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HIGH-ORDER TIME-ACCURATE SCHEMES FOR SINGULARLY PERTURBED PARABOLIC CONVECTION-DIFFUSION PROBLEMS WITH ROBIN BOUNDARY CONDITIONS [1]

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Abstract — The boundary-value problem for a singularly perturbed parabolic PDE with convection is considered on an interval in the case of the singularly perturbed Robin boundary condition; the highest space derivatives in the equation and in the boundary condition are multiplied by the perturbation parameter ε. In contrast to the Dirichlet boundary-value problem, for the problem under consideration the errors of the well-known classical methods, generally speaking, grow without bound as $\varepsilon \ll N^{-1}$ where N defines the number of mesh points with respect to x. The order of convergence for the known ε-uniformly convergent schemes does not exceed 1. In this paper, using a defect correction technique, we construct ε-uniformly convergent schemes of high-order time-accuracy. The efficiency of the new defect-correction schemes is confirmed by numerical experiments. A new original technigue for experimental studying of convergence orders is developed for the cases where the orders of convergence in the x-direction and in the t-direction can be substantially different.

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1. Introduction

In this paper we consider the boundary-value problem on an interval for a singularly perturbed parabolic PDE with convection in the case of the singularly perturbed Robin boundary condition. The Robin condition is given on the inflow and outflow boundary. The highest space derivative in the equation and the derivatives in the boundary condition are

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multiplied by an arbitrarily small parameter ε. When the perturbation parameter ε tends to zero, the solution of such a problem typically exhibits a boundary layer in a neighborhood of the outflow boundary. This gives rise to difficulties when classical discretization methods are applied, because the errors in the approximate solution mainly depend on the value of ε: the errors of standard methods can even exceed many times the solution itself for small values of the parameter ε. Moreover, in contrast to Dirichlet conditions (errors for Dirichlet's problem are ε-uniformly bounded), in the case of Neumann conditions as the special case of the Robin condition the errors of discrete solutions grow without bound as ε tends to zero (see, for example, the remark to Theorem 6.1 in Section 3). Thus, in connection with such a behavior of the errors for standard numerical methods applied to the problem in question with the Robin boundary condition, it is of interest to develop special numerical methods whose errors are independent of the parameter ε and depend only on the number of mesh points, i.e., ε-uniformly convergent methods. Such methods have been proposed in the literature for a number of boundary-value problems for singularly perturbed elliptic and parabolic equations with Dirichlet conditions (see, for example, [1–7] and also the bibliography therein). It should be noted that the rate of ε-uniform convergence of known special schemes for parabolic equations with convection terms is $\mathcal{O}(N^{-1}\ln N + K^{-1})$, i.e., it is of the order not exceeding one, where N and K define the number of nodes in the grids with respect to x and t. However, the well-known classical difference methods of high-order accuracy with respect to x and/or t for the same problems (see, for example, [8], [9] and also the bibliography therein), generally speaking, do not converge ε-uniformly. Thus, it is necessary to construct ε-uniformly convergent schemes of high-order accuracy with respect to x and/or t for a class of singularly perturbed convection-diffusion problems, including the case of singularly perturbed Robin boundary conditions. Besides, a higher order accuracy in time can considerably reduce computational expenses.

Defect correction techniques proved to be efficient for constructing ε-uniformly convergent schemes of high-order accuracy with respect to t in the case of singularly perturbed reaction-diffusion problems (see, for example, [10–12]). Therefore, this method seems attractive to be used for the new class of singular perturbation problems under consideration.

In the present paper, ε-uniformly convergent schemes of high-order accuracy in time are constructed, also based on the defect correction principle, for a singularly perturbed parabolic convection-diffusion equation with the singularly perturbed Robin boundary condition. Note that the Robin condition admits both the Dirichlet condition and the singularly perturbed Neumann condition.

Theoretical investigations, as a rule, make it possible to evaluate only asymptotic orders of ε-uniform convergence for anew constructed schemes. However, actual errors of the constructed schemes can be significantly large for these schemes to be of practical use. Therefore, experimental study of both errors and convergence orders would be an interesting and important adjunct to the construction of special ε-uniform schemes. It should also be noted that, for high-order time-accurate schemes, errors due to the discretization of the space derivatives can be considerably greater than errors due to the time discretization (by a few orders; see, for example, Section 7). This behavior of the errors leads to difficulties in the experimental study of orders of ε-uniform convergence. For such cases, in the present paper we apply the original technique which has been first developed by the authors in [13]. Using this elegant technique, we give a sufficiently accurate analysis of the errors in the numerical solutions and of the convergence orders, which convincingly verifies the theoretical results.

2. The studied class of initial boundary-value problems

On the domain $G = D \times (0, T]$, $D = (0, 1)$ with the boundary $S = \overline{G} \setminus G$ we consider the following singularly perturbed parabolic equation with Robin boundary conditions [2]:

$$L_{(2.1)} u(x,t) \equiv \left\{ \varepsilon\, a(x,t) \frac{\partial^2}{\partial x^2} + b(x,t) \frac{\partial}{\partial x} - c(x,t) - p(x,t) \frac{\partial}{\partial t} \right\} u(x,t) = f(x,t), \quad (x,t) \in G, \tag{2.1a}$$

$$l_{(2.1)} u(x,t) \equiv \left\{ \varepsilon\, \alpha(x,t) \frac{\partial}{\partial n} + \beta(x,t) \right\} u(x,t) = \psi(x,t), \quad (x,t) \in S^L, \tag{2.1b}$$

$$u(x,t) = \varphi(x), \quad (x,t) \in S_0. \tag{2.1c}$$

For $S = S_0 \cup S^L$, we distinguish the lateral boundary $S^L = \{(x,t) : \ x = 0 \text{ or } x = 1, \ 0 < t \leqslant T\}$ and the initial boundary $S_0 = \{(x,t) : \ x \in [0,1], \ t = 0\}$; here $\partial/\partial n$ is the derivative in the direction of the outward normal to S^L. In (2.1) $a(x,t)$, $b(x,t)$, $c(x,t)$, $p(x,t)$, $f(x,t)$, $(x,t) \in \overline{G}$, $\alpha(x,t)$, $\beta(x,t)$, $\psi(x,t)$, $(x,t) \in S^L$, and $\varphi(x)$, $x \in \overline{D}$ are sufficiently smooth and bounded functions which satisfy

$$\begin{aligned} &0 < a_0 \leqslant a(x,t), \quad 0 < b_0 \leqslant b(x,t), \quad 0 < p_0 \leqslant p(x,t), \quad c(x,t) \geqslant 0, \quad (x,t) \in \overline{G}, \\ &\alpha(x,t), \ \beta(x,t) \geqslant 0, \quad \alpha(x,t) + \beta(x,t) \geqslant \alpha_0 > 0, \qquad (x,t) \in S^L. \end{aligned} \tag{2.1d}$$

The real parameter ε in (2.1a) and (2.1b) may take any values from the half-open unit interval

$$\varepsilon \in (0, 1]. \tag{2.1e}$$

When the parameter ε tends to zero, the solution exhibits a layer in the neighborhood of the outflow boundary $S_1^L = \{(x,t) : \ x = 0, \ 0 \leqslant t \leqslant T\}$, i.e., the left side of the lateral boundary. This layer is described by an ordinary differential equation (an ordinary boundary layer).

We have the Dirichlet problem if $\alpha(x,t) \equiv 0$, $(x,t) \in S^L$, and the Neumann problem if $\beta(x,t) \equiv 0$, $(x,t) \in S^L$. For simplicity, we assume that the following conditions are satisfied on the inflow (S_2^L) and outflow boundaries [3]:

$$\beta(x,t) \geqslant m, \quad (x,t) \in S_2^L, \quad \text{and} \quad \left\{ \begin{array}{ll} \text{or} & \alpha(x,t) = 0 \\ \text{or} & \alpha(x,t) \geqslant m \end{array} \right\}, \quad (x,t) \in S_k^L, \quad k = 1, 2; \tag{2.2}$$

β can equal zero on S_1^L.

3. Difference scheme on an arbitrary mesh

To solve problem (2.1) we first consider a classical finite difference method. On the set $\overline{G}$ we introduce the rectangular mesh

$$\overline{G}_h = \overline{\omega} \times \overline{\omega}_0, \tag{3.1}$$

[2] The notation is such that the operator $L_{(a.b)}$ is first introduced in equation $(a.b)$.

[3] Here and below we denote by M (or m) sufficiently large (or small) positive constants which do not depend on the value of the parameter ε and on the discretization parameters.

where $\overline{\omega}$ is the (possibly) nonuniform mesh of nodal points, x^i, in $[0,1]$, $\overline{\omega}_0$ is a uniform mesh on the interval $[0,T]$; N and K are the numbers of intervals in the meshes $\overline{\omega}$ and $\overline{\omega}_0$, respectively. We define $\tau = T/K$, $h^i = x^{i+1} - x^i$, $h = \max_i h^i$, $h \leqslant M/N$, $G_h = G \cap \overline{G}_h$, $S_h = S \cap \overline{G}_h$.

For problem (2.1) we use the difference scheme [8]

$$\Lambda_{(3.2)} z(x,t) = f(x,t), \qquad (x,t) \in G_h, \tag{3.2a}$$

$$\lambda_{(3.2)} z(x,t) = \psi(x,t), \qquad (x,t) \in S_h^L, \tag{3.2b}$$

$$z(x,t) = \varphi(x), \qquad (x,t) \in S_{0h}. \tag{3.2c}$$

Here

$$\Lambda_{(3.2)} z(x,t) \equiv \left\{ \varepsilon\, a(x,t) \delta_{\overline{x}\widehat{x}} + b(x,t)\delta_x - c(x,t) - p(x,t)\delta_{\overline{t}} \right\} z(x,t), \quad (x,t) \in G_h,$$

$$\lambda_{(3.2)} z(x,t) \equiv \varepsilon\, \alpha(x,t) \left\{ \begin{array}{ll} -\delta_x z(x,t), & (x,t) \in S_{1h}^L, \\ \delta_{\overline{x}} z(x,t), & (x,t) \in S_{2h}^L \end{array} \right\} + \beta(x,t) z(x,t), \quad (x,t) \in S_h^L,$$

$$\begin{aligned} \delta_{\overline{x}\widehat{x}} z(x^i,t) &= 2\left(h^{i-1} + h^i\right)^{-1} \left(\delta_x z(x^i,t) - \delta_{\overline{x}} z(x^i,t)\right), \\ \delta_{\overline{x}} z(x^i,t) &= \left(h^{i-1}\right)^{-1} \left(z(x^i,t) - z(x^{i-1},t)\right), \\ \delta_x z(x^i,t) &= \left(h^i\right)^{-1} \left(z(x^{i+1},t) - z(x^i,t)\right), \\ \delta_{\overline{t}} z(x^i,t) &= \tau^{-1} \left(z(x^i,t) - z(x^i,t-\tau)\right), \end{aligned}$$

$\delta_x z(x,t)$ and $\delta_{\overline{x}} z(x,t)$, $\delta_{\overline{t}} z(x,t)$ are the forward and backward differences, and the difference operator $\delta_{\overline{x}\widehat{x}} z(x,t)$ is an approximation of the operator $\dfrac{\partial^2}{\partial x^2} u(x,t)$ on the nonuniform mesh.

From [8] we know that the difference scheme (3.2), (3.1) is monotone. By means of the maximum principle and taking into account *a priori* estimates of the derivatives (see Theorem 8.1 in Section 8) we find that the solution of the difference scheme (3.2), (3.1) converges for a fixed value of the parameter ε:

$$|\, u(x,t) - z(x,t) \,| \leqslant M \left(\varepsilon^{-2} N^{-1} + \tau \right), \quad (x,t) \in \overline{G}_h. \tag{3.3}$$

This error bound for the classical difference scheme is clearly not ε-uniform.

The proof of (3.3) follows the lines of the classical convergence proof for monotone difference schemes (see [2,8]). This results in the following theorem.

Theorem 3.1. *Assume in equation* (2.1) *that*

$$a \in H^{(\vartheta+2n-1)}(\overline{G}); \quad b,\ c,\ p,\ f \in H^{(\vartheta+2n-2)}(\overline{G});$$
$$\varphi \in H^{(\vartheta+2n)}(\overline{D}); \quad \alpha,\ \beta,\ \psi \in H^{(\vartheta+2n)}(\overline{S}^L); \quad \vartheta > 4,\ n = 0,$$

and let conditions (2.2) *and also the compatibility conditions* (8.1) *with* $n=0$ *be satisfied. Then, for a fixed value of the parameter* ε*, the solution of* (3.2), (3.1) *converges to the solution of* (2.1) *with an error bound given by* (3.3).

Remark 3.1. The consideration of model examples shows that on uniform meshes the error of the mesh solution grows without bound similarly to $\varepsilon^{-1} N^{-1}$ for $\varepsilon \ll N^{-1}$ if $\beta(x,t) = 0$, $(x,t) \in S_1^{0L}$ (that is, in the case of the Neumann condition given on the set S_1^0) where S_1^{0L} is a subset of the set S_1^L.

4. The ε-uniformly convergent scheme

Here we discuss an ε-uniformly convergent method for (2.1) by taking a special mesh condensed in the neighborhood of the boundary layer. The location of the nodes is determined properly from the *a priori* estimates of the solution and its derivatives. The way to construct the mesh for problem (2.1) is the same as in [10, 11, 14]. More specifically, we take

$$\overline{G}_h^{\,*} = \overline{\omega}^*(\sigma) \times \overline{\omega}_0 \,, \tag{4.1}$$

where $\overline{\omega}_0$ is the uniform mesh with the step size $\tau = T/K$, i.e., $\overline{\omega}_0 = \overline{\omega}_{0(3.1)}$, and $\overline{\omega}^* = \overline{\omega}^*(\sigma)$ is a special *piecewise* uniform mesh depending on the parameter $\sigma \in \mathbb{R}$, which depends on ε and N. We take

$$\sigma = \sigma_{(4.1)}(\varepsilon, N) = \min\left\{ 1/2,\, m^{-1}\varepsilon \ln N \right\},$$

where m is an arbitrary number from the interval $(0, m_0)$, $m_0 = \min_{\overline{G}} [a^{-1}(x,t)\, b(x,t)]$. The mesh $\overline{\omega}^*(\sigma)$ is constructed as follows. The interval $[0,1]$ is divided in two parts $[0,\sigma]$, $[\sigma,1]$, $\sigma \leqslant 1/2$. In each part we use a uniform mesh, with $N/2$ subintervals in $[\,0,\sigma\,]$ and $[\,\sigma,1\,]$.

Theorem 4.1. *Let the hypotheses of Theorem* 3.1 *be fulfilled. Then the solution of* (3.2), (4.1) *converges ε-uniformly to the solution of* (2.1) *and the following estimate holds:*

$$|\, u(x,t) - z(x,t)\,| \;\leqslant\; M\left(N^{-1}\ln N + \tau\right), \quad (x,t) \in \overline{G}_h^*. \tag{4.2}$$

The proof of this theorem can be found in [2, 15].

5. Numerical results for scheme (3.2), (4.1)

To see the effect of the special mesh in practice, we take the model problem

$$L_{(5.1)}u(x,t) \equiv \left\{\varepsilon\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right\} u(x,t) = f(x,t), \quad (x,t) \in G, \tag{5.1}$$

$$l_{(5.1)}u(x,t) \equiv \left\{ \begin{array}{rl} -\varepsilon \dfrac{\partial}{\partial x}u(x,t), & (x,t) \in S_1^L, \\ u(x,t), & (x,t) \in S_2^L \end{array} \right\} = \psi(x,t), \quad (x,t) \in S^L,$$

$$u(x,t) = \varphi(x), \quad (x,t) \in S_0,$$

where $f(x,t) = -4t^3$, $(x,t) \in \overline{G}$, $\psi(0,t) = t^2$, $\psi(1,t) = 0$, $t \in [0,T]$, $T = 1$; $\varphi(x) = 0$, $x \in \overline{D}$.

For the approximation of problem (5.1) we use the scheme (3.2), (4.1), where $m = 2^{-1}$, $\overline{G}_h = \overline{G}_h^*$.

Since the exact solution for this problem is unknown, we replace it by the numerical solution $U_\varepsilon^{4096}(x,t)$ computed on the finest available mesh $\overline{G}_h$ with $N = K = 4096$ for each value of ε. Then the computed maximum pointwise error is defined by

$$E(N,K,\varepsilon) = \max_{(x,t)\in\overline{G}_h} |\, z(x,t) - u^*(x,t)\,|\,. \tag{5.2}$$

Here $u^*(x,t)$ is the linear interpolation obtained from the reference numerical solution $U_\varepsilon^{4096}(x,t)$ of problem (3.2), (4.1). We compute $E(N,K,\varepsilon)$ for various values of ε, N, K. Note that no special interpolation is needed along the t-axis.

The results are given in Table 1. From the analysis of these numerical results we conclude that, in accordance with (4.2), the order of convergence for large $N = K$ is $\mathcal{O}(N^{-1} \ln N + K^{-1})$, i.e., almost one with respect to the space and time variables, which corresponds to the theoretical results.

Table 1. Errors $E(N = K, \varepsilon)$ for the model problem (5.1) solved by the special scheme (3.2), (4.1)

$\varepsilon \setminus N$	8	16	32	64	128	256	512	1024	2048
1.0	9.356-2	5.266-2	2.774-2	1.414-2	7.052-3	3.435-3	1.608-3	6.904-4	2.303-4
2^{-1}	2.447-1	1.293-1	6.616-2	3.324-2	1.646-2	7.989-3	3.734-3	1.602-3	5.341-4
2^{-2}	5.048-1	2.584-1	1.302-1	6.491-2	3.202-2	1.551-2	7.244-3	3.106-3	1.035-3
2^{-3}	9.973-1	5.027-1	2.514-1	1.250-1	6.154-2	2.979-2	1.390-2	5.959-3	1.987-3
2^{-4}	1.984+0	9.945-1	4.961-1	2.463-1	1.212-1	5.867-2	2.738-2	1.173-2	3.912-3
2^{-5}	2.172+0	1.409+0	8.643-1	4.916-1	2.419-1	1.171-1	5.464-2	2.342-2	7.806-3
2^{-6}	2.240+0	1.448+0	8.821-1	5.174-1	2.931-1	1.598-1	8.261-2	3.875-2	1.420-2
2^{-7}	2.283+0	1.476+0	8.965-1	5.248-1	2.969-1	1.616-1	8.326-2	3.873-2	1.380-2
2^{-8}	2.307+0	1.493+0	9.049-1	5.291-1	2.993-1	1.629-1	8.391-2	3.904-2	1.391-2
2^{-9}	2.320+0	1.502+0	9.095-1	5.314-1	3.005-1	1.636-1	8.426-2	3.920-2	1.397-2
2^{-10}	2.326+0	1.507+0	9.119-1	5.326-1	3.011-1	1.639-1	8.443-2	3.928-2	1.399-2
2^{-12}	2.331+0	1.511+0	9.137-1	5.335-1	3.016-1	1.642-1	8.457-2	3.934-2	1.402-2
2^{-14}	2.332+0	1.512+0	9.142-1	5.338-1	3.018-1	1.642-1	8.460-2	3.936-2	1.402-2
2^{-16}	2.333+0	1.512+0	9.143-1	5.338-1	3.018-1	1.642-1	8.461-2	3.936-2	1.402-2
$\overline{E}(N)$	2.333+0	1.512+0	9.143-1	5.338-1	3.018-1	1.642-1	8.461-2	3.936-2	1.402-2

In this table the function $E(N, K, \varepsilon)$ is defined by (5.2). Here $K = N$. In the bottom line $\overline{E}(N)$ gives the computed maximum pointwise errors for each column.

6. Improved time-accuracy

6.1. A scheme based on defect correction

In this section we construct a new discrete method based on defect correction, which also converges ε-uniformly to the solution of the boundary-value problem, but with an order of accuracy (with respect to τ) higher than in (4.2).

The technique used in this paper to improve time-accuracy is based on that from [10, 11]. For the difference scheme (3.2), (4.1) the error in the approximation of the partial derivative $(\partial/\partial t)\, u(x,t)$ is caused by the divided difference $\delta_{\overline{t}}\, z(x,t)$ and is associated with the truncation error given by

$$\frac{\partial u}{\partial t}(x,t) - \delta_{\overline{t}}\, u(x,t) = 2^{-1}\, \tau \frac{\partial^2 u}{\partial t^2}(x,t) - 6^{-1}\, \tau^2 \frac{\partial^3 u}{\partial t^3}(x, t-\theta), \quad \theta \in [0, \tau]. \tag{6.1}$$

Therefore, for the approximation of $(\partial/\partial t)\, u(x,t)$ we now use the expression

$$\delta_{\overline{t}}\, u(x,t) + \tau \delta_{\overline{t}\,\overline{t}}\, u(x,t)/2, \quad \text{where} \quad \delta_{\overline{t}\,\overline{t}}\, u(x,t) \equiv \delta_{t\,\overline{t}}\, u(x, t-\tau).$$

Notice that $\delta_{t\,\overline{t}}\, u(x,t)$ is the second central divided difference. We can evaluate a better approximation than (3.2a) by defect correction

$$\Lambda_{(3.2)} z^c(x,t) = f(x,t) + 2^{-1} p(x,t) \tau \frac{\partial^2 u}{\partial t^2}(x,t), \tag{6.2}$$

with $x \in \overline{\omega}$ and $t \in \overline{\omega}_0$, where $\overline{\omega}$ and $\overline{\omega}_0$ are as in (3.1); τ is the step size of the mesh $\overline{\omega}_0$; $z^c(x,t)$ is the "corrected" solution. Instead of $(\partial^2/\partial t^2)\,u(x,t)$ we shall use $\delta_{\bar{t}\,\bar{t}}\, z(x,t)$, where $z(x,t)$, $(x,t) \in G_{h(4.1)}$ is the solution of the difference scheme (3.2), (4.1). We may expect that the new solution $z^c(x,t)$ has a consistency error $\mathcal{O}(\tau^2)$. This is true, as will be shown in Section 6.2.

Moreover, in a similar way we can construct a difference approximation with a convergence order higher than two (with respect to the time variable) and $\mathcal{O}(N^{-1}\ln N)$ with respect to the space variable ε-uniformly.

6.2. The defect correction scheme of second-order accuracy in time

We denote by $\delta_{k\bar{t}} z(x,t)$ the backward difference of order k:

$$\delta_{k\bar{t}}\, z(x,t) = \left(\delta_{k-1\,\bar{t}}\, z(x,t) - \delta_{k-1\,\bar{t}}\, z(x,t-\tau)\right)/\tau, \quad t \geqslant k\tau, \quad k \geqslant 1;$$
$$\delta_{0\bar{t}}\, z(x,t) = z(x,t), \quad (x,t) \in \overline{G}_h.$$

To construct the difference schemes of second-order accuracy in τ in (6.2), instead of $(\partial^2/\partial t^2)u(x,t)$ we use $\delta_{2\,\bar{t}}\, z(x,t)$, the second divided difference of the solution to the discrete problem (3.2), (4.1). On the mesh $\overline{G}_h$ we write the finite difference scheme (3.2) in the form

$$\begin{aligned} \Lambda_{(3.2)} z^{(1)}(x,t) &= f(x,t), \quad (x,t) \in G_h, \\ \lambda_{(3.2)} z^{(1)}(x,t) &= \psi(x,t), \quad (x,t) \in S_h^L, \\ z^{(1)}(x,t) &= \varphi(x), \quad (x,t) \in S_{0h}, \end{aligned} \tag{6.3}$$

where $z^{(1)}(x,t)$ is the uncorrected solution.

For the corrected solution $z^{(2)}(x,t)$ we solve the problem for $(x,t) \in G_h$

$$\Lambda_{(3.2)} z^{(2)}(x,t) = f(x,t) + \left\{ \begin{array}{ll} p(x,t) 2^{-1}\,\tau \dfrac{\partial^2}{\partial t^2} u(x,0), & t = \tau, \\ p(x,t) 2^{-1}\,\tau \delta_{2\,\bar{t}}\, z^{(1)}(x,t), & t \geqslant 2\tau \end{array} \right\}, \quad (x,t) \in G_h,$$

$$\lambda_{(3.2)} z^{(2)}(x,t) = \psi(x,t), \quad (x,t) \in S_h^L, \tag{6.4}$$

$$z^{(2)}(x,t) = \varphi(x), \quad (x,t) \in S_{0h}.$$

Here the derivative $\dfrac{\partial^2 u}{\partial t^2}(x,0)$ can be obtained from equation (2.1a). We shall call $z^{(2)}(x,t)$ the solution of difference scheme (6.4), (6.3), (4.1) (or shortly, (6.4), (4.1)).

For simplicity, in the remainder of this section we suppose that the coefficients $a(x,t)$, $b(x,t)$ do not depend on t:

$$a(x,t) = a(x), \quad b(x,t) = b(x), \quad (x,t) \in \overline{G}, \tag{6.5}$$

and we take a homogeneous initial condition

$$\varphi(x) = 0, \quad x \in \overline{D}. \tag{6.6}$$

Under conditions (6.5), (6.6), the following estimate holds for the solution of problem (6.4), (4.1):

$$\left|\, u(x,t) - z^{(2)}(x,t) \,\right| \leqslant M \left[\, N^{-1} \ln N + \tau^2 \,\right], \quad (x,t) \in \overline{G}_h. \tag{6.7}$$

Theorem 6.1. *Let conditions* (6.5), (6.6) *hold and assume in equation* (2.1) *that* a, b, c, p, $f \in H^{(\vartheta+2n-2)}(\overline{G})$, $\varphi \in H^{(\vartheta+2n)}(\overline{D})$, α, β, $\psi \in H^{(\vartheta+2n)}(\overline{S}^L)$, $\vartheta > 4$, $n = 1$, *and let conditions* (2.2) *and also the compatibility conditions* (8.1) *for* $n = 1$ *be satisfied. Then for the solution of difference scheme* (6.4), (4.1) *estimate* (6.7) *is valid.*

Proof. The proof of Theorem 6.1 is given in Section 9.2. □

Remark 6.1. The conclusion of Theorem 6.1 remains also valid in cases where the coefficients a and b depend on x, t, for example, where the condition $a^{-1}(x,t)\,b(x,t) = g(x)$, $(x,t) \in \overline{G}$ is satisfied. This remark holds for Theorem 6.2 as well.

6.3. The defect correction scheme of third-order accuracy in time

The above procedure can be used to obtain an arbitrarily large order of accuracy in time. Here we only show how to construct the difference scheme of third-order accuracy. On the grid $\overline{G}_h$ we consider the difference scheme

$$\Lambda_{(3.2)}\, z^{(3)}(x,t) \;=\; f(x,t) \;+ \tag{6.8a}$$

$$+\left\{\begin{array}{ll} p(x,t)\left(C_{11}\tau\dfrac{\partial^2}{\partial t^2}u(x,0) + C_{12}\tau^2\dfrac{\partial^3}{\partial t^3}u(x,0)\right), & t=\tau, \\[2ex] p(x,t)\left(C_{21}\tau\dfrac{\partial^2}{\partial t^2}u(x,0) + C_{22}\tau^2\dfrac{\partial^3}{\partial t^3}u(x,0)\right), & t=2\tau, \\[2ex] p(x,t)\left(C_{31}\tau\delta_{2\bar t}z^{(2)}(x,t) + C_{32}\tau^2\delta_{3\bar t}z^{(1)}(x,t)\right), & t\geqslant 3\tau \end{array}\right\}, \quad (x,t)\in G_h,$$

$$\lambda_{(3.2)} z^{(3)}(x,t) = \psi(x,t), \quad (x,t) \in S_h^L,$$

$$z^{(3)}(x,t) = \varphi(x,t), \quad (x,t) \in S_{0h}.$$

Here $z^{(1)}(x,t)$ and $z^{(2)}(x,t)$ are the solutions of problems (6.3), (4.1) and (6.4), (4.1), respectively; the derivatives $(\partial^2/\partial t^2)u(x,0)$, $(\partial^3/\partial t^3)u(x,0)$ are again obtained from equation (2.1a). The coefficients C_{ij} are chosen such that they satisfy the following conditions:

$$\frac{\partial}{\partial t}u(x,t) = \delta_{\bar t}u(x,t) + C_{11}\tau\frac{\partial^2}{\partial t^2}u(x,t-\ \tau) + C_{12}\tau^2\frac{\partial^3}{\partial t^3}u(x,t-\ \tau) + \mathcal{O}(\tau^3),$$

$$\frac{\partial}{\partial t}u(x,t) = \delta_{\bar t}u(x,t) + C_{21}\tau\frac{\partial^2}{\partial t^2}u(x,t-2\tau) + C_{22}\tau^2\frac{\partial^3}{\partial t^3}u(x,t-2\tau) + \mathcal{O}(\tau^3),$$

$$\frac{\partial}{\partial t}u(x,t) = \delta_{\bar t}u(x,t) + C_{31}\tau\delta_{2\bar t}u(x,t) + C_{32}\tau^2\delta_{3\bar t}u(x,t) + \mathcal{O}(\tau^3).$$

It follows that

$$C_{11} = C_{21} = C_{31} = 1/2, \quad C_{12} = C_{32} = 1/3, \quad C_{22} = 5/6. \tag{6.8b}$$

By $z^{(3)}(x,t)$ we denote the solution of the difference scheme (6.8), (4.1) and again, for simplicity, we assume the homogeneous initial condition to take place:

$$\varphi(x) = 0, \quad f(x,0) = 0, \quad x \in \overline{D}. \tag{6.9}$$

Under conditions (6.5), (6.9) the following estimate holds for the solution of difference scheme (6.8), (4.1):

$$\big| \, u(x,t) - z^{(3)}(x,t) \, \big| \; \leqslant \; M \left[\, N^{-1} \ln N + \tau^3 \, \right], \quad (x,t) \in \overline{G}_h. \tag{6.10}$$

Theorem 6.2. *Let conditions* (6.9) *hold and assume in equation* (2.1) *that* a, b, c, p, $f \in H^{(\vartheta+2n-2)}(\overline{G})$, $\varphi \in H^{(\vartheta+2n)}(\overline{D})$, α, β, $\psi \in H^{(\vartheta+2n)}(\overline{S}^L)$, $\vartheta > 4$, $n = 2$, *and let conditions* (2.2) *and also the compatibility conditions* (8.1) *with* $n = 2$ *be satisfied. Then for the solution of scheme* (6.8), (4.1) *the estimate* (6.10) *is valid.*

Proof. The proof of Theorem 6.2 is given in Section 9.2. □

In a similar way we could construct difference schemes with an arbitrarily high order of accuracy

$$\mathcal{O}(N^{-1} \ln N + \tau^{n+1}), \quad n > 2.$$

7. Numerical results for the time-accurate schemes

We find the solution of the following boundary-value problem:

$$L_{(5.1)} u(x,t) = 0, \quad 0 < x < 1, \;\; 0 < t \leqslant T, \;\; T = 1. \tag{7.1}$$

$$l_{(5.1)} u(x,t) = \left\{ \begin{array}{ll} t^5, & x = 0, \\ 0, & x = 1 \end{array} \right\}, \quad (x,t) \in S^L,$$

$$u(x,t) = 0, \quad (x,t) \in S_0.$$

It should be noted that the solution of this problem is singular.

It is very attractive to use the analytical solution of problem (7.1) for the computation of errors in the approximate solution, as was done in [10, 11]. But here the suitable (for computation) representation of the solution $u(x,t)$ is unknown. Instead of the exact solution, it is possible to use the solution of the discrete problem on a very fine mesh. But this method is not effective because the analysis of the order of accuracy for a defect-correction scheme requires a very dense mesh that leads not only to large computational expenses but also to large round-off errors.

Here we use the method from [16], different from the above-mentioned techniques. The solution of problem (7.1) is represented in the form of the sum

$$u(x,t) = V^{(1)}(x,t) + v(x,t), \quad (x,t) \in \overline{G}, \tag{7.2}$$

where $V^{(1)}(x,t)$ is the main singular part (two first terms) of the asymptotic expansion of the solution $u(x,t)$, and $v(x,t)$ is the remainder term, which is a sufficiently small smooth function. The function $V^{(1)}(x,t)$ has a sufficiently simple analytical representation

$$V^{(1)}(x,t) = t^4 [t - 5x - 5\varepsilon] \exp(-\varepsilon^{-1} x), \qquad \big| V^{(1)}(x,t) \big| \leqslant M, \quad (x,t) \in \overline{G}.$$

The function $v(x,t)$ is the solution of the problem

$$L_{(5.1)} v(x,t) = f_0(x,t), \quad (x,t) \in G, \tag{7.3}$$

$$-\varepsilon \, \frac{\partial}{\partial x} \, v(0,t) = 0, \;\; v(1,t) = -V^{(1)}(1,t), \;\; 0 < t \leqslant T, \qquad v(x,0) = 0, \;\; 0 < x < 1.$$

Here

$$f_0(x,t) = -20\,t^3\,(x+\varepsilon)\,\exp(-\varepsilon^{-1}\,x).$$

For the function $v(x,t)$ the following estimate holds:

$$\left|\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} v(x,t)\right| \leqslant M\varepsilon^2[1+\varepsilon^{-k}], \quad (x,t)\in\overline{G}, \quad k+2k_0 \leqslant 4, \quad k\leqslant 3. \tag{7.4}$$

Table 2. Errors $E(N,K)$ for $\varepsilon = 2^{-10}$

$K \setminus N$	4	8	16	32	64	128	256	512	1024
					$z^{(1)}$				
4	**2.56+0**	2.12+0	1.41+0	8.74-1	5.22-1	3.04-1	1.75-1	9.91-2	5.59-2
8	2.43+0	**2.07+0**	1.40+0	8.71-1	5.20-1	3.03-1	1.74-1	9.82-2	5.50-2
16	2.36+0	2.05+0	**1.40+0**	8.70-1	5.19-1	3.03-1	1.73-1	9.76-2	5.45-2
32	2.33+0	2.03+0	1.40+0	**8.69-1**	5.19-1	3.02-1	1.73-1	9.73-2	5.42-2
64	2.31+0	2.03+0	1.39+0	8.68-1	**5.19-1**	3.02-1	1.73-1	9.72-2	5.40-2
128	2.30+0	2.02+0	1.39+0	8.68-1	5.19-1	**3.02-1**	1.73-1	9.71-2	5.40-2
256	2.30+0	2.02+0	1.39+0	8.68-1	5.19-1	3.02-1	**1.73-1**	9.70-2	5.39-2
512	2.29+0	2.02+0	1.39+0	8.68-1	5.19-1	3.02-1	1.73-1	**9.70-2**	5.39-2
1024	2.29+0	2.02+0	1.39+0	8.68-1	5.19-1	3.02-1	1.72-1	9.70-2	**5.39-2**
					$z^{(2)}$				
4	**2.37+0**	2.05+0	1.40+0	8.70-1	5.20-1	3.03-1	1.73-1	9.79-2	5.47-2
8	2.31+0	**2.03+0**	1.40+0	8.69-1	5.19-1	3.02-1	1.73-1	9.73-2	5.42-2
16	2.30+0	2.02+0	**1.39+0**	8.68-1	5.19-1	3.02-1	1.73-1	9.71-2	5.40-2
32	2.29+0	2.02+0	1.39+0	**8.68-1**	5.19-1	3.02-1	1.72-1	9.70-2	5.39-2
64	2.29+0	2.02+0	1.39+0	8.68-1	**5.18-1**	3.02-1	1.72-1	9.70-2	5.39-2
128	2.29+0	2.02+0	1.39+0	8.68-1	5.18-1	**3.02-1**	1.72-1	9.70-2	5.39-2
256	2.29+0	2.02+0	1.39+0	8.68-1	5.18-1	3.02-1	**1.72-1**	9.70-2	5.39-2
512	2.29+0	2.02+0	1.39+0	8.68-1	5.18-1	3.02-1	1.72-1	**9.70-2**	5.39-2
1024	2.29+0	2.02+0	1.39+0	8.68-1	5.18-1	3.02-1	1.72-1	9.70-2	**5.39-2**
					$z^{(3)}$				
4	**2.33+0**	2.03+0	1.40+0	8.69-1	5.19-1	3.02-1	1.73-1	9.74-2	5.42-2
8	2.30+0	**2.02+0**	1.39+0	8.68-1	5.19-1	3.02-1	1.73-1	9.71-2	5.39-2
16	2.29+0	2.02+0	**1.39+0**	8.68-1	5.18-1	3.02-1	1.72-1	9.70-2	5.39-2
32	2.29+0	2.02+0	1.39+0	**8.68-1**	5.18-1	3.02-1	1.72-1	9.70-2	5.39-2
64	2.29+0	2.02+0	1.39+0	8.68-1	**5.18-1**	3.02-1	1.72-1	9.70-2	5.39-2
128	2.29+0	2.02+0	1.39+0	8.68-1	5.18-1	**3.02-1**	1.72-1	9.70-2	5.39-2
256	2.29+0	2.02+0	1.39+0	8.68-1	5.18-1	3.02-1	**1.72-1**	9.70-2	5.39-2
512	2.29+0	2.02+0	1.39+0	8.68-1	5.18-1	3.02-1	1.72-1	**9.70-2**	5.39-2
1024	2.29+0	2.02+0	1.39+0	8.68-1	5.18-1	3.02-1	1.72-1	9.70-2	**5.39-2**

Then the function $v(x,t)$ and the product $\varepsilon(\partial^3/\partial x^3)\,v(x,t)$ are ε-uniformly bounded. Thus, we can consider $v(x,t)$ as the regular part of this solution and, moreover, $v(x,t)$ is of order $\mathcal{O}(\varepsilon^2)$, according to (7.4).

(1.) For the chosen value of ε, we solve the discrete problem approximating the model problem (7.3) on the finest available mesh $\overline{G}_h = \overline{G}^*_{h(4.1)}$ for $N = K = 2048$, and there are no difficulties to find the function $v(x,t) = v_\varepsilon^{2048}(x,t)$ and the reference solution $u_{(7.2)}(x,t)$ which can be practically taken as the exact solution

$$u_{(7.2)}(x,t) = u_\varepsilon^{2048}(x,t) = V^{(1)}(x,t) + v_\varepsilon^{2048}(x,t).$$

(2.) Further for solving problem (7.1), we consecutively use scheme (6.3), (4.1) and the defect correction schemes (6.4), (4.1) and (6.8), (4.1) to find the functions $z^{(1)}(x,t)$, $z^{(2)}(x,t)$, and $z^{(3)}(x,t)$, respectively. Note that $z^{(1)}(x,t)$ is the uncorrected solution, $z^{(2)}(x,t)$ and $z^{(3)}(x,t)$ are the corrected solutions. In these cases we compute the maximum pointwise errors $E(N,K,\varepsilon)$ by formula (5.2), where $u^*(x,t)$ is the linear interpolation obtained from the reference solution $u_\varepsilon^{2048}(x,t)$ corresponding to the numerical solution $z^{(k)}(x,t)$, $k = 1,2,3$ for the values $N = 2^i$, $i = 2,3,\dots,10$, $K = 2^j$, $j = 2,3,\dots,10$.

Table 3. Space errors $E^{(s)}(N,K)$ for $\varepsilon = 2^{-10}$

$K \setminus N$	8	16	32	64	128	256	512
				$z^{(1)}$			
4	7.01-1	5.41-1	3.52-1	2.17-1	1.30-1	7.56-2	4.32-2
8	6.66-1	5.34-1	3.51-1	2.17-1	1.30-1	7.56-2	4.32-2
16	6.46-1	5.29-1	3.50-1	2.17-1	1.30-1	7.55-2	4.31-2
32	6.36-1	5.27-1	3.50-1	2.17-1	1.30-1	7.55-2	4.31-2
64	6.31-1	5.26-1	3.50-1	2.17-1	1.30-1	7.55-2	4.31-2
128	6.29-1	5.25-1	3.50-1	2.17-1	1.30-1	7.55-2	4.31-2
256	6.27-1	5.25-1	3.49-1	2.17-1	1.29-1	7.55-2	4.31-2
512	6.27-1	5.25-1	3.49-1	2.17-1	1.29-1	7.55-2	4.31-2
1024	6.26-1	5.25-1	3.49-1	2.17-1	1.29-1	7.55-2	4.31-2
				$z^{(2)}$			
4	6.52-1	5.31-1	3.51-1	2.17-1	1.30-1	7.55-2	4.32-2
8	6.34-1	5.27-1	3.50-1	2.17-1	1.30-1	7.55-2	4.31-2
16	6.28-1	5.25-1	3.50-1	2.17-1	1.30-1	7.55-2	4.31-2
32	6.27-1	5.25-1	3.49-1	2.17-1	1.29-1	7.55-2	4.31-2
64	6.26-1	5.25-1	3.49-1	2.17-1	1.29-1	7.55-2	4.31-2
...	...	...	...	...	...	...	...
1024	6.26-1	5.25-1	3.49-1	2.17-1	1.29-1	7.55-2	4.31-2
				$z^{(3)}$			
4	6.37-1	5.28-1	3.50-1	2.17-1	1.30-1	7.55-2	4.31-2
8	6.28-1	5.25-1	3.49-1	2.17-1	1.29-1	7.55-2	4.31-2
16	6.26-1	5.25-1	3.49-1	2.17-1	1.29-1	7.55-2	4.31-2
...	...	...	...	...	...	...	...
1024	6.26-1	5.25-1	3.49-1	2.17-1	1.29-1	7.55-2	4.31-2

The computational process (1.) and (2.) is repeated for all values of $\varepsilon = 2^{-n}$, $n = 0,2,4,\dots,12$. As a result, we get $E(N,K,\varepsilon)$ for various values of ε, N, K for each of the functions $z^{(1)}(x,t)$, $z^{(2)}(x,t)$, $z^{(3)}(x,t)$. Analyzing these results, we observe convergence of the solutions for increasing $N = K$ for any of the functions $z^{(1)}(x,t)$, $z^{(2)}(x,t)$, $z^{(3)}(x,t)$ and for all values of ε used. In order to show this result we give Table 2 only for $\varepsilon = 2^{-10}$. The error tables for the other values of ε are similar.

In Table 2 the values of $E(N,K)$ are given separately for the functions $z^{(1)}(x,t)$, $z^{(2)}(x,t)$, and $z^{(3)}(x,t)$. For each of them we see decreasing errors for $N=K$, i.e., we have ε-uniform convergence. But the order of convergence, which we observe, is approximately equal to one for all the functions. All errors corresponding to the same values of N, K but to different $z^{(k)}(x,t)$ are similar.

We know that the error of approximation consists of two parts. One part is due to the discretization of the space derivatives and the second is due to the time discretization. We briefly call these components the space error and the time error. Since by the defect correction we improve only the accuracy with respect to time, we expect a decreasing time error. It can be much smaller than the space error and, therefore, the observed error in Table 2 corresponds only to the space error. In order to show this fact, we split the combined error into the space error (Table 3) and the time error (Table 5). The structure of Table 3 is similar to that of Table 2.

Table 4. Ratios of space errors $R^{(s)}(N,K)$ for $\varepsilon = 2^{-10}$

$K \setminus N$	8	16	32	64	128	256
			$z^{(1)}$			
4	1.30	1.54	1.62	1.67	1.72	1.75
8	1.25	1.52	1.62	1.67	1.72	1.75
16	1.22	1.51	1.62	1.67	1.72	1.75
32	1.21	1.51	1.61	1.67	1.72	1.75
64	1.20	1.50	1.61	1.67	1.72	1.75
128	1.20	1.50	1.61	1.67	1.72	1.75
256	1.19	1.50	1.61	1.67	1.72	1.75
512	1.19	1.50	1.61	1.67	1.72	1.75
1024	1.19	1.50	1.61	1.67	1.72	1.75
			$z^{(2)}$			
4	1.23	1.52	1.62	1.67	1.72	1.75
8	1.20	1.51	1.61	1.67	1.72	1.75
16	1.20	1.50	1.61	1.67	1.72	1.75
32	1.19	1.50	1.61	1.67	1.72	1.75
...	...	...	...	...	...	...
1024	1.19	1.50	1.61	1.67	1.72	1.75
			$z^{(3)}$			
4	1.21	1.51	1.62	1.67	1.72	1.75
8	1.20	1.50	1.61	1.67	1.72	1.75
16	1.19	1.50	1.61	1.67	1.72	1.75
...	...	...	...	...	...	...
1024	1.19	1.50	1.61	1.67	1.72	1.75

Table 3 contains the values of the space errrors computed from the formula

$$E^{(s)}(N_i,K) = E(N_i,K) - E(N_{i+1},K), \quad i = 3,4,\ldots,9, \quad N_i = 2^i.$$

We see that the errors are the same for all different K. The errors in Table 2 and Table 3 have the same order.

From Table 3 we deduce Table 4, where the ratios of the space errors is given by

$$R^{(s)}(N_i, K) = E^{(s)}(N_i, K)/E^{(s)}(N_{i+1}, K), \quad i = 3, 4, \ldots, 8.$$

In Table 4 we see the first order of the convergence with respect to the space variable up to a small logarithmic factor.

In a similar way we construct Table 5 for the time errors

$$E^{(t)}(N, K_j) = E(N, K_j) - E(N, K_{j+1}), \quad j = 2, 3, \ldots, 9$$

and Table 6 for their ratios

$$R^{(t)}(N, K_j) = E^{(t)}(N, K_j)/E^{(t)}(N, K_{j+1}), \quad j = 2, 3, 4, \ldots, 8, \quad K_j = 2^j.$$

At last, we now observe very interesting results in Table 5:

Table 5. Time errors $E^{(t)}(N, K)$ for $\varepsilon = 2^{-10}$

$K \setminus N$	4	8	16	32	64	128	256	512	1024
					$z^{(1)}$				
4	1.30-1	4.55-2	9.96-3	2.48-3	1.33-3	1.07-3	9.59-4	8.97-4	8.61-4
8	6.71-2	2.51-2	5.80-3	1.49-3	8.10-4	6.54-4	5.85-4	5.47-4	5.25-4
16	3.38-2	1.31-2	3.13-3	8.19-4	4.46-4	3.60-4	3.22-4	3.01-4	2.89-4
32	1.69-2	6.69-3	1.62-3	4.28-4	2.34-4	1.89-4	1.69-4	1.58-4	1.52-4
64	8.48-3	3.38-3	8.26-4	2.19-4	1.20-4	9.66-5	8.64-5	8.08-5	7.76-5
128	4.24-3	1.70-3	4.17-4	1.11-4	6.05-5	4.89-5	4.37-5	4.09-5	3.93-5
256	2.12-3	8.51-4	2.09-4	5.57-5	3.04-5	2.46-5	2.20-5	2.06-5	1.98-5
512	**1.06-3**	**4.26-4**	**1.05-4**	**2.79-5**	**1.53-5**	**1.23-5**	**1.10-5**	**1.03-5**	**9.90-6**
					$z^{(2)}$				
4	5.89-2	2.47-2	6.16-3	1.65-3	9.07-4	7.32-4	6.55-4	6.13-4	5.88-4
8	1.67-2	7.50-3	1.97-3	5.34-4	2.92-4	2.36-4	2.11-4	1.97-4	1.90-4
16	4.42-3	2.06-3	5.53-4	1.50-4	8.22-5	6.64-5	5.94-5	5.55-5	5.33-5
32	1.14-3	5.40-4	1.47-4	3.99-5	2.18-5	1.76-5	1.57-5	1.47-5	1.41-5
64	**2.88-4**	**1.38-4**	**3.77-5**	**1.03-5**	**5.60-6**	**4.51-6**	**4.04-6**	**3.78-6**	**3.63-6**
128	7.25-5	3.50-5	9.57-6	2.60-6	1.42-6	1.14-6	1.02-6	9.57-7	9.19-7
256	1.82-5	8.79-6	2.41-6	6.55-7	3.57-7	2.88-7	2.58-7	2.41-7	2.31-7
512	4.55-6	2.20-6	6.04-7	1.64-7	8.96-8	7.23-8	6.46-8	6.04-8	5.81-8
					$z^{(3)}$				
4	3.02-2	1.26-2	3.13-3	8.14-4	4.38-4	3.53-4	3.16-4	2.95-4	2.84-4
8	4.18-3	1.95-3	5.06-4	1.29-4	6.81-5	5.47-5	4.89-5	4.57-5	4.39-5
16	**5.77-4**	**2.77-4**	**7.20-5**	**1.79-5**	**9.35-6**	**7.50-6**	**6.71-6**	**6.27-6**	**6.02-6**
32	7.60-5	3.70-5	9.59-6	2.34-6	1.22-6	9.79-7	8.75-7	8.18-7	7.86-7
64	9.75-6	4.77-6	1.24-6	3.00-7	1.56-7	1.25-7	1.12-7	1.04-7	1.00-7
128	1.24-6	6.06-7	1.57-7	3.79-8	1.97-8	1.58-8	1.41-8	1.32-8	1.27-8
256	1.55-7	7.64-8	1.98-8	4.77-9	2.47-9	1.98-9	1.77-9	1.66-9	1.59-9
512	1.95-8	9.59-9	2.48-9	5.98-10	3.10-10	2.49-10	2.22-10	2.08-10	1.99-10

1. We see that the time error is considerably smaller than the space error. This explains the fact that we could not see the effect of the time error in Table 2.

2. The errors for $z^{(1)}$ are larger than those for $z^{(2)}$ and, in turn, the errors for $z^{(2)}$ are larger than those for $z^{(3)}$.

3. We see that approximately the same error ($\approx 10^{-6}$) is obtained for $z^{(1)}$ at $K = 512$, for $z^{(2)}$ at $K = 64$, and for $z^{(3)}$ at $K = 16$. Because the computational work is proportional to K, we conclude that the amount of work is reduced by the defect correction.

4. Table 6, which shows the ratios of the errors, in fact confirms the theoretical order of convergence as derived in Section 6. From the theory, the solution $z^{(1)}(x,t)$ of problem (6.3), (4.1) converges with the rate $\mathcal{O}(\tau)$ (see estimate (4.2) and Theorem 4.1). The solution $z^{(2)}(x,t)$ of problem (6.4), (4.1), where $z^{(1)}(x,t)$ is the solution of problem (6.3), (4.1), converges with the rate $\mathcal{O}(\tau^2)$ (see estimate (6.7) and Theorem 6.1). The solution $z^{(3)}(x,t)$ of problem (6.8), (4.1), where $z^{(2)}(x,t)$ and $z^{(1)}(x,t)$ are the solutions of problems (6.4), (4.1) and (6.3), (4.1), respectively, converges with the rate $\mathcal{O}(\tau^3)$ (see estimate (6.10) and Theorem 6.2). The corresponding reduction factors are easily recognized in Table 6.

Table 6. Ratios of time errors $R^{(t)}(N,K)$ for $\varepsilon = 2^{-10}$

$K \setminus N$	4	8	16	32	64	128	256	512	1024
					$z^{(1)}$				
4	1.94	1.81	1.72	1.66	1.64	1.64	1.64	1.64	1.64
8	1.98	1.91	1.86	1.82	1.82	1.82	1.82	1.82	1.82
16	2.00	1.96	1.93	1.91	1.91	1.91	1.91	1.91	1.91
32	2.00	1.98	1.96	1.96	1.95	1.95	1.95	1.95	1.95
64	2.00	1.99	1.98	1.98	1.98	1.98	1.98	1.98	1.98
128	2.00	2.00	1.99	1.99	1.99	1.99	1.99	1.99	1.99
256	2.00	2.00	2.00	1.99	1.99	1.99	1.99	1.99	1.99
					$z^{(2)}$				
4	3.53	3.29	3.14	3.10	3.10	3.10	3.10	3.10	3.10
8	3.78	3.64	3.55	3.55	3.55	3.56	3.56	3.56	3.56
16	3.89	3.82	3.77	3.77	3.78	3.78	3.78	3.78	3.78
32	3.95	3.91	3.89	3.89	3.89	3.89	3.89	3.89	3.89
64	3.97	3.95	3.94	3.94	3.94	3.95	3.95	3.95	3.95
128	3.99	3.98	3.97	3.97	3.97	3.97	3.97	3.97	3.97
256	3.99	3.99	3.99	3.99	3.99	3.99	3.99	3.99	3.99
					$z^{(3)}$				
4	7.23	6.48	6.19	6.32	6.44	6.45	6.46	6.46	6.46
8	7.25	7.03	7.02	7.21	7.29	7.29	7.30	7.30	7.30
16	7.59	7.49	7.51	7.62	7.66	7.66	7.66	7.66	7.66
32	7.79	7.74	7.76	7.81	7.83	7.83	7.83	7.83	7.83
64	7.90	7.87	7.88	7.91	7.92	7.92	7.92	7.92	7.92
128	7.95	7.94	7.94	7.95	7.96	7.96	7.96	7.96	7.96
256	7.97	7.97	7.97	7.98	7.98	7.98	7.98	7.98	7.98

8. *A priori* estimates of the solution and its derivatives

In this Section we rely on the *a priori* estimates for the solution of problem (2.1) on the domain $G = D \times [0,T]$, and its derivatives as derived for elliptic and parabolic equations in [2, 14, 17].

We denote by $H^{(\vartheta)}(\overline{G}) = H^{\vartheta,\vartheta/2}(\overline{G})$ the Hölder space, where ϑ is an arbitrary positive number [18]. We suppose that the functions $f(x,t)$ and $\varphi(x)$, $\psi(x,t)$ satisfy compatibility conditions at the corner points, so that the solution of the boundary-value problem is smooth for each fixed value of the parameter ε.

For simplicity, we assume that the following conditions hold at the end points of the interval $\overline{D}$ and at the corner points $S_0 \cap \overline{S}_1$:

$$\begin{aligned} &\frac{\partial^k}{\partial x^k}\varphi(x) = 0, \quad \frac{\partial^{k_0}}{\partial t^{k_0}}\psi(x,t) = 0, \quad k + 2k_0 \leqslant [\ \vartheta\ ] + 2n, \\ &\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} f(x,t) = 0, \qquad\qquad\qquad k + 2k_0 \leqslant [\ \vartheta\ ] + 2n - 2, \end{aligned} \tag{8.1}$$

where $[\ \vartheta\ ]$ is the integer part of a number ϑ, $\vartheta > 0$, $n \geqslant 0$ is an integer. We also suppose that $[\ \vartheta\ ] + 2n \geqslant 2$.

Using interior *a priori* estimates and estimates up to the boundary for the regular function $\widetilde{u}(\xi,t)$ (see [18]), where $\widetilde{u}(\xi,t) = u(x(\xi),t)$, $\xi = x/\varepsilon$, we find for $(x,t) \in \overline{G}$ the estimate

$$\left| \frac{\partial^{k+k_0}}{\partial x^k\, \partial t^{k_0}} u(x,t) \right| \leqslant M\, \varepsilon^{-k}, \quad k + 2k_0 \leqslant 2n + 4, \quad n \geqslant 0. \tag{8.2}$$

This estimate holds, for example, for

$$u \in H^{(2n+4+\nu)}(\overline{G}), \quad \nu > 0, \tag{8.3}$$

where ν is some small number.

For example, (8.3) is guaranteed for the solution of (2.1) if the coefficients satisfy inclusions a, c, p, $f \in H^{(\vartheta+2n-2)}(\overline{G})$, $\varphi \in H^{(\vartheta+2n)}(\overline{D})$, α, β, $\psi \in H^{(\vartheta+2n)}(\overline{S}^L)$, $\vartheta > 4$, $n \geqslant 0$ and condition (8.1) is fulfilled.

In fact we need a more accurate estimate than (8.2). Therefore, we represent the solution of the boundary-value problem (2.1) in the form of the sum

$$u(x,t) = U(x,t) + W(x,t), \quad (x,t) \in \overline{G}, \tag{8.4}$$

where $U(x,t)$ represents the regular part, and $W(x,t)$ the singular part, i.e., the parabolic boundary layer. The function $U(x,t)$ is the smooth solution of equation (2.1a) satisfying conditions (2.1c) for $t = 0$ and (2.1b) for $x = 1$. For example, under suitable assumptions for the data of the problem, we can consider the solution of the boundary-value problem for equation (2.1a) smoothly continued onto the domain $\overline{G}^*$ extended beyond of S_1^L ($\overline{G}^*$ is a sufficiently large neighborhood of $\overline{G}$ beyond of S_1^L). On the domain $\overline{G}$ the coefficients and the initial value of the extended problem are the same as for (2.1). Then the function $U(x,t)$ is the restriction (on $\overline{G}$) of the solution to the extended problem, and $U \in H^{(2n+4+\nu)}(\overline{G})$, $\nu > 0$. The function $W(x,t)$ is the solution of a boundary-value problem for the parabolic equation

$$\begin{aligned} L_{(2.1)}\, W(x,t) &= 0, \quad (x,t) \in G, && (8.5)\\ l_{(2.1)}\, W(x,t) &= l\,\big(u(x,t) - U(x,t)\big), \quad (x,t) \in S^L, \\ W(x,t) &= u(x,t) - U(x,t) = 0, \quad (x,t) \in S_0. \end{aligned}$$

If (8.3) is true, then $W \in H^{(2n+4+\nu)}(\overline{G})$. Now, for the functions $U(x,t)$ and $W(x,t)$ we derive the estimates

$$\left| \frac{\partial^{k+k_0}}{\partial x^k\,\partial t^{k_0}} U(x,t) \right| \leqslant M, \tag{8.6}$$

$$\left| \frac{\partial^{k+k_0}}{\partial x^k\,\partial t^{k_0}} W(x,t) \right| \leqslant M\,\varepsilon^{-k} \exp(-m_{(8.7)}\varepsilon^{-1} r(x,\gamma)\,), \tag{8.7}$$

$$(x,t) \in \overline{G}, \quad k + 2k_0 \leqslant 2n+2,$$

where $r(x,\gamma)$ is the distance between the point $x \in [0,1]$ and the set γ which is the endpoints of the segment $[0,1]$, $m_{(8.7)}$ is a sufficiently small positive number. Estimates (8.6) and (8.7) hold, for example, when

$$U,\ W \in H^{\,(2n+4+\nu)}(\overline{G}), \quad \nu > 0. \tag{8.8}$$

Inclusions (8.8) are guaranteed if $a,\ c,\ p,\ f \in H^{\,(\vartheta+2n-2)}(\overline{G})$, $\varphi \in H^{\,(\vartheta+2n)}(\overline{D})$, $\alpha,\ \beta,\ \psi \in H^{\,(\vartheta+2n)}(\overline{S}^{\,L})$, $\vartheta > 4$, $n \geqslant 0$ and condition (8.1) is fulfilled. We summarize these results in the following theorem.

Theorem 8.1. *Assume in equation* (2.1) *that*

$$a,\ b,\ c,\ p,\ f \in H^{\,(\vartheta+2n-2)}(\overline{G}), \quad \varphi \in H^{\,(\vartheta+2n)}(\overline{D})$$
$$\alpha,\ \beta,\ \psi \in H^{\,(\vartheta+2n)}(\overline{S}^{\,L}), \quad \vartheta > 4,\ n \geqslant 0$$

and let conditions (2.2)*,* (8.1) *be fulfilled. Then, for the solution* $u(x,t)$ *of problem* (2.1) *and for its components from the representation* (8.4)*, it follows that* $u,\ U,\ W \in H^{\,(\vartheta+2n)}(\overline{G})$ *and that estimates* (8.2)*,* (8.6)*,* (8.7) *hold.*

The proof of the theorem is similar to the proof in [2], where the equation

$$\varepsilon\, a(x,t)\,\frac{\partial^2}{\partial x^2} u(x,t) + b(x,t)\,\frac{\partial}{\partial x} u(x,t) - c(x,t)u(x,t) - p(x,t)\,\frac{\partial u}{\partial t}(x,t) = f(x,t)$$

was considered in the case of the Dirichlet boundary conditions.

9. The proof of Theorems 6.1 and 6.2

9.1. The proof of Theorem 6.1

Let us show that the function $\delta_{\overline{t}}\, z(x,t)$, where $z(x,t) = z_{(6.3)}(x,t)$ is the solution of the difference problem (6.3), approximates the function $\delta_{\overline{t}}\, u(x,t)$ ε-uniformly. For simplicity we assume $a(x,t)$, $b(x,t)$ and $\alpha(x,t)$ to be constant on $\overline{G}$ and S^L. The function $\delta_{\overline{t}}\, z(x,t)$ is the solution of the difference problem

$$\Lambda_{(9.1)}\, \delta_{\overline{t}}\, z(x,t) = f_{(9.1)}(x,t), \qquad (x,t) \in G_h^{[1]}, \tag{9.1a}$$

$$\lambda_{(9.1)}\, \delta_{\overline{t}}\, z(x,t) = \psi_{(9.1)}(x,t), \qquad (x,t) \in S_h^{[1]L}, \tag{9.1b}$$

$$\delta_{\overline{t}}\, z(x,t) = \varphi_{(9.1)}(x,t), \qquad (x,t) \in S_{0h}^{[1]}. \tag{9.1c}$$

Here $\overline{G}_h^{[k]} = \overline{G}_h \cap \{t \geqslant k\tau\}, \quad G_h^{[k]} = G_h \cap \{t > k\tau\}, \quad S_h^{[k]} = \overline{G}_h^{[k]} \setminus G_h^{[k]},$

$$S_h^{[k]} = S_{0h}^{[k]} \cup S_h^{[k]L}, \quad S_{0h}^{[k]} = \overline{G}_h \cap \{t > k\tau\}, \quad k \geqslant 1,$$

$$\Lambda_{(9.1)}\delta_{\bar{t}} z(x,t) \equiv \{ \varepsilon\, a\delta_{\overline{x}\widehat{x}} + b\delta_x - \check{c}(x,t) - p_{\bar{t}}(x,t) - \check{p}(x,t)\,\delta_{\bar{t}} \}\; \delta_{\bar{t}} z(x,t),$$

$$f_{(9.1)}(x,t) = f_{\bar{t}}(x,t) + c_{\bar{t}}(x,t)\, z(x,t), \quad (x,t) \in G_h^{[1]}$$

$$\lambda_{(9.1)}\, \delta_{\bar{t}}\, z(x,t) \equiv \left\{ \varepsilon\, \alpha \left\{ \begin{array}{ll} -\delta_x, & x = 0, \\ \delta_{\overline{x}}, & x = 1 \end{array} \right\} + \check{\beta}(x,t) \right\} \delta_{\bar{t}}\, z(x,t),$$

$$\psi_{(9.1)}(x,t) = \psi_{\bar{t}}(x,t) - \beta_{\bar{t}}(x,t)\; z(x,t), \quad (x,t) \in S_h^{[1]L},$$

$$\varphi_{(9.1)}(x,t) = \varphi^0_{(9.1)}(x) \equiv \tau^{-1}\, [\, z(x,\tau) - \varphi(x)\,]\,, \quad t = \tau, \quad (x,t) \in S_{0h}^{[1]},$$

$\check{v}(x,t) = v(x,t-\tau)$ where $\check{v}(x,t)$ is one of the functions $\check{c}(x,t)$, $\check{p}(x,t)$, $\check{\beta}(x,t)$.

The function $\delta_{\bar{t}}\, u(x,t) \equiv [u(x,t) - u(x,t-\tau)]/\tau$, $(x,t) \in \overline{G}$, $t \geqslant \tau$ is the solution of the differential problem

$$L_{(9.2)}\, \delta_{\bar{t}}\, u(x,t) = f_{(9.2)}(x,t), \quad (x,t) \in G^{[1]}, \tag{9.2a}$$

$$l_{(9.2)}\, \delta_{\bar{t}}\, u(x,t) = \psi_{(9.2)}(x,t), \quad (x,t) \in S^{[1]L}, \tag{9.2b}$$

$$\delta_{\bar{t}}\, u(x,t) = \varphi_{(9.2)}(x,t), \quad (x,t) \in S_0^{[1]}. \tag{9.2c}$$

Here

$$\overline{G}^{[k]} = \overline{G} \cap \{t \geqslant k\tau\}\,, \quad G^{[k]} = G \cap \{t > k\tau\}\,, \quad S^{[k]} = \overline{G}^{[k]} \setminus G^{[k]},$$
$$S^{[k]} = S_0^{[k]} \cup S^{[k]L}, \quad S_0^{[k]} = \overline{G} \cap \{t > k\tau\}\,, \quad k \geqslant 1,$$

$$L_{(9.2)}\delta_{\bar{t}} u(x,t) \equiv \left\{ \varepsilon a \frac{\partial^2}{\partial x^2} + b\frac{\partial}{\partial x} - \check{c}(x,t) - p_{\bar{t}}(x,t) - \check{p}(x,t)\frac{\partial}{\partial t} \right\} \delta_{\bar{t}}\, u(x,t),$$

$$l_{(9.2)}\, \delta_{\bar{t}}\, u(x,t) \equiv \left\{ \varepsilon\, \alpha \left\{ \begin{array}{ll} -(d/dx), & x = 0, \\ (d/dx), & x = 1 \end{array} \right\} + \check{\beta}(x,t) \right\} \delta_{\bar{t}}\, u(x,t),$$

$$f_{(9.2)}(x,t) = f_{\bar{t}}(x,t) + c_{\bar{t}}(x,t)u(x,t) + p_{\bar{t}}(x,t) \left(\frac{\partial u}{\partial t}(x,t) - \delta_{\bar{t}}\, u(x,t) \right),$$

$$\psi_{(9.2)}(x,t) = \psi_{\bar{t}}(x,t) - \beta_{\bar{t}}(x,t)\, u(x,t), \quad (x,t) \in S^{[1]L},$$

$$\varphi_{(9.2)}(x,t) = \varphi^0_{(9.2)}(x) \equiv \tau^{-1}\, [\, u(x,\tau) - \varphi(x)\,]\,, \quad t = \tau, \quad (x,t) \in S_0^{[1]}.$$

Let us estimate

$$\varphi^0_{(9.2)}(x) - \varphi^0_{(9.1)}(x) = \tau^{-1}\omega(x,\tau),$$

where

$$\omega(x,t) = u(x,t) - z(x,t), \quad (x,t) \in \overline{G}_h.$$

The function $\omega(x,t)$ is the solution of the problem

$$\Lambda_{(6.3)}\,\omega(x,t) = (\Lambda_{(6.3)} - L_{(2.1)})\,u(x,t), \quad (x,t) \in G_h,$$
$$\lambda_{(6.3)}\,\omega(x,t) = (\lambda_{(6.3)} - l_{(2.1)})\,u(x,t), \quad (x,t) \in S_h^L, \quad \omega(x,t) = 0, \quad (x,t) \in S_{0h}.$$

The above assumptions and Theorem 8.1 lead to the estimates of the truncation error (the deduction technique for these estimates are shown, for example, in [2,3,7])

$$\left|\left(\Lambda_{(6.3)} - L_{(2.1)}\right) U(x,t)\right| \leqslant M\,\left[N^{-1}\ln\,N + \tau\right], \quad (x,t) \in G_h$$

$$\left|\left(\Lambda_{(6.3)} - L_{(2.1)}\right) W(x,t)\right| \leqslant M\left[\varepsilon^{-1}N^{-1}\ln N\exp(-m\varepsilon^{-1}x) + \tau\right], \quad (x,t) \in G_h, \quad x \leqslant \sigma,$$

where $U(x,t)$ and $W(x,t)$ are the regular and singular parts of the solution from (8.4); $\sigma = \sigma_{(4.1)}$, $m = m_{(8.7)}$. For the components $W(x,t)$ and $W^h(x,t)$ the following estimate is also satisfied

$$|W(x,t)|\,, \quad \left|W^h(x,t)\right| \leqslant M\,N^{-1}, \quad (x,t) \in \overline{G}_h, \quad x \geqslant \sigma.$$

Here $W^h(x,t)$ is the solution of the problem

$$\Lambda_{(6.3)}\,W^h(x,t) = 0, \qquad (x,t) \in G_h,$$
$$\lambda_{(6.3)}\,W^h(x,t) = l_{(2.1)}\,W(x,t), \quad (x,t) \in S^L, \quad W^h(x,t) = W(x,t), \quad (x,t) \in S_{0h}.$$

Using the maximum principle we estimate $\omega(x,t)$

$$|\,\omega(x,t)\,| \leqslant M\left[N^{-1}\ln N + \tau\right]\,t, \quad (x,t) \in \overline{G}_h.$$

Further, for the derivatives we proceed similarly. On the boundary we have

$$|\,\delta_{\bar t}u(x,\tau) - \delta_{\bar t}z(x,\tau)\,| = \left|\,\varphi^0_{(9.2)}(x) - \varphi^0_{(9.1)}(x)\,\right| \leqslant M\,\left[N^{-1}\ln N + \tau\right], \; (x,t) \in S_{0h}^{[1]}, \; t = \tau,$$

i.e. the function $\delta_{\bar t}z(x,\tau)$ approximates $\delta_{\bar t}u(x,\tau)$ ε-uniformly. Now, it is easy to see that the solution of the difference problem (9.1) approximates the solution of the differential problem (9.2) for the divided difference. Thus, using the same argument as above, we derive the estimate

$$|\,\delta_{\bar t}u(x,t) - \delta_{\bar t}z(x,t)\,| \leqslant M\,\left[N^{-1}\ln\,N + \tau\right], \quad (x,t) \in \overline{G}_h^{[1]}. \tag{9.3}$$

Now, for the 2nd difference derivative we show that under condition (6.6) the function $\delta_{2\bar t}\,z(x,t)$ approximates the function $\delta_{2\bar t}\,u(x,t)$ ε-uniformly on the set $\overline{G}_h^{[2]}$. So, the functions $\delta_{2\bar t}z(x,t)$ and $\delta_{2\bar t}u(x,t)$ are solutions of the equations

$$\Lambda_{(9.4)}\delta_{2\bar t}z(x,t) = f_{(9.4)}(x,t), \quad (x,t) \in G_h^{[2]}, \tag{9.4a}$$

$$L_{(9.5)}\delta_{2\bar t}u(x,t) = f_{(9.5)}(x,t), \quad (x,t) \in G_h^{[2]}. \tag{9.5a}$$

The equations are found by applying the operator $\delta_{\bar t}$ to equations (9.1a) and (9.2a). At the left and the right boundaries the following conditions are satisfied:

$$\lambda_{(9.4)}\,\delta_{2\bar t}\,z(x,t) = \psi_{(9.4)}(x,t), \quad (x,t) \in S_h^{[2]L}, \tag{9.4b}$$

$$l_{(9.5)}\,\delta_{2\bar t}u(x,t) = \psi_{(9.5)}(x,t), \quad (x,t) \in S_h^{[2]L}, \tag{9.5b}$$

where

$$\lambda_{(9.4)}\,\delta_{2\bar t}\,z(x,t) \equiv \left\{\varepsilon\,\alpha \left\{\begin{array}{ll} -\delta_x, & x=0, \\ \delta_{\overline{x}}, & x=1 \end{array}\right\} + \check{\check{\beta}}(x,t)\right\}\delta_{2\bar t}\,z(x,t), \quad (x,t)\in S_h^{[2]}, \tag{9.4c}$$

$$\psi_{(9.4)}(x,t) = \delta_{2\bar t}\,\psi(x,t) - 2\,\delta_{\bar t}\,\check\beta(x,t)\,\delta_{\bar t}\,z(x,t) - \delta_{2\bar t}\,\beta(x,t)\,z(x,t), \quad (x,t)\in S_h^{[2]},$$

$$l_{(9.5)}\,\delta_{2\bar t}\,u(x,t) \equiv \left\{\varepsilon\,\alpha \left\{\begin{array}{ll} -(\partial/\partial x), & x=0, \\ (\partial/\partial x), & x=1 \end{array}\right\} + \check{\check{\beta}}(x,t)\right\}\delta_{2\bar t}\,u(x,t), \quad (x,t)\in S^{[2]}. \tag{9.5c}$$

$$\psi_{(9.5)}(x,t) = \delta_{2\bar t}\,\psi(x,t) - 2\,\delta_{\bar t}\,\check\beta(x,t)\,\delta_{\bar t}\,u(x,t) - \delta_{2\bar t}\,\beta(x,t)\,u(x,t), \quad (x,t)\in S^{[2]},$$

First we estimate

$$\varphi^0_{(9.5)}(x) - \varphi^0_{(9.4)}(x) \equiv \delta_{2\bar t}\,u(x,t) - \delta_{2\bar t}\,z(x,t), \quad t=2\tau.$$

For this purpose we write the function $u(x,t)$ in a Taylor expansion in t

$$u(x,t) = a^{(1)}(x)t + a^{(2)}(x)t^2 + v_2(x,t) \equiv u^{[2]}(x,t) + v_2(x,t), \quad (x,t)\in\overline{G}, \tag{9.6}$$

where the coefficients $a^{(1)}(x)$, $a^{(2)}(x)$ should be determined. Inserting $u(x,t)$, in its form (9.6), into equation (2.1a), we come to the system

$$-p(x,0)a^{(1)}(x) = f(x,0),$$
$$-2p(x,0)a^{(2)}(x) + \varepsilon a\frac{\partial^2}{\partial x^2}a^{(1)}(x) + b\frac{\partial}{\partial x}a^{(1)}(x) - \left(c(x,0) + \frac{\partial}{\partial t}p(x,0)\right)a^{(1)}(x) = \frac{\partial}{\partial t}f(x,0)$$

from which the functions $a^{(1)}(x)$, $a^{(2)}(x)$ can be found successively. The function $v_2(x,t)$ is the solution of the boundary-value problem

$$L_{(2.1)}v_2(x,t) = f_{(9.7)}(x,t) \equiv f(x,t) - L_{(2.1)}u^{[2]}(x,t), \quad (x,t)\in G, \tag{9.7}$$
$$l_{(2.1)}v_2(x,t) = \psi_{(9.7)}(x,t) \equiv \psi(x,t) - l_{(2.1)}u^{[2]}(x,t), \quad (x,t)\in S^L,$$
$$v_2(x,t) = \varphi_{(9.7)}(x,t) \equiv \varphi(x) - u^{[2]}(x,t), \quad (x,t)\in S_0.$$

Estimating $f_{(9.7)}(x,t)$, $\psi_{(9.7)}(x,t)$, and $\varphi_{(9.7)}(x,t)$, and using the maximum principle, we derive the estimate

$$|\,v_2(x,t)\,| \leqslant M\,t^3, \quad (x,t)\in\overline{G}. \tag{9.8}$$

Further we have to construct the function $z(x,t)$ in the form

$$z(x,t) = \left(b_0^{(1)}(x) + b_1^{(1)}(x)\tau\right)t + b_0^{(2)}(x)\,t^2 + v_2^h(x,t) \equiv z^{[2]}(x,t) + v_2^h(x,t), \quad (x,t)\in\overline{G}_h,$$

i.e., as an expansion in powers of τ and t. Inserting $z(x,t)$ into equation (6.3), we arrive at the equations

$$-p(x,0)b_0^{(1)}(x) = f(x,0), \quad b_0^{(2)}(x) + b_1^{(1)}(x) = 0,$$
$$b_0^{(1)}(x) + b\frac{\partial}{\partial x}b_0^{(1)}(x) - \left(c(x,0) + \frac{\partial}{\partial t}p(x,0)\right)b_0^{(1)}(x) = \frac{\partial}{\partial t}f(x,0).$$

So, we have

$$z^{[2]}(x,t) = u^{[2]}(x,t) + b_1^{(1)}(x)\tau t, \quad (x,t) \in \overline{G}_h. \tag{9.9}$$

The function $v_2^h(x,t)$ is the solution of the discrete boundary-value problem

$$\begin{aligned} \Lambda_{(6.3)} v_2^h(x,t) &= f_{(9.10)}(x,t) \equiv f(x,t) - \Lambda_{(6.3)} z^{[2]}(x,t), \quad (x,t) \in G_h, \\ \lambda_{(6.3)} v_2^h(x,t) &= \psi_{(9.10)}(x,t) \equiv \psi(x,t) - \lambda_{(6.3)} z^{[2]}(x,t), \quad (x,t) \in S_h^L, \\ v_2^h(x,t) &= \varphi_{(9.10)}(x,t) \equiv \varphi(x,t) - z^{[2]}(x,t), \quad (x,t) \in S_{0h}. \end{aligned} \tag{9.10}$$

Taking into account estimates of the functions $f_{(9.10)}(x,t)$ and $\varphi_{(9.10)}(x,t)$, we derive the estimate

$$\left| v_2^h(x,t) \right| \leqslant M \left[N^{-1} \ln N + t \right] t^2, \quad (x,t) \in \overline{G}_h. \tag{9.11}$$

By virtue of relations (9.8), (9.9) and (9.11), the following inequality is valid:

$$\begin{aligned} \left| \varphi^0_{(9.5)}(x) - \varphi^0_{(9.4)}(x) \right| &= | \; \delta_{2\bar{t}} u(x,t) - \delta_{2\bar{t}} z(x,t) \; | \\ &\leqslant M \left[N^{-1} \ln N + \tau \right], \quad (x,t) \in \overline{G}_h, \; t = 2\tau. \end{aligned} \tag{9.12}$$

We continue by estimating $\delta_{2\bar{t}} u(x,t) - \delta_{2\bar{t}} z(x,t)$ for $t > 2\tau$. Note that the functions $\delta_{2\bar{t}} u(x,t)$ and $\delta_{2\bar{t}} z(x,t)$ are solutions of the differential and difference equations, obtained from equations (2.1) and (6.3), respectively, by applying the operator $\delta_{2\bar{t}}$. Moreover, the difference equation (9.4a) for $\delta_{2\bar{t}} z(x,t)$ approximates the differential equation (9.5a) for $\delta_{2\bar{t}} u(x,t)$ ε-uniformly. On the boundary S_h^L we have equations (9.4b), (9.5b). Taking into account estimates (9.12) and (4.2), (9.3), we find

$$| \; \delta_{2\bar{t}} u(x,t) - \delta_{2\bar{t}} z(x,t) \; | \leqslant M \left[N^{-1} \ln N + \tau \right], \quad (x,t) \in \overline{G}_h, \quad t \geqslant 2\tau. \tag{9.13}$$

So, we come to the estimates

$$\begin{aligned} \left| \delta_{\bar{t}} u(x,t) - \delta_{\bar{t}} z^{(1)}(x,t) \right| &\leqslant M \left[N^{-1} \ln N + \tau \right], \quad (x,t) \in \overline{G}_h, \quad t \geqslant \tau, \\ \left| \delta_{2\bar{t}} u(x,t) - \delta_{2\bar{t}} z^{(1)}(x,t) \right| &\leqslant M \left[N^{-1} \ln N + \tau \right], \quad (x,t) \in \overline{G}_h, \quad t \geqslant 2\tau, \\ \left| u(x,t) - z^{(2)}(x,t) \right| &\leqslant M \left[N^{-1} \ln N + \tau^2 \right], \quad (x,t) \in \overline{G}_h. \end{aligned} \tag{9.14}$$

This completes the proof.

Now, as a direct consequence of the theorem, we make two remarks to prepare the proof of Theorem 6.2.

Remark 9.1. Above we have found (9.13) for $z^{(k)}(x,t)$, $k = 1$. In completely the same way we derive this bound for $k = 2$, so that we obtain

$$\left| \delta_{2\bar{t}} u(x,t) - \delta_{2\bar{t}} z^{(k)}(x,t) \right| \leqslant M \left[N^{-1} \ln N + \tau^k \right], \quad (x,t) \in \overline{G}_h, \quad t \geqslant k\tau, \quad k \leqslant 2. \tag{9.15}$$

Remark 9.2. Making use of (9.15), similar to the derivation of estimate (9.14), we also find

$$\left| \delta_{3\bar{t}} u(x,t) - \delta_{3\bar{t}} z^{(1)}(x,t) \right| \leqslant M \left[N^{-2} \ln N + \tau \right], \quad (x,t) \in \overline{G}_h, \quad t \geqslant 3\tau. \tag{9.16}$$

We briefly indicate the differences with the proof given above for (9.14). To estimate the difference between $\delta_{3\bar{t}}\, u(x,t)$ and $\delta_{3\bar{t}}\, z(x,t)$ for $t = 3\tau$, we represent the function $u(x,t)$ (with condition (6.9)) in the form

$$u(x,t) = a^{(2)}(x)t^2 + a^{(3)}(x)t^3 + v_3(x,t) \equiv u^{[3]}(x,t) + v_3(x,t), \quad (x,t) \in \overline{G},$$

and the function $z(x,t)$ in the form

$$z(x,t) = u^{[3]}(x,t) + (b_1^{(1)}(x)\tau + b_2^{(1)}(x)\tau^2)t + b_1^{(2)}(x)\tau t^2 + v_3^h(x,t) \equiv$$
$$\equiv z^{[3]}(x,t) + v_3^h(x,t), \quad (x,t) \in \overline{G}_h.$$

The coefficients of these expansions are found using equations (2.1) and (6.3), respectively. For the coefficients we have the system

$$-2p(x,0)a^{(2)}(x) = \frac{\partial}{\partial t} f(x,0), \quad -b_1^{(1)}(x) + a^{(2)}(x) = 0, \quad -b_2^{(1)}(x) - a^{(3)}(x) + b_1^{(2)}(x) = 0,$$
$$-3p(x,0)a^{(3)}(x) + \varepsilon a \frac{\partial^2}{\partial x^2} a^{(2)}(x) + b\frac{\partial}{\partial x} a^{(2)}(x) - \left(c(x,0) + 2\frac{\partial}{\partial t} p(x,0)\right) a^{(2)}(x) = \frac{1}{2}\frac{\partial^2}{\partial t^2} f(x,0),$$
$$-2p(x,0)b_1^{(2)}(x) + \frac{\partial}{\partial t} p(x,0) a^{(2)}(x) + 3p(x,0)a^{(3)}(x) -$$
$$-\left(\frac{\partial}{\partial t} p(x,0) + c(x,0)\right) b_1^{(1)}(x) + \varepsilon a \frac{\partial^2}{\partial x^2} b_1^{(1)}(x) + b\frac{\partial}{\partial x} b_1^{(1)}(x) = 0.$$

The unknown functions $a^{(2)}$, $a^{(3)}$, $b_1^{(1)}$, $b_1^{(2)}$, $b_2^{(1)}$ can be found successively. For the function $v_3(x,t)$ and $v_3^h(x,t)$ the following estimates are derived

$$|\, v_3(x,t)\,| \leqslant M\, t^4, \quad (x,t) \in \overline{G},$$
$$\left|\, v_3^h(x,t)\,\right| \leqslant M\left[N^{-1}\ln N + t\right] t^3, \quad (x,t) \in \overline{G}_h.$$

From these inequalities and the expression for $z^{[3]}(x,t)$ it follows that (9.16) holds ε-uniformly for $t = 3\tau$. The remainder of the proof of the estimate (9.16) repeats with small variations the proof of the estimate (9.14).

9.2. The proof of Theorem 6.2

Notice that, if the following relations hold for the functions $z^{(1)}(x,t)$ and $z^{(2)}(x,t)$:

$$\left|\, \delta_{3\bar{t}}\, u(x,t) - \delta_{3\bar{t}}\, z^{(1)}(x,t)\,\right| \leqslant M\left[N^{-1}\ln N + \tau\right], \quad (x,t) \in G_h, \quad t \geqslant 3\tau, \qquad (9.17)$$
$$\left|\, \delta_{2\bar{t}}\, u(x,t) - \delta_{2\bar{t}}\, z^{(2)}(x,t)\,\right| \leqslant M\left[N^{-1}\ln N + \tau^2\right], \quad (x,t) \in G_h, \quad t \geqslant 2\tau,$$

then for the difference $u(x,t) - z^{(3)}(x,t) \equiv \omega^{(3)}(x,t)$ we obtain

$$\left|\, \Lambda_{(6.3)}\omega^{(3)}(x,t)\,\right| \leqslant M\left[N^{-1}\ln N + \tau^3\right], \quad (x,t) \in G_h, \quad \omega^{(3)}(x,t) = 0, \quad (x,t) \in S_h.$$

Hence we have

$$\left|\, u(x,t) - z^{(3)}(x,t)\,\right| \leqslant M\left[N^{-1}\ln N + \tau^3\right], \quad (x,t) \in \overline{G}_h.$$

Thus, for the proof of the theorem it is sufficient to show inequalities (9.17). These inequalities follow from (9.15), (9.16). This completes the proof of Theorem 6.2.

Conclusion

In this paper we have shown theoretically that the use of a defect correction technique for the class of boundary-value problems in the case of a singularly perturbed parabolic convection-diffusion equation with the singularly perturbed Robin boundary condition allows us to construct effectively ε-uniformly convergent schemes with the second and third orders of accuracy with respect to t, still preserving the ε-uniform first-order accuracy in space. The same technique can be applied in order to construct similar schemes with the order of time-accuracy more than three.

The original technigue for experimental study of the convergence parameters of ε - uniformly convergent schemes have been developed which can be applied to the cases where the error components due to the discretization of the space and time derivatives can be essentially (many times) different. In particular, as has been observed in the paper for a model problem, the time error is a quantity of the order $10^{-4} - 10^{-10}$, whereas the space error is of the order $10^{-1} - 10^{-2}$. Thus, the time error is considerably smaller than the space error, and the total error is practically equal to the latter.

It is shown with numerical experiments that the use of the defect correction technique in practice does not affect the magnitude of the error component due to the discretization in x. The magnitude of the error component due to the discretization in t decreases essentially for schemes with a higher order of accuracy in t.

The numerical example is given when the passage to the scheme of third-order accuracy in t make it possible to decrease the number of the time steps from 512 to 16 with preservation of the ε-uniform accuracy of the approximate solution. As a practical result, this means the substantial decrease in the amount of computational work with preserving (moreover, improving) the ε-uniform global accuracy of the final solution.

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