} = 0, \quad v_{h,\tau} = \frac{1}{\widehat{\omega}_j}\phi_{n,j}(t) v_h \text{ with arbitrary } v_h \in V_h,$$

and

$$v_{h,\tau} = 0, \quad q_{h,\tau} = \frac{1}{\widehat{\omega}_j}\phi_{n,j}(t) q_h \text{ with arbitrary } q_h \in Q_h,$$

for the test functions lead to the following system of linear equations:

Find the coefficients $(U_{n,h}^j, P_{n,h}^j) \in V_h \times Q_h$, $j = 1, \ldots k+1$, such that

$$\sum_{j=1}^{k+1} \alpha_{ij}\big(U_{n,h}^j, v_h\big) + \frac{\tau_n}{2} A_h\big((U_{n,h}^i, P_{n,h}^i); (v_h, q_h)\big) = \beta_i\big(U_{n,h}^0, v_h\big) + \frac{\tau_n}{2}\big(f(t_{n,i}), v_h\big) \tag{34}$$

for $i = 1, \dots, k+1$ and for all $(v_h, q_h) \in V_h \times Q_h$ where

$$\alpha_{ij} := \hat{\phi}_j'(\hat{t}_i) + \beta_i \hat{\phi}_j(-1), \qquad \beta_i := \frac{1}{\widehat{\omega}_i} \hat{\phi}_i(-1),$$

see [32]. The initial condition $U^0_{n,h}$ on I_n is given by

$$U^0_{n,h} := \begin{cases} j_h u_0, & n = 1, \\ U^{k+1}_{n-1,h}, & n > 1. \end{cases}$$

Note that no initial pressure is required.

4.2. Stability of the method

In this section, we study the stability properties of the fully discrete scheme (25). The following result provides the unconditional stability of velocity and pressure in the $\|\cdot\|_{\mathrm{S}}$-norm. However, the L^2-bound of the pressure solution depends on the inverse of the time step length.

Theorem 9. *The solution $(u_{h,\tau}, p_{h,\tau}) \in X_k \times Y_k$ of the fully discrete scheme (25) is uniquely determined and satisfies the stability estimates*

$$\|(u_{h,\tau}, p_{h,\tau})\|_{\mathrm{S}}^2 \le C\Big(\|j_h u_0\|_0^2 + Q\left[\|f\|_0^2\right]\Big), \tag{35}$$

$$\int_0^T \|p_{h,\tau}\|_0^2 \le C\Big(Q\left[\|f\|_0^2\right] + \max\{1, \tau_{\min}^{-2}\}\|(u_{h,\tau}, p_{h,\tau})\|_{\mathrm{S}}^2 + \tau_1^{-1}\|j_h u_0\|_0^2\Big) \tag{36}$$

where C is a constant independent of h and $\tau_{\min}$ defined in (21).

Proof. Setting $(v_{h,\tau}, q_{h,\tau}) = (u_{h,\tau}, p_{h,\tau})$ in (25) and using the coercivity (26) of the bilinear form B_h, the first statement of the lemma follows by means of the Cauchy–Schwarz inequality, the Friedrichs' inequality, Young's inequality, and the fact that the Gauß–Radau quadrature is exact for $|u_{h,\tau}|_1^2$.

We shall use the representation

$$\int_0^T \|p_{h,\tau}\|_0^2 = \sum_{n=1}^N Q_n\left[\|p_{h,\tau}\|_0^2\right] = \sum_{n=1}^N \frac{\tau_n}{2}\sum_{i=1}^{k+1} \widehat{\omega}_i \|p_{h,\tau}(t_{n,i})\|_0^2 = \sum_{n=1}^N \frac{\tau_n}{2}\sum_{i=1}^{k+1} \widehat{\omega}_i \|P^i_{n,h}\|_0^2$$

to prove the second estimate. It follows from (13) that for each $P^i_{n,h} \in Q_h \subset Q \cap H^1(\Omega)$ there exists a discrete velocity field $W^i_{n,h} \in V_h$ such that

$$\beta\|P^i_{n,h}\|_0 \le \frac{(P^i_{n,h}, \nabla \cdot W^i_{n,h})}{|W^i_{n,h}|_1} + C S_h(P^i_{n,h}, P^i_{n,h})^{1/2}. \tag{37}$$

In order to get a bound for the first term on the right-hand side of (37), we use $(v_h, q_h) = (W^i_{n,h}, 0)$ as a test function in the i-th equation of (34) and rearrange the terms to get

$$\big(P^i_{n,h}, \nabla \cdot W^i_{n,h}\big) = -\frac{2}{\tau_n}\beta_i(U^0_{n,h}, W^i_{n,h}) + \frac{2}{\tau_n}\sum_{j=1}^{k+1}\alpha_{ij}(U^j_{n,h}, W^i_{n,h}) - (f(t_{n,i}), W^i_{n,h}) + (\nabla U^i_{n,h}, \nabla W^i_{n,h}). \tag{38}$$

In the following, we will bound all terms on the right-hand side of (38) separately. The bounds for the third and fourth terms on the right-hand side of (38) follow by using the Cauchy–Schwarz inequality. We obtain

$$\big|(f(t_{n,i}), W^i_{n,h})\big| \le \|f(t_{n,i})\|_0 \, \|W^i_{n,h}\|_0 \le \|f(t_{n,i})\|_0 \, \|W^i_{n,h}\|_1,$$
$$\big|(\nabla U^i_{n,h}, \nabla W^i_{n,h})\big| \le \|\nabla U^i_{n,h}\|_0 \, \|\nabla W^i_{n,h}\|_0 \le |U^i_{n,h}|_1 \, \|W^i_{n,h}\|_1.$$

Using the properties of Lagrange interpolation and the representation (33) of $u_{h,\tau}$, one obtains

$$\sum_{j=1}^{k+1} \alpha_{ij} U_{n,h}^j = \frac{\tau_n}{2} u_{h,\tau}'(t_{n,i}) + \beta_i u_{h,\tau}(t_{n-1}^+).$$

This implies for the first two terms on the right-hand side of (38)

$$\begin{aligned} \left| \frac{2}{\tau_n} \left[\beta_i \left(U_{n,h}^0, W_{n,h}^i \right) - \sum_{j=1}^{k+1} \alpha_{ij} \left(U_{n,h}^j, W_{n,h}^i \right) \right] \right| &\le \left| \left(u_{h,\tau}'(t_{n,i}), W_{n,h}^i \right) \right| + \left| \frac{2\beta_i}{\tau_n} \left([u_{h,\tau}]_{n-1}, W_{n,h}^i \right) \right| \\ &\le \left[\| u_{h,\tau}'(t_{n,i}) \|_0 + \frac{2\beta_i}{\tau_n} \left\| [u_{h,\tau}]_{n-1} \right\|_0 \right] \| W_{n,h}^i \|_0. \end{aligned}$$

Inserting these bounds into (38), we get

$$(P_{n,h}^i, \nabla \cdot W_{n,h}^i) \le \left[\| u_{h,\tau}'(t_{n,i}) \|_0 + \frac{2\beta_i}{\tau_n} \| [u_{h,\tau}]_{n-1} \|_0 + |U_{n,h}^i|_1 + \| f(t_{n,i}) \|_0 \right] \| W_{n,h}^i \|_1.$$

Together with (37) and Friedrichs' inequality, we obtain

$$\| P_{n,h}^i \|_0^2 \le C \left[\| u_{h,\tau}'(t_{n,i}) \|_0^2 + \frac{\beta_i^2}{\tau_n^2} \| [u_{h,\tau}]_{n-1} \|_0^2 + |U_{n,h}^i|_1^2 + \| f(t_{n,i}) \|_0^2 + S_h(P_{n,h}^i, P_{n,h}^i) \right].$$

After multiplication by $\hat{\omega}_i \tau_n / 2$ and summing over $i = 1, \ldots, k+1$, we get

$$\begin{aligned} \int_{I_n} \| p_{h,\tau} \|_0^2 &= \frac{\tau_n}{2} \sum_{i=1}^{k+1} \widehat{\omega}_i \| P_{n,h}^i \|_0^2 \\ &\le C \sum_{i=1}^{k+1} \widehat{\omega}_i \left[\frac{\tau_n}{2} \| u_{h,\tau}'(t_{n,i}) \|_0^2 + \frac{\beta_i^2}{2\tau_n} \| [u_{h,\tau}]_{n-1} \|_0^2 + \frac{\tau_n}{2} |U_{n,h}^i|_1^2 + \frac{\tau_n}{2} \| f(t_{n,i}) \|_0^2 + \frac{\tau_n}{2} S_h(P_{n,h}^i, P_{n,h}^i) \right] \\ &\le C \int_{I_n} \left[\| u_{h,\tau}' \|_0^2 + |u_{h,\tau}|_1^2 + S_h(p_{h,\tau}, p_{h,\tau}) \right] + Q_n \left[\| f \|_0^2 \right] + C \sum_{i=1}^{k+1} \frac{\beta_i^2}{\tau_n} \| [u_{h,\tau}]_{n-1} \|_0^2 \end{aligned} \tag{39}$$

where $\widehat{\omega}_1 + \cdots + \widehat{\omega}_{k+1} = 2$ and the exactness of the Gauß–Radau quadrature were used. Applying an inverse inequality in time and Friedrichs' inequality, the estimate

$$\int_{I_n} \| u_{h,\tau}' \|_0^2 \le \frac{C}{\tau_n^2} \int_{I_n} \| u_{h,\tau} \|_0^2 \le \frac{C}{\tau_{\min}^2} \int_{I_n} |||(u_{h,\tau}, p_{h,\tau})|||^2$$

is obtained. Inserting this into (39) and summing over n proves the second statement of this Lemma. Note that the last term in (39) can not be bounded by $\tau_{\min}^{-1} \| (u_{h,\tau}, p_{h,\tau}) \|_{\mathrm{S}}^2$ for $n = 1$. Here one uses

$$\| [u_{h,\tau}]_0 \|_0^2 \le 2 \| u_{h,\tau}(0^+) \|_0^2 + 2 \| u_{h,\tau}(0^-) \|_0^2 \le 4 |||(u_{h,\tau}, p_{h,\tau})|||_{\mathrm{S}}^2 + 2 \| j_h u_0 \|_0^2$$

where the triangle inequality yields the first inequality. □

4.3. Error analysis

This section presents the error analysis of the fully discrete LPS/dG method (25). Let $\widetilde{w}$ denote the Gauß–Radau interpolant of a time-continuous function w, i.e., we have

$$\widetilde{w}|_{I_n}(t) := \sum_{j=1}^{k+1} w(t_{n,j}) \phi_{n,j}(t),$$

with $\phi_{n,j}$ defined in (32). Moreover, we set $\widetilde{w}_0^- := w_0^-$. Note that $\tilde{u}$ and $\tilde{p}$ will be on each time interval I_n, $n = 1, \ldots, N$, polynomials of degree less than or equal to k with values in V and Q, respectively. Furthermore, we define u^I on each I_n as the Lagrange interpolant of u with respect to the nodes $t_{n-1}, t_{n,1}, \ldots, t_{n,k+1}$. Hence, u^I is a time continuous, piecewise polynomial of degree less than or equal to $k+1$ with values in V.

Using the identity of the interpolants $\tilde{w}$ and w^I in $t_{n,j}$, an integration by parts, and the exactness of the quadrature rule for polynomials of degree $2k$ multiple times, shows that

$$\begin{aligned} Q_n \left[\big((w^I - \tilde{w})', v_{h,\tau}\big)\right] &= \int_{I_n} \big((w^I - \tilde{w})', v_{h,\tau}\big)\, dt \\ &= -\int_{I_n} (w^I - \tilde{w}, v'_{h,\tau})\, dt + \big((w^I - \tilde{w})(t_n^-), v_n^-\big) - \big((w^I - \tilde{w})(t_{n-1}^+), v_{n-1}^+\big) \\ &= -Q_n\left[\big(w^I - \tilde{w}, v'_{h,\tau}\big)\right] - \big(\tilde{w}(t_{n-1}^-) - \tilde{w}(t_{n-1}^+), v_{n-1}^+\big) \\ &= \big([\tilde{w}]_{n-1}, v_{n-1}^+\big) \end{aligned} \tag{40}$$

for all $v_{h,\tau} \in X_k$.

The standard interpolation theory leads to the error estimates

$$\sup_{0\le t\le T} |w^{(i)}(t) - \widetilde{w}^{(i)}(t)|_j \le C\tau^{k+1-i} \sup_{0\le t\le T} |w^{(k+1)}(t)|_j, \tag{41}$$

$$\int_{I_n} |w^{(i)}(t) - \widetilde{w}^{(i)}(t)|_j^2\, dt \le C\tau_n^{2(k+1-i)} \int_{I_n} |w^{(k+1)}(t)|_j^2\, dt, \tag{42}$$

$$\sup_{0\le t\le T} |u(t) - u^I(t)|_j \le C\tau^{k+2} \sup_{0\le t\le T} |u^{(k+2)}(t)|_j \tag{43}$$

with $i, j = 0, 1$.

The following lemma presents an estimate for the difference of the fully discrete solution and the space-time interpolation of the solution of the continuous problem.

Lemma 10. *Suppose Assumptions 1, 2, and $\mu_K \sim h_K^2$ for all $K \in \mathcal{T}_h$. Let $(u_{h,\tau}, p_{h,\tau})$ and (u, p) be the solutions of the fully discrete problem* (25) *and the continuous problem* (2)*, respectively. Moreover, let $u \in C^1(H^{r+1}(\Omega)^d) \cap C^{k+2}(H^1(\Omega)^d)$ and $p \in C(H^r(\Omega))$. Then, the estimate*

$$\big\|(u_{h,\tau} - j_h\tilde{u}, p_{h,\tau} - j_h\tilde{p})\big\|_{\mathrm{S}} \le Ch^r\Big[h\|u\|_{C^1(H^{r+1})} + \|u\|_{C(H^{r+1})} + \|p\|_{C(H^r)}\Big] + C\tau^{k+1}\|u\|_{C^{k+2}(H^1)} \tag{44}$$

holds true where C is a constant independent of h and τ.

Proof. For $\xi_{h,\tau} := u_{h,\tau} - j_h\tilde{u} \in X_k$ and $\vartheta_{h,\tau} := p_{h,\tau} - j_h\tilde{p} \in Y_k$, we have from (26)

$$\begin{aligned} \|(\xi_{h,\tau}, \vartheta_{h,\tau})\|_{\mathrm{S}}^2 = B_h\big((\xi_{h,\tau}, \vartheta_{h,\tau}); (\xi_{h,\tau}, \vartheta_{h,\tau})\big) &= B_h\big((u_{h,\tau} - j_h u, p_{h,\tau} - j_h p); (\xi_{h,\tau}, \vartheta_{h,\tau})\big) \\ &\quad + B_h\big((j_h(u - \tilde{u}), j_h(p - \tilde{p})); (\xi_{h,\tau}, \vartheta_{h,\tau})\big). \end{aligned} \tag{45}$$

We will bound the two terms on the right-hand side of (45) separately. For the first term, we have from (24) and (25)

$$\begin{aligned} &B_h\big((u_{h,\tau} - j_h u, p_{h,\tau} - j_h p); (\xi_{h,\tau}, \vartheta_{h,\tau})\big) \\ &\quad = Q\big[(u' - j_h u', \xi_{h,\tau}) + (\nabla(u - j_h u), \nabla\xi_{h,\tau}) - (p - j_h p, \nabla\cdot\xi_{h,\tau}) \\ &\qquad + (\vartheta_{h,\tau}, \nabla\cdot(u - j_h u)) - S_h(j_h p, \vartheta_{h,\tau})\big]. \end{aligned} \tag{46}$$

Note that no jump terms and no initial term occur since u and $j_h u$ are continuous in time. Adapting the techniques used to bound the terms on the right-hand side of (20), we obtain

$$\begin{aligned} &Q\left[(u' - j_h u', \xi_{h,\tau}) + (\nabla(u - j_h u), \nabla\xi_{h,\tau}) - (p - j_h p, \nabla\cdot\xi_{h,\tau}) + (\vartheta_{h,\tau}, \nabla\cdot(u - j_h u)) - S_h\left(j_h p, \vartheta_{h,\tau}\right)\right] \\ &\qquad \le Ch^r Q\left[h^2\|u'\|_{r+1}^2 + \|u\|_{r+1}^2 + \|p\|_r^2\right]^{1/2} Q\left[|||(\xi_{h,\tau}, \vartheta_{h,\tau})|||^2\right]^{1/2} \\ &\qquad \le Ch^r Q\left[h^2\|u'\|_{r+1}^2 + \|u\|_{r+1}^2 + \|p\|_r^2\right]^{1/2} \|(\xi_{h,\tau}, \vartheta_{h,\tau})\|_{\mathrm{S}}. \end{aligned}$$

Taking into consideration that (u,p) and $(\tilde u,\tilde p)$ coincide in all quadrature points, we obtain for the second term in (45)

$$\begin{aligned} &B_h\big((j_h(u-\tilde u), j_h(p-\tilde p)); (\xi_{h,\tau},\vartheta_{h,\tau})\big)\\ &\qquad = Q\left[(j_h(u-\tilde u)', \xi_{h,\tau})\right] + \sum_{n=1}^{N-1}\big([j_h(u-\tilde u)]_n, \xi_n^+\big) + (j_h(u-\tilde u)(t_0^+), \xi_0^+)\\ &\qquad = Q\left[(j_h(u-u^I)', \xi_{h,\tau})\right] \end{aligned}$$

thanks to (40) for $w = j_h u$ and the fact that the interpolation operators in time and space commute. Hence, we get

$$\Big| B_h\big((j_h(u-\tilde u), j_h(p-\tilde p)); (\xi_{h,\tau},\vartheta_{h,\tau})\big)\Big| \le C\tau^{k+1} Q\left[\|u^{(k+2)}\|_1^2\right]^{1/2} \|(\xi_{h,\tau},\vartheta_{h,\tau})\|_{\mathrm{S}}$$

by exploiting the stability of j_h and (43). We conclude with the statement of this Lemma by collecting the above estimates. ☐

Some error estimates for the interpolations in space and time are given in the next lemma.

Lemma 11. *Suppose Assumptions 1, 2, and $\mu_K \sim h_K^2$ for all $K \in \mathcal{T}_h$. Let $u \in H^1\big(H^{r+1}(\Omega)^d\big) \cap H^{k+1}\big(H^1(\Omega)^d\big)$ and $p \in L^2\big(H^r(\Omega)\big) \cap H^{k+1}\big(H^1(\Omega)\big)$ be the solution of problem (2) with $u_0 \in H^{r+1}(\Omega)^d$. Then, the estimates*

$$\begin{aligned} \big\|\big(j_h(\tilde u - u), j_h(\tilde p - p)\big)\big\|_{\mathrm{S}} &\le C\tau^{k+1/2}\big\{\|u\|_{H^{k+1}(H^1)} + h\tau^{1/2}\|p\|_{H^{k+1}(H^1)}\big\},\\ \big\|\big(j_h u - u, j_h p - p\big)\big\|_{\mathrm{S}} &\le Ch^r\big\{h\|u\|_{H^1(H^{r+1})} + \|u\|_{L^2(H^{r+1})} + \|p\|_{L^2(H^r)}\big\} \end{aligned}$$

hold true.

Proof. Using the definition of $\|\cdot\|_{\mathrm{S}}$ and the property $\tilde u(t_n^-) = u(t_n^-)$, $n = 1,\ldots,N$, we get

$$\begin{aligned} \|(j_h(\tilde u - u), j_h(\tilde p - p))\|_{\mathrm{S}}^2 &= \int_0^T \big\{|j_h(\tilde u - u)|_1^2 + S_h\big(j_h(\tilde p - p), j_h(\tilde p - p)\big)\big\} + \frac{1}{2}\sum_{n=0}^{N-1}\|j_h(\tilde u - u)(t_n^+)\|_0^2\\ &\le C\int_0^T \big\{\|\tilde u - u\|_1^2 + h^2\|\tilde p - p\|_1^2\big\} + C\sum_{n=0}^{N-1}\|(\tilde u - u)(t_n^+)\|_1^2 \end{aligned}$$

where $\mu_K \le Ch_K^2$ and the stability properties of j_h and κ_k were exploited. The interpolation error estimate (42) is used to bound the first two terms. Using

$$\sum_{n=0}^{N-1}\|(\tilde u - u)(t_n^+)\|_1^2 = -\sum_{n=1}^{N}\int_{I_n}\frac{d}{dt}\|\tilde u - u\|_1^2 = -2\sum_{n=1}^{N}\int_{I_n}(\tilde u - u, \tilde u' - u') + (\nabla(\tilde u - u), \nabla(\tilde u' - u'))$$

and the Cauchy–Schwarz inequality, the estimate for the last term is obtained. Putting these estimates together, the first statement of this Lemma follows.

In order to prove the second statement, we start with

$$\big\|(j_h u - u, j_h p - p)\big\|_{\mathrm{S}}^2 = \int_0^T \big\{|j_h u - u|_1^2 + S_h(j_h p - p, j_h p - p)\big\} + \frac{1}{2}\|(j_h u - u)(t_0^+)\|_0^2 + \frac{1}{2}\|(j_h u - u)(t_N^-)\|_0^2$$

which follows from the fact that u and $j_h u$ are continuous in time. With the interpolation error estimate (12), one gets

$$\big\|(j_h u - u, j_h p - p)\big\|_{\mathrm{S}}^2 \le Ch^{2r}\big\{\|u\|_{L^2(H^{r+1})}^2 + \|p\|_{L^2(H^r)}^2 + h^2\|u_0\|_{r+1}^2 + h^2\|u_N^-\|_{r+1}^2\big\}$$

and the second estimate of this Lemma follows from the embedding $H^1\big(H^{r+1}(\Omega)^d\big) \subset C\big(H^{r+1}(\Omega)^d\big)$. ☐

The next theorem provides the final error estimate as the main result of this section.

Theorem 12. *Suppose Assumptions 1, 2, and $\mu_K \sim h_K^2$ for all $K \in \mathcal{T}_h$. Let $(u_{h,\tau}, p_{h,\tau})$ and (u,p) be the solutions of the fully discrete problem (25) and of the continuous problem (2), respectively. Moreover, let $u \in C^1\big(H^{r+1}(\Omega)^d\big) \cap C^{k+2}\big(H^1(\Omega)^d\big) \cap H^{k+1}\big(H^1(\Omega)^d\big)$ and $p \in C\big(H^r(\Omega)\big) \cap H^{k+1}\big(H^1(\Omega)\big)$. Then, there is a positive constant C independent of h and τ such that the error estimates*

$$\begin{aligned}
\big\|(u_{h,\tau}-u, p_{h,\tau}-p)\big\|_{\mathrm{S}} \le{}& Ch^r\Big\{h\|u\|_{C^1(H^{r+1})} + \|u\|_{C(H^{r+1})} + \|p\|_{C(H^r)}\Big\} \\
&+ C\tau^{k+1/2}\Big\{\tau^{1/2}\|u\|_{C^{k+2}(H^1)} + \|u\|_{H^{k+1}(H^1)} + h\tau^{1/2}\|p\|_{H^{k+1}(H^1)}\Big\}, \\
\big\|p_{h,\tau}-p\big\|_{L^2(L^2)} \le{}& C\max\big\{1, \tau_{\min}^{-1}\big\}\big\|(u_{h,\tau}-u, p_{h,\tau}-p)\big\|_{\mathrm{S}} \\
&+ C\tau^k\|u\|_{C^{k+1}(L^2)} + C\tau^{k+1}\Big\{\|u\|_{C^{k+1}(H^1)} + \|u\|_{H^{k+1}(H^1)}\Big\} \\
&+ C\frac{h^{r+1}}{\sqrt{\tau_1}}\|u_0\|_{r+1}
\end{aligned}$$

hold true.

Proof. The first statement of this Theorem follows from the triangle inequality applied to the splittings

$$\begin{aligned}
u_{h,\tau} - u &= u_{h,\tau} - j_h\tilde{u} + j_h\tilde{u} - j_h u + j_h u - u, \\
p_{h,\tau} - p &= p_{h,\tau} - j_h\tilde{p} + j_h\tilde{p} - j_h p + j_h p - p,
\end{aligned} \tag{47}$$

and the estimates from Lemma 10 and Lemma 11.

In the second step, an estimate for the pressure error is derived. Consider $\vartheta_{h,\tau} := p_{h,\tau} - j_h\tilde{p} \in Y_k$ with

$$\vartheta_{h,\tau}|_{I_n}(t) = \sum_{i=1}^{k+1} Q_{n,h}^i \phi_{n,i}(t).$$

Using (13), we have the existence of $W_{n,h}^i \in V_h$, $i = 1, \dots, k+1$, such that

$$\beta\|Q_{n,h}^i\|_0 \le \frac{(Q_{n,h}^i, \nabla\cdot W_{n,h}^i)}{|W_{n,h}^i|_1} + CS_h(Q_{n,h}^i, Q_{n,h}^i)^{1/2}. \tag{48}$$

We obtain

$$Q_{n,h}^i = P_{n,h}^i - j_h\tilde{p}(t_{n,i}) = P_{n,h}^i - p(t_{n,i}) + p(t_{n,i}) - j_h p(t_{n,i}) \tag{49}$$

with the help of $\tilde{p}(t_{n,i}) = p(t_{n,i})$. Hence, we have

$$\big|\big(Q_{n,h}^i, \nabla\cdot W_{n,h}^i\big)\big| \le \big|\big(P_{n,h}^i - p(t_{n,i}), \nabla\cdot W_{n,h}^i\big)\big| + \big|\big(p(t_{n,i}) - j_h p(t_{n,i}), \nabla\cdot W_{n,h}^i\big)\big|.$$

Using (2) and (34), we get

$$\begin{aligned}
&(p(t_{n,i}) - P_{n,h}^i, \nabla\cdot W_{n,h}^i) \\
&\qquad = \frac{2}{\tau_n}\beta_i\big([u-u_{h,\tau}]_{n-1}, W_{n,h}^i\big) + \big(u'(t_{n,i}) - u_{h,\tau}'(t_{n,i}), W_{n,h}^i\big) + \big(\nabla(u(t_{n,i}) - U_{n,h}^i), \nabla W_{n,h}^i\big)
\end{aligned}$$

and

$$\begin{aligned}
&\big|(P_{n,h}^i - p(t_{n,i}), \nabla\cdot W_{n,h}^i)\big| \\
&\qquad \le C\left(\frac{2}{\tau_n}\beta_i\big\|[u-u_{h,\tau}]_{n-1}\big\|_0 + \|u'(t_{n,i}) - u_{h,\tau}'(t_{n,i})\|_0 + |u(t_{n,i}) - u_{h,\tau}(t_{n,i})|_1\right)|W_{n,h}^i|_1
\end{aligned}$$

where Friedrichs' inequality was used. Putting this into (48) leads together with (49) to

$$\|Q^i_{n,h}\|_0 \le C\Big(\|p(t_{n,i}) - j_h p(t_{n,i})\|_0 + \frac{2}{\tau_n}\beta_i \big\|[u-u_{h,\tau}]_{n-1}\big\|_0 \\ + \|u'(t_{n,i}) - u'_{h,\tau}(t_{n,i})\|_0 + |u(t_{n,i}) - u_{h,\tau}(t_{n,i})|_1 + S_h(Q^i_{n,h}, Q^i_{n,h})^{1/2}\Big).$$

After squaring, multiplying by $\widehat{\omega}_j \tau_n/2$, and summing over $j = 1, \dots, k+1$, we obtain

$$\int_{I_n} \|\vartheta_{h,\tau}\|_0^2 \le C\Big(Q_n\left[\|p - j_h p\|_0^2 + |u - u_{h,\tau}|_1^2 + \|u' - u'_{h,\tau}\|_0^2 + S_h(\vartheta_{h,\tau}, \vartheta_{h,\tau})\right] \\ + \frac{1}{\tau_n}\big\|[u - u_{h,\tau}]_{n-1}\big\|_0^2\Big). \tag{50}$$

The first term in (50) can be estimated using the interpolation properties (12) of j_h. Since $p_{h,\tau} - j_h\tilde{p} = \vartheta_{h,\tau} \in Y_k$, the quadrature formula Q_n is exact. Hence, the fourth term in (50) can be written as an integral contained in $\|(u_{h,\tau} - j_h\tilde{u}, p_{h,\tau} - j_h\tilde{p})\|_{\mathrm{S}}$ which has been already bounded in Lemma 10. The last term in (50) can be estimated using the first statement of this Theorem since the jump term is included in $\|(u_{h,\tau} - u, p_{h,\tau} - p)\|_{\mathrm{S}}$ when $n \ne 1$. We have to bound for $n = 1$ additionally the term $\|(u - u_{h,\tau})(0^-)\|_0$. The choice of the initial condition and (12) yield

$$\|(u - u_{h,\tau})(0^-)\|_0 = \|u_0 - j_h u_0\|_0 \le C h^{r+1}\|u_0\|_{r+1} \le C h^{r+1}\|u\|_{C(H^{r+1})}.$$

In order to estimate the second term in (50), we proceed as follows

$$Q_n\left[|u - u_{h,\tau}|_1^2\right] \le 2Q_n\left[|u - \tilde{u}|_1^2\right] + 2Q_n\left[|\tilde{u} - u_{h,\tau}|_1^2\right] = 2Q_n\left[|u - \tilde{u}|_1^2\right] + 2\int_{I_n} |\tilde{u} - u_{h,\tau}|_1^2 \\ \le 2Q_n\left[|u - \tilde{u}|_1^2\right] + 4\int_{I_n} |\tilde{u} - u|_1^2 + 4\int_{I_n} |u - u_{h,\tau}|_1^2.$$

The first two terms can be estimated using the interpolation properties (41) and (42) while the last term is bounded by the first statement of this Theorem. Using the same ideas combined with an inverse inequality in time, the third term in (50) is handled. We obtain

$$Q_n\left[\|u' - u'_{h,\tau}\|_0^2\right] \le 2Q_n\left[\|u' - \tilde{u}'\|_0^2\right] + 2Q_n\left[\|\tilde{u}' - u'_{h,\tau}\|_0^2\right] \le 2Q_n\left[\|u' - \tilde{u}'\|_0^2\right] + \frac{2\,C_{\mathrm{inv}}^2}{\tau_n^2}\int_{I_n} \|\tilde{u} - u_{h,\tau}\|_0^2 \\ \le 2Q_n\left[\|u' - \tilde{u}'\|_0^2\right] + \frac{4\,C_{\mathrm{inv}}^2}{\tau_n^2}\int_{I_n} \|\tilde{u} - u\|_0^2 + \frac{4\,C_{\mathrm{inv}}^2}{\tau_n^2}\int_{I_n} \|u - u_{h,\tau}\|_0^2$$

and, after applying Friedrichs' inequality, the appearing terms can be bounded by the interpolation properties (41), (42), and the first statement of this Theorem.

Finally, to prove the second statement of this Theorem, we use the splitting (47) and the triangle inequality. We have already estimated the first difference. By using the stability of j_h, the interpolation property (42), and the approximation property (12) to estimate the two remaining differences, the proof is completed. □

Remark 1. The estimate of the pressure shows an instability in the limit of small time step length since the error bound in Thm. 12 is proportional to $\tau_{\min}^{-1}$. The effect is less pronounced for higher order methods in space. Our numerical results are in agreement with these theoretical observations, see Section 5.3.

Choosing the discrete initial condition for the velocity as a Stokes projection allows to improve the pressure estimate in Thm. 12 since the negative powers of the time step length can be prevented, see [33].

4.4. Algebraic formulation and post processing

We will present at the beginning of this section the algebraic formulation of the fully discrete problem. For simplicity, we restrict ourselves to the two-dimensional case since the three-dimensional case is obtained in a straightforward manner.

Let $\{\phi_1,\dots,\phi_{m_h}\}$ be a finite element basis of $Y_h \cap H_0^1(\Omega)$ and $\boldsymbol{\xi}^j_{n,1}, \boldsymbol{\xi}^j_{n,2} \in \mathbb{R}^{m_h}$ denote the nodal vectors associated with the components of the finite element function $U^j_{n,h} \in V_h = \big(Y_h \cap H_0^1(\Omega)\big)^2$, i.e.,

$$U^j_{n,h}(x) = \sum_{l=1}^{2} \left(\sum_{\nu=1}^{m_h} \big(\boldsymbol{\xi}^j_{n,l}\big)_\nu \phi_\nu(x) \right) e^l, \qquad x \in \Omega,$$

where $e^1, e^2 \in \mathbb{R}^2$ are the canonical unit vectors. Similarly for the pressure, let $\{\psi_1,\dots,\psi_{n_h}\}$ denote a finite element basis functions of the space $Q_h = Y_h \cap L_0^2(\Omega)$ and $\boldsymbol{\eta}^j_n \in \mathbb{R}^{n_h}$ the nodal vector of $P^j_{n,h} \in Q_h$ such that

$$P^j_{n,h}(x) = \sum_{\nu=1}^{n_h} \big(\boldsymbol{\eta}^j_n\big)_\nu \psi_\nu(x), \qquad x \in \Omega.$$

Furthermore, we define by

$$\begin{aligned} M_{s,\nu} &:= (\phi_\nu, \phi_s), & A_{s,\nu} &:= \big(\nabla\phi_\nu, \nabla\phi_s\big), \quad C_{s,\nu} := -S_h(\psi_\nu, \psi_s), \\ (B_i)_{s,\nu} &:= -\big(\psi_s, \nabla\cdot(\psi_\nu e^i)\big), & (F^j_{n,i})_\nu &:= \big(f(t_{n,j}), \phi_\nu e^i\big), \qquad i = 1,2, \end{aligned}$$

the mass matrix $M \in \mathbb{R}^{m_h \times m_h}$, the stiffness matrix $A \in \mathbb{R}^{m_h \times m_h}$, the pressure stabilization matrix $C \in \mathbb{R}^{n_h \times n_h}$, the velocity-pressure coupling matrices $B_i \in \mathbb{R}^{n_h \times m_h}$, and the right-hand side vectors $F^j_{n,i}$, $i = 1,2$.

Using the block matrices and the block vectors

$$\mathcal{M} = \begin{bmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathcal{A} = \begin{bmatrix} A & 0 & B_1^T \\ 0 & A & B_2^T \\ B_1 & B_2 & C \end{bmatrix}, \qquad F^j_n = \begin{bmatrix} F^j_{n,1} \\ F^j_{n,2} \\ 0 \end{bmatrix}, \qquad \boldsymbol{\zeta}^j_n = \begin{bmatrix} \boldsymbol{\xi}^j_{n,1} \\ \boldsymbol{\xi}^j_{n,2} \\ \boldsymbol{\eta}^j_n \end{bmatrix},$$

the fully discrete I_n-problem (34) results in the following $(k+1)\times(k+1)$-block system:

Find $\boldsymbol{\zeta}^j_n \in \mathbb{R}^{2m_h+n_h}$ for $j = 1,\dots,k+1$ such that

$$\sum_{j=1}^{k+1} \alpha_{ij} \mathcal{M}\boldsymbol{\zeta}^j_n + \frac{\tau_n}{2}\mathcal{A}\boldsymbol{\zeta}^i_n = \beta_i \mathcal{M}\boldsymbol{\zeta}^0_n + \frac{\tau_n}{2} F^i_n, \qquad i = 1,\dots,k+1.$$

Although $\boldsymbol{\zeta}^0_n$ contains a pressure part, no initial pressure is needed since $\mathcal{M}$ has a last column of zero matrix blocks. After solving this system, we enter the next time interval and set the initial value of the time interval I_{n+1} to $\boldsymbol{\zeta}^0_{n+1} := \boldsymbol{\zeta}^{k+1}_n$. In the following, we present the schemes for the cases $k = 0$ and $k = 1$.

The dG(0)-method. The 1-point Gauß–Radau formula with point $t_{n,1} = t_n$ and weight $\widehat{\omega}_1 = 2$ gives the well-known implicit Euler method, i.e., the I_n-problem is the following one-block equation for $\boldsymbol{\zeta}^1_n \in \mathbb{R}^{2m_h+n_h}$:

$$\Big(\mathcal{M} + \tau_n \mathcal{A}\Big)\boldsymbol{\zeta}^1_n = \mathcal{M}\boldsymbol{\zeta}^0_n + \tau_n F^1_n.$$

The dG(1)-method. The 2-point Gauß–Radau formula with points $t_{n,1} = t_{n-1} + \tau_n/3$, $t_{n,2} = t_n$ and weights $\widehat{\omega}_1 = 3/2$, $\widehat{\omega}_2 = 1/2$ yields on the time interval I_n the following coupled (2×2)-block system for $\boldsymbol{\zeta}^1_n, \boldsymbol{\zeta}^2_n \in \mathbb{R}^{2m_h+n_h}$:

$$\begin{bmatrix} \frac{3}{4}\mathcal{M} + \frac{\tau_n}{2}\mathcal{A} & \frac{1}{4}\mathcal{M} \\ -\frac{9}{4}\mathcal{M} & \frac{5}{4}\mathcal{M} + \frac{\tau_n}{2}\mathcal{A} \end{bmatrix} \begin{bmatrix} \boldsymbol{\zeta}^1_n \\ \boldsymbol{\zeta}^2_n \end{bmatrix} = \begin{bmatrix} \mathcal{M}\boldsymbol{\zeta}^0_n + \frac{\tau_n}{2} F^1_n \\ -\mathcal{M}\boldsymbol{\zeta}^0_n + \frac{\tau_n}{2} F^2_n \end{bmatrix}.$$

In [32], a simple post-processing for systems of ordinary differential equations was presented which allows to construct numerical approximations being in the integral-based norms $\|\cdot\|_{\mathrm{dG}}$ and $\|\cdot\|_{L^2(L^2)}$ at least one order better than the originally obtained numerical solution provided the exact solution is sufficiently smooth in time.

We will here generalize the idea to the Stokes problem. Define

$$\vartheta_n(t) = \frac{\tau_n}{2}\hat{\vartheta}(\hat{t}), \qquad \hat{t} := T_n^{-1}(t),\ t \in I_n,$$

with T_n from (23). The polynomial $\hat{\vartheta} \in \mathbb{P}_{k+1}$ is uniquely defined by $\hat{\vartheta}(\hat{t}_j) = 0$ for all Gauß–Radau points $\hat{t}_j$, $j = 1, \ldots, k+1$, and $\hat{\vartheta}'(1) = 1$. Let $(u_{h,\tau}, p_{h,\tau})$ denote the solution of (31). The post-processed solution $(\Pi u_{h,\tau}, \Pi p_{h,\tau})$ on the time interval I_n is given by

$$(\Pi u_{h,\tau})(t) = u_{h,\tau}(t) + g_n \vartheta_n(t), \quad (\Pi p_{h,\tau})(t) = p_{h,\tau}(t) + d_n \vartheta_n'(t), \qquad t \in I_n,$$

with finite element functions $g_n \in V_h$ and $d_n \in Q_h$. The corresponding coefficient vectors $\boldsymbol{\gamma}_n = (\boldsymbol{\gamma}_{n,1}, \boldsymbol{\gamma}_{n,2})^T \in \mathbb{R}^{2m_h}$ and $\boldsymbol{\delta}_n \in \mathbb{R}^{n_h}$ are the solution of the saddle-point problem

$$\begin{bmatrix} M & 0 & B_1^T \\ 0 & M & B_2^T \\ B_1 & B_2 & C \end{bmatrix} \begin{bmatrix} \boldsymbol{\gamma}_{n,1} \\ \boldsymbol{\gamma}_{n,2} \\ \boldsymbol{\delta}_n \end{bmatrix} = \begin{bmatrix} F_{n,1}^{k+1} \\ F_{n,2}^{k+1} \\ 0 \end{bmatrix} - \begin{bmatrix} A & 0 & B_1^T \\ 0 & A & B_2^T \\ B_1 & B_2 & C \end{bmatrix} \begin{bmatrix} \boldsymbol{\xi}_{n,1}^{k+1} \\ \boldsymbol{\xi}_{n,2}^{k+1} \\ \boldsymbol{\eta}_n^{k+1} \end{bmatrix} - \begin{bmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\chi}_{n,1}^{k+1} \\ \boldsymbol{\chi}_{n,2}^{k+1} \\ 0 \end{bmatrix}$$

where $\boldsymbol{\chi}_n^{k+1} = \left(\boldsymbol{\chi}_{n,1}^{k+1}, \boldsymbol{\chi}_{n,2}^{k+1}\right)^T \in \mathbb{R}^{2m_h}$ denotes the nodal representation of $u_{h,\tau}'(t_n) \in V_h$. Note that the post-processing can be easily generalized to the three-dimensional case.

5. Numerical studies

We present in this section some numerical experiments to verify the theoretical results from the previous sections. All computations were performed with the finite element code MooNMD [34].

We shall consider three different examples of time-dependent Stokes problems defined in the domain $\Omega = (0,1)^2$. The first two example taken from [35] and [16] possess smooth solutions. To observe the behavior in the small time step limit, a steady Stokes problem taken from [16] is considered as third example.

Our calculations were carried out on uniform quadrilateral grids where the coarsest grid (level 1) is obtained by dividing the unit square into four congruent squares. We used in our numerical computations mapped finite element spaces [36] where the enriched spaces on the reference cell $\widehat{K} = (-1,1)^2$ are given by

$$\mathbb{Q}_s^{\text{bubble}}(\widehat{K}) := \mathbb{Q}_s(\widehat{K}) + \operatorname{span}\left\{\hat{b}_\Box \hat{x}_i^{s-1},\ i = 1,2\right\}$$

with the biquadratic bubble function $\hat{b}_\Box = (1 - \hat{x}_1^2)(1 - \hat{x}_2^2)$ on the reference square $\widehat{K}$. Together with the choice $D(K) = \mathbb{P}_{r-1}(K)$, these spaces are suited for local projection methods, see [8].

In addition to the $\|\cdot\|_{\mathrm{S}}$-norm, convergence results in the integrated L^2-norm, the integrated H^1-semi-norm, and the discrete ℓ^∞-norm which are defined by

$$\|v\|_{L^2(L^2)} := \left\{ \int_0^T \|v(t)\|_0^2 \, dt \right\}^{1/2}, \qquad \|v\|_{L^2(H^1)} := \left\{ \int_0^T |v(t)|_1^2 \, dt \right\}^{1/2}, \qquad \|v\|_\infty := \max_{1 \le n \le N} \|v(t_n^-)\|_0$$

will be shown. Let

$$e_u := u - u_{h,\tau}, \qquad e_p := p - p_{h,\tau}$$

denote the velocity and pressure errors between the solutions of the continuous problem (2) and the fully discrete problem (25).

5.1. *Test example 1*

Our first example is to test the predicted rates of convergence in time. The right-hand side f and the initial condition u_0 are chosen such that

$$\begin{aligned} u_1(x,y,t) &= x^2(1-x)^2\big(2y(1-y)^2 - 2y^2(1-y)\big)\sin(10\pi t),\\ u_2(x,y,t) &= -\big(2x(1-x)^2 - 2x^2(1-x)\big)y^2(1-y)^2\sin(10\pi t),\\ p(x,y,t) &= -\big(x^3+y^3-0.5\big)\big(1.5+0.5\,\sin(10\pi t)\big) \end{aligned}$$

is the solution of problem (1) with homogeneous Dirichlet boundary conditions.

In order to illustrate the convergence order in time, we excluded the error in space. To this end, the numerical tests are performed with $Y_h = Q_4^{\text{bubble}}$ and $D(K) = P_3(K)$ for all $K \in \mathcal{T}_h$, the stabilization parameters $\mu_K = 0.1h_K^2$, and a mesh which consists of 8×8 squares. The calculations were done for dG(1) and dG(2) with time step lengths $\tau = 0.1\times 10^i$, $i = 0,\ldots 6$.

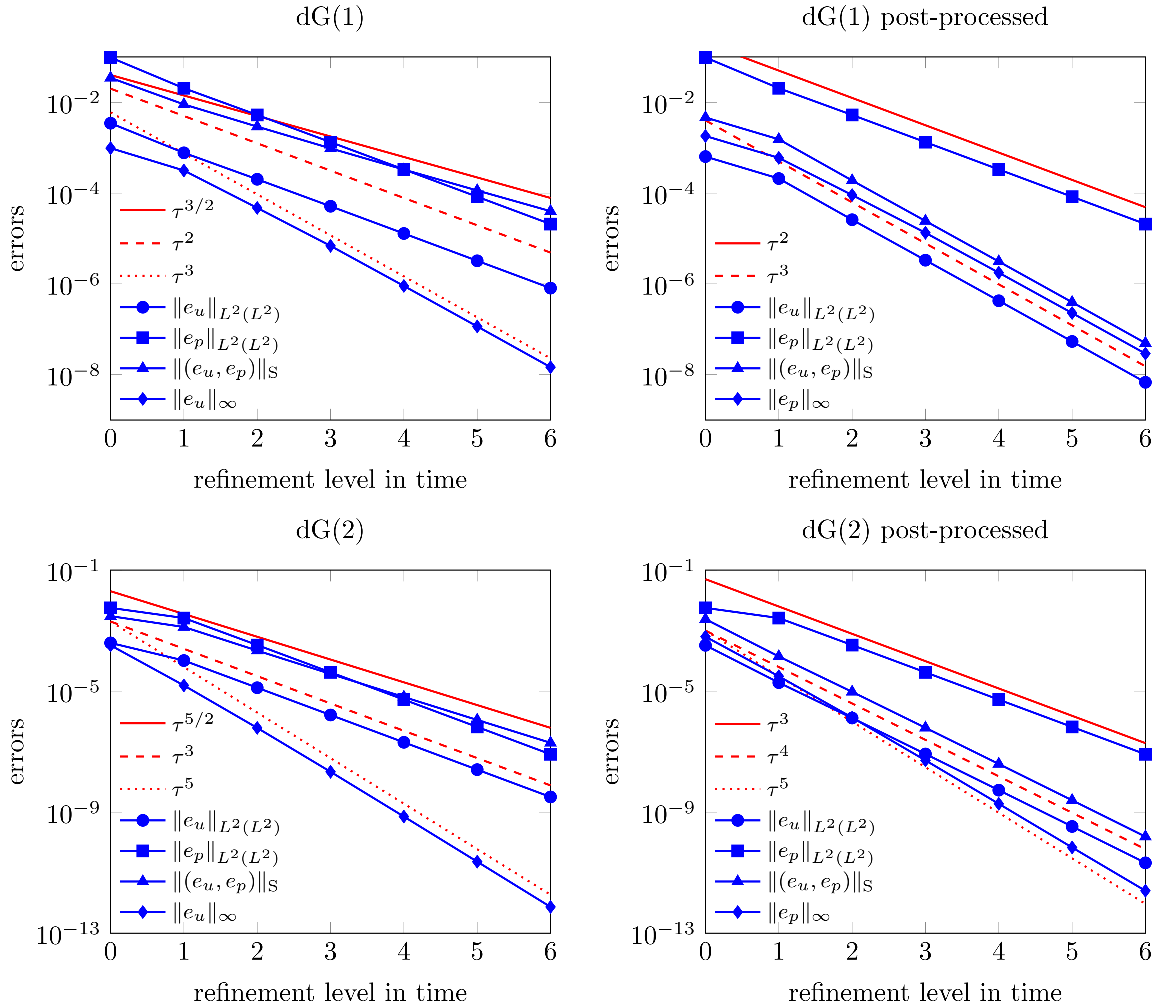


Figure 1: Example 1 with smooth pressure. Simulations for dG(1) (top row) and dG(2) (bottom row). Convergence orders for different errors of the solution (left column) and the post-processed solution (right column).

The results of the numerical studies are presented in Figure 1 where the errors in the different norms are plotted against the different refinement levels in time. It is expected from Thm. 12 that the error in the

$\|\cdot\|_S$-norm is of order $k+1/2$. This expectation can be well observed in the numerical simulations. It can also be seen that the dG(k) methods are accurate of order $k+1$ in the $L^2(L^2)$-norm of the velocity. It is known from [37] that the dG(k) methods applied to scalar problems are super-convergent of order $2k+1$ at the discrete time points t_n, $n = 1, \dots N$. Comparing the errors in the discrete ℓ^∞-norm of velocity and pressure, it can be seen that the dG(k) methods are accurate of order $2k+1$. Furthermore, the post-processing allows to get solutions which provide the convergence order $k+2$ in the integral-based norms. One can see the improved accuracy by comparing the convergence results in the $L^2(L^2)$-norm and the $\|\cdot\|_S$-norm shown in the plots of Figure 1. Note that the mapping of line style and convergence order may differ from plot to plot.

Table 1: Example 1 with rough pressure: errors of the post-processed solution and convergence orders for dG(1) and dG(2) methods in the $L^2(L^2)$-norm of velocity and pressure.

	dG(1)				dG(2)			
	$\|u-\Pi u_{h,\tau}\|_{L^2(L^2)}$		$\|p-\Pi p_{h,\tau}\|_{L^2(L^2)}$		$\|u-\Pi u_{h,\tau}\|_{L^2(L^2)}$		$\|p-\Pi p_{h,\tau}\|_{L^2(L^2)}$	
τ	error	order	error	order	error	order	error	order
1/10	3.465-3		9.979-4		3.916-4		6.278-4	
1/20	7.717-4	3.27	4.580-4	1.12	1.024-4	1.94	6.847-5	3.20
1/40	2.016-4	1.94	9.312-5	2.98	1.294-5	2.98	2.149-5	1.67
1/80	5.125-5	1.98	3.163-5	1.56	1.623-6	3.00	8.927-6	1.27
1/160	1.288-5	1.99	1.294-5	1.29	2.030-7	3.00	3.752-6	1.25
1/320	3.224-6	2.00	5.426-6	1.25	2.539-8	3.00	1.578-6	1.25
1/640	8.062-7	2.00	2.281-6	1.25	3.173-9	3.00	6.633-7	1.25
1/1280	2.016-7	2.00	9.589-7	1.25	3.967-10	3.00	2.789-7	1.25
theory		2		1		3		1

The convergence order for the pressure in the $L^2(L^2)$-norm is one order better than predicted by our theory, see Thm. 12. This is caused by the smoothness of the exact pressure. If we consider the current problem where the pressure is replaced by the rough function

$$p(x, y, t) = -\left(x^3 + y^3 - 0.5\right)\left(1.5 + 0.5\, t^{3/4}\right)$$

then the convergence order for the pressure is limited by its smoothness, see Tab. 1. However, the convergence order of the velocity is not influenced by the smoothness of the pressure.

5.2. Test example 2

Our next numerical example is to study the influence of the parameters μ_K in the stabilization term. The right-hand side f and the initial condition u_0 are chosen such that

$$\begin{aligned} u_1(x, y, t) &= g(t)\sin(\pi x - 0.7)\sin(\pi y + 0.2), \\ u_2(x, y, t) &= g(t)\cos(\pi x - 0.7)\cos(\pi y + 0.2), \\ p(x, y, t) &= g(t)\Big(\sin(x)\cos(y) + \big(\cos(1) - 1\big)\sin(1)\Big), \end{aligned}$$

with $g(t) = 1 + t^5 + \exp(-0.1t) + \sin(t)$, is the exact solution of the problem (1) with non-homogeneous boundary conditions.

We restrict ourselves to dG(1) in time and $Y_h = Q_r^{\text{bubble}}$ with $D_K = P_{r-1}(K)$, $r = 1, 2$, for all $K \in \mathcal{T}_h$ as spatial discretization. The computations were carried out on the refinement levels 3–6 (corresponding to 16×16 up to 128×128 mesh cells) and with a time step length $\tau = 1/180$. To see the influence of the stabilization parameters, we set $\mu_K = \mu_0 h_K^2$ where the constant μ_0 varies from 10^{-6} to 10^6. Let

$$\|q\|^2_{L^2(S)} = \int_0^T S_h(q, q)$$

a semi-norm measuring the pressure error in the stabilization term.

Our calculations showed that the velocity errors were almost constant in the investigated range of μ_0. However, we plot the error in the integrated H^1-semi-norm in order to illustrate its influence on $\|(e_u, e_p)\|_{\mathrm{S}}$.

We start with the consideration of Q_1^{bubble} finite elements and projection onto P_0. Figure 2 plots the errors in the norms $\|e_p\|_{L^2(L^2)}$, $\|e_p\|_{L^2(S)}$, $\|e_u\|_{L^2(H^1)}$, and $\|(e_u, e_p)\|_{\mathrm{S}}$ versus the constant μ_0 inside the definition $\mu_K = \mu_0 h_K^2$ of the stabilization parameters. We observe that the pressure error measured in the $L^2(L^2)$-norm is large for both very small and very large values of μ_0. This is caused by under-stabilization and over-stabilization, respectively. Since the $L^2(S)$-semi-norm includes μ_0 as a factor, the corresponding error rises with increasing μ_0. The error in the semi-norm $\|\cdot\|_{\mathrm{S}}$ is dominated by the $L^2(H^1)$-error. Hence, it is almost constant for small and moderate values of μ_0. In contrast, the increasing pressure stabilization term affects $\|(e_u, e_p)\|_{\mathrm{S}}$ for larger μ_0.

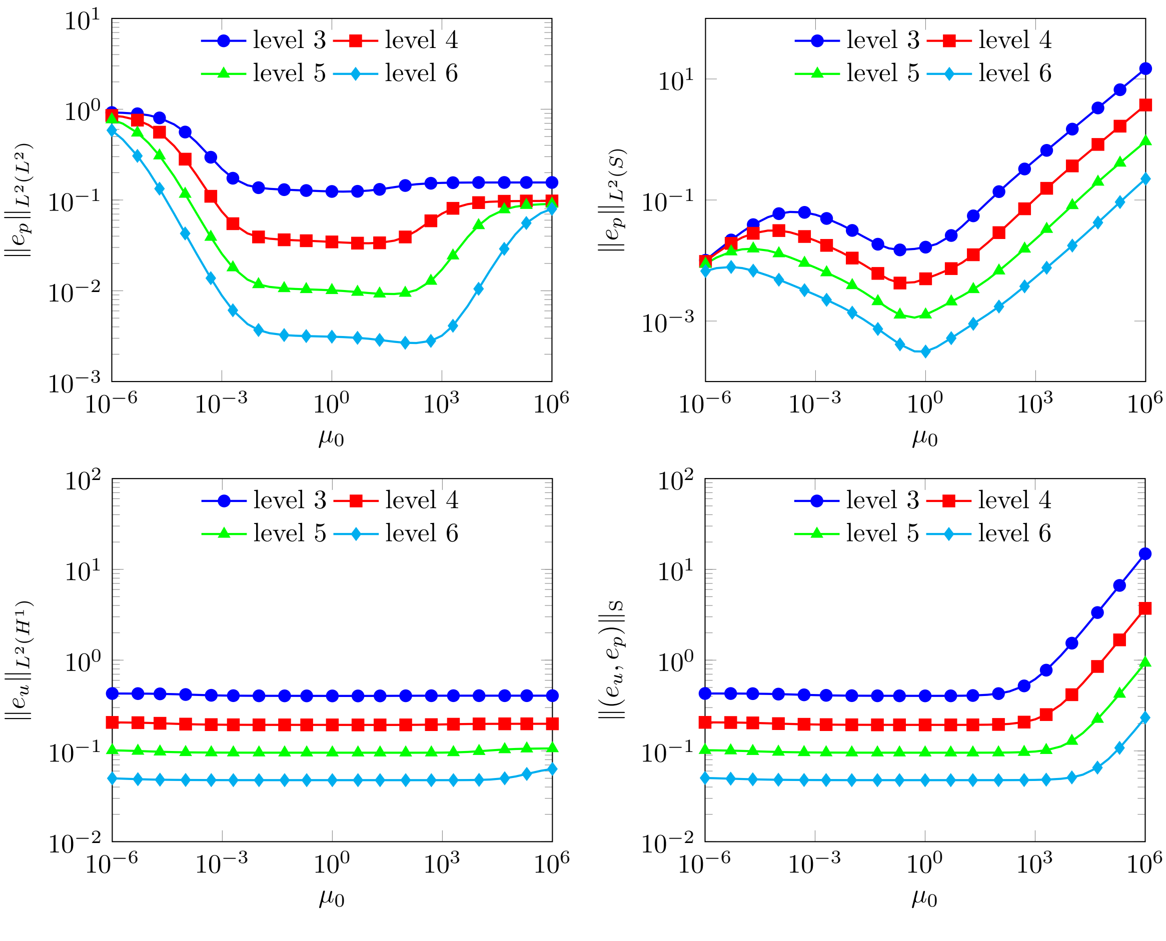


Figure 2: Example 2 with Q_1^{bubble} and $D(K) = P_0(K)$. Errors $\|e_p\|_{L^2(L^2)}$ (upper left), $\|e_p\|_{L^2(S)}$ (upper right), $\|e_u\|_{L^2(H^1)}$ (lower left), and $\|(e_u, e_p)\|_{\mathrm{S}}$ (lower right) versus the constant μ_0 in $\mu_K = \mu_0 h_K^2$.

We consider now the influence of the constant μ_0 on the errors for Q_2^{bubble} finite elements and projection onto P_1. Results for the error norms under consideration are presented in Figure 3. One can see that the pressure errors in the $\|\cdot\|_{L^2(L^2)}$-norm is almost independent of μ_0 in a wide range. However, the corresponding error increases for small values of μ_0 due to under-stabilization. As for first order elements, the pressure error in the $\|\cdot\|_{L^2(S)}$-norm gets larger with μ_0. Also the error in the $\|\cdot\|_{\mathrm{S}}$-norm behaves similar as for Q_1^{bubble}.

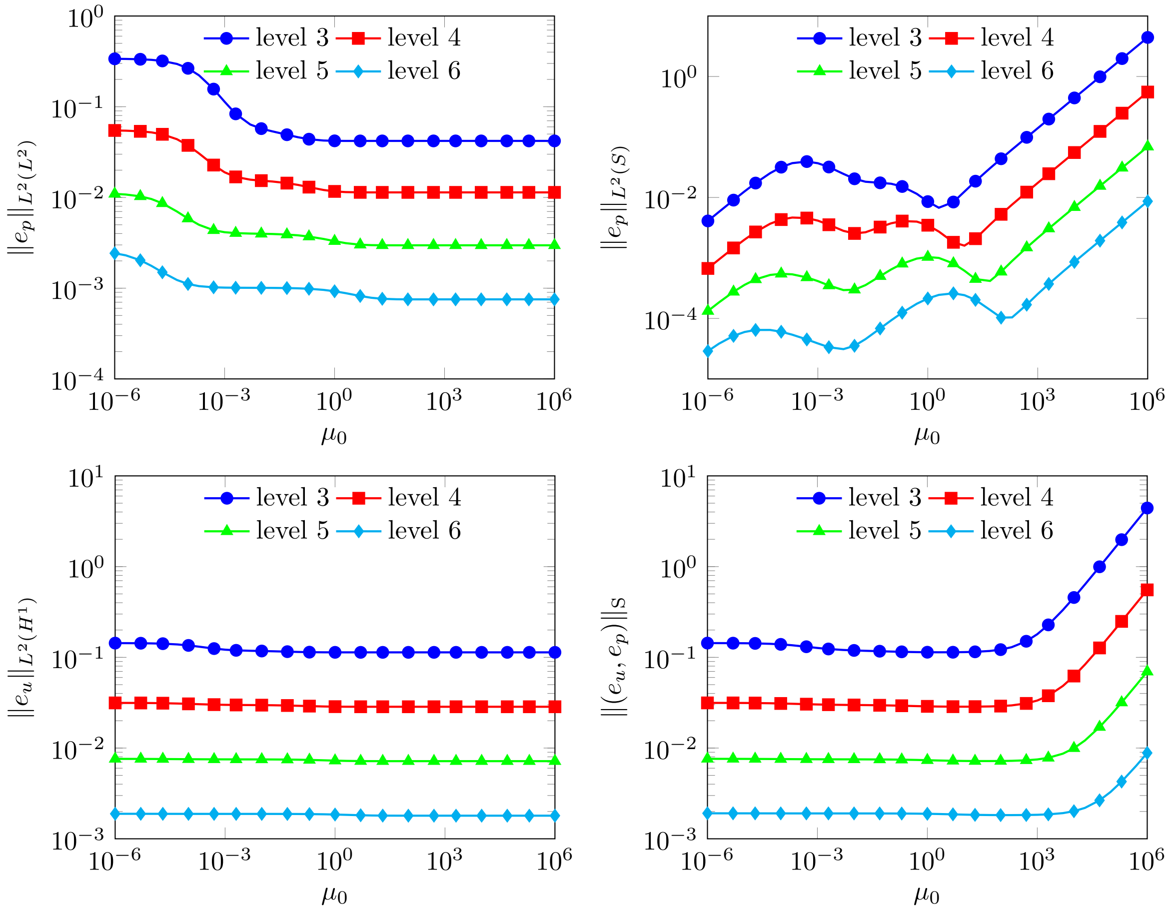


Figure 3: Example 2 with Q_2^{bubble} and $D(K) = P_1(K)$. Errors $\|e_p\|_{L^2(L^2)}$ (upper left), $\|e_p\|_{L^2(S)}$ (upper right), $\|e_u\|_{L^2(H^1)}$ (lower left), and $\|(e_u, e_p)\|_S$ (lower right) versus the constant μ_0 in $\mu_K = \mu_0 h_K^2$.

5.3. Test example 3

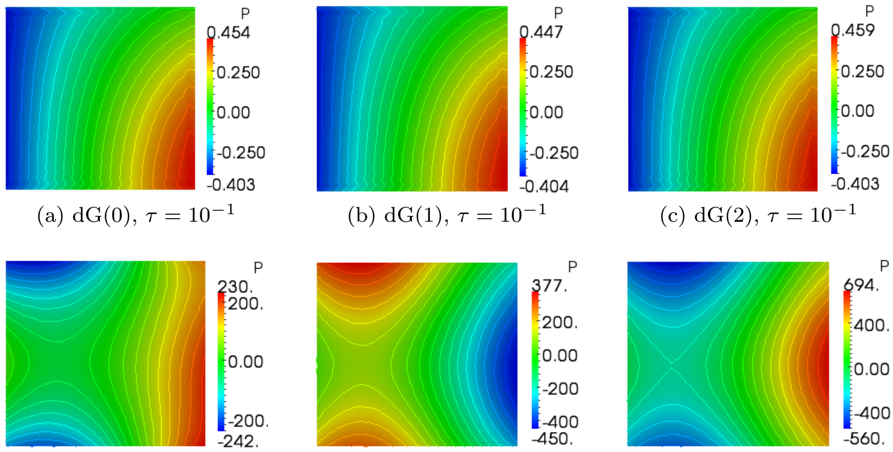
We consider the problem where the source term f, the initial condition u_0, and the boundary condition are generated by the time-independent solution

$$\begin{aligned} u_1(x, y, t) &= \sin(\pi x - 0.7)\sin(\pi y + 0.2), \\ u_2(x, y, t) &= \cos(\pi x - 0.7)\cos(\pi y + 0.2), \\ p(x, y, t) &= \Big(\sin(x)\cos(y) + \big(\cos(1) - 1\big)\sin(1)\Big). \end{aligned}$$

This problem has been already studied numerically for finite element discretizations in [16].

The aim of this example is to study the behavior of stabilized finite element methods in combination with discontinuous Galerkin method scheme in time applied to transient Stokes problems in the limit of small time step length. To this end, the numerical tests are performed using the methods dG(k), $k = 0, 1, 2$, with time step lengths $\tau = 10^{-1}$ and $\tau = 10^{-6}$. A uniform grid consisting of 32×32 squares was used in the simulations. The numerical tests have used $Y_h = Q_r^{\text{bubble}}$ with $D(K) = P_{r-1}(K)$ for all $K \in \mathcal{T}_h$, $r = 1, 2, 3$, and stabilization parameters $\mu_K = 0.1 h_K^2$.

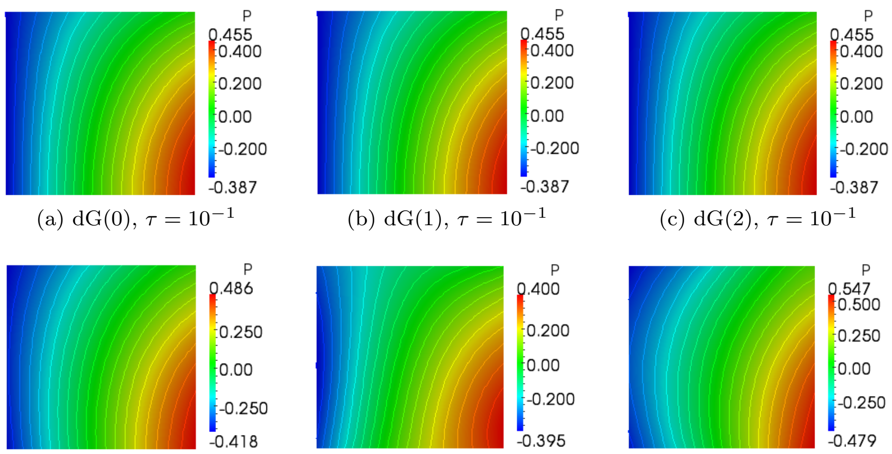
The approximate pressure $p_{h,\tau}$ is plotted in Fig. 4 for $r = 1$ after a single time step of dG(0), dG(1), and dG(2), respectively. Fig. 4a–4c show almost no substantial variation in the approximated solution $p_{h,\tau}$



(a) dG(0), $\tau = 10^{-1}$ (b) dG(1), $\tau = 10^{-1}$ (c) dG(2), $\tau = 10^{-1}$

(d) dG(0), $\tau = 10^{-6}$ (e) dG(1), $\tau = 10^{-6}$ (f) dG(2), $\tau = 10^{-6}$

Figure 4: Example 3 with Q_1^{bubble} and $D(K) = P_0(K)$. Pressure approximations after a single time step with $\tau = 10^{-1}$ (top row) and $\tau = 10^{-6}$ (bottom row).



(a) dG(0), $\tau = 10^{-1}$ (b) dG(1), $\tau = 10^{-1}$ (c) dG(2), $\tau = 10^{-1}$

(d) dG(0), $\tau = 10^{-6}$ (e) dG(1), $\tau = 10^{-6}$ (f) dG(2), $\tau = 10^{-6}$

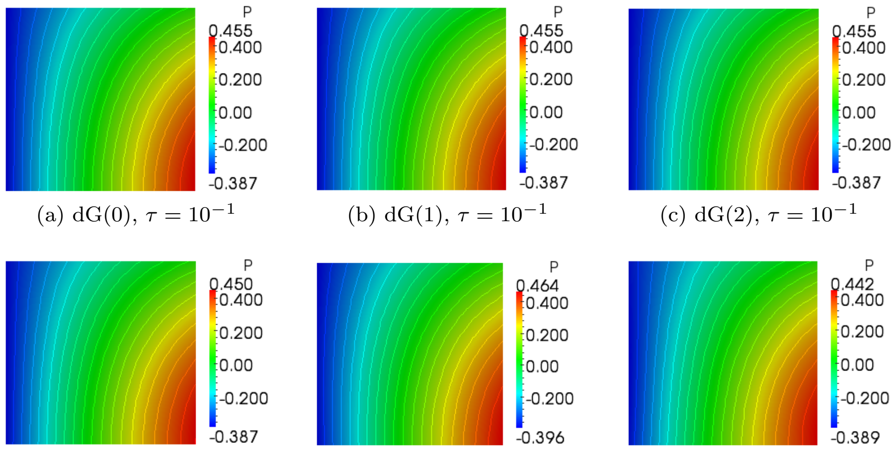
Figure 5: Example 3 with Q_2^{bubble} and $D(K) = P_1(K)$. Pressure approximations after a single time step with $\tau = 10^{-1}$ (top row) and $\tau = 10^{-6}$ (bottom row).

for the time step length $\tau = 10^{-1}$. However, reducing the time step length to $\tau = 10^{-6}$, the pressure approximation converge to a solution that is not an approximation of the exact solution.

The approximated pressure for $r = 2$ and $r = 3$ is plotted in Fig. 5 and 6, respectively. Using the large time step length $\tau = 10^{-1}$ gives also for the higher order spatial discretizations discrete pressure solutions which show almost no deviations. Moreover, one step of dG(0), dG(1), and dG(2) with the small time step length $\tau = 10^{-6}$ leaves the fully discrete pressure solution $p_{h,\tau}$ essentially unchanged. However, a small deviation can be seen for dG(2) with $\tau = 10^{-6}$, see Fig. 5f.

Hence, we conclude that the pressure approximation $p_{h,\tau}$ obtained for the first order spatial approximation starts to deviate from the exact solution for small time step lengths. However, the numerical solutions for the higher order approximations ($r = 2, 3$) in space remain remarkably stable even in the case of small time step lengths.

The obtained numerical results are in agreement with the theoretical predictions given in Thm. 12 since the upper bound in the pressure estimates is proportional to $h^r/\tau_{\min}$. This means that the effect of the instability in the limit of small time step length is reduced for higher order elements in space.



(a) dG(0), $\tau = 10^{-1}$ (b) dG(1), $\tau = 10^{-1}$ (c) dG(2), $\tau = 10^{-1}$

(d) dG(0), $\tau = 10^{-6}$ (e) dG(1), $\tau = 10^{-6}$ (f) dG(2), $\tau = 10^{-6}$

Figure 6: Example 3 with Q_3^{bubble} and $D(K) = P_2(K)$. Pressure approximations after a single time step with $\tau = 10^{-1}$ (top row) and $\tau = 10^{-6}$ (bottom row).

6. Conclusion

We have considered discontinuous Galerkin time discretizations applied to transient Stokes problems where finite element approximations based on equal-order interpolation were used. Optimal error estimates have been proven for the semi-discretization in space. For the fully discrete problem, the unconditional stability of the discrete velocity solution was shown. The stability bound for the L^2-norm of the pressure depends on the inverse time step length but is independent of the spatial mesh size h. For the velocity, an optimal error estimate in space and time was given. The obtained error bound for the pressure shows an instability in the limit of small time step length. However, the impact is reduced when higher order approximation in space is used. This theoretical behavior is also observed in Example 3.

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