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Averaging of time-periodic dissipation potentials in rate-independent processes

Dedicated to Tomáš Roubíček on the occasion of his sixtieth birthday

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Abstract

We study the existence and well-posedness of rate-independent systems (or hysteresis operators) with a dissipation potential that oscillates in time with period ε. In particular, for the case of quadratic energies in a Hilbert space, we study the averaging limit $\varepsilon \to 0$ and show that the effective dissipation potential is given by the minimum of all friction thresholds in one period, more precisely as the intersection of all the characteristic domains. We show that the rates of the process do not converge weakly, hence our analysis uses the notion of energetic solutions and relies on a detailed estimates to obtain a suitable equi-continuity of the solutions in the limit $\varepsilon \to 0$.

1 Introduction

In most applications of hysteresis or rate-independent systems the hysteresis operator or the dissipation potential is time-independent while the system is driven by a time-dependent external loading, see [Vis94, BrS96, Kre99, MiR15]. However, there are also systems where the internal dissipative mechanism depends on time in a prescribed manner, see [Mor77, KrL09, AlK11] and the references below for mathematical treatments of this case. Moreover, there are mechanical devices where friction is modulated time-periodically by using a rotating unbalance, as in a vibratory plate compactor used in construction areas, see Figure 1.1.

In this paper we are interested in cases where the dissipation processes is oscillating periodically on a much faster time scale than the driving of the system by an external loading. Similar, time-dependent friction mechanisms occur during walking or crawling of animals or mechanical devices. Typically, there is a periodic gait, where the contact pressure of the different extremities oscillates periodically, and only those legs are moved for which the normal pressure is minimal. Simple mechanical toys, where this interplay can easily be studied, are so-called the descending woodpecker, the toy ramp walkers, and the rocking toy animals, see Figure 1.2. We refer to [GND14, DGN15, GiD16b, GiD16a] for models on locomotion for micro-machines or animals and to [RaN14] for the slip-stick dynamics of polymers on inhomogeneous surfaces. Application to time-dependent hysteresis in piezo-ceramic actuators are given in [AlK11, Al13].

Another application arises by moving an elastic body like a rubber over a flat surface, where the surface is prepared such that the friction coefficient changes periodically. Then, the system under consideration might serve as a model how microstructures on surfaces give rise to kinetic friction which is smaller than the static friction (also called stiction). Of course, this model does not account for the true microstructure of the surface, being in general of a stochastic nature.

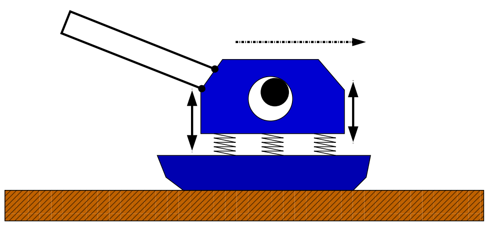

Figure 1.1: Because of the in-built unbalance, the plate compactor vibrates vertically leading to an oscillatory normal pressure. When pushing the plate compactor horizontally it will move only when the normal pressure is very low.

(A) (B) (C)

Figure 1.2: (A) In rest, the woodpecker sticks to the metal rod by dry friction, when oscillating the reduction in friction allows for a slow sliding downwards, cf. [Pfe84]. (B) Toy ramp walker: the frog walks down only, when alternating the weight between the rigid downhill leg and the hinged uphill leg. (C) Rocking animal: A weight beyond the table edge pulls the cow forward, while the perpendicular rocking motions allows the lifted legs to swing forward because of the reduced normal pressure.

In the present work, we will not investigate how the periodic oscillation is generated by the system itself. Instead, we will assume the friction is induced by a given time-periodic dissipation mechanism. More precisely, we consider a rate-independent system $(\mathbf{Y}, \mathcal{E}, \mathcal{R}_\varepsilon)$, where $\mathbf{Y}$ is a reflexive Banach space. The energy $\mathcal{E} : [0,T]\times\mathbf{Y} \to \mathbb{R}$ is the energy functional, where the Gateaux differential $\mathrm{D}\mathcal{E}(t,y) \in \mathbf{Y}^*$ is the static restoring force and $\partial_t\mathcal{E}(t,y) \in \mathbb{R}$ is the power of the external loadings. In this introduction we restrict ourself to the quadratic case, where $\mathbf{Y}$ is a Hilbert space and

$$\mathcal{E}(t,y) = \frac{1}{2}\langle Ay, y\rangle - \langle \ell(t), y\rangle\,, \tag{1.1}$$

for a positive definite, symmetric, bounded operator $A : \mathbf{Y} \to \mathbf{Y}^*$ and $\ell \in \mathrm{W}^{1,2}(0,T;\mathbf{Y}^*)$. Then, $\mathrm{D}\mathcal{E}(t,y) = Ay - \ell(t) \in \mathbf{Y}^*$.

For each $t \in [0,T]$, the functional $\mathcal{R}_\varepsilon(t,\cdot) : \mathbf{Y} \to [0,\infty]$ is a 1-homogeneous dissipation potential, i.e.

$$\begin{aligned} &\mathcal{R}_\varepsilon(t,\cdot) : \mathbf{Y} \to [0,\infty] \text{ is convex and lower semi-continuous,}\\ &\forall\,\gamma > 0,\ v \in \mathbf{Y} : \quad \mathcal{R}_\varepsilon(t,\gamma v) = \gamma\mathcal{R}_\varepsilon(t,v). \end{aligned}$$

The dissipation forces are given by the set-valued subdifferential $\partial\mathcal{R}_\varepsilon(t,v) \subset \mathbf{Y}^*$ of the convex function $\mathcal{R}_\varepsilon(t,\cdot)$. We are interested in the case where the temporal behavior is characterized by the microscopic period $2\pi\varepsilon > 0$ through a functional $\Phi : [0,T]\times\mathbb{R}\times\mathbf{Y} \to [0,\infty]$ via

$$\mathcal{R}_\varepsilon(t,v) = \Phi(\,t,\, t/\varepsilon,\, v), \tag{1.2}$$

where now $\Phi(t,s,\cdot)$ is a 1-homogeneous dissipation potential, and $\Phi(t,\cdot,v)$ is periodic with period 2π on the real line.

In Section 4 we provide a general existence result for the Cauchy-problem

$$0 \in \partial\mathcal{R}_\varepsilon(t,\dot{y}) + \mathrm{D}\mathcal{E}(t,y(t)) \quad \text{for a.a. } t \in [0,T], \qquad y(0) = y_0\,. \tag{1.3}$$

Indeed, we establish the existence of energetic solutions for general rate-independent systems with non-convex energies in Section 4. This part is a suitable generalization of the general existence theory based on incremental minimization developed in [Mie05, MiR15].

For this we develop a suitable calculus to generalize the definition

$$\mathrm{Diss}_{\mathcal{R}_\varepsilon}(y,[t_1,t_2]) := \int_{t_1}^{t_2} \mathcal{R}_\varepsilon(t,\dot y(t))\,\mathrm{d}t$$

from functions in $\mathrm{W}^{1,1}([0,T],\mathbf{Y})$ to all functions of bounded variation, see Section 3.

Section 5 then provides the main result concerning the limit of fast oscillatory dissipation structures, i.e. $\varepsilon \searrow 0$ where $\mathcal{R}_\varepsilon$ is given in the form (1.2). This result states that the solutions $y^\varepsilon : [0,T] \to \mathbf{Y}$ of (1.3) converge uniformly to a function $y^0 : [0,T] \to \mathbf{Y}$, which is again a solution to a rate-independent system $(\mathbf{Y},\mathcal{E},\mathcal{R}_{\mathrm{eff}})$, where the effective dissipation potential is given by an infinite-dimensional inf-convolution, namely

$$\mathcal{R}_{\mathrm{eff}}(t,v) := \inf\Big\{ \frac{1}{2\pi}\int_0^{2\pi} \Phi\big(t,s,\dot y(s)\big)\,\mathrm{d}s \,\Big|\, y \in \mathrm{W}^{1,1}([0,2\pi];\mathbf{Y}), \quad y(0)=0,\ y(2\pi)=v \Big\}. \tag{1.4}$$

In fact, using the 1-homogeneity of the dissipation potential $\Phi(t,s,\cdot)$ we see that the dual dissipation potential has the form $\Phi^*(t,s,\xi) = \chi_{K(t,s)}(\xi)$, where $K(t,s) = \partial\Phi(t,s,0) \subset \mathbf{Y}^*$ is a convex set containing $0 \in \mathbf{Y}^*$. Here $\chi_A(\xi) = 0$ for $\xi \in A$ and $\chi_A(\xi) = \infty$ for $\xi \notin A$. Under suitable assumptions we show (see Proposition 3.6) that

$$\mathcal{R}^*_{\mathrm{eff}}(t,\xi) = \chi_{K_{\mathrm{eff}}(t)}(\xi) \quad \text{with } K_{\mathrm{eff}}(t) := \bigcap\nolimits_{s\in[0,2\pi]} K(t,s)\,. \tag{1.5}$$

This can be understood in the sense that the effective dissipation potential is given in terms of the minimum of all the possible friction thresholds. Because of the rate-independent nature, the system can take immediate advantage of a low threshold and move as far as necessary, see e.g. the solution y^ε in Figure 2.1, which moves with fast velocity $O(1/\varepsilon)$ on tiny intervals of length $O(\varepsilon^2)$, or the zigzag pattern of the solution $t \mapsto (y_1(t),y_2(t))$ in Figure 2.2, where $\dot y_1(t)\dot y_2(t) \equiv 0$.

To be more precise, we that $\mathbf{Y} = \mathbf{F}{\times}\mathbf{Z}$ for Hilbert spaces $\mathbf{F}$ and $\mathbf{Z}$, where the component $\phi \in \mathbf{F}$ acts as a "purely elastic part" of the state $y = (\phi,z)$, while z is the "dissipative part". Consequently, we will assume that the dissipation potential $\Phi(t,s,(\dot\phi,\dot z))$ is independent of $\dot\phi$. Moreover, we assume that there exists a convex, lower semi-continuous positive 1-homogeneous functional ψ_0 on $\mathbf{Z}$ such that the following holds

$$\exists\, C_\psi > 0 \,:\, \forall\, z \in \mathbf{Z},\ \psi_0(z) < \infty: \quad \psi_0(z) \le C_\psi \|z\|_{\mathbf{Z}}\,, \tag{1.6a}$$

$$\begin{aligned}&\exists\, \omega_1,\omega_2 \in \mathrm{C}([0,\infty[)\ \omega_1(0)=\omega_2(0)=0 \ :\ \forall\, t_1,t_2 \in [0,T]\ \forall\, s_1,s_2 \in \mathbb{R}\ \forall\, v \in \mathbf{Z}:\\ &\qquad |\Phi(t_1,s_1,v) - \Phi(t_2,s_2,v)| \le \Big(\omega_1\big(|t_1-t_2|\big) + \omega_2\big(|s_1-s_2|\big)\Big)\psi_0(v),\end{aligned} \tag{1.6b}$$

$$\exists\, \overline{\alpha} > \underline{\alpha} > 0\ \forall\, (t,s,v) \in [0,T]{\times}[0,2\pi]{\times}\mathbf{Z}: \quad \underline{\alpha}\psi_0(v) \le \Phi(t,s,v) \le \overline{\alpha}\psi_0(v), \tag{1.6c}$$

where ω_1 and ω_2 are moduli of continuity, i.e. continuous, nondecreasing functions with $\omega_j(0) = 0$.

Our main result is the following convergence result that states that the solutions $y^\varepsilon : [0,T] \to \mathbf{Y}$ converge to the unique solution $y : [0,T] \to \mathbf{Y}$ of the effective rate-independent system $(\mathbf{F}{\times}\mathbf{Z},\mathcal{E},\mathcal{R}_{\mathrm{eff}})$.

Theorem 1.1. *Let the rate-independent system* $(\mathbf{F}{\times}\mathbf{Z},\mathcal{E},\mathcal{R}_\varepsilon)$ *satisfy* (1.1), (1.2), *and* (1.6). *Moreover, assume that the initial condition* $y_0 \in \mathbf{Y}$ *satisfies*

$$0 \in \partial\Phi(0,s,0) + \mathrm{D}\mathcal{E}(0,y_0) \quad \textit{for all } s \in [0,2\pi]\,. \tag{1.7}$$

Then, for every $\varepsilon > 0$ there exists a unique energetic solution $y^\varepsilon \in C([0,T];\mathbf{Y})$ to (1.3) *in the sense of (4.6)–(4.7).*

Moreover, for $\varepsilon \searrow 0$ we have $y^\varepsilon(t) \rightharpoonup y(t)$ for all $t \in [0,T]$, where $y \in C([0,T];\mathbf{Y})$ is the unique energetic solution of the effective rate-independent system $(\mathbf{F}{\times}\mathbf{Y}, \mathcal{E}, \mathcal{R}_{\text{eff}})$ with $y(0) = y_0$. In particular, if Φ is Lipschitz in the first variable, we have

$$0 \in \partial\mathcal{R}_{\text{eff}}(t, \dot{y}(t)) + \mathrm{D}\mathcal{E}(t, y(t)) \text{ for a.e. } t \in [0,T]\,,$$

where $\mathcal{R}_{\text{eff}}$ is defined in (1.4) *and characterized in* (1.5).

The proof of this result strongly uses the theory of energetic rate-independent systems as developed in [Mie05, MiR15]. The main point that we cannot pass to the limit $\varepsilon \searrow 0$ in the equation (1.3) is that the derivatives $\dot{y}^\varepsilon$ do not exist in the sense of $\mathrm{L}^p(0,T;\mathbf{Y})$ and even if they exists, they do not converge weakly in any L^p space. The problem of the very weak convergence is already seen in the simple scalar model

$$0 \in \big(2{+}\cos(t/\varepsilon)\big)\,\mathrm{Sign}(\dot{y}^\varepsilon) + y^\varepsilon - \ell(t)\,,$$

which is studied in Section 2.1 for illustrative purposes, see Figure 2.1. In Section 2.2 we study a two-dimensional case (i.e. $\mathbf{Y} = \mathbb{R}^2$) which can be seen as a strongly simplified model for a two-leg walker, where the weight of the body is periodically relocated from one leg to the other such that their motion occurs alternatingly.

Instead of weak convergence of the solutions of $\dot{y}^\varepsilon$ we will rather rely on equicontinuity properties of the family $(y^\varepsilon)_{\varepsilon\in(0,1)}$, see Proposition 5.2. Thus, the derivative-free notion of energetic solutions is ideally suited for the limit passage in the proof of Theorem 1.1.

Despite the fact that hysteresis operators and rate-independent systems have been studied by many works (e.g. [Vis94, BrS96, Kre99, MiR15]), there seems to be only few work on time-dependent dissipation potentials, even though a first result for time dependent $\mathcal{R}$ was already obtained by Moreau in 1977, see [Mor77] and the follow up paper [KuM98].

The conceptually closest existence result to our work has been obtained by Krejčí and Liero in [KrL09], who combined the framework of Kurzweil-integrals with the concept of energetic solutions. Instead of assuming continuity of $\mathcal{R}(t,\cdot)$ with respect to t, they consider $\mathcal{R}^*(t,\cdot) = \chi_{K(v(t)}$ and assume Lipschitz continuity of $v \mapsto K(v) \subset \mathbf{Y}^*$ in the Hausdorff distance, where v lies in a Banach space $\mathbf{V}$. They then obtain existence and uniqueness of solutions for data $\ell \in \mathrm{BV}([0,T],\mathbf{Y}^*)$ and $v \in \mathrm{BV}([0,T],\mathbf{V})$. Note that in that case the mapping $t \mapsto \mathcal{R}^*(t,\cdot) = \chi_{K(v(t))}(\cdot)$ may even have jumps. More recent generalizations are given in [Rec11, KrR11, Roc12, KKR15].

Our work uses similar ideas for establishing continuous dependence on the data, but restricts to the continuous case. Note that the time-periodic setting with $\varepsilon \searrow 0$ does not allow for uniform bounds in BV, since even Lipschitz continuity of $(t,s) \mapsto \Phi(t,s,\cdot)$ will give a BV bound for $t \mapsto \mathcal{R}_\varepsilon(t,\cdot) = \Phi(t,t/\varepsilon,\cdot)$ of order $O(1/\varepsilon)$. So, we will explicitly exploit the periodicity of $s \mapsto \Phi(t,s,\cdot)$ to prove the equicontinuity in Proposition 5.2.

2 Two low-dimensional examples

Before we give a general theory, we provide a two examples that illustrate the concept and the question of convergence.

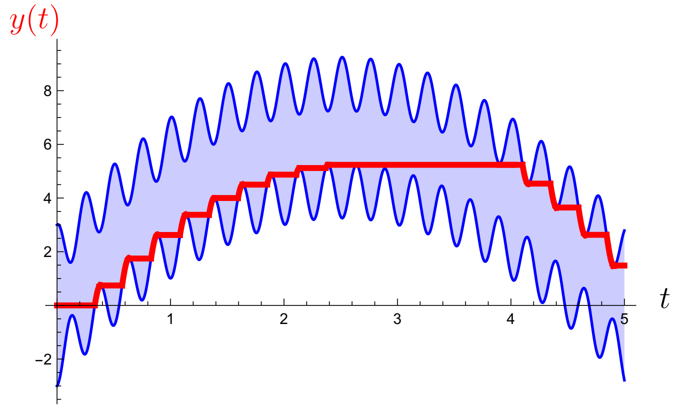

Figure 2.1: The red curve is the solution of $0 \in \rho(t/\varepsilon)\mathrm{Sign}(\dot{y}(t)) + y(t) - 5t + t^2$ with $y(0) = 0$ for $\varepsilon = 0.04$. The blue, wavy area indicates the stable regions

2.1 A scalar hysteresis operator

In the simplest setting we choose $\mathbf{Y} = \mathbf{Z} = \mathbb{R}$ and define the rate-independent system in the form

$$\mathcal{E}(t, y) = \frac{1}{2}y^2 - \ell(t)y \quad \text{and} \quad \Phi(t, s, \dot{y}) = \rho(s)|\dot{y}|,$$

where we choose for definiteness $\ell(t) = 5t - t^2$ and $\rho(s) = 2 + \cos(s)$.

This leads to the simple equation

$$0 \in \rho(t/\varepsilon)\,\mathrm{Sign}\big(\dot{y}^\varepsilon\big) + y^\varepsilon - \ell(t), \qquad y(0) = y_0, \tag{2.1}$$

which implies that the solution $y^\varepsilon(t)$ has to lie in $[\ell(t){-}\rho(t/\varepsilon), \ell(t){+}\rho(t/\varepsilon)]$. The unique solution for this hysteresis model with initial condition $y(0) = y_0 = 0$ is shown in Figure 2.1. We see that for $t \in [0, 4]$ the solution is nondecreasing and hence it is given by the explicit formula $y^\varepsilon(t) = \max\big\{\ \max\{0, \ell(\tau){-}\rho(\tau/\varepsilon)\} \,\big|\, s \in [0, t]\ \big\}$.

In particular, y^ε uniformly converges to the unique solution y^0 of the limit system

$$0 \in \rho_{\min}\,\mathrm{Sign}(\dot{y}) + y - \ell(t), \qquad y(0) = 0,$$

which is given by $y^0(t) = \min\big\{5t{-}t^2{+}1, \max\{0, 5t{-}t^2{-}1\}\big\}$.

Moreover, we see that the derivative $\dot{y}^\varepsilon : [0, 4] \to \mathbb{R}$ is either 0 (namely on flat parts) or $\dot{\ell}(t) - \frac{1}{\varepsilon}\sin(t/\varepsilon)$. On each interval $[k\pi/\varepsilon, (k{+}2)\pi/\varepsilon]$ the solution has one increasing region whose length is $O(\varepsilon^2)$, while $\dot{y}^\varepsilon$ is of order $O(1/\varepsilon)$. We observe the basic principle that y^ε waits until $\rho(t/\varepsilon) = 2 + \cos(t/\varepsilon)$ gets very close to $\rho_{\min} = 1$ and then moves very quickly. Thus, the flat regions dominate and $\dot{y}^\varepsilon$ is not bounded in $\mathrm{L}^p([0, T])$ for any $p > 1$. We only have $\dot{y}^\varepsilon \stackrel{\mathrm{M}}{\rightharpoonup} \dot{y}^0$ (convergence in measure when testing with continuous test functions). More precisely, the sequence $\dot{y}^\varepsilon$ is bounded $\mathrm{L}^1([0, T])$ but not weakly convergent.

Nevertheless, one can establish an asymptotic equicontinuity estimate in the form

$$\exists\, C > 0\ \forall\, \varepsilon \in [0, 1]\ \forall\, t_1, t_2 \in [0, T]: \quad |y^\varepsilon(t_1) - y^\varepsilon(t_2)| \le C\big(\varepsilon + |t_2{-}t_1|\big).$$

Below, we will derive similar estimates in the general setting, see Proposition 5.2.

2.2 A two-dimensional model for walking

We now consider $\mathbf{Y} = \mathbf{Z} = \mathbb{R}^2$, where $y = (y_1, y_2)$ contains the coordinates of the two legs walking on a one-dimensional line. This may serve as a model for a toy ramp walker

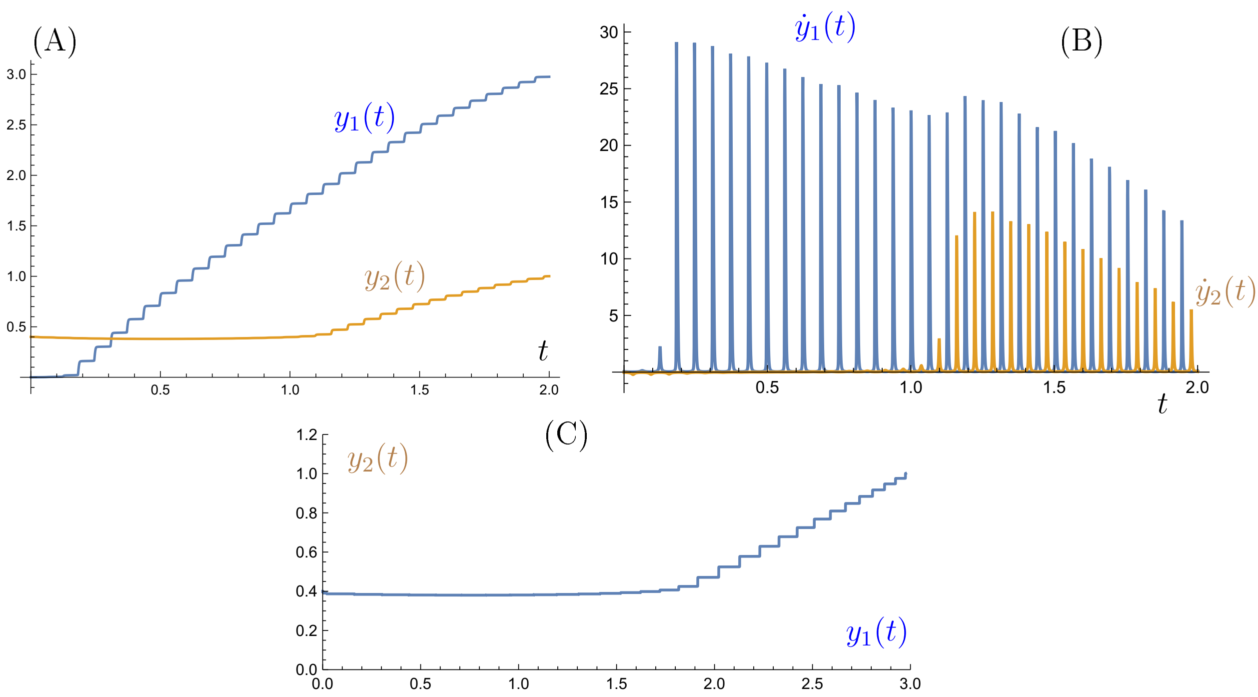

Figure 2.2: Plots for the solution of (2.2). (A) The positions $y_j(t)$ of the two legs move by alternating between plateaus (sticking phase) and fast motion. (B) The derivatives $\dot{y}_j(t)$ show that the motion is alternating, i.e. at most one of the legs moves at a time. (C) The path $t \mapsto y(t) = (y_1(t), y_2(t)) \in \mathbb{R}^2$ shows a microscopic zigzag pattern.

as well as for a rocking animal, if one restricts to the only relevant case where the two left and the two right legs always move together. We take

$$\mathcal{E}(t,y) = \frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 + \frac{\kappa}{2}(y_2 - y_1)^2 - \ell(t)y_1 \quad \text{and} \quad \Phi(t,s,\dot{y}) = \rho_1(s)|\dot{y}_1| + \rho_2(s)|\dot{y}_2|.$$

For a walker with symmetric legs one would assume $\rho_1(s+\frac{1}{2}) = \rho_2(s)$ and $\rho_1(s) + \rho_2(s) = \text{const.} = \rho_*$, where ρ_* is the constant normal pressure induced by the total weight. However, this is not important for our purpose, we only need that $\rho_j^{\min} := \min\{\rho_j(s) \,|\, s \in \mathbb{R}\} \geq \underline{\alpha} > 0$. (Note that for a walker, when moving the free leg, there is always some small friction in the joints.) Moreover, we want to impose that the two minima are not attained for the same phase $s \in [0, 2\pi]$.

The associated differential inclusion takes the form

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \begin{pmatrix} \rho_1(t/\varepsilon)\,\mathrm{Sign}(\dot{y}_1) \\ \rho_2(t/\varepsilon)\,\mathrm{Sign}(\dot{y}_2) \end{pmatrix} + \begin{pmatrix} 1+\kappa & -\kappa \\ -\kappa & 1+\kappa \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - \begin{pmatrix} \ell(t) \\ 0 \end{pmatrix}, \quad y(0) = (y_1^0, y_2^0). \tag{2.2}$$

In Figure 2.2 we display a numerical simulation for the solution with $\varepsilon = 0.01$ for the case $\kappa = 1$, $\ell(t) = 5t - t^2$, and $\rho_j(s) = 2 + (-1)^j \cos s$.

Here the effective equation for $\varepsilon \searrow 0$ is obtained with

$$\mathcal{R}_{\text{eff}}(\dot{y}) = \rho_1^{\min}|\dot{y}_1| + \rho_2^{\min}|\dot{y}_2|.$$

This is most easily seen by using the dual characterization via

$$\Phi^*(t,s,\xi) = \chi_{K(s)}(\xi), \quad \text{where } K(s) = [-\rho_1(s), \rho_1(s)] \times [-\rho_2(s), \rho_2(s)] \subset \mathbb{R}^2.$$

Then, our formula (1.5) and the assumption $\rho_j^{\min} = \rho_j(s_j)$ for $s_1 \neq s_2$ give

$$\mathcal{R}_{\text{eff}}^*(\xi) = \chi_{K_{\text{eff}}}(\xi) \quad \text{with } K_{\text{eff}} = \bigcap_{s\in\mathbb{R}} K(s) = [-\rho_1^{\min}, \rho_1^{\min}] \times [-\rho_2^{\min}, \rho_2^{\min}].$$

3 Functions of bounded variation

For a Banach space $\mathbf{Z}$, we consider functions $u : [0,T] \to \mathbf{Z}$. For an interval $[s,t] \subset [0,T]$ and a convex, positive, lower semi-continuous and 1-homogeneous functional $\psi : X \to [0,\infty]$, we define the (pointwise) variation

$$\operatorname{var}(u;\psi;s,t) := \sup\left\{ \sum_{i=1}^{N-1} \psi\left(u(t_{i+1}) - u(t_i)\right) \,:\, N \geq 2\,,\ s \leq t_1 < t_2 < \cdots < t_N \leq t \right\}.$$

We remind at this point, that convex 1-homogeneous functionals automatically fulfill a triangle inequality. We denote by $\mathrm{BV}_\psi(0,T;X)$ the set of all functions u with finite variation $\operatorname{var}(u;\psi;0,T)$. In case $\psi(\cdot) = \|\cdot\|_X$, we simply write $\mathrm{BV}(0,T;X) := \mathrm{BV}_{\|\cdot\|_X}(0,T;X)$.

In what follows, we say that a sequence of functions $(u_n)_{n\in\mathbb{N}} \subset \mathrm{BV}(0,T;\mathbf{Z})$ converges weakly to $u \in \mathrm{BV}(0,T;\mathbf{Z})$ if $\sup_n \operatorname{var}(u_n;\psi;0,T) + \|u_n\|_{L^1} < \infty$ and $u_n(t) \rightharpoonup u(t)$ for all $t \in [0,T]$.

Definition 3.1 (ψ_0-regular dissipation potentials)**.** Let $\psi_0 : \mathbf{Z} \to \mathbb{R}$ be a convex, lower-semicontinuous and positive 1-homogeneous functional. A map $\Psi : [0,T] \times \mathbf{Z} \to \mathbb{R}$ is called a ψ_0-regular dissipation potential if the following holds: For all $t \in [0,T]$, the functional $\Psi(t,\cdot)$ is convex, lower semi-continuous, positive 1-homogeneous and

$$\exists\, \overline{\alpha} \geq \underline{\alpha} > 0 \text{ such that } \forall\, t \in [0,T],\ z \in \mathbf{Z} \text{ holds } \underline{\alpha}\psi_0(z) \leq \Psi(t,z) \leq \overline{\alpha}\psi_0(z)\,, \tag{3.1}$$

$$\begin{aligned} &\exists\, \omega \in C([0,+\infty)) \text{ with } \omega(0) = 0 \text{ s.t.} \\ &\qquad \forall\, t, t_0 \in [0,T] : \quad \sup_{z\in\mathbf{Z}\setminus\{0\}} |\Psi(t,z) - \Psi(t_0,z)| \leq \omega(|t-t_0|)\psi_0(z)\,. \end{aligned} \tag{3.2}$$

The proof of the following results is postponed to Appendix A.

Lemma 3.2 (Definition of total dissipation)**.** *Let $\psi_0 : \mathbf{Z} \to \mathbb{R}$ be a convex, positive, lower semi-continuous 1-homogeneous functional and let $\Psi : [0,T] \times \mathbf{Z} \to \mathbb{R}$ be a ψ_0-regular dissipation potential. For every $K \in \mathbb{N}$, let $\mathcal{T}_K \subset [0,T]$ be a finite set of isolated points, i.e. $\mathcal{T}_K = \left\{t_0^K, t_1^K, \dots t_{N_K}^K\right\}$ where $0 \leq t_0^K < t_1^K < \cdots < t_{N_K}^K \leq T$ with the property $\tau_K := \sup_{k\in\{0,N_K\}}\left(t_{k+1}^K - t_k^K\right) \to 0$ as $K \to \infty$. Then, writing $\mathcal{T}_K[s,t) := \mathcal{T}_K \cap [s,t] \cup \{s\}$ for all $u \in BV_{\psi_0}(0,T;\mathbf{Z})$, the limit*

$$\mathrm{Diss}_\Psi(u;s,t) = \lim_{K\to\infty} \sum_{t_i\in\mathcal{T}_K[s,t)} \operatorname{var}(u, \Psi(t_i), t_i, t_{i+1})$$

is independent from the choice of $\mathcal{T}_K$ and

$$\begin{aligned} \mathrm{Diss}_\Psi(u;s,t) &= \lim_{K\to\infty} \sum_{t_i\in\mathcal{T}_K[s,t)} \fint_{t_i}^{t_{i+1}} \Psi\left(r, u(t_{i+1}) - u(t_i)\right) dr\,, \\ &= \lim_{K\to\infty} \sum_{t_i\in\mathcal{T}_K[s,t)} \Psi\left(t_i, u(t_{i+1}) - u(t_i)\right) dr\,. \end{aligned} \tag{3.3}$$

The quantity $\mathrm{Diss}_\Psi(u;s,t)$ is the total dissipation of the function u with respect to Ψ over the time-interval $[s,t]$.

Lemma 3.3 (Properties of total dissipation)**.** *Let $\psi_0 : \mathbf{Z} \to \mathbb{R}$ be a convex, positive, lower semi-continuous 1-homogeneous functional and let $\Psi_1, \Psi_2 : [0,T] \times \mathbf{Z} \to \mathbb{R}$ be ψ_0-regular dissipation potentials. Assume $\Psi_1(t,u) \le \Psi_2(t,u)$ for all $t \in [0,T]$, $u \in \mathbf{Z}$. Then,*

$$\mathrm{Diss}_{\Psi_1}(u;s,t) \le \mathrm{Diss}_{\Psi_2}(u;s,t)\,.$$

If $\Psi(t,u) = \psi_0(u)$ for all $s,t \in [0,T]$ and $u \in \mathbf{Z}$, we find

$$\mathrm{Diss}_\Psi(u;s,t) = \mathrm{var}(u;\psi_0;0,T) = \lim_{K\to\infty} \sum_{k=0}^{N_K-1} \psi_0\left(u(t_{k+1}^K) - u(t_k^K)\right)\,. \tag{3.4}$$

If $|\Psi_1(t,u) - \Psi_2(t,u)| \le \beta\psi_0(u)$ for all $t \in [0,T]$, then

$$\forall\, s,t \in [0,T] : \quad \Big|\mathrm{Diss}_{\Psi_1}(u;s,t) - \mathrm{Diss}_{\Psi_2}(u;s,t)\Big| \le \beta\,\mathrm{Diss}_{\psi_0}(u;s,t).$$

Corollary 3.4. *If $\mathrm{Diss}_\Psi(u;0,T) < \infty$, then for all $0 \le t \le \tau \le T$ and $s \in [0,T]$ we have $\Psi(s,u(\tau){-}u(t)) < \infty$.*

Proof. We have $\Psi(t,u(\tau){-}u(t)) \le \overline{\alpha}\psi_0(u(\tau){-}u(t)) \le \overline{\alpha}\mathrm{Diss}_{\psi_0}(u;t,\tau) \le \frac{\overline{\alpha}}{\underline{\alpha}}\mathrm{Diss}_\Psi(u;t,\tau)$, which gives the desired result. □

Lemma 3.5 (Lower semi-continuity of total dissipation)**.** *Let $\psi_0 : \mathbf{Z} \to \mathbb{R}$ be a convex, positive, lower semi-continuous 1-homogeneous functional and let $\Psi : [0,T] \times \mathbf{Z} \to \mathbb{R}$ be a ψ_0-regular dissipation potential. Let $u_n \in BV_{\psi_0}(0,T;\mathbf{Z})$, $n \in \mathbb{N}$, be a sequence with $u_n \rightharpoonup u \in BV_{\psi_0}(0,T;\mathbf{Z})$ as $n \to \infty$ in the sense that $u_n(t) \rightharpoonup u(t)$ weakly in $\mathbf{Z}$ for all $t \in [0,T]$. Then, for all $s,t \in [0,T]$ it holds*

$$\mathrm{Diss}_\Psi(u;s,t) \le \liminf_{n\to\infty} \mathrm{Diss}_\Psi(u_n;s,t)\,. \tag{3.5}$$

We now consider $\mathcal{R}_\varepsilon(t,\dot z) = \Phi(t,\frac{t}{\varepsilon},\dot z)$ and recall the definition of $\mathcal{R}_{\mathrm{eff}}$ in (1.4) and provide the useful characterization (1.5), which will be proved in Appendix A.4.

Proposition 3.6 (Characterization of effective dissipation)**.** *Let ψ_0 and Φ satisfy* (1.6a)–(1.6c)*. Then, for all $t \in [0,T]$, $s \in \mathbb{R}$, and all $z \in \mathbf{Z}$ we have*

$$\mathcal{R}_{\mathrm{eff}}(t,z) \le \Phi(t,s,z)\,, \quad \textit{and} \quad \partial\mathcal{R}_{\mathrm{eff}}(t,0) = \bigcap_{\widehat{s}\in[0,2\pi]} \partial\Phi(t,\widehat{s},0)\,. \tag{3.6}$$

4 Existence of Energetic Solutions

In this section, we will provide two existence results. Theorem 4.1 is more general than needed for the proof of Theorem 1.1, but it could also be useful in other contexts. It can also be proved in a metric setting. This can be achieved by replacing $\|z_1 - z_2\|_\mathbf{Z}$ by $d(z_1,z_2)$, the weak convergence with a topology $\mathcal{T}_\mathbf{Z}$ that is weaker than $d(\cdot,\cdot)$ and $\Psi(t,z_1,z_2)$ is lower semi-continuous with respect to $\mathcal{T}_\mathbf{Z}$ in the proof below. Theorem 4.7 deals with the special case of quadratic energies in a Hilbert space, which is the basis of the proof of Theorem 1.1.

4.1 The general case

We assume that $\mathbf{F}$ and $\mathbf{Z}$ are separable and reflexive Banach spaces and consider $\mathbf{Y} = \mathbf{F} \times \mathbf{Z}$ equipped with the product norm. We write $y \in \mathbf{Y}$ as $y = (\phi, z)$. Assume we are given a functional $\mathcal{E} : [0,T] \times \mathbf{Y} \to \mathbb{R}$ continuous in the first and lower semicontinuous in the second variable. We furthermore assume that

$$\begin{aligned}
&\text{There exist } c_E^{(1)}, c_E^{(0)} > 0 \text{ such that for all } y_* \in \mathcal{Y}: \\
&\mathcal{E}(t, y_*) < \infty \implies \begin{cases} \mathcal{E}(\cdot, y_*) \in W^{1,\infty}([0,T]) \text{ and} \\ |\partial_t \mathcal{E}(\cdot, y_*)| \leq c_E^{(1)} \left(\mathcal{E}(\cdot, y_*) + c_E^{(0)} \right) ; \end{cases} && (4.1) \\
&\forall\, t \in [0,T] : \ \mathcal{E}(t, \cdot) : \mathbf{Y} \to \mathbb{R}_\infty \text{ has bounded sublevels.} && (4.2)
\end{aligned}$$

Gronwall's inequality applied to Assumption (4.1) yields

$$\mathcal{E}(t,y) + c_E^{(0)} \leq \left(\mathcal{E}(s,y) + c_E^{(0)} \right) \mathrm{e}^{c_E^{(1)} |t-s|} \quad \forall\, y \in \mathbf{Y} . \tag{4.3}$$

Given $\theta \in [0,1]$ and $y_0, y_1 \in \mathbf{Y}$, we define

$$[y_0, y_1]_\theta := \{ y \in \mathbf{Y} \,:\, \|y - y_0\|_{\mathbf{Y}} = \theta \, \|y_1 - y_0\|_{\mathbf{Y}} \text{ and } \|y - y_1\|_{\mathbf{Y}} = (1-\theta) \, \|y_1 - y_0\|_{\mathbf{Y}} \} .$$

The functional $\mathcal{E}$ is called λ-convex if there exists $\lambda > 0$ such that

$$\begin{aligned}
&\forall\, y_0, y_1 \in \mathbf{Y} \ \forall \theta \in (0,1) \ \forall\, t \in [0,T] \ \exists y \in [y_0, y_1]_\theta : \\
&\qquad \mathcal{E}(t,y) \leq (1-\theta)\mathcal{E}(t,y_0) + \theta \mathcal{E}(t,y_1) - \frac{\lambda}{2} \theta (1-\theta) \, \|y_1 - y_0\|_{\mathbf{Y}}^2 . \qquad (4.4)
\end{aligned}$$

Clearly, the sum of a λ-convex functional and a convex functional is λ-convex.

Finally, in order to prove continuity of solutions, we will also need Lipschitz continuity of the power $\partial_t \mathcal{E}(t, \cdot)$, namely

$$\begin{aligned}
&\text{There exists } C_E^{(3)} > 0 \text{ such that } |\partial_t \mathcal{E}(t,y_0) - \partial_t \mathcal{E}(t,y_1)| \leq C_E^{(3)} \, \|y_1 - y_0\|_{\mathbf{Y}} \\
&\text{for all } t \in [0,T] \text{ and all } y_0, y_1 \in \mathbf{Y} .
\end{aligned} \tag{4.5}$$

The functional $\psi_0 : \mathbf{Z} \to \mathbb{R}$ is a convex, non-negative, lower semi-continuous 1-homogeneous functional and $\Psi : [0,T] \times \mathbf{Z} \to \mathbb{R}$ is a ψ_0-regular dissipation potential. Similar to [MiR15, MiT04] it can be shown that a suitable weak formulation of the inclusion

$$0 \in \partial \Psi(t, \dot{y}) + \mathrm{D}\mathcal{E}(t, y(t))$$

is given by the notion of *energetic solutions* defined via

$$\forall \hat{y} \in \mathbf{Y} : \qquad \mathcal{E}(t, y(t)) \leq \mathcal{E}(t, \hat{y}) + \Psi(t, \hat{y} - y(t)) , \tag{4.6}$$

$$\mathcal{E}(t, y(t)) + \mathrm{Diss}_{\Psi_\varepsilon}(y, [0,t]) = \mathcal{E}(0, y(0)) + \int_0^t \partial_s \mathcal{E}(s, y(s)) \, \mathrm{d}s . \tag{4.7}$$

Inequality (4.6) is called the stability condition, while (4.7) is the energy balance equation. According to (4.6) we define the sets

$$\mathcal{S}(t) := \{ y \in \mathbf{Y} \,:\, \mathcal{E}(t,y) \leq \mathcal{E}(t,\hat{y}) + \Psi(t, \hat{y} - y) \ \forall \hat{y} \in \mathbf{Y} \}$$

and the set of stable points $\mathcal{S}_{[0,T]} := \bigcup_{t \in [0,T]} \{t\} \times \mathcal{S}(t)$.

Theorem 4.1. *Let Assumptions (4.1)–(4.2) hold. Furthermore, assume either one of the following three conditions:*

$$\psi_0 : \mathbf{Z} \to \mathbb{R} \textit{ is weakly continuous.} \tag{4.8a}$$

$$\begin{aligned} &\textit{The set } \mathcal{S}_{[0,T]} \textit{ of stable states is closed in } [0,T]\times\mathbf{Y} \textit{ and} \\ &\quad \forall\, E_0>0: \quad \partial_t\mathcal{E} : \{(t,y) \,:\, \mathcal{E}(t,y)\le E_0\} \to \mathbb{R} \textit{ is weakly continuous.} \end{aligned} \tag{4.8b}$$

$$\begin{aligned} &t_n \to t \textit{ and } y_n \rightharpoonup y \;\;\Rightarrow\;\; \mathrm{D}\mathcal{E}(t_n,y_n) \rightharpoonup \mathrm{D}\mathcal{E}(t,y) \textit{ and} \\ &\quad \forall\, E_0>0: \quad \partial_t\mathcal{E} : \{(t,y) \,:\, \mathcal{E}(t,y)\le E_0\} \to \mathbb{R} \textit{ is weakly continuous.} \end{aligned} \tag{4.8c}$$

Then there exists a solution $y=(\phi,z)\in C([0,T];\mathbf{Y})$ with $z\in BV_{\psi_0}(0,T;\mathbf{Z})$ to (4.6)–(4.7). Furthermore, if $\mathcal{E}$ is λ-convex, if there exists $C_\psi^{(1)}<\infty$ such that

$$\psi_0(z)<\infty \quad\Rightarrow\quad \psi_0(z)\le C_\psi^{(1)}\left\|(0,z)\right\|_{\mathbf{Y}}\,, \tag{4.9}$$

and if (4.5) holds, then for all $t,\tau\in[0,T]$ it holds

$$\|y(\tau)-y(t)\| \le \frac{2C_E^{(3)}}{\lambda}|\tau{-}t| + \frac{2C_\psi}{\underline{\alpha}}\omega\big(|\tau{-}t|\big)\,, \tag{4.10}$$

with $\underline{\alpha}$ and $\overline{\alpha}$ from (3.1). In particular, $y\in C([0,T];\mathbf{Y})$ with modulus of continuity $\sigma\mapsto C\big(\sigma{+}\omega(\sigma)\big)$.

In case that Ψ does not depend on time, we obtain the well-known result that the solution is Lipschitz continuous in time. Thus, we suppose that our result is optimal with regard to the regularity of the solution.

Preliminary results. The following lemma will turn out to be very useful in the proof of Theorem 4.1 and Proposition 5.2.

Lemma 4.2. *Let $\mathcal{E}:[0,T]\times\mathbf{Y}\to\mathbb{R}$ be continuous in the first variable, lower semi-continuous in the second variable and λ-convex such that (4.5) holds and let Ψ be a ψ_0-regular dissipation potential. Then every solution y to (4.6)–(4.7) satisfies for every $t,\tau\in[0,T]$:*

$$\frac{\lambda}{2}\left\|y(\tau)-y(t)\right\|^2 \le \int_t^\tau C_E^{(3)}\left\|y(s)-y(\tau)\right\|\,\mathrm{d}s + \frac{\omega(|\tau-t|)}{\underline{\alpha}}\Psi(t,z(\tau)-z(t))\,.$$

Proof. Since $\mathcal{E}$ is λ-convex, we obtain that $f_t(y):=\mathcal{E}(t,y)+\Psi(t,z-z(t))$ is λ-convex for every $t\in[0,T]$. In particular, for any minimizer y_* of f_t there holds

$$\forall\, y\in\mathbf{Y}: \qquad f_t(y)\ge f_t(y_*)+\frac{\lambda}{2}\left\|y-y_*\right\|^2\,. \tag{4.11}$$

Let $\tau>t$. From (3.2) we conclude that $\Psi(\tau,z)\ge\left(1-\frac{\omega(|\tau-t|)}{\underline{\alpha}}\right)\Psi(t,z)$. Hence

$$\mathrm{Diss}_\Psi(z;t,\tau)\ge\left(1-\frac{\omega(|\tau-t|)}{\underline{\alpha}}\right)\mathrm{Diss}_{\Psi(t)}(z;t,\tau)\ge\left(1-\frac{\omega(|\tau-t|)}{\underline{\alpha}}\right)\Psi(t,z(\tau)-z(t))\,. \tag{4.12}$$

Estimates (4.11)–(4.12) together with stability of $y(t)$ (cf. (4.6)) imply:

$$\begin{aligned}
\frac{\lambda}{2}\left\|y(\tau)-y(t)\right\|^2 &\leq \mathcal{E}(t,y(\tau))+\Psi(t,z(\tau)-z(t))-\mathcal{E}(t,y(t))\\
&\leq \mathcal{E}(\tau,y(\tau))-\mathcal{E}(t,y(t))-\int_t^\tau \partial_s\mathcal{E}(s,y(\tau))\,\mathrm{d}s+\Psi(t,z(\tau)-z(t))\\
&\overset{(4.7)}{\leq} \int_t^\tau \left(\partial_s\mathcal{E}(s,y(s))-\partial_s\mathcal{E}(s,y(\tau))\right)\,\mathrm{d}s+\Psi(t,z(\tau)-z(t))-\mathrm{Diss}_\Psi(z;t,\tau)\\
&\leq \int_t^\tau \left(\partial_s\mathcal{E}(s,y(s))-\partial_s\mathcal{E}(s,y(\tau))\right)\,\mathrm{d}s+\frac{\omega(|\tau-t|)}{\underline{\alpha}}\Psi(t,z(\tau)-z(t))\,.
\end{aligned}$$

Applying (4.5) gives the assertion of the lemma. □

The following Lemma will be used to prove continuity of solutions.

Lemma 4.3. *Let $\alpha>0$, $\mu:(0,T)\to\mathbb{R}$ be measurable and $\beta\in C([0,\infty))$. If*

$$\forall\, t\in(0,T):\qquad \mu(\tau)^2\leq\alpha\int_0^\tau \mu(s)\,\mathrm{d}s+\beta(\tau)\mu(\tau)$$

then, for every $\tau\in(0,T)$ it holds

$$0\leq\mu(\tau)\leq\alpha\tau+\sup\left\{\beta(s)\mid s\in[0,\tau]\right\}\,.$$

Proof. Assume μ is a simple function. Let $\overline{\mu}(t):=\sup\left\{\mu(s)\mid s\in[0,t]\right\}$ be the essential maximum of μ over $[0,t]$ and let $s_t:=\mathrm{argmax}\overline{\mu}(t)$. Then

$$\begin{aligned}
\overline{\mu}(t)^2=\mu(s_t)^2 &\leq \alpha\int_0^{s_t}\mu(s)\mathrm{d}s+\beta(\mathrm{s_t})\mu(\mathrm{s_t})\\
&\leq \alpha t\overline{\mu}(t)+\sup\left\{\beta(s)\mid s\in[0,t]\right\}\,\overline{\mu}(t)\,.
\end{aligned}$$

Hence, we have

$$\mu(t)\leq\overline{\mu}(t)\leq\alpha t+\sup\left\{\beta(s)\mid s\in[0,t]\right\}\,.$$

The general statement follows from approximating μ pointwise by simple functions. □

Proof of Theorem 4.1.

We follow the approach of [Mie05, MiR15].

Discretization scheme. We first consider the case that (4.8a) holds and afterwards discuss how the proof has to be modified if, instead, (4.8b) or (4.8c) hold. Remark, that continuity of ψ_0 implies boundedness of ψ_0 on bounded subsets of $\mathbf{Z}$ since ψ_0 is 1-homogeneous. This in turn implies continuity of $z\mapsto\Psi(t,z)$ for all $t\in[0,T]$.

We consider the following sequence of partitions of the interval $[0,T]$. For every $K\in\mathbb{N}$, we set $t_k^K:=\frac{k}{2^K}T$, $0\leq k\leq 2^K$ and define for $k\geq 1$

$$\mathcal{D}_k^K\,:\,\mathbf{Z}\times\mathbf{Z}\to\mathbb{R}\,,\qquad \mathcal{D}_k^K(z,\hat{z})=\fint_{t_{k-1}^K}^{t_k^K}\Psi(s,\hat{z}-z)\,\mathrm{d}s\,.$$

Clearly, $\mathcal{D}_k^K(z,\hat{z})$ is weakly lower semicontinuous (respectively continuous if ψ_0 is continuous). We will use $\mathcal{D}_k^K$ in the discretization scheme (4.13) in order to be able to apply Lemma 3.5 in the limit $K\to\infty$ in (4.20).

Lemma 4.4. *Let ψ_0 be weakly continuous. Let $t \in [0,T]$, $z \in \mathbf{Z}$ and $z_K \rightharpoonup z$ weakly as $K \to \infty$. For $K \in \mathbb{N}$ choose $k(K,t)$ such that $t \in [t^K_{k(K,t)}, t^K_{k(K,t)} + 2^{-K})$. Then, we have*

$$\forall\, \hat{z} \in \mathbf{Z}: \quad \lim_{K\to\infty} \mathcal{D}^K_{k(K,t)}(z_K, \hat{z}) = \Psi(t, \hat{z} - z)\,.$$

Proof. Due to (3.2) there holds for all $s \in [0,T]$ and all $\hat{z} \in \mathbf{Z}$ that

$$|\Psi(s, \hat{z} - z_K) - \Psi(t, \hat{z} - z_K)| \le \omega\left(|t-s|\right) \psi_0\left(\hat{z} - z_K\right).$$

Therefore, we find

$$\begin{aligned}
\lim_{K\to\infty} \mathcal{D}^K_{k(K,t)}(z_K, \hat{z}) &= \lim_{K\to\infty} \left(\fint_{t_{k(K,t)}}^{t_{k(K,t)+1}} \Psi(s, \hat{z} - z_K) \right) \\
&\le \lim_{K\to\infty} \left(\Psi(t, \hat{z} - z_K) + \omega\left(2^{-K}\right) \psi_0\left(\hat{z} - z_K\right) \right) \\
&\le \lim_{K\to\infty} \left(\Psi(t, \hat{z} - z(t)) + \Psi(t, z(t) - z_K) + \omega\left(2^{-K}\right) \psi_0\left(\hat{z} - z_K\right) \right) \\
&\le \lim_{K\to\infty} \left(\Psi(t, \hat{z} - z(t)) + \overline{\alpha}\psi_0\left(z(t) - z_K\right) + \omega\left(2^{-K}\right) \psi_0\left(\hat{z} - z_K\right) \right) \\
&= \Psi(t, \hat{z} - z(t)).
\end{aligned}$$

Similarly, we obtain $\lim_{K\to\infty} \mathcal{D}^K_{k(K,t)}(z_K, \hat{z}) \ge \Psi(t, \hat{z} - z(t))$. This concludes the proof. $\square$

Given the initial value $y_0 \in \mathbf{Y}$ and $K \in \mathbb{N}$, we look for $y^K_1, \dots y^K_{2^K} \in \mathbf{Y}$ such that

$$y^K_k \in \operatorname{Argmin}\left\{ \mathcal{E}(t^K_k, y) + \mathcal{D}^K_k(z^K_{k-1}, z) \,|\, y \in \mathcal{V} \right\}. \tag{4.13}$$

The existence of the minimizers y^K_k follows from (4.2) and the lower semicontinuity of $\mathcal{E}$ and Ψ.

Step 1: A priori estimates To simplify the notation in this step, we fix K and write $t_k = t^K_k$, $(\phi_k, z_k) = y_k = y^K_k$ and $\mathcal{D}_k(\cdot,\cdot) = \mathcal{D}^K_k(\cdot,\cdot)$. We use (4.13) and the triangle inequality to find in a first step:

$$\begin{aligned}
\mathcal{E}(t_k, \hat{y}) + \mathcal{D}_k(z_k, \hat{z}) &= \mathcal{E}(t_k, \hat{y}) + \mathcal{D}_k(z_{k-1}, \hat{z}) + \mathcal{D}_k(z_k, \hat{z}) - \mathcal{D}_k(z_{k-1}, \hat{z}) \\
&\ge \mathcal{E}(t_k, y_k) + \mathcal{D}_k(z_{k-1}, z_k) + \mathcal{D}_k(z_k, \hat{z}) - \mathcal{D}_k(z_{k-1}, \hat{z}) \\
&\ge \mathcal{E}(t_k, y_k)\,.
\end{aligned} \tag{4.14}$$

Using again the minimization property (4.13) of y_k, we obtain the upper energy inequality

$$\begin{aligned}
&\mathcal{E}(t_k, y_k) - \mathcal{E}(t_{k-1}, y_{k-1}) + \mathcal{D}_k(z_{k-1}, z_k) \\
&\qquad \le \mathcal{E}(t_k, y_{k-1}) - \mathcal{E}(t_{k-1}, y_{k-1}) + \mathcal{D}_k(z_{k-1}, z_{k-1}) = \int_{t_{k-1}}^{t_k} \partial_s \mathcal{E}(s, y_{k-1})\, \mathrm{d}s\,.
\end{aligned} \tag{4.15}$$

Like on page 52–53 in [MiR15] or page 489 in [Mie05], we find

$$c^{(0)}_E + \mathcal{E}(t_k, y_k) \le (c^{(0)}_E + \mathcal{E}(0, y_0))\, \mathrm{e}^{c^{(1)}_E t_k}\,, \tag{4.16}$$

$$\sum_{j=1}^{k} \mathcal{D}_j(z_{j-1}, z_j) \le (c^{(0)}_E + \mathcal{E}(0, y_0))\, \mathrm{e}^{c^{(1)}_E t_k}\,. \tag{4.17}$$

We finally observe that a combination of (4.1) and (4.3) yields

$$\begin{aligned}|\partial_t\mathcal{E}(t,y_{k-1})| &\le c_E^{(1)}\left(\mathcal{E}(t,y_{k-1})+c_E^{(0)}\right)\\ &\le c_E^{(1)}\left(\mathcal{E}(t_{k-1},y_{k-1})+c_E^{(0)}\right)\mathrm{e}^{c_E^{(1)}|t-t_{k-1}|}\qquad \forall\, t\in[t_{k-1},t_k)\,. \end{aligned}\tag{4.18}$$

In what follows, let

$$Y_K=(\Phi_K,Z_K)$$

denote the right-continuous piecewise constant interpolation of $y_k^K=(\phi_k^K,z_k^K)$ defined by (4.13) and let $\widehat{Z}_K$ denote the piecewise linear interpolation of z_k^K. Furthermore, we define

$$\Theta_K:\ t\mapsto \partial_t\mathcal{E}(t,Y_K(t))\,.$$

Due to (4.16) and (4.18), we find that Θ_K is uniformly bounded in $L^\infty(0,T)$. Since Ψ is a ψ_0-regular dissipation potential and since Z_K are piecewise constant in time, we obtain from estimate (4.17) and property (3.1) that

$$\begin{aligned}\operatorname{var}(Z_K;\psi_0;0,T)=\operatorname{var}(\hat{Z}_K;\psi_0;0,T)&=\sum_{j=1}^{2^K}\psi_0(z_j^K-z_{j-1}^K)\\ &\le \underline{\alpha}^{-1}\sum_{j=1}^{2^K}\mathcal{D}_j^K(z_{j-1}^K,z_j^K)\le\underline{\alpha}^{-1}(c_E^{(0)}+\mathcal{E}(0,y_0))\,e^{c_E^{(1)}T}\,.\end{aligned}$$

Furthermore, from (4.16) and (4.3) we obtain that

$$\mathcal{E}(0,Y_K(t))+c_E^{(0)}\le\left(\mathcal{E}(0,y_0)+c_E^{(0)}\right)e^{2c_E^{(1)}T}$$

and hence $\|Y_K\|_{L^\infty(0,T;\mathbf{Y})}$ is bounded by (4.2).

Summing up (4.15) over k, we obtain

$$\mathcal{E}(t_k,y_k)+\mathrm{Diss}_\Psi(\widehat{Z}_K;0,t_k)\le\mathcal{E}(0,y_0)+\int_0^{t_k}\partial_s\mathcal{E}(s,Y_K(s))\,\mathrm{d}s\tag{4.19}$$

Step 2: Selection of subsequence and passing to the limit From the generalized Helly selection principle in [Mie05, Thm. 5.1] and Step 1 we infer that there exists $z\in\mathrm{BV}_{\psi_0}(0,T;\mathbf{Z})$ and a subsequence of Z_K (still indexed by K) such that $Z_K(t)\rightharpoonup z(t)$ and $\widehat{Z}_K(t)\rightharpoonup z(t)$ pointwise for all $t\in[0,T]$. From Lemma 3.5 we obtain that

$$\mathrm{Diss}_\Psi(z;s,t)\le\liminf_{n\to\infty}\mathrm{Diss}_\Psi(\widehat{Z}_K;s,t)\,.\tag{4.20}$$

Furthermore, there exists $\theta\in L^\infty(0,T)$ such that for a further subsequence (still indexed by K) it holds

$$\Theta_K\rightharpoonup^*\theta\qquad\text{weakly}^*\text{ in }L^\infty(0,T)\,.$$

We furthermore define the function $\theta_{\sup}:t\mapsto\limsup_{K\to\infty}\Theta_K(t)$, for which we find $\theta_{\sup}\in L^\infty(0,T)$ by Fatou's Lemma. For a fixed $t\in[0,T]$ we chose a subsequence K_t^n such that for some $\phi(t)\in\mathbf{F}$ it holds

$$\Theta_{K_t^n}(t)\to\theta_{\sup}(t)\qquad\text{and}\qquad\Phi_{K_t^n}(t)\rightharpoonup\phi(t)\in\mathbf{F}\qquad\text{as }n\to\infty\,.$$

Hence, $z(t)$ and $\phi(t)$ are defined for all $t\in[0,T]$.

Step 3: Stability of the limit function Given $t \in [0,T]$ and $y(t) = (z(t), \phi(t))$, let $k_t^n := \max\left\{k \in \mathbb{N} : t_k^K \le t\right\}$. Since $\mathcal{E}$ is continuous in the first and lower semicontinuous in the second variable, from (4.14) and Lemma 4.4 we obtain

$$\begin{aligned}\mathcal{E}(t, y(t)) &\le \liminf_{K_t^n \to \infty} \mathcal{E}(t_{k_t^n}^{K_t^n}, Y_{K_t^n}(t)) \le \limsup_{K_t^n \to \infty} \mathcal{E}(t_{k_t^n}^{K_t^n}, Y_{K_t^n}(t)) \\ &\le \limsup_{K_t^n \to \infty} \left(\mathcal{E}(t_{k_t^n}^{K_t^n}, \hat{y}) + \mathcal{D}_{k_t^n}^{K_t^n}(Z_{K_t^n}(t), \hat{z}) \right) \\ &\le \mathcal{E}(t, \hat{y}) + \Psi(t, \hat{z} - z(t)) \,. \end{aligned} \tag{4.21}$$

where we have used that $Z_{K_t^n}(t) \rightharpoonup z(t)$ as $K_t^n \to \infty$.

Step 4: Upper energy estimate The choice $\hat{y} = y(t)$ in (4.21) yields $\mathcal{E}(t, y(t)) = \lim_{K_t^n \to \infty} \mathcal{E}(t_{k_t^n}^{K_t^n}, Y_{K_t^n}(t))$. Since Θ_K is uniformly bounded, this yields

$$\mathcal{E}(t, y(t)) = \lim_{K_t^n \to \infty} \mathcal{E}(t, Y_{K_t^n}(t)) \,. \tag{4.22}$$

Therefore, we can apply [Mie05] Proposition 5.6 and obtain that

$$\theta_{\sup}(t) = \lim_{n \to \infty} \Theta_{K_t^n}(t) = \partial_t \mathcal{E}(t, y(t)) \,. \tag{4.23}$$

Taking together (4.19) and (4.20)–(4.23) and Lemma 3.5, we obtain for all $t \in [0,T]$

$$\begin{aligned}\mathcal{E}(t, y(t)) + \mathrm{Diss}_\Psi(z; [s,t]) &\le \mathcal{E}(0, y(0)) + \int_0^t \theta(s) \,\mathrm{d}s \le \mathcal{E}(0, y(0)) + \int_0^t \theta_{\sup}(s) \,\mathrm{d}s \\ &\le \mathcal{E}(0, y(0)) + \int_0^t \partial_t \mathcal{E}(s, y(s)) \,\mathrm{d}s \,. \end{aligned}$$

Step 5: Lower energy estimate Let $t \in [0,T]$. We follow the proof of [MiR15, Prop. 2.1.23]. Since $\theta : t \mapsto \partial_t \mathcal{E}(t, y(t))$ is integrable, we can apply the methods from [DFT05, Sec. 4.4] and obtain a sequence of partitions $0 \le t_0^K < t_1^K < \cdots < t_K^K = T$ with $\sup_{j \in \{0,\dots,K-1\}} t_{j+1}^K - t_j^K \to 0$ as $K \to \infty$ such that

$$\int_r^s \theta(t) \mathrm{d}t = \lim_{K \to \infty} \sum_{j=0}^{K-1} \theta(t_j^K)(t_{j+1}^K - t_j^K) \,.$$

We use (4.21) and observe that for all $K \in \mathbb{N}$ and $j = 1, \dots, K$ we have

$$\mathcal{E}(t_{j-1}^K, y(t_{j-1}^K)) \le \mathcal{E}(t_{j-1}^K, y(t_j^K)) + \Psi(t_{j-1}^K, z(t_j^K) - z(t_{j-1}^K))$$

or

$$\mathcal{E}(t_{j-1}^K, y(t_{j-1}^K)) + \int_{t_{j-1}}^{t_j} \partial_t \mathcal{E}(s, y(t_j^K)) \,\mathrm{d}s \le \mathcal{E}(t_j^K, y(t_j^K)) + \Psi(t_{j-1}^K, z(t_j^K) - z(t_{j-1}^K)) \,.$$

Summing j over $1, \dots, K$, letting $K \to \infty$ and using (3.3) we obtain

$$\mathcal{E}(t, y(t)) + \mathrm{Diss}_\Psi(z; [s,t]) \ge \mathcal{E}(0, y(0)) + \int_0^t \partial_t \mathcal{E}(s, y(s)) \,\mathrm{d}s \,.$$

This finishes the proof of existence of energetic solutions.

Step 6: Continuity Properties Lemma 4.2 yields for $0 \le \tau < t \le T$

$$\frac{\lambda}{2} \|y(\tau) - y(t)\|_{\mathbf{Y}}^2 \le \int_t^\tau C_E^{(3)} \|y(s) - y(\tau)\|_{\mathbf{Y}} \, \mathrm{d}s + \frac{\omega(|\tau - t|)}{\underline{\alpha}} \Psi(t, z(\tau) - z(t)) \,.$$

From Corollary 3.4, we infer that $\Psi(t, z(\tau) - z(t)) < \infty$. Hence

$$\Psi(t, z(\tau) - z(t)) \le C_\psi \overline{\alpha} \|z(\tau) - z(t)\|_{\mathbf{Z}} \le C_\psi \overline{\alpha} \|y(\tau) - y(t)\|_{\mathbf{Y}} \,.$$

Therefore, using Lemma 4.3 we obtain (4.10).

The case of (4.8b) Steps 1,2 and 5–6 work the same way as above. In Step 4, the identity $\theta_{\sup}(t) = \partial_t \mathcal{E}(t, y(t))$ can be obtained directly from the continuity of $\partial_t \mathcal{E}$ assumed in (4.8b). Step 3 is a consequence of the first assumption in (4.8b).

The case of (4.8c) It only remains to prove Step 3. We will use that if $\psi_1, \psi_2 : \mathbf{Z} \to [0, T]$ are convex, positive and 1-homogeneous, then

$$\psi_1 \le \psi_2 \quad \Rightarrow \quad \partial\psi_1(0) \subseteq \partial\psi_2(0) \,.$$

Let $t \in [0, T]$ and for $K \in \mathbb{N}$ choose $t_{K_t} = \frac{k}{2^K} T$ such that $t \in [t_{K_t}, t_{K_t} + 2^{-K})$. We define the 1-homogeneous functional $\overline{\Psi}_K(t, z) := \fint_{t_{K_t}}^{t_{K_t} + 2^{-K}} \Psi(s, z) \, \mathrm{d}s$.

Due to Assumptions (3.1)–(3.2) there holds $\left|\overline{\Psi}_K(t, z) - \Psi(t, z)\right| \le \underline{\alpha}^{-1} \omega\left(2^{-K}\right) \Psi(t, z)$. This in particular implies that $\partial \overline{\Psi}_K(t, 0) \subseteq \left(1 + \underline{\alpha}^{-1} \omega\left(2^{-K}\right)\right) \partial\Psi(t, 0)$. Now, we rewrite (4.14) as

$$-\mathrm{D}\mathcal{E}(t_{K_t} + 2^{-K}, \, y_{K_t}) \in \partial \overline{\Psi}_K(t, 0) \subseteq \left(1 + \underline{\alpha}^{-1} \omega\left(2^{-K}\right)\right) \partial\Psi(t, 0) \,. \tag{4.24}$$

In the limit $K \to \infty$, we find that $t_{K_t} + 2^{-K} \to t$, $y_{K_t} \rightharpoonup y(t)$ and $\underline{\alpha}^{-1} \omega\left(2^{-K}\right) \to 0$. By Assumption (4.8c), this implies $\mathrm{D}\mathcal{E}(t_{K_t} + 2^{-K}, \, y_{K_t}) \rightharpoonup \mathrm{D}\mathcal{E}(t, \, y(t))$ and hence by convexity of $\partial\Psi(t, 0)$ we conclude

$$-\mathrm{D}\mathcal{E}(t, \, y(t)) \in \partial\Psi(t, 0) \,.$$

This is equivalent to the stability condition (4.6). This finishes the proof of our main existence Theorem 4.1.

4.2 The case of a quadratic energy

In this section, we make the following assumptions.

Assumption 4.5. The spaces $\mathbf{F}$, $\mathbf{Z}$ and $\mathbf{Y} = \mathbf{F} \otimes \mathbf{Z}$ are Hilbert spaces, and the energy has the form

$$\mathcal{E}(t, y) = \frac{1}{2} \langle Ay, y \rangle_{\mathbf{Y}} - \langle l(t), y \rangle_{\mathbf{Y}} \,, \tag{4.25}$$

where $A : \mathbf{Y} \to \mathbf{Y}^*$ is positive definite, symmetric and bounded, and where $l \in W^{1,2}(0, T; \mathbf{Y})$. The positive definiteness of A implies that $\mathcal{E}(t, \cdot)$ is λ-convex with λ independent from t. We write $\|y\|_A^2 := \langle Ay, y \rangle$. Concerning ψ_0 and Ψ, we assume $\psi_0 : \mathbf{Z} \to \mathbb{R}$ is a lower semi-continuous, convex, and positively 1-homogeneous functional satisfying (4.9) and $\Psi : [0, T] \times \mathbf{Z} \to \mathbb{R}$ is a ψ_0-regular dissipation potential.

We start with a result on the dependence of the solutions on the right-hand side and the dissipation potential. This result is close to the continuous dependence result in [KrL09, Thm. 2.3], which is more general as it allows for more general uniformly convex energies as well as for temporal jumps which are treated by the Kurzweil integral. Our result is slightly more general in a different direction, because we do not need any a priori bounds on the temporal BV norm of $t \mapsto \Psi(t, \cdot)$. This generalization is crucial for our application to dissipations $\mathcal{R}_\varepsilon(t, \cdot) = \Psi(t, t/\varepsilon, \cdot)$ where no uniform bound is available.

Under the additional assumption that the solutions are differentiable almost everywhere the following result would be easily derived, however it still holds in the general case, see Appendix B for a discussion and the full proof, which is done within the concept of energetic solutions.

Proposition 4.6 (Dependence of solutions on data). *Let $\mathbf{Y} = \mathbf{F} \otimes \mathbf{Z}$ be a Hilbert space. Let $\psi_0 : \mathbf{Z} \to \mathbb{R}$ be a convex, positive 1-homogeneous functional satisfying (4.9) and let $\Psi_1, \Psi_2 : [0, T] \times \mathbf{Z} \to \mathbb{R}$ be ψ_0-regular dissipation potentials with a modulus of continuity ω. Let $A : \mathbf{Y} \to \mathbf{Y}$ be positive definite, symmetric and bounded, $l_1, l_2 \in \mathrm{W}^{1,2}(0, T; \mathbf{Y})$ and let $\mathcal{E}_i$ have the form (4.25) for $l = l_i$ respectively. For $j = 1, 2$ let $y_j : [0, T] \to \mathbf{Y}$ be a solution for the rate-independent system $(\mathbf{Y}, \mathcal{E}_j, \Psi_j)$, i.e. (4.6)–(4.7) hold for parameters $(l, \Psi) = (l_j, \Psi_j)$. Then, for all $t \in [0, T]$ we have the estimate*

$$\begin{aligned} \frac{1}{2}\left\|y_1(t)-y_2(t)\right\|_A^2 \leq \frac{1}{2}\left\|y_1(0)-y_2(0)\right\|_A^2 &+ \int_0^t \langle \dot{l}_1(s)-\dot{l}_2(s), y_1(s)-y_2(s)\rangle \,\mathrm{d}s \\ &+ \langle l_1(t)-l_2(t), y_1(t)-y_2(t)\rangle - \langle l_1(0)-l_2(0), y_1(0)-y_2(0)\rangle \\ &+ \sum_{j=1}^{2}\Big(\mathrm{Diss}_{\Psi_{3-j}}(y_j; 0, t) - \mathrm{Diss}_{\Psi_j}(y_j; 0, t)\Big). \end{aligned} \tag{4.26}$$

As an application of this proposition, we first obtain the following well-posedness and Lipschitz continuity result.

Theorem 4.7. *Let $A : \mathbf{Y} \to \mathbf{Y}^*$ be as above, consider $l \in W^{1,\infty}([0, T]; \mathbf{Y}^*)$ and $y_0 \in \mathbf{Y}$ satisfying (4.6). Then there exists a unique energetic solution $y : [0, T] \to \mathbf{Y}$ for (4.6)–(4.7) with $y(0) = y_0$. Furthermore, for any two solutions y_1 and y_2 we have $\|y_1(t)-y_2(t)\|_A \leq \|y_1(s)-y_2(s)\|_A$ for all $t > s \geq 0$, i.e. we have a contraction semigroup.*

Proof. Existence and continuity properties of solutions follow from Theorem 4.1 observing that (4.8c) is satisfied. The uniqueness of solutions and the contraction property are a direct consequence of (4.26) with $l = l_1 = l_2$ and $\Psi = \Psi_1 = \Psi_2$. □

A second result is obtained if we give a specific estimate between the two dissipation potentials, namely

$$\exists\, \delta > 0\ \forall\, t \in [0, T],\ z \in \mathbf{Z} : \quad |\Psi_1(t, z) - \Psi_2(t, z)| \leq \delta \psi_0(z)\,. \tag{4.27}$$

For the difference of the loadings $l_1 - l_2$ we use the adapted norm

$$\|l_1(t)-l_2(t)\|_* := \left\|A^{-1}\big(l_1(t)-l_2(t)\big)\right\|_A,$$

and similarly for the derivative. We obtain the following explicit estimate.

Corollary 4.8. *Consider the situation of Proposition 4.6 and assume additionally* (4.27) *and $y_1(0) = y_2(0)$, then for all $t > 0$ we obtain the estimate*

$$\|y_1(t)-y_2(t)\|_A \leq 2\,\mathrm{e}^t\Big(\big\|\dot{l}_1-\dot{l}_2\big\|_{L^2(0,t;\mathbf{Y}^*)} + \big\|l_1-l_2\big\|_{L^\infty(0,t;\mathbf{Y}^*)} + \Delta^{1/2}\Big), \tag{4.28}$$

where $\Delta = \delta\big(\mathrm{Diss}_{\psi_0}(y_1; 0, t) + \mathrm{Diss}_{\psi_0}(y_2; 0, t)\big)$.

Proof. We start from (4.26), where we set $\mu(t) = \|y_1(t){-}y_2(t)\|_A^2$, $\lambda(t) = \|l_1(t){-}l_2(t)\|_*^2$, and $\eta(t) = \|\dot l_1(t){-}\dot l_2(t)\|_*^2$. Using $\mu(0) = 0$ we find $\mu(t) \le \int_0^t \big(\eta(s){+}\mu(s)\big)\,\mathrm{d}s + 2\lambda(t) + \frac{1}{2}\mu(t) + \Delta$. This leads us to

$$\mu(t) \le \int_0^t 2\mu(s)\,\mathrm{d}s + K(t) \quad \text{ with } K(t) = 4\Big(\int_0^t \eta(s)\,\mathrm{d}s + \sup_{s\in[0,t]} \lambda(s) + \Delta\Big).$$

Since $K(t)$ is non-decreasing in t, Gronwall's lemma gives $\mu(t) \le 4\mathrm{e}^{2t} K(t)$ and taking the square root gives the result. $\square$

5 Proof of Theorem 1.1

We now return back to the case that $\mathcal{R}_\varepsilon$ is given in the oscillatory form $\mathcal{R}_\varepsilon(t,v) = \Phi(t,t/\varepsilon,v)$. We first show that it is easy to pass to the limit in the energetic formulation if we are able to extract a weakly convergent subsequence. While in previous evolutionary Γ-convergence results for rate-independent systems (cf. [MRS08] or [MiR15, Sec 2.4]) it was sufficient to use a uniform a priori bound for the dissipation and apply Helly's selection principle, this is not enough in the present case, since the oscillatory behavior of the dissipation potential destroys the usual arguments.

5.1 Convergence to the effective equation

Due to Theorem 4.7, for every $\varepsilon > 0$ there exists a unique solution $y^\varepsilon \in C([0,T];\mathbf{Y})$ to (1.3) satisfying the following stability condition and energy equality:

$$\forall\, \hat y \in \mathbf{Y}: \quad \mathcal{E}(t,y^\varepsilon(t)) \le \mathcal{E}(t,\hat y) + \mathcal{R}_\varepsilon(t,\hat y - y^\varepsilon(t))\,, \tag{5.1}$$

$$\mathcal{E}(t,y^\varepsilon(t)) + \mathrm{Diss}_{\mathcal{R}_\varepsilon}(y^\varepsilon,0,t) = \mathcal{E}(0,y_0) + \int_0^t \partial_s\mathcal{E}(s,y^\varepsilon(s))\,\mathrm{d}s\,. \tag{5.2}$$

We now postulate the asymptotic equicontinuity which will be established in the next section in Proposition 5.2:

$$\begin{aligned} &\exists\,\text{modulus of continuity } \omega_\text{equi}\ \forall\,\varepsilon \in (0,1)\\ &\qquad \forall\, t,\tau \in [0,T]: \quad \|y^\varepsilon(t){-}y^\varepsilon(\tau)\| \le \omega_\text{equi}\big(|t{-}\tau|\big) + \omega_\text{equi}(\varepsilon). \end{aligned} \tag{5.3}$$

Recall that a modulus of continuity is a continuous, nondecreasing function $\omega : [0,\infty) \to [0,\infty)$ with $\omega(0) = 0$.

Using $y^\varepsilon(0) = y_0$ which is independent of ε, we also have a uniform bound and may apply the Arzela-Ascoli theorem in the weak topology of $\mathbf{Y}$ restricted to a large ball. Thus, we find a sequence $\varepsilon_k \to 0$ such that

$$\forall\, t \in [0,T]: \quad y^{\varepsilon_k}(t) \rightharpoonup y(t) \tag{5.4}$$

for a limit function $y : [0,T] \to \mathbf{Y}$. The aim is now to show that this limit y is indeed a solution of the effective rate-independent system $(\mathbf{Y},\mathcal{E},\mathcal{R}_\text{eff})$ with $y(0) = y_0$. Since this solution is unique, we know that the whole family y^ε converges (without selecting a subsequence).

From the weak lower semi-continuity of the norm we obtain $\|y(t){-}y(\tau)\| \le \omega_\text{equi}(|t{-}\tau|)$ as well as the continuous convergence

$$t_k \to t_* \implies y^{\varepsilon_k}(t_k) \rightharpoonup y(t_*). \tag{5.5}$$

This is easily seen using $y^{\varepsilon_k}(t_k) - y(t_*) = \big(y^{\varepsilon_k}(t_k) - y^{\varepsilon_k}(t_*)\big) + \big(y^{\varepsilon_k}(t_*) - y(t_*)\big)$, where the first term converges in norm like $\omega_{\mathrm{equi}}(|t_k - t_*|)$ while the second converges weakly.

To show that the limit $y \in C([0,T], \mathbf{Y})$ is an energetic solution, we have to establish the stability (4.6) and the energy balance (4.7), but now with $\mathcal{R}_{\mathrm{eff}}$ instead of Ψ.

Stability condition. Before establishing the result, we recall Proposition 3.6 where $\mathcal{R}_{\mathrm{eff}}(t,\cdot)$ is characterized via

$$\partial\mathcal{R}_{\mathrm{eff}}(t,\cdot) = \bigcap_{s\in[0,2\pi]} \partial\Phi(t,s,0) \ \subset \mathbf{Y}^*.$$

Inequality (5.1) is equivalent with $Ay^\varepsilon(t) - l(t) \in \partial\mathcal{R}_\varepsilon(t,0) = \partial\Phi(t, t/\varepsilon, 0)$.

For a fixed $t_* \in [0,T]$ and a fixed $s \in [0,2\pi)$ we choose a sequence t_k with $t_k \to t_*$ by setting $t_k := \varepsilon_k\,(2\pi\lfloor t_*/(2\pi\varepsilon_k)\rfloor + s)$. By (5.5) we have $y^\varepsilon(t_k) \rightharpoonup y(t)$ and $\mathcal{R}_\varepsilon(t_k,\cdot) = \Phi(t_k,s,\cdot)$. Moreover, the stability of y^ε at time t_k gives $Ay^\varepsilon(t_k) - l(t_k) \in \partial\Phi(t_k,s,0)$. Calculations similar to (4.24) yield that

$$Ay^\varepsilon(t_k) - l(t_k) \in \partial\Phi(t_k,s,0) \subseteq (1 + \omega_1(|t_k - t_*|))\,\partial\Phi(t_*,s,0),$$

and hence the limit $k \to \infty$ (first on the left-hand side, and then on the right-hand side) gives $Ay(t_*) - l(t_*) \in \partial\Phi(t_*,s,0)$. Since $s \in [0,2\pi)$ was arbitrary, we conclude $Ay(t_*) - l(t_*) \in \partial\mathcal{R}_{\mathrm{eff}}(t_*,0)$, which is the stability of y since $t_* \in [0,T]$ was arbitrary as well.

Upper energy inequality. By the first relation in (3.6) we have the lower estimate

$$\mathrm{Diss}_{\mathcal{R}_\varepsilon}(y^\varepsilon;0,T) \geq \mathrm{Diss}_{\mathcal{R}_{\mathrm{eff}}}(y^\varepsilon;0,T)$$

for all $\varepsilon > 0$ and all $t \in (0,T]$. Using the lower semicontinuity of the total dissipation as stated in Lemma 3.5 and the weak lower semi-continuity of the energy $\mathcal{E}(t,\cdot)$, we can pass to the limit $\varepsilon_k \to 0$ in (5.2) to obtain

$$\mathcal{E}(t,y(t)) + \mathrm{Diss}_{\mathcal{R}_{\mathrm{eff}}}(y,0,t) \leq \mathcal{E}(0,y_0) + \int_0^t \partial_s\mathcal{E}(s,y(s))\,\mathrm{d}s\,.$$

Note that for the power integral on the right-hand side we can use the linearity of $-\int_0^t \langle \dot{l}(s), y^{\varepsilon_k}(s)\rangle\,\mathrm{d}s$, the weak convergence and Lebesgue's dominated convergence theorem.

Lower energy inequality. This can be obtained from the stability like in Step 5 of the proof of Theorem 4.1.

Taking the above three points together we have shown that the limit $y : [0,T] \to \mathbf{Y}$ is an energetic solution for the effective rate-independent system.

Uniqueness. Since the uniqueness of the solution with the initial value $y(0) = y_0$ follows from Theorem 4.7, we see that this solution is the only possible accumulation point of the family $(y^\varepsilon)_{\varepsilon\in(0,1)}$. Hence we conclude the convergence as stated in Theorem 1.1. In particular, the only missing point in the proof of the theorem is the equicontinuity stated in (5.3).

5.2 Uniform equicontinuity of solutions

In what follows, we write $\|y\|_A^2 := \langle y,\, Ay\rangle$ and $\|y\|_\infty := \|y\|_{L^\infty(0,T;\mathbf{Y})}$. The first result is a basic lemma showing that we have a uniform L^∞ bound for all $\varepsilon \in (0,1)$.

Lemma 5.1. *Let $y^\varepsilon \in C([0,T];\mathbf{Y})$ be the unique solution to (1.3) for a given initial data y_0 and a forcing $l \in W^{1,\infty}(0,T;\mathbf{Y})$. Then, there exists a constant $C_* = C(y_0, l(\cdot))$ such that*

$$\forall\, \varepsilon \in (0,1): \quad \|y^\varepsilon(\cdot)\|_{L^\infty(0,T;\mathbf{Y})} + \mathrm{Diss}_{\psi_0}(z^\varepsilon; 0, T) \leq C_*\,.$$

Proof. We want to use the a priori estimates (4.16) and (4.17). For this we choose the constant $c_E^{(0)} = 1 + \|l\|^2_{L^\infty(0,T;\mathbf{Y}^*)}$ which implies

$$\mathcal{E}(t,y) + c_E^{(0)} = \frac{1}{2}\|y\|_A^2 - \|l(t)\|_*\|y\|_A + c_E^{(0)} \geq \frac{1}{4}\|y\|_A^2 + 1 \geq \|y\|_A.$$

Hence, we find the power control

$$|\partial_t \mathcal{E}(t,y)| = |\langle \dot{l}(t), y\rangle| \leq \|\dot{l}(t)\|_*\|y\|_A \leq c_E^{(1)}\big(\mathcal{E}(t,y) + c_E^{(0)}\big) \text{ with } c_E^{(1)} := \|\dot{l}\|_{L^\infty(0,T;\mathbf{Y}^*)}$$

Now, (4.16) and (4.17) imply the desired result. □

The next result is the fundamental equicontinuity result, the proof of which is delicate, since there cannot be any uniform a priori bounds for the derivatives $\dot{y}^\varepsilon$ in any L^p space for $p > 1$, see the examples in Section 2. The idea of the proof is to use the microscopic periodicity which provides good bounds if we shift compare a solution with itself but shifted by integer multiples of the period $2\pi\varepsilon$. For this we can use the continuous data dependence derived in Proposition 4.6. In a second step we then control the maximal oscillations in intervals of length $2\pi\varepsilon$.

Proposition 5.2 (Asymptotic equicontinuity of y^ε)**.** *For $\varepsilon \in (0,1)$ let $y^\varepsilon \in C([0,T];\mathbf{Y})$ be the unique solution to (1.3) with initial datum $y^\varepsilon(0) = y_0$ and loading $l \in W^{1,\infty}(0,T;\mathbf{Y})$ where y_0 satisfies (1.7). Then, there exists a modulus of continuity ω_{equi} such that the asymptotic equicontinuity* (5.3) *holds.*

Proof. Since we want to compare y^ε with shifted version of y^ε, we extend l and Φ to the larger interval $[-T, 2T]$ constantly, i.e.

$$l(t) = \begin{cases} l(0) & \text{for } t \in [-T,0], \\ l(T) & \text{for } t \in [T,2T]; \end{cases} \quad \text{and} \quad \Phi(t,s,\cdot) = \begin{cases} \Phi(0,s,\cdot) & \text{for } t \in [-T,0], \\ \Phi(T,s,\cdot) & \text{for } t \in [T,2T]. \end{cases}$$

Hence, $\mathcal{E}(t,y)$ is now defined for $t \in [-T,2T]$ as well. Extending each y^ε by $y^\varepsilon(t) = y_0$ for $t \in [-T,0]$ and using (1.7), we see that y^ε is the unique solution to (4.6)–(4.7) on $[-T,T]$ with initial value $y^\varepsilon(-T) = y_0$.

We now consider τ and t with $0 \leq \tau < t \leq T$ and want to estimate $\|y^\varepsilon(t) - y^\varepsilon(\tau)\|_A$ uniformly in $\varepsilon \in (0,1)$. There are unique $k \in \mathbb{N}_0$ and $\theta \in [0, 2\pi\varepsilon)$ with

$$t = \tau + 2\pi\varepsilon k + \theta.$$

By $y_k^\varepsilon : [-T,T] \to \mathbf{Y}$ we denote the shifted function $y^\varepsilon(t - 2\pi k\varepsilon)$ such that $y_k^\varepsilon(t) = y^\varepsilon(\tau + \theta)$. Hence, we can estimate our equicontinuity term via

$$\|y^\varepsilon(t) - y^\varepsilon(\tau)\|_A \leq \|y^\varepsilon(t) - y_k^\varepsilon(t)\|_A + \|y^\varepsilon(\tau+\theta) - y^\varepsilon(\tau)\|_A\,. \tag{5.6}$$

Since y^ε and y^ε_k both are energetic solutions with the same initial datum y_0 at $t=-T$, we can compare them with our estimate from Corollary 4.8. Note that y^ε_k is obtained with the shifted loading $l_k(t)=l(t{-}2\pi k\varepsilon)$ and the shifted dissipation potential $\Phi(t{-}2\pi k\varepsilon,s,\cdot)$. Using the a priori bound for the dissipation from Lemma 5.1 we obtain

$$\|y^\varepsilon(t)-y^\varepsilon_k(t)\|_A \le 2\mathrm{e}^{t+T}\Big(\|\dot l{-}\dot l_k\|_{L^2(-T,t;\mathbf{Y}^*)}+\|l{-}l_k\|_{L^\infty(-T,t;\mathbf{Y}^*)}+\big(\omega_1(2\pi k\varepsilon)2C_*\big)^{1/2}\Big),$$

where ω_1 is the modulus of continuity of $\Phi(\cdot,s,v)$.

Clearly, we have $\|l(s)-l_k(s)\|\le 2\pi k\varepsilon\|\dot l\|_{L^\infty(0,T;\mathbf{Y}^*)}$. Moreover, for $\rho>0$ we set

$$\omega^l(\rho)^2 := \int_{-T}^{T}\|\dot l(s)-\dot l(s{+}\rho)\|_*^2\,\mathrm{d}s$$

and see that ω^l is continuous with $\omega^l(0)=0$. Hence, $\widehat\omega(s)=\sup_{\rho\in[0,s]}\omega^l(\rho)$ is a modulus of continuity and we find

$$\|y^\varepsilon(t)-y^\varepsilon_k(t)\|_A\le C_4\Big(\widehat\omega(2\pi k\varepsilon)+2\pi k\varepsilon+\omega_1(2\pi k\varepsilon)\Big)=:\omega_*(2\pi k\varepsilon)\le\omega_*(t{-}\tau),\qquad(5.7)$$

where ω_* is still a modulus of continuity.

Now we want to estimate the second term on the right-hand side in (5.6), namely $\|y^\varepsilon(\tau{+}\theta){-}y^\varepsilon(\tau)\|_A$, where $\theta\in[0,2\pi\varepsilon]$. We will not be able to show equicontinuity, but we will obtain a uniform bound that vanishes for $\varepsilon\searrow 0$. To achieve this we first show that the dissipation in intervals of the length $2\pi\varepsilon$ is uniformly bounded. Indeed, using the energy balance (5.2) we have

$$\begin{aligned}\mathrm{Diss}_{\mathcal{R}_\varepsilon}(z^\varepsilon;t,t{+}2\pi\varepsilon)&=\mathcal{E}(t,y^\varepsilon(t))-\mathcal{E}(t{+}2\pi\varepsilon,y^\varepsilon(t{+}2\pi\varepsilon))-\int_t^{t+2\pi\varepsilon}\partial_s\mathcal{E}(s,y^\varepsilon(s))\,\mathrm{d}s\\&\le\|y^\varepsilon(t){-}y^\varepsilon(t{+}2\pi\varepsilon)\|_A\,\big(\,\|y^\varepsilon\|_\infty+\|l\|_\infty\,\big)+2\,\|y^\varepsilon\|_\infty\int_t^{t+2\pi\varepsilon}\|\dot l(s)\|_*\,\mathrm{d}s\\&\le C_5\Big(\omega_*(2\pi\varepsilon)+2\pi\varepsilon\Big),\end{aligned}\qquad(5.8)$$

where we used the uniform a priori bound from Lemma 5.1, $l\in W^{1,\infty}([0,T];\mathbf{Y}^*)$, and the fact that we already have control over shifts by integer multiples of $2\pi\varepsilon$.

Finally we exploit the λ-convexity estimate from Lemma 4.2, where we can use $\lambda=1$, if we use the norm $\|\cdot\|_A$. Indeed, since $\mathcal{R}_\varepsilon(t,\cdot)=\Phi(t,t/\varepsilon,\cdot)$ has the modulus of continuity $\omega_{(\varepsilon)}(r)=\omega_1(r)+\omega_2(r/\varepsilon)$ we obtain, for $0\le\tau<\tau{+}\theta\le T$ with $\theta<2\pi\varepsilon$, the estimate

$$\begin{aligned}\frac12\,\|y^\varepsilon(\tau{+}\theta)-y^\varepsilon(\tau)\|_A^2&\le\int_\tau^{\tau+\theta}\|\dot l(s)\|_*\,\|y^\varepsilon(s)-y^\varepsilon(\tau)\|\;\mathrm{d}s+\frac{\omega_{(\varepsilon)}(\theta)}{\underline{\alpha}}\psi_0\big(\tau,z^\varepsilon(\tau{+}\theta)-z^\varepsilon(\tau)\big)\\&\le\theta2\|\dot l\|_\infty\,\|y^\varepsilon\|_\infty+\big(\omega_1(\theta){+}\omega_2(\theta/\varepsilon)\big)\frac{\overline{\alpha}}{\underline{\alpha}}\mathrm{Diss}_{\mathcal{R}_\varepsilon}(z^\varepsilon;\tau,\tau{+}\theta)\\&\le4\pi\varepsilon\|\dot l\|_\infty\,\|y^\varepsilon\|_\infty+\big(\omega_1(2\pi\varepsilon){+}\omega_2(2\pi)\big)\frac{\overline{\alpha}}{\underline{\alpha}}\mathrm{Diss}_{\mathcal{R}_\varepsilon}(z^\varepsilon;\tau,\tau{+}2\pi\varepsilon)\,.\end{aligned}$$

Combining this with (5.8) we arrive at

$$\|y^\varepsilon(\tau{+}\theta)-y^\varepsilon(\tau)\|_A\le C_6\big(2\pi\varepsilon+\omega_*(2\pi\varepsilon)\big)^{1/2}=:\omega_\circ(\varepsilon),$$

where $\omega_\circ$ is again a modulus of continuity.

Now letting $\omega_{\mathrm{equi}}(r)=\max\{\omega_*(r),\omega_\circ(r)\}$ we see that the last estimate together with (5.6) and (5.7) give the desired assertion. $\square$

A Proofs

Let $\psi_0 : X \to \mathbb{R}$ be a convex, positive 1-homogeneous functional and let $\Psi : [0,T] \times X \to \mathbb{R}$ be a ψ_0-regular dissipation potential. Then, by definition of the variation and by (3.2), we make the following general observations for $u \in \mathrm{BV}_{\psi_0}(0,T;X)$, $0 \le t_0 < t_1 \le T$:

$$\mathrm{var}(u;\Psi(t);t_0,t_1) = \mathrm{var}(u;\Psi(t);t_0,t) + \mathrm{var}(u;\Psi(t);t,t_1) \qquad \forall\, t \in [0,T], \tag{A.1}$$

$$|\mathrm{var}(u;\Psi(s);t_0,t_1) - \mathrm{var}(u;\Psi(t);t_0,t_1)| \le \omega\left(|s-t|\right) \mathrm{var}(u;\psi_0;t,t_1) \forall\, s,t \in [0,T]\,. \tag{A.2}$$

A.1 Proof of Lemma 3.2

Recalling the notation $\mathcal{T}_K[s,t) := \mathcal{T}_K \cap [s,t) \cup \{s\}$,we define

$$D(u;\Psi;\mathcal{T}_K;s,t) := \sum_{t_i \in \mathcal{T}_K[s,t)} \mathrm{var}(u,\Psi(t_i),t_i,t_{i+1})\,.$$

If we define $\tilde{\mathcal{T}}_K := \bigcup_{k=1}^K \mathcal{T}_K$, inequality (A.2) implies

$$\left| D(u;\Psi;\mathcal{T}_K;s,t) - D(u;\Psi;\tilde{\mathcal{T}}_K;s,t) \right| \le \omega(\tau_K)\, \mathrm{var}(u,\psi_0,s,t)\,.$$

Since $\tau_K \to 0$ for $K \to 0$, we obtain that

$$\lim_{K\to\infty} D(u;\Psi;\mathcal{T}_K;s,t) = \lim_{K\to\infty} D(u;\Psi;\tilde{\mathcal{T}}_K;s,t) \tag{A.3}$$

provided one of these limits exists. However, given $K_1, K_2 \in \mathbb{N}$, we observe

$$\left| D(u;\Psi;\tilde{\mathcal{T}}_{K_1};s,t) - D(u;\Psi;\tilde{\mathcal{T}}_{K_2};s,t) \right| \le \left(\omega(\tau_{K_1}) + \omega(\tau_{K_2})\right)\, \mathrm{var}(u,\psi_0,s,t)$$

and thus the limits (A.3) exist. Given two families of partitions $\mathcal{T}_K^1$ and $\mathcal{T}_K^2$ with $\tau_K^1 \to 0$ and $\tau_K^2 \to 0$ as $K \to \infty$, we can follow the above lines and obtain for $\hat{\mathcal{T}}_K := \mathcal{T}_K^1 \cup \mathcal{T}_K^2$

$$\lim_{K\to\infty} D(u;\Psi;\tilde{\mathcal{T}}_K^1;s,t) = \lim_{K\to\infty} D(u;\Psi;\hat{\mathcal{T}}_K;s,t) = \lim_{K\to\infty} D(u;\Psi;\tilde{\mathcal{T}}_K^2;s,t)\,.$$

Now, choose $K_0 \in \mathbb{N}$. For $K > K_0$ and $t_i \in \mathcal{T}_K$ we write $\lfloor t_i \rfloor_{K_0} := \mathrm{argmax}\,\{t \in \mathcal{T}_{K_0} :\ t \le t_i\}$. Then, we obtain

$$\begin{aligned}
&\liminf_{K\to\infty} \sum_{t_i \in \mathcal{T}_K[s,t]\setminus\{t\}} ⨍_{t_i}^{t_{i+1}} \Psi\left(r, u(t_{i+1}) - u(t_i)\right) dr \\
&\qquad \ge \lim_{K\to\infty} \sum_{t_i \in \mathcal{T}_K[s,t)} \Psi\left(\lfloor t_i \rfloor_{K_0}, u(t_{i+1}) - u(t_i)\right) - \omega(\tau_{K_0})\mathrm{var}(u,\psi_0,s,t) \\
&\qquad \to \sum_{t_k \in \mathcal{T}_{K_0}[s,t)} \mathrm{var}(u,\Psi(t_k),t_k,t_{k+1}) - \omega(\tau_{K_0})\mathrm{var}(u,\psi_0,s,t)
\end{aligned}$$

as $K \to \infty$. A similar estimate for the $\limsup_{K\to\infty} \sum_{t_i \in \mathcal{T}_K[s,t)} ⨍_{t_i}^{t_{i+1}} \Psi\left(r, u(t_{i+1}) - u(t_i)\right) dr$ yields the first equality of (3.3). The second follows from the uniform modulus of continuity.

A.2 Proof of Lemma 3.3

The first statement is obviously true. The inequality

$$\mathrm{var}(u;\psi;s,t) \geq \limsup_{K\to\infty} \sum_{k=0}^{N_K-1} \psi\left(u(t_{k+1}^K) - u(t_k^K)\right)$$

is an immediate consequence of the definition of var(·). On the other hand, for $n \in \mathbb{N}$ we can chose a partition $\mathcal{T}_n$ such that $\sum_{k=0}^{N_K-1} \psi\left(u(t_{k+1}^K) - u(t_k^K)\right) \geq \mathrm{var}(u;\psi;s,t) - \frac{1}{n}$. Due to the triangle inequality, we can assume $\mathcal{T}_n \supset \mathcal{T}_{n-1}$.

The third statement follows from $\Psi_1(t,u) \leq \Psi_2(t,u) + \beta\psi_0(u)$ for all $t \in [0,T]$.

A.3 Proof of Lemma 3.5

Let $(\mathcal{T}_K)_{K\in\mathbb{N}}$ be a sequence of partitions of $[0,T]$ such that $\mathcal{T}_K \subset \mathcal{T}_{K+1}$, i.e. $\tau_K \searrow 0$. Then we obtain

$$\begin{aligned}
&\sum_{i=0}^{N_{K_0}-1} \int_{t_i^{K_0}}^{t_{i+1}^{K_0}} \Psi(t_i, u_n(t_{i+1}^{K_0}) - u_n(t_i^{K_0})) \\
&\quad\leq \sum_{i=0}^{N_{K_0}-1} \sum_{s_k\in\mathcal{T}_K[t_i^{K_0}, t_{i+1}^{K_0})} \int_{s_k}^{s_{k+1}} \Psi(s_k, u_n(s_{k+1}) - u_n(s_k)) + \omega(\tau_{K_0})\,\mathrm{var}(u_n, \psi_0; s,t)
\end{aligned}$$

Passing to the limit $K \to \infty$ on the right hand side, we obtain

$$\sum_{i=0}^{N_{K_0}-1} \int_{t_i}^{t_{i+1}} \Psi(t_i, u_n(t_{i+1}) - u_n(t_i)) \leq \mathrm{Diss}_\Psi(u_n; s,t) + \omega(\tau_{K_0})\,\mathrm{var}(u_n, \psi_0; s,t)\,. \tag{A.4}$$

Since $\Psi(t_i,\cdot)$ is lower semicontinuous and convex, it is also weakly lower semicontinuous and we obtain

$$\begin{aligned}
\sum_{i=0}^{N_{K_0}-1} \int_{t_i}^{t_{i+1}} \Psi(t_i, u(t_{i+1}) - u(t_i)) &\leq \liminf_{n\to\infty} \sum_{i=0}^{N_{K_0}-1} \int_{t_i}^{t_{i+1}} \Psi(t_i, u_n(t_{i+1}) - u_n(t_i)) \\
&\leq \mathrm{Diss}_\Psi(u_n; s,t) + \omega(\tau_{K_0})\,\mathrm{var}(u_n, \psi_0; s,t)\,. \qquad (A.5)
\end{aligned}$$

Passing to the limit $K_0 \to 0$, we find from (A.4) and (A.5) that (3.5) holds.

A.4 Proof of Proposition 3.6

Let $\frac{1}{2} > \delta > 0$ and define $\dot{x}_\delta : [\hat{t} - 1/2, \hat{t} + 1/2] \to \mathbf{Y}$ through

$$\dot{x}_\delta(s) := \begin{cases} \frac{1}{\delta^2}\left(\delta - \left|s - \hat{t}_\varepsilon\right|\right) y & \text{if } \left|s-\hat{t}\right| < \delta \\ 0 & \text{else} \end{cases}.$$

We continue $\dot{x}_\delta$ in a 2π-periodic way and define $x_\delta(t) := \int_0^t \dot{x}_\delta(s)\,\mathrm{d}s$. For $\delta \to 0$ we find by continuity of $\Psi_t(\cdot,y) : \hat{t} \mapsto \Phi(t,\hat{t},y)$ and 1-homogeneity of $\Phi(t,s,\cdot)$ that

$$\begin{aligned}
\Phi(t,\hat{t},y) &= \lim_{\delta\to 0} \int_{\hat{t}-\delta}^{\hat{t}+\delta} \left(\frac{1}{\delta} - \frac{\left|s-\hat{t}\right|}{\delta^2}\right) \Phi(t,s,y)\,\mathrm{d}s = \lim_{\delta\to 0}\int_0^{2\pi} \Phi(t,s,\dot{x}_\delta(s))\,\mathrm{d}s \\
&= \lim_{\delta\to 0} \mathrm{Diss}_{\Psi_t}(x_\delta; 0, 2\pi) \geq \mathcal{R}_{\mathrm{eff}}(t,y)\,.
\end{aligned}$$

This immediately yields $\partial\mathcal{R}_{\text{eff}}(t,0) \subseteq \partial\Phi(t,s,0)$ for all $s\in[0,2\pi]$.

On the other hand, let $\xi \notin \partial\mathcal{R}_{\text{eff}}(t,0)$ but $\xi\in\partial\Phi(t,s,0)$ for all $s\in[0,2\pi]$. Then

$$\Phi(t,s,\tilde z) \geq \langle \xi,\tilde z\rangle \text{ for all } \tilde z\in\mathbf{Z}\,, \qquad \text{and} \qquad \exists z\in\mathbf{Z}\,:\, \mathcal{R}_{\text{eff}}(t,z) < \langle \xi,z\rangle\ .$$

The latter means that there exists $x:[0,2\pi]\to\mathbf{Z}$ such that $x(0)=0$, $x(2\pi)=z$ and $\text{Diss}_{\Psi_t}(x;0,2\pi) < \langle\xi,z\rangle$. This implies for some finite partition $\mathcal{T}_K[0,2\pi]$ of the interval $[0,2\pi]$ that

$$\langle\xi,z\rangle > \sum_{s_i\in\mathcal{T}_K[0,2\pi]\setminus\{2\pi\}} \text{var}(x,\Psi_t(s_i),s_i,s_{i+1}) \geq \sum_{s_i\in\mathcal{T}_K[0,2\pi]\setminus\{2\pi\}} \langle\xi,x(s_{i+1})-x(s_i)\rangle = \langle\xi,z\rangle\ ,$$

which is a contradiction.

B Proof of Proposition 4.6

The proof of Proposition 4.6 is quite technical although the idea behind is very simple and is well-known for proving our result in the case that $\Psi_1=\Psi_2=\psi_0$. The case of time-dependent and different $\Psi_j(t,\cdot)$ is contained in [KrL09] even for cases where Ψ_j may jump in time. We motivate the result in the case that the solution are sufficiently smooth. A function $y\in W^{1,\infty}(0,T;\mathbf{Y})$ is a solution to (4.6)–(4.7) if and only if

$$\forall_{\text{a.a.}} t\in[0,T]\ \forall\, v\in\mathbf{Y}:\quad \langle \mathrm{D}\mathcal{E}(t,y),v-\dot y\rangle + \Psi(t,v)-\Psi(t,(\dot y)\geq 0, \tag{B.1}$$

(see e.g. [Kre99, Mie05, KrL09, MiR15]). Given two loadings l_1 and l_2 and two dissipation potentials Ψ_1 and Ψ_2 we obtain (choosing $v=\dot y_{3-j}$ and adding)

$$\langle \mathrm{D}\mathcal{E}_1(t,y_1){-}\mathrm{D}\mathcal{E}_2(t,y_2),\dot y_1{-}\dot y_2\rangle + \Psi_1(t,\dot y_1)-\Psi_2(t,\dot y_1)+\Psi_2(t,\dot y_2)-\Psi_1(t,\dot y_2)\leq 0\,.$$

Integrating over time and some integration by parts leads us then to the estimate

$$\begin{aligned}\frac12\,\|y_1(t){-}y_2(t)\|_A^2 \leq \frac12\,\|y_1(0){-}y_2(0)\|_A^2 &+ \int_0^t \Big\langle \dot l_1(s){-}\dot l_2(s),y_1(s){-}y_2(s)\Big\rangle\,\mathrm{d}s\\ &+\langle l_1(t){-}l_2(t),y_1(t){-}y_2(t)\rangle - \langle l_1(0){-}l_2(0),y_1(0){-}y_2(0)\rangle\\ &+\int_0^t\big(\Psi_1(s,\dot y_1(s)){+}\Psi_2(s,\dot y_1(s)){-}\Psi_1(s,\dot y_1(s)){-}\Psi_2(s,\dot y_2(s))\big)\,\mathrm{d}s,\end{aligned}$$

which is essentially the desired estimate (4.26) as stated in Proposition 4.6. However, due to the temporal fluctuations of Ψ_i and the low temporal regularity of y_i, we have to carry out all of these calculations in a time-discrete setting.

Proof of Proposition 4.6. As above we write $\|y\|_A^2 := \langle Ay,y\rangle$, and for fixed $t\in(0,T]$ and $N\in\mathbb{N}$ we define the partition $t_k=\frac{kt}{N}$. For all continuous functions $a:[0,T]\to\mathbf{X}$ we let $a^k := a(t_k)$ and $a^{k-1/2}:=\frac12(a^k{+}a^{k-1})$. In addition to y_j and l_j we will also use $E_j(t)=\mathcal{E}_j(t,y_j(t))$ and $\sigma_j(t)=\mathrm{D}\mathcal{E}_j(t,y_j(t))=Ay_j(t)-l_j(t)$,

Subsequently using the quadratic structure of $\mathcal{E}(t,y)$ and (4.7) we obtain the relation

$$\begin{aligned}\frac12\|y_j^k\|_A^2-\frac12\|y_j^{k-1}\|_A^2 &= E_j^k-E_j^{k-1}+\big\langle l_j^k,y_j^k\big\rangle - \big\langle l_j^{k-1},y_j^{k-1}\big\rangle\\ &=\big\langle l_j^k,y_j^k\big\rangle - \big\langle l_j^{k-1},y_j^{k-1}\big\rangle - \int_{t_{k-1}}^{t_k}\Big\langle \dot l_j,y_j\Big\rangle\,\mathrm{d}s - \text{Diss}_{\Psi_j}(y_j;t_{k-1},t_k).\end{aligned}$$

For comparing the two solutions y_1 and y_2 we add the above identities and, after some quadratic rearrangements, we obtain the relation

$$\begin{aligned}
&\frac{1}{2}\|y_1^k{-}y_2^k\|_A^2 - \frac{1}{2}\|y_1^{k-1}{-}y_2^{k-1}\|_A^2 \\
&\quad = \frac{1}{2}\|y_1^k\|_A^2 - \frac{1}{2}\|y_1^{k-1}\|_A^2 \;+\; \frac{1}{2}\|y_2^k\|_A^2 - \frac{1}{2}\|y_2^{k-1}\|_A^2 \\
&\qquad - \Big\langle y_1^{k-1/2}, y_2^k{-}y_2^{k-1}\Big\rangle - \Big\langle y_2^{k-1/2}, y_1^k{-}y_1^{k-1}\Big\rangle \\
&\quad = \sum_{j=1}^{2}\Big(\langle l_j^k, y_j^k\rangle - \langle l_j^{k-1}, y_j^{k-1}\rangle - \int_{t_{k-1}}^{t_k}\Big\langle \dot l_j, y_j\Big\rangle\,\mathrm{d}s - \mathrm{Diss}_{\Psi_j}(y_j;t_{k-1},t_k)\Big) \\
&\qquad - \Big\langle \sigma_1^{k-1/2}{+}l_1^{k-1/2}, y_2^k{-}y_2^{k-1}\Big\rangle - \Big\langle \sigma_2^{k-1/2}{+}l_2^{k-1/2}, y_1^k{-}y_1^{k-1}\Big\rangle .
\end{aligned}$$

We now insert the stability condition (4.6) which takes the form

$$\forall\, v\in \mathbf{Y}: \quad -\big\langle \sigma_j^k, v\big\rangle \le \Psi_j(t_k, v),$$

where we choose $v = y_{3-j}^k - y_{3-j}^{k-1}$. This leads us to the estimate

$$\begin{aligned}
&\frac{1}{2}\|y_1^k{-}y_2^k\|_A^2 - \frac{1}{2}\|y_1^{k-1}{-}y_2^{k-1}\|_A^2 \\
&\quad \le \sum_{j=1}^{2}\Big(\langle l_j^k{-}l_{3-j}^k, y_j^k\rangle - \langle l_j^{k-1}{-}l_{3-j}^{k-1}, y_j^{k-1}\rangle + \int_{t_{k-1}}^{t_k}\Big\langle \dot l_j(s), y_{3-j}^{k-1/2}{-}y_j(s)\Big\rangle\,\mathrm{d}s\Big) \\
&\qquad + \sum_{j=1}^{2}\Big(\frac{1}{2}\Psi_j(t_k, y_{3-j}^k{-}y_{3-j}^{k-1}) + \frac{1}{2}\Psi_j(t_{k-1}, y_{3-j}^k{-}y_{3-j}^{k-1}) - \mathrm{Diss}_{\Psi_j}(y_j;t_{k-1},t_k)\Big).
\end{aligned}$$

Summing this inequality over $k = 1, \ldots, N$ we see that many terms cancel by the telescoping effect. Moreover, taking the limit $N \to \infty$, we use $t_N = t$ and $t_0 = 0$ and can employ Lemma 3.2 to obtain the desired estimate (4.26). Hence the proof of Proposition 4.6 is complete. □

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