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Geometric Methods of Complex Analysis

Organised by
Mats Andersson, Göteborg
Bo Berndtsson, Göteborg
John Erik Fornaess, Trondheim
Nikolay Shcherbina, Wuppertal

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Abstract. The purpose of this workshop was to discuss recent results in Several Complex Variables, Complex Geometry and Complex Dynamical Systems with a special focus on the exchange of ideas and methods among these areas. The main topics of the workshop included Holomorphic Dynamics and Foliations; L^2-methods and Cohomologies; Plurisubharmonic Functions and Pluripotential Theory; Singular Metrics on Vector Bundles, Chern Forms and Residue Currents; Geometric Questions of Complex Analysis (including Uniformization, Polynomial Convexity, Levi-flat Surfaces etc.).

Mathematics Subject Classification (2010): 32xx, 53xx, 14xx, 37Fxx.

Introduction by the Organisers

The workshop *Geometric Methods of Complex Analysis* attracted 52 researchers from 14 countries. Both, leading experts in the field and young researchers (including four Ph. D. students and three postdocs) were well represented in the meeting and gave talks. There was 9 female researchers among the participants of the workshop. A rather wide spectrum of topics related to Complex Analysis (and this was one of the aims of the workshop) was covered by the talks and informal discussions. All 21 lectures presented on the meeting can be conditionally divided into the following groups.

Holomorphic Dynamics and Foliations were represented by talks of E. Bedford, H. Peters, N. Sibony and B. Jöricke. Bedford discussed some open problems arising in the complex dynamics of volume-preserving polynomial maps and their

relation to the well-known results. Peters showed that a transcendental entire function has infinite topological entropy. Sibony gave new versions of the classical theorems, Nevanlinna's second main Theorem, Bloch's Theorem, Ax Lindemann's Theorem, in the case when the source is an open Parabolic Riemann surface. Jöricke presented finiteness theorems for bundles over connected finite Riemann surfaces with some specified types of fibers.

L^2-methods and Cohomologies were represented by the talks of T. Ohsawa, B.-Y. Chen, F. Deng, X. Zhou and N. Tardini. Ohsawa explained how L^2-extension technique can be applied to analytic families. In particular, he showed how using L^2-extension theorem one can prove Nishino's theorem on Stein submersions and, moreover, how using this method one can extend the results of Takegoshi and Takayama on the Levi problem on complex manifolds. Chen presented relations of the directional derivative of the weighted Bergman kernel corresponding to a given plurisubharmonic function to certain directional Lelong numbers associated to this function. Deng gave a new characterization of plurisubharmonic functions and Griffiths positivity of holomorphic vector bundles with singular Finsler metrics, and presented a proof of the Griffiths positivity of the direct image bundles of the twisted relative canonical bundle associated to a holomorphic family of Stein manifolds or compact Kähler manifolds, by applying this characterization and Ohsawa-Takegoshi type extension theorems. Zhou explained how to obtain new vanishing and finiteness theorems for multiplier ideal sheaves using the strong openness property of the multiplier ideal sheaves. He also gave some generalisations of Siu's lemma which can be proved using ideas and results contained in the proof of the strong openness conjecture. Tardini explained how to characterize the $\partial\overline{\partial}$-lemma using the Bott-Chern cohomology.

Plurisubharmonic Functions and Pluripotential Theory were represented by the talks of Z. Błocki, C.H. Lu and E.A. Poletsky. Błocki presented a new definition of the Monge-Ampère operator for plurisubharmonic functions with analytic singularities on Hermitian manifolds. The advantage of this definition is that it gives no loose of the total mass. Lu showed monotonicity property of total mass for natural classes of θ-plurisubharmonic functions. This allows him to prove concavity of the volume function $T \mapsto \log \mathrm{Vol}(T)$ and hence confirm a conjecture of Boucksom-Eyssidieux-Guedj-Zeriahi. Poletsky discussed the question of separability of points in a complex manifold M by bounded above continuous plurisubharmonic functions $PSH^{cb}(M)$. In particular, he explained how to prove that the *core* $\mathbf{c}(M)$ of an arbitrary complex manifold M can be decomposed as the disjoint union of the sets E_j, $j \in J$, that are 1-pseudoconcave in the sense of Rothstein and have the following Liouville property: every function from $PSH^{cb}(M)$ is constant on each of E_j.

Singular Metrics on Vector Bundles, Chern Forms and Residue Currents were represented by the talks of R. Lärkäng, E. Wulcan and J. Ruppenthal. Lärkäng explained how to generalize the classical *Poincaré-Lelong formula* to the case of

residue current associated to a complex of vector bundles. Wulcan presented a definition of the Chern forms of hermitian metrics with analytic singularities which uses so-called Segre forms. Ruppenthal explained how to give a natural meaning to the k-th Chern form of a singular Griffiths semi-positive hermitian metric as a closed (k,k)-current of order 0, as long as the set where the metric degenerates is small enough. He has also shown how these results can be extended to Chern forms of arbitrary degree if the metric has analytic singularities.

Geometric Questions of Complex Analysis (including Uniformization, Polynomial Convexity, Levi-flat Surfaces etc.) were represented by the talks of S. Nemirovski, P. Gupta, B. Stensønes, J. Winkelmann, J. Brinkschulte and K.-T. Kim. Nemirovski discussed relations of uniformization and steinness. He explained that the universal cover of a Stein stictly pseudoconvex domain with non-spherical boundary cannot cover a complex manifold containing a compact analytic subset of positive dimension. In particular, it follows that any other strictly pseudoconvex domain with the same universal cover is also Stein. Gupta presented results on optimal (with respect to the dimension of the target complex space) polynomially convex embeddings of compact real manifolds. Stensønes gave an example of a bounded (pseudoconvex) domain in $\mathbb{C}^2$ with boundary of class $\mathcal{C}^{1,1}$ which has a Stein neighborhood basis, but is *not* s-H-convex for any real number $s \geq 1$. Winkelmann defined the notion of a *tame* set for a subset of an arbitrary complex manifold. In the case when this manifold is a semisimple Lie group he investigated the properties of tame subsets in more details. Brinkschulte presented a non-existence result for smooth compact Levi-flat real hypersurfaces in a complex manifold such that their normal bundle admits a Hermitian metric with positive curvature along the leaves. Kim explained that a bounded symmetric domain in a complex Banach space is holomorphic homogeneous regular if and only if it is of finite rank. He also gave a complete characterization of such domains and provided an explicit formula for their rank.

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Workshop: Geometric Methods of Complex Analysis

Table of Contents

Eric Bedford
Some interesting domains that arise in the complex dynamics 2259

Berit Stensønes (joint with Lars Simon)
Stein neighborhood basis 2261

Elizabeth Wulcan (joint with Richard Lärkäng, Hossein Raufi and Martin Sera)
Chern forms of hermitian metrics with analytic singularities 2263

Han Peters (joint with Anna Miriam Benini, John Erik Fornæss)
Entropy of transcendental entire functions 2266

Chinh H. Lu
Complex Monge-Ampère equations with prescribed singularity 2267

Takeo Ohsawa
Applications of the L^2 extension theorem to analytic families 2269

Kang-Tae Kim (joint with Cho-Ho Chu, Sejun Kim)
HHR/USQ domains and squeezing constants for bounded symmetric domains 2270

Evgeny A. Poletsky (joint with Nikolay Shcherbina)
Plurisubharmonically separable complex manifolds 2272

Richard Lärkäng (joint with Elizabeth Wulcan)
Residue currents and cycles of complexes of vector bundles 2275

Purvi Gupta (joint with Rasul Shafikov)
Polynomially convex embeddings of compact real manifolds 2277

Zbigniew Błocki
Monge-Ampère operator for plurisubharmonic functions with analytic singularities on Hermitian manifolds 2278

Nessim Sibony
The Second Main Theorem in the hyperbolic case. 2280

Nicoletta Tardini (joint with Daniele Angella)
Cohomological properties of complex non-Kähler manifolds 2282

Bo-Yong Chen
Weighted Bergman kernel, directional Lelong number and John-Nirenberg exponent 2284

Jörg Winkelmann
Tame discrete subsets in Stein manifolds 2285

Burglind Jöricke
Riemann surfaces of second kind and finiteness theorems 2286

Fusheng Deng (joint with Zhiwei Wang, Liyou Zhang and Xiangyu Zhou)
New characterization of plurisubharmonic functions and positivity of direct image bundles 2288

Jean Ruppenthal
Chern Forms of Singular Metrics on Vector Bundles 2291

Xiangyu Zhou
Some results on multiplier ideal sheaves 2294

Judith Brinkschulte
Curvature properties and nonexistence of compact Levi-flat hypersurfaces 2296

Stefan Nemirovski (joint with Rasul Shafikov)
Uniformization and Steinness 2297

Abstracts

Some interesting domains that arise in the complex dynamics

Eric Bedford

We consider the iteration of polynomial automorphisms of $\mathbb{C}^2$. The prime example of such a map is a complex Hénon map

$$f(x,y) = (y, y^2 + c - \delta x)$$

We will consider this as a generalization of the one-dimensional map $y \mapsto q_c(y)$, where $q_c(y) = y^2 + c$. The set where the iterates of q_c remain bounded is the *filled Julia set* K_c. If c belongs to the Main Cardioid of the Mandelbrot set, then q_c has a unique attracting fixed point, and K_c is the closure of the basin of attraction.

In the case of a complex Hénon map, we consider the set K^+ of the points whose forward orbits are bounded. Similarly, we define K^- in terms of backward orbits, and we set $K := K^+ \cap K^-$.

Theorem (Hubbard-ObersteVorth, Fornæss-Sibony)**.** *If c belongs to the Main Cardioid of the Mandelbrot set, then for $|\delta|$ sufficiently small, the complex Hénon map f has a unique attracting fixed point, and K^+ is the closure of the basin of attraction.*

A set of primary importance is the *Fatou set* $\mathcal{F}^+$, which is defined as the largest open set on which the iterates $\{f^n : n \geq 0\}$ are a normal family. This is the same as the set of equicontinuity of the forward iterates. Equivalently, it is the same as the set of points where f is Lyapunov stable. It is not hard to show that

$$\mathcal{F}^+ = U^+ \cup \operatorname{int}(K^+), \quad U^+ := \mathbb{C}^2 - K^+$$

It is known that the (2-dimensional) Hénon map is closely related to the 1-dimensional map q_c in the case where q_c is hyperbolic, and $|\delta| \ll 1$. And the case $|\delta| = 1$ is the case where the 2-dimensional case is expected to be most different from the 1-dimensional case. The case $|\delta| = 1$ is called *conservative* because such maps preserve volume. In general, the Lebesgue volume of a Borel set satisfies:

$$\text{Volume}(f(E)) = \text{Volume}(E)$$

so the term "conservative" means the same as "volume-preserving". The case $|\delta| = 1$ differs from the case $|\delta| \neq 1$ in several respects. For instance:

Theorem (Friedland-Milnor)**.** *If $|\delta| = 1$, then $\operatorname{int}(K^+) = \operatorname{int}(K^-) = \operatorname{int}(K)$, and this set is bounded.*

At the moment, a number of basic questions about conservative Hénon maps remain unanswered. Recall that for c in the Main Cardioid, K_c, the filled Julia set of q_c, is the closure of its (connected) interior. Can there be an analogue of this for volume-preserving maps?

Question 1. *If f is conservative, is it possible that the interior of K^+ is connected, and its closure is equal to K?*

We will discuss the Fatou components, i.e., connected components of $\mathcal{F}^+$. The set U^+ is the *escape locus*, points escaping to infinity, and it is quite different from the other components, which will be seen to be *rotation domains*. Thus by *Fatou component* we will mean the connected components of the interior of K^+.

If Ω is a component of the interior of K, then it follows from the Friedland-Milnor Theorem that $f^N(\Omega) = \Omega$ for some $N \geq 1$. Thus we may replace f by f^N and suppose that $f(\Omega) = \Omega$. Let us consider the restriction $f|_\Omega$ and define the set of all sub-sequential limits of its iterates:

$$\mathcal{G} = \{g = \lim_{j\to\infty} (f|_\Omega)^{n_j} : \Omega \to \Omega\}$$

where the limit is taken over all sequences $\{n_j\}$ for which the limit g exists. Using Theorems of H. Cartan, we see that $\mathcal{G}$ consists of automorphisms of Ω, and it is a compact, Abelian Lie group. Thus if $\mathcal{G}_0$ denotes the connected component of the identity, we have that $\mathcal{G}_0 \cong \mathbb{T}^\rho$, where $\mathbb{T}^\rho$ is a real torus of dimension ρ. Thus f generates a torus action on Ω, and we call Ω a *rotation domain of rank* ρ.

Theorem (Bedford-Smillie)**.** *The only possible values of ρ are 1 or 2.*

Theorem (Barrett-Bedford-Dadok)**.** *If $\rho = 2$, then $(f|_\Omega, \Omega)$ is biholomorphically conjugate to (L, D), where $D \subset \mathbb{C}^2$ is a Reinhardt domain, with $L = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ a linear transformation of D, and $|\lambda| = |\mu| = 1$, $\lambda^j \mu^k \neq 1$ for all $j, k \in \mathbb{Z}$, $(j, k) \neq (0, 0)$.*

We can ask whether the rank 1 case is always given by a (p, q)-domain:

Question 2. *Suppose that Ω is a rotation domain of rank 1. Is $(f|_\Omega, \Omega)$ biholomorphically conjugate to a linear map (L, D), where $D \subset \mathbb{C}^2$ is a domain, and $L = \begin{pmatrix} \alpha^p & 0 \\ 0 & \alpha^q \end{pmatrix}$, with $(p, q) = 1$, $|\alpha| = 1$ and α not a root of unity?*

In this case, we think of D as the *linear model* of the rotation domain Ω. If D is a Reinhardt domain (the case where rank is 2), then D is logarithmically convex, and thus the boundary is quite "tame". On the other hand, we would expect that the boundary of Ω can be a rather "wild" fractal, corresponding to the case of Siegel disks in dimension 1.

In some sense the "nicest" case is when $f|_\Omega$ has a fixed point. Suppose $f(0) = 0$, and let $L = Df(0)$. By a linear conjugation, we may suppose that $L = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ with $|\lambda| = |\mu| = 1$. Since Ω is an invariant Fatou component, it follows that the limit

$$\Psi = \frac{1}{N} \sum_{n=0}^{N-1} L^{-n} \circ f^n \tag{1}$$

converges on Ω, and Ψ gives a conjugacy from $(f|_\Omega, \Omega)$ to $(L, \Psi(\Omega))$. Thus, when there is a fixed point, there is always a conjugacy to a linear action.

Let us revisit the case of a rotation domain Ω with rank 2. Thus we have a rotation L of a Reinhardt domain $D \subset \mathbb{C}^2$.

Question 3. *What sorts of Reinhardt domains can arise as linear models of rotation domains?*

These Fatou components should be especially interesting to us as Complex Analysts because they present us with very concrete/abstract examples of Reinhardt domains for which we have not yet answered the basic questions:

Can D be biholomorphically equivalent to the bidisk?
and/or
Can D be biholomorphically equivalent to the ball?

Remark. If we wish to examine these questions more concretely, we may choose f to have a fixed point with $Df = L = \begin{pmatrix} e^{\sqrt{2}i\pi} & 0 \\ 0 & e^{\sqrt{3}i\pi} \end{pmatrix}$. By a celebrated Theorem of C.L. Siegel, the map f will be linearizable in a neighborhood of the fixed point. That is, the map Ψ defined in (1) will be convergent in a neighborhood of the fixed point. In fact, the Reinhardt model D will be the region of convergence for the power series for Ψ^{-1}.

A rather different question is the following:

Question 4. *Is it possible that the Reinhardt model is incomplete (i.e., it does not contain the origin)?*

Question 4 is equivalent to asking: *Is it possible that Ω does not contain a fixed point?* We note that since all Fatou components are Runge domains and polynomially convex, it follows that $H_2(\Omega; \mathbb{Z}) = 0$. It follows that the Reinhardt model, also, must satisfy $H_2(D; \mathbb{Z}) = 0$. Thus if D does not contain the origin, it must intersect one of the coordinate axes, and the intersection of D with this coordinate axis is a special invariant annulus.

S. Ushiki has done computer experiments to address Question 4 and has found numerical examples which seem to be strong evidence that the answer to Question 4 is "yes". We present and discuss one of Ushiki's examples. This appears to give a very curvy and complicated embedding of an incomplete Reinhardt domain into $\mathbb{C}^2$.

Stein neighborhood basis

Berit Stensønes
(joint work with Lars Simon)

The notion of s-H-convexity was introduced by J. Chaumat and A.-M. Chollet and goes back to work by A. Dufresnoy. Given a real number $s \geq 1$, a compact set $\emptyset \neq K \subseteq \mathbb{C}^n$ is called s-H-convex, if there exists a $C > 0$ with $C \leq 1$, such that for all ϵ, $0 < \epsilon \leq 1$, there exists an open pseudoconvex subset Ω_ϵ of $\mathbb{C}^n$ satisfying

$$\{z \in \mathbb{C}^n : d(z, K) < C\epsilon^s\} \subseteq \Omega_\epsilon \subseteq \{z \in \mathbb{C}^n : d(z, K) < \epsilon\},$$

where $d(\cdot, K)$ denotes the Euclidean distance to K.

J. Chaumat and A.-M. Chollet obtain various $\overline{\partial}$-results for such sets. Another result in that spirit is due to A.-M. Chollet.

Furthermore, the notion of s-H-convexity is related to the Mergelyan property. Specifically, there exists a $k_0(s,n) > 0$, such that $\mathcal{O}(\overline{\Omega})$ is dense in $\mathcal{C}^k(\overline{\Omega}) \cap \mathcal{O}(\Omega)$, whenever k is an integer $\geq k_0(s,n)$ and $\Omega \subseteq \mathbb{C}^n$ is a bounded pseudoconvex domain, satisfying suitable assumptions, whose closure is s-H-convex.

Given these $\overline{\partial}$-results and the connection to the Mergelyan property, it becomes desirable to identify sets which are s-H-convex for some $s \geq 1$. Specifically, given a bounded (pseudoconvex) domain in $\mathbb{C}^n$ whose closure admits a Stein neighborhood basis, one can ask under which additional assumptions said closure is necessarily s-H-convex for some $s \geq 1$.

To our knowledge, it is unknown whether there exists a bounded (pseudoconvex) domain Ω in $\mathbb{C}^2$ with boundary of class $\mathcal{C}^2$ (or $\mathcal{C}^\infty$), such that $\overline{\Omega}$ has a Stein neighborhood basis, but is *not* 1-H-convex. In this paper we show that, if the smoothness assumption on the boundary is relaxed appropriately, there exists a bounded domain whose closure admits a Stein neighborhood basis, but is not s-H-convex for *any* $s \geq 1$. This is achieved by modifying the construction of the classical Diederich-Fornæss worm domain. A precise statement of the main result of this paper goes as follows:

Theorem. *There exists a bounded (pseudoconvex) domain $\Omega \neq \emptyset$ in $\mathbb{C}^2$ with boundary of class $\mathcal{C}^{1,1}$, such that:*

- *$\overline{\Omega}$ has a Stein neighborhood basis,*
- *$\overline{\Omega}$ is* not *s-H-convex for any real number $s \geq 1$.*

We give here an *informal* explanation of the intuition behind our constructions:

A classical worm domain admits a Stein neighborhood basis if the duration of the rotation at maximal radius is less than π. If the duration is exactly π this fails to be true, as can be seen by refining the classical argument by K. Diederich and J. E. Fornæss. In the case of the domain Ω defined above, we prevent this argument from working by drastically increasing the speed of the round-off, which leads to the boundary regularity dropping to $\mathcal{C}^{1,1}$. Using the fact that the function g vanishes to infinite order in $0 \in \mathbb{R}$, one can apply the Kontinuitätssatz for annuli to open pseudoconvex neighborhoods of the closure of Ω to show that $\overline{\Omega}$ is not s-H-convex for any $s \geq 1$.

It is easy to construct a neighborhood basis for $\overline{\Omega}$ (*not* a Stein one) by taking appropriate worm domains and increasing the radii of the rotating discs without changing the centers. This increase of the radii of course destroys pseudoconvexity. We counteract this by "chopping off" the "bad part", which is done by intersecting with a domain of half planes rotating around 0 in the w-plane. This, however, leads to these sets not being neighborhoods anymore, as can be seen by considering 0 in the w-plane. We finally resolve this issue by moving the center of the rotation from 0 slightly in the direction of $-i$ and slightly slowing down the rotation (symmetrically around the angle $\pi/2$), which intuitively speaking amounts

to introducing a small tilt. In the w-plane, $-i$ represents the "out direction" of Ω, which exists because the duration of the rotation at maximal radius does not exceed π. Since g is positive on $\mathbb{R}_{>0}$, one actually leaves the *closure* of Ω, when going from 0 slightly in the direction of $-i$ in the w-plane, which is of course crucial for our construction to work. Since the purpose of the domain of rotating half planes is to help with the pseudoconvexity of the neighborhoods we are constructing, we have to apply these changes to both of the domains we are intersecting.

Chern forms of hermitian metrics with analytic singularities

Elizabeth Wulcan

(joint work with Richard Lärkäng, Hossein Raufi and Martin Sera)

I will discuss a joint work [8] with Richard Lärkäng, Hossein Raufi and Martin Sera. The overall goal of this project is to define Chern forms or rather currents for singular hermitian metrics on holomorphic vector bundles. In this talk I will describe how to do this for metrics with so-called analytic singularities. The general strategy is to define Chern forms through so-called Segre forms, following ideas from classical intersection theory. The Segre forms are defined as the push-forwards of certain Monge-Ampère type products introduced by Andersson and me. Our work is inspired by the previous work [7] by my coauthors and Jean Ruppenthal, where they define Segre forms for singular metrics in a different way, using regularisations.

Let E be a holomorphic vector bundle of rank r over a manifold X of dimension n and let h be a smooth hermitian metric on E. Then the associated *Chern forms* $c_k(E,h)$ are by definition the coefficents in the characteristic polynomial of the curvature of h. More precisely if $\Theta(E,h)$ is the curvature of h, then

$$\sum c_k(E,h)t^k := \det\big(\mathrm{Id} + \frac{i}{2\pi i}t\Theta(E,h)\big).$$

The de Rham cohomology classes, the *Chern classes* of, $c_k(E)$ are topological invariants of E and in particular independent of the metric h. Alternatively Chern forms can be defined in the following way: Let $\pi : \mathbf{P}(E) \to X$ be the projective bundle of lines in E^*, and let $e^{-\varphi}$ be the metric induced on the dual $\mathcal{O}_{\mathbf{P}(E)}(1)$ of the tautological line bundle $\mathcal{O}_{\mathbf{P}(E)}(-1) \subset \pi^*E^*$. Then following ideas from intersection theory one can define the associated *kth Segre form* as

$$s_k(E,h) := (-1)^k \pi_*(dd^c\varphi)^{k+r-1}, \tag{1}$$

where $d^c = (i/4\pi)(\partial - \bar\partial)$. Then the total Segre form $s(E,h) = 1 + s_1(E,h) + s_2(E,h) + \cdots$ is the multiplicative inverse of the total Chern form $c(E,h) = 1 + c_1(E,h) + c_2(E,h) + \cdots$, i.e.,

$$c(E,h) \wedge s(E,h) = 1. \tag{2}$$

On cohomology level this identity is a classical intersection theory result whereas on form level this was first proved in [9]. Note that from (2) one can recover $c(E,h)$ from $s(E,h)$ and vice versa.

Next, let h be a singular hermitian metric on a vector bundle E in the sense of Berndtsson-Păun, [5]; then the associated metric $e^{-\varphi}$ on $\mathcal{O}_{\mathbf{P}(E)}(1)$ is singular.

Example. Let $L \to X$ be a line bundle with a global holomorphic section locally defined by holomorphic functions f. Then $e^{-\log|f|^2}$ is a singular metric on L that is smooth outside $\{f = 0\}$. In this case the only non-trivial Chern form is a well-defined positive current

$$c_1(L, e^{-\log|f|^2}) = dd^c \log|f|^2 = [f = 0],$$

where $[f = 0]$ denotes the current of integration along $\{f = 0\}$ (with multiplicities) and the last equality is the classical Poincaré-Lelong formula.

Our goal is to define Chern forms of higher rank vector bundles. However, if h is singular the curvature $\Theta(E, h)$ is a current in general as is the first Chern form $dd^c\varphi$ of the associated line bundle on $\mathcal{O}_{\mathbf{P}(E)}(1)$. Therefore expressions like $\det\big(\mathrm{Id} + \frac{i}{2\pi i} t\Theta(E, h)\big)$ and $(dd^c\varphi)^m$ do not make sense, since one cannot multiply currents in general.

In the example above, however, φ is nice in a certain way. It is plurisubharmonic (psh) and moreover it has analytic singularities, which means that locally it is of the form

$$\varphi = c \log|F|^2 + v, \tag{3}$$

where $c > 0$, F is a tuple of holomorphic functions f_j, $|F|^2 = \sum |f_j|^2$, and v is bounded. In [6] Hosono generalized the metric in the example above to vector bundles of higher rank; also in this case the associated line bundle metric φ is psh with analytic singularities. That φ is psh is equivalent to that h is positive in a certain sense, namely Griffiths positive. We say that h has analytic singularities if φ has.

Given a psh function φ with analytic singularities, together with Andersson [4], building on ideas from [1], we defined generalized Monge-Ampère products $(dd^c\varphi)^m$ recursively as

$$(dd^c\varphi)^k := dd^c(\varphi \mathbf{1}_{X\setminus Z}(dd^c\varphi)^{k-1}), \tag{4}$$

where Z is the unbounded locus of φ, i.e., locally defined as $\{F = 0\}$ where φ is given by (3). The current $(dd^c\varphi)^m$ is positive and closed and of bidegree (m, m), and for $m \le \operatorname{codim} Z$, it coincides with Bedford-Taylor-Demailly's classically defined $(dd^c u)^m$.

Now assume that h is a Griffiths positive metric with analytic singularities and let θ be the first Chern form of a smooth metric on $\mathcal{O}_{\mathbf{P}(E)}(1)$. Since the difference of two local weights φ is of the form $\log|f|^2$, where f is a nonvanishing holomorphic function, $(dd^c\varphi)^m$ is a globally defined current on $\mathbf{P}(E)$. Inspired by [2, Theorem 1.2], in order to keep track of what is lost in the recursive definition (4) we introduce

$$[dd^c\varphi]^m_\theta := (dd^c\varphi)^m + \sum_{\ell=0}^{m-1} \theta^{m-\ell} \mathbf{1}_Z (dd^c\varphi)^\ell,$$

and, inspired by (1), we define

$$s_k(E,h,\theta) := (-1)^k \pi_* [dd^c\varphi]_\theta^{k+r-1}.$$

If the φ are smooth, then clearly $s_k(E,h,\theta)$ coincides with $s_k(E,h)$ defined by (1). By extending ideas in [4] and [3] we give meaning to mixed Monge-Ampère products

$$[dd^c\varphi_t]_{\theta_t}^{m_t} \wedge \cdots \wedge [dd^c\varphi_1]_{\theta_1}^{m_1}$$

and we use these to define currents $s_{k_t}(E,h,\theta) \wedge \cdots \wedge s_{k_1}(E,h,\theta)$ and through (2) currents $c_k(E,h,\theta)$. Now these currents have nice properties. They are closed normal (k,k)-currents; more precisely they are locally differences of closed positive currents so that in particular they have well-defined Lelong numbers. Moreover

(1) $c_k(E,h,\theta)$ and $s_k(E,h,\theta)$ represent the kth Chern and Segre classes $c_k(E)$ and $s_k(E)$ of E, respectively,
(2) $c_k(E,h,\theta)$ and $s_k(E,h,\theta)$ coincide with the Chern and Segre forms $c_k(E,h)$ and $s_k(E,h)$, respectively, where h is smooth,
(3) $c_k(E,h,\theta)$ and $s_k(E,h,\theta)$ coincide with the Chern and Segre currents introduced in [7] when these are defined,
(4) the Lelong numbers of $c_k(E,h,\theta)$ and $s_k(E,h,\theta)$ at each $x \in X$ are independent of θ.

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Entropy of transcendental entire functions

Han Peters
(joint work with Anna Miriam Benini, John Erik Fornæss)

In this work with Anna Miriam Benini and John Erik Fornæss we prove that a transcendental entire function has infinite topological entropy. In the presentation I will focus on the special and simpler case where the function omits some value $\beta \in \mathbb{C}$.

Our main theorem is the following.

Theorem 1. *Let f be a transcendental entire function with an omitted value β, and let $N \in \mathbb{N}$. For $R > 0$ define the annulus*

$$A_R := \{R/2 < |z - \beta| < 2R\}.$$

Then there exists $\delta > 0$ and $R = R(n)$ such that $f(A_R) \supset A_R$ and every point in A_R has at least N preimages in A_R which are at Euclidean distance at least δ from each other.

Theorem 1 has the following corollary:

Theorem 2. *Let f be a transcendental entire function with an omitted value. Then f has infinite topological entropy.*

The fact that entire transcendental functions should have infinite entropy is no surprise. Indeed, it has been know for decades that rational maps of degree d acting on the Riemann sphere have topological entropy equal to $\log d$. The lower bound was shown by Misiurewicz and Przytycki, and the upper bound by Gromov and independently by Lyubich.

One of the reasons why the problem of topological entropy for entire transcendental maps has not been addressed for so long is that there are several non equivalent definitions of topological entropy on non-compact metric spaces. Observe that transcendental maps are not uniformly continuous on $\mathbb{C}$ and that they do not extend continuously to its one-point compactification: the Riemann sphere $\hat{\mathbb{C}}$. The definition of topological entropy that we will use is the following.

Definition 3 (Definition of topological entropy)**.** Let $f : Y \to Y$ be a self-map of a metric space (Y, d). Let X be a compact subset of Y. Let $n \in \mathbb{N}$ and $\delta > 0$. A set $E \subset X$ is called (n, δ)-*separated* if

- for any $z \in E$, its orbit $\{z, f(z), \dots, f^{n-1}(z)\} \subset X$;
- for any $z \neq w \in E$ there exists $k \leq n - 1$ such that $d(f^k(z), f^k(w)) > \delta$.

Let $K(n, \delta)$ be the maximal cardinality of an (n, δ)-separated set. Then the *topological entropy* $h_{\text{top}}(X, f)$ is defined as

$$h_{\text{top}}(X, f) := \sup_{\delta > 0} \left\{ \limsup_{n \to \infty} \frac{1}{n} \log K(n, \delta) \right\}.$$

We define the topological entropy $h_{\text{top}}(f)$ of f on Y as the supremum of $h_{\text{top}}(X, f)$ over all compact subsets $X \subset Y$.

In general this definition depends on the metric d. In our setting, the natural metrics on $\mathbb{C}$ with respect to the dynamics of transcendental entire functions are the spherical metric and the Euclidean metric. Since they are comparable on compact subsets of $\mathbb{C}$, both choices yield the same result with respect to Definition 3.

The proof of Theorem 1 is short and elementary, and relies on the following well known ingredients:

- By Picard's Theorem, every value apart from the omitted value is obtained infinitely often.
- The maximal value on a circle of radius R grows faster than any polynomial in R.
- The twice punctured plane is hyperbolic, and equipped with the Poincaré metric gives a complex metric space.

Complex Monge-Ampère equations with prescribed singularity

Chinh H. Lu

Let (X, ω) be a compact Kähler manifold of dimension n and θ be a closed smooth real $(1,1)$-form on X representing a big cohomology class. A function $u : X \to \mathbb{R} \cup \{-\infty\}$ is quasi plurisubharmonic (quasi-psh for short) if it is locally the sum of a smooth and a plurisubharmonic function. We let $PSH(X, \theta)$ denote the set of all quasi-psh functions $u \not\equiv -\infty$ such that $\theta_u := \theta + dd^c u \geq 0$. The De Rham cohomology class $\{\theta\}$ is big if $PSH(X, \theta - \varepsilon\omega)$ is non empty for some $\varepsilon > 0$.

Given two θ-psh functions u, v we say that u is more singular than v (and we write $u \prec v$) if $u \leq v + C$ on X, where C is a constant (possibly dependent on u, v). We say that u has the same singularity as v (and we write $u \simeq v$) if $u \prec v$ and $v \prec u$. If $\phi \in PSH(X, \theta)$ is less singular than all $u \in PSH(X, \theta)$ then we say that ϕ has minimal singularity. An example of such potentials is

$$V_\theta(x) := \sup\{u(x) \mid u \in PSH(X, \theta),\ u \leq 0\}.$$

It follows from Demailly's approximation theorem that there is a Zariski open set Ω, called the ample locus of θ in which V_θ is locally bounded. Given a θ-psh function u the sequence of positive measures

$$\mathbf{1}_{\{u > V_\theta - k\}} (\theta + dd^c \max(u, V_\theta - k))^n$$

is increasing and defines the non-pluripolar Monge-Ampère measure of u, that we denote by $(\theta + dd^c u)^n$. By definition $\int_X (\theta + dd^c u)^n$ takes values in $[0, V]$ where $V = \int_X (\theta + dd^c V_\theta)^n$. We let $\mathcal{E}(X, \theta)$ denote the set of all θ-psh functions such that $\int_X (\theta + dd^c u)^n = V$. These are called θ-psh functions with full Monge-Ampère mass.

There is a satisfactory (global) pluripotential theory developed for this class of potentials by [1, 2] (see also [5]). The analogous theory for potentials with non-full mass, i.e. for θ-psh functions u with $\int_X (\theta + dd^c u)^n < V$, was recently developed in [3, 4], stemming from the following monotonicity property of total mass.

Theorem 1. *Assume $\theta_1, ..., \theta_n$ are big classes and $u_j \leq v_j$ are θ_j-psh functions, $j = 1, ..., n$. Then*

$$\int_X (\theta_1 + dd^c u_1) \wedge ... \wedge (\theta_n + dd^c u_n) \leq \int_X (\theta_1 + dd^c v_1) \wedge ... \wedge (\theta_n + dd^c v_n).$$

This result, conjectured to be true in [2], was proved by Witt Nyström for the case when $\theta_1 = ... = \theta_n$ and $u_1 = ... = u_n$. The general case was established in [3].

Theorem 1 opens the door for a (global) pluripotential theory in relative full mass classes. Several pluripotential tools such as comparison principle, convergence of non-pluripolar measures, stability of sub/super solutions were established in [3, 4], leading to the following resolution of complex Monge-Ampère equations with prescribed singularity:

Theorem 2. *Assume $\phi = P[\phi]$, $0 \leq f \in L^p(X), p > 1$ with $\int_X (\theta + dd^c \phi)^n = \int_X f dV > 0$. Then there exists a unique $u \in PSH(X, \theta)$ such that $\sup_X u = 0$, $u \simeq \phi$ and $(\theta + dd^c u)^n = f dV$.*

The envelope $P[\phi]$, introduced by Ross and Witt Nyström [6] is defined as

$$P[\phi] := \left(\lim_{t \to +\infty} P_\theta(\min(\phi + t, 0)) \right)^*.$$

An important consequence of Theorem 2 is the following result confirming a conjecture of Boucksom-Eyssidieux-Guedj-Zeriahi [2] :

Theorem 3. *The volume function $T \mapsto \log \mathrm{Vol}(T)$ is concave on the convex set of closed positive $(1,1)$-currents.*

Here given T a positive closed current of type $(1,1)$ on X we can find $\theta \in \{T\}$, which is a smooth closed real $(1,1)$-form on X and $u \in PSH(X, \theta)$ such that $T = \theta + dd^c u$. If $\{\theta\}$ is big then $\mathrm{Vol}(T)$ is defined to be $\int_X (\theta + dd^c u)^n$, otherwise $\mathrm{Vol}(T) = 0$.

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Applications of the L^2 extension theorem to analytic families

Takeo Ohsawa

Geometric invariants of complex manifolds are encoded in the L^2-space of holomorphic sections of vector bundles. They are compressed in the Bergman kernel as the works of Kodaira, Hörmander and Fefferman have shown, so that relations between analysis and geometry on complex manifolds are suggested in the results on the Bergman kernels. Given an analytic family of complex manifolds, say $\pi : M \to T$, the parameter dependence of the Bergman kernel $K_t = K_{M_t}$ of $M_t = \pi^{-1}(t)$ reflects how the complex structure of M_t deforms. It is known by Berndtsson that $\log K_t$ depends plurisubharmonically in t if M is weakly 1-complete ($=C^\infty$ plurisubharmonic) and Kählerian. An immediate consequence of Berndtsson's theorem is that such a family is locally analytically trivial if $\log K_t \in C^\infty$ and $\partial\bar\partial \log K_t$ annihilates a horizontal distribution (a subbundle of $T^{1,0}_M$ which bijects to $T^{1,0}_T$ by π). This generalizes a result of Maitani and Yamaguchi for Stein families of Jordan domains. Roughly speaking, the Bergman kernel detects the rigidity of analytic families. On the other hand, it was proved by Nishino that a Stein submersion over the unit disc is trivial if the fibers are $\mathbb{C}$. Although this rigidity does not follow directly from $K_{\mathbb{C}} \equiv 0$, it turned out that an L^2 extension theorem is available to give its alternate proof (cf. [1]).

Indeed, Nishino's theorem is an immediate consequence of the following L^2 extension theorem in [3], up to simple or well-known assertions on holomorphic functions.

Theorem 1. *Let M be a Stein manifold of dimension n, let φ be a plurisubharmonic function on M, let s be a holomorphic function on M and let $X = s^{-1}(0)$. Assumme that $X_0 := X \cap \{ds \neq 0\}$ is a dense subset of X. Then, for any holomorphic $(n-1)$-form on X_0 satisfying*

$$\left| \int_{X_0} e^{-\varphi} f \wedge \overline{f} \right| < \infty,$$

one can find a holomorphic n-form F on M such that $F = f \wedge ds$ on X_0 and

$$\left| \int_M e^{-\varphi} (1 + |s|^2)^{-2} F \wedge \overline{F} \right| \leq C_0 \int_{X_0} e^{-\varphi} f \wedge \overline{f},$$

where $C_0 \leq 1620\pi$.

Besides the rigidity, a variant of the L^2 extension theorem can be applied to extend the results of Takegoshi and Takayama on the Levi problem on complex manifolds (cf. [2]).

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HHR/USQ domains and squeezing constants for bounded symmetric domains

Kang-Tae Kim
(joint work with Cho-Ho Chu, Sejun Kim)

1. Basic notions

Let Ω be a complex manifold of dimension n with a 1-1 holomorphic mapping $f\colon \Omega \to B^n(0,1)$ into the unit open ball $B^n(0,1)$ in $\mathbb{C}^n$. Then Ω is biholomorphic to a subdomain of $B^n(0,1)$, as a consequence. So we are naturally lead to focus upon the case of bounded domains.

For $p \in \Omega$, let $\mathcal{F}_p(\Omega) := \{f\colon \Omega \to B^n(0,1) \mid f(p) = 0 \ \&\ \text{1-1 holomorphic}\}$. Then for each $f \in \mathcal{F}_p(\Omega)$, let

$$\sigma_f(p) := \sup\{r > 0\colon B^n(0,r) \subset f(\Omega)\},$$

and let

$$\sigma_\Omega(p) := \sup\{\sigma_f(p)\colon f \in \mathcal{F}_p(\Omega)\},$$

which we call the *squeezing function* of Ω. (Of course, $B^n(p,r) = \{z\colon \|z-p\| < r\}$.) Moreover, Ω is called *holomorphic homogeneous regular*, or *uniformly squeezing*, if the squeezing function admits a positive lower bound [6, 7, 9].

2. Infinite dimensions

This consideration can be extended to the infinite dimensional complex Banach spaces. Let V be a Banach space. Then a domain Ω in V is with a holomorphic embedding $\psi\colon \Omega \to B_H(0,1)$ into the unit open ball in a Hilbert space H, with $\psi(p) = 0$ is associated with the *squeezing function* $\sigma_\Omega\colon \Omega \to \mathbb{R}$ defined much the same way as follows: let $\mathcal{F}_p(\Omega, H) := \{f\colon \Omega \to H \mid \text{holomorphic embedding with } f(p) = 0 \ \&\ f(\Omega) \subset B_H(0,1)\}$. Then put

$$\sigma_f(p) = \sup\{r > 0\colon B_H(0;r) \subset f(\Omega)\}$$

for each $f \in \mathcal{F}_p(\Omega, H)$. Moreover,

$$\sigma_\Omega(p) := \sup_{f \in \mathcal{F}_p(\Omega,H)} \sigma_f(p).$$

As in the finite dimensional case, Ω is called *holomorphic homogeneous regular* (HHR) if the sqeezing function has a positive lower bound.

Proposition 1 ([3])**.** *The concept of squeezing function is independent of choice of the Hilbert space H.*

Theorem 2 ([3])**.** *A domain in V is HHR if, and only if, V is of type 2 and cotype 2, or equivalently it is linearly homeomorphic to a Hilbert space. In fact, any uniformly elliptic domains in Hilbert spaces are HHR.*

For the concept of type and cotype for Banach spaces, we refer to [8].

On the other hand, unlike finite dimensions, the following question is yet to be answered:

Question 5. *Is a convex bounded domain in an infinite dimensional Hilbert space HHR?*

In $\mathbb{C}^n$ the answer is known to be affirmative [4]. But that method does not seem to generalize directly.

3. Squeezing functions for bounded symmetric domains

First we discuss the finite dimensional bounded symmetric domains.

According to [2], the complete list of bounded symmetric domains divides into 6 types: 4 classical types and 2 exceptional domains which are called type V and type VI. Since these domains are homogeneous, the squeezing function is a constant function. The value of this constants were sought after.

Kubota [5] discovered these constants, in fact, even before the squeezing function was extensively studied following H. Alexander's analysis on the polydiscs [1].

We have discovered the following:

Proposition 3 ([3])**.** *The squeezing constnat of the exceptional domain D_{27} of type V is $1/\sqrt{3}$. Moreover the squeezing constant of the bounded symmetric domein D_{16} of type VI is $1/\sqrt{2}$. In fact, the squeezing for every irreducible BSD's of finite dimensions is $1/\sqrt{rank}$, regardless classical or exceptional.*

Regardless of dimensions we have the following:

Theorem 4 ([3])**.** *Let D be a bounded symmetric domain in a complex Banach space. Then D is HHR if, and only if, it is of finite rank. In this case, D is biholomorphic to a finite product $D_1 \times \cdots \times D_k$ of irreducible bounded symmetric domains and it follows that*

$$\sigma_D = \left(\frac{1}{\sigma_{D_1}^2} + \cdots + \frac{1}{\sigma_{D_k}^2} \right)^{-\frac{1}{2}}.$$

If $\dim V$ is finite, then each D_j is one of the domains of six types by Cartan. If $\dim V = \infty$, then each D_j is either a Lie ball or a Type I domain of finite rank. For the Lie ball the squeezing constant is $1/\sqrt{2}$. For the Type I domain, it is again $1/\sqrt{rank}$.

Then we pose, finally:

Question 6. *Can it be possible to compute squeezing function value (constant) for the bounded homogeneous domains?*

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Plurisubharmonically separable complex manifolds

Evgeny A. Poletsky
(joint work with Nikolay Shcherbina)

Let M be a complex manifold and $PSH^{cb}(M)$ be the space of bounded continuous plurisubharmonic functions on M. In this paper we study obstructions to separation of points in a complex manifold M by functions from $PSH^{cb}(M)$.

There are complex manifolds, for example compact manifolds, where all plurisubharmonic functions are constants. There are parabolic manifolds, for example $\mathbb{C}^n$, where all bounded plurisubharmonic functions are constants. For their characterization see [6] and [1]. But there are plenty of complex manifolds like $\mathbb{D} \times \mathbb{C}$, $\mathbb{D}$ is the unit disk, where the space $PSH^{cb}(M)$ is large but, nevertheless, does not separate points.

The first main results of our paper can be summarized in the following theorem.

Theorem 1. *For a complex manifold M the following statements are equivalent:*

(1) *the functions from $PSH^{cb}(M)$ separate points of M;*
(2) *for every point $w_0 \in M$ there is a function $u \in PSH^{cb}(M)$ that is smooth and strictly plurisubharmonic near w_0 ;*
(3) *for every point $w_0 \in M$ there are a negative continuous plurisubharmonic function v on M and constants C_1 and C_2 such that $\log|z - w_0| + C_1 < v(z) < \log|z - w_0| + C_2$ near w_0.*

The main obstruction to separation is the set $\mathbf{c}(M)$ of all points $w \in M$, where every function of $PSH^{cb}(M)$ fails to be smooth and strictly plurisubharmonic near w. It was first introduced and systematically studied by Harz–Shcherbina–Tomassini in [2]–[4] and was called the *core* of M. Observe that directly from the definition one concludes that $\mathbf{c}(M)$ is a closed subset of M. Among the main properties of the core established in these papers we mention here the following result that will be one of the important technical tools in the present paper.

Theorem 2 (see [2, Theorem 3.2]). *Let M be a complex manifold. Then the set $\mathbf{c}(M)$ is 1-pseudoconcave in the sense of Rothstein. In particular, $\mathbf{c}(M)$ is pseudoconcave in M if $\dim_{\mathbb{C}} M = 2$.*

Our next main result is the following theorem that was proved in [3] when the dimension of M is two.

Theorem 3. *Let M be a complex manifold. Then the set $\mathbf{c}(M)$ is the disjoint union of the sets E_j, $j \in J$, that are 1-pseudoconcave in the sense of Rothstein and have the following Liouville property: every function from $PSH^{cb}(M)$ is constant on each of E_j.*

A closed set $E \subset M$ is called *1-pseudoconcave in the sense of Rothstein* if for any $z_0 \in E$ and for any strictly plurisubharmonic function ρ defined on a neighborhood V of z_0 at any neighborhood $U \subset\subset V$ containing z_0 there is a point $z \in E \cap U$ where $\rho(z) > \rho(z_0)$.

By their definition these sets are perfect, i.e., have no isolated points. But it may happen that such a set is compact. For example, take the unit ball B in $\mathbb{C}^2$ and blow-up a complex projective line E at the origin. We get a complex manifold M. Clearly, E is a set that is 1-pseudoconcave in the sense of Rothstein. But if M is Stein, then any connected component X of E is non-compact.

Observe that if $E \subset M$ is a set that is 1-pseudoconcave in the sense of Rothstein in M which has the property that each bounded above continuous plurisubharmonic function φ on M is constant on E, then $E \subset \mathbf{c}(M)$ (for details see Lemma 3.1 in [3]). Hence, mentioned above Theorem 3 clarify the phenomenon of both, existence and the structure of the core.

Note that in this paper we are dealing mainly with the core defined using continuous plurisubharmonic on M functions (the core $\mathbf{c}^0(M)$ in terminology of [4]), while the main object of the study in [2]-[4] was the core $\mathbf{c}(M)$ defined using smooth plurisubharmonic on M functions. Note also, that another proof of Theorem 3 for cores defined by smooth plurisubharmonic functions was obtained by Slodkowski [8] using essentially different methods. His proof also covers the case of *minimal kernels* which are defined and studied in [9].

Let w_0 be a point in M. A point $A^{cb}(w_0)$ if $u(z) \leq u(w_0)$ for any $u \in PSH^{cb}(M)$.

Let us list some easily derived properties of the sets $A^b(w_0)$ and $A^{cb}(w_0)$.

Proposition 4. (1) *The set $A^{cb}(w_0)$ is closed.*
(2) *If $z_0 \in A^{cb}(w_0)$, then $A^{cb}(z_0) \subset A^{cb}(w_0)$.*
(3) *A point $w_0 \notin \mathbf{c}(M)$ if and only if $A^{cb}(w_0) = \{w_0\}$.*

In the theorem 5 below we prove the major property of the sets $A^{cb}(w_0)$.

Theorem 5. *If $A^{cb}(w_0) \neq \{w_0\}$, then the set $A^{cb}(w_0)$ is 1-pseudoconcave in the sense of Rothstein.*

Following the terminology of [7] we say that a closed set X in a complex manifold M has the *local maximum property* if X is perfect (i.e. has no isolated points) and

for any $z_0 \in X$ there is an open neighborhood V of z_0 in M with compact closure such that if an open set $U \subset\subset V$ contains z_0 and the set $L = X \cap \partial U$ is non-empty, then

$$\sup_{X \cap \overline{U}} u \leq \sup_{L} u$$

for any plurisubharmonic function u on V.

Theorem 6. *If M is a complex manifold, then any closed set $X \subset M$ has the local maximum property if and only if it is 1-pseudoconcave in the sense of Rothstein.*

Let $A_e^{cb}(w_0)$ be the set of $w \in M$ such that $u(w) = u(w_0)$ for any $u \in PSH^{cb}(M)$.

Theorem 7. *If M is a complex manifold and $w_0 \in \mathbf{c}(M)$, then the set $A_e^{cb}(w_0)$ has the local maximum property.*

As a corollary to the latter result we obtain the following theorem.

Theorem 8. *Let M be a complex manifold with non-empty core $\mathbf{c}(M)$. Then:*

1. *for every $w_0 \in M$ the set $A_e^{cb}(w_0) \neq \{w_0\}$ if and only if $w_0 \in \mathbf{c}(M)$;*
2. *if $w_0 \in \mathbf{c}(M)$, then the set $A_e^{cb}(w_0)$ is a is 1-pseudoconcave in the sense of Rothstein, lies in $\mathbf{c}(M)$ and all functions $u \in PSH^{cb}(M)$ are constants on $A_e^{cb}(w_0)$;*
3. *the core $\mathbf{c}(M)$ of M can be decomposed into the disjoint union of closed sets E_j, $j \in J$, that are 1-pseudoconcave in the sense of Rothstein and have the following Liouville property: Every function $\varphi \in PSH^{cb}(M)$ is constant on each of the sets E_j.*

If $A_e^{cb}(w_0) \cap A_e^{cb}(w_1) \neq \emptyset$, then $A_e^{cb}(w_0) = A_e^{cb}(w_1)$. Hence $\mathbf{c}(M)$ is the disjoint union of the sets $E_j = A_e^{cb}(w)$.

If the manifold M is not Stein, it may happen that the sets $A^{cb}(w_0)$ and $A_e^{cb}(w_0)$ are compact. For example, take the unit ball B in $\mathbb{C}^2$ and blow-up a complex projective line X at the origin. We get a complex manifold M and a holomorphic mapping F of M onto B such that $F(X) = \{0\}$. If u is a plurisubharmonic function on B, then the function $v = u \circ F$ is plurisubharmonic on M while any plurisubharmonic function on M is constant on X. Hence $\mathbf{c}(M) = X$ and $A^{cb}(w_0) = A_e^{cb}(w_0) = X$ for $w_0 \in X$.

The following theorem shows that such phenomenon can exist only on non-Stein manifolds.

Theorem 9. *Let M be a complex manifold and let $X \neq \{w_0\}$ be the closed connected component of $A^{cb}(w_0)$ containing w_0. If X is compact, then $A^{cb}(w_0) = X$ and M is not Stein.*

Open problems

1. Is $A^{cb}(w_0) = A_e^{cb}(w_0)$?
2. Does $A^{cb}(w_0) \subset \mathbf{c}(M)$?
3. Is there a non-negative $u \in PSH^{cb}(M)$ such that the set $\{u = 0\} = \mathbf{c}(M)$?

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Residue currents and cycles of complexes of vector bundles

Richard Lärkäng
(joint work with Elizabeth Wulcan)

The classical *Poincaré-Lelong formula* states that

$$\bar\partial\partial \log|f|^2 = 2\pi i[f=0], \tag{1}$$

where $[f=0]$ is the intergration current along $\{f=0\}$, counted with appropriate multiplicities. Formally, the left-hand side of (1) equals $\bar\partial(1/f)\wedge df$, and indeed, if one defines $\bar\partial(1/f) := \lim_{\epsilon\to 0}\bar\partial\chi(|f|^2/\epsilon)/f$, then

$$\bar\partial\frac{1}{f}\wedge df = 2\pi i[f=0]. \tag{2}$$

If $f=(f_1,\dots,f_p)$ is a tuple of holomorphic functions such that $\operatorname{codim}\{f=0\}=p$, i.e., such that f is a complete intersection, then (2) can be generalized to the following formula,

$$\bar\partial\frac{1}{f_p}\wedge\cdots\wedge\bar\partial\frac{1}{f_1}\wedge df_1\wedge\cdots\wedge df_p = 2\pi i[f_1=\cdots=f_p=0], \tag{3}$$

where $\bar\partial(1/f_p)\wedge\cdots\wedge\bar\partial(1/f_1)$ is the so-called Coleff-Herrera product, [2].

For a coherent analytic sheaf $\mathcal{F}$, the *cycle of* $\mathcal{F}$ is

$$[\mathcal{F}] = \sum m_i[Z_i],$$

where Z_i runs over the irreducible components of $\operatorname{supp}\mathcal{F}$, $[Z_i]$ is the integration current along Z_i, and m_i is the geometric multiplicity of Z_i in $\mathcal{F}$. For generic $z\in Z_i$, $\mathcal{F}$ can locally be given the structure of a free $\mathcal{O}_{Z_i}$-module of constant rank, and m_i is this rank.

Consider now a complex

$$0 \to E_N \xrightarrow{\varphi_N} E_{N-1} \to \cdots \to E_1 \xrightarrow{\varphi_1} E_0 \to 0 \tag{4}$$

of vector bundles on X. We let the *cycle* of (E,φ) be the current

$$[E] := \sum (-1)^\ell [\mathcal{H}_\ell(E)],$$

where $\mathcal{H}_\ell(E)$ is the homology group of (E,φ) at level ℓ. If (E,φ) is a locally free resolution of a coherent sheaf $\mathcal{F}$, then $[E] = [\mathcal{F}]$.

Given a generically exact complex (E,φ), with $E_0,\dots,E_N$ equipped with hermitian metrics, in [1] Andersson and Wulcan introduced an associated $\operatorname{End} E$-valued residue current R. More precisely, one has $R = \sum_{\ell<k} R^\ell_k$, where R^ℓ_k is a $\operatorname{Hom}(E_\ell, E_k)$-valued $(0,k-\ell)$-current. More concretely, if $\operatorname{rank} E_j = r_j$, then R^ℓ_k is in a local frame an $r_k \times r_\ell$-matrix of $(0,k-\ell)$-currents.

Our main result is the following variant of the Poincaré-Lelong formula.

Theorem 1. *Let (E,φ) be a complex of vector bundles (4), such that all the homology groups $\mathcal{H}_\ell(E)$ have pure codimension $p>0$ or vanish, and let D be the connection on $\operatorname{End} E$ induced by arbitrary connections on $E_0,\dots,E_N$. Assume $E_0,\dots,E_N$ are equipped with hermitian metrics, and let $R = \sum_{\ell<k} R^\ell_k$ be the associated current. Then*

$$\frac{1}{(2\pi i)^p p!} \sum_{\ell=0}^{N-p} (-1)^\ell \operatorname{tr} D\varphi_{\ell+1}\cdots D\varphi_{\ell+p} R^\ell_{\ell+p} = [E]. \tag{5}$$

Here, connections $D_0,\dots,D_N$ on $E_0,\dots,E_N$ induce a connection D on $\operatorname{End}(E)$, and in particular, $D\varphi_k = D_{k-1}\circ\varphi_k + \varphi_k \circ D_k$. Note that there is no relation required between the hermitian metrics on $E_0,\dots,E_N$ used to define R and the connections $D_0,\dots,D_N$ on $E_0,\dots,E_N$.

In case $H_k(E) = 0$ for $k>0$, then (E,φ) is a locally free resolution of $\mathcal{F} := H_0(E)$, and in this case, $R^\ell_{\ell+p} = 0$ for $\ell > 0$, so (5) becomes

$$\frac{1}{(2\pi i)^p p!} \operatorname{tr} D\varphi_1 \cdots D\varphi_p R^0_p = [\mathcal{F}]. \tag{6}$$

In the case $\mathcal{F} = \mathcal{O}_Z$, this is an earlier result of ours, [3], but we give a new, simpler proof of (6) than the proof in [3]. In the special case when $\mathcal{F} = \mathcal{O}_Z = \mathcal{O}_X/(f_1,\dots,f_p)$, where f defines a complete intersection, and one takes (E,φ) to be the Koszul complex of f, then

$$R^0_p = \bar\partial\frac{1}{f_p}\wedge\cdots\bar\partial\frac{1}{f_1} \text{ and } D\varphi_1\cdots D\varphi_p = p! df_1\wedge\cdots df_p,$$

so (6) in this case becomes the Poincaré-Lelong formula (3).

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Polynomially convex embeddings of compact real manifolds

Purvi Gupta

(joint work with Rasul Shafikov)

A compact set $X \subset \mathbb{C}^n$ is said to be polynomially convex if it coincides with its polynomial hull, given by $\widehat{X} = \{z \in \mathbb{C}^n : |p(z)| \leq \sup_X |p|, \text{ for all holomorphic polynomials } p\}$. Polynomial convexity is an important notion owing to the Oka-Weil theorem which states that holomorphic functions in a neighbourhood of a polynomially convex set M can be approximated uniformly on M by holomorphic polynomials. Although it is not a topological property, polynomial convexity imposes topological constraints on the underlying set. For instance, it is known that no m-dimensional compact manifold without boundary can be polynomially convex in $\mathbb{C}^m$. This raises the following question.

Question. *What is the least n so that all m-dimensional compact real manifolds admit a polynomially convex embedding into $\mathbb{C}^n$?*

It is known that all m-manifolds admit a polynomially convex *topological* embedding in $\mathbb{C}^{m+1}$, which is optimal (see [4]). In the context of smooth embeddings, this question is open. It is related to the question of the minimum number of smooth generators required to generate the algebra of continuous function on any m-dimensional compact manifold. If M is a nonmaximally totally real submanifold of $\mathbb{C}^n$, it can be deformed via a small perturbation into a polynomially convex one, as proved by Forstnerič-Rosay [6], Forstnerič [5], and Løw-Wold [7]. The condition that *any* abstract m-dimensional compact real manifold admits a totally real embedding into $\mathbb{C}^n$ is well understood: one must have $\lfloor \frac{3m}{2} \rfloor \leq n$. Thus, any m-dimensional compact manifold can be embedded as a totally real polynomially convex submanifold of $\mathbb{C}^n$ if $n \geq \lfloor \frac{3m}{2} \rfloor$ and $(m, n) \neq (1, 1)$. If $n < \lfloor \frac{3m}{2} \rfloor$, then certain m-dimensional compact manifolds necessarily acquire complex tangent directions when embedded into $\mathbb{C}^n$. The points where these occur are the CR-singularities of M. When $n = m$, certain CR-singularities prevent the manifold from being polynomially convex, and these cannot be avoided due to topological reasons. However, when $n > m$, it is not known whether the same phenomenon takes place.

We discuss a technique of constructing polynomially convex embeddings of manifolds in the case when CR-singularities will necessarily occur, thus allowing us to reduce the known bound for the optimal n sought in the question above. This

method involves replacing CR-singularities with suitable local models based on the Beloshapka-Coffman normal form (see [2] and [3]), and then perturbing M away from the new set of CR-singularities using a general result established in Arosio-Wold [1]. In particular, this method works when the CR-singularities are either isolated, or — under additional assumptions — one-dimensional. Similar techniques can also be applied to produce improved bounds for yet another question involving a minimum embedding dimension: what is the least n so that every m-dimensional real compact manifold can be embedded into $\mathbb{C}^n$ to have no analytic discs in its polynomial hull?

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Monge-Ampère operator for plurisubharmonic functions with analytic singularities on Hermitian manifolds

Zbigniew Błocki

A plurisubharmonic (psh) function u defined on a complex manifold X of dimension n is said to have *analytic singularities* if locally it can be written of the form

$$u = c \log |F| + v,$$

where $c \geq 0$ is a constant, $F = (f_1, \ldots, f_m)$ is a tuple of holomorphic functions not vanishing everywhere and v is bounded. By Z we will denote its singular set. If $m = 1$ then we say that u has *divisorial singularities*; then v has to be a bounded psh function.

The Monge-Ampère operator for such functions was defined by Andersson and Wulcan [2] as follows: for $k = 1, \ldots, n$ set inductively

$$(dd^c u)^k := dd^c \left(u \mathbf{1}_{X \setminus Z} (dd^c u)^{k-1} \right).$$

It was shown in [2] that $T_{k-1} := \mathbf{1}_{X \setminus Z} (dd^c u)^{k-1}$ extends across Z as a closed current on X and that uT_{k-1} has locally finite mass. If $u = c \log |f| + v$ has divisorial singularities then

$$(dd^c u)^k = dd^c u \wedge (dd^c v)^{k-1}.$$

In general, u does not belong to the domain of definition $\mathcal{D}$ defined in [3, 4] which is the maximal subclass of the class of plurisubharmonic functions where the complex Monge-Ampère operator can be defined in such a way that it is continuous (in the weak* topology) for decreasing sequences. This is because $\mathcal{D} \subset W^{1,2}_{loc}$ and usually functions with analytic singularities do not have gradient in L^2_{loc}.

This means that one cannot expect good continuity properties for arbitrary regularizations of u. In [1] it was shown however that this definition is continuous for certain special regularizations:

Theorem 1. *Let u be a negative psh function with analytic singularities and assume that χ_j is a sequence of bounded nondecreasing convex functions on $(-\infty, 0]$ such that $\chi_j(t)$ decreases to t as j increases to ∞. Then for $k = 1, \dots, n$*

$$(dd^c(\chi_j \circ u))^k \to (dd^c u)^k$$

weakly as $j \to \infty$.

This can be treated as an alternative definition of the Monge-Ampère operator.

If ω is a Kähler form on X and φ is an ω-psh function with analytic singularities then the Monge-Ampère operator $(\omega + dd^c\varphi)^k$ was defined locally in [1] as $(dd^c(g + \varphi))^k$ where g is a local potential for ω (i.e. $\omega = dd^c g$). This definition has two weaknesses however: first of all if X is compact then it may happen that

$$\int_X (\omega + dd^c\varphi)^n < \int_X \omega^n$$

(that is we loose some mass) and secondly it cannot be repeated if ω is only Hermitian.

We propose instead an alternative definition:

Theorem 2. *Let ω be a Hermitian form on X and φ a negative ω-psh function with analytic singularities. Then for $k = 1, \dots, n$ one can uniquely define a (k,k)-current $(\omega + dd^c\varphi)^k$ on X in such a way that if χ_j are as in Theorem 1 with $\chi_j' \le 1$ then*

$$(\omega + dd^c(\chi_j \circ \varphi))^k \to (\omega + dd^c\varphi)^k$$

weakly as $j \to \infty$.

Since we may for example take $\chi_j(t) = \max\{t, -j\}$, we immediately get the following:

Corollary 3. *If ω is Kähler and X is compact then*

$$\int_X (\omega + dd^c\varphi)^n = \int_X \omega^n.$$

The way to prove Theorem 2 is to write the Newton expansion

$$(\omega + dd^c\varphi)^k = \sum_{l=1}^{k} \binom{k}{l} (dd^c\varphi)^l \wedge \omega^{k-l}$$

and to generalize Theorem 1 to quasi-psh functions, that is functions that can be locally written as $\varphi = u + \psi$ where u is psh and ψ is smooth.

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The Second Main Theorem in the hyperbolic case.

Nessim Sibony

Most results in Nevanlinna's theory study the behavior of maps from $\mathbb{C}$ to a compact complex manifold, mostly projective. In a joint work with Mihai Păun, we question, the nature of the source space. We give versions of the classical theorems, Nevanlinna's second main Theorem, Bloch's Theorem, Ax Lindemann's Theorem, when the source is an open Parabolic Riemann surface (i.e all bounded subharmonic functions are constant)see [4].

With Min-Ru,[5], we consider the case when the source space is the unit disc $\mathbb{D}$. We have to assume that the map $f : \mathbb{D} \to (M, \omega)$ satisfy a certain growth condition expressed in terms of the Nevanlinna's characteristic of the map f. We require that

$$c_{f,\omega} := \inf \left\{ c > 0 \;\Big|\; \int_0^R \exp(cT_{f,\omega}(r))dr = \infty \right\},$$

is finite.

In [5] one shows that the Cartan, Ahlfors, Notchka Theorem can be extended to this context.

Theorem 1. *Let $H_1, \dots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. Let $f : \mathbb{D} \to \mathbb{P}^n(\mathbb{C})$ be a linearly non-degenerate holomorphic curve (i.e. its image is not contained in any proper subspace of $\mathbb{P}^n(\mathbb{C})$) with $c_f < \infty$, where $c_f = c_{f,\omega_{FS}}$ and $0 < R \leq \infty$. Then, for any $\epsilon > 0$, the inequality*

$$\sum_{j=1}^{q} m_f(r, H_j) + N_W(r, 0) \leq (n+1)T_f(r) + \frac{n(n+1)}{2}(1+\epsilon)(c_f+\epsilon)T_f(r)$$

$$+O(\log T_f(r)) + \frac{n(n+1)}{2}\epsilon \log r$$

holds for all $r \in (0, R)$ outside a set E with $\int_E \exp((c_f+\epsilon)T_f(r))dr < \infty$. Here W denotes the Wronskian of f.

It turns out that the condition is satisfied for generic leaves of foliations by Riemann surfaces in compact Kähler manifolds, [1],[3]. We give a simple example of the situation we have in mind. Let $\mathcal{F}$ be a foliation in $\mathbb{P}^2$, let L be a leave

with $\phi \to L$ the universal covering map. For any line $\Lambda \subset \mathbb{P}^2$, one can study the behavior of the family of measures

$$\frac{1}{T_\phi(r)} \sum_{\phi(a)\in\Lambda, |a|<r} \delta_a \log^+ \frac{r}{|a|}$$

which describe the intersection of the line Λ and the leaf L. where δ_a is the Dirac measure at a. We get a family of measures on the unit disc, whose behavior is related to the wiggling of the leave. The previous Theorem, says that with "few" exceptions the cluster points are probability measures.

In [2], the following geometrical ergodic theorem is proved. Let $\mathscr{F}$ be a foliation, possibly singular, by Riemann surfaces on a compact Kähler manifold M. The manifold M is endowed with a fixed Kähler form ω. Let T be an extremal positive dd^c-closed current directed by the foliation $\mathscr{F}$. Assume for simplicity that T has full mass on the hyperbolic leaves, i.e. leaves covered by the unit disc.

If $\phi_a : \mathbb{D} \to L$ is the universal covering of a leaf L with $\phi_a(0) = a$, we consider for $r < 1$ the *Nevanlinna current*

$$\tau_r^a := \frac{1}{T(r)} (\phi_a)_* \Big(\log^+ \frac{r}{|\xi|} \Big) = \frac{1}{T(r)} \int_0^r \frac{dt}{t} (\phi_a)_* [\mathbb{D}(t)].$$

Here, $[\mathbb{D}(t)]$ is the current of integration on the disc $\mathbb{D}(t)$ of radius t centered at 0, $\log^+ := \max(\log, 0)$ and $T(r)$ is the *Nevanlinna characteristic* for ϕ_a which is given by the formula

$$T(r) := \int_0^r \frac{dt}{t} \int_{\mathbb{D}(t)} (\phi_a)^*(\omega).$$

Theorem 2 ([2])**.** *In the above situation, assume that all singularities of $\mathscr{F}$ are linearizable. Then, for T-almost every a, we have $\tau_r^a \to T$ as r tends to 1, in the sense of currents. More precisely, for any smooth test form θ of bi-degree $(1,1)$ on $\mathbb{P}^k$, we have*

$$\langle \tau_r^a, \theta \rangle \to \langle T, \theta \rangle \quad as \quad r \to 1.$$

A consequence of Theorem 2 is that for almost every leaf, with respect to the current T, the Nevanlinna characteristic $T(r)$ for the leaf parametrized by ϕ_a, satisfies the inequality

$$T(r) \geq c \log \frac{1}{(1-r)}$$

for r close to 1, where c is a positive constant depending on a. In particular, it satisfies the growth conditions needed for a second main Theorem for maps with the unit disc as the source.

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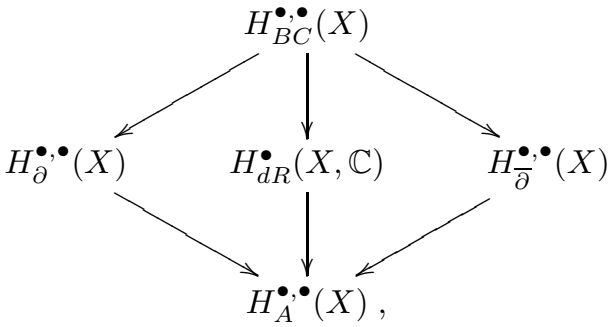
Cohomological properties of complex non-Kähler manifolds

Nicoletta Tardini
(joint work with Daniele Angella)

On a complex manifold X the triple $(A^{\bullet,\bullet}(X), \partial, \overline{\partial})$ given by the space of (p,q)-forms on X and the differential operators ∂ and $\overline{\partial}$ is a double complex. One can define the classical de Rham and Dolbeault cohomologies but it turns out that in complex non-Kähler geometry they do not suffice in studying a complex manifold. Many informations are indeed contained in the Bott-Chern [5] and Aeppli [1] cohomologies, defined, on a complex manifold X, respectively as

$$H^{\bullet,\bullet}_{BC}(X) := \frac{\text{Ker } \partial \cap \text{Ker } \overline{\partial}}{\text{Im } \partial\overline{\partial}}, \qquad H^{\bullet,\bullet}_{A}(X) := \frac{\text{Ker } \partial\overline{\partial}}{\text{Im } \partial + \text{Im } \overline{\partial}}.$$

These two cohomologies represent a bridge between a topological invariant (the de Rham cohomology) and a complex invariant (the Dolbeault cohomology), indeed we have that the identity induces natural maps



while there is no natural map between the Dolbeault and de Rham cohomologies. Generally such maps are neither injective nor surjective but when the map $H^{\bullet,\bullet}_{BC}(X) \longrightarrow H^{\bullet}_{dR}(X)$ is injective, namely every ∂-closed, $\overline{\partial}$-closed and d-exact form is $\partial\overline{\partial}$-exact, the manifold X is said to satisfy the *$\partial\overline{\partial}$-lemma*. Moreover, the injectivity of $H^{\bullet,\bullet}_{BC}(X) \longrightarrow H^{\bullet}_{dR}(X)$ is equivalent to all the maps in the diagram being isomorphisms [6, Lemma 5.15]. Every compact Kähler manifold satisfies the $\partial\overline{\partial}$-lemma [6] but the converse is not true. More generally, manifolds in class $\mathcal{C}$ of Fujiki and Moishezon manifolds (namely manifolds that can be respectively modified to a Kähler manifold and a projective manifold) satisfy the $\partial\overline{\partial}$-lemma.
As a consequence of Hodge theory, all these cohomology groups on compact complex manifolds are finite dimensional vector spaces [8] but differently from the Dolbeault cohomology, Hermitian duality does not preserve the Bott-Chern cohomology, in fact it realizes an isomorphism with the Aeppli cohomology. In fact

in [3] we prove that if X is a compact complex manifold, then X satisfies the *qualitative Kodaira-Spencer-Schweitzer property*, i.e., the natural pairing

$$H_{BC}^{\bullet,\bullet}(X) \times H_{BC}^{\bullet,\bullet}(X) \to \mathbb{C}\,, \qquad ([\alpha],[\beta]) \mapsto \int_X \alpha \wedge \beta$$

is non-degenerate if and only if X satisfies the $\partial\overline{\partial}$-Lemma.

On the other side, many informations can be obtained investigating quantitative properties of the Bott-Chern and Aeppli cohomologies (namely, relations between their dimensions in terms of the Betti and Hodge numbers) towards the study of their qualitative properties (namely, their algebraic structure).
In [4, Theorem A, Theorem B] it is proven a Frölicher-type inequality [7], namely on X for every $k \in \mathbb{Z}$ one has

$$\Delta^k(X) \;:=\; \sum_{p+q=k} (\dim_{\mathbb{C}} H_{BC}^{p,q}(X) + \dim_{\mathbb{C}} H_A^{p,q}(X)) - 2\,b_k \;\geq\; 0\,.$$

Moreover, the equalities characterize the validity of the $\partial\overline{\partial}$-lemma on X, meaning that from a purely quantitative information we are able to understand whether the manifold is cohomologically Kähler or not. Moreover, with D. Angella in [3] we prove that the Bott-Chern cohomology is an invariant strong enough to characterize the $\partial\overline{\partial}$-lemma; more precisely, a compact complex manifold X of complex dimension n satisfies the $\partial\overline{\partial}$-Lemma if and only if, for any p, $q \in \mathbb{Z}$, one has

$$\dim_{\mathbb{C}} H_{BC}^{p,q}(X) = \dim_{\mathbb{C}} H_{BC}^{n-p,n-q}(X)\,.$$

This means that a central simmetry in the Bott-Chern diamond forces the natural maps among all the cohomology groups to be isomorphisms.
Similar considerations can be done on (locally-conformally-)symplectic manifolds for the analogue of the Bott-chern cohomology groups in these settings (cf [10], [9], [2]).

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Weighted Bergman kernel, directional Lelong number and John-Nirenberg exponent

Bo-Yong Chen

Let B_1 be the unit ball in $\mathbb{C}^n$ and $PSH(B_1)$ the set of plurisubharmonic functions on B_1. For each $\psi \in PSH(B_1)$ we define $K_{t\psi}(z,w)$ to be the weighted Bergman kernel of the Hilbert space

$$A^2_{t\psi} = \left\{ f \in \mathcal{O}(B_1) : \int_{B_1} |f|^2 e^{-t\psi} < \infty \right\}, \quad t \geq 0.$$

Set $K_{t\psi}(z) = K_{t\psi}(z,z)$. A cerebrated theorem of Demailly states that

$$\psi_t := \frac{1}{t} \log K_{t\psi}(z) \to \psi(z) \quad (t \to +\infty)$$

and

$$\nu(\psi, z) - 2n/t \leq \nu(\psi_t, z) \leq \nu(\psi, z)$$

where $\nu(\varphi, z)$ denotes the Lelong number for a psh function φ at z.

In this talk we consider the case when t is fixed and z approaches the boundary ∂B_1. The main results are the following

Theorem 1. *Let ψ be a psh function in a neighborhood of the closed ball $\overline{B}_R := \{|z| \leq R\}$ where $R > 1$. For each $0 \leq t < \tilde{\varepsilon}_{B_R}(\psi)$ and each $\zeta \in \partial B_1$ we have*

$$\lim_{r \to 0} \frac{\log K_{t\psi}((1-r)\zeta)}{\log 1/r} = n + 1 - t\tilde{\nu}(\psi_\zeta).$$

Here $\tilde{\varepsilon}_{B_R}(\psi)$ is the John-Nirenberg exponent of ψ associated to certain family of *nonisotropic* balls in B_R, and $\tilde{\nu}(\psi_\zeta)$ is certain directional Lelong number associated to ψ.

Theorem 2. *Let ψ be as above. Then $\tilde{\varepsilon}_{B_R}(\psi) > 0$.*

Tame discrete subsets in Stein manifolds

Jörg Winkelmann

For discrete subsets in $\mathbf{C}^n$ the notion of being "tame" was defined by Rosay and Rudin. A discrete subset $D \subset \mathbf{C}^n$ is called "tame" if and only if there exists an automorphism ϕ of $\mathbf{C}^n$ such that $\phi(D) = \mathbb{N} \times \{0\}^{n-1}$. (Here a subset D of a topological space X is called a "discrete subset" if every point p in X admits an open neighbourhood W such that $W \cap D$ is finite.)

We want to introduce and study a similar notion for complex manifolds other than $\mathbf{C}^n$.

Therefore we propose a new definition, show that it is equivalent to that of Rosay and Rudin if the ambient manifold is $\mathbf{C}^n$ and deduce some standard properties.

To obtain good results, we need some knowledge on the automorphism group of the respective complex manifold. For this reason we get our best results in the case where the manifold is biholomorphic to a complex Lie group. We concentrate on semisimple complex Lie groups, since every simply-connected complex Lie group is biholomorphic to a direct product of $\mathbf{C}^n$ and a semisimple complex Lie group.

Definition 1. Let X be a complex manifold. An infinite discrete subset D is called (weakly) *tame* if for every exhaustion function $\rho : X \to \mathbb{R}^+$ and every map $\zeta : D \to \mathbb{R}^+$ there exists an automorphism ϕ of X such that $\rho(\phi(x)) \geq \zeta(x)$ for all $x \in D$.

Andrist and Ugolini have proposed a different notion, namely the following:

Definition 2. Let X be a complex manifold. An infinite discrete subset D is called (strongly) *tame* if for every injective map $f : D \to D$ there exists an automorphism ϕ of X such that $\phi(x) = f(x)$ for all $x \in D$.

It is easily verified that "strongly tame" implies "weakly tame". For $X \simeq \mathbf{C}^n$ and $X \simeq SL_n(\mathbf{C})$ both tameness notions coincide. Furthermore, for $X = \mathbf{C}^n$ both notions agree with tameness as defined by Rosay and Rudin.

However, for arbitrary manifolds "strongly tame" and "weakly tame" are not equivalent. (For instance this happens for $\Delta \times \mathbf{C}$.)

In this article, unless explicitly stated otherwise, tame always means weakly tame.

Comparison between $\mathbf{C}^n$ and semisimple complex Lie groups. For tame discrete sets in $\mathbf{C}^n$ in the sense of Rosay and Rudin, the following facts are well-known:

(1) Any two tame sets are equivalent.
(2) Every discrete subgroup of $(\mathbf{C}^n, +)$ is tame as a discrete set.
(3) Every discrete subset of $\mathbf{C}^n$ is the union of two tame ones.
(4) There exist non-tame subsets in $\mathbf{C}^n$.
(5) Every injective self-map of a tame discrete subset of $\mathbf{C}^n$ extends to a biholomorphic self-map of $\mathbf{C}^n$.

(6) If v_k is a sequence in $\mathbf{C}^n$ with $\sum_{k=1}^{\infty} \frac{1}{||v_k||^{2n-1}} < \infty$, then $\{v_k : k \in \mathbb{N}\}$ is a tame discrete subset.

For discrete subsets in semisimple complex Lie groups we are able to prove the following properties:

(1) Any two tame discrete subsets in a semisimple complex Lie group are equivalent. The notions "strongly tame" and "(weakly) tame" coincide.
(2) Certain discrete subgroups may be verified to be tame discrete subsets. In particular, $SL_n(\mathbb{Z}[i])$ is a tame discrete subset, and also every discrete subgroup of a one-dimensional Lie subgroup of $SL_n(\mathbb{C})$ and every discrete subgroup of a maximal torus.
(3) Every discrete subset of complex linear algebraic group is the union of 2 tame discrete subsets.
(4) Every semisimple complex Lie group admits a non-tame discrete subset.
(5) Every injective self-map of a tame discrete subset of a semisimple complex Lie group extends to a biholomorphic self map of this Lie group.
(6) For every semisimple complex Lie group S there exists a "threshold sequence", i.e., there exists a sequence of numbers $R_k > 0$ and an exhaustion function τ such that every sequence g_k with $\tau(g_k) > R_k$ defines a tame discrete subset.

Riemann surfaces of second kind and finiteness theorems

Burglind Jöricke

The Geometric Shafarevich Conjecture (now a theorem) states that over a closed or punctured Riemann surface there are only finitely many locally (holomorphically) non-trivial holomorphic fiber bundles with fibers a punctured Riemann surface of given hyperbolic type. We prove finiteness theorems in this spirit in case the base is an arbitrary finite open Riemann surface, maybe, of second kind. For example, over an arbitrary finite Riemann surface there are up to isotopy no more than finitely many irreducible holomorphic torus bundles. Moreover, there are up to homotopy no more than finitely many holomorphic mappings of an open Riemann surface to the three times punctured Riemann sphere that are not contractible to a point and not contractible to a puncture.

The Geometric Shafarevich conjecture was stated in connection with research related to Fermat's last theorem and to the Mordell conjecture. It was proved by Parshin in the case of compact base and fibers of type $(g, 0)$, $g \geq 2$, and by Arakelov for punctured Riemann surfaces as base and fibers of type $(g, 0)$. Imayoshi and Shiga gave a proof of the quoted version using Teichmüller theory. Here g is the genus of the Riemann surface and m is the number of punctures.

For fiber bundles over punctured Riemann surfaces the following two statements hold. First, each isotopy class of smooth (g, m) bundles contains no more

than finitely many holomorphic fiber bundles that are locally holomorphically non-trivial. Secondly, there are no reducible genus g fiber bundles over a punctured Riemann surface that are not locally holomorphically trivial.

The notion of reducible fiber bundles is related to Thurston's theory of surface homeomorphisms. It is defined as follows.

A finite non-empty set of mutually disjoint closed Jordan curves $C = \{C_1, \ldots, C_\alpha\}$ on a punctured Riemann surface S is called admissible if no C_i is homotopic to a point in S, or to a puncture, or to a C_j with $i \neq j$. Thurston calls an isotopy class of self-homeomorphism of S reducible if there exists an admissible system of curves $C = \{C_1, \ldots, C_\alpha\}$ on S so that some homeomorphism in the class (and hence each homeomorphism in the class) maps C to a homotopic system of curves. In this case we say that C reduces the class. If there is no such system the class is called irreducible.

Irreducible mapping classes can be classified and studied. Reducible mapping classes can be decomposed in some sense into irreducible mapping classes.

A (g, m) bundle over a finite Riemann surface is called reducible if there is an admissible system of curves in the fiber over a base point that reduces all monodromy maps simultaneously. Otherwise the bundle is called irreducible.

The afore mentioned statements for bundles over a punctured Riemann surface are not true if the base is a finite Riemann surface of second kind. We have the following conjecture which is equivalent to the geometric Shafarevich conjecture in the case when the base is a punctured Riemann surface.

Conjecture. *Consider an arbitrary connected finite Riemann surface X, maybe, of second kind. Up to isotopy there are no more than finitely many irreducible holomorphic (g, m) fiber bundles over X.*

The following theorem holds.

Theorem 1. *The conjecture is true for torus bundles (i.e. bundles of type $(1, 0)$) and for bundles with fiber being the three times punctured complex plane.*

Notice that each locally holomorphically non-trivial holomorphic torus bundle over an open Riemann surface is isotopic to such a bundle with a holomorphic section. Hence, in the theorem we may consider $(1, 1)$ bundles instead of $(1, 0)$ fiber bundles. Notice also that bundles with fibers being the three times punctured complex plane are special cases of $(0, 4)$ bundles.

The theorem reduces to the following finiteness theorem for holomorphic mappings.

Theorem 2. *Let Y be equal to the three punctured Riemann sphere $\mathbb{P}^1 \setminus \{-1, 1, \infty\}$. For each finite open Riemann surface X there are up to homotopy only finitely many holomorphic mappings from X into Y that are not contractible to a point in Y and not contractible to a puncture.*

There are no topological obstructions for the existence of isotopies (or homotopies, respectively) in the theorems. Indeed, the following theorem is known.

Theorem. *Let X be a connected smooth oriented open surface with finitely generated fundamental group. The set of isotopy classes of smooth oriented (g, m) fiber bundles on X is in one-to-one correspondence to the set of conjugacy classes of homomorphisms from the fundamental group $\pi_1(X, q_0)$ into the modular group* $\mathrm{Mod}(g, m)$ *of a Riemann surface of genus g with m punctures.*

The modular group $\mathrm{Mod}(g, m)$ is the group of isotopy classes of self-homeomorphisms of a Riemann surface of genus g with m punctures.

The obstructions for the existence of isotopies in the theorems are of conformal nature. In the first theorem they rely on a braid invariant, the extremal length (equivalently, on the conformal module) of braids, in the second theorem they rely on the extremal length of elements of the fundamental group of the twice punctured complex plane.

New characterization of plurisubharmonic functions and positivity of direct image bundles

Fusheng Deng

(joint work with Zhiwei Wang, Liyou Zhang and Xiangyu Zhou)

This note gives a report on a recent work joint with Zhiwei Wang, Liyou Zhang, and Xiangyu Zhou.

The aim of the work is to give a new characterization of plursisubharmonic (p.s.h for abbreviation) functions and Griffiths positivity of holomorphic vector bundles with singular Finsler metrics, and present a proof of the Griffiths positivity of the direct image bundles of the twisted relative canonical bundle associated to a holomorphic family of Stein manifolds or compact Kähler manifolds, by applying this characterization and Ohsawa-Takegoshi type extension theorems. The work is inspired by Demailly's method to the regularization of plurisubharmonic functions and Berndtsson's proof of the integral form of the minimum principle for p.s.h functions.

Let us first take a look at Demailly's method to the regularization of p.s.h functions. Let φ be a p.s.h function on a bounded pseudoconvex domain $D \subset \mathbb{C}^n$. Let $K_m(z, \bar{z})$ be the weighted Bergman kernel of D with respect to the weight $e^{-m\varphi}$. Applying the Ohsawa-Takegoshi extension theorem, Demailly showed that $\log K_m/m$ converges (in certain sense) to φ on D as $m \to \infty$, and then got a regularization of the original p.s.h function φ.

Our first main observation is as follows. We start from an u.s.c (upper semi-continuous) function φ in D which is not assumed to be p.s.h. at the beginning. From the argument of Demailly, we see that the convergence for $\log K_m/m$ to φ is also valid, provided that the Ohsawa-Takegoshi extension theorem holds on D with the weight $e^{-m\varphi}$, with the constant C_m in the estimate growing mildly as m goes to infinity. Note that $\log K_m/m$ is always p.s.h, so φ is p.s.h. Precisely we prove the following:

Theorem 1. *Let $\varphi : D \to [-\infty, +\infty)$ be an upper semicontinuous function on $D \subset \mathbb{C}^n$ that is not identically $-\infty$. If for any $z_0 \in D$ with $\varphi(z_0) > -\infty$ and any $m > 0$, there is $f \in \mathcal{O}(D)$ such that $f(z_0) = 1$ and*

$$\int_D |f|^p e^{-m\varphi} \leq C_m e^{-m\varphi(z_0)},$$

where $p > 0$ is a fixed constant, and C_m are constants independent of z_0 and satisfy $\log C_m/m \to 0$, then φ is plurisubharmonic.

Our second main observation is to relate Theorem 1 with the positivity of Hodge type bundles, via giving a trivial but important geometric interpretation of Theorem 1 as follows. We view $e^{-\varphi}$ as a (singular) hermitian metric on the trivial line bundle $L = D \times \mathbb{C}$. Let $\pi = Id : D \to D' = D$ be the trivial fibration with each fiber being one single point. It is obvious that the associated twisted relative canonical bundle $K_{D/D'} \otimes L$ is isomorphic to L and its direct image $L' := \pi_*(K_{D/D'} \otimes L) \to D'$ is also canonically isomorphic to L. The Hodge-type metric (defined by integration along fibers) on L' is given by $e^{-\varphi'}$ with $\varphi' = \varphi$. Therefore Theorem 1 implies that: if $(mL, e^{-m\varphi})$ satisfies the Ohsawa-Takegoshi extension theorem, then the direct image $\pi_*(K_{D/D'} \otimes L)$ is positively curved with respect to the Hodge-type metric. This observation leads us to a rough principle of expectation that *Ohsawa-Takegoshi type extensions implies Griffiths positivity of the associated (0-th) direct image bundles.* In connection to this direction, we get a generalization of Theorem 1 and a characterization of the Griffiths positivity of a holomorphic vector bundle.

Theorem 2. *Let (E, h) be a Hermitian holomorphic vector bundle over a domain $D \subset \mathbb{C}^n$. Assume that for any $z_0 \in D$, any nonzero element $a \in E_{z_0}$ with finite norm, and any $m \geq 1$, there is a holomorphic section f_m of $E^{\otimes m}$ on D such that $f(z_0) = a^{\otimes m}$ and satisfies the following estimate:*

$$\int_D |f_m|^{2p} \leq C_m |a^{\otimes m}|^{2p} = C_m |a|^{2mp},$$

where $p > 0$ is a fixed constant and C_m are constants independent of z_0 and satisfying $\log C_m/m \to 0$ as $m \to \infty$. Then (E, h) is positive in the sense of Griffiths.

In fact Theorem 2 can be generalized to singular Finsler metrics.

In recent years, there are several important works on positivity of direct image sheaves of twisted relative canonical bundles associated to holomorphic families of certain complex manifolds, based on L^2-method in complex analysis.

The starting point is Berndtsson's work on the integral form of Kiselman's minimum principle for p.s.h functions [1]. Following this work, Berndtsson shows that the relative weighted Bergman kernel associated to a family of pseudoconvex domains varies plurisubharmonically [2]. This idea was further developed by Berndtsson himself who finally shows that the direct image sheaf of the twisted relative canonical bundle with a semipositive twist associated to a holomorphic family of pseudoconvex domains or compact Kähler manifolds is Griffiths positive

[3]. This result was generalized by Berndtsson and Păun to projective family of compact manifolds with singular twist [5]. The main tool in the works of Berndtsson and Berndtsson-Păun is Hörmander's L^2-estimate of $\bar{\partial}$ and Kohn's regularity of the $\bar{\partial}$-Neumann problem for strongly pesudoconvex domains.

In another direction, Guan-Zhou shows that Berndtsson's plurisubharmonic variation of Bergman kenels can be deduced from the Ohsawa-Takegoshi extension theorem with optimal estimate that was established by Błocki and Guan-Zhou [6]. In connection to this result, Berndtsson and Lempert show that the Ohsawa-Takegoshi extension theorem with optimal estimate (at least for the case of pesudoconvex domains) can be deduced from Berndtsson's result of positivity of direct image sheaves [4].

With the works of Guan-Zhou and Berndtsson-Lempert, it is widely believed by experts that the Ohsawa-Takegoshi extension theorem with optimal estimate and the positivity of direct image sheaves should be equivalent. Recently, by developing the method of Guan-Zhou, Hacon, Popa, and Schnell showed that, for a projective family of compact manifolds, the positivity of the the direct image sheaf of the twisted relative canonical bundle can be deduced from the Ohsawa-Takegoshi extension theorem with optimal estimate [7], and hence provide stronger evidence of the equivalence for the two conclusions.

Although it is widely believed that the Ohsawa-Takegoshi extension theorem with optimal estimate and the positivity of direct image sheaves should be equivalent, it seems that, as far as we know, no one has even expected that the positivity of direct image sheaves can be deduced from the *ordinary* Ohsawa-Takegoshi extension theorem (namely without optimal estimate).

However, using the two observations mentioned above, we show that this is indeed the case, that is, the positivity of the direct image sheaf of the twisted relative canonical bundle (with singular weight) associated to a holomorphic family of pseudoconvex domains or compact Kähler manifolds can be deduced from the ordinary Ohsawa-Takegoshi extension theorem, in a very transparent way.

By the same method, we also prove the plurisubharmonic variation of the so called relative m-Bergman kernel associated to a holomorphic family of pseudoconvex domains or compact Kähler manifolds (the case that $m = 1$ corresponds to the usual Bergman kernel and was proved by Berndtsson).

In addition to the above two observations, our other main technique is rising the powers of complex manifolds, which was inspired by Berndtsson's method to the minimum principle for p.s.h functions in [1].

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Chern Forms of Singular Metrics on Vector Bundles

Jean Ruppenthal

Let X be a complex manifold of dimension n, let $E \to X$ be a rank r holomorphic vector bundle over X, and let h denote a hermitian metric on E. The classical differential geometric study of X through (E, h) revolves heavily around the notion of the curvature associated with h. This approach requires the metric to be smooth (i.e. twice differentiable). However, for line bundles Demailly introduced the notion of *singular* hermitian metrics, and in a series of influential papers he and others showed how these are a fundamental tool for giving complex algebraic geometry an analytic interpretation. He showed e.g. that a holomorphic line bundle is pseudo-effective in the algebraic sense if and only if it carries a singular hermitian metric which is Griffiths semi-positive.

For holomorphic vector bundles of higher rank, things are much more sophisticated, even when considering smooth metrics. One reason is that Griffiths positivity and Nakano positivity do not coincide any more. In view of the connection between algebraic geometry and analysis mentioned above, it is, however, very interesting to study also singular hermitian metrics of holomorphic vector bundles. A suitable notion of singular hermitian metrics for vector bundles of higher rank and Griffiths semi-positivity/negativity has been introduced by Berndtsson-Păun in [1]. It turned out to be extremely useful in the study of positivity of direct images of twisted canonical bundles (see [1] and [5]).

For vector bundles with such a Griffiths semi-positive singular metric, there is a naturally defined first Chern form which is a positive closed $(1,1)$-current, but there are examples where the full curvature matrix is not of order 0 (so that it is not clear how to define higher Chern forms).

In [2], it is shown that one can give a natural meaning to the k-th Chern form of a singular Griffiths semi-positive hermitian metric as a closed (k,k)-current of order 0, as long as the set where the metric degenerates is small enough. It is shown in [3] that the results can be extended to Chern forms of arbitrary degree if the metric has analytic singularities, which is a very natural condition. Let us explain the results in [2] more precisely.

In [1] Berndtsson and Păun introduced the following notion of singular metrics for vector bundles:

Definition 1. Let $E \to X$ be a holomorphic vector bundle over a complex manifold X. A *singular hermitian metric* h on E is a measurable map from the base

space X to the space of hermitian forms on the fibers. The hermitian forms are allowed to take the value ∞ at some points in the base (i.e. the norm function $\|\xi\|_h$ is a measurable function with values in $[0,\infty]$), but for any fiber E_x the subset $E_0 := \{\xi \in E_x \ ; \ \|\xi\|_{h(x)} < \infty\}$ has to be a linear subspace, and the restriction of the metric to this subspace must be an ordinary hermitian form.

They also defined what it means for these types of metrics to be curved in the sense of Griffiths:

Definition 2. Let $E \to X$ be a holomorphic vector bundle over a complex manifold X and let h be a singular hermitian metric. We say that h is *Griffiths negative* if $\|u\|_h^2$ is plurisubharmonic for any (local) holomorphic section u. Furthermore, we say that h is *Griffiths positive* if the dual metric h^* is Griffiths negative.

Strictly speaking, [1] define h to be Griffiths negative if $\log \|u\|_h$ is plurisubharmonic for any holomorphic section u. It is, however, not too difficult to show that these two definitions are equivalent (see e.g. [6], section 2). Any singular hermitian metric on a vector bundle E induces a dual metric on the dual bundle E^* (see [2], Lemma 3.1). This justifies the notion of Griffiths positivity in Definition 2 in terms of duality. Definition 2 is very natural as these conditions are well-known equivalent properties for smooth metrics.

Although Definition 1 is very liberal, as it basically puts no restriction on the metrics, it turns out that Definition 2 rules out most of the pathological behaviour. We have e.g. the following proposition ([6], Proposition 1.3 (ii)):

Proposition 3. *Let h be a singular, Griffiths negative, hermitian metric. If $\det h \not\equiv 0$, then $i\partial\overline{\partial} \log \det h$ is a closed, positive $(1,1)$-current.*

The proof uses the well-known fact that if h is a metric on E, then $\det h$ is a metric on $\det E$. For smooth metrics it is also well-known that the curvature of $\det h$, i.e. $-\partial\overline{\partial} \log \det h = \partial\overline{\partial} \log \det h^*$, is the trace of the curvature of h, i.e. $2\pi i$ times the first Chern form $c_1(E,h)$. Thus, a simple consequence of Proposition 3 is that for a singular metric which is curved in the sense of Definition 2, it is possible to define the first Chern form in a meaningful way as a closed, positive or negative $(1,1)$-current.

However, despite this, one of the main results in [6] (Theorem 1.5) is a counterexample that shows that the curvature requirement of Definition 2 is not enough to define the curvature of a singular metric as a current with measure coefficients. This rather surprising fact, given the existence of the first Chern form, leads to the question of which differential geometric concepts one can obtain from Definition 2. A main result in [2] (Theorems 1.11 and 1.13) is as follows:

Theorem 4. *Let $E \to X$ be a holomorphic vector bundle over a complex manifold X, and let h denote a singular, Griffiths positive, hermitian metric on E. Assume that there is some subvariety V of X with $codim(V) \geq k$ such that $L(\log \det h^*) \subseteq V$, where $L(\log \det h^*)$ denotes the unbounded locus of $\log \det h^*$.*

Then the k-th Chern current of E associated with h, $c_k(E,h)$, can be defined as a closed (k,k)-current of order 0 with locally finite mass in X. If X is compact,

then $[c_k(E,h)] = c_k(E)$ *where* $c_k(E) = [c_k(E,h_0)]$ *is the usual Chern class defined by any smooth metric* h_0 *on* E.

If h *is instead a singular, Griffiths negative, hermitian metric, then the same result holds if* $L(\log\det h^*)$ *is replaced by* $L(\log\det h)$ *throughout the statement.*

If the metric h is smooth, then our $c_k(E,h)$ coincides by construction with the usual Chern form. If h fulfils the assumptions of Theorem 4 and is continuous outside the variety V, then for any local regularizing sequence $\{h_\varepsilon\}$ of h, with $h_\varepsilon \to h$ locally uniformly outside of V, we have that

$$c_k(E,h_\varepsilon) \to c_k(E,h)$$

in the sense of currents. This is important for applications, because local regularization is one of the main tools to deal with singular metrics, and it is possible for singular hermitian metrics which are Griffiths positive or negative by convolution with an approximate identity (see [1], Proposition 3.1, and [6], Proposition 6.2). If h is continuous outside V, then the Chern currents defined in Theorem 4 satisfy all the usual properties known for smooth Chern forms in terms of *duality, pull-backs, direct sums, direct products* (see [2], Corollary 1.9).

The main idea behind the proof of Theorem 4 is as follows. Let $\pi : \mathbb{P}(E) \to X$ be the projective bundle of lines in E^*. Then the dual metric h^* induces a metric on the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(-1) \subset \pi^*E^*$. Let $e^{-\varphi}$ be dual metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$. If h is Griffiths positive, then $e^{-\varphi}$ is a positive metric, i.e., the local weights φ are plurisubharmonic and the first Chern form of $e^{-\varphi}$ is given as $dd^c\varphi$. If h is smooth, Mourougane [4] showed that the k-th Segre-form of h can be recovered as

$$s_k(E,h) = (-1)^k\pi_*\big(dd^c\varphi\big)^{k+r-1},$$

where r is the rank of E. We use this approach to define Segre forms also for singular hermitian metrics. But the Chern forms can then be understood in terms of the Segre forms, as the total Chern form is the multiplicative inverse of the total Segre form.

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Some results on multiplier ideal sheaves

Xiangyu Zhou

In the present talk, we'll recall the basics of multiplier ideal sheaves, then present our recent solution of Demailly's strong openness conjecture on multiplier ideal sheaves, and some applications in complex geometry, e.g. some new results related to the vanishing, finiteness theorems of analytic cohomology groups with multiplier ideal sheaves for the pseudo-effective line bundles over holomorphically convex manifolds, and generalized Siu's lemma and pseudoeffectiveness of the twisted relative pluricanonical bundles and their direct images.

1. Introduction

Associated to a plurisubharmonic function, one may define multiplier ideal sheaf. The first basic properties of multiplier ideal sheaf includes that it is a coherent analytic sheaf and integral closed, and it holds Nadel vanishing theorem.

Guan-Zhou (Ann. of Math. 2015) showed that Demailly's strong openness conjecture holds.

2. Application 1

2.1. **Vanishing theorems and finiteness theorem.** Meng-Zhou (JAG, to appear) obtained some vanishing and finiteness theorems about multiplier ideal sheaves, using the strong openness property of the multiplier ideal sheaves. For holomorphically convex manifolds, the Kähler assumption can be removed in the Nadel vanishing theorem.

Theorem 1. *Let (X,ω) be a hermitian holomorphically convex manifold, and let L be a holomorphic line bundle over X equipped with a singular hermitian metric h. Assume that $i\Theta_{L,h} \geq \varepsilon\omega$ for some continuous positive function ε on X. Then*

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(h)) = 0 \quad \textit{for} \quad q \geq 1.$$

Theorem 2. *Let X be a strongly 1-convex manifold and let L be a holomorphic line bundle over X equipped with a singular hermitian metric h such that $i\Theta_{L,h} \geq 0$ in the sense of currents. Then*

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(h)) = 0 \quad \textit{for} \quad q \geq 1.$$

We present the following "singular" version of Ohsawa's finiteness theorem.

Theorem 3. *Let X be a holomorphically convex manifold, and let L be a holomorphic line bundle over X equipped with a singular hermitian metric h. Assume the curvature current $i\Theta_{L,h} \geq \gamma$ for some continuous real $(1,1)$-form γ on X and γ is strictly positive outside a compact subset K of X. Then we have*

$$\dim H^q(X, K_X \otimes L \otimes \mathcal{I}(h)) < +\infty \quad \textit{for} \quad q \geq 1,$$

and the corresponding restriction maps are bijective.

The proof is based on Theorem 1 and the Remmert reduction theorem.

3. Application 2

Using ideas and results in the proof of the strong openness conjecture, Zhou and Zhu prove:

Generalized Siu's lemma 1. (Zhou, Zhu 2016)

Let $\varphi(z', z'')$ be a plurisubharmonic function, h be a nonnegative continuous function on $\mathbb{B}_r^1 \times \mathbb{B}_r^{n-1}$ $(1 \leq m \leq n)$.

$$\begin{aligned} & \lim_{\varepsilon \to 0^+} \frac{1}{\lambda(\mathbb{B}_\varepsilon^1)} \int_{\mathbb{B}_\varepsilon^1 \times \mathbb{B}_r^{n-1}} h(z', z'') e^{-\varphi(z', z'')} d\lambda_n \\ = & \int_{z'' \in \mathbb{B}_r^{n-1}} h(0, z'') e^{-\varphi(0, z'')} d\lambda_{n-1} \end{aligned}$$

Generalized Siu's lemma 2. (Zhou, Zhu 2017)

Assume that

$$I_{f,\varphi} := \int_{z'' \in \mathbb{B}_r^{n-m}} |f(z'')|^2 e^{-\varphi(0, z'')} d\lambda_{n-m} < +\infty$$

Assume that $\varepsilon, r_1, r_2 \in (0, r)$ and $r_1 < r_2$.

Then there exists a holomorphic function $F(z', z'')$ on $\mathbb{B}_{r_2}^m \times \mathbb{B}_{r_2}^{n-m}$ such that $F(0, z'') = f(z'')$ on $\mathbb{B}_{r_2}^{n-m}$,

$$\int_{\mathbb{B}_{r_2}^m \times \mathbb{B}_{r_2}^{n-m}} |F(z', z'')|^2 e^{-\varphi(z', z'')} d\lambda_n < +\infty,$$

and

$$\begin{aligned} & \lim_{\varepsilon \to 0^+} \frac{1}{\lambda(\mathbb{B}_\varepsilon^m)} \int_{\mathbb{B}_\varepsilon^m \times \mathbb{B}_{r_1}^{n-m}} h(z', z'') |F(z', z'')|^2 e^{-\varphi(z', z'')} d\lambda_n \\ = & \int_{z'' \in \mathbb{B}_{r_1}^{n-m}} h(0, z'') |f(z'')|^2 e^{-\varphi(0, z'')} d\lambda_{n-m} \end{aligned}$$

Guan-Zhou (Ann. Math. 2015) established the optimal L^2 extension theorem in the almost Stein setting and discovered that it implies log-psh property of the relative Bergman kernel and Griffiths of the relative canonical bundles.

The generalized versions of Siu's lemma together with optimal L^2 extension theorem in the Kähler setting by Zhou-Zhu (JDG, to appear) implies the optimal $L^{\frac{2}{m}}$ theorem for the twisted pluricanonical bundles in the Kähler setting. Using Guan-Zhou's discovery, Zhou-Zhu (2017) obtained pseudoeffectiveness of the twisted relative pluricanonical bundles and their direct images for the Kähler fibration (maybe non smooth).

Curvature properties and nonexistence of compact Levi-flat hypersurfaces

Judith Brinkschulte

A real hypersurface M (of class at least $\mathcal{C}^2$) in a complex manifold is called Levi-flat if its Levi-form vanishes identically or, eqivalently, if it admits a foliation by complex hypersurfaces. Another equivalent formulation is that M is locally pseudoconvex from both sides.

Levi-flat hypersurfaces can be of quite different nature and therefore one would like to classify compact Levi-flat real hypersurfaces. On the other hand, the study of Levi-flat real hypersurfaces is related to basic questions in dynamical systems and foliation theory: Levi-flats arise as stable sets of holomorphic foliations, and a real-analytic Levi-flat real hypersurface extends to a holomorphic foliation leaving M invariant. Relating to this, a famous open problem is whether or not $\mathbb{CP}^2$ contains a smooth Levi-flat real hypersurface.

Given a Levi-flat real hypersurface M in a complex manifold X of dimension n, we call $N_M^{1,0} = (T_X^{1,0})_{|M}/T^{1,0}M$ the holomorphic normal bundle of M. The restriction of $N_M^{1,0}$ to each $(n-1)$-dimensional complex submanifold of M has a structure of a holomorphic line bundle induced from that of $T_X^{1,0}$.

In this talk, I discuss curvature properties of the holomorphic normal bundle of a compact Levi-flat real hypersurface.

For $n \geq 3$, however, it is known that there does not exist any smooth real Levi-flat hypersurface M in $\mathbb{CP}^n$. This was first proved by LinsNeto for real-analytic M and by Siu for $\mathcal{C}^{12}$-smooth M. The proofs of these results essentially exploited the positivity of $T^{1,0}\mathbb{CP}^n$. Brunella observed that the positivity of the normal bundle itself is enough to ensure that the complement of M is pseudoconvex. This led Brunella to prove that if X is a compact Kähler manifold with $\dim X \geq 3$, and if M is a smooth Levi-flat real hypersurface such that there exists a holomorphic foliation on a neighborhood of M leaving M invariant, then the normal bundle of this foliation does not admit any fiber metric with positive curvature.

In a recent paper, I proved the following result:

Theorem. *Let X be a complex manifold of dimension $n \geq 3$. Then there does not exist a smooth compact Levi-flat real hypersurface M in X such that the normal bundle to the Levi foliation admits a Hermitian metric with positive curvature along the leaves.*

This theorem is a generalized version of Brunella's result in the sense that we are able to drop the compact Kähler assumption on the ambient X.

The following example due to Brunella shows that the above result cannot hold for $n = 2$, even for X compact Kähler:

Let Σ be a compact Riemann surface of genus $g \geq 2$. Let $\mathbb{D}$ be the open unit disc, and let Γ be a discrete subgroup of $\mathrm{Aut}\mathbb{D} \subset \mathrm{Aut}\mathbb{CP}^1$ such that $\Sigma \simeq \mathbb{D}/\Gamma$. Then Γ also acts on $\mathbb{D} \times \mathbb{CP}^1$ by

$$(z, w) \mapsto (\gamma(z), \gamma(w)), \ \gamma \in \Gamma.$$

The quotient $X = (\mathbb{D} \times \mathbb{CP}^1)/\Gamma$ is a compact complex surface, ruled over Σ (and hence projective). From the horizontal foliation on $\mathbb{D} \times \mathbb{CP}^1$, we get a holomorphic foliation on X, leaving invariant a real analytic Levi-flat hypersurface M induced from the Γ-invariant $\mathbb{D} \times \mathbb{S}^1$. The Bergman metric induces a metric with positive curvature on the normal bundle of M.

Uniformization and Steinness

Stefan Nemirovski
(joint work with Rasul Shafikov)

A Stein strictly pseudoconvex domain D is universally covered by the unit ball $\mathbb{B}^n$ if and only if its boundary is spherical, that is, locally CR-diffeomorphic to the unit sphere $\partial\mathbb{B}^n$. The 'only if' implication holds also for strictly pseudoconvex domains that are not necessarily Stein. This suggests two related questions:

(1) How to tell that a given strictly pseudoconvex ball quotient is Stein?
(2) Which non-Stein strictly pseudoconvex domains with spherical boundary are ball quotients?

For instance, Christian Miebach observed that a strictly pseudoconvex quotient of $\mathbb{B}^n$ by a discrete group Γ such that $\mathrm{H}^{2p}(\Gamma, \mathbb{R}) = 0$ for $1 \le p \le n-1$ must be Stein. If this condition doesn't hold (e.g. if Γ is the fundamental group of a closed orientable surface of genus ≥ 2), both Stein and non-Stein quotients may occur.

Another sufficient condition for such a quotient to be Stein was pointed out in the seminal paper of Burns–Shnider. Namely, it is enough to assume that the limit set of the action on the boundary is polynomially convex. It is not clear at present whether a condition of this type might also be necessary.

As for the second question, it is known only that a domain with spherical boundary need not be a blow-up of a ball quotient. This can be seen by considering ramified coverings of the quotients constructed by Goldman–Kapovich–Leeb.

The situation for strictly pseudoconvex domains with non-spherical boundary is strikingly different:

Theorem. *The universal cover of a Stein stictly pseudoconvex domain with non-spherical boundary cannot cover a complex manifold containing a compact analytic subset of positive dimension. In particular, any other strictly pseudoconvex domain with the same universal cover is also Stein.*

The proof of this result is based on a version of the Wong–Rosay theorem for coverings of strictly pseudoconvex domains.

Reporter: Nikolay Shcherbina

Participants

Prof. Dr. Marco Abate
Dipartimento di Matematica
Università di Pisa
Largo Bruno Pontecorvo, 5
56127 Pisa
ITALY

Prof. Dr. Mats Andersson
Department of Mathematics
Chalmers University of Technology
412 96 Göteborg
SWEDEN

Prof. Dr. Eric D. Bedford
Department of Mathematics
Indiana University
Rawles Hall
Bloomington IN 47405
UNITED STATES

Prof. Dr. Zbigniew Blocki
Institute of Mathematics
Jagiellonian University
Lojasiewicza 6
30-348 Kraków
POLAND

Dr. Judith Brinkschulte
Mathematisches Institut
Universität Leipzig
Postfach 10 09 20
04009 Leipzig
GERMANY

Michael Brünnig
Fakultät 4 - Mathematik u. Naturwissenschaften
Bergische Universität Wuppertal
Gaußstrasse 20
42119 Wuppertal
GERMANY

Prof. Dr. Bo-Yong Chen
School of Mathematical Sciences
Fudan University
220 Hadan Road
Shanghai Shi, 200433
CHINA

Prof. Dr. Jean-Pierre Demailly
Institut Fourier
Laboratoire de Mathématiques
UMR 5582 du CNRS
100 rue des Maths
38610 Gières
FRANCE

Prof. Dr. Fusheng Deng
School of Mathematical Sciences
University of Chinese Academy of Sciences
19A Yuquan Road
Beijing 100049
CHINA

Dr. Xin Dong
Department of Mathematics
University of California, Irvine
Irvine, CA 92697
UNITED STATES

Prof. Dr. Barbara Drinovec Drnovšek
Institute of Mathematics, Physics and Mechanics
University of Ljubljana
Jadranska 21
1000 Ljubljana
SLOVENIA

Prof. Dr. Peter Ebenfelt
Department of Mathematics
University of California, San Diego
9500 Gilman Drive
La Jolla, CA 92093-0112
UNITED STATES

Prof. Dr. John Erik Fornaess
Department of Mathematics
The Norwegian University of Science and Technology
Sentralbygg 2
Alfred Getz vei 1
7491 Trondheim
NORWAY

Prof. Dr. Erlend Fornaess Wold
Department of Mathematics
University of Oslo
P. O. Box 1053 - Blindern
0316 Oslo
NORWAY

Prof. Dr. Josip Globevnik
Faculty of Mathematics and Physics
University of Ljubljana
Jadranska 19
1000 Ljubljana
SLOVENIA

Prof. Dr. Purvi Gupta
Department of Mathematics
Rutgers University
Busch Campus, Hill Center
New Brunswick, NJ 08854-8019
UNITED STATES

Tobias Harz
Fachbereich Mathematik
Bergische Universität Wuppertal
Gaußstrasse 20
42119 Wuppertal
GERMANY

Hendrik Herrmann
Mathematisches Institut
Universität zu Köln
Weyertal 86 - 90
50931 Köln
GERMANY

Prof. Dr. Burglind Joericke
Matematisk Institut
Aarhus Universitet
Ny Munkegade 118
8000 Aarhus C
DENMARK

Seungro Joo
The Center for Geometry and its Applications (GAIA)
Mathematical Sciences Building
Pohang University of Science and Technology
Gyeong-sang-bugdo
Pohang City 790-784
KOREA, REPUBLIC OF

Prof. Dr. Kang-Tae Kim
Department of Mathematics
Pohang University of Science and Technology
CheongAmRo 77, Hyojadong, Namgu
Pohang City, Gyeongsangbukdo 37673
KOREA, REPUBLIC OF

Prof. Dr. Slawomir Kolodziej
Institute of Mathematics
Jagiellonian University
Lojasiewicza 6
30-348 Kraków
POLAND

Prof. Dr. Frank Kutzschebauch
Mathematisches Institut
Universität Bern
Sidlerstrasse 5
3012 Bern
SWITZERLAND

Dr. Richard Lärkäng
Department of Mathematics
Chalmers University of Technology
and University of Göteborg
412 96 Göteborg
SWEDEN

Prof. Dr. Laszlo Lempert
Department of Mathematics
Purdue University
West Lafayette, IN 47907-1395
UNITED STATES

Prof. Dr. Erik Low
Department of Mathematics
University of Oslo
P. O. Box 1053 - Blindern
0316 Oslo
NORWAY

Dr. Hoang Chinh Lu
Laboratoire de Mathématiques
Université Paris Sud (Paris XI)
Batiment 307
91405 Orsay Cedex
FRANCE

Dr. Stefan Nemirovski
V.A. Steklov Institute of Mathematics
Russian Academy of Sciences
8, Gubkina St.
119991 Moscow GSP-1
RUSSIAN FEDERATION

Prof. Dr. Takeo Ohsawa
Graduate School of Mathematics
Nagoya University
Chikusa-ku
Nagoya 464-8602
JAPAN

Dr. Han Peters
Korteweg-de Vries Instituut
Universiteit van Amsterdam
Postbus 94248
1090 GE Amsterdam
NETHERLANDS

Prof. Dr. Evgeny A. Poletsky
Department of Mathematics
Syracuse University
900 South Crouse Avenue
Syracuse, NY 13244-1150
UNITED STATES

Jasmin Raissy
Institut de Mathématiques de Toulouse
Université Paul Sabatier
118, route de Narbonne
31062 Toulouse Cedex 9
FRANCE

Prof. Dr. Feng Rong
Department of Mathematics
Shanghai Jiao Tong University
800 Dong Chuan Road
Shanghai Shi, 200240
CHINA

Prof. Dr. Jean Ruppenthal
Fachbereich Mathematik
Bergische Universität Wuppertal
Gaußstrasse 20
42119 Wuppertal
GERMANY

Prof. Dr. Hakan Samuelsson Kalm
Department of Mathematics
Chalmers University of Technology
and University of Göteborg
412 96 Göteborg
SWEDEN

Prof. Dr. Rasul Shafikov
Department of Mathematics
The University of Western Ontario
London ON N6A 5B7
CANADA

Prof. Dr. Nikolay Shcherbina
FB C: Mathematik u.
Naturwissenschaften
Bergische Universität Wuppertal
Gaußstrasse 20
42119 Wuppertal
GERMANY

Prof. Dr. Nessim Sibony
Laboratoire de Mathématiques
Université Paris Sud (Paris XI)
Batiment 307
91405 Orsay Cedex
FRANCE

Prof. Dr. Berit Stensones
Department of Mathematics
The Norwegian University of Science
and Technology
Sentralbygg 2
Alfred Getz vei 1
7491 Trondheim
NORWAY

Prof. Dr. Alexander Sukhov
U. F. R. de Mathématiques
Université de Lille 1
Bât. M2
59655 Villeneuve-d'Ascq. Cedex
FRANCE

Nicoletta Tardini
Dipartimento di Matematica e
Informatica
Università degli Studi di Firenze
Viale Giovanni Battista Morgagni 67A
50134 Firenze
ITALY

Prof. Dr. Alexander Tumanov
Department of Mathematics
University of Illinois at
Urbana-Champaign
273 Altgeld Hall, MC-382
1409 West Green Street
Urbana, IL 61801-2975
UNITED STATES

Dr. Liz Vivas
Department of Mathematics
The Ohio State University
100 Mathematics Building
231 West 18th Avenue
Columbus, OH 43210-1174
UNITED STATES

Prof. Dr. Xueyuan Wan
Department of Mathematics
Chalmers University of Technology and
University of Göteborg
412 96 Göteborg
SWEDEN

Prof. Dr. Xu Wang
Department of Mathematical Sciences
Norwegian University of Science and
Technology
Sentralbygg 2
Alfred Getz vei 1
7491 Trondheim
NORWAY

Prof. Dr. Jörg Winkelmann
Fakultät für Mathematik
Ruhr-Universität Bochum
44780 Bochum
GERMANY

Prof. Dr. David Witt Nyström
Department of Mathematics
Chalmers University of Technology and
University of Göteborg
412 96 Göteborg
SWEDEN

Jingcao Wu
Department of Mathematics
Chalmers University of Technology and
University of Göteborg
412 96 Göteborg
SWEDEN

Dr. Elizabeth Wulcan
Department of Mathematics
Chalmers University of Technology and
University of Göteborg
412 96 Göteborg
SWEDEN

Prof. Dr. Liyou Zhang
Department of Mathematics
Capital Normal University
Haidian District
105 West Third Ring Road N
Beijing 100048
CHINA

Prof. Dr. Xiangyu Zhou
Institute of Mathematics
Academy of Mathematics and Systems
Science
Chinese Academy of Sciences
No. 55, Zhongguancun East Road
Beijing 100190
CHINA

Prof. Dr. Andrew M. Zimmer
Department of Mathematics
College of William and Mary
Sadler Center
P.O. Box 8795
Williamsburg VA 23185
UNITED STATES