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Existence of weak solutions for Cahn-Hilliard systems coupled with elasticity and damage

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Abstract

A typical phase field approach for describing phase separation and coarsening phenomena in alloys is the Cahn-Hilliard model. This model has been generalized to the so-called Cahn-Larché system by combining it with elasticity to capture non-neglecting deformation phenomena, which occurs during phase separation and coarsening processes in the material. Evolving microstructures such as phase separation and coarsening processes have a strong influence on damage initiation and propagation in alloys. In order to account for damage effects we develop the existing framework of Cahn-Hilliard and Cahn-Larché systems by incorporating an internal damage variable of local character in the sense that damage of a material point is influenced by its local surrounding. The damage process is described by a unidirectional evolution inclusion for the internal variable.

For Cahn-Hilliard and Cahn-Larché systems coupled with rate-dependent damage processes, we establish a suitable notion of weak solutions. We prove existence of weak solutions of the introduced systems. The result is based on a time-incremental minimization problem for a regularized model with constraints due to the unidirectionality of the damage. By passing to the limit in the regularized version we show existence of weak solutions of the introduced Cahn-Hilliard and Cahn-Larché systems coupled with damage.

1 Introduction

Due to the ongoing miniaturization in the area of micro-electronics the demands on strength and lifetime of the materials used is considerably rising, while the structural size is continuously being reduced. Materials, which enable the functionality of technical products, change the microstructure over time. Phase separation and coarsening phenomena take place and the complete failure of electronic devices like motherboards or mobile phones often results from micro–cracks in solder joints.

Solder joints, for instance, are essential components in electronic devices since they form the electrical and the mechanical bond between electronic components like micro–chips and the circuit–board. The Figures 1 and 2 illustrate the typical morphology in the interior of solder materials. At high temperatures, one homogeneous phase consisting of different components of the alloy is energetically favourable. If the temperature is decreased below a critical value a fine microstructure of two or more phases (different compositions of the components of the material) arises on a very short time scale. The formation of microstructures, also called phase separation or spinodal decomposition, take place to reduce the bulk chemical free energy. Then coarsening phenomena occur, which are mainly driven by decreasing interfacial energy. Due to the misfit of the crystal lattices, the different heat expansion coefficients and the different elastic moduli of the components, very high mechanical stresses occur preferably at the interfaces of the phases. These stress concentrations initiate the nucleation of micro–cracks, whose propagation finally can lead to the failure of the whole electronic device.

The knowledge of the mechanisms inducing phase separation, coarsening and damage phenomena is of great importance for technological applications. A uniform distribution of the original materials is aimed to guarantee evenly distributed material properties of the sample. For instance, mechanical properties, such as the strength and the stability of the material, depend on how finely regions of the original materials are mixed. The control of the evolution of the microstructure and therefore of the lifetime of materials relies on the ability to understand phase separation, coarsening and damage processes. This shows the importance of developing reliable mathematical models to describe such effects.

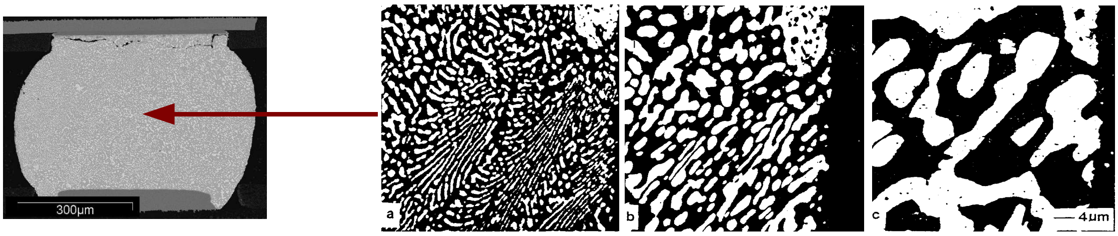

Figure 1: Left: Solder ball and micro–structural coarsening in eutectic Sn–Pb; Right: a) directly after solidification, b) after 3 hours, and c) after 300 hours [HCW91];

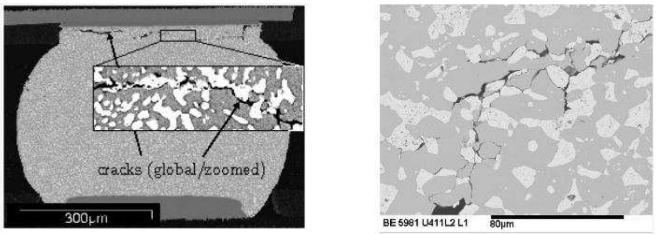

Figure 2: Initiation and propagation along the phase boundary [FBFD06].

In the mathematical literature, coarsening and damage processes are treated in general separated. Phase separation and coarsening phenomena are usually described by phase–field models of Cahn-Hilliard type. The evolution is modeled by a parabolic diffusion equation for the phase fractions. To include elastic effects, resulting from stresses caused by different elastic properties of the phases, Cahn-Hilliard systems are coupled with an elliptic equation, describing the quasi-static balance of forces. Such coupled Cahn-Hilliard systems with elasticity are also called Cahn-Larché systems. Since in general the mobility, stiffness and surface tension coefficients depend on the phases (see for instance [BDM07] and [BDDM07] for the explicite structure deduced by the embedded atom method), the mathematical analysis of the coupled problem is very complex. Existence results were derived for special cases in [Gar00, CMP00, BP05] (constant mobility, stiffness and surface tension coefficients), in [BCD$^+$02] (concentration dependent mobility, two space dimensions) and in [PZ08] in an abstract measure-valued setting (concentration dependent mobility and surface tension tensor). For numerical results and simulations we refer [Wei01, Mer05, BM10].

The mathematical investigation of models for damage evolution in elastic materials has started in the last ten years. In the simplest case, the damage variable is a scalar function and describes the local accumulation of damage in the body. The damage process is typically modeled as a unidirectional evolution, which means that damage can increase, but not decrease. Based on the model developed in [FN96], the damage evolution is described by the balance of force equation which is coupled with a unidirectional parabolic [BSS05, FK09, Gia05] or rate–independent [MR06, MRZ10] evolution inclusion for the damage variable. The models studied in [FK09, MR06, Gia05] also include the effect that the applied forces have to pass over a threshold before the damage starts to increase.

In this work we introduce a mathematical model describing both phenomena, phase separation/coarsening and damage processes, in a *unifying* model. We focus on the analytical modeling on the meso– and macroscale. To this end we couple phase–field models of Cahn-Larché type with damage models. The evolution system consists of a balance of force equation which is coupled with a parabolic evolution equation for the phase fractions and a unidirectional evolution inclusion for the damage variable. The evolution inclusion includes the phenomenon that a threshold for the loads has to be passed before the damage increases.

The main aim of the present work is to show existence of weak solutions of the introduced model for rate-dependent damage processes. To this end we first study the model with regularization terms. A crucial step has been to establish a suitable notion of weak solutions. The existence result is based on a time–incremental minimization problem for the regularized model with constraints due to the unidirectionality of the damage. The major task has been to prove convergence of the time incremental solutions for the regularized model when the discretization fineness tends to zero. In this context, several approximation results have been established to handle the damage evolution inclusion. The main results are stated in Sections 5.1 and 5.2, see Theorems 5.4 and 5.6.

To the best of our knowledge, phase separation processes coupled with damage are not studied yet in the mathematical literature. However, promising simulations were carried out in the context of phase field models of Cahn-Hilliard and Cahn-Larché type, see [USG07, GUaMM$^+$07].

The paper is organized as follows: We start with introducing a phase field model of Cahn-Larché type coupled with damage, cf. Section 2. Then we state some assumptions and preliminary results for this model, see Section 3 and Section 4. In Section 5, we establish a suitable notation for weak formulations of solutions for the introduced model and a regularized version of the model and state the main results. Section 6.1 is devoted to the existence proof for the regularized Cahn-Larché system coupled with damage. Finally, we pass to the limit in the regularized version, which shows the existence of weak solutions of the original model, see Section 6.2.

2 Model

We consider a material of two components occupying a bounded domain $\Omega \subseteq \mathbb{R}^3$. The state of the system at a fixed time point is specified by a triple $q = (u, c, z)$. The displacement field $u : \Omega \to \mathbb{R}^3$ determines the current position $x + u(x)$ of an undeformed material point x. Throughout this paper we will work with the linearized strain tensor $e(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$, which is an adequate assumption only when small strains occur in the material.

However, this assumption is justified for phase-separation processes in alloys since the deformation usually has a small gradient. The function $c : \Omega \to \mathbb{R}$ is a phase-field variable where c indicates a scaled concentration difference of the two components. To account for damage effects, we choose an isotropic damage parameter $z : \Omega \to \mathbb{R}$. This parameter can be considered as the reduction of the effective volume of the material due to void nucleation, growth, and coalescence. No damage at a material point $x \in \Omega$ is described by $z(x) = 1$, whereas $z(x) = 0$ stands for a completely damaged material point $x \in \Omega$. We require that even a damaged material can store a small amount of elastic energy. Plastic effects are not considered in our model. One mathematical challenge is located in the *unidirectionality* of damage processes. In other words, z is monotonically decreasing and allowed to jump with respect to the time variable for any material point. This restriction forbids self-healing processes in the material.

2.1 Energies and evolutionary equations

Here we qualify our model formally and postpone a rigorous treatment to Section 5. The presented model is based on two functionals, i.e. a generalized Ginzburg-Landau free energy functional $\mathcal{E}$ and a damage dissipation potential $\mathcal{R}$. The free energy density φ of the system is given by

$$\varphi(e,c,\nabla c,z,\nabla z) := \frac{\gamma}{2}|\nabla c|^2 + \frac{\delta}{p}|\nabla z|^p + W_{\mathrm{ch}}(c) + W_{\mathrm{el}}(e,c,z), \qquad \gamma,\delta > 0, \tag{1}$$

where the gradient terms penalize spatial changes of the variables c and z, W_{ch} denotes the chemical energy density and W_{el} is the elastically stored energy density accounting for elastic deformations and damage effects. For simplicity of notation we set $\gamma = \delta = 1$.

The *chemical free energy density* W_{ch} has usually the form of a double well potential for a two phase system. For a rigorous treatment we only need the assumption (GC6), see Section 3.1. Hence, in particular, classical ansatzes such as

$$W_{\mathrm{ch}} = (1-c^2)^2$$

fit in our framework.

The *elastically stored energy density* $\hat{W}_{\mathrm{el}}$ due to stresses and strains, which occur in the material, is typically of quadratic form, i.e.

$$\hat{W}_{\mathrm{el}}(c,e) = \frac{1}{2}\big(e - e^*(c)\big) : \mathbb{C}(c)\big(e - e^*(c)\big). \tag{2}$$

Here, $e^*(c)$ denotes the *eigenstrain*, which is usually linear in c and $\mathbb{C}(c)$ is the elasticity tensor, which is symmetric and positive definite. If the elasticity tensor does not depend on the concentration, i. e. $\mathbb{C}(c) = \mathbb{C}$, we refer to *homogeneous* elasticity.

To incorporate the effect of damage on the elastic response of the material, $\hat{W}_{\mathrm{el}}$ is replaced by

$$W_{\mathrm{el}} = (\Phi(z) + \tilde{\eta})\, \hat{W}_{\mathrm{el}}, \tag{3}$$

where $\Phi : [0,1] \to \mathbb{R}_+$ is a continuous and monotonically increasing function with $\Phi(0) = 0$ and $\tilde{\eta} > 0$ is a small value. The small value $\tilde{\eta} > 0$ in (3) is introduced for analytical reasons, see for instance (GC1).

Rigorous results in the present work are obtained under certain growth conditions for the elastic energy density W_{el}, see Section 3.1. These conditions are, however, only satisfied for W_{el} as in (3) in the case of homogeneous elasticity or if $e^*(c)$ does not depend linearly on c.

The overall stored energy $\mathcal{E}$ of Ginzburg-Landau type has the following structure:

$$\begin{aligned} \mathcal{E}(u,c,z) &:= \tilde{\mathcal{E}}(u,c,z) + \int_\Omega \iota_{[0,\infty)}(z)\,\mathrm{d}x, \\ \tilde{\mathcal{E}}(u,c,z) &:= \int_\Omega \varphi(e(u),c,\nabla c,z,\nabla z)\,\mathrm{d}x. \end{aligned} \tag{4}$$

Here, $\iota_{[0,\infty)}$ signifies the indicator function on the set $[0,\infty)$. We assume that the energy dissipation for the damage process is triggered by a dissipation potential $\mathcal{R}$ of the form

$$\begin{aligned} \mathcal{R}(\dot{z}) &:= \tilde{\mathcal{R}}(\dot{z}) + \int_\Omega \iota_{(-\infty,0]}(\dot{z})\,\mathrm{d}x, \\ \tilde{\mathcal{R}}(\dot{z}) &:= \int_\Omega -\alpha\dot{z} + \frac{1}{2}\beta\dot{z}^2\,\mathrm{d}x \text{ for } \alpha > 0 \text{ and } \beta \geq 0. \end{aligned} \tag{5}$$

If $\beta > 0$ then the damage process is referred to as *rate-dependent* while in the case $\beta = 0$ the process is called *rate-independent*. We refer for rate-independent processes to [EM06, MT99, MR06, MRZ10, Rou10] and in particular to [Mie05] for a survey.

The governing evolutionary equations for a system state $q = (u, c, z)$ can be expressed by virtue of the functionals (4) and (5). The evolution is driven by the following elliptic-parabolic system of differential equations and differential inclusion:

$$\begin{aligned}
&\textit{Diffusion}: && \partial_t c = -\mathrm{div}(J(u, c, z)), && \text{(6a)}\\
&\textit{Mechanical equilibrium}: && \mathrm{div}(\sigma(e(u), c, z)) = 0, && \text{(6b)}\\
&\textit{Damage evolution}: && 0 \in \mathrm{d}_z\mathcal{E}(u, c, z) + \mathrm{d}_{\dot z}\mathcal{R}(\partial_t z), && \text{(6c)}
\end{aligned}$$

where $\sigma(e, c, z) := \partial_e W_{\mathrm{el}}(e, c, z)$ denotes the elastic stress tensor and $J(u, c, z)$ designates the diffusion flux given by $-\nabla\mu$ with the chemical potential $\mu := \mathrm{d}_c\mathcal{E}(u, c, z)$. Equation (6a) is a fourth order quasi-linear parabolic equation of Cahn-Hilliard type and describes the phase separation process for the concentration c while the elliptic equation (6b) constitutes a quasi-static equilibrium for u. This means physically that we neglect kinetic energies and instead assume that mechanical equilibrium is attained at any time. The doubly nonlinear differential inclusion (6c) specifies the flow rule of the damage profile z according to the constraints $0 \leq z \leq 1$ and $\partial_t z \leq 0$ (in space and time). The inclusion (6c) has to be read in terms of generalized sub-differentials (*Clarke sub-differentials*).

We choose Dirichlet conditions for the displacements u on a subset Γ of the boundary $\partial\Omega$ with $\mathcal{H}^{n-1}(\Gamma) > 0$. Let $b : [0, T] \times \Gamma \to \mathbb{R}^n$ be a function which defines the displacements on Γ on a fixed chosen time interval $[0, T]$. The imposed boundary and initial conditions and constraints are as follows:

$$\begin{aligned}
&\textit{Boundary displacements}: && u(t) = b(t) \text{ on } \Gamma \text{ for all } t \in [0, T], && \text{(BC1)}\\
&\textit{Initial concentration}: && c(0) = c^0 \text{ in } \Omega, && \text{(BC2)}\\
&\textit{Initial damage}: && z(0) = z^0 \leq 1 \text{ in } \Omega, && \text{(BC3)}\\
&\textit{Mass conservation}: && \int_\Omega c(t) - c^0 \,\mathrm{d}x = 0 \text{ for all } t \in [0, T]. && \text{(BC4)}
\end{aligned}$$

Moreover, we use Neumann boundary conditions for the remaining variables on (parts of) the boundary:

$$\begin{aligned}
&\sigma \cdot \nu = 0 \text{ on } \partial\Omega \setminus \Gamma, && \text{(BC5)}\\
&\nabla\mu(t) \cdot \nu = 0 \text{ on } \partial\Omega, && \text{(BC6)}\\
&\nabla c(t) \cdot \nu = 0 \text{ on } \partial\Omega, && \text{(BC7)}\\
&\nabla z(t) \cdot \nu = 0 \text{ on } \partial\Omega, && \text{(BC8)}
\end{aligned}$$

where ν stands for the outer unit normal to $\partial\Omega$.

We like to mention that the mass conservation follows already from the diffusion equation (6a) and (BC7).

3 Assumptions and Notation

3.1 General assumptions

In this section we collect all assumptions and constants which are used for a rigorous analysis in the subsequent sections:

(i) *General assumptions.* Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with Lipschitz boundary where the space dimension n is not greater than 3. The exponent p in equation (1) has to be strictly greater than n and the factor β greater than 0. The functions $W_{\mathrm{el}} : \mathbb{R}^{n\times n}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}^+$ and $W_{\mathrm{ch}} : \mathbb{R}\to\mathbb{R}^+$ are assumed to be continuously differentiable and W_{el} satisfies the symmetric condition $W_{\mathrm{el}}(e, c, z) = W_{\mathrm{el}}(e^t, c, z)$ for all $e \in \mathbb{R}^{n\times n}$ and $c, z \in \mathbb{R}$. The value $T \in (0, \infty)$ specifies the time interval $[0, T]$ of interest. Further, $C > 0$ always denotes a constant, which may vary from estimate to estimate.

(ii) *Convexity and growth assumptions.* The function W_{el} is assumed to satisfy the following estimates:

$$\eta|e_1 - e_2|^2 \leq (\partial_e W_{\mathrm{el}}(e_1, c, z) - \partial_e W_{\mathrm{el}}(e_2, c, z)) : (e_1 - e_2), \tag{GC1}$$
$$W_{\mathrm{el}}(e, c, z) \leq C(|e|^2 + |c|^2 + 1), \tag{GC2}$$
$$|\partial_e W_{\mathrm{el}}(e_1 + e_2, c, z)| \leq C(W_{\mathrm{el}}(e_1, c, z) + |e_2| + 1), \tag{GC3}$$
$$|\partial_c W_{\mathrm{el}}(e, c, z)| \leq C(|e| + |c|^2 + 1), \tag{GC4}$$
$$|\partial_z W_{\mathrm{el}}(e, c, z)| \leq C(|e|^2 + |c|^2 + 1) \tag{GC5}$$

for arbitrary $c \in \mathbb{R}$ and $z \in [0, 1]$, symmetric $e, e_1, e_2 \in \mathbb{R}^{n\times n}$ and fixed constants $\eta, C > 0$. The chemical energy density function W_{ch} is of the type

$$|\partial_c W_{\mathrm{ch}}(c)| \leq \hat{C}(|c|^{2^\star/2} + 1) \tag{GC6}$$

for some constant $\hat{C} > 0$. For dimension $n = 3$ the constant $2^\star$ denotes the Sobolev critical exponent given by $\frac{2n}{n-2}$. In the lower dimensional cases $n < 3$, the constant $2^\star$ can be an arbitrary large number in this context.

(iii) *Boundary displacements.* Let Γ be a subset of $\partial\Omega$ with $\mathcal{H}^{n-1}(\Gamma) > 0$.
For further considerations, we assume that the boundary displacement $b : [0, T] \times \Gamma \to \mathbb{R}^n$ may be extended by $\hat{b} \in W^{1,1}([0, T]; W^{1,\infty}(\Omega; \mathbb{R}^n))$ such that $b(t)|_\Gamma = \hat{b}(t)|_\Gamma$ in the sense of traces for a.e. $t \in [0, T]$. We will also write b instead of $\hat{b}$.

Remark 3.1 *Conditions* (GC1), (GC2) *and* (GC3) *imply the following estimates*

$$|\partial_e W_{\mathrm{el}}(e, c, z)| \leq C(|e| + |c|^2 + 1), \tag{11a}$$
$$\eta|e|^2 - C(|c|^4 + 1) \leq W_{\mathrm{el}}(e, c, z) \tag{11b}$$

for some appropriate constants $\eta > 0$ and $C > 0$, cf. [Gar00, Section 3.2] for (11b).

3.2 Solution spaces

In this section we define the spaces where the solution curves of our evolutionary problem will be constructed. First of all, we define the space $\mathcal{Q}$ fulfilling certain regularity requirements which will be used for the limit problem:

$$\mathcal{Q} := \left\{ \begin{array}{l} u \in L^\infty([0, T]; H^1(\Omega; \mathbb{R}^n)), \\ c \in L^\infty([0, T]; H^1(\Omega)), \\ z \in L^\infty([0, T]; W^{1,p}(\Omega)) \\ \quad \cap\, \mathrm{AC}^2([0, T]; L^2(\Omega)) \end{array} \quad \text{with} \quad \begin{array}{l} u(t)|_\Gamma = b(t)|_\Gamma \text{ as traces for a.e. } t \in [0, T], \\ 0 \leq z(t) \leq 1 \text{ a.e. in } \Omega \text{ for all } t \in [0, T], \\ z \text{ monotonically decreasing with respect to } t \end{array} \right\}.$$

Based on $\mathcal{Q}$, the set of admissible solutions of the viscous problem (see Section 5) is

$$\mathcal{Q}^{\mathrm{v}} := \big\{ q = (u, c, z) \in \mathcal{Q} \,|\, c \in \mathrm{AC}^2([0, T]; L^2(\Omega)) \text{ and } \nabla u \in L^\infty([0, T]; L^4(\Omega)) \big\},$$

where AC^2 designates the space of absolutely continuous functions with quadratically integrable derivatives.

It will be convenient for the variational formulation to define Sobolev spaces with functions taking only non-negative and non-positive values, respectively, and Sobolev spaces consisting of functions with vanishing traces on the boundary Γ:

$$\begin{aligned}
W_+^{1,r}(\Omega) &:= \big\{\zeta \in W^{1,r}(\Omega)\,\big|\,\zeta \geq 0 \text{ a.e. in } \Omega\big\},\\
W_-^{1,r}(\Omega) &:= \big\{\zeta \in W^{1,r}(\Omega)\,\big|\,\zeta \leq 0 \text{ a.e. in } \Omega\big\},\\
W_\Gamma^{1,r}(\Omega;\mathbb{R}^n) &:= \big\{\zeta \in W^{1,r}(\Omega;\mathbb{R}^n)\,\big|\,\zeta|_\Gamma = 0 \text{ in the sense of traces}\big\}
\end{aligned}$$

for $r \in [1, \infty]$.

In Cahn-Hilliard systems, the integral mean value of the concentration variable c is conserved and its time derivative still has $(H^1(\Omega))^*$-regularity. In this context we will work in the following spaces:

$$\begin{aligned}
H_0 &:= \left\{\zeta \in H^1(\Omega)\,\Big|\,\int_\Omega \zeta\,\mathrm{d}x = 0\right\},\\
\tilde{H}_0 &:= \left\{\zeta \in (H^1(\Omega))^*\,\Big|\,\langle\zeta, \mathbf{1}\rangle_{(H^1)^*\times H^1} = 0\right\}.
\end{aligned}$$

This permits us to define the operator $(-\Delta)^{-1} : \tilde{H}_0 \to H_0$ as the inverse of the operator $-\Delta : H_0 \to \tilde{H}_0$, $u \mapsto \langle\nabla u, \nabla\cdot\rangle_{L^2(\Omega)}$. The space $\tilde{H}_0$ will be endowed with the scalar product $\langle u, v\rangle_{\tilde{H}_0} := \langle\nabla(-\Delta)^{-1}u, \nabla(-\Delta)^{-1}v\rangle_{L^2(\Omega)}$.

4 Preliminaries

This section is devoted to some frequently used approximation features in this paper. The expression $B_R(K)$ designates the open neighborhood with width $R > 0$ of a subset $K \subseteq \mathbb{R}^n$. Whenever we consider the zero set of a function $\zeta \in W^{1,p}(\Omega)$ for $p > n$ denoted in the following by $\{\zeta = 0\}$ we mean $\{x \in \overline{\Omega}\,|\,\zeta(x) = 0\}$ by taking the embedding $W^{1,p}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ into account. We adapt the convention that for two given functions $\zeta, \xi \in L^1([0,T]; W^{1,p}(\Omega))$ the inclusion $\{\zeta = 0\} \supseteq \{\xi = 0\}$ is an abbreviation for $\{\zeta(t) = 0\} \supseteq \{\xi(t) = 0\}$ for a.e. $t \in [0,T]$.

Lemma 4.1 (Approximation Ia) *Let $p > n$ and $f, \zeta \in W_+^{1,p}(\Omega)$ with $\{\zeta = 0\} \supseteq \{f = 0\}$. Furthermore, let $\{f_M\}_{M\in\mathbb{N}} \subseteq W_+^{1,p}(\Omega)$ be a sequence with $f_M \rightharpoonup f$ in $W^{1,p}(\Omega)$ as $M \to \infty$. Then there exist sequences $\{\zeta_M\}_{M\in\mathbb{N}} \subseteq W_+^{1,p}(\Omega)$ and constants $\nu_M > 0$ such that*

(i) $\zeta_M \to \zeta$ in $W^{1,p}(\Omega)$ as $M \to \infty$,

(ii) $\zeta_M \leq \zeta$ a.e. in Ω for all $M \in \mathbb{N}$,

(iii) $\nu_M\zeta_M \leq f_M$ a.e. in Ω for all $M \in \mathbb{N}$.

Proof. Without loss of generality we may assume $\zeta \not\equiv 0$ on $\overline{\Omega}$.

Let $\{\delta_k\}$ be a sequence with $\delta_k \searrow 0$ as $k \to \infty$ and $\delta_k > 0$. Define for every $k \in \mathbb{N}$ the approximation function $\tilde{\zeta}_k \in W_+^{1,p}(\Omega)$ as

$$\tilde{\zeta}_k := [\zeta - \delta_k]^+,$$

where $[\cdot]^+$ stands for $\max\{0,\cdot\}$. Let $0<\alpha<1-\frac{n}{p}$ be a fixed constant. Then $\tilde{\zeta}_k \in C^{0,\alpha}(\overline{\Omega})$ due to $W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$. Furthermore, set the constant R_k, $k\in\mathbb{N}$, to

$$R_k := \Big(\delta_k/\|\zeta\|_{C^{0,\alpha}(\overline{\Omega})}\Big)^{1/\alpha} > 0.$$

It follows $\{\tilde{\zeta}_k = 0\} \supseteq \overline{\Omega}\cap B_{R_k}(\{\zeta=0\}) \supseteq \overline{\Omega}\cap B_{R_k}(\{f=0\})$. Without loss of generality we may assume $\overline{\Omega}\setminus B_{R_k}(\{f=0\}) \neq \emptyset$ for all $k\in\mathbb{N}$. Furthermore, there exists a strictly increasing sequence $\{M_k\}\subseteq\mathbb{N}$ such that we find for all $k\in\mathbb{N}$:

$$f_M \geq \eta_k/2 \text{ a.e. on } \overline{\Omega}\setminus B_{R_k}(\{f=0\}) \text{ for all } M\geq M_k$$

with $\eta_k := \inf\{f(x)\,|\,x\in\overline{\Omega}\setminus B_{R_k}(\{f=0\})\} > 0$, $k\in\mathbb{N}$, (note that $f_M\to f$ in $C^{0,\alpha}(\overline{\Omega})$ as $M\to\infty$). This implies $\tilde{\nu}_k\tilde{\zeta}_k \leq f_M$ a.e. on $\overline{\Omega}$ for all $M\geq M_k$ by setting $\tilde{\nu}_k := \eta_k/(2\|\zeta\|_{L^\infty(\Omega)}) > 0$. The claim follows with $\zeta_M := 0$ and $\nu_k := 1$ for $M\in\{1,\ldots,M_1-1\}$ and $\zeta_M := \tilde{\zeta}_{\delta_k}$ and $\nu_M := \tilde{\nu}_k$ for each $M\in\{M_k,\ldots,M_{k+1}-1\}$, $k\in\mathbb{N}$. ∎

Lemma 4.2 (Approximation Ib) *Let $p>n$ and $q\geq 1$ and $f,\zeta\in L^q([0,T];W^{1,p}_+(\Omega))$ with $\{\zeta=0\}\supseteq\{f=0\}$. Furthermore, let $\{f_M\}_{M\in\mathbb{N}}\subseteq L^q([0,T];W^{1,p}_+(\Omega))$ be a sequence with $f_M(t)\rightharpoonup f(t)$ in $W^{1,p}(\Omega)$ as $M\to\infty$ for a.e. $t\in[0,T]$. Then there exists a sequence $\{\zeta_M\}_{M\in\mathbb{N}}\subseteq L^q([0,T];W^{1,p}_+(\Omega))$ and constants $\nu_{M,t}>0$ such that*

(i) $\zeta_M\to\zeta$ in $L^q([0,T];W^{1,p}(\Omega))$ as $M\to\infty$,

(ii) $\zeta_M\leq\zeta$ a.e. in Ω_T for all $M\in\mathbb{N}$ (in particular $\{\zeta_M=0\}\supseteq\{\zeta=0\}$),

(iii) $\nu_{M,t}\zeta_M(t)\leq f_M(t)$ a.e. in Ω for a.e. $t\in[0,T]$ and for all $M\in\mathbb{N}$.

If, in addition, $\zeta\leq f$ a.e. in Ω_T then condition (iii) can be refined to

(iii)' $\zeta_M\leq f_M$ a.e. in Ω_T for all $M\in\mathbb{N}$.

Proof. Let $\{\delta_k\}$ with $\delta_k\searrow 0$ as $k\to\infty$ and $\delta_k>0$ be a sequence and $0<\alpha<1-\frac{n}{p}$ be a fixed constant. We construct the approximations $\{\zeta_M\}\subseteq L^q([0,T];W^{1,p}_+(\Omega))$ as follows:

$$\zeta_M(t) := \sum_{k=1}^{M} \chi_{A^k_M}(t)[\zeta(t)-\delta_k]^+, \tag{12}$$

where $\chi_{A^k_M}:[0,T]\to\{0,1\}$ is defined as the characteristic function on the measurable set A^k_M given by

$$A^k_M := \begin{cases} P^k_M \setminus \Big(\bigcup_{i=k+1}^M P^i_M\Big), & \text{if } k<M,\\ P^M_M, & \text{if } k=M,\end{cases}$$

with

$$P^k_M := \Big\{t\in[0,T]\,\big|\,\overline{\Omega}\setminus B_{R_k(t)}(\{f(t)=0\})\neq\emptyset \\ \text{and } f_M(t)\geq\eta_k(t)/2 \text{ a.e. on } \overline{\Omega}\setminus B_{R_k(t)}(\{f(t)=0\})\Big\}, \tag{13}$$

where the functions $R_k, \eta_k : [0,T] \to \mathbb{R}^+$ are defined by

$$R_k(t) = \left(\delta_k / \|\zeta(t)\|_{C^{0,\alpha}(\overline{\Omega})}\right)^{1/\alpha},$$
$$\eta_k(t) = \inf\{f(t,x) \,|\, x \in \overline{\Omega} \setminus B_{R_k(t)}(\{f(t)=0\})\}.$$

Here we use the convention $R_k(t) := \infty$ for $\zeta(t) \equiv 0$. Note that $\{A_M^k\}$, $1 \le k \le M$, are by construction, pairwise disjoint.

Consider a $t \in [0,T]$ with $f_M(t) \rightharpoonup f(t)$ in $W^{1,p}(\Omega)$ and $\zeta(t) \not\equiv 0$ with $\{\zeta(t) = 0\} \supseteq \{f(t) = 0\}$. Let $K \in \mathbb{N}$ be arbitrary but large enough such that $\overline{\Omega} \setminus B_{R_K(t)}(\{f(t)=0\}) \neq \emptyset$ holds. It follows the existence of an $\tilde{M} \ge K$ with $t \in P_M^K$ for all $M \ge \tilde{M}$. Therefore, for each $M \ge \tilde{M}$ exists a $k \ge K$ such that $t \in A_M^k$, i.e. $\zeta_M(t) = [\zeta(t) - \delta_k]^+$. Thus $\zeta_M(t) \to \zeta(t)$ in $W^{1,p}(\Omega)$ as $K \to \infty$. Lebesgue's convergence theorem shows (i).
Property (ii) follows immediately from (12). It remains to show (iii). Let $M \in \mathbb{N}$ be arbitrary. If $\zeta_M(t) \equiv 0$ we set $\nu_{M,t} = 1$. Otherwise we find a unique $1 \le k \le M$ with $t \in A_M^k$ and $\zeta_M(t) = [\zeta(t) - \delta_k]^+$. This, in turn, implies the existence of a $\nu_{M,t} > 0$ with $\nu_{M,t}\zeta_M \le f_M$ (see proof of Lemma 4.1).

In the case $\zeta \le f$ we use instead of (13) the sets:

$$P_M^k := \left\{ t \in [0,T] \,\middle|\, \|f_M(t) - f(t)\|_{C^0(\overline{\Omega})} \le \delta_k \right\}.$$

With a similar argumentation $\{\zeta_M\}$ fulfills (i), (ii) and (iii)'. ■

Lemma 4.3 (Approximation II) *Let $p > n$ and $f \in L^1([0,T]; W^{1,p}(\Omega))$. Then*

$$\mathcal{L}^{n+1}\left(\left\{(t,x) \in \overline{\Omega}_T \,\middle|\, x \in B_\delta(\{f(t)=0\}) \setminus \{f(t)=0\}\right\}\right) \to 0 \ \textit{as}\ \delta \searrow 0.$$

Proof. Define the sets

$$N_{\delta,t} := \overline{\Omega}_T \cap (B_\delta(\{f(t)=0\}) \setminus \{f(t)=0\})$$

and

$$N_\delta := \{(t,x) \in \overline{\Omega}_T \,|\, x \in N_{\delta,t}\}.$$

We first show the statement

$$\mathcal{L}^n(N_{\delta,t}) \to 0 \text{ as } \delta \searrow 0 \text{ for } t \in [0,T] \backslash S, \tag{14}$$

where $S \subset [0,T]$ is a Lebesgue set of measure zero.

Assume $\lim_{\delta \searrow 0} \mathcal{L}^n(N_{\delta,t}) > 0$ as $\delta \searrow 0$ for some $t \in [0,T]\backslash S$. From the monotonicity of the sets $N_{\delta,t}$, i.e. $N_{\delta_1,t} \subseteq N_{\delta_2,t}$ whenever $\delta_1 \le \delta_2$, we get $\mathcal{L}^n\left(\bigcap_{\delta>0} N_{\delta,t}\right) = \lim_{\delta\searrow 0} \mathcal{L}^n(N_{\delta,t}) > 0$ as $\delta \searrow 0$. Let $x \in \bigcap_{\delta>0} N_{\delta,t}$. Then $\mathrm{dist}(x, \{f(t)=0\}) = 0$ and the closeness of $\{f(t)=0\}$ in $\overline{\Omega}$ (due to $f(t) \in C^0(\overline{\Omega})$) implies the contradiction $x \in \{f(t)=0\}$.

Now we show the claim. Assume $\lim_{\delta\searrow 0} \mathcal{L}^{n+1}(N_\delta) > 0$ as $\delta \searrow 0$. Then $\lim_{\delta\searrow 0} \mathcal{L}^{n+1}(N_\delta) = \mathcal{L}^{n+1}\left(\bigcap_{\delta>0} N_\delta\right) > 0$. Therefore, we find a $t \in [0,T]\backslash S$ with $\mathcal{L}^n\left(\left(\bigcap_{\delta>0} N_{\delta,t}\right)\right) > 0$. This contradicts (14). ■

5 Weak formulation and existence theorems

On the one hand, existence results for multi-phase Cahn-Larché systems without considering damage phase fields are shown in [Gar00] provided that the chemical energy density W_{ch} can be decomposed into $W_{\mathrm{ch}}^1 + W_{\mathrm{ch}}^2$ with convex W_{ch}^1 and linear growth behavior of $\partial_c W_{\mathrm{ch}}^2$ ([Gar00, Section 3.2] provides a detailed explanation). See also [Gar05a] for a survey. Logarithmic free energies are also studied in [Gar00] as well as in [Gar05b]. Further variants are investigated in [CMP00], [BP05] and [BCD$^+$02].

Purely mechanical systems with rate-independent damage processes, on the other hand, are analytically considered and reviewed for instance in [MR06] and [MRZ10]. The rate-independence enables the concept of the so-called global energetic solutions (see Remark 5.2 (i)) to such systems.

Coupling rate-independent systems with other (rate-dependent) processes (such as with inertial or thermal effects) can bring, however, serious mathematical difficulties as pointed out in [Rou10].

In our situation where the Cahn-Larché system is coupled with rate-dependent damage, we will treat our model problem analytically by a regularization method that gives better regularity property for c and integrability for u in the first instance. A passage to the limit will finally give us solutions to the original problem. In doing so, the notion of a weak solution consists of variational equalities and inequalities as well as an energy estimate, inspired by the concept of energetic solutions in the framework of rate-independent systems.

5.1 Regularization

The regularization we want to consider here is achieved by adding the term $\varepsilon\Delta\partial_t c$ to the Cahn-Hilliard equations (such regularization also occurs in [BP05]) and the 4-Laplacian $\varepsilon\mathrm{div}(|\nabla u|^2\nabla u)$ to the quasi-static equilibrium equation of the model problem. The formulation of the regularized problem for $\varepsilon > 0$ now reads as

$$\partial_t c = \Delta(-\Delta c + \partial_c W_{\mathrm{ch}}(c) + \partial_c W_{\mathrm{el}}(e(u), c, z) + \varepsilon\partial_t c), \tag{15a}$$

$$\mathrm{div}(\sigma(e(u), c, z)) + \varepsilon\mathrm{div}(|\nabla u|^2\nabla u) = 0, \tag{15b}$$

$$0 \in \mathrm{d}_z\mathcal{E}_\varepsilon(u, c, z) + \mathrm{d}_{\dot z}\mathcal{R}(\partial_t z) \tag{15c}$$

with viscous energy

$$\mathcal{E}_\varepsilon(u, c, z) := \mathcal{E}(u, c, z) + \varepsilon\int_\Omega \frac{1}{4}|\nabla u|^4\,\mathrm{d}x,$$

$$\tilde{\mathcal{E}}_\varepsilon(u, c, z) := \tilde{\mathcal{E}}(u, c, z) + \varepsilon\int_\Omega \frac{1}{4}|\nabla u|^4\,\mathrm{d}x.$$

In the following we consider $q \in \mathcal{Q}^{\mathrm{v}}$. For every $t \in [0, T]$ the equations (15a)-(15c) can be translated in a weak formulation as follows:

$$\int_\Omega (\partial_t c(t))\zeta\,\mathrm{d}x = -\int_\Omega \nabla\mu(t)\cdot\nabla\zeta\,\mathrm{d}x \tag{16}$$

for all $\zeta \in H^1(\Omega)$ and

$$\int_\Omega \mu(t)\zeta\,\mathrm{d}x = \int_\Omega \nabla c(t)\cdot\nabla\zeta + \partial_c W_{\mathrm{ch}}(c(t))\zeta + \partial_c W_{\mathrm{el}}(e(u(t)), c(t), z(t))\zeta + \varepsilon(\partial_t c(t))\zeta\,\mathrm{d}x \tag{17}$$

for all $\zeta \in H^1(\Omega)$. In the same spirit we translate (15b) as

$$\int_\Omega \partial_e W_{\mathrm{el}}(e(u(t)), c(t), z(t)) : e(\zeta) + \varepsilon|\nabla u(t)|^2\nabla u(t) : \nabla\zeta\,\mathrm{d}x = 0 \tag{18}$$

for all $\zeta \in W^{1,4}_\Gamma(\Omega;\mathbb{R}^n)$. The differential inclusion (15c) is equivalent to the variational inequality

$$\iota_{W^{1,p}_-(\Omega)}(\partial_t z(t)) - \left\langle \mathrm{d}_z\tilde{\mathcal{E}}_\varepsilon(q(t)) + r(t) + \mathrm{d}_{\dot z}\tilde{\mathcal{R}}(\partial_t z(t)), \zeta - \partial_t z(t)\right\rangle \le \iota_{W^{1,p}_-(\Omega)}(\zeta) \tag{19}$$

for all $\zeta \in W^{1,p}(\Omega)$ (see (5) for the definition of $\tilde{\mathcal{R}}$) and an $r(t)$ lying in the Fréchet normal cone $N_{\mathrm{F}}(W^{1,p}_+(\Omega); z(t)) \subseteq (W^{1,p}(\Omega))^*$, i.e. $r(t)$ satisfies

$$\iota_{W^{1,p}_+(\Omega)}(z(t)) + \langle r(t), \zeta - z(t)\rangle \le \iota_{W^{1,p}_+(\Omega)}(\zeta) \tag{20}$$

for all $\zeta \in W^{1,p}(\Omega)$.

Since $q \in \mathcal{Q}^{\mathrm{v}}$, property (19) is for test-functions $\zeta \in W^{1,p}_-(\Omega)$ equivalent to the following inequality system

$$0 \le \left\langle \mathrm{d}_z\tilde{\mathcal{E}}_\varepsilon(q(t)) + r(t) + \mathrm{d}_{\dot z}\tilde{\mathcal{R}}(\partial_t z(t)), \zeta\right\rangle, \tag{21a}$$

$$0 \le -\left\langle \mathrm{d}_z\tilde{\mathcal{E}}_\varepsilon(q(t)) + r(t) + \mathrm{d}_{\dot z}\tilde{\mathcal{R}}(\partial_t z(t)), \partial_t z(t)\right\rangle. \tag{21b}$$

In general, due to the lack of regularity of q, (21b) cannot be justified rigorously. To overcome this difficulty, we use a formal calculation coming from energetic formulations introduced in [MT99].

Proposition 5.1 (Energetic characterization) *Let $q \in \mathcal{Q}^{\mathrm{v}} \cap C^2(\Omega_T;\mathbb{R}^n\times\mathbb{R}\times\mathbb{R})$ be a smooth solution of* (16)-(18)*. Furthermore, let $r(t) \in N_{\mathrm{F}}(W^{1,p}_+(\Omega); z(t))$ for every $t \in [0,T]$. Then*

(i) $\langle r(t), \partial_t z(t)\rangle = 0$ *for all* $t \in [0,T]$

and the following two conditions are equivalent:

(ii) (21b) *for all* $t \in [0,T]$,

(iii) for all $0 \le t_1 \le t_2 \le T$*:*

$$\begin{aligned}\mathcal{E}_\varepsilon(q(t_2)) + \int_{t_1}^{t_2} \langle \mathrm{d}_{\dot z}\tilde{\mathcal{R}}(\partial_t z), \partial_t z\rangle\,\mathrm{d}s + \int_{t_1}^{t_2}\int_\Omega |\nabla\mu|^2 + \varepsilon|\partial_t c|^2\,\mathrm{d}x\mathrm{d}s - \mathcal{E}_\varepsilon(q(t_1))\\ \le \int_{t_1}^{t_2}\int_\Omega \partial_e W_{\mathrm{el}}(e(u),c,z) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}s + \varepsilon\int_{t_1}^{t_2}\int_\Omega |\nabla u|^2\nabla u : \nabla\partial_t b\,\mathrm{d}x\mathrm{d}s\end{aligned} \tag{22}$$

Proof.

(i) The inequality $0 \le \langle r(t), \partial_t z(t)\rangle$ follows directly from (20) by putting $\zeta = z(t) - \partial_t z(t)$. The '$\ge$' - part can be shown by an approximation argument. Applying Lemma 4.1 with $f_M = z(t)$ and $f = z(t)$ and $\zeta = -\partial_t z(t)$. We obtain a sequence $\{\zeta_M\} \subseteq W^{1,p}_+(\Omega)$ and constants $\nu_M > 0$ such that

(a) $-\zeta_M \to \partial_t z(t)$ in $W^{1,p}(\Omega)$ as $M \to \infty$,

(b) $0 \le z(t) - \nu_M\zeta_M$ a.e. in Ω for all $M \in \mathbb{N}$.

Testing (20) with $\zeta = z(t) - \nu_M\zeta_M$ shows $\langle r(t), -\zeta_M\rangle \le 0$. Passing to $M \to \infty$ gives $\langle r(t), \partial_t z(t)\rangle \le 0$.

To $(iii) \Rightarrow (ii)$: We remark that (17) and (18) can be written in the following form:

$$\int_\Omega \mu(t)\zeta_1 - \varepsilon(\partial_t c(t))\zeta_1 \,\mathrm{d}x = \langle \mathrm{d}_c\tilde{\mathcal{E}}_\varepsilon(q(t)), \zeta_1\rangle, \tag{23a}$$

$$\langle \mathrm{d}_u\tilde{\mathcal{E}}_\varepsilon(q(t)), \zeta_2\rangle = 0, \tag{23b}$$

for all $t \in [0,T]$, all $\zeta_1 \in H^1(\Omega)$ and all $\zeta_2 \in W^{1,4}_\Gamma(\Omega;\mathbb{R}^n)$.

Let $t_0 \in [0,T)$. It follows

$$\begin{aligned}&\frac{\mathcal{E}_\varepsilon(q(t_0+h)) - \mathcal{E}_\varepsilon(q(t_0))}{h} + \fint_{t_0}^{t_0+h}\langle \mathrm{d}_{\dot z}\tilde{\mathcal{R}}(\partial_t z), \partial_t z\rangle \,\mathrm{d}t + \fint_{t_0}^{t_0+h}\int_\Omega |\nabla\mu|^2 + \varepsilon|\partial_t c|^2 \,\mathrm{d}x\mathrm{d}t\\ &\quad\le \fint_{t_0}^{t_0+h}\int_\Omega \partial_e W_{\mathrm{el}}(e(u),c,z) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}t + \varepsilon\fint_{t_0}^{t_0+h}\int_\Omega |\nabla u|^2\nabla u : \nabla\partial_t b\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Letting $h \searrow 0$ gives

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}_\varepsilon(q(t_0)) + \langle \mathrm{d}_{\dot z}\tilde{\mathcal{R}}(\partial_t z(t_0)), \partial_t z(t_0)\rangle + \int_\Omega |\nabla\mu(t_0)|^2 + \varepsilon|\partial_t c(t_0)|^2 \,\mathrm{d}x\\ &\quad\le \int_\Omega \partial_e W_{\mathrm{el}}(e(u(t_0)),c(t_0),z(t_0)) : e(\partial_t b(t_0))\mathrm{d}x + \varepsilon\int_\Omega |\nabla u(t_0)|^2\nabla u(t_0) : \nabla\partial_t b(t_0)\,\mathrm{d}x\\ &\quad= \langle \mathrm{d}_u\tilde{\mathcal{E}}_\varepsilon(q(t_0)), \partial_t b(t_0)\rangle.\end{aligned}$$

Using the chain rule and (16)-(18) yield

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}_\varepsilon(q(t_0)) &= \underbrace{\langle \mathrm{d}_u\tilde{\mathcal{E}}_\varepsilon(q(t_0)), \partial_t u(t_0)\rangle}_{\text{apply (23b)}} + \underbrace{\langle \mathrm{d}_c\tilde{\mathcal{E}}_\varepsilon(q(t_0)), \partial_t c(t_0)\rangle}_{\text{apply (23a) and (16)}} + \langle \mathrm{d}_z\tilde{\mathcal{E}}_\varepsilon(q(t_0)), \partial_t z(t_0)\rangle\\ &= \langle \mathrm{d}_u\tilde{\mathcal{E}}_\varepsilon(q(t_0)), \partial_t b(t_0)\rangle + \int_\Omega -|\nabla\mu(t_0)|^2 - \varepsilon|\partial_t c(t_0)|^2 \,\mathrm{d}x + \langle \mathrm{d}_z\tilde{\mathcal{E}}_\varepsilon(q(t_0)), \partial_t z(t_0)\rangle.\end{aligned}$$

In consequence, property (ii) follows with (i). The case $t_0 = T$ can be derived similarly by considering the difference quotient between t_0 and $t_0 - h$.

To $(ii) \Rightarrow (iii)$: Follows from the relation $\mathcal{E}_\varepsilon(q(t_2)) - \mathcal{E}_\varepsilon(q(t_1)) = \int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}_\varepsilon(q(t))\,\mathrm{d}t$ as well as the equations (16)-(18) and (i). ∎

Remark 5.2

(i) In the rate-independent case $\beta = 0$ and for convex $\mathcal{E}_\varepsilon$ with respect to z, condition (21a) *can be characterized by a stability condition which reads as*

$$\mathcal{E}_\varepsilon(u(t),c(t),z(t)) \le \mathcal{E}_\varepsilon(u(t),c(t),\zeta) + \mathcal{R}(\zeta - z(t)) \tag{24}$$

for all $t \in [0,T]$ and all test-functions $\zeta \in W^{1,p}_+(\Omega)$. Thereby, (22) *and* (24) *give an equivalent description of the differential inclusion* (15c) *for smooth solutions. This concept of solutions are referred to as global energetic solutions and was introduced in [MT99]. We emphasize that the damage variable z in the rate-independent case $\beta = 0$ is a function of bounded variation and is allowed to exhibit jumps. For a comprehensive introduction we refer to [AFP00]. To tackle rate-dependent systems and non-convexity of $\mathcal{E}_\varepsilon$ with respect to z, we can not use formulation* (24) *(cf. [MRS09]).*

(ii) *For smooth solutions q, satisfying* (16)-(18)*, the energy inequality* (22) *and the variational inequality* (21a)*, one can even show energy balance:*

$$\mathcal{E}_\varepsilon(q(t_2)) + \int_{t_1}^{t_2} \langle \mathrm{d}_{\dot z}\tilde{\mathcal{R}}(\partial_t z), \partial_t z\rangle \,\mathrm{d}s + \int_{t_1}^{t_2}\int_\Omega |\nabla \mu|^2 + \varepsilon|\partial_t c|^2 \,\mathrm{d}x\mathrm{d}s$$
$$= \mathcal{E}_\varepsilon(q(t_1)) + \int_{t_1}^{t_2}\int_\Omega \partial_e W_{\mathrm{el}}(e(u),c,z) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}s + \varepsilon\int_{t_1}^{t_2}\int_\Omega |\nabla u|^2\nabla u : \nabla\partial_t b\,\mathrm{d}x\mathrm{d}s$$

for all $0 \le t_1 \le t_2 \le T$.

(iii) *In the case $\beta = 0$ we only know that $\partial_t z$ is a $L^1(\Omega)$-valued Radon measure and the term $\int_{t_1}^{t_2}\langle \mathrm{d}_{\dot z}\tilde{\mathcal{R}}(\partial_t z(t)), \partial_t z(t)\rangle_{L^2(\Omega)}\,\mathrm{d}t = \int_{t_1}^{t_2}\int_\Omega -\alpha\partial_t z\,\mathrm{d}x\mathrm{d}t$ in* (22) *should be read as $\int_\Omega \alpha z(t_1))\,\mathrm{d}x - \int_\Omega \alpha z(t_2)\,\mathrm{d}x$.*

This motivates the definition of a solution in the following sense:

Definition 5.3 (Weak solution - viscous problem) *A triple $q = (u,c,z) \in \mathcal{Q}^{\mathrm{v}}$ with $c(0) = c^0$ and $z(0) = z^0$ is called a weak solution of the viscous system* (15a)-(15c) *with initial-boundary data and constraints* (BC1)-(BC8) *if it satisfies the following conditions:*

(i) *integral equality*

$$\int_{\Omega_T} (\partial_t c)\zeta \,\mathrm{d}x\mathrm{d}t = -\int_{\Omega_T} \nabla\mu\cdot\nabla\zeta\,\mathrm{d}x\mathrm{d}t$$

for all $\zeta \in L^2([0,T];H^1(\Omega))$ where $\mu \in L^2([0,T];H^1(\Omega))$ is given by the integral equality

$$\int_{\Omega_T} \mu\zeta\,\mathrm{d}x\mathrm{d}t = \int_{\Omega_T} \nabla c\cdot\nabla\zeta + \partial_c W_{\mathrm{ch}}(c)\zeta + \partial_c W_{\mathrm{el}}(e(u),c,z)\zeta + \varepsilon(\partial_t c)\zeta\,\mathrm{d}x\mathrm{d}t$$

for all $\zeta \in L^2([0,T];H^1(\Omega))$;

(ii) *integral equality*

$$\int_{\Omega_T} \partial_e W_{\mathrm{el}}(e(u),c,z) : e(\zeta) + \varepsilon|\nabla u|^2\nabla u : \nabla\zeta\,\mathrm{d}x\mathrm{d}t = 0$$

for all $\zeta \in L^4([0,T];W^{1,4}_\Gamma(\Omega;\mathbb{R}^n))$;

(iii) *integral inequality*

$$0 \le \int_{\Omega_T} |\nabla z|^{p-2}\nabla z\cdot\nabla\zeta + \partial_z W_{\mathrm{el}}(e(u),c,z)\zeta - \alpha\zeta + \beta(\partial_t z)\zeta\,\mathrm{d}x\mathrm{d}t + \int_0^T \langle r(t),\zeta(t)\rangle\,\mathrm{d}t,$$

for all $\zeta \in L^p([0,T];W^{1,p}_-(\Omega)) \cap L^\infty(\Omega_T)$ where $r \in L^1(\Omega_T) \subset L^1\big([0,T];(W^{1,p}(\Omega))^*\big)$ satisfies

$$\langle r(t), \zeta - z(t)\rangle \le 0$$

for a.e. $t \in [0,T]$ and for all $\zeta \in W^{1,p}_+(\Omega)$;

(iv) energy inequality

$$\mathcal{E}_\varepsilon(q(t_2)) + \int_\Omega \alpha(z(t_1) - z(t_2))\,\mathrm{d}x + \int_{t_1}^{t_2}\int_\Omega \beta|\partial_t z|^2\,\mathrm{d}x\mathrm{d}s + \int_{t_1}^{t_2}\int_\Omega |\nabla\mu|^2 + \varepsilon|\partial_t c|^2\,\mathrm{d}x\mathrm{d}s$$
$$\leq \mathcal{E}_\varepsilon(q(t_1)) + \int_{t_1}^{t_2}\int_\Omega \partial_e W_{\mathrm{el}}(e(u),c,z) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}s + \varepsilon\int_{t_1}^{t_2}\int_\Omega |\nabla u|^2\nabla u : \nabla\partial_t b\,\mathrm{d}x\mathrm{d}s$$

for a.e. $0 \leq t_1 \leq t_2 \leq T$*.*

Theorem 5.4 (Existence theorem - viscous problem) *Let the assumptions in Section 3.1 be satisfied and let* $c^0 \in H^1(\Omega)$*,* $z^0 \in W^{1,p}(\Omega)$ *with* $0 \leq z^0 \leq 1$ *a.e. in* Ω *and a viscosity factor* $\varepsilon \in (0,1]$ *be given. Then there exists a weak solution* $q \in \mathcal{Q}^{\mathrm{v}}$ *of the viscous system* (15a)-(15c) *in the sense of Definition 5.3. In addition:*

$$r = -\chi_{N_z}[\partial_z W_{\mathrm{el}}(e(u),c,z)]^+,$$

where χ_{N_z} *denotes the characteristic function of the level set* $N_z := \{z = 0\}$ *and* $[\cdot]^+ := \max\{0,\cdot\}$*.*

5.2 Limit problem

Our main objective in this work is an existence result for the system (15a)-(15c) with vanishing ε-terms, i.e. with $\varepsilon = 0$. In the same fashion as in Section 5.1 we introduce a weak notion of (6a)-(6c) as follows.

Definition 5.5 (Weak solution - limit problem) *A triple* $q = (u,c,z) \in \mathcal{Q}$ *with* $z(0) = z^0$ *is called a weak solution of the system* (6a)-(6c) *with boundary and initial conditions* (BC1)-(BC8) *if it satisfies the following conditions:*

(i) integral equality

$$\int_{\Omega_T} (c - c^0)\partial_t\zeta\,\mathrm{d}x\mathrm{d}t = \int_{\Omega_T} \nabla\mu\cdot\nabla\zeta\,\mathrm{d}x\mathrm{d}t$$

for all $\zeta \in L^2([0,T];H^1(\Omega))$ *with* $\partial_t\zeta \in L^2(\Omega_T)$ *and* $\zeta(T) = 0$ *where* $\mu \in L^2([0,T];H^1(\Omega))$ *is given by the integral equality*

$$\int_{\Omega_T} \mu\zeta\,\mathrm{d}x\mathrm{d}t = \int_{\Omega_T} \nabla c\cdot\nabla\zeta + \partial_c W_{\mathrm{ch}}(c)\zeta + \partial_c W_{\mathrm{el}}(e(u),c,z)\zeta\,\mathrm{d}x\mathrm{d}t$$

for all $\zeta \in L^2([0,T];H^1(\Omega))$*;*

(ii) integral equality

$$\int_{\Omega_T} \partial_e W_{\mathrm{el}}(e(u),c,z) : e(\zeta)\,\mathrm{d}x\mathrm{d}t = 0,$$

for all $\zeta \in L^2([0,T];H^1_\Gamma(\Omega;\mathbb{R}^n))$*;*

(iii) integral inequality

$$0 \leq \int_{\Omega_T} |\nabla z|^{p-2}\nabla z\cdot\nabla\zeta + \partial_z W_{\mathrm{el}}(e(u),c,z)\zeta - \alpha\zeta + \beta(\partial_t z)\zeta\,\mathrm{d}x\mathrm{d}t + \int_0^T \langle r(t),\zeta(t)\rangle\,\mathrm{d}t,$$

for all $\zeta \in L^p([0,T];W^{1,p}_-(\Omega)) \cap L^\infty(\Omega_T)$ *where* $r \in L^1(\Omega_T)$ *satisfies*

$$\langle r(t),\zeta - z(t)\rangle \leq 0$$

for a.e. $t \in [0,T]$ *and for all* $\zeta \in W^{1,p}_+(\Omega)$*;*

(iv) energy inequality

$$\mathcal{E}(q(t_2)) + \int_\Omega \alpha(z(t_1) - z(t_2))\,\mathrm{d}x + \int_{t_1}^{t_2}\int_\Omega \beta|\partial_t z|^2\,\mathrm{d}x\mathrm{d}s + \int_{t_1}^{t_2}\int_\Omega |\nabla\mu|^2\,\mathrm{d}x\mathrm{d}s$$
$$\leq \mathcal{E}(q(t_1)) + \int_{t_1}^{t_2}\int_\Omega \partial_e W_{\mathrm{el}}(e(u), c, z) : e(\partial_t b)\mathrm{d}x\,\mathrm{d}s$$

for a.e. $0 \leq t_1 \leq t_2 \leq T$.

Theorem 5.6 (Existence theorem - limit problem) *Let the assumptions in Section 3.1 be satisfied and let* $c^0 \in H^1(\Omega)$, $z^0 \in W^{1,p}(\Omega)$ *with* $0 \leq z^0 \leq 1$ *a.e. in* Ω *be given. Then there exists a weak solution* $q \in \mathcal{Q}$ *of the system* (6a)-(6c) *in the sense of Definition 5.5.*

Remark 5.7 *We want to emphasize that* z *obtained in Theorem 5.4 as well as in Theorem 5.6 is monotonically decreasing with respect to* t *and* $0 \leq z \leq 1$. *See the definition of* $\mathcal{Q}$ *in Section 3.2.*

6 Proof of the existence theorems

6.1 Viscous case

The proof of Theorem 5.4 is based on recursive functional minimization that comes from an implicit Euler scheme of system (15a)-(15c).

We first consider the initial values. The initial displacement u_ε^0 is chosen to be a minimizer of the functional $u \mapsto \mathcal{E}_\varepsilon(u, c^0, z^0)$ defined on the space $W^{1,4}(\Omega)$ with the constraint $u|_\Gamma = b(0)|_\Gamma$ (the existence proof is based on direct methods in the calculus of variations - see the proof of Lemma 6.1 below). The discretization fineness is given by $\tau := \frac{T}{M}$, where $M \in \mathbb{N}$. We set $q_{M,\varepsilon}^0 := (u_{M,\varepsilon}^0, c_{M,\varepsilon}^0, z_{M,\varepsilon}^0) := (u_\varepsilon^0, c^0, z^0)$ and construct $q_{M,\varepsilon}^m$ for $m \in \{1, \ldots, M\}$ recursively by considering the functional

$$\mathbb{E}_{M,\varepsilon}^m(u,c,z) := \tilde{\mathcal{E}}_\varepsilon(u,c,z) + \tilde{\mathcal{R}}\left(\frac{z - z_{M,\varepsilon}^{m-1}}{\tau}\right)\tau + \frac{1}{2\tau}\|c - c_{M,\varepsilon}^{m-1}\|_{\tilde{H}_0}^2 + \frac{\varepsilon}{2\tau}\|c - c_{M,\varepsilon}^{m-1}\|_{L^2(\Omega)}^2.$$

The set of admissible states for $\mathbb{E}_{M,\varepsilon}^m$ is set to

$$\mathcal{Q}_{M,\varepsilon}^m := \Big\{ q = (u,c,z) \in W^{1,4}(\Omega;\mathbb{R}^n) \times H^1(\Omega) \times W^{1,p}(\Omega)$$
$$\text{with } u|_\Gamma = b(m\tau)|_\Gamma,\ \int_\Omega c - c^0\,\mathrm{d}x = 0 \text{ and } 0 \leq z \leq z_{M,\varepsilon}^{m-1} \text{ a.e. in } \Omega \Big\}.$$

A minimization problem with the weighted $(H^1(\Omega, \mathbb{R}^n))^\star$-scalar product $\langle \cdot, \cdot \rangle_{\tilde{H}_0}$ has been considered for $\mathbb{E}_{M,\varepsilon}^m(u,c,z) = \mathbb{E}_{M,\varepsilon}^m(u,c) = \int_\Omega \frac{1}{2}|\nabla c|^2 + W_{\mathrm{ch}}(c) + W_{\mathrm{el}}(e(u), c)\,\mathrm{d}x + \frac{1}{2\tau}\|c - c_{M,\varepsilon}^{m-1}\|_{\tilde{H}_0}^2$ in [Gar00]. However, due to the additional internal variable z, our minimization procedure becomes much more involved.

In the following, we will omit the ε-dependence in the notation since $\varepsilon \in (0, 1]$ is fixed until Section 6.2.

Lemma 6.1 *The functional* $\mathbb{E}_M^m$ *has a minimizer* $q_M^m = (u_M^m, c_M^m, z_M^m) \in \mathcal{Q}_M^m$.

Proof. The existence is shown by direct methods in calculus of variations. We can immediately see that $\mathcal{Q}_M^m$ is closed with respect to the weak topology in $W^{1,4}(\Omega;\mathbb{R}^n)\times H^1(\Omega)\times W^{1,p}(\Omega)$. Furthermore, we need to show coercivity and sequentially weakly lower semi-continuity of $\mathbb{E}_M^m$ defined on $\mathcal{Q}_M^m$.

(i) *Coercivity.* We have the estimate

$$\mathbb{E}_M^m(q)\geq\frac{1}{2}\|\nabla c\|_{L^2(\Omega)}^2+\frac{1}{p}\|\nabla z\|_{L^p(\Omega)}^p+\frac{\varepsilon}{4}\|\nabla u\|_{L^4(\Omega)}^4.$$

Therefore, given a sequence $\{q_k\}_{k\in\mathbb{N}}$ in $\mathcal{Q}_M^m$ with the boundedness property $\mathbb{E}_M^m(q_k)<C$ for all $k\in\mathbb{N}$, we obtain the boundedness of u_k in $W^{1,4}(\Omega)$ by Korn's inequality and the continuous embedding $H^1(\Omega)\hookrightarrow L^4(\Omega)$, the boundedness of c_k in $H^1(\Omega)$ by Poincaré's inequality ($\int_\Omega c_k\,\mathrm{d}x$ is conserved) and the boundedness of z_k in $W^{1,p}(\Omega)$ by the restriction $0\leq z_k\leq 1$ a.e. in Ω.

(ii) *Sequentially weakly lower semi-continuity.* All terms in $\mathbb{E}_M^m$ except $\int_\Omega W_{\mathrm{ch}}(c)\,\mathrm{d}x$ and $\int_\Omega W_{\mathrm{el}}(e(u),c,z)\,\mathrm{d}x$ are convex and continuous and therefore sequentially weakly l.s.c.. Now let $(u_k,c_k,z_k)\rightharpoonup(u,c,z)$ be a weakly converging sequence in $\mathcal{Q}_M^m$. In particular, $z_k\to z$ in $L^p(\Omega)$, $z_k\to z$ a.e. in Ω and $c_k\to c$ in $L^r(\Omega)$ as $k\to\infty$ for all $1\leq r<2^\star$ and $c_k\to c$ a.e. in Ω for a subsequence. Lebesgue's generalized convergence theorem yields $\int_\Omega W_{\mathrm{ch}}(c_k)\,\mathrm{d}x\to\int_\Omega W_{\mathrm{ch}}(c)\,\mathrm{d}x$ using (GC6). The remaining term can be treated by employing the uniform convexity of $W_{\mathrm{el}}(\cdot,c,z)$ (see (GC1)):

$$\begin{aligned}
&\int_\Omega W_{\mathrm{el}}(e(u_k),c_k,z_k)-W_{\mathrm{el}}(e(u),c,z)\,\mathrm{d}x\\
&\quad=\int_\Omega W_{\mathrm{el}}(e(u),c_k,z_k)-W_{\mathrm{el}}(e(u),c,z)\,\mathrm{d}x+\int_\Omega W_{\mathrm{el}}(e(u_k),c_k,z_k)-W_{\mathrm{el}}(e(u),c_k,z_k)\,\mathrm{d}x\\
&\quad\geq\underbrace{\int_\Omega W_{\mathrm{el}}(e(u),c_k,z_k)-W_{\mathrm{el}}(e(u),c,z)\,\mathrm{d}x}_{\to 0\text{ by Lebesgue's gen. conv. theorem and (GC2)}}+\int_\Omega\partial_e W_{\mathrm{el}}(e(u),c_k,z_k)(e(u_k)-e(u))\,\mathrm{d}x
\end{aligned}$$

The second term also converges to 0 because of $\partial_e W_{\mathrm{el}}(e(u),c_k,z_k)\to\partial_e W_{\mathrm{el}}(e(u),c,z)$ in $L^2(\Omega)$ (by Lebesgue's generalized convergence theorem and (11a)) and $e(u_k)-e(u)\rightharpoonup 0$ in $L^2(\Omega)$.

Thus there exists $q_M^m=(u_M^m,c_M^m,z_M^m)\in\mathcal{Q}_M^m$ such that $\mathbb{E}_M^m(q_M^m)=\inf_{q\in\mathcal{Q}_M^m}\mathbb{E}_M^m(q)$. ■

The minimizers q_M^m for $m\in\{0,\ldots,M\}$ are used to construct approximate solutions q_M and $\hat{q}_M$ to our viscous problem by a piecewise constant and linear interpolation in time, respectively. More precisely,

$$\begin{aligned}
q_M(t)&:=q_M^m,\\
\hat{q}_M(t)&:=\beta q_M^m+(1-\beta)q_M^{m-1}
\end{aligned}$$

with $t\in((m-1)\tau,m\tau]$ and $\beta=\frac{t-(m-1)\tau}{\tau}$. The retarded function q_M^- is set to

$$q_M^-(t):=\begin{cases}q_M(t-\tau), & \text{if } t\in[\tau,T],\\ q_\varepsilon^0, & \text{if } t\in[0,\tau).\end{cases}$$

The functions b_M and b_M^- are analogously defined adapting the notation $b_M^m := b(m\tau)$. Furthermore, the discrete chemical potential is given by (note that $\partial_t \hat{c}_M(t) \in H_0$)

$$\mu_M(t) := -(-\Delta)^{-1} (\partial_t \hat{c}_M(t)) + \lambda_M(t) \tag{25}$$

with the Lagrange multiplier λ_M originating from mass conservation:

$$\lambda_M(t) := \fint_\Omega \partial_c W_{\mathrm{ch}}(c_M(t)) + \partial_c W_{\mathrm{el}}(e(u_M(t)), c_M(t), z_M(t))\,\mathrm{d}x. \tag{26}$$

The discretization of the time variable t will be expressed by the functions

$$\begin{aligned} d_M(t) &:= \min\{m\tau \,|\, m \in \mathbb{N}_0 \text{ and } m\tau \geq t\}, \\ d_M^-(t) &:= \min\{(m-1)\tau \,|\, m \in \mathbb{N}_0 \text{ and } m\tau \geq t\}. \end{aligned}$$

The following lemma clarifies why the functions q_M, q_M^- and $\hat{q}_M$ are approximate solutions to our problem.

Lemma 6.2 (Euler-Lagrange equations) *The functions q_M, q_M^- and $\hat{q}_M$ satisfy the following properties:*

(i) for all $t \in (0,T)$ and all $\zeta \in H^1(\Omega)$:

$$\int_\Omega (\partial_t \hat{c}_M(t))\zeta \,\mathrm{d}x = -\int_\Omega \nabla \mu_M(t) \cdot \nabla \zeta \,\mathrm{d}x \tag{27}$$

(ii) for all $t \in (0,T)$ and all $\zeta \in H^1(\Omega)$:

$$\begin{aligned} \int_\Omega \mu_M(t)\zeta \,\mathrm{d}x = &\int_\Omega \nabla c_M(t) \cdot \nabla \zeta + \partial_c W_{\mathrm{ch}}(c_M(t))\zeta \,\mathrm{d}x \\ &+ \int_\Omega \partial_c W_{\mathrm{el}}(e(u_M(t)), c_M(t), z_M(t))\zeta + \varepsilon(\partial_t \hat{c}_M(t))\zeta \,\mathrm{d}x \end{aligned} \tag{28}$$

(iii) for all $t \in [0,T]$ and for all $\zeta \in W^{1,4}_\Gamma(\Omega;\mathbb{R}^n)$:

$$0 = \int_\Omega \partial_e W_{\mathrm{el}}(e(u_M(t)), c_M(t), z_M(t)) : e(\zeta) + \varepsilon|\nabla u_M(t)|^2 \nabla u_M(t) : \nabla \zeta \,\mathrm{d}x \tag{29}$$

(iv) for all $t \in (0,T)$ and all $\zeta \in W^{1,p}(\Omega)$ such that there exists a constant $\nu > 0$ with $0 \leq \nu\zeta + z_M(t) \leq z_M^-(t)$ a.e. in Ω:

$$0 \leq \int_\Omega |\nabla z_M(t)|^{p-2} \nabla z_M(t) \cdot \nabla \zeta + \partial_z W_{\mathrm{el}}(e(u_M(t)), c_M(t), z_M(t))\zeta - \alpha\zeta + \beta(\partial_t \hat{z}_M(t))\zeta \,\mathrm{d}x \tag{30}$$

(v) for all $t \in [0,T]$:

$$\begin{aligned} &\mathcal{E}_\varepsilon(q_M(t)) + \int_0^{d_M(t)} \mathcal{R}(\partial_t \hat{z}_M)\,\mathrm{d}s + \int_0^{d_M(t)} \int_\Omega \frac{\varepsilon}{2}|\partial_t \hat{c}_M|^2 + \frac{1}{2}|\nabla \mu_M|^2 \,\mathrm{d}x\mathrm{d}s \\ &\quad \leq \mathcal{E}_\varepsilon(q_\varepsilon^0) + \int_0^{d_M(t)} \int_\Omega \partial_e W_{\mathrm{el}}(e(u_M^- + b - b_M^-), c_M^-, z_M^-) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}s \\ &\qquad + \varepsilon \int_0^{d_M(t)} \int_\Omega |\nabla u_M^- + \nabla b - \nabla b_M^-|^2 \nabla(u_M^- + b - b_M^-) : \nabla \partial_t b \,\mathrm{d}x\mathrm{d}s \end{aligned} \tag{31}$$

Proof. Using Lebesgue's generalized convergence theorem, the mean value theorem of differentiability and growth conditions (11a), (GC4)-(GC6) we obtain the variational derivatives of $\tilde{\mathcal{E}}_\varepsilon$ with respect to u, c and z:

$$\langle \mathrm{d}_u\tilde{\mathcal{E}}_\varepsilon(q),\zeta\rangle = \int_\Omega \partial_e W_{\mathrm{el}}(e(u),c,z) : e(\zeta) + \varepsilon|\nabla u|^2\nabla u : \nabla\zeta \,\mathrm{d}x \text{ for } \zeta \in W^{1,4}(\Omega;\mathbb{R}^n), \tag{32a}$$

$$\langle \mathrm{d}_c\tilde{\mathcal{E}}_\varepsilon(q),\zeta\rangle = \int_\Omega \nabla c\cdot\nabla\zeta + \partial_c W_{\mathrm{ch}}(c)\zeta + \partial_c W_{\mathrm{el}}(e(u),c,z)\zeta \,\mathrm{d}x \text{ for } \zeta \in H^1(\Omega), \tag{32b}$$

$$\langle \mathrm{d}_z\tilde{\mathcal{E}}_\varepsilon(q),\zeta\rangle = \int_\Omega |\nabla z|^{p-2}\nabla z\cdot\nabla\zeta + \partial_z W_{\mathrm{el}}(e(u),c,z)\zeta \,\mathrm{d}x \text{ for } \zeta \in W^{1,p}(\Omega). \tag{32c}$$

To (i)-(v):

(i) This follows from (25).

(ii) q_M^m fulfills $\langle \mathrm{d}_c\mathbb{E}_M^m(q_M^m),\zeta_1\rangle = 0$ for all $\zeta_1 \in H_0$ and all $m \in \{1,\dots,M\}$. Therefore,

$$0 = \langle \mathrm{d}_c\tilde{\mathcal{E}}_\varepsilon(q_M(t)),\zeta_1\rangle + \langle \partial_t\hat{c}_M(t),\zeta_1\rangle_{\tilde{H}_0} + \varepsilon\langle\partial_t\hat{c}_M(t),\zeta_1\rangle_{L^2(\Omega)}.$$

On the one hand definition (25) implies

$$\begin{aligned}\langle \partial_t\hat{c}_M(t),\zeta_1\rangle_{\tilde{H}_0} &= \langle (-\Delta)^{-1}\left(\partial_t\hat{c}_M(t)\right),\zeta_1\rangle_{L^2(\Omega)}\\ &= \langle -\mu_M(t)+\lambda_M(t),\zeta_1\rangle_{L^2(\Omega)}\\ &= -\langle \mu_M(t),\zeta_1\rangle_{L^2(\Omega)}\end{aligned}$$

and consequently

$$0 = \langle \mathrm{d}_c\tilde{\mathcal{E}}_\varepsilon(q_M(t)),\zeta_1\rangle - \langle\mu_M(t),\zeta_1\rangle_{L^2(\Omega)} + \varepsilon\langle\partial_t\hat{c}_M(t),\zeta_1\rangle_{L^2(\Omega)} \quad \text{for all } \zeta_1 \in H_0. \tag{33}$$

On the other hand definitions (25) and (26) yield for $\zeta_2 \equiv C$ with $C \in \mathbb{R}$:

$$\begin{aligned}&\langle \mathrm{d}_c\tilde{\mathcal{E}}_\varepsilon(q_M(t)),\zeta_2\rangle - \langle\mu_M(t),\zeta_2\rangle_{L^2(\Omega)} + \varepsilon\langle\partial_t\hat{c}_M(t),\zeta_2\rangle_{L^2(\Omega)}\\ &\quad = C\mathcal{L}^n(\Omega)\lambda_M(t) + \underbrace{\langle(-\Delta)^{-1}\left(\partial_t\hat{c}_M(t)\right),\zeta_2\rangle_{L^2(\Omega)}}_{=0} - \underbrace{\langle\lambda_M(t),\zeta_2\rangle_{L^2(\Omega)}}_{C\mathcal{L}^n(\Omega)\lambda_M(t)} + 0\\ &\quad = 0.\end{aligned} \tag{34}$$

Setting $\zeta_1 = \zeta - \fint\zeta$ and $\zeta_2 = \fint\zeta$, inserting (32b) into (33) and (34), and adding (33) to (34) shows finally (ii) (cf. [Gar00, Lemma 3.2]).

(iii) This property follows from (32a) and $0 = \langle \mathrm{d}_u\mathbb{E}_M^m(q_M^m),\zeta\rangle = \langle \mathrm{d}_u\tilde{\mathcal{E}}_\varepsilon(q_M^m),\zeta\rangle$ for all $\zeta \in W_\Gamma^{1,4}(\Omega;\mathbb{R}^n)$.

(iv) By construction, z_M^m minimizes $\mathbb{E}_M^m(u_M^m,c_M^m,\cdot)$ with the constraints $0 \le z$ and $z - z_M^{m-1} \le 0$ a.e. in Ω. This implies

$$-\langle \mathrm{d}_z\tilde{\mathcal{E}}_\varepsilon(q_M^m),\zeta - z_M^m\rangle - \langle \mathrm{d}_{\dot z}\tilde{\mathcal{R}}\left(\frac{z_M^m - z_M^{m-1}}{\tau}\right),\zeta - z_M^m\rangle_{L^2(\Omega)} \le 0 \tag{35}$$

for all $\zeta \in W^{1,p}(\Omega)$ with $0 \le \zeta \le z_M^{m-1}$ a.e. in Ω. Now, let the functions $\zeta \in W^{1,p}(\Omega)$ and $\nu > 0$ with $0 \le \nu\zeta + z_M(t) \le z_M^-(t)$ a.e. in Ω be given. Since $\nu > 0$, we obtain from (35):

$$-\langle \mathrm{d}_z\tilde{\mathcal{E}}_\varepsilon(q_M(t)),\zeta(t)\rangle - \langle \mathrm{d}_{\dot z}\tilde{\mathcal{R}}\left(\partial_t\hat{z}_M(t)\right),\zeta(t)\rangle_{L^2(\Omega)} \le 0$$

This and (32c) gives (iv).

(v) Testing $\mathbb{E}_M^m$ with $q = (u_M^{m-1} + b_M^m - b_M^{m-1}, c_M^{m-1}, z_M^{m-1})$ and using the chain rule yields:

$$
\begin{aligned}
&\mathcal{E}_\varepsilon(q_M^m) + \mathcal{R}\left(\frac{z_M^m - z_M^{m-1}}{\tau}\right)\tau + \frac{1}{2\tau}\|c_M^m - c_M^{m-1}\|_{\tilde{H}_0}^2 + \frac{\varepsilon}{2\tau}\|c_M^m - c_M^{m-1}\|_{L^2(\Omega)}^2 \\
&\quad \leq \mathcal{E}_\varepsilon(u_M^{m-1} + b_M^m - b_M^{m-1}, c_M^{m-1}, z_M^{m-1}) \\
&\quad = \mathcal{E}_\varepsilon(q_M^{m-1}) + \mathcal{E}_\varepsilon(u_M^{m-1} + b_M^m - b_M^{m-1}, c_M^{m-1}, z_M^{m-1}) - \mathcal{E}_\varepsilon(q_M^{m-1}) \\
&\quad = \mathcal{E}_\varepsilon(q_M^{m-1}) + \int_{(m-1)\tau}^{m\tau} \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}_\varepsilon(u_M^{m-1} + b(s) - b_M^{m-1}, c_M^{m-1}, z_M^{m-1})\,\mathrm{d}s \\
&\quad = \mathcal{E}_\varepsilon(q_M^{m-1}) \\
&\qquad + \int_{(m-1)\tau}^{m\tau}\int_\Omega \partial_e W_{\mathrm{el}}(e(u_M^{m-1} + b(s) - b_M^{m-1}), c_M^{m-1}, z_M^{m-1}) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}s \\
&\qquad + \varepsilon\int_{(m-1)\tau}^{m\tau}\int_\Omega |\nabla u_M^{m-1} + \nabla b(s) - \nabla b_M^{m-1}|^2\nabla(u_M^{m-1} + b(s) - b_M^{m-1}) : \nabla\partial_t b\,\mathrm{d}x\mathrm{d}s
\end{aligned}
$$

Summing this inequality for $k = 1, \ldots, m$ one gets:

$$
\begin{aligned}
&\mathcal{E}_\varepsilon(q_M^m) + \sum_{k=1}^m \tau\left(\mathcal{R}\left(\frac{z_M^k - z_M^{k-1}}{\tau}\right) + \frac{1}{2}\left\|\frac{c_M^k - c_M^{k-1}}{\tau}\right\|_{\tilde{H}_0}^2 + \frac{\varepsilon}{2}\left\|\frac{c_M^k - c_M^{k-1}}{\tau}\right\|_{L^2(\Omega)}^2\right) \\
&\quad \leq \mathcal{E}_\varepsilon(q_\varepsilon^0) + \int_0^{m\tau}\int_\Omega \partial_e W_{\mathrm{el}}(e(u_M^- + b - b_M^-), c_M^-, z_M^-) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}s \\
&\qquad + \varepsilon\int_0^{m\tau}\int_\Omega |\nabla u_M^- + \nabla b - \nabla b_M^-|^2\nabla(u_M^- + b - b_M^-) : \nabla\partial_t b\,\mathrm{d}x\mathrm{d}s
\end{aligned}
$$

Because of $\left\|\frac{c_M^k - c_M^{k-1}}{\tau}\right\|_{\tilde{H}_0}^2 = \|\nabla\mu_M^k\|_{L^2(\Omega)}^2$ by (25), above estimate shows (v). ■

The discrete energy inequality (31) gives rise to a-priori estimates for the approximate solutions:

Lemma 6.3 (Energy boundedness) *There exists a constant $C > 0$ independent of M, t and ε such that*

$$
\mathcal{E}_\varepsilon(q_M(t)) + \int_0^{d_M(t)} \mathcal{R}(\partial_t \hat{z}_M)\,\mathrm{d}s + \int_0^{d_M(t)}\int_\Omega \frac{\varepsilon}{2}|\partial_t \hat{c}_M|^2 + \frac{1}{2}|\nabla\mu_M|^2\,\mathrm{d}x\mathrm{d}s \leq C(\mathcal{E}_\varepsilon(q_\varepsilon^0) + 1).
$$

Proof. Exploiting (GC3) yields the estimate ($C > 0$ denotes a context-dependent constant independent of M, t and ε):

$$
\begin{aligned}
&\int_\Omega \partial_e W_{\mathrm{el}}(e(u_M^-(s) + b(s) - b_M^-(s)), c_M^-(s), z_M^-(s)) : e(\partial_t b(s))\,\mathrm{d}x \\
&\qquad \leq C\|\nabla\partial_t b(s)\|_{L^\infty(\Omega)}\int_\Omega W_{\mathrm{el}}(e(u_M^-(s)), c_M^-(s), z_M^-(s)) + |e(b(s) - b_M^-(s))| + 1\,\mathrm{d}x. \quad (36)
\end{aligned}
$$

In addition,

$$
\begin{aligned}
&\int_\Omega |\nabla u_M^-(s) + \nabla b(s) - \nabla b_M^-(s)|^2\nabla(u_M^-(s) + b(s) - b_M^-(s)) : \nabla\partial_t b(s)\,\mathrm{d}x \\
&\qquad \leq C\|\nabla\partial_t b(s)\|_{L^\infty(\Omega)}\int_\Omega |\nabla u_M^-(s)|^3 + |\nabla(b(s) - b_M^-(s))|^3\,\mathrm{d}x. \quad (37)
\end{aligned}
$$

To simplify notation we define the function:

$$\gamma(t) := \begin{cases} \mathcal{E}_\varepsilon(q_M(t)) + \int_0^{d_M(t)} \mathcal{R}(\partial_t \hat{z}_M)\,\mathrm{d}s + \int_0^{d_M(t)} \int_\Omega \frac{\varepsilon}{2}|\partial_t \hat{c}_M|^2 + \frac{1}{2}|\nabla \mu_M|^2\,\mathrm{d}x\mathrm{d}s, & \text{if } t \in [0,T], \\ \mathcal{E}_\varepsilon(q_\varepsilon^0), & \text{if } t \in [-\tau, 0). \end{cases}$$

Using (36) and (37) the discrete energy inequality (31) can be estimated as follows:

$$\begin{aligned} \gamma(t) &\leq \mathcal{E}_\varepsilon(q_\varepsilon^0) + C\int_0^{d_M(t)} \|\nabla \partial_t b(s)\|_{L^\infty(\Omega)} \mathcal{E}_\varepsilon(q_M^-(s))\,\mathrm{d}s \\ &\quad + C\big\|\nabla\partial_t b\big\|_{L^1([0,T];L^\infty(\Omega))}\big\||\nabla(b-b_M^-)|^3 + |e(b-b_M^-)| + 1\big\|_{L^\infty([0,T];L^1(\Omega))} \\ &\leq \mathcal{E}_\varepsilon(q_\varepsilon^0) + C\int_{-\tau}^{d_M^-(t)} \|\nabla\partial_t b(s+\tau)\|_{L^\infty(\Omega)}\mathcal{E}_\varepsilon(q_M(s))\,\mathrm{d}s + C \\ &\leq \mathcal{E}_\varepsilon(q_\varepsilon^0) + C\int_{-\tau}^{t} \|\nabla\partial_t b(s+\tau)\|_{L^\infty(\Omega)}\gamma(s)\,\mathrm{d}s + C. \end{aligned}$$

Gronwall's inequality shows $\gamma(t) \leq C(\mathcal{E}_\varepsilon(q_\varepsilon^0) + 1)$ for all $t \in [0,T]$. ■

Corollary 6.4 (A-priori estimates) *There exists a constant $C > 0$ independent of M such that*

(i) $\|u_M\|_{L^\infty([0,T];W^{1,4}(\Omega;\mathbb{R}^n))} \leq C$

(ii) $\|c_M\|_{L^\infty([0,T];H^1(\Omega))} \leq C$

(iii) $\|z_M\|_{L^\infty([0,T];W^{1,p}(\Omega))} \leq C$

(iv) $\|\partial_t \hat{c}_M\|_{L^2(\Omega_T)} \leq C$

(v) $\|\partial_t \hat{z}_M\|_{L^2(\Omega_T)} \leq C$

(vi) $\|\mu_M\|_{L^2([0,T];H^1(\Omega))} \leq C$

for all $M \in \mathbb{N}$.

Proof. We use Lemma 6.3. The boundedness of $\{\nabla u_M(t)\}$ in $L^4(\Omega;\mathbb{R}^n)$ and Korn's inequality yield (i). The boundedness of $\{\nabla c_M(t)\}$ in $L^2(\Omega)$ and mass conservation imply (ii) by Poincaré's inequality. The boundedness of $\{\nabla z_M(t)\}$ in $L^p(\Omega)$ and $0 \leq z_M(t) \leq 1$ a.e. in Ω for all M and all $t \in [0,T]$ show (iii). The properties (iv) and (v) follow immediately. The boundedness of $\{\nabla \mu_M\}$ in $L^2(\Omega_T)$ and $\{\int_\Omega \mu_M(t)\,\mathrm{d}x\}$ with respect to M and t show (vi) by Poincaré's inequality. Indeed, $\{\int_\Omega \mu_M(t)\,\mathrm{d}x\}$ is bounded with respect to M and t because of (28) and (27) tested with $\zeta \equiv 1$. ■

Due to the a-priori estimates we can select weakly (weakly-$\star$) convergent subsequences (see Lemma 6.5). Furthermore, exploiting the Euler-Lagrange equations of the approximate solutions we even attain strong convergence properties (see Lemma 6.6 and Lemma 6.8).

Lemma 6.5 (Weak convergence of the approximate solutions) *There exists a subsequence $\{M_k\}$ and an element $(u,c,z) = q \in \mathcal{Q}^{\mathrm{v}}$ with $c(0) = c^0$ and $z(0) = z^0$ such that the following properties are satisfied:*

(i) $z_{M_k}, z^-_{M_k} \overset{\star}{\rightharpoonup} z$ *in* $L^\infty([0,T]; W^{1,p}(\Omega))$,
$z_{M_k}(t), z^-_{M_k}(t) \rightharpoonup z(t)$ *in* $W^{1,p}(\Omega)$ *for a.e.* $t \in [0,T]$,
$z_{M_k}, z^-_{M_k} \to z$ *a.e. in* Ω_T *and*
$\hat{z}_{M_k} \rightharpoonup z$ *in* $H^1([0,T]; L^2(\Omega))$

(ii) $c_{M_k}, c^-_{M_k} \overset{\star}{\rightharpoonup} c$ *in* $L^\infty([0,T]; H^1(\Omega))$,
$c_{M_k}(t), c^-_{M_k}(t) \rightharpoonup c(t)$ *in* $H^1(\Omega)$ *for a.e.* $t \in [0,T]$,
$c_{M_k}, c^-_{M_k} \to c$ *a.e. in* Ω_T *and*
$\hat{c}_{M_k} \rightharpoonup c$ *in* $H^1([0,T]; L^2(\Omega))$

(iii) $u_{M_k} \overset{\star}{\rightharpoonup} u$ *in* $L^\infty([0,T]; W^{1,4}(\Omega))$

(iv) $\mu_{M_k} \rightharpoonup \mu$ *in* $L^2([0,T]; H^1(\Omega))$

as $k \to \infty$.

Proof. To simplify notation we omit the index k in the proof.

(ii) Since $\hat{c}_M$ is bounded in $L^2([0,T]; H^1(\Omega))$ and $\partial_t \hat{c}_M$ is bounded in $L^2(\Omega_T)$, we obtain $\hat{c}_M \to \hat{c}$ in $L^2(\Omega_T)$ as $M \to \infty$ for a subsequence by a compactness result from Aubin and Lions (see [Sim86]). Therefore we can extract a subsequence such that $\hat{c}_M(t) \to \hat{c}(t)$ in $L^2(\Omega)$ for a.e. $t \in [0,T]$ and $\hat{c}_M \to \hat{c}$ a.e. on Ω_T. We denote this subsequence also with $\{\hat{c}_M\}$. The boundedness of $\{\hat{c}_M(t)\}_{M\in\mathbb{N}}$ in $H^1(\Omega)$ even shows $\hat{c}_M(t) \rightharpoonup \hat{c}(t)$ in $H^1(\Omega)$ for a.e. $t \in [0,T]$. In addition, the boundedness of $\{\hat{c}_M\}$ in $L^\infty([0,T]; H^1(\Omega))$ shows $\hat{c}_M \overset{\star}{\rightharpoonup} \hat{c}$ in $L^\infty([0,T]; H^1(\Omega))$. Furthermore, we obtain from the boundedness of $\{\partial_t \hat{c}_M\}$ in $L^2(\Omega_T)$ for every $t \in [0,T]$:

$$\begin{aligned}\|c_M(t) - \hat{c}_M(t)\|_{L^1(\Omega)} &= \|\hat{c}_M(d_M(t)) - \hat{c}_M(t)\|_{L^1(\Omega)} \\ &\le \int_t^{d_M(t)} \|\partial_t \hat{c}_M(s)\|_{L^1(\Omega)} \,\mathrm{d}s \\ &\le C(d_M(t) - t)^{1/2} \|\partial_t \hat{c}_M\|_{L^2(\Omega_T)} \to 0 \text{ as } M \to \infty\end{aligned}$$

Lebesgue's convergence theorem yields $\|c_M - \hat{c}_M\|_{L^1(\Omega_T)} \to 0$ as $M \to \infty$. Analogously we obtain $\|c_M - c^-_M\|_{L^1(\Omega_T)} \to 0$ as $M \to \infty$. Thus the convergence properties for $\hat{c}_M$ also holds for c_M and c^-_M with the same limit $c = c^- = \hat{c}$ a.e. . The boundedness of $\{\hat{c}_M\}$ in $H^1([0,T]; L^2(\Omega))$ shows $\hat{c}_M \rightharpoonup c$ in $H^1([0,T]; L^2(\Omega))$ for a subsequence.

(i) We obtain the convergence properties for $\{z_M\}$ with the same argumentation. Note that the limit function is monotonically decreasing with respect to t.

(iii) This property follows from the boundedness of $\{u_M\}$ in $L^\infty([0,T]; H^1(\Omega; \mathbb{R}^n))$.

(iv) This property follows from the boundedness of $\{\mu_M\}$ in $L^2([0,T]; H^1(\Omega))$. ■

Lemma 6.6 *There exists a subsequence* $\{M_k\}$ *such that* $u_{M_k}, u^-_{M_k} \to u$ *in* $L^4([0,T]; W^{1,4}(\Omega; \mathbb{R}^n))$ *as* $k \to \infty$.

Proof. We omit the index k in the proof.
Applying (GC1), Lemma A.1 with $p=4$, and considering (29) with the test-function $\zeta = u_M(t)-u(t)-b_M(t)+b(t)$ we get

$$\begin{aligned}
&\eta\|e(u_M)-e(u)\|^2_{L^2(\Omega_T;\mathbb{R}^{n\times n})}+\varepsilon C_{\text{ineq}}^{-1}\|\nabla u_M-\nabla u\|^4_{L^4(\Omega_T;\mathbb{R}^{n\times n})}\\
&\quad\le\int_{\Omega_T}(\partial_e W_{\text{el}}(e(u_M),c_M,z_M)-\partial_e W_{\text{el}}(e(u),c_M,z_M)):(e(u_M)-e(u))\,\mathrm{d}x\mathrm{d}t\\
&\qquad+\varepsilon\int_{\Omega_T}(|\nabla u_M|^2\nabla u_M-|\nabla u|^2\nabla u):(\nabla u_M-\nabla u)\,\mathrm{d}x\mathrm{d}t\\
&\quad=\underbrace{\int_{\Omega_T}\partial_e W_{\text{el}}(e(u_M),c_M,z_M):e(\zeta)+\varepsilon|\nabla u_M|^2\nabla u_M:\nabla\zeta\,\mathrm{d}x\mathrm{d}t}_{=0\text{ by (29)}}\\
&\qquad+\underbrace{\int_{\Omega_T}\partial_e W_{\text{el}}(e(u_M),c_M,z_M):(e(b_M)-e(b))\,\mathrm{d}x\mathrm{d}t}_{(\star)}+\underbrace{\varepsilon\int_{\Omega_T}|\nabla u_M|^2\nabla u_M:(\nabla b_M-\nabla b)\,\mathrm{d}x\mathrm{d}t}_{(\star\star)}\\
&\qquad-\underbrace{\int_{\Omega_T}(\partial_e W_{\text{el}}(e(u),c_M,z_M):(e(u_M)-e(u))\,\mathrm{d}x\mathrm{d}t}_{(\star\star\star)}-\underbrace{\varepsilon\int_{\Omega_T}|\nabla u|^2\nabla u:(\nabla u_M-\nabla u)\,\mathrm{d}x\mathrm{d}t}_{(\star\star\star\star)}.
\end{aligned}\tag{38}$$

Since $\partial_e W_{\text{el}}(e(u_M),c_M,z_M)$ is bounded in $L^2(\Omega_T;\mathbb{R}^{n\times n})$ (by (11a) and Corollary 6.4) as well as $e(b_M)\to e(b)$ in $L^2(\Omega_T;\mathbb{R}^{n\times n})$, we obtain $(\star)\to 0$ as $M\to\infty$. The boundedness of $|\nabla u_M|^2\nabla u_M$ in $L^{4/3}(\Omega_T;\mathbb{R}^{n\times n})$ by Corollary 6.4 and $\nabla b_M\to\nabla b$ in $L^4(\Omega_T;\mathbb{R}^{n\times n})$ lead to $(\star\star)\to 0$. We also have $\partial_e W_{\text{el}}(e(u),c_M,z_M)\to\partial_e W_{\text{el}}(e(u),c,z)$ in $L^2(\Omega_T;\mathbb{R}^{n\times n})$ by (11a) and Lebesgue's generalized convergence theorem. Furthermore, $e(u_M)\rightharpoonup e(u)$ in $L^2(\Omega_T;\mathbb{R}^n\times\mathbb{R}^n)$ by Lemma 6.5. This gives $(\star\star\star)\to 0$. Since $\nabla u_M\rightharpoonup\nabla u$ in $L^4(\Omega_T;\mathbb{R}^n)$ by Lemma 6.5, we obtain $(\star\star\star\star)\to 0$. Therefore, (38) implies $e(u_M)\to e(u)$ in $L^2(\Omega_T;\mathbb{R}^{n\times n})$ and $\nabla u_M\to\nabla u$ in $L^4(\Omega_T;\mathbb{R}^{n\times n})$ as $M\to\infty$. Korn's inequality finally shows $u_M\to u$ in $L^4([0,T];W^{1,4}(\Omega;\mathbb{R}^n))$. Now, we choose a subsequence such that $u_M(t)\to u(t)$ in $W^{1,4}(\Omega;\mathbb{R}^n)$ for a.e. $t\in[0,T]$ and $u_M\to u$ a.e. in Ω_T. We also denote this subsequence with $\{u_M\}$.

Analogously we obtain a $u^-\in L^4([0,T];W^{1,4}(\Omega))$ satisfying $u_M^-\to u^-$ with the same convergence properties. We will show $u=u^-$ a.e. . Consider (29) for $q_M(t)$ and for $q_M^-(t)$:

$$0=\int_{\Omega_T}\partial_e W_{\text{el}}(e(u_M),c_M,z_M):e(\zeta)+\varepsilon|\nabla u_M|^2\nabla u_M:\nabla\zeta\,\mathrm{d}x\mathrm{d}t,\tag{39a}$$

$$0=\int_{\Omega_T}\partial_e W_{\text{el}}(e(u_M^-),c_M^-,z_M^-):e(\zeta)+\varepsilon|\nabla u_M^-|^2\nabla u_M^-:\nabla\zeta\,\mathrm{d}x\mathrm{d}t\tag{39b}$$

We choose the test-function $\zeta(t)=u_M(t)-u_M^-(t)-b_M(t)+b_M^-(t)\in W_\Gamma^{1,4}(\Omega)$. An estimate similar to (38) gives:

$$\begin{aligned}
&\eta\|e(u_M)-e(u_M^-)\|^2_{L^2(\Omega_T)}+\varepsilon C_{\text{ineq}}^{-1}\|\nabla u_M-\nabla u_M^-\|^4_{L^4(\Omega_T)}\\
&\qquad\le\int_{\Omega_T}(\partial_e W_{\text{el}}(e(u_M),c_M,z_M)-\partial_e W_{\text{el}}(e(u_M^-),c_M,z_M)):(e(u_M)-e(u_M^-))\,\mathrm{d}x\mathrm{d}t\\
&\qquad\quad+\varepsilon\int_{\Omega_T}(|\nabla u_M|^2\nabla u_M-|\nabla u_M^-|^2\nabla u_M^-):(\nabla u_M-\nabla u_M^-)\,\mathrm{d}x\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&= \underbrace{\int_{\Omega_T} \partial_e W_{\mathrm{el}}(e(u_M), c_M, z_M) : e(\zeta) + \varepsilon |\nabla u_M|^2 \nabla u_M : \nabla \zeta \,\mathrm{d}x\mathrm{d}t}_{=0 \text{ by (39a)}} \\
&\quad - \underbrace{\int_{\Omega_T} \partial_e W_{\mathrm{el}}(e(u_M^-), c_M^-, z_M^-) : e(\zeta) + \varepsilon |\nabla u_M^-|^2 \nabla u_M^- : \nabla \zeta \,\mathrm{d}x\mathrm{d}t}_{=0 \text{ by (39b)}} \\
&\quad + \int_{\Omega_T} (\partial_e W_{\mathrm{el}}(e(u_M^-), c_M^-, z_M^-) - \partial_e W_{\mathrm{el}}(e(u_M^-), c_M, z_M)) : (e(u_M) - e(u_M^-)) \,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_{\Omega_T} (\partial_e W_{\mathrm{el}}(e(u_M), c_M, z_M) - \partial_e W_{\mathrm{el}}(e(u_M^-), c_M^-, z_M^-)) : (e(b_M) - e(b_M^-)) \,\mathrm{d}x\mathrm{d}t \\
&\quad + \varepsilon \int_{\Omega_T} (|\nabla u_M|^2 \nabla u_M - |\nabla u_M^-|^2 \nabla u_M^-) : (\nabla b_M - \nabla b_M^-) \,\mathrm{d}x\mathrm{d}t
\end{aligned}
$$

Observe that $\partial_e W_{\mathrm{el}}(e(u_M^-), c_M^-, z_M^-) - \partial_e W_{\mathrm{el}}(e(u_M^-), c_M, z_M) \to 0$ in $L^2(\Omega_T)$ by Lebesgue's generalized convergence theorem (using growth condition (11a), Lemma 6.5 and convergence properties of u_M and u_M^-) as well as $e(b_M) - e(b_M^-) \to 0$ in $L^2(\Omega_T; \mathbb{R}^{n\times n})$ and $\nabla b_M - \nabla b_M^- \to 0$ in $L^4(\Omega_T; \mathbb{R}^{n\times n})$. So each term on the right hand side converges to 0 as $M \to \infty$ ■

Lemma 6.7 *There exists a subsequence $\{M_k\}$ such that $c_{M_k}, c_{M_k}^- \to c$ in $L^2([0,T]; H^1(\Omega))$ as $k \to \infty$.*

Proof. We omit the index k in the proof.
Lemma 6.5 implies $c_M(t) \to c(t)$ in $L^{2^\star/2+1}(\Omega)$ for a.e. $t \in [0,T]$. Using Corollary 6.4 and Lebesgue's convergence theorem, we get $c_M \to c$ in $L^{2^\star/2+1}(\Omega_T)$. Testing (28) with $\zeta = c_M(t)$ and integrate from $t = 0$ to $t = T$ and using Lebesgue's generalized convergence theorem, growth conditions (GC4) and (GC6) and Lemma 6.5:

$$
\int_{\Omega_T} |\nabla c_M|^2 \,\mathrm{d}x\mathrm{d}t \to - \int_{\Omega_T} \partial_c W_{\mathrm{ch}}(c)c + \partial_c W_{\mathrm{el}}(e(u), c, z)c + \varepsilon(\partial_t c)c - \mu c \,\mathrm{d}x\mathrm{d}t
$$

as $M \to \infty$. On the other hand testing (28) with $c(t)$ and integrate from $t = 0$ to $t = T$ (note that $c \in L^{2^\star}(\Omega_T)$ and using $\partial_c W_{\mathrm{ch}}(c_M) \to \partial_c W_{\mathrm{ch}}(c)$ in $L^{2^\star/(2^\star-1)}(\Omega_T)$ as $M \to \infty$ by Lebesgue's generalized convergence theorem) we obtain by passing to $M \to \infty$ on each side:

$$
\int_{\Omega_T} |\nabla c|^2 \,\mathrm{d}x\mathrm{d}t = - \int_{\Omega_T} \partial_c W_{\mathrm{ch}}(c)c + \partial_c W_{\mathrm{el}}(e(u), c, z)c + \varepsilon(\partial_t c)c - \mu c \,\mathrm{d}x\mathrm{d}t
$$

Therefore $c_M \to c$ in $L^2([0,T]; H^1(\Omega))$ as $M \to \infty$. The convergence $\|c_M\|_{L^2([0,T];H^1(\Omega))} \to \|c\|_{L^2([0,T];H^1(\Omega))}$ implies $\|c_M^-\|_{L^2([0,T];H^1(\Omega))} \to \|c\|_{L^2([0,T];H^1(\Omega))}$. We also have $c_M^- \rightharpoonup c$ in $L^2([0,T]; H^1(\Omega))$ (by Lemma 6.5 (ii)) and consequently $c_M^- \to c$ in $L^2([0,T]; H^1(\Omega))$ as $M \to \infty$. ■

Note that in connection with Corollary 6.4 we even get for each $q \in \mathbb{N}$

$$
c_M, c_M^- \to c \text{ in } L^q([0,T]; H^1(\Omega))
$$

for a subsequence as $M \to \infty$.

Lemma 6.8 *There exists a subsequence $\{M_k\}$ such that $z_{M_k}, z_{M_k}^- \to z$ in $L^p([0,T]; W^{1,p}(\Omega))$ as $k \to \infty$.*

Proof. To simplify notation we omit the index k in the proof.
Applying Lemma 4.2 with $f = \zeta = z$ and $f_M = z_M^-$ gives a sequence of approximations $\{\zeta_M\}_{M\in\mathbb{N}} \subseteq L^p([0,T];W_+^{1,p}(\Omega)) \cap L^\infty(\Omega_T)$ with the properties (note that we have $z_M^-(t) \rightharpoonup z(t)$ in $W^{1,p}(\Omega)$ for a.e. $t \in [0,T]$ by Lemma 6.5):

$$\zeta_M \to z \text{ in } L^p([0,T];W^{1,p}(\Omega)) \text{ as } M \to \infty \tag{40a}$$

$$0 \le \zeta_M \le z_M^- \text{ a.e. on } \Omega_T \text{ for all } M \in \mathbb{N} \tag{40b}$$

Testing (30) with $\zeta = \zeta_M(t) - z_M(t)$ for $\nu = 1$ (possible due to (40b)) integrating from $t = 0$ to $t = T$, and using the elementary inequality in Lemma A.1 yields:

$$\begin{aligned}
&C_{\mathrm{ineq}}^{-1} \int_{\Omega_T} |\nabla z_M - \nabla z|^p \,\mathrm{d}x\mathrm{d}t \\
&\le \int_{\Omega_T} (|\nabla z_M|^{p-2}\nabla z_M - |\nabla z|^{p-2}\nabla z) \cdot \nabla(z_M - z) \,\mathrm{d}x\mathrm{d}t \\
&\le \int_{\Omega_T} |\nabla z_M|^{p-2}\nabla z_M \cdot \nabla(z_M - \zeta_M) \,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_{\Omega_T} |\nabla z_M|^{p-2}\nabla z_M \cdot \nabla(\zeta_M - z) - |\nabla z|^{p-2}\nabla z \cdot \nabla(z_M - z) \,\mathrm{d}x\mathrm{d}t \\
&\le \int_{\Omega_T} (\partial_z W_{\mathrm{el}}(e(u_M), c_M, z_M) - \alpha + \beta\partial_t \hat{z}_M)(\zeta_M - z_M) \,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_{\Omega_T} |\nabla z_M|^{p-2}\nabla z_M \cdot \nabla(\zeta_M - z) - |\nabla z|^{p-2}\nabla z \cdot \nabla(z_M - z) \,\mathrm{d}x\mathrm{d}t \\
&\le \underbrace{\|\partial_z W_{\mathrm{el}}(e(u_M), c_M, z_M) - \alpha + \beta\partial_t \hat{z}_M\|_{L^2(\Omega_T)}}_{\text{bounded by (GC5) and Cor. 6.4}} \|\zeta_M - z_M\|_{L^2(\Omega_T)} \\
&\quad + \underbrace{\|\nabla z_M\|_{L^p(\Omega_T)}^{p-1}}_{\text{bounded by Cor. 6.4}} \|\nabla \zeta_M - \nabla z\|_{L^p(\Omega_T)} - \int_{\Omega_T} |\nabla z|^{p-2}\nabla z \cdot \nabla(z_M - z) \,\mathrm{d}x\mathrm{d}t
\end{aligned}$$

Observe that $\nabla\zeta_M - \nabla z \to 0$ in $L^p(\Omega_T;\mathbb{R}^n)$ and $\zeta_M - z_M \to 0$ in $L^2(\Omega_T)$ (by property (40a) and by Lemma 6.5) as well as $\nabla z_M - \nabla z \rightharpoonup 0$ in $L^p(\Omega_T;\mathbb{R}^n)$ by Lemma 6.5. Using these properties, each term on the right hand side converges to 0 as $M \to \infty$.

We also obtain $\|z_M^-\|_{L^p([0,T];W^{1,p}(\Omega))} \to \|z\|_{L^p([0,T];W^{1,p}(\Omega))}$ from $\|z_M\|_{L^p([0,T];W^{1,p}(\Omega))} \to \|z\|_{L^p([0,T];W^{1,p}(\Omega))}$. Because of $z_M^- \rightharpoonup z$ in $L^p([0,T];W^{1,p}(\Omega))$ (by Lemma 6.5 (i)) we even have $z_M^- \to z$ in $L^p([0,T];W^{1,p}(\Omega))$ as $M \to \infty$. ∎

In conclusion, Corollary 6.4, Lemma 6.5, Lemma 6.6, Lemma 6.7 and Lemma 6.8 imply the following convergence properties:

Corollary 6.9 *There exists subsequence $\{M_k\}$ and an element $(u,c,z) = q \in \mathcal{Q}^{\mathrm{v}}$ with $c(0) = c^0$ and $z(0) = z^0$ such that*

(i) $z_{M_k}, z_{M_k}^- \to z$ *in* $L^p([0,T];W^{1,p}(\Omega))$,
$z_{M_k}(t), z_{M_k}^-(t) \to z(t)$ *in* $W^{1,p}(\Omega)$ *for a.e.* $t \in [0,T]$,
$z_{M_k}, z_{M_k}^- \to z$ *a.e. in* Ω_T *and*
$\hat{z}_{M_k} \rightharpoonup z$ *in* $H^1([0,T];L^2(\Omega))$

(ii) $c_{M_k}, c_{M_k}^- \to c$ *in* $L^{2^\star}([0,T];H^1(\Omega))$,
$c_{M_k}(t), c_{M_k}^-(t) \to c(t)$ *in* $H^1(\Omega)$ *for a.e.* $t \in [0,T]$,

$c_{M_k}, c^-_{M_k} \to c$ *a.e. in* Ω_T *and*
$\hat{c}_{M_k} \rightharpoonup c$ *in* $H^1([0,T];L^2(\Omega))$

(iii) $u_{M_k}, u^-_{M_k} \to u$ *in* $L^4([0,T];W^{1,4}(\Omega;\mathbb{R}^n))$,
$u_{M_k}(t), u^-_{M_k}(t) \to u(t)$ *in* $W^{1,4}(\Omega;\mathbb{R}^n)$ *for a.e.* $t \in [0,T]$,
$u_{M_k}, u^-_{M_k} \to u$ *a.e. in* Ω_T

(iv) $\mu_{M_k} \rightharpoonup \mu$ *in* $L^2([0,T];H^1(\Omega))$

(v) $\partial_c W_{\mathrm{ch}}(c_{M_k}) \to \partial_c W_{\mathrm{ch}}(c)$ *in* $L^2(\Omega_T)$

as $k \to \infty$.

The above convergence properties allow us to establish an energy estimate, which is in an asymptotic sense stronger than the one in Lemma 6.2 (v). We emphasize that (31) has in comparison with (41) no factor 1/2 in front of the terms $\beta|\partial_t \hat{z}_M|^2$, $\varepsilon|\partial_t \hat{c}_M|^2$ and $|\nabla \mu_M|^2$.

Lemma 6.10 (Precise energy inequality) *For every* $0 \le t_1 < t_2 \le T$*:*

$$\begin{aligned}
&\mathcal{E}_\varepsilon(q_M(t_2)) + \int_{d^-_M(t_1)}^{d_M(t_2)} \int_\Omega -\alpha\partial_t \hat{z}_M + \beta|\partial_t \hat{z}_M|^2 + \varepsilon|\partial_t \hat{c}_M|^2 + |\nabla \mu_M|^2 \,\mathrm{d}x\mathrm{d}s - \mathcal{E}_\varepsilon(q^-_M(t_1)) \\
&\quad \le \int_{d^-_M(t_1)}^{d_M(t_2)} \int_\Omega \partial_e W_{\mathrm{el}}(e(u^-_M + b - b^-_M), c^-_M, z^-_M) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}s \\
&\qquad + \varepsilon \int_{d^-_M(t_1)}^{d_M(t_2)} \int_\Omega |\nabla u^-_M + \nabla b - \nabla b^-_M|^2 \nabla(u^-_M + b - b^-_M) : \nabla \partial_t b \,\mathrm{d}x\mathrm{d}s + \kappa_M
\end{aligned} \tag{41}$$

with $\kappa_M \to 0$ *as* $M \to \infty$.

Proof. We know $\mathbb{E}^m_M(q^m_M) \le \mathbb{E}^m_M(u^{m-1}_M + b^m_M - b^{m-1}_M, c^m_M, z^m_M)$. The regularity properties of the functions b, $\hat{c}_M$ and $\hat{z}_M$ ensure that the chain rule can be applied and the following integral terms are well defined:

$$\begin{aligned}
&\mathcal{E}_\varepsilon(u^m_M, c^m_M, z^m_M) \\
&\quad \le \mathcal{E}_\varepsilon(u^{m-1}_M + b^m_M - b^{m-1}_M, c^m_M, z^m_M) \\
&\quad = \mathcal{E}_\varepsilon(u^{m-1}_M, c^{m-1}_M, z^{m-1}_M) \\
&\qquad + \mathcal{E}_\varepsilon(u^{m-1}_M + b^m_M - b^{m-1}_M, c^{m-1}_M, z^{m-1}_M) - \mathcal{E}_\varepsilon(u^{m-1}_M, c^{m-1}_M, z^{m-1}_M) \\
&\qquad + \mathcal{E}_\varepsilon(u^{m-1}_M + b^m_M - b^{m-1}_M, c^m_M, z^{m-1}_M) - \mathcal{E}_\varepsilon(u^{m-1}_M + b^m_M - b^{m-1}_M, c^{m-1}_M, z^{m-1}_M) \\
&\qquad + \mathcal{E}_\varepsilon(u^{m-1}_M + b^m_M - b^{m-1}_M, c^m_M, z^m_M) - \mathcal{E}_\varepsilon(u^{m-1}_M + b^m_M - b^{m-1}_M, c^m_M, z^{m-1}_M) \\
&\quad = \mathcal{E}_\varepsilon(u^{m-1}_M, c^{m-1}_M, z^{m-1}_M) \\
&\qquad + \int_{(m-1)\tau}^{m\tau} \langle \mathrm{d}_u \tilde{\mathcal{E}}_\varepsilon(u^{m-1}_M + b(s) - b^{m-1}_M, c^{m-1}_M, z^{m-1}_M), \partial_t b(s)\rangle_{(H^1)^* \times H^1}\,\mathrm{d}s \\
&\qquad + \int_{(m-1)\tau}^{m\tau} \langle \mathrm{d}_c \tilde{\mathcal{E}}_\varepsilon(u^{m-1}_M + b^m_M - b^{m-1}_M, \hat{c}_M(s), z^{m-1}_M), \partial_t \hat{c}_M(s)\rangle_{(H^1)^* \times H^1}\,\mathrm{d}s \\
&\qquad + \int_{(m-1)\tau}^{m\tau} \langle \mathrm{d}_z \tilde{\mathcal{E}}_\varepsilon(u^{m-1}_M + b^m_M - b^{m-1}_M, c^m_M, \hat{z}_M(s)), \partial_t \hat{z}_M(s)\rangle_{(W^{1,p})^* \times W^{1,p}}\,\mathrm{d}s
\end{aligned}$$

Summing from $m = \frac{d_M^-(t_1)}{\tau} + 1$ to $\frac{d_M(t_2)}{\tau}$ yields:

$$
\begin{aligned}
&\mathcal{E}_\varepsilon(q_M(t_2)) - \mathcal{E}_\varepsilon(q_M^-(t_1)) \\
&\quad \leq \varepsilon \int_{d_M^-(t_1)}^{d_M(t_2)} \int_\Omega |\nabla(u_M^- + b - b_M^-)|^2 \nabla(u_M^- + b - b_M^-) : \nabla \partial_t b \,\mathrm{d}x\mathrm{d}s \\
&\qquad + \int_{d_M^-(t_1)}^{d_M(t_2)} \int_\Omega \partial_e W_{\mathrm{el}}(e(u_M^- + b - b_M^-), c_M^-, z_M^-) : e(\partial_t b) \,\mathrm{d}x\mathrm{d}s \\
&\qquad + \underbrace{\int_{d_M^-(t_1)}^{d_M(t_2)} \int_\Omega \partial_c W_{\mathrm{el}}(e(u_M^- + b_M - b_M^-), \hat{c}_M, z_M^-) \partial_t \hat{c}_M \,\mathrm{d}x\mathrm{d}s}_{(\star)} \\
&\qquad + \underbrace{\int_{d_M^-(t_1)}^{d_M(t_2)} \int_\Omega \nabla \hat{c}_M \cdot \nabla \partial_t \hat{c}_M + \partial_c W_{\mathrm{ch}}(\hat{c}_M) \partial_t \hat{c}_M \,\mathrm{d}x\mathrm{d}s}_{(\star\star)} \\
&\qquad + \underbrace{\int_{d_M^-(t_1)}^{d_M(t_2)} \int_\Omega \partial_z W_{\mathrm{el}}(e(u_M^- + b_M - b_M^-), c_M, \hat{z}_M) \, \partial_t \hat{z}_M + |\nabla \hat{z}_M|^{p-2} \nabla \hat{z}_M \cdot \nabla \partial_t \hat{z}_M \,\mathrm{d}x\mathrm{d}s}_{(\star\star\star)}
\end{aligned}
\tag{42}
$$

Putting $a := \nabla z_M^-(t,x)$ and $b := \nabla z_M(t,x)$ in Lemma A.2 we obtain the following elementary inequality

$$(|\nabla \hat{z}_M(t,x)|^{p-2} \nabla \hat{z}_M(t,x) - |\nabla z_M(t,x)|^{p-2} \nabla z_M(t,x)) \cdot \nabla \partial_t \hat{z}_M(t,x) \leq 0.$$

This and (30) tested with $\zeta = -\partial_t \hat{z}_M(t)$ for $\nu = \tau$ and integrated from $t = 0$ to $t = T$ leads to the estimate:

$$
\begin{aligned}
(\star\star\star) \leq\ & - \int_{d_M^-(t_1)}^{d_M(t_2)} \int_\Omega -\alpha \partial_t \hat{z}_M + \beta |\partial_t \hat{z}_M|^2 \,\mathrm{d}x\mathrm{d}s \\
& + \underbrace{\int_{d_M^-(t_1)}^{d_M(t_2)} \int_\Omega (\partial_z W_{\mathrm{el}}(e(u_M^- + b_M - b_M^-), c_M, \hat{z}_M) - \partial_z W_{\mathrm{el}}(e(u_M), c_M, z_M)) \partial_t \hat{z}_M \,\mathrm{d}x\mathrm{d}s}_{=:\kappa_M^3}
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
(\star) \leq\ & \int_{d_M^-(t_1)}^{d_M(t_2)} \int_\Omega \partial_c W_{\mathrm{el}}(e(u_M), c_M, z_M) \partial_t \hat{c}_M \,\mathrm{d}x\mathrm{d}s \\
& + \underbrace{\int_{d_M^-(t_1)}^{d_M(t_2)} \int_\Omega (\partial_c W_{\mathrm{el}}(e(u_M^- + b_M - b_M^-), \hat{c}_M, z_M^-) - \partial_c W_{\mathrm{el}}(e(u_M), c_M, z_M)) \partial_t \hat{c}_M \,\mathrm{d}x\mathrm{d}s}_{=:\kappa_M^1} .
\end{aligned}
$$

Using the elementary estimate $(\nabla\hat{c}_M - \nabla c_M)\nabla\partial_t\hat{c}_M \le 0$:

$$(\star\star) \le \int_{d_M^-(t_1)}^{d_M(t_2)}\int_\Omega \nabla c_M\cdot\nabla\partial_t\hat{c}_M + \partial_c W_{\mathrm{ch}}(c_M)\partial_t\hat{c}_M\,\mathrm{d}x\mathrm{d}s \\ + \underbrace{\int_{d_M^-(t_1)}^{d_M(t_2)}\int_\Omega (\partial_c W_{\mathrm{ch}}(\hat{c}_M) - \partial_c W_{\mathrm{ch}}(c_M))\partial_t\hat{c}_M\,\mathrm{d}x\mathrm{d}s}_{=:\kappa_M^2}$$

Hence, applying equations (28) and (27) shows

$$\int_{d_M^-(t_1)}^{d_M(t_2)}\langle \mathrm{d}_c\tilde{\mathcal{E}}_\varepsilon(q_M),\partial_t\hat{c}_M\rangle_{(H^1)^*\times H^1}\,\mathrm{d}s = \int_{d_M^-(t_1)}^{d_M(t_2)}\int_\Omega \mu_M\partial_t\hat{c}_M - \varepsilon|\partial_t\hat{c}_M|^2\,\mathrm{d}x\mathrm{d}s \\ = \int_{d_M^-(t_1)}^{d_M(t_2)}\int_\Omega -|\nabla\mu_M|^2 - \varepsilon|\partial_t\hat{c}_M|^2\,\mathrm{d}x\mathrm{d}s.$$

Thus

$$(\star) + (\star\star) \le \int_{d_M^-(t_1)}^{d_M(t_2)}\int_\Omega -|\nabla\mu_M|^2 - \varepsilon|\partial_t\hat{c}_M|^2\,\mathrm{d}x\mathrm{d}s + \kappa_M^1 + \kappa_M^2.$$

Lebesgue's generalized convergence theorem, the growth conditions (GC4), (GC5), (GC6) and Corollary 6.9 ensure that κ_M^1, κ_M^2 and κ_M^3 converges to 0 as $M\to\infty$. Here we want to emphasize that we need boundedness of $\partial_t\hat{c}_M$ and $\partial_t\hat{z}_M$ in $L^2(\Omega_T)$ and the convergence $e(u_M)\to e(u)$ in $L^4(\Omega_T)$ which we have only due to the regularization for every fixed $\varepsilon>0$ as $M\to\infty$ (see Corollary 6.9). To finish the proof, set $\kappa_M := \kappa_M^1 + \kappa_M^2 + \kappa_M^3$. ∎

We are now in the position to prove the existence theorem for viscous systems.

Proof of Theorem 5.4. The proof is divided into several steps:

(i) Using growth conditions (GC4), (GC6), (11a), Corollary 6.9 and Lebesgue's generalized convergence theorem, we can pass to $M\to\infty$ in the time integrated version of the integral equations (27), (28) and (29). This shows (i) and (ii) of Definition 5.3.

(ii) Let $0\le t_1 < t_2 \le T$ be arbitrary. Because of $d_M^-(t_1)\le t_1 < t_2 \le d_M(t_2)$, Lemma 6.10 particularly implies

$$\mathcal{E}_\varepsilon(q_M(t_2)) + \int_{t_1}^{t_2}\int_\Omega -\alpha\partial_t\hat{z}_M + \beta|\partial_t\hat{z}_M|^2 + \varepsilon|\partial_t\hat{c}_M|^2 + |\nabla\mu_M|^2\,\mathrm{d}x\mathrm{d}t - \mathcal{E}_\varepsilon(q_M^-(t_1)) \\ \le \int_{d_M^-(t_1)}^{d_M(t_2)}\int_\Omega \partial_e W_{\mathrm{el}}(e(u_M^- + b - b_M^-), c_M^-, z_M) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}t \\ + \varepsilon\int_{d_M^-(t_1)}^{d_M(t_2)}\int_\Omega |\nabla u_M^- + \nabla b - \nabla b_M^-|^2\nabla(u_M^- + b - b_M^-) : \nabla\partial_t b\,\mathrm{d}x\mathrm{d}t + \kappa_M \tag{43}$$

with $\kappa_M\to 0$ as $M\to\infty$. Growth conditions (GC2), (GC6), Corollary 6.9 and Lebesgue's generalized convergence theorem yield:

$$\mathcal{E}_\varepsilon(q_M(t))\to\mathcal{E}_\varepsilon(q(t)) \text{ and } \mathcal{E}_\varepsilon(q_M^-(t))\to\mathcal{E}_\varepsilon(q(t)) \tag{44}$$

as $M \to \infty$ for a.e. $t \in [0,T]$. A sequentially weakly lower semi-continuity argument based on Corollary 6.9 shows:

$$\begin{aligned}\liminf_{M\to\infty} \int_{t_1}^{t_2} \int_\Omega -\alpha\partial_t \hat{z}_M + \beta|\partial_t \hat{z}_M|^2 + \varepsilon|\partial_t \hat{c}_M|^2 + |\nabla\mu_M|^2 \,\mathrm{d}x\mathrm{d}t \\ \geq \int_\Omega \alpha(z(t_1) - z(t_2))\,\mathrm{d}x + \int_{t_1}^{t_2}\int_\Omega \beta|\partial_t z|^2 + \varepsilon|\partial_t c|^2 + |\nabla\mu|^2 \,\mathrm{d}x\mathrm{d}t \end{aligned} \tag{45}$$

Growth condition (11a), Corollary 6.9 and Lebesgue's generalized convergence theorem show:

$$\begin{aligned}\partial_e W_{\mathrm{el}}(e(u_M^- + b - b_M^-), c_M^-, z_M) &\overset{\star}{\rightharpoonup} \partial_e W_{\mathrm{el}}(e(u), c, z) && \text{in } L^\infty([0,T]; L^2(\Omega)),\\ |\nabla u_M^- + \nabla b - \nabla b_M^-|^2 \nabla(u_M^- + b - b_M^-) &\overset{\star}{\rightharpoonup} |\nabla u|^2\nabla u && \text{in } L^\infty([0,T]; L^{4/3}(\Omega))\end{aligned}$$

Since $e(\partial_t b) \in L^1([0,T]; L^2(\Omega))$ and $\nabla\partial_t b \in L^1([0,T]; L^4(\Omega))$, we get:

$$\begin{aligned}&\int_{d_M^-(t_1)}^{d_M(t_2)} \int_\Omega \partial_e W_{\mathrm{el}}(e(u_M^- + b - b_M^-), c_M^-, z_M) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}t \\ &\qquad\qquad \to \int_{t_1}^{t_2}\int_\Omega \partial_e W_{\mathrm{el}}(e(u), c, z) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}t,\\ &\int_{d_M^-(t_1)}^{d_M(t_2)} \int_\Omega |\nabla u_M^- + \nabla b - \nabla b_M^-|^2\nabla(u_M^- + b - b_M^-) : \nabla\partial_t b\,\mathrm{d}x\mathrm{d}t \\ &\qquad\qquad \to \int_{t_1}^{t_2}\int_\Omega |\nabla u|^2\nabla u : \nabla\partial_t b\,\mathrm{d}x\mathrm{d}t\end{aligned} \tag{46}$$

Now, using (44), (45) and (46) gives (iv) of Definition 5.3 by passing to $M \to \infty$ in (43) for a subsequence.

(iii) Let $\tilde{\zeta} \in L^p([0,T]; W_-^{1,p}(\Omega)) \cap L^\infty(\Omega_T)$ be a test-function with $\{\tilde{\zeta} = 0\} \supseteq \{z = 0\}$. Applying Lemma 4.2 with $f = z$ and $f_M = z_M$ and $\zeta = -\tilde{\zeta}$ gives a sequence of approximations $\{\zeta_M\}_{M\in\mathbb{N}} \subseteq L^p([0,T]; W_+^{1,p}(\Omega)) \cap L^\infty(\Omega_T)$ with the properties:

$$\zeta_M \to -\tilde{\zeta} \text{ in } L^p([0,T]; W^{1,p}(\Omega)) \text{ as } M \to \infty, \tag{47a}$$

$$0 \leq \nu_{M,t}\zeta_M(t) \leq z_M(t) \text{ a.e. in } \Omega \text{ for a.e. } t \in [0,T] \text{ and all } M \in \mathbb{N}. \tag{47b}$$

Let $\tilde{\zeta}_M$ denote the function $-\zeta_M$. Then (47b) in particular implies $0 \leq \nu_{M,t}\tilde{\zeta}_M(t) + z_M(t) \leq z_M^-(t)$ a.e. in Ω for a.e. $t \in [0,T]$. Now (30) holds for $\zeta = \tilde{\zeta}_M(t)$. Integration from $t = 0$ to $t = T$ and using growth condition (GC5), Corollary 6.9 and Lebesgue's generalized convergence theorem, as well as the strong convergence (47a) yield for $M \to \infty$:

$$-\int_{\Omega_T} |\nabla z|^{p-2}\nabla z \cdot \nabla\tilde{\zeta} + \partial_z W_{\mathrm{el}}(e(u), c, z)\tilde{\zeta} - \alpha\tilde{\zeta} + \beta(\partial_t z)\tilde{\zeta}\,\mathrm{d}x\mathrm{d}t \leq 0. \tag{48}$$

(iv) Define the function

$$r := -\chi_{\{z=0\}}[\partial_z W_{\mathrm{el}}(e(u), c, z)]^+.$$

It follows directly for any $\zeta \in W_+^{1,p}(\Omega)$ and a.e. $t \in [0,T]$:

$$\langle r(t), \zeta - z(t)\rangle = -\int_{\{z(t)=0\}} [\partial_z W_{\mathrm{el}}(e(u(t)), c(t), z(t))]^+(\zeta - z(t))\,\mathrm{d}x \leq 0$$

Let $\zeta \in L^p([0,T];W_-^{1,p}(\Omega)) \cap L^\infty(\Omega_T)$ be a test-function. We define the subset $T_\delta \subseteq [0,T]$ as

$$T_\delta := \{t \in [0,T] \,|\, \overline{\Omega} \setminus B_\delta(\{z(t)=0\}) \neq \emptyset\}, \qquad \delta > 0,$$

which is measurable. To proceed, we construct a sequence of approximations $\{\zeta_\delta\}_{\delta \in (0,1]} \subseteq L^p([0,T];W_-^{1,p}(\Omega)) \cap L^\infty(\Omega_T)$ in the following way:

$$\zeta_\delta(t) := \begin{cases} \max\left\{\zeta(t), -z(t)\frac{\|\zeta(t)\|_{L^\infty(\Omega)}}{C_{\delta,t}}\right\} & \text{if } t \in T_\delta, \\ 0 & \text{else.} \end{cases}$$

with the constant $C_{\delta,t} := \inf\{z(t,x) \,|\, x \in \overline{\Omega} \setminus B_\delta(\{z(t)=0\})\}$, $t \in T_\delta$. Note that for every $\delta \in (0,1]$ and every $t \in T_\delta$ the constant $C_{\delta,t}$ is greater than 0 since $z(t)$ is continuous on $\overline{\Omega}$. For each $t \in T_\delta$ and each $\delta \in (0,1]$ we construct a partition of $\overline{\Omega}$ of the form

$$\overline{\Omega} = A_{\delta,t} \cup B_{\delta,t}^{\leq} \cup B_{\delta,t}^{>} \cup \{z(t)=0\},$$

where

$$\begin{aligned} A_{\delta,t} &:= \overline{\Omega} \setminus B_\delta(\{z(t)=0\}), \\ B_{\delta,t} &:= (\overline{\Omega} \cap B_\delta(\{z(t)=0\})) \setminus \{z(t)=0\}, \\ B_{\delta,t}^{\leq} &:= B_{\delta,t} \cap \left\{\zeta(t) \leq -z(t)\frac{\|\zeta(t)\|_{L^\infty(\Omega)}}{C_{\delta,t}}\right\}, \\ B_{\delta,t}^{>} &:= B_{\delta,t} \cap \left\{\zeta(t) > -z(t)\frac{\|\zeta(t)\|_{L^\infty(\Omega)}}{C_{\delta,t}}\right\}. \end{aligned}$$

We obtain the following properties for the sequence $\{\zeta_\delta\}_{\delta \in (0,1]}$:

$$\zeta_\delta = 0 \text{ on } \{z=0\}, \tag{49a}$$

$$\zeta_\delta = 0 \text{ on } ([0,T] \setminus T_\delta) \times \overline{\Omega}, \tag{49b}$$

$$\zeta_\delta = \zeta \text{ on } \{(t,x) \in \overline{\Omega}_T \,|\, x \in A_{\delta,t}\}, \tag{49c}$$

$$\zeta_\delta = \zeta \text{ on } \{(t,x) \in \overline{\Omega}_T \,|\, t \in T_\delta \text{ and } x \in B_{\delta,t}^{>}\}, \tag{49d}$$

$$\zeta_\delta = -z\frac{\|\zeta(t)\|_{L^\infty(\Omega)}}{C_{\delta,t}} \text{ on } \{(t,x) \in \overline{\Omega}_T \,|\, t \in T_\delta \text{ and } x \in B_{\delta,t}^{\leq}\}, \tag{49e}$$

$$\zeta_\delta \overset{\star}{\rightharpoonup} \zeta \text{ in } L^\infty(\{z>0\}) \text{ as } \delta \searrow 0. \tag{49f}$$

The last property follows from $\zeta_\delta \to \zeta$ a.e. in $\{z>0\}$ (because of (49c) and since Lemma 4.3 implies $\mathcal{L}^{n+1}\left(\{z>0\} \setminus \{(t,x) \in \overline{\Omega}_T \,|\, x \in A_{\delta,t}\}\right) \to 0$ as $\delta \searrow 0$) and the boundedness of $\{\zeta_\delta\}$ in the space $L^\infty(\Omega_T)$ with respect to $\delta \in (0,1]$. Since $\zeta(t)$ and $z(t)$ are functions in $W^{1,p}(\Omega)$, property (49d) and (49e) imply for almost every $t \in T_\delta$ and every $\delta \in (0,1]$ (cf. [Zie89]):

$$\nabla\zeta_\delta(t) = \nabla\zeta(t) \text{ a.e. in } B_{\delta,t}^{>}, \tag{50a}$$

$$\nabla\zeta_\delta(t) = -\nabla z(t)\frac{\|\zeta(t)\|_{L^\infty(\Omega)}}{C_{\delta,t}} \text{ a.e. in } B_{\delta,t}^{\leq}. \tag{50b}$$

and (49c) implies for a.e. $t \in [0,T]$ and every $\delta \in (0,1]$:

$$\nabla\zeta_\delta(t) = \nabla\zeta(t) \text{ a.e. in } A_{\delta,t}. \tag{51}$$

We get the estimates:

$$
\begin{aligned}
&-\int_{\Omega_T}|\nabla z|^{p-2}\nabla z\cdot\nabla\zeta+\partial_z W_{\mathrm{el}}(e(u),c,z)\zeta-\alpha\zeta+\beta\partial_t z\zeta\,\mathrm{d}x\mathrm{d}t-\int_0^T\langle r(t),\zeta(t)\rangle\,\mathrm{d}t\\
&\quad\le-\int_{\Omega_T}|\nabla z|^{p-2}\nabla z\cdot\nabla(\zeta-\zeta_\delta)\,\mathrm{d}x\mathrm{d}t-\int_{\Omega_T}|\nabla z|^{p-2}\nabla z\cdot\nabla\zeta_\delta\,\mathrm{d}x\mathrm{d}t\\
&\qquad\underbrace{-\int_{\Omega_T}(\partial_z W_{\mathrm{el}}(e(u),c,z)-\alpha+\beta\partial_t z)(\zeta-\zeta_\delta)\,\mathrm{d}x\mathrm{d}t+\int_{\{z=0\}}[\partial_z W_{\mathrm{el}}(e(u),c,z)]^+\zeta\,\mathrm{d}x\mathrm{d}t}_{\le-\int_{\{z>0\}}(\partial_z W_{\mathrm{el}}(e(u),c,z)-\alpha+\beta\partial_t z)(\zeta-\zeta_\delta)\,\mathrm{d}x\mathrm{d}t\text{ by (49a)}}\\
&\qquad-\int_{\Omega_T}(\partial_z W_{\mathrm{el}}(e(u),c,z)-\alpha+\beta\partial_t z)\zeta_\delta\,\mathrm{d}x\mathrm{d}t\\
&\quad\le-\int_{\{z>0\}}|\nabla z|^{p-2}\nabla z\cdot\nabla(\zeta-\zeta_\delta)\,\mathrm{d}x\mathrm{d}t\\
&\qquad-\int_{\{z>0\}}(\partial_z W_{\mathrm{el}}(e(u),c,z)-\alpha+\beta\partial_t z)(\zeta-\zeta_\delta)\,\mathrm{d}x\mathrm{d}t\\
&\qquad\underbrace{-\int_{\Omega_T}|\nabla z|^{p-2}\nabla z\cdot\nabla\zeta_\delta\,\mathrm{d}x\mathrm{d}t-\int_{\Omega_T}(\partial_z W_{\mathrm{el}}(e(u),c,z)-\alpha+\beta\partial_t z)\zeta_\delta\,\mathrm{d}x\mathrm{d}t}_{\le 0\text{ by (48) with }\zeta_\delta}\\
&\quad\le-\int_{\{z>0\}}|\nabla z|^{p-2}\nabla z\cdot\nabla(\zeta-\zeta_\delta)\,\mathrm{d}x\mathrm{d}t\\
&\qquad-\int_{\{z>0\}}(\partial_z W_{\mathrm{el}}(e(u),c,z)-\alpha+\beta\partial_t z)(\zeta-\zeta_\delta)\,\mathrm{d}x\mathrm{d}t
\end{aligned}
\tag{52}
$$

The second term on the right hand side converges to 0 as δ goes to 0 because of (49f). The first term can be estimated in the following way:

$$
\begin{aligned}
&-\int_{\{z>0\}}|\nabla z|^{p-2}\nabla z\cdot\nabla(\zeta-\zeta_\delta)\,\mathrm{d}x\mathrm{d}t\\
&\quad=\underbrace{-\int_0^T\int_{A_{\delta,t}}|\nabla z|^{p-2}\nabla z\cdot\nabla(\zeta-\zeta_\delta)\,\mathrm{d}x\mathrm{d}t}_{=0\text{ by (51)}}-\int_0^T\int_{B_{\delta,t}}|\nabla z|^{p-2}\nabla z\cdot\nabla(\zeta-\zeta_\delta)\,\mathrm{d}x\mathrm{d}t\\
&\quad=-\int_{T_\delta}\int_{B_{\delta,t}}|\nabla z|^{p-2}\nabla z\cdot\nabla(\zeta-\zeta_\delta)\,\mathrm{d}x\mathrm{d}t\\
&\qquad-\int_{[0,T]\setminus T_\delta}\int_{B_{\delta,t}}|\nabla z|^{p-2}\nabla z\cdot\nabla\zeta\,\mathrm{d}x\mathrm{d}t+\underbrace{\int_{[0,T]\setminus T_\delta}\int_{B_{\delta,t}}|\nabla z|^{p-2}\nabla z\cdot\nabla\zeta_\delta\,\mathrm{d}x\mathrm{d}t}_{=0\text{ by (49b)}}\\
&\quad=-\int_{T_\delta}\int_{B_{\delta,t}^{\le}}|\nabla z|^{p-2}\nabla z\cdot\nabla(\zeta-\zeta_\delta)\,\mathrm{d}x\mathrm{d}t\underbrace{-\int_{T_\delta}\int_{B_{\delta,t}^{>}}|\nabla z|^{p-2}\nabla z\cdot\nabla(\zeta-\zeta_\delta)\,\mathrm{d}x\mathrm{d}t}_{=0\text{ by (50a)}}\\
&\qquad-\int_{[0,T]\setminus T_\delta}\int_{B_{\delta,t}}|\nabla z|^{p-2}\nabla z\cdot\nabla\zeta\,\mathrm{d}x\mathrm{d}t
\end{aligned}
$$

$$= \underbrace{-\int_{T_\delta}\int_{B^{\le}_{\delta,t}} |\nabla z|^{p-2}\nabla z\cdot\nabla\zeta\,\mathrm{d}x\mathrm{d}t}_{(\star)} + \underbrace{\int_{T_\delta}\int_{B^{\le}_{\delta,t}} |\nabla z|^{p-2}\nabla z\cdot\nabla\zeta_\delta\,\mathrm{d}x\mathrm{d}t}_{=\int_{T_\delta}\int_{B^{\le}_{\delta,t}}|\nabla z|^{p-2}\nabla z\cdot\left(-\nabla z\frac{\|\zeta(t)\|_{L^\infty(\Omega)}}{C_{\delta,t}}\right)\mathrm{d}x\mathrm{d}t\le 0\text{ by (50b)}}$$

$$\underbrace{-\int_{[0,T]\setminus T_\delta}\int_{B_{\delta,t}} |\nabla z|^{p-2}\nabla z\cdot\nabla\zeta\,\mathrm{d}x\mathrm{d}t}_{(\star\star)}$$

Both, $(\star)$ as well as $(\star\star)$ can be estimated above by $\int_0^T\int_{B_{\delta,t}}|\nabla z|^{p-1}|\nabla\zeta|\,\mathrm{d}x\mathrm{d}t$ which converges to 0 as $\delta\searrow 0$ by Vitali's convergence theorem. Indeed, Lemma 4.3 shows $\mathcal{L}^{n+1}(\{(t,x)\in\overline{\Omega}_T\,|\,x\in B_{\delta,t}\})\to 0$ as $\delta\searrow 0$. Hence, passing to $\delta\searrow 0$ in (52) shows (iii) of Definition 5.3. ■

6.2 Vanishing viscosity: $\varepsilon\searrow 0$

For each $\varepsilon\in(0,1]$ we denote with $q_\varepsilon=(u_\varepsilon,c_\varepsilon,z_\varepsilon)\in\mathcal{Q}^{\mathrm{v}}$ a viscous solution according to Theorem 5.4. By the use of Lemma 6.12, Lemma 6.13 and Lemma 6.14 below we identify a suitable subsequence where we can pass to the limit. Let us first summarize the equalities and inequalities which hold for q_ε.

Summary 6.11 *For $q_\varepsilon=(u_\varepsilon,c_\varepsilon,z_\varepsilon)\in\mathcal{Q}^{\mathrm{v}}$, $\varepsilon\in(0,1]$, the following properties are satisfied:*

(i) for all $\zeta\in L^2([0,T];H^1(\Omega))$:

$$\int_{\Omega_T}(\partial_t c_\varepsilon)\zeta\,\mathrm{d}x\mathrm{d}t = -\int_{\Omega_T}\nabla\mu_\varepsilon\cdot\nabla\zeta\,\mathrm{d}x\mathrm{d}t \tag{53}$$

(ii) for all $\zeta\in L^2([0,T];H^1(\Omega))$:

$$\int_{\Omega_T}\mu_\varepsilon\zeta\,\mathrm{d}x\mathrm{d}t = \int_{\Omega_T}\nabla c_\varepsilon\cdot\nabla\zeta + (\partial_c W_{\mathrm{ch}}(c_\varepsilon)+\partial_c W_{\mathrm{el}}(e(u_\varepsilon),c_\varepsilon,z_\varepsilon)+\varepsilon(\partial_t c_\varepsilon))\zeta\,\mathrm{d}x\mathrm{d}t \tag{54}$$

(iii) for all $\zeta\in L^4([0,T];W^{1,4}_\Gamma(\Omega;\mathbb{R}^n))$:

$$\int_{\Omega_T}\partial_e W_{\mathrm{el}}(e(u_\varepsilon),c_\varepsilon,z_\varepsilon):e(\zeta)+\varepsilon|\nabla u_\varepsilon|^2\nabla u_\varepsilon:\nabla\zeta\,\mathrm{d}x\mathrm{d}t=0 \tag{55}$$

(iv) for all $\zeta\in L^p([0,T];W^{1,p}_-(\Omega))\cap L^\infty(\Omega_T)$:

$$0\le\int_{\Omega_T}|\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon\cdot\nabla\zeta+(\partial_z W_{\mathrm{el}}(e(u_\varepsilon),c_\varepsilon,z_\varepsilon)-\alpha+\beta\partial_t z_\varepsilon+r_\varepsilon)\zeta\,\mathrm{d}x\mathrm{d}t \tag{56}$$

with

$$r_\varepsilon=-\chi_{\{z_\varepsilon=0\}}[\partial_z W_{\mathrm{el}}(e(u_\varepsilon),c_\varepsilon,z_\varepsilon)]^+ \tag{57}$$

(v) for a.e. $t \in [0,T]$ and for all $\zeta \in W^{1,p}_+(\Omega)$:

$$\int_\Omega r_\varepsilon(t)(\zeta - z_\varepsilon(t))\,\mathrm{d}x \le 0 \tag{58}$$

(vi) for a.e. $0 \le t_1 \le t_2 \le T$:

$$\begin{aligned}\mathcal{E}_\varepsilon(q_\varepsilon(t_2)) + \int_\Omega \alpha(z_\varepsilon(t_1) - z_\varepsilon(t_2))\,\mathrm{d}x + \int_{t_1}^{t_2}\int_\Omega \beta|\partial_t z_\varepsilon|^2 + |\nabla\mu_\varepsilon|^2 + \varepsilon|\partial_t c_\varepsilon|^2\,\mathrm{d}x\mathrm{d}t - \mathcal{E}_\varepsilon(q_\varepsilon(t_1))\\ \le \int_{t_1}^{t_2}\int_\Omega \partial_e W_{\mathrm{el}}(e(u_\varepsilon), c_\varepsilon, z_\varepsilon) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}t + \varepsilon\int_{t_1}^{t_2}\int_\Omega |\nabla u_\varepsilon|^2 \nabla u_\varepsilon : \nabla\partial_t b\,\mathrm{d}x\mathrm{d}t\end{aligned} \tag{59}$$

Lemma 6.12 (A-priori estimates) *The following estimates hold ($C > 0$ is independent of $\varepsilon > 0$):*

(i) $\|u_\varepsilon\|_{L^\infty([0,T];H^1(\Omega;\mathbb{R}^n))} \le C$

(ii) $\varepsilon^{1/4}\|u_\varepsilon\|_{L^\infty([0,T];W^{1,4}(\Omega;\mathbb{R}^n))} \le C$

(iii) $\|c_\varepsilon\|_{L^\infty([0,T];H^1(\Omega))} \le C$

(iv) $\|z_\varepsilon\|_{L^\infty([0,T];W^{1,p}(\Omega))} \le C$

(v) $\varepsilon^{1/2}\|\partial_t c_\varepsilon\|_{L^2(\Omega_T)} \le C$

(vi) $\|\partial_t z_\varepsilon\|_{L^2(\Omega_T)} \le C$

(vii) $\|\mu_\varepsilon\|_{L^2([0,T];H^1(\Omega))} \le C$

for all $\varepsilon \in (0,1]$.

Proof. According to Lemma 6.3 the discretization $q_{M,\varepsilon}$ of q_ε fulfills

$$\mathcal{E}_\varepsilon(q_{M,\varepsilon}(t)) + \int_0^{d_M(t)} \mathcal{R}(\partial_t \hat{z}_{M,\varepsilon})\,\mathrm{d}s + \int_0^{d_M(t)}\int_\Omega \frac{\varepsilon}{2}|\partial_t \hat{c}_{M,\varepsilon}|^2 + \frac{1}{2}|\nabla\mu_{M,\varepsilon}|^2\,\mathrm{d}x\mathrm{d}s \le C(\mathcal{E}_\varepsilon(q^0_\varepsilon) + 1). \tag{60}$$

where C is independent of M, t, ε. By the minimizing property of q^0_ε we also obtain $\mathcal{E}_\varepsilon(q^0_\varepsilon) \le \mathcal{E}_\varepsilon(q^0_1) \le \mathcal{E}_1(q^0_1)$ for all $\varepsilon \in (0,1]$. Therefore, the left hand side of (60) is bounded with respect to $M \in \mathbb{N}$, $t \in [0,T]$ and $\varepsilon \in (0,1]$. This leads to the boundedness of

$$\mathcal{E}_\varepsilon(q_\varepsilon(t)) + \int_0^t \mathcal{R}(\partial_t z_\varepsilon)\,\mathrm{d}s + \int_0^t\int_\Omega \frac{\varepsilon}{2}|\partial_t c_\varepsilon|^2 + \frac{1}{2}|\nabla\mu_\varepsilon|^2\,\mathrm{d}x\mathrm{d}s \le C \tag{61}$$

for a.e. $t \in [0,T]$ and for all $\varepsilon \in (0,1]$. We immediately obtain (iv), (v) and (vi). Due to $\int c_\varepsilon(t)\,\mathrm{d}x = \mathrm{const}$ and the boundedness of $\|\nabla c_\varepsilon(t)\|_{L^2(\Omega)}$, Poincaré's inequality yields (iii). Now using (61), growth conditions (11b) and Korn's inequality we attain the desired a-priori estimates (i) and (ii). Using (54) and (53) show boundedness of $\int_\Omega \mu_\varepsilon(t)\,\mathrm{d}x$. Since $\|\nabla\mu_\varepsilon(t)\|_{L^2(\Omega_T)}$ is also bounded, Poincaré's inequality yields (vii). ■

Lemma 6.13 (Weak convergence of the viscous solutions) *There exists a subsequence of $\{q_\varepsilon\}$ (which is also denoted by $\{q_\varepsilon\}$) and an element $(u,c,z) = q \in \mathcal{Q}$ with $z(0) = z^0$ such that:*

(i) $z_\varepsilon \overset{\star}{\rightharpoonup} z$ *in* $L^\infty([0,T];W^{1,p}(\Omega))$,
$z_\varepsilon(t) \rightharpoonup z(t)$ *in* $W^{1,p}(\Omega)$ *for a.e.* $t \in [0,T]$,
$z_\varepsilon \to z$ *a.e. in* Ω_T *and*
$z_\varepsilon \rightharpoonup z$ *in* $H^1([0,T];L^2(\Omega))$

(ii) $c_\varepsilon \overset{\star}{\rightharpoonup} c$ *in* $L^\infty([0,T];H^1(\Omega))$,
$c_\varepsilon(t) \rightharpoonup c(t)$ *in* $H^1(\Omega)$ *for a.e.* $t \in [0,T]$ *and*
$c_\varepsilon \to c$ *a.e. in* Ω_T

(iii) $u_\varepsilon \overset{\star}{\rightharpoonup} u$ *in* $L^\infty([0,T];H^1(\Omega;\mathbb{R}^n))$

(iv) $\mu_\varepsilon \rightharpoonup \mu$ *in* $L^2([0,T];H^1(\Omega))$

as $\varepsilon \searrow 0$.

Proof.

(i) This property follows from the boundedness of $\{z_\varepsilon\}$ in $L^\infty([0,T];W^{1,p}(\Omega))$ and in $H^1([0,T];L^2(\Omega))$ (see proof of Lemma 6.12). The function z obtained in this way is monotonically decreasing with respect to t.

(ii) We know from the boundedness of $\{\nabla\mu_\varepsilon\}$ in $L^2(\Omega_T)$ that $\{\partial_t c_\varepsilon\}$ is also bounded in $L^2([0,T];(H^1(\Omega))^*)$ with respect to ε by using equation (53). This and the boundedness of $\{c_\varepsilon\}$ in $L^2([0,T];H^1(\Omega))$ (see Lemma 6.12) shows that c_ε converges strongly to an element c in $L^2(\Omega_T)$ as $\varepsilon \searrow 0$ for a subsequence by a compactness results from J. P. Aubin and J. L. Lions (see [Sim86]). Thus we can extract a subsequence such that $c_\varepsilon(t) \rightharpoonup c(t)$ in $H^1(\Omega)$ for a.e. $t \in [0,T]$ and $c_\varepsilon \to c$ a.e. in Ω_T as well as $c_\varepsilon \overset{\star}{\rightharpoonup} c$ in $L^\infty([0,T];H^1(\Omega))$ by the boundedness of $\{c_\varepsilon\}$ in $L^\infty([0,T];H^1(\Omega))$.

(iii) This property follows from the boundedness of $\{u_\varepsilon\}$ in $L^\infty([0,T];H^1(\Omega;\mathbb{R}^n))$.

(iv) This property follows from the boundedness of $\{\mu_\varepsilon\}$ in $L^2([0,T];H^1(\Omega))$. ■

Lemma 6.14 (Strong convergence of the viscous solutions) *The following convergence properties are satisfied:*

(i) $u_\varepsilon \to u$ *in* $L^2([0,T];H^1(\Omega;\mathbb{R}^n))$

(ii) $c_\varepsilon \to c$ *in* $L^2([0,T];H^1(\Omega))$

(iii) $z_\varepsilon \to z$ *in* $L^p([0,T];W^{1,p}(\Omega))$

as $\varepsilon \searrow 0$ *for a subsequence of* $\{q_\varepsilon\}$.

Proof.

(i) We consider an approximation sequence $\{\tilde{u}_\delta\}_{\delta\in(0,1]} \subseteq L^4([0,T];W^{1,4}(\Omega))$ with

$$\tilde{u}_\delta \to u \text{ in } L^2([0,T];H^1(\Omega)) \text{ as } \delta \searrow 0, \tag{62a}$$
$$\tilde{u}_\delta - b \in L^4([0,T];W^{1,4}_\Gamma(\Omega)) \text{ for all } \delta > 0. \tag{62b}$$

Since ε and δ are independent, we consider a sequence $\{\delta_\varepsilon\}_{\varepsilon\in(0,1]}$ with

$$\varepsilon^{1/4}\|\nabla\tilde{u}_{\delta_\varepsilon}\|_{L^4(\Omega_T)} \to 0 \text{ and } \delta_\varepsilon \searrow 0 \text{ as } \varepsilon \searrow 0. \tag{63}$$

Testing (55) with $\zeta = u_\varepsilon - \tilde{u}_{\delta_\varepsilon}$ (possible due to (62b)) and applying uniform monotonicity of $\partial_e W_{\mathrm{el}}$ (assumption (GC1)) and Lemma A.1 (compare with (38)):

$$\begin{aligned}
&\frac{\eta}{2}\|e(u_\varepsilon)-e(u)\|^2_{L^2(\Omega_T)}\\
&\quad\le \eta\|e(u)-e(\tilde{u}_{\delta_\varepsilon})\|^2_{L^2(\Omega_T)} + \eta\|e(u_\varepsilon)-e(\tilde{u}_{\delta_\varepsilon})\|^2_{L^2(\Omega_T)} + \varepsilon C_{\mathrm{ineq}}^{-1}\|\nabla u_\varepsilon - \nabla\tilde{u}_{\delta_\varepsilon}\|^4_{L^4(\Omega_T)}\\
&\quad\le \eta\|e(u)-e(\tilde{u}_{\delta_\varepsilon})\|^2_{L^2(\Omega_T)}\\
&\qquad + \int_{\Omega_T}(\partial_e W_{\mathrm{el}}(e(u_\varepsilon),c_\varepsilon,z_\varepsilon) - \partial_e W_{\mathrm{el}}(e(\tilde{u}_{\delta_\varepsilon}),c_\varepsilon,z_\varepsilon)) : (e(u_\varepsilon)-e(\tilde{u}_{\delta_\varepsilon}))\,\mathrm{d}x\mathrm{d}t\\
&\qquad + \varepsilon\int_{\Omega_T}(|\nabla u_\varepsilon|^2\nabla u_\varepsilon - |\nabla\tilde{u}_{\delta_\varepsilon}|^2\nabla\tilde{u}_{\delta_\varepsilon}) : (\nabla u_\varepsilon - \nabla\tilde{u}_{\delta_\varepsilon})\,\mathrm{d}x\mathrm{d}t\\
&\quad= \eta\|e(u)-e(\tilde{u}_{\delta_\varepsilon})\|^2_{L^2(\Omega_T)}\\
&\qquad \underbrace{+\int_{\Omega_T}\partial_e W_{\mathrm{el}}(e(u_\varepsilon),c_\varepsilon,z_\varepsilon) : (e(u_\varepsilon)-e(\tilde{u}_{\delta_\varepsilon})) + \varepsilon|\nabla u_\varepsilon|^2\nabla u_\varepsilon : (\nabla u_\varepsilon - \nabla\tilde{u}_{\delta_\varepsilon})\,\mathrm{d}x\mathrm{d}t}_{=0\text{ by (55)}}\\
&\qquad -\int_{\Omega_T}\partial_e W_{\mathrm{el}}(e(\tilde{u}_{\delta_\varepsilon}),c_\varepsilon,z_\varepsilon) : (e(u_\varepsilon)-e(\tilde{u}_{\delta_\varepsilon}))\,\mathrm{d}x\mathrm{d}t\\
&\qquad \underbrace{-\varepsilon\int_{\Omega_T}|\nabla\tilde{u}_{\delta_\varepsilon}|^2\nabla\tilde{u}_{\delta_\varepsilon} : (\nabla u_\varepsilon - \nabla\tilde{u}_{\delta_\varepsilon})\,\mathrm{d}x\mathrm{d}t}_{(\star)}.
\end{aligned} \tag{64}$$

Finally

$$\begin{aligned}
(\star) &\le \varepsilon\|\nabla\tilde{u}_{\delta_\varepsilon}\|^3_{L^4(\Omega_T)}\|\nabla u_\varepsilon - \nabla\tilde{u}_{\delta_\varepsilon}\|_{L^4(\Omega_T)}\\
&\le \Big(\underbrace{\varepsilon^{1/4}\|\nabla\tilde{u}_{\delta_\varepsilon}\|_{L^4(\Omega_T)}}_{\to 0\text{ as }\varepsilon\searrow 0\text{ by (63)}}\Big)^3\Big(\underbrace{\varepsilon^{1/4}\|\nabla u_\varepsilon\|_{L^4(\Omega_T)}}_{\le C\text{ by Lemma 6.12}} + \underbrace{\varepsilon^{1/4}\|\nabla\tilde{u}_{\delta_\varepsilon}\|_{L^4(\Omega_T)}}_{\to 0\text{ as }\varepsilon\searrow 0\text{ by (63)}}\Big).
\end{aligned}$$

From growth condition (11a), Lemma 6.13 and Lebesgue's generalized convergence theorem, we obtain

$$\partial_e W_{\mathrm{el}}(e(\tilde{u}_{\delta_\varepsilon}),c_\varepsilon,z_\varepsilon) \to \partial_e W_{\mathrm{el}}(e(u),c,z) \text{ in } L^2(\Omega_T)$$

for a subsequence $\varepsilon \searrow 0$. By $u_\varepsilon \overset{\star}{\rightharpoonup} u$ in $L^\infty([0,T];H^1(\Omega;\mathbb{R}^n))$ for a subsequence $\varepsilon \searrow 0$ (Lemma 6.13 (iii)) as well as (62a) we also have

$$e(u_\varepsilon) - e(\tilde{u}_{\delta_\varepsilon}) \rightharpoonup 0 \text{ in } L^2(\Omega_T)$$

as $\varepsilon \searrow 0$ for a subsequence. Therefore every term on the right hand side of (64) converges to 0 as $\varepsilon \searrow 0$ for a subsequence. This shows $u_\varepsilon \to u$ in $L^2([0,T];H^1(\Omega;\mathbb{R}^n))$ as $\varepsilon \searrow 0$ for a subsequence.

(ii) Testing (54) with c_ε and c and passing to $\varepsilon \searrow 0$ for a subsequence eventually shows strong convergence $c_\varepsilon \to c$ in $L^2([0,T];H^1(\Omega))$ (see the argumentation in Lemma 6.7 and notice that $\int_{\Omega_T}\varepsilon(\partial_t c_\varepsilon)c_\varepsilon\,\mathrm{d}x\mathrm{d}t \le \varepsilon\|\partial_t c_\varepsilon\|_{L^2(\Omega_T)}\|c_\varepsilon\|_{L^2(\Omega_T)} \to 0$ as $\varepsilon \searrow 0$).

(iii) According to Lemma 4.2 with $f = \zeta = z$ and $f_M = z_{\varepsilon_M}$ (here we choose $\varepsilon_M = 1/M$) we find an approximation sequence $\{\zeta_{\varepsilon_k}\} \subseteq L^p([0,T]; W^{1,p}_+(\Omega)) \cap L^\infty(\Omega_T)$ with $\varepsilon_k \searrow 0$ and the properties:

$$\zeta_{\varepsilon_k} \to z \text{ in } L^p([0,T]; W^{1,p}(\Omega)) \text{ as } k \to \infty, \tag{65a}$$

$$0 \le \zeta_{\varepsilon_k} \le z_{\varepsilon_k} \text{ a.e. in } \Omega_T \text{ for all } k \in \mathbb{N}. \tag{65b}$$

We denote the subsequences also with $\{z_\varepsilon\}$ and $\{\zeta_\varepsilon\}$, respectively. The desired property $z_\varepsilon \to z$ in $L^p([0,T]; W^{1,p}(\Omega))$ as $\varepsilon \searrow 0$ follows with the same estimate as in the proof of Lemma 6.8 by using Lemma A.1 and the integral inequality (56) with $\zeta := \zeta_\varepsilon - z_\varepsilon$ (note that $\langle r_\varepsilon, \zeta_\varepsilon - z_\varepsilon \rangle = 0$ holds by (57) and (65b)). Indeed, we obtain

$$\begin{aligned} C_{\text{ineq}}^{-1} \int_{\Omega_T} |\nabla z_\varepsilon - \nabla z|^p \,\mathrm{d}x\mathrm{d}t \\ \le \underbrace{\|\partial_z W_{\text{el}}(e(u_\varepsilon), c_\varepsilon, z_\varepsilon) - \alpha + \beta \partial_t z_\varepsilon\|_{L^2([0,T];L^1(\Omega))}}_{\text{bounded}} \underbrace{\|\zeta_\varepsilon - z_\varepsilon\|_{L^2([0,T];L^\infty(\Omega))}}_{\to 0} \\ + \underbrace{\|\nabla z_\varepsilon\|^{p-1}_{L^p(\Omega_T)}}_{\text{bounded}} \underbrace{\|\nabla \zeta_\varepsilon - \nabla z\|_{L^p(\Omega_T)}}_{\to 0} - \underbrace{\int_{\Omega_T} |\nabla z|^{p-2} \nabla z \cdot \nabla(z_\varepsilon - z) \,\mathrm{d}x\mathrm{d}t}_{\to 0}. \end{aligned}$$

as $\varepsilon \searrow 0$ for a subsequence. We used here $z_\varepsilon \to z$ and $\zeta_\varepsilon \to z$ in $L^2([0,T]; L^\infty(\Omega))$ as $\varepsilon \searrow 0$ for a subsequence due to Lemma 6.13 and the compact embedding $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$. ■

Corollary 6.15 *The following convergence properties are fulfilled:*

(i) $z_\varepsilon \to z$ *in* $L^p([0,T]; W^{1,p}(\Omega))$,
$z_\varepsilon(t) \to z(t)$ *in* $W^{1,p}(\Omega)$ *for a.e.* $t \in [0,T]$,
$z_\varepsilon \to z$ *a.e. in* Ω_T *and*
$z_\varepsilon \rightharpoonup z$ *in* $H^1([0,T]; L^2(\Omega))$

(ii) $c_\varepsilon \to c$ *in* $L^{2^\star}([0,T]; H^1(\Omega))$,
$c_\varepsilon(t) \to c(t)$ *in* $H^1(\Omega)$ *for a.e.* $t \in [0,T]$ *and*
$c_\varepsilon \to c$ *a.e. in* Ω_T

(iii) $u_\varepsilon \to u$ *in* $L^2([0,T]; H^1(\Omega; \mathbb{R}^n))$,
$u_\varepsilon(t) \to u(t)$ *in* $H^1(\Omega; \mathbb{R}^n)$ *for a.e.* $t \in [0,T]$ *and*
$u_\varepsilon \to u$ *a.e. in* Ω_T

(iv) $\mu_\varepsilon \rightharpoonup \mu$ *in* $L^2([0,T]; H^1(\Omega))$

(v) $\partial_c W_{\text{ch}}(c_\varepsilon) \to \partial_c W_{\text{ch}}(c)$ *in* $L^2(\Omega_T)$

as $\varepsilon \searrow 0$ *for a subsequence of* $\{q_\varepsilon\}$.

Now we are well prepared to prove the main result of this work.

Proof of Theorem 5.6. We can pass to $\varepsilon \searrow 0$ in (54) and (55) by the already known convergence features (see Corollary 6.15) noticing that $\int_{\Omega_T} \varepsilon |\nabla u_\varepsilon|^2 \nabla u_\varepsilon : \nabla \zeta \,\mathrm{d}x\mathrm{d}t$ and $\int_{\Omega_T} \varepsilon (\partial_t c_\varepsilon) \zeta \,\mathrm{d}x\mathrm{d}t$ converge to 0 as $\varepsilon \searrow 0$. We get

$$\int_{\Omega_T} \partial_e W_{\text{el}}(e(u), c, z) : e(\zeta) \,\mathrm{d}x\mathrm{d}t = 0 \tag{66}$$

for all $\zeta \in L^4([0,T]; W^{1,4}_\Gamma(\Omega;\mathbb{R}^n))$. A density argument shows that (66) also holds for all $\zeta \in L^2([0,T]; H^1_\Gamma(\Omega;\mathbb{R}^n))$. Writing (53) in the form

$$\int_{\Omega_T} (c_\varepsilon - c^0)\partial_t\zeta \,\mathrm{d}x\mathrm{d}t = \int_{\Omega_T} \nabla\mu_\varepsilon\cdot\nabla\zeta \,\mathrm{d}x\mathrm{d}t,$$

by only allowing test-functions $\zeta \in L^2([0,T]; H^1(\Omega))$ with $\partial_t\zeta \in L^2(\Omega_T)$ and $\zeta(T) = 0$, we can also pass to $\varepsilon \searrow 0$ by using Corollary 6.15.

To obtain a limit equation in (56) and (58), observe that

$$\begin{aligned} [\partial_z W_{\mathrm{el}}(e(u_\varepsilon), c_\varepsilon, z_\varepsilon)]^+ &\to [\partial_z W_{\mathrm{el}}(e(u), c, z)]^+ &&\text{in } L^1(\Omega_T),\\ \chi_{\{z_\varepsilon=0\}} &\overset{\star}{\rightharpoonup} \chi, &&\text{in } L^\infty(\Omega_T) \end{aligned}$$

for a subsequence $\varepsilon \searrow 0$ and an element $\chi \in L^\infty(\Omega_T)$. Set $r := -\chi[\partial_z W_{\mathrm{el}}(e(u), c, z)]^+$. Keeping (57) into account we find for all $\zeta \in L^\infty(\Omega_T)$:

$$\int_{\Omega_T} r_\varepsilon\zeta \,\mathrm{d}x\mathrm{d}t \to \int_{\Omega_T} r\zeta \,\mathrm{d}x\mathrm{d}t \tag{67}$$

for a subsequence $\varepsilon \searrow 0$. Thus we can also pass to $\varepsilon \searrow 0$ for a subsequence in (56) by using Lebesgue's generalized convergence theorem, growth condition (GC5) and Corollary 6.15 and (67). Let $\xi \in L^\infty([0,T])$ with $\xi \geq 0$ a.e. on $[0,T]$ be a further test-function. Then (58) and (57) imply

$$\begin{aligned} 0 \geq \int_0^T \left(\int_\Omega r_\varepsilon(t)(\zeta - z_\varepsilon(t))\,\mathrm{d}x\right)\xi(t)\,\mathrm{d}t = \int_{\Omega_T} r_\varepsilon(\zeta - z_\varepsilon)\xi \,\mathrm{d}x\mathrm{d}t\\ \to \int_{\Omega_T} r(\zeta - z)\xi \,\mathrm{d}x\mathrm{d}t = \int_0^T \left(\int_\Omega r(t)(\zeta - z(t))\,\mathrm{d}x\right)\xi(t)\,\mathrm{d}t. \end{aligned}$$

This shows $\int_\Omega r(t)(\zeta - z(t))\,\mathrm{d}x \leq 0$ for a.e. $t \in [0,T]$.

It remains to show that (59) also yields to a limit inequality. First observe, that (59) implies:

$$\begin{aligned} &\mathcal{E}_\varepsilon(q_\varepsilon(t_2)) + \int_\Omega \alpha(z_\varepsilon(t_1) - z_\varepsilon(t_2))\,\mathrm{d}x + \int_{t_1}^{t_2}\int_\Omega \beta|\partial_t z_\varepsilon|^2 + |\nabla\mu_\varepsilon|^2 \,\mathrm{d}x\mathrm{d}t - \mathcal{E}_\varepsilon(q_\varepsilon(t_1))\\ &\quad\leq \int_{t_1}^{t_2}\int_\Omega \partial_e W_{\mathrm{el}}(e(u_\varepsilon), c_\varepsilon, z_\varepsilon) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}t + \varepsilon\int_{t_1}^{t_2}\int_\Omega |\nabla u_\varepsilon|^2\nabla u_\varepsilon : \nabla\partial_t b\,\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{68}$$

To proceed, we need to prove $\varepsilon\int_\Omega |\nabla u_\varepsilon(t)|^4\,\mathrm{d}x \to 0$ as $\varepsilon \searrow 0$ for a.e. $t \in [0,T]$. Indeed, testing (55) with $\zeta := u_\varepsilon - b$:

$$\varepsilon\int_{\Omega_T} |\nabla u_\varepsilon|^4\,\mathrm{d}x\mathrm{d}t = \varepsilon\int_{\Omega_T} |\nabla u_\varepsilon|^2\nabla u_\varepsilon : \nabla b\,\mathrm{d}x\mathrm{d}t - \int_{\Omega_T} \partial_e W_{\mathrm{el}}(e(u_\varepsilon), c_\varepsilon, z_\varepsilon) : e(u_\varepsilon - b)\,\mathrm{d}x\mathrm{d}t$$

We immediately see that the first term converges to 0 as $\varepsilon \searrow 0$. The second term also converges to 0 because of $\int_{\Omega_T} \partial_e W_{\mathrm{el}}(e(u), c, z) : e(u - b)\,\mathrm{d}x\mathrm{d}t = 0$ (equation (66)). This, together with Corollary 6.15, proves $\mathcal{E}_\varepsilon(q_\varepsilon(t)) \to \mathcal{E}(q(t))$ for a.e. $t \in [0,T]$. In conclusion, we can pass to $\varepsilon \searrow 0$ in (68) for a.e. $0 \leq t_1 < t_2 \leq T$ by using Corollary 6.15 together with Lebesgue's generalized convergence theorem and growth condition (GC2), (11a) and (GC6) as well as by using a sequentially weakly lower semi-continuity argument for $\int_\Omega \beta|\partial_t z_\varepsilon|^2\,\mathrm{d}x$ and for $\int_\Omega |\nabla\mu_\varepsilon|^2\,\mathrm{d}x$. ■

7 Discussion

The existence results in this paper can also be generalized to multi-phase systems. For instance, instead of (6a)-(6c) we may consider the system

$$\begin{aligned}
&\partial_t c = \operatorname{div}(\mathbf{M}\nabla w),\\
&w = \mathbf{P}(-\operatorname{div}(\mathbf{\Gamma}\nabla c) + \partial_c W_{\mathrm{ch}}(c) + \partial_c W_{\mathrm{el}}(e(u), c, z)),\\
&\operatorname{div}(\sigma(e(u), c, z)) = 0,\\
&0 \in \mathrm{d}_z \mathcal{E}(u, c, z) + \mathrm{d}_{\dot z} \mathcal{R}(\partial_t z),
\end{aligned}$$

describing an alloy of N-components, i.e. $c : [0, T] \times \Omega \to \mathbb{R}^N$, with a gradient energy tensor $\mathbf{\Gamma}$ (constant, symmetric and positive mapping from $\mathbb{R}^{N\times n}$ into itself) as well as a mobility matrix $\mathbf{M}$ (constant, symmetric, positive definite on $T := \{x \in \mathbb{R}^N \mid \sum_{k=1}^N x_k = 0\}$) with $\sum_{l=1}^N \mathbf{M}_{kl} = 0$ for all $k = 1, \ldots, n$. The mapping $\mathbf{P}$ is the orthogonal projection from $\mathbb{R}^N$ to the subspace T. Additionally, the constraint $\sum_{k=1}^N c_k = 1$ is imposed to this system.

The existence proofs in this work, however, cannot directly be generalized to the physically important exponent case $p = 2$ due to the lack of the compact embedding $W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$ for $0 \leq \alpha < 1 - \frac{n}{p}$ and $n > 1$, which is an essential feature in Lemma 4.2 and Lemma 4.3.

A Appendix

Lemma A.1 (Uniform convexity) *Let $n, p \in \mathbb{N}$ with $p \geq 2$. Then there exists a constant $C_{\mathrm{ineq}} > 0$ such that*

$$|x - y|^p \leq C_{\mathrm{ineq}}(|x|^{p-2}x - |y|^{p-2}y) \cdot (x - y) \quad \textit{for all } x, y \in \mathbb{R}^n$$

and (matrix-version)

$$|x - y|^p \leq C_{\mathrm{ineq}}(|x|^{p-2}x - |y|^{p-2}y) : (x - y) \quad \textit{for all } x, y \in \mathbb{R}^{n\times n}.$$

Proof. Substitution $y = x + h$ leads to the claim:

$$|h|^p \leq C_{\mathrm{ineq}} h \cdot (|x + h|^{p-2}(x + h) - |x|^{p-2}x) \text{ for all } x, h \in \mathbb{R}^n \tag{69}$$

Moreover, it suffices to prove (69) only for $x, h \in \mathbb{R}^n$ with $|h| = 1$. Then (69) is equivalent to

$$\frac{1}{C_{\mathrm{ineq}}} \leq |x + h|^{p-2} + (h \cdot x)(|x + h|^{p-2} - |x|^{p-2}) \text{ for all } x, h \in \mathbb{R}^n \text{ with } |h| = 1. \tag{70}$$

Now the implications $|x + h| \leq |x| \;\Rightarrow x \cdot h \leq -\frac{1}{2}|h|^2$ as well as $|x + h| \geq |x| \;\Rightarrow x \cdot h \geq -\frac{1}{2}|h|^2$ give the estimate:

$$\begin{aligned}
|x + h|^{p-2} + (h \cdot x)(|x + h|^{p-2} - |x|^{p-2}) &\geq |x + h|^{p-2} + \frac{1}{2}|h|^2(|x|^{p-2} - |x + h|^{p-2})\\
&= \frac{1}{2}|x + h|^{p-2} + \frac{1}{2}|x|^{p-2}
\end{aligned}$$

Since $|h| = 1$, the right hand side is bounded from below by a positive constant and therefore (70) follows. ∎

Lemma A.2 *Let $n, p \in \mathbb{N}$ with $p \geq 2$. It holds for every $a, b \in \mathbb{R}^n$ and every $\lambda \in [0,1]$:*

$$(|\lambda b + (1-\lambda)a|^{p-2}(\lambda b + (1-\lambda)a) - |b|^{p-2}b) \cdot (b-a) \leq 0.$$

Proof. This can be derived from the convexity of the map $x \mapsto |x|^p$. Alternatively, putting $x := \lambda b + (1-\lambda)a$ and $y := b$ in Lemma A.1 gives

$$(1-\lambda)C_{\text{ineq}}(|\lambda b + (1-\lambda)a|^{p-2}(\lambda b + (1-\lambda)a) - |b|^{p-2}b) \cdot (b-a) \leq -|\lambda b + (1-\lambda)a - b|^p,$$

and hence the claim follows. ∎

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