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Mini-Workshop: Projective Normality of Smooth Toric Varieties

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Abstract. The mini-workshop on "Projective Normality of Smooth Toric Varieties" focused on the question of whether every projective embedding of a smooth toric variety is projectively normal. Equivalently, this question asks whether every lattice point in kP is the sum of k lattice points in P when P is a smooth (lattice) polytope. The workshop consisted of morning talks on different aspects of the problem, and afternoon discussion groups where participants from a variety of different backgrounds worked on specific examples and approaches.

Mathematics Subject Classification (2000): 14M25, 52B20.

Introduction by the Organisers

The mini-workshop on *Projective normality of smooth toric varieties*, organized by Christian Haase (Berlin), Takayuki Hibi (Osaka), and Diane Maclagan (New Brunswick), was held August 12th-18th, 2007. A small group of researchers with backgrounds in combinatorics, commutative algebra, and algebraic geometry worked on the conjecture that embeddings of smooth toric varieties are projectively normal. This very basic question appears in different guises in algebraic geometry, commutative algebra, and integer programming, but specific cases also arise in additive number theory, representation theory, and statistics. See the summary by Diane Maclagan for three versions of the same question.

There were a limited number of contributed talks in the mornings, setting the theme for the afternoon working groups. Monday morning began with Diane Maclagan describing the problem, and Winfried Bruns surveying the known results in the polyhedral formulation. This was followed on Tuesday morning by Benjamin J. Howard and Hidefumi Ohsugi on special cases of normality, and an introductory talk by Milena Hering on the geometric vanishing theorem approach to the problem. On Wednesday morning Hal Schenck described a commutative algebra approach developed on site together with Greg Smith, while Sam Payne explained the Frobenius splitting approach. The commutative algebra approach, with optimization notes, continued in the talk of Ngô Viêt Trung on Thursday morning. Najmuddin Fakhruddin also explained his proof of the extended two-dimensional case on Thursday morning. Finally, on Friday we heard from Christian Haase and Andreas Paffenholz on some techniques for showing normality in special cases, and Francisco Santos on lattice Delaunay simplices which are potential starting points in search for a counterexample.

In the afternoons we split into working groups which then reported on their findings before dinner. These discussions continued through breaks, and in gaps between talks. The atmosphere of the group was very energetic, and we hope that the momentum generated during the meeting will continue with some of the ideas developed being pursued by the participants.

As a direct outcome of the workshop, we would like to mention

- many examples of very-ample-yet-non-normal polytopes found by Winfried Bruns,
- a joint effort of Christian Haase, Benjamin Nill, Andreas Paffenholz, and Francisco Santos to (finally) settle the ample+nef additivity question in dimension two, as well as
- a dynamic survey on projective normality and related questions to be edited by Diane Maclagan.

The organizers and participants sincerely thank the institute for providing excellent working conditions and the unique Oberwolfach spirit. We are also grateful for funding from the NSF grant supporting young US-based participants, which allowed an extra participant to attend.

In what follows we present, in addition to summaries of the talks, brief accounts on the outcome of brainstorming sessions and working groups.

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Mini-Workshop: Projective Normality of Smooth Toric Varieties

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Abstracts

Introduction to the problem

Diane Maclagan

The goal of this workshop was to consider the following problem.

Definition 1. A smooth lattice polytope is a simple lattice polytope where the first lattice points on each ray of a top-dimensional cone of the normal fan generate the lattice.

Question 1. (Projective Normality) Let P be a smooth lattice polytope in $\mathbb{R}^d$. Is every lattice point in kP for $k \geq 2$ a sum of k lattice points in P?

We note that the smoothness hypothesis is essential here; for example, consider the polytope $P = \text{conv}((0,0,0),(1,0,0),(0,1,0),(1,1,3),(2,1,3),(1,2,3)) \subset \mathbb{R}^3$. The lattice point $(1,1,1)$ lies in $2P$, but is not the sum of two lattice points in P. Question 1 has an affirmative answer in dimension $d = 2$, since every two-dimensional lattice polytope has a unimodular triangulation (a triangulation of P with lattice points as vertices such that each simplex has normalized volume one).

A variant of Question 1 is the following.

Question 2. (Oda) Let P be a smooth lattice polytope and let Q be a lattice polytope whose normal fan is a coarsening of that of P. Is every lattice point z in the Minkowski sum $P + Q = \{x + y : x \in P, y \in Q\}$ a sum $z = u + v$ where u is a lattice point in P and v is a lattice point in Q?

In the case that P and Q are lattice polytopes with the same normal fan, a positive answer to Question 1 in dimension $d + 1$ implies a positive answer to Question 2 in dimension d. To see this, consider the polytope $R = \text{conv}(P \times \{0\}, Q \times \{1\}) \subseteq \mathbb{R}^{d+1}$. This is smooth if P and Q are smooth with the same normal fan, and the lattice points slice of $2R$ with last coordinate one are in bijection with the lattice points in $P+Q$. This observation seems to be originally due to Mustaţă. Question 1 is also known in dimension two, due to work of Fakhruddin [1].

These questions also have formulations in commutative algebra and algebraic geometry. Let $S = \mathbb{k}[x_1, \dots, x_n]$, where $\mathbb{k}$ is a field, and grade S by $\deg(x_i) = \mathbf{a}_i \in \mathbb{Z}^r$. The vectors $\mathbf{a}_i$ divide $\text{pos}(\mathbf{a}_i : 1 \leq i \leq n)$ into open chambers.

Definition 2. The chamber $\sigma \subset \text{pos}(\mathbf{a}_i : 1 \leq i \leq n)$ containing a point $\mathbf{b} \in \text{pos}(\mathbf{a}_i)$ is the collection of those $\mathbf{c} \in \text{pos}(\mathbf{a}_i$ that can be written as a rational combination of the same collections of r of the $\mathbf{a}_i$ as $\mathbf{b}$. The chamber σ is smooth if whenever $\sigma \subset \text{pos}(\mathbf{a}_1, \dots, \mathbf{a}_r)$ we have $\det([\mathbf{a}_1 \dots \mathbf{a}_r]) = \pm 1$.

Question 3. Let the chamber of $\mathbf{b} \in \text{pos}(\mathbf{a}_i)$ be smooth. Is the multiplication map

$$S_{\mathbf{b}} \times \dots S_{\mathbf{b}} \to S_{k\mathbf{b}}$$

surjective for all $k \geq 2$? If $\mathbf{b}$ lives in a smooth chamber, and $\mathbf{c}$ lives in its closure, is the multiplication map

$$S_{\mathbf{b}} \times S_{\mathbf{c}} \to S_{\mathbf{b}+\mathbf{c}}$$

surjective?

The connection with Questions 1 and 2 is seen by taking $P_{\mathbf{b}} = \mathrm{conv}(\mathbf{u} : \deg(x^{\mathbf{u}}) = \mathbf{b})$. Then the first sentence of Question 3 is Question 1, and the second is Question 2.

In algebraic geometry these questions have the following formulation.

Question 4. Let X_Σ be a smooth projective toric variety, and let $\mathcal{L}$ be an ample line bundle on X_Σ. Is the embedding given by the complete linear series of $\mathcal{L}$ projectively normal? In other words, is the multiplication map

$$H^0(X_\Sigma, \mathcal{L}) \otimes \cdots \otimes H^0(X_\Sigma, \mathcal{L}) \to H^0(X_\Sigma, L^{\otimes k})$$

surjective? If $\mathcal{N}$ is a nef line bundle, is the multiplication map

$$H^0(X_\Sigma, \mathcal{L}) \otimes H^0(X_\Sigma, \mathcal{N}) \to H^0(X_\Sigma, \mathcal{L} \otimes \mathcal{N})$$

surjective?

The first of these is a reformulation of Question 1, while the second is a reformulation of Question 2.

All these questions appear in an unpublished manuscript of Oda [2].

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Covering properties of affine monoids

Winfried Bruns

Let $C \subset \mathbb{R}^d$ be a finitely generated rational cone, i. e. the set of all linear combinations $a_1x_1 + \cdots + a_nx_n$ of rational vectors $x_1, \dots, x_d$ with coefficients from $\mathbb{R}_+$. We can of course assume that $x_i \in \mathbb{Z}^d$, $i = 1, \dots, n$. In this note a *cone* is always supposed to be rational and finitely generated. Moreover, we will assume that C is *pointed*: if $x, -x \in C$, then $x = 0$. Finally, it is tacitly understood that C has full dimension d.

The monoid $M(C) = C \cap \mathbb{Z}^d$ is finitely generated by Gordan's lemma (for example, see [4, Section 2.A]). Since C is pointed, M is a *positive* monoid so that 0 is the only invertible element in $M(C)$.

It is not hard to see that $M(C)$ has a unique minimal system of generators that we call its *Hilbert basis*, denoted by $\mathrm{Hilb}(M(C))$ or simply $\mathrm{Hilb}(C)$. It consists of those elements $z \neq 0$ of $M(C)$ that have no decomposition $z = x + y$ in $M(C)$ with $y, z \neq 0$.

We have investigated combinatorial conditions on $\mathrm{Hilb}(C)$ expressing that C or $M(C)$ is covered by certain "simple" subcones or submonoids, respectively. To this end we define a *u-subcone* of C to be a subcone generated by vectors $x_1, \dots, x_d \in \mathrm{Hilb}(C)$ that form a basis of the group $\mathbb{Z}^d$. In particular, $x_1, \dots, x_d$ are linearly independent, and if just this weaker condition is satisfied, then the cone S generated by $x_1, \dots, x_d$ is called an *f-subcone*. In this case we let $\Sigma(S)$ denote the submonoid of $\mathbb{Z}^d$ generated by $x_1, \dots, x_d$.

One says that C satisfies (UHC) if C is the union of its u-subcones. A weaker condition than (UHC) is the *integral Carathéodory property* (ICP): C has (ICP) if every element of $M(C)$ can be written as a linear combination of at most d elements $x_i \in \mathrm{Hilb}(C)$ with coefficients $a_i \in \mathbb{Z}_+$.

Both (UHC) and (ICP) can be formulated more generally for positive affine monoids $M \subset \mathbb{Z}^d$. However, by a theorem of Bruns and Gubeladze [3, Theorem 6.1] (see also [4, 2.B]) a monoid M must be normal if it satisfies (ICP), i. e. it is of the form $M(C)$. In loc. cit. it is also shown that (ICP) is equivalent to the formally stronger condition that M is the union of its submonoids $\Sigma(S)$ where S runs over the f-subcones. (This condition is called (FHC) in [3].) The equivalence is of crucial importance for the algorithm checking (ICP).

While we view (UHC) and (ICP) as structural properties of (normal) affine monoids, these properties have first been discussed in the context of integer programming: see Cook, Fonlupt and Schrijver [6] and Sebő [9].

It was asked by Sebő [9] whether every cone C has (ICP) or (UHC), and he proved that (UHC) holds if $d \leq 3$. A counterexample to (UHC) in dimension 6, called C_{10} in the following, was found by Bruns and Gubeladze [3], and then verified to violate (ICP), too, in cooperation with Henk, Martin, and Weismantel [5]. The counterexample has a Hilbert basis of 10 elements that lie in a hyperplane. Therefore it is the cone over a 5-dimensional polytope P_5, and the lattice points of P_5 form the Hilbert basis of C_{10}. The symmetry group of P_5 is remarkably large: it has 20 elements and acts transitively on the lattice points; see loc. cit.

Another noteworthy property of C_{10} was discovered by Santos [8]: the underlying polytope P_5 is the projection of the Ohsugi-Hibi polytope that has a unimodular triangulation, but no regular unimodular triangulation.

For a long time it remained an open problem whether (UHC) is strictly stronger than (ICP), but in the fall of 2006 the author found examples that satisfy (ICP) but fail (UHC). The smallest of them has a Hilbert basis of 12 elements, again lying in a hyperplane.

The details of the search strategy and the algorithms that decide (UHC) and (ICP) are described in [1]. The article discusses also some aspects of the actual implementation, such as memory requirements and computing times.

Despite the existence of the counterexamples, one can fairly say, at least heuristically, that almost all cones satisfy (UHC). All our experiments seem to indicate that C_{10} is the core counterexample to (ICP) and (UHC). In fact, all counterexamples to these properties that we have been found contain it.

Since the positive results cover cones of dimension 3 and the counterexamples live in dimension 6, it is unclear whether all cones in dimensions 4 and 5 have (UHC). It would be very desirable indeed to clarify the situation.

A nonnormal, very ample polytope of dimension 3. It has been known for a long time that there exist nonnormal lattice polytopes $P \subset \mathbb{R}^d$ with the following property: the set $(P \cap \mathbb{Z}^d) - x$ generates the monoid $\mathbb{R}_+(P - x) \cap \mathbb{Z}^d$ for every vertex x of P. (We assume that $\mathbb{Z}^d$ is the smallest affine lattice containing $P \cap \mathbb{Z}^d$.) It is justified to call such polytopes *very ample* since they correspond to a very ample line bundle $\mathcal{L}$ on a normal projective toric variety V. The embedding of V into projective space afforded by $\mathcal{L}$ is projectively normal if and only if the monoid generated by the vectors $(y, 1) \in \mathbb{Z}^{d+1}$, $y \in P \cap \mathbb{Z}^d$, is normal, in which case P is called *normal*. While every normal polytope is very ample, the converse is disproved by an example of Bruns and Gubeladze [2]: the polytope spanned by the 10 facet-vertex incidence vectors of the minimal triangulation of the projective plane is very ample, but not normal (in the affine lattice generated by them).

While all polytopes of dimension 2 are normal, the author has meanwhile found a nonnormal very ample polytope of dimension 3. All its 8 lattice points are vertices, given by

$$(1, 7, 2),\ (1, 5, 3),\ (1, 4, 4),\ (1, 6, 3),$$
$$(0, 2, 3),\ (0, 0, 4),\ (0, 9, 0),\ (0, 7, 1).$$

It seems very likely that this is the smallest such example. Its integral symmetry group is of order 8.

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Edge Unimodular Polytopes

Benjamin Howard

For simplicity we assume that the lattice is $\mathbb{Z}^d \subset \mathbb{R}^d$. Suppose that $P \subset \mathbb{R}^d$ is a lattice polytope such that the edge directions of P form a unimodular system. This means there is a matrix $M \in \mathbb{Z}^{d\times n}$, where all d by d determinant minors of M are either 0, 1, or -1, and for each edge $\overline{uv}$ of P (where u and v are adjacent vertices of P) there exists a column w of M such that $u - v = kw$ for some $k \in \mathbb{Z}$. If P has this property, we say that P is *edge-unimodular.*

Both Theorem 1 and Corollary 2 below were proven in [3]. The application in [3] was to flag matroid polytopes (see [1]) which are always edge-unimodular, since the edges of flag matroid polytopes are parallel to roots of $SL(n, \mathbb{C})$, and the roots of $SL(n, \mathbb{C})$ are a unimodular system.

Theorem 1. Suppose that M is a unimodular matrix, and that P and Q are lattice polytopes with edges parallel to columns of M. Then, $P \cap \mathbb{Z}^d + Q \cap \mathbb{Z}^d = (P + Q) \cap \mathbb{Z}^d$.

Corollary 2. If P is edge unimodular, then P is normal.

Proof. Suppose that P is edge-unimodular. Let $Q = (k - 1)P$. Then P and Q meet the criterion of Theorem 1, so $P \cap \mathbb{Z}^d + (k - 1)P \cap \mathbb{Z}^d = kP \cap \mathbb{Z}^d$. □

Clearly $P \cap \mathbb{Z}^d + Q \cap \mathbb{Z}^d = (P + Q) \cap \mathbb{Z}^d$ if and only if for all lattice points w, $P \cap (w - Q) \cap \mathbb{Z}^d$ is nonempty whenever $P \cap (w - Q)$ is nonempty. We now consider strengthening this condition in stages. Given a face F of a lattice polytope P, let $\Lambda(F)$ denote the sublattice of $\mathbb{Z}^d$ spanned by the edge directions in F. Given two lattice polytopes P and Q, we say they are *pairwise face unimodular* if for any face F of P and face G of Q, the abelian group $\mathbb{Z}^d/(\Lambda(F) + \Lambda(G))$ is torsion-free. The following list shows successively stronger conditions on the pair P, Q of lattice polytopes:

(1) For all $w \in \mathbb{Z}^d$, if $P \cap (w - Q) \neq \emptyset$ then $P \cap (w - Q) \cap \mathbb{Z}^d \neq \emptyset$.
(2) For all $w \in \mathbb{Z}^d$, the intersection $P \cap (w - Q)$ is a lattice polytope (possibly empty).
(3) The polytopes P, Q are pairwise face unimodular.
(4) The edge directions of P together with the edge directions of Q all fit into one unimodular matrix, as in the premise of Theorem 1.

One can easily prove that

$$(4) \implies (3) \implies (2) \implies (1).$$

Here we will list examples (all due to Francisco Santos) where (1) does not imply (2) and where (3) does not imply (4). However I still don't know an example where (2) does not imply (3), but I doubt that (2) and (3) are equivalent.

Example 3. (The stop sign) Let $P = Q$ be the smooth polygon which is the convex hull of the points $(1, 0)$, $(2, 0)$, $(3, 1)$, $(3, 2)$, $(2, 3)$, $(1, 3)$, $(0, 2)$, $(0, 1)$. Let $w = (3, 1)$. Then $P \cap (w - Q)$ has vertices $(1, 0)$, $(2, 0)$, $(5/2, 1/2)$, $(2, 1)$, $(1, 1)$, and $(1/2, 1/2)$, two of which are non-integral. This example shows that (1) does not imply (2), since all lattice polygons are normal.

Example 4. (The cube with three truncated edges) Start with the 3-cube with side lengths 3. We position the cube with one vertex at the origin and the opposite vertex at (3,3,3). Let $\mathbf{e} = \mathrm{conv}((0,0,0),(0,0,3))$, $\mathbf{f} = \mathrm{conv}((3,0,0),(3,3,0))$, and $\mathbf{g} = \mathrm{conv}((0,3,3),(3,3,3))$. Now truncate these three edges by cutting at a 45° angle to the cube. For example, truncation of edge $\mathbf{e}$ has the effect of removing the vertices of $\mathbf{e}$ and replacing them with $(1,0,0)$,$(1,0,1)$,$(0,1,0)$,$(0,1,1)$. This results in a polytope with nine facets. Let P be this polytope. Now, the edge directions of P are all parallel to columns of the matrix

$$M = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{pmatrix},$$

which is not unimodular since the determinant of the right-most 3 by 3 minor is equal to 2. However any other minor has determinant ± 1. Furthermore for any pair F, G of proper faces of P there is a column among the final three columns of M which isn't parallel to any edge of F or G. On the other hand if F or G is equal to the entire polytope P then $\Lambda(F) + \Lambda(G) = \mathbb{Z}^3$. In either case the quotient group $\mathbb{Z}^d/(\Lambda(F) + \Lambda(G))$ is torsion-free. Taking $Q = P$, this shows that (3) does not imply (4).

There was a debate at Oberwolfach as to whether the condition that P is edge-unimodular is equivalent to P being facet-unimodular, which means that the directions perpendicular to the facets of P are unimodular. It has been shown that any facet-unimodular polytope is normal. In fact, any facet-unimodular polytope has a unimodular triangulation [2, Prop 2.4, p 60].

However, it turns out that edge-unimodular does not imply facet-unimodular, and neither does facet-unimodular imply edge-unimodular. The following two examples are both due to Francisco Santos:

Example 5. (Edge-unimodular but not facet-unimodular) Let P be the three dimensional permutahedron, which is the convex hull of all 24 component-permutations of the vector $(3, 2, 1, 0)$. This a flag matroid polytope (corresponding to the full flag variety $SL(4)/B$) and so it is edge-unimodular. However, P is not facet-unimodular.

Example 6. (Facet-unimodular but not edge-unimodular) Let P be the Birkhoff polytope B_3, equal to the convex hull of the six permutation matrices in $\mathbb{R}^{3\times 3}$. It is known that P is a transportation polytope, and all transportation polytopes are facet-unimodular. But P is not edge-unimodular.

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Smooth edge polytopes

Hidefumi Ohsugi and Takayuki Hibi

Let G be a finite graph on the vertex set $V(G) = \{1, \ldots, d\}$ allowing loops and having no multiple edges, and $E(G) = \{e_1, \ldots, e_n\}$ the set of its edges (and loops). If $e = \{i, j\}$ is an edge of G between $i \in V(G)$ and $j \in V(G)$, then we define $\rho(e) = \mathbf{e}_i + \mathbf{e}_j$. Here $\mathbf{e}_i$ is the ith unit coordinate vector of $\mathbb{R}^d$. In particular, for a loop $e = \{i, i\}$ at $i \in V(G)$, one has $\rho(e) = 2\mathbf{e}_i$. The *edge polytope* of G is the convex polytope $\mathcal{P}_G$ ($\subset \mathbb{R}^d$) which is the convex hull of the finite set $\{\rho(e_1), \ldots, \rho(e_n)\}$.

If $e = \{i, j\}$ is an edge of G, then $\rho(e)$ cannot be a vertex of $\mathcal{P}_G$ if and only if $i \neq j$ and G has a loop at each of the vertices i and j. With considering this fact, we assume that G satisfies the following condition:

($*$) If i and j are vertices of G and if G has a loop at each of i and j, then the edge $\{i, j\}$ belongs to G.

Let $K[\mathbf{t}] = K[t_1, \ldots, t_d]$ denote the polynomial ring in d variables over K. If $e = \{i, j\}$ is an edge of G, then $\mathbf{t}^e$ stands for the monomial $t_i t_j$ belonging to $K[\mathbf{t}]$. Thus in particular, if $e = \{i, i\}$ is a loop of G at $i \in V(G)$, then $\mathbf{t}^e = t_i^2$. The *edge ring* of G is the affine semigroup ring $K[G]$ ($\subset K[\mathbf{t}]$) which is generated by $\mathbf{t}^{e_1}, \ldots, \mathbf{t}^{e_n}$ over K.

Let $K[\mathbf{x}] = K[x_1, \ldots, x_n]$ denote the polynomial ring in n variables over K. The *toric ideal* of G is the ideal I_G ($\subset K[\mathbf{x}]$) which is the kernel of the surjective ring homomorphism $\pi : K[\mathbf{x}] \to K[G]$ defined by setting $\pi(x_i) = \mathbf{t}^{e_i}$ for $i = 1, \ldots, n$.

By using combinatorial technique, we determine all graphs G for which $\mathcal{P}_G$ is simple. From this classification, it follows that

Theorem. *Let G be a finite graph allowing loops and having no multiple edge, and suppose that G satisfies the condition ($*$). Then the following conditions are equivalent:*

(i) *$\mathcal{P}_G$ is smooth ;*
(ii) *$\mathcal{P}_G$ is simple.*

Moreover if $\mathcal{P}_G$ is simple, then the toric ideals I_G possesses a squarefree quadratic initial ideal. (In particular, $K[G]$ is normal and Koszul.)

Vanishing theorems

Milena Hering

The purpose of this lecture is to review basic properties of line bundles on projective varieties, and to present some vanishing theorems that are helpful for understanding properties of embeddings induced by line bundles, such as projective normality and quadratic generation of the ideal cutting out the image.

We first recall some basic notions for line bundles on projective varieties. Let L be a line bundle on a projective variety X. Recall that L is called *globally generated*, if for every point p in X, there exists a global section of L not vanishing at p. This implies that the global sections $H^0(X, L)$ induce a morphism, $\phi_L : X \to \mathbb{P}\left(H^0\left(X, Ł\right)\right)$. If this morphism is an embedding, L is called *very ample*. Moreover, L is called *ample* if $L^{\otimes m}$ is very ample for some m. We are interested in the question when a line bundle is *normally generated*, i.e., the natural map $\mathrm{Sym}^m H^0(X, L) \to H^0(X, L^m)$ is surjective for all $m \geq 0$. An ample and normally generated line bundle is very ample, and if X is normal, it gives rise to a projectively normal embedding, i.e., the homogeneous coordinate ring of $\phi_L(X) \subset \mathbb{P}\left(H^0\left(X, L\right)\right)$ is integrally closed.

These properties translate into the toric world as follows. Let $M \cong \mathbb{Z}^d$ be a lattice, and let $P \subset M_{\mathbb{Z}} \otimes \mathcal{R}$ be a lattice polytope. Then P corresponds to a normal toric variety $X = X_P$, together with an ample line bundle L. An arbitrary lattice polytope Q corresponds to a globally generated (ample) line bundle on X, if its normal fan is refined by (equal to) the normal fan to P. Moreover, if Q corresponds to an ample line bundle on X, it corresponds to a very ample line bundle if and only if for every vertex v of Q, the Hilbert basis of the cone generated by $\{u - v \mid u \in Q\}$ is contained in $(Q - v) \cap M$. In particular, every ample line bundle on a smooth toric variety is very ample. Morever, the line bundle corresponding to Q is normally generated, if the natural map

$$\underbrace{Q \cap M + \cdots Q \cap M}_{m} \to mQ \cap M$$

is surjective for all $m \geq 0$.

In the following we review some very basic cases of classical vanishing theorems guaranteeing projective normality. Their proof is due to Lazarsfeld and relies on the theory of Koszul cohomology developed by Mark Green in [2]. For more details see for example . [3].

First, we give a criterion for projective normality of a very ample line bundle in terms of vanishing of twists of the ideal sheaf of the corresponding line bundle.

Lemma 7. Let L be very ample, and let $\mathcal{I}_X$ be the ideal sheaf of the embedding induced by L. Then L is normally generated if and only if

$$H^1(\mathbb{P}(H^0(X, L)), \mathcal{I}_X(m)) = 0 \text{ for all } m \geq 0.$$

However, the main point of this note is to relate projective normality and quadratic generation to cohomology vanishing of certain vector bundles associated to a globally generated line bundle. Let L be globally generated. Then there exists a short exact sequence

$$0 \to M_L \to H^0(X, L) \otimes \mathcal{O}_X \to L \to 0.$$

Tensoring this sequence with a line bundle L', and taking long exact sequence of cohomology, we obtain the following condition for multiplication maps to be surjective.

Lemma 8. Let L, L' be globally generated line bundles on a projective variety X. If $H^1(X, M_L \otimes L') = 0$, then the natural map $H^0(X, L) \otimes H^0(X, L') \to H^0(X, L \otimes L')$ is surjective. Moreover, if $H^1(X, L') = 0$, the converse holds.

In particular, we get the following criterion for projective normality.

Proposition 9. Let L be a globally generated line bundle. If $H^1(X, M_L \otimes L^m) = 0$ for all $m \geq 1$, then L is normally generated. Moreover, if $H^1(X, L^m) = 0$ for all m, the converse holds.

There is a similar criterion governing the degrees of the generators of the ideal I_X of the image of X under the map induced by L.

Lemma 10. Let L be a globally generated line bundle on X. Suppose

$$H^1(X, \bigwedge^2 M_L \otimes L^j) = 0$$

for $j \geq \ell$. Then I_X is generated in degree $\ell + 1$.

In particular, for $\ell = 1$, the lemma implies that the ideal I_X is generated by quadric equations.

When we work of a field of characteristic zero, it suffices to prove the vanishing of $H^1(X, M_L^{\otimes 2} \otimes L^j)$ for $j \geq \ell$. Or that the natural map

$$H^0(X, L) \otimes H^0(X, M_L \otimes L^j) \to H^0(X, M_L \otimes L^{j+1})$$

is surjective for all $j \geq \ell$.

Remark 11. There exist similar vanishing theorems guaranteeing that the homogeneous coordinate ring of the embedding is Koszul, see for example [4]; the above vanishing theorems extend to criteria for the line bundle to satisfy Greens property N_p, see [3].

Using this vanishing theorem, Ein and Lazarsfeld prove the following theorem.

Theorem 12 ([1])**.** Let X be a smooth projective variety of dimension n, A very ample on X, and N nef. Assume that $A \ncong \mathcal{O}_{\mathbb{P}^n}(1)$. Then $\mathrm{L} = K_X \otimes A^{n+1} \otimes N$ is normally generated, and the ideal of the embedding induced by L is generated by quadratic equations.

For lattice polytopes this has the following consequences. We call a lattice polytope smooth, if it corresponds to an ample line bundle on a smooth toric variety. Equivalently, the primitive lattice vectors spanning the rays of each maximal cone of the normal fan form a lattice basis.

Corollary 13. Let P be a smooth polytope of dimension n, and let Q be a polytope whose normal fan is refined by that of P. Then $\mathrm{conv}\langle \mathrm{int}(nP) + Q\rangle$ is a normal polytope.

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A syzygy approach to projective normality

Hal Schenck

For any variety X, if D is a very ample divisor with $h0(D) = n+1$, then for all $i \geq 1$ there are isomorphisms

$$H1(\mathcal{I}_X(t)) \simeq H^{i+1}_m(I)_t \simeq Ext^n(S/I, S)_{-t-n-1},$$

where $S = k[x_0, \ldots, x_n]$, H^i_m is the local cohomology functor at $m = \langle x_0, \ldots, x_n\rangle$, and I is the ideal of X in the embedding induced by D.

Hochster gives a beautiful recipe for computing $Tor_i(S/I, k)_b$, where b is a multidegree, in particular one can associate a simplicial complex Δ_b:

$$Tor_i(I, k)_b \simeq \widetilde{H}^i(\Delta_b).$$

An explicit description of Δ_b may be found in Sturmfels, "Gröbner bases and Convex polytopes". Translating, we find that failure of projective normality is encoded by the existence of a semigroup weight b which corresponds to a copy of S^{n-1} on $n+1$ vertices. For the Bruns-Gubeladze example of a triangulation of $RP2$, this gives a very concrete description of failure of projective normality. Question: is this criterion actually useful in hunting counterexamples (or finding a proof)? (Joint work with Greg Smith).

• Report of session on resolving the singularities of the polytope

$$P = conv\{(0,0,0), (1,0,0), (0,1,0), (1,1,2)\},$$

(Bruns, Gubeladze, Santos, Schenck, Trung)

The normal fan of P consists of 4 cones, each of which is simplicial. A check shows that each cone has multiplicity four, hence each cone needs to be subdivided into four cells. This can be done (minimally) in two ways, and results in two different smooth polytopes (remark: we chose to move from the fan approach to dilating P and truncating to desingularize).

One of the two resulting smooth polytopes was obtained by taking a cube, and pruning at the vertices. In this case, the resulting normal fan has 14 rays. The second polytope consists of a pair of plane octagons, embedded in parallel planes at different heights. In this case, the resulting normal fan has 10 rays. Both of these families can be shown to have unimodular covers, and so for this specific example, there were no counterexamples to the conjecture.

Frobenius splittings of toric varieties

Sam Payne

We discuss Frobenius splittings of toric varieties with a view toward questions about projective normality and quadratic generation for projective embeddings. Frobenius splittings are notorious in the context of toric varieties for their role in unsuccessful attempts to prove that smooth projective toric varieties embedded by complete linear series are always projectively normal and cut out by quadrics. See, for instance, [2]. Nevertheless, these unsuccessful attempts do inspire hope that Frobenius splittings may be a useful tool for studying questions about projective normality and quadratic generation of toric varieties.

Frobenius splittings are a positive characteristic technique developed by Mehta, Ramanathan, and their collaborators in the 1980s. The original paper of Mehta and Ramanathan [4] is exceedingly well written and remains an excellent first introduction to the subject. For a more comprehensive exposition, see the recent book of Brion and Kumar [1]. Frobenius splittings were rapidly applied to give elegant unified proofs that all ample line bundles on generalized Schubert varieties of all types are very ample and give projectively normal embeddings whose images are cut out by quadrics [6, 5]. In characteristic zero, these results are deduced from the positive characteristic case using general semicontinuity theorems. On toric varieties, the Frobenius endomorphisms lift to endomorphisms over $\mathbb{Z}$, and it is easiest to work independently of the characteristic using these lifted endomorphisms, as follows.

Fix an integer $m \geq 2$. Let T be a torus with character lattice M, and let $N = \mathrm{Hom}(M, \mathbb{Z})$ be the dual lattice. Let Σ be a complete fan in $N_{\mathbb{R}}$, with $X = X(\Sigma)$ the associated toric variety with dense torus T. Multiplication by m on $N_{\mathbb{R}}$ preserves the fan Σ and maps the lattice N into itself, and therefore gives an endomorphism

$$F : X \to X.$$

The restriction of F to the dense torus T is given by $t \mapsto t^m$, and this determines F uniquely. In the special case where the base field is $\mathbb{F}_p$ and $m = p$, F is the absolute Frobenius morphism on X. Pulling back functions by F gives a natural inclusion of $\mathcal{O}_X$-algebras

$$F^* : \mathcal{O}_X \hookrightarrow F_*\mathcal{O}_X.$$

Definition 3. A splitting of X is an $\mathcal{O}_X$-module map $\pi : F_*\mathcal{O}_X \to \mathcal{O}_X$ such that the composition $\pi \circ F^*$ is the identity on $\mathcal{O}_X$.

From the existence of a splitting, standard arguments show that for every ample line bundle L on X, $H^i(X, L) = 0$ for $i > 0$ [4, Proposition 1]. These arguments are simple, using only the projection formula, that sheaf cohomology commutes with direct sums, and that $F^*L \cong L^m$ for all line bundles L on X. A remarkable feature of this approach is that one can obtain information on all ample line bundles on X simultaneously from a single map of coherent $\mathcal{O}_X$-modules.

Proposition 14. Every toric variety has a canonical splitting.

Proof. (Sketch.) Suppose $X = U_\sigma$ is affine. Then $k[U_\sigma]$ and $F_*k[U_\sigma]$ are naturally identified with the semigroup rings $k[\sigma^\vee \cap M]$ and $k[\sigma^\vee \cap \frac{1}{m}M]$, respectively. Then the canonical splitting π_0 is given for $u \in \frac{1}{m}M$ by

$$\pi_0(x^u) = \begin{cases} x^u & \text{if } u \in M, \\ 0 & \text{otherwise.} \end{cases}$$

The general result follows from the affine case by gluing. □

Properties of this canonical splitting π_0 are closely related to Smith's proof that toric varieties are globally F-regular [7, Proposition 6.3]. It follows from the existence of a splitting that the higher cohomology of ample line bundles on toric varieties must vanish. The standard proof of this vanishing is quite different and uses a topological interpretation of the cohomology of line bundles on toric varieties and the convexity of the support functions associated to ample toric line bundles. See [3] for details.

To apply the standard machinery of Frobenius splittings to questions about projective normality and quadratic generation, we will need to look at splittings of toric varieties other than the canonical splitting π_0. Typically, we will be interested in splittings of $X \times X$ that are compatible with the diagonal, and splittings of $X \times X \times X$ that are compatible with large semidiagonals, in the following sense.

Let $Y \subset X$ be a subvariety cut out by an ideal sheaf I.

Definition 4. A splitting $\pi : F_*\mathcal{O}_X \to \mathcal{O}_X$ is compatible with Y if $\pi(F_*I) = I$.

Since π is a splitting, the image of $F_*(I)$ must contain I, so compatibility with Y is the requirement that the image of $F_*(I)$ must be contained in I. It follows from the short exact sequence $0 \to I \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$ that a splitting of X compatible with Y induces a splitting of Y; this motivates the definition. Standard arguments show that if X is split compatibly with Y and L is an ample line bundle on X, then the restriction map $H^0(X, L) \to H^0(Y, L|_Y)$ is surjective. See [4, Proposition 3].

Example 15. The canonical splitting π_0 is compatible with all T-invariant subvarieties of X.

Example 16. If X is positive dimensional, the canonical splitting of $X \times X$ is not compatible with the diagonal Δ. To see this, observe that if $u \in \frac{1}{m}M$ is not in M, then $1 - x^u \otimes x^{-u}$ is in F_*I_Δ, but $\pi_0(1 - x^u \otimes x^{-u}) = 1$, which is not in I_Δ.

Suppose $X \times X$ is split compatibly with the diagonal, and let L and L' be ample line bundles on X. It follows from the existence of a compatible splitting that the restriction map

$$H^0(X \times X, p_1^*L) \otimes H^0(X \times X, p_2^*L') \to H^0(\Delta, (p_1^*L \otimes p_2^*L')|_\Delta)$$

is surjective. Since this restriction agrees with product map

$$H^0(X, L) \otimes H^0(X, L') \to H^0(X, L \otimes L'),$$

it follows by taking $L = L'$ that every ample line bundle on X gives a projectively normal embedding [6]. A slightly more intricate argument shows that if $X \times X \times X$ is split compatibly with the union of the large semidiagonals $\Delta \times X$ and $X \times \Delta$, then the images of all such embeddings of X are cut out by quadrics [5]. Since the canonical splitting of $X \times X$ is not compatible with Δ, to apply these standard techniques we must look for other splittings of $X \times X$ and $X \times X \times X$ and ask which of these, if any, are compatible with the diagonal and the union of the large semidiagonals, respectively.

Some progress has been made toward giving combinatorial characterizations of toric varieties X such that $X \times X$ is split compatibly with the diagonal and such that $X \times X \times X$ is split compatibly with the union of the large semidiagonals, and these efforts and some partial results were discussed during the workshop. Details may appear elsewhere.

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Normality and Integer Linear Programming

Ngo Viet Trung

Theme: Methods of Integer Linear Programming can be used to study the normality of certain polytopes.

1. Integer Round-up Property

Let $P \subset \mathbb{R}^n$ be a rational polyhedron. We say that P has the integer decomposition property (ID) if $x \in k\dot{P} \cap \mathbb{Z}^n, k \geq 1$ implies $x = x_1 + \cdots + x_k$ for $x_i \in P \cap \mathbb{Z}^n$. If P is a lattice polytope, then ID means nothing else than P is normal.

It is known that ID of certain rational polyhedra can be characterized by the so-called integer rounding properties of Integer Linear Programming.

Let $v_1, ..., v_m$ be non-negative integral vectors in $\mathbb{R}^n$ such that $v_i \not\leq v_j$ for all $i \neq j$. Let

$$P := \operatorname{conv}\{x \in \mathbb{N}^n : \ \exists\, i \text{ s.t. } x \leq v_i\}.$$

The normality of P can be studied by means of the matrix $A = (v_1, ..., v_m)$.

We say that A has the integer round-up property (IRU) if

$$\min\{|y| : \ y \geq 0 \text{ integral}, Ay \geq c\} = \lceil \min\{|y| : \ y \geq 0, Ay \geq c\} \rceil$$

for all $c \in \mathbb{Z}^n$. Note that IRU can be tested in polynomial time.

Theorem 17. [1] P has ID iff A has the IRU.

An instance of the above class of polytopes is the knapsack polytope:

$$P = \operatorname{conv}\{x \in \mathbb{N}^n |\ ax \leq \lambda\},$$

where $a \in \mathbb{N}^n$ and $\lambda \in \mathbb{N}$ are given. There have been a lot works in finding knapsack polytopes which have or don't have IRU.

Now we will present a class of matrices with IRU.

Let G be a simple graph. A colouring of G is an assignment of colours to the vertices such that adjacent vertices have different colours. Let $c(G)$ denote the minimal number of colours of colourings of G. It is clear that

$$c(G) \geq \max\{|K| : \ K \text{ is a clique of } G\}.$$

One calls G perfect if equality holds above for all induced subgraphs of G.

Let A now be the incidence matrix of the maximal cliques of G. It is known that G is perfect iff the system $xA \leq 1$ is tottally dual integral (TDI), i.e. the equation

$$\max\{cx : x \geq 0, xA \leq 1\} = \min\{|y| : \ y \geq 0, Ay \geq c\}$$

has an integral optimal solution y for all $c \in \mathbb{Z}^n$. It is obvious that TDI implies IRU.

Corollary 18. P has ID if A is the incidence matrix of the maximal cliques of a perfect graph.

2. Normal square-free monomial ideals

Let $R = K[X]$ be a polynomial ring. Let $X^{v_1}, ..., X^{v_m}$ be square-free monomials and $I = (X^{v_1}, ..., X^{v_m})$. We want give a combinatorial condition for I to be a normal ideal, that is $\overline{I^k} = I^k$ for all $k \geq 1$. It is obvious that I is normal iff the Rees algebra $R[It]$ is normal.

If the monomials $X^{v_1}, ..., X^{v_m}$ have the same degree, say d, we denote by Q the lattice polytope spanned by the vectors $(e_1, 0), ..., (e_n, 0), (v_1, 1), ..., (v_m, 1)$ in the hyperplane $x_1 + \cdots x_n = (d-1)x_{n+1} + 1$ of $\mathbb{R}^{n+1}$, where $e_1, ..., e_n$ are the unit vectors of $\mathbb{R}^n$. In this case, $R[It]$ is the polytopal ring of Q. In particular, if P is the polytopes spanned by the vectors $v_1, ..., v_m$, we may identify P with a facet of Q. Therefore, if I is normal, then Q and hence P is normal.

Since I is generated by square-free monomials, I is an intersection of prime ideals: $I = P_1 \cap \cdots \cap P_s$. One calls the ideal $I^{(k)} := P_1^k \cap \cdots \cap P_s^k$ the kth symbolic power of I. We have

$$I^k \subseteq \overline{I^k} \subseteq I^{(k)}.$$

Therefore, I is normal if $I^{(k)} = I^k$ for all $k \geq 1$.

Let $A = (v_1, ..., v_m)$. We say that A has the max-flow min-cut property (MFMC) if

$$\max\{|y| : \; y \geq 0 \text{ integral}, Ay \leq c\} = \min\{cx : \; x \geq 0 \text{ integral}, xA \geq 1\}$$

for all $c \in \mathbb{N}^n$.

Theorem 19. [2] [3] $I^{(k)} = I^k$ for all $k \geq 1$ iff A has MFMC.

For instance, A has MFMC if A is a balanced matrix, i.e. A has no square submatrix of odd size of the form

$$\begin{pmatrix} 1 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 1 & 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ \cdot & & & & & 0 & 1 \\ 0 & \cdot & \cdot & \cdot & \cdot & 1 & 1 \end{pmatrix}$$

In general, we have

$$\begin{aligned} &\max\{|y| : \; y \geq 0 \text{ integral}, Ay \leq c\} \leq \max\{|y| : \; y \geq 0, Ay \leq c\} \\ &\qquad = \min\{cx : \; x \geq 0, xA \geq 1\} \leq \min\{cx : \; x \geq 0 \text{ integral}, xA \geq 1\}. \end{aligned}$$

Theorem 20. [2] [6] $I^{(k)} = \overline{I^k}$ for all $k \geq 1$ iff

$$\min\{cx : \; x \geq 0, xA \geq 1\} = \min\{cx : \; x \geq 0 \text{ integral}, xA \geq 1\}$$

for all $c \in \mathbb{N}^r$.

One may guess that $\overline{I^k} = I^k$ for all $k \geq 1$ iff

$$\max\{|y| : \; y \geq 0 \text{ integral}, Ay \leq c\} = \max\{|y| : \; y \geq 0, Ay \leq c\}$$

for all $c \in \mathbb{N}^r$. But that is not true. Instead of that we find the following condition.

We say that A has the integer round-down property (IRD) if

$$\max\{|y|: \; y \geq 0 \text{ integral}, Ay \leq c\} = \lfloor \max\{|y|: \; y \geq 0, Ay \leq c\} \rfloor$$

for all $c \in \mathbb{N}^n$.

Theorem 21. [7] *I is normal iff A has IRD.*

If I is generated by monomials of degree 2, then I is the edge ideal of a graph. In this case, one can give a characterization for the normality of I in terms of the underlying graph [4], [5].

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Lattice points in Minkowski sums of lattice polygons

Najmuddin Fakhruddin

The following theorem was proved in [1]:

Theorem 1. *Let X be a smooth projective toric surface, $\mathcal{L}$ an ample line bundle on X, and $\mathcal{M}$ a line bundle on X which is generated by global sections. Then the multiplication map $H^0(X, \mathcal{L}) \otimes H^0(X, \mathcal{M}) \to H^0(X, \mathcal{L} \otimes \mathcal{M})$ is surjective.*

The combinatorial description of toric varieties and base point free line bundles on them shows that the above theorem is equivalent to the following purely combinatorial result:

Theorem 2. *Let P and Q be convex lattice polygons with the normal fan at each vertex of P being unimodular and the normal cone of P refining that of Q. Then*

$$P_{\mathbb{Z}} + Q_{\mathbb{Z}} = (P + Q)_{\mathbb{Z}} \; .$$

In the above $P + Q$ denotes the Minkowski sum of P and Q and the subscript $\mathbb{Z}$ on a polygon denotes the lattice points in that polygon.

The proof consists in reducing to the case that Q is a triangle of a specific type which allows one to check the equality explicitly. The vertex unimodularity of P is used to show that the triangle produced by the reduction step has vertices which are lattice points.

During the workshop S. Payne suggested that the Oda conjecture could be extended to all convex lattice polytopes by putting an appropriate condition on lattice lengths of edges. In the case of polygons there are no extra conditions; this suggests that Theorem 2 should hold without the unimodularity condition on P or, in the geometric language of Theorem 1, without the nonsingularity hypothesis on the toric surface.

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On Fanos and Chimneys

Christian Haase and Andreas Paffenholz

A lattice polytope which has a (regular) unimodular triangulation is normal – see the hierarchy of properties listed later in this volume, in the section "What else do we know?". We use this fact to describe a surprisingly effective method to prove projective normality (and more) for smooth reflexive polytopes.

Let $P \subset \mathbb{R}^d \times \{1\}$ be a lattice polytope. To each such polytope we can associate a toric ideal I_P in the following way. Let $A = \{a_1, \ldots, a_n\}$ be the set of lattice points in P. for any $a \in A$ there is a monomial $t^a = t^{a^1} \cdots t^{a^d}$. The *toric ideal* I_P is defined as the kernel of the map

$$\begin{array}{ccc} \Bbbk[x_1, \ldots, x_n] & \longrightarrow & \Bbbk[t^{a_1}, \ldots, t^{a_n}] \\ x_i & \longmapsto & t^{a_i}\ . \end{array}$$

A triangulation T of P is *unimodular* if every simplex has normalized volume 1. T is *regular* if there exists a height function $h := \mathcal{V}(P) \to \mathbb{R}$ such that the projection of the lower hull of $\operatorname{conv}\{(v, h(v)) \mid v \in \mathcal{V}(P)\}$ is T. A *non-face* in a triangulation is a set $S \subseteq A$ of the vertices of T that does not define a face of the triangulation. It is *minimal* if any proper subset defines a face.

For $S \subseteq A$ we write x^S for the monomial $\prod_{a_i \in S} x_i$. The *Stanley-Reisner ideal* of a triangulation is the monomial ideal

$$\mathcal{I}_T := \langle x^S \mid S \text{ is a non-face of } T \rangle.$$

The connection between regular unimodular triangulations and toric ideals is given by the following theorem.

Theorem 22 (Sturmfels [4])**.** The toric ideal I_P has a square-free initial ideal if and only if P has a regular unimodular triangulation.

In this case, the initial ideal coincides with the Stanley-Reisner ideal of the triangulation.

Hence, to prove that some toric ideal has a square-free initial ideal it suffices to construct a regular unimodular triangulation of the associated polytope. Not only does such a triangulation imply projective normality, knowledge about the minimal non-faces also yields degree bounds for Gröbner bases.

We applied this method to the classes of d-dimensional smooth reflexive polytopes for $d \leq 7$. A lattice polytope is called *reflexive*, if it contains 0 and its polar polytope is again a lattice polytope. The polar of a smooth reflexive polytope is a *Fano polytope*. Explicit representations of these Fano polytopes for $d \leq 7$ were recently computed by Mikkel Øbro [3]. There are $5, 18, 124, 866, 7622$ and 72256 smooth reflexive polytopes in dimensions $2, 3, 4, 5, 6$ and 7 respectively.

Theorem 23. All smooth reflexive d-polytopes for $d \leq 7$ are normal. All but 8 (out of 7622) of the 6-dimensional and 120 (out of 72256) of the 7-dimensional smooth reflexive polytopes have a regular unimodular triangulation.

Our construction of a regular unimodular triangulation proves normality (in fact, quadratic generation) for all but the 128 exceptional polytopes. They may well have such a triangulation. For the remaining ones we checked normality using the program `enormalz` by Bruns and Koch [1]. Previously it was shown by Lindsay Piechnik that all smooth reflexive d-polytopes have a regular unimodular triangulation for $d \leq 4$.

We sketch our method. Let $Q \subset \mathbb{R}^d$ be a lattice polytope. For two integral linear functionals l, u with $l \leq u$ along Q we define the *chimney polytope*

$$P := \{(x, y) \in \mathbb{R}^d \times \mathbb{R} \mid x \in Q, l(x) \leq y \leq u(x)\}$$

associated to Q, l and u. This is again a lattice polytope (see Figure 1).

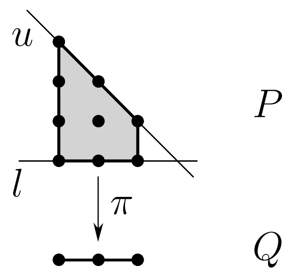

FIGURE 1. An example of the chimney construction. Here $l \equiv 0$ and $u(x) = 3 - x$.

Assume that the polytope Q has a regular unimodular triangulation. This defines a subdivision of the polytope P by intersecting P with the infinite prisms over the simplices of the triangulation. Any maximal triangulation of such prisms

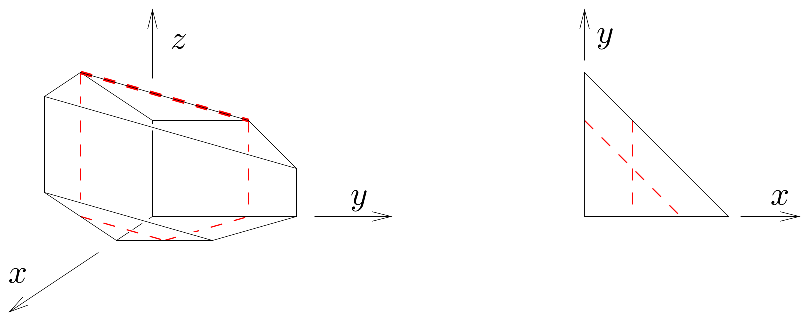

FIGURE 2. The projections P_{xyz} and P_{xy}

is unimodular. Subdividing all of P in this way we obtain a regular unimodular triangulation of P.

This gives a simple method to check whether a lattice polytope admits a regular unimodular triangulation. Namely, given a lattice polytope P, we can search for unimodular transformations Φ of P such that $\Phi(P)$ has the above form. Project to Q and check whether Q has a regular unimodular triangulation. Iterate.

More generally, in the above construction we can allow more than one functional bounding from below or above. We have to use a refinement of this subdivision in the projection. To this end we define a *push-forward* of a subdivision.

Here is an example. Consider the following polytope given by eight inequalities in variables x, y, z, w.

$$\begin{array}{rcccl} 0 & \le & x & & \\ 0 & \le & y & \le & 3-x \\ 0 & \le & z & & \\ x-1 & \le & z & & \\ 0 & \le & w & \le & 2+x-z \\ & & w & \le & 4-y-z \end{array} \tag{1}$$

We have ordered the inequalities so that each variable is bounded above or below by integral linear functionals in the previous variables. We want to project P to x-y-z-space. This projection P_{xyz} has the representation (see Figure 2 on the left)

$$\begin{array}{rcccl} 0 & \le & x & & \\ 0 & \le & y & \le & 3-x \\ 0 & \le & z & \le & 2+x \\ x-1 & \le & z & \le & 4-y\,. \end{array}$$

Observe that P_{xyz} has facets $z \le 2+x$ and $z \le 4-y$ whose pull-backs are not facets of P. They are implied by the inequalities $0 \le w$ and $w \le 2+x-z$, respectively $w \le 4-y-z$.

The push-forward of the trivial subdivision of P divides P_{xyz} along the plane $x+y=2$, the projection of the ridge formed by the two upper bounds on w in (1),

$$\left.\begin{array}{rcccl} 0 & \le & w & \le & 2+x-z \\ & & w & \le & 4-y-z \end{array}\right\} x+y=2\,.$$

This is a lattice subdivision, as the intersection of this hyperplane with P_{xyz} is the convex hull of the lattice points $(1,1,0)$, $(0,2,0)$, $(0,2,2)$, $(2,0,4)$, and $(2,0,1)$.

We can project this again to obtain a subdivided polytope P_{xy} in the x-y-plane given by the inequalities $0 \le x$ and $0 \le y \le 3-x$ (see Figure 2 on the right). Any (regular and unimodular) triangulation of this subdivision can be used to construct a triangulation of P.

We implemented this method using the software package `polymake` by Gawrilow and Joswig [2]. We applied it iteratively to each polytope and checked whether we can find a sequence of projections down to dimension 2, where we know that any lattice polytope admits a regular unimodular triangulation.

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Normality and Hadamard simplices

Francisco Santos

In this note we explore how to construct non-normal polytopes and smooth polytopes based on the Hadamard simplices. Unfortunately, we do not get polytopes with both properties at the same time.

1. Hadamard simplices; an introduction

Let C be the ± 1 cube in $\mathbb{R}^d$. That is, $C = [-1,1]^d$. Let Δ be any regular simplex with vertices contained in those of C (Δ exists only for certain values of d. See below). We call Δ a *Hadamard simplex*. It is easy to check that:

Lemma 24. A subset $\{v_1, \ldots, v_{d+1}\} \subset \{-1,+1\}^d$ is the vertex set of a Hadamard simplex if and only if any of the following equivalent conditions holds:

(1) The $(d+1)\times(d+1)$ matrix with columns $\{(v_1,1),\ldots,(v_{d+1},1)\}$ equals $\sqrt{d+1}$ times an orthogonal matrix. Such a ± 1 matrix is called a *Hadamard matrix* (cf. [1], Chapter 3, 2.13; for a recent survey on Hadamard matrices see [3]).

(2) The Hamming distance between any two of the v_i's equals $(d+1)/2$. As usual, the Hamming distance between two vectors is the number of coordinates on which they differ. In $\{-1,+1\}^d$ the Hamming distance is half the L_1-distance.

These descriptions imply the following basic properties of Hadamard simplices:

- For a Hadamard simplex to exist in dimension $d > 1$, $d+1$ must be a multiple of four. Indeed, the three Hamming distances between three vertices v_i, v_j and v_k of Δ should be equal to $(d+1)/2$. But at least one of them is even. The *Hadamard conjecture* is that this condition on d is also sufficient. It has been verified up to $d+1 = 664$ [3, 2].
- Since the tensor product of Hadamard matrices is a Hadamard matrix, Hadamard simplices exist in at least all dimensions of the form $2^k - 1$.
- Hadamard simplices are reflexive polytopes. Actually, the polar of a Hadamard simplex Δ is $-\Delta$.
- The determinant of a Hadamard matrix is $(d+1)^{(d+1)/2}$. Hence, the volume of a Hadamard simplex, normalized to the unimodular simplex in the lattice generated by $\{-1,1\}^d$, equals $(d+1)^{(d+1)/2}/2^d$.

The last property, saying that Hadamard simplices are "big", is what makes us believe they could be a starting point to the construction of smooth non-normal polytopes. Of course, "big" empty simplices are easy to construct directly even in $\mathbb{Z}^3$, but Hadamard simplices do have a property that cannot arise in dimension three. It is known that lattices of dimension up to four only have unimodular Delone simplices. However:

Proposition 25 ([5])**.** For every dimension $d = 2^k - 1$ there is a 0/1-Hadamard simplex that is a Delaunay simplex in a sublattice $\Lambda \subseteq \mathbb{Z}^d$ of index $2^{d-k} = 2^d/(d+1)$. That is, Λ is a lattice with Delone simplices of normalized volume $(d+1)^{(d+3)/2}/4^d$.

2. Smooth and not-normal truncations of Hadamard simplices

We omit the proof of the following statement:

Lemma 26. For $m < (d+1)/4$, the dilated Hadamard simplex $m\Delta$ does not contain two antipodal points from the boundary of the cube $[-1,1]^d$.

In particular, it does not contain two opposite non-zero lattice points.

Corollary 27. Let $m < (d+1)/4$ be an odd natural number. Considered as a polytope in the affine lattice $\Lambda = (1+2\mathbb{Z})^d$, the dilated Hadamard simplex $m\Delta$ is not normal.

Proof. Let $P = m\Delta$. Then, the origin is in $(P+P)\cap(\Lambda+\Lambda)$ but it is not in $(P\cap\Lambda)+(P\cap\Lambda)$; the latter is by the previous lemma. □

This result is interesting in the light of the following fact: for any lattice polytope P in $\mathbb{R}^d$, dP is normal (considered in the lattice $d\Lambda$, for any affine lattice containing the vertices of P).

We are finally interested in the polytope $m\Delta \cap nC$, for $m,n \in \mathbb{N}$. We will always assume that $n < m < nd$, since for $n \geq m$ we have $m\Delta\cap nC = m\Delta$ and for $m \geq nd$ we have $m\Delta \cap nC = nC$. Also, we assume that m and n are both odd, so that the lattices $(m+2\mathbb{Z})^d$ and $(n+2\mathbb{Z})^d$ containing their vertices coincide with $\Lambda := (n+2\mathbb{Z})^d$.

We propose the following questions:

Question 5. (1) Is $m\Delta \cap nC$ always a lattice polytope in Λ?
(2) For which values of m and n is it normal?
(3) For which values of m and n is it smooth?

We believe the answer to the first question to be yes. The same argument of Corollary 27 shows that for $m < (d+1)/4$ the polytope $m\Delta \cap nC$ is not normal. An easy argument shows that for $m > nd - n\frac{d+1}{4}$ it is smooth: at every vertex all but one of the facets are coming from the cube. But between those bounds we know nothing.

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Strategies for proving projective normality of ample line bundles on smooth projective toric varieties

Problem session of the Mini-Workshop on projective normality

Introduction. In this session one half of the participants gathered ideas and approaches to find positive results for projective normality. We first separated suggestions that might work only in special cases from the ones that should be more useful for the general situation. Then we focused on collecting methods and strategies that seemed promising to pursuit. Contributions came from diverse areas as enumerative combinatorics, convex geometry, commutative algebra, and algebraic geometry. Finally, we made notes of open questions and conjectures whose solution or disprove should shed some light on the problems addressed.

In what follows, $P \subset \mathcal{R}^d$ is a d-dimensional lattice polytope. We say P is *smooth*, if the normal fan of P consists of unimodular cones, or equivalently, the toric variety X associated to the normal fan of P is smooth. We call P *normal*, if the ample line bundle on X associated to P is projectively normal, or equivalently, every lattice point in kP is the sum of k lattice points in P (for all $k \in \mathbb{Z}_{\geq 1}$). We remark that it suffices to check this condition for $2 \leq k \leq d-1$ (for instance, this can be derived from the fact that any lattice polytope can be triangulated into empty lattice simplices).

The open question we are going to refer to as the main conjecture is the following: *Are smooth polytopes normal?*

Summary of possible approaches. Let P be smooth.

(1) $\mathbf{d = 3.}$ The case $d = 2$ being elementary, the first unsettled situation is the three-dimensional case. Proofs of the main conjecture have been announced however none has been confirmed by now, so the problem is still considered to be open. Here, the question can be easily formulated: Is every lattice point in $2P$ the sum of two lattice points in P?

(2) **Vanishing theorems and onion skins.** In cohomology theory there exist many vanishing theorems that were recently successfully applied (keywords: adjoint line bundles, multigraded regularity of line bundles) to show that line bundles are normal (respectively, nef, very ample, N_p). In particular, a theorem due to Ein & Lazarsfeld says that $A^{d+2} + B + K_X$ is projectively normal, if A is an ample and B a nef line bundle. One may even take $d + 1$ instead of $d + 2$, if X is not projective space.

From the combinatorial point of view these ideas are closely related to taking so-called onion skins of lattice polytopes. This means to "shrink" the polytope by moving a facet one integral distance more to the inside. Since under suitable assumptions the smaller polytope is going to be smooth again, we might be able to use some induction.

(3) **Close lattice points.** The attempt that seems to be the most natural one from the viewpoint of convex geometry is the following: Let x be a lattice point of height k in the cone over $P \times 1$ (we may assume $k \leq d-1$). Now, we should look for a lattice point y in $P \times 1$ that is "close" to x/k, meaning that $x - y$ is also in the cone over $P \times 1$. This would give the nice decomposition $x = y + (x - y)$, where $x - y$ is a lattice point of height $k - 1$.

(4) **Generating functions.** It is suggested to use results of Brion and Barvinok & Woods on the generating functions enumerating lattice points in lattice polytopes. Comparing the generating function for the lattice points in $2P = P + P$ with the one enumerating the sum of lattice points in P, normality turns out to be equivalent to the vanishing of an explicit but complicated series.

(5) **Frobenius splitting.** There is a condition for general smooth projective varieties, not necessarily toric, that implies that *every* ample line bundle is projectively normal. This is the assumption that $X \times X$ is split compatibly with the diagonal $\Delta \subset X \times X$. However, in the toric case only a rather limited class of smooth projective varieties is expected to admit such a Frobenius splitting. Still, it may be useful to identify the associated smooth (and necessarily normal) lattice polytopes, since it might turn out that more lattice polytopes could appear as faces of these normal polytopes.

Summary of related open questions.

(1) Find a purely combinatorial proof of a result stated in (2) above which has an algebro-geometric proof: X smooth implies $D := A^{d+2} + B + K_X$ being

normal. Does the lattice polytope associated to D have even a unimodular covering? Here is another related question: if P is any lattice d-polytope, is $(d-1)P$ not only always a normal polytope, but does it also have a unimodular covering?

(2) We expect that simple lattice polytopes with "long" edges are normal, where "long" means some invariant uniform in the dimension. More precisely, we suggest the following conjecture: Let P be a simple lattice polytope. Let k be the maximum over the heights of Hilbert basis elements of tangent cones to vertices of P. Then, if any edge of P has length $\geq k$, the polytope P should be normal. Note that this generalizes the main conjecture on smooth polytopes.

(3) We are interested in finding a good description of the region R in the ample cone of a projective normal toric variety consisting of projectively normal line bundles. In particular, we would like to know the answers to the following questions:
   (a) Is R a semigroup? Equivalently, are Minkowski sums of normal lattice polytope with the same normal fan again normal?
   (b) Is R a module over the nef cone? Equivalently, is the Minkowski sum of a normal lattice polytope and a lattice polytope with a coarser normal fan again normal?
   (c) Is R convex?
   (d) Are there only finitely many ample (respectively, nef) line bundles not in R?

   If the variety is smooth, the last point is a weaker variant of the main conjecture.

problem session reported by Benjamin Nill

Searching for a counterexample, Monday afternoon

Problem session of the Mini-Workshop on projective normality

The suggestions made are listed below, where the marked ones are those on which the participants plan to focus more deeply in the coming days:

(1*) Try to use known strange polytopes to generate candidates for a counterexample. In particular, one could try to resolve the projective toric variety, corresponding to the very ample non-normal polytope associated with the triangulation of the real projective plane, and then find a small projective embedding of the obtained smooth variety.

(2) There was a suggestion to generate explicit large class of non-normal polytopes, with a potential possibility to make them smooth by some sort of polytope modification, without forcing the normality property.

(3) Find smooth polytopes every vertex of which admits an opposite facet not too far from the vertex (in the lattice distance sense). Such polytopes have better chances to fail the normality because there is not much room to propagate unimodular covers from the vertices deep inside the polytope.

(4) A Dueck-Hosten result was mentioned: every normal lattice d-polytope with $d+2$ or $d+3$ lattice points has a unimodular triangulation.

(5*) How can one find lattice polytopes P with the property that the number of lattice points in it is essentially smaller than that in $2P$? If P is also smooth then the we have a candidate. This question is related to the size of the degree 2 part of the corresponding toric ideal: the more such degree 2 relations the better the chances for the desired inequality.

(6*) Test the smooth polytopes, coming from smooth Fano toric varieties and their small projective embeddings.

PS. Bruns-Gubeladze-Serkan have a rudimentary implementation of a more general algorithm which will be refined in the near future.

(7) Are there some nice invariants that detect the non-normality of a polytope? Do the Ehrhart functions help?

(8) Filter out smooth polytopes by the Hilbert connectivity property: every two lattice points are linked by such a broken line inside the polytope that the directed edges are Hilbert basis elements of the corner cones. In general this property may not be related to the normality property, but for smooth polytopes the relationship may be strong enough to lead to a real candidate.

(9*) Let P be a smooth polytope and $F \subset P$ be a face. If all edges of P, meeting F at a vertex, have lattice length at least 2 then $conv(L(P \setminus F))$ is also smooth and is smaller than P. Iterate this process as many times as possible to arrive at a "tight smooth polytopes".

(10) Start by a polytope with an adjacent pair of unimodular vertices and close up the corresponding pair of unimodular corner cones to a small smooth polytope.

problem session reported by Joseph Gubeladze

Summary of Discussion, Tuesday Afternoon

Problem session of the Mini-Workshop on projective normality

This note records the discussion of a working group comprising Paffenholz, Craw, Hasse and Smith. The afternoon's task was to produce a counterexample to the conjecture that every ample bundle on a smooth toric variety defines a projectively normal embedding. The method assigned to our group was to modifying the projective embedding arising from a normally generated line bundle; in combinatorial terms, our task was to produce a smooth, non-normal polytope by modifying in some way a normal one.

Rather than tackle this problem head-on, we chose to modify a nonsmooth polytope that is 'close' to being non-normal. More specifically, our starting point was a 3×3 Birkoff transportation polytope B_3. While non-smooth, the toric ideal arising from embedding by the sections of this line bundle is not quadratically generated and hence is close in some sense to being non-normal.

The polytope that we considered is the convex hull of those lattice points in $\mathbb{Z}^9$ such that, putting the coordinates into a 3×3 matrix, each row-sum and column-sum must equal 3. The resulting four-dimensional polytope has 6 vertices and 9 facets, where every vertex is incident to 6 facets. We chose to *shrink* by deleting the vertex

$$v = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

i.e., we consider the polytope obtained from the convex hull of the remaining lattce points. The neighbouring lattice points on the edges emanating from v are

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

Using `Polymake`, we deduced that the resulting polytope has 10 vertices, 24 edges, 25 faces, 11 facets, and it is normal. Thus, we had not yet achieved our aim. Repeating the process on the remaining 5 vertices of the original polytope led to a smaller polytope that is nevertheless not normal. Going further still, we shrunk the polytope as far as possible with the hope of producing a non-normal example, but this proved not to be the case.

Thus, we this brief study, our repeated shrinking of a relatively small four-dimensional polytope failed to produce a non-normal polytope.

problem session reported by Alastair Craw

Report on the session "Investigating the Ohsugi–Hibi example"

Problem session of the Mini-Workshop on projective normality

This group consisted of Ohsugi, Hibi, Santos, and Schenck. The Ohsugi-Hibi example (OH, "Toric ideals generated by quadratic binomials" J.Alg 218, p. 509–527 (1999)) is a non-normal, Koszul semigroup ring, generated by the monomials (given by subscripts).

$$123, 134, 145, 125, 236, 456, 347, 257$$

The resulting variety is a projective four-fold $X \subseteq \mathbb{P}7$, which is in fact a complete intersection, with ideal generated by

$$x_2x_8 - x_4x_7, x_1x_6 - x_3x_5, x_1x_3 - x_2x_4.$$

Two natural questions are

(1) Is X smooth?
(2) Is the divisor corresponding to the embedding very ample?

It is easy to write down the Jacobian matrix for the equations of the embedded variety. One row contains only the variables $x_1, \ldots, x_4$, and it is easy to check that all points of the form $(0:0:0:0:*:*:*:*)$ lie on X. Since X is a complete intersection, the singular locus of the embedded object consists of those points where the Jacobian matrix drops rank, which implies that $X \subseteq \mathbb{P}7$ is singular. The group also investigated triangulations of the polytope; by passing to the Gale dual diagram we showed that there are 20 of them.

problem session reported by Hal Schenck

What else is known?

Christian Haase
(joint work with all workshop participants)

This is a report on two brainstorming sessions where we tried to list as many results as possible related to the projective normality question. Please accept my apologies as this list may seem chaotic (and it is), and in places there are contributions which I have not or falsely attributed.

1. A hierarchy

Let $P \subset \mathcal{R}^d$ be a lattice polytope and let $C \subset \mathcal{R}^{d+1}$ be the cone it generates. Then we have the following hierarchy of properties listed in decreasing strength. Compare [MFO04, p. 2097f].

(1) $P \cap \mathbb{Z}^d$ is totally unimodular
(2) P is compressed (any pulling triangulation is unimodular)
(3) P has a regular unimodular triangulation (RUT)
(4) P has a unimodular triangulation (UT)
(5) P has a unimodular binary cover (a $\mathbb{Z}_2$ cycle generating $H_d(P, \partial P; \mathbb{Z}_2)$ formed by unimodular simplices)
(6) P has a unimodular cover (UC)
(7) C has a free Hilbert cover (FHC: every lattice point is a $\mathbb{Z}_{\geq 0}$-linear combination of linearly independent lattice points in $P \times \{1\}$)
(8) C has the integral Carathéodory property (ICP: every lattice point is a $\mathbb{Z}_{\geq 0}$-linear combination of $\dim C$ many lattice points in $P \times \{1\}$)
(9) P is normal

Most of the implications $(i) \Rightarrow (i+1)$ are strict. The 3-dimensional 0/1-cube is compressed but not unimodular. Example 28 below has a unimodular triangulation, yet not a regular one. There are tetrahedra which have a unimodular cover but fail to have a unimodular binary cover [KS03]. The most recent example is the one by Bruns proving FHC $\not\Rightarrow$ UC [Bru07], while FHC and ICP are, in fact, equivalent. The first normal polytope without ICP is described in [BGH+99].

There are infinitely many properties which fit between properties (8) and (9), as we now explain. Define the Carathéodory rank $\mathrm{CR}(C)$ of a cone C to be the least number k so that every lattice point in C is a $\mathbb{Z}_{\geq 0}$-linear combination of k Hilbert basis elements. Then ICP is just the statement $\mathrm{CR}(C) = \dim C$. The maximal Carathéodory rank CR_d of a d-dimensional cone is between $7d/6$ and $2d - 2$ by [BGH+99] and by [Seb90] respectively. Gubeladze believes that the function CR_d / d is increasing with limit equal to 2.

2. Examples of normal polytopes

Faces of normal polytopes, products and dilations of normal polytopes are again normal. The same is true for the other properties in our hierarchy.

The polytope is very ample if and only if its semigroup has only finitely many holes.

Example 28 ([OH99])**.** Let $G = (V, E)$ be a graph. Then the edge polytope $P(G)$ is the convex hull in $\mathcal{R}^V$ of the sum of unit vectors $e_i + e_j$ for every edge $ij \in E$.

Ohsugi and Hibi describe a polytope which has a unimodular triangulation but not a regular one — see Ohsugi's abstract in this report. Santos observed that it projects to the polytope disproving ICP [MFO04, p. 2097f].

Another class of normal polytopes coming from combinatorial structures are

- base polytopes of discrete polymatroids,
- the convex hull of the 0/1 incidence vectors of branchings of a directed graph,
- the anti-blocking polytope of the incidence vectors of cliques of a perfect graph.

Here are more examples of (classes of) normal polytopes.

Theorem 29. For any lattice d-polytope P, and $c \in \mathbb{Z}_{\geq d-1}$, the dilate cP is normal. For $c \geq O(d^{2.1d+5})$, cP even has a unimodular cover [BG02].

It is conceivable (though certainly not easy) that this super exponential bound can be reduced significantly when one restricts to smooth polytopes.

The following names were invented during the conference.

Definition 5. A lattice polytope is edge unimodular if there is a totally unimodular collection $\mathcal{V}$ of vectors so that every edge of P is parallel to an element of $\mathcal{V}$.

We call P pairwise face unimodular if for any two faces $F, F' \prec P$, we have $(V + V') \cap \mathbb{Z}^d = (V \cap \mathbb{Z}^d) + (V' \cap \mathbb{Z}^d)$ where V and V' are the linear spaces parallel to F and F' respectively.

Eventually, P is facet unimodular if the primitive facet normals form a totally unimodular collection.

Edge unimodular implies pairwise face unimodular, but there are examples showing that edge unimodular and facet unimodular are independent notions. The zonotope generated by the A_n root system, also known as the permutahedron is not facet normal starting in dimension 3. On the other hand, a deformation of the rhombic dodecahedron $\mathrm{conv}([0,1]^3 \cap [-1,0]^3)$ will not be edge unimodular. Paco Santos came up with these examples.

Matroid polytopes are edge unimodular. The following theorem was inspired by the study of flag matroid polytopes in type A, corresponding to torus orbit closures in $\mathrm{SL}(n)/P$.

Theorem 30 (Ben Howard, this report)**.** Pairwise face unimodular $\Rightarrow$ normal.

The proof does not yield UC or any stronger property.

Theorem 31 (Santos 97 unpublished, [OH01, Sul04])**.** P is compressed if and only if P can be realized as the intersection of a linear space with the unit cube.

Examples of such polytopes are order polytopes, hypersimplices or stable polytopes of perfect graphs.

Theorem 32. Facet unimodular $\Rightarrow$ RUT.

The main source of such polytopes are flow polytopes: given a graph with an acyclic orientation $G = (V, \vec{E})$, and a demand vector $d \in \mathbb{Z}^V$, form the polytope

$$\Big\{ f \in \mathcal{R}_{\geq 0}^{\vec{E}} \; : \; \sum_{e \in \delta_+(v)} f_e - \sum_{e \in \delta_-(v)} f_e \; = \; d_v \text{ for all } v \in V \Big\}.$$

If G is the equioriented complete bipartite graph, then this flow polytope is also known under the name transportation polytope.

Example 33. The Minkowski sum of linearly independent segments is an affine cube. These polytopes are normal. Hence, general zonotopes are normal, as they can be subdivided into affine cubes.

What about unimodular cover in either case?

Theorem 34 (Haase after Kaibel and Wolf)**.** Let P be smooth so that all lattice points in P are vertices. Then P is a product of unimodular simplices.

Work in progress: all smooth reflexive polytopes in dimension up to 5 have RUTs. In dimension 6 all but 10 (out of $\sim$8000) have RUTs, and in dimension 7 all but 200 (out of $\sim$75000).

3. Examples of non-normal polytopes

The prism $\operatorname{conv}\left[\begin{smallmatrix} 0&1&0&1&2&1 \\ 0&0&1&1&1&2 \\ 0&0&0&3&3&3 \end{smallmatrix}\right]$ is actually the Minkowski sum of two unimodular simplices, but not normal. From it one can construct two 4-dimensional unimodular simplices in 4-space whose Minkowski sum has it as a facet. Also, this prism appears as a facet of a reflexive 4-polytope. (Three dimensional reflexive polytopes have a RUT.)

The triangle-vertex incidence polytope of the 6-vertex real projective plane is very ample, but not normal. The hole has coordinates $(1, 1, 1, 1, 1, 1)$.

The triangle-vertex incidence polytope of the simplicial complex on the vertex set $\{0, \ldots, 6\}$ with facets 012, 023, 034, 041, 125, 345, 236, 146 is Koszul despite not being very ample, and not having a quadratic Gröbner basis.

Many Frobenius/knapsack simplices $\operatorname{conv}[0, n_1e_1, \ldots, n_de_d]$ are not normal. (See references O. Marcotte & Scheithauser/Terno, and many others.)

Anti-blocking polyhedra which do not satisfy the rounding up condition (compare Trung's abstract in this report).

4. Related questions

A necessary numerical criterion for normality is that the h^* vector be an M-sequence. Is at least this true for smooth polytopes?

Meta question: what does the set of normal (very) ample line bundles in the nef cone look like? Here are a few instances.

- Is the set a semigroup or even a module over the nef cone? That is,

  P, Q normal with the same normal fan $\overset{?}{\Rightarrow}$ $P+Q$ normal (which $kP+lQ$ are normal?)

  Stronger: $\mathcal{N}_Q$ coarsens $\mathcal{N}_P$
- In dimension two, does Fakhruddin's theorem need the smoothness assumption? (See Fakhruddin's and Santos' abstracts in this report.)
- X smooth toric $\overset{?}{\rightsquigarrow}$ finitely many non projectively normal (very) ample line bundles

  More vaguely: what does the set of normal (very) ample line bundles in the nef cone look like?
- If P has all edge lengths $\geq d$ (?) $\overset{?}{\Rightarrow}$ P normal

Close cousins of the projective normality question, e.g., in the context of Green-Lazarsfeld's properties N_p, are the following.

- Normal smooth toric $\overset{?}{\Rightarrow}$ quadratically generated
- Normal toric Koszul $\overset{?}{\Rightarrow}$ quadratic Gröbner basis

Here are some questions related to the diagonally split property. (Compare Payne's abstract in this report.)

- Which polytopes arise as faces of diagonally split polytopes?
- Does diagonally split for $p = 2$ imply diagonally split for all p? (Answer obtained during the workshop: yes for $d = 2$, no for $d \geq 3$.)
- Give a combinatorial proof for "diagonally split $\Rightarrow$ normal".

Suppose we have a set of lattice points (not necessarily all lattice points in the convex hull) which yield a very ample embedding. Can one bound the height of the highest hole in the semigroup by the normalized volume of the polytope? This is related to the Eisenbud-Goto conjecture, and to Herzog's multiplicity question.

Which anti-blocking polyhedra are smooth? (Compare Trung's abstract in this report).

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