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Classical and Quantum Mechanical Models of Many-Particle Systems

Organized by
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Abstract. The workshop focused on the collective behavior of many-particle systems in various application fields: physics (gas dynamics, plasmas, quantum mechanics), mathematical biology (cell mobility, evolution of trait-structured species), and social sciences (wealth distribution). This includes famous models such as the Boltzmann equation of gas dynamics, Vlasov equation for plasmas, Fokker–Planck equations, Smoluchowski and related equations, Keller–Segel system of chemotaxis.

Mathematics Subject Classification (2020): 76P05, 35Q40, 35Q92, 82C22, 35K65, 82C70, 45K05, 82D10, 81S30, 81S22.

Introduction by the Organizers

The workshop *Classical and Quantum Mechanical Models of Many-Particle Systems*, organized by Klemens Fellner (University of Graz), Isabelle Gallagher (Ecole Normale Supérieure, Paris), Pierre-Emmanuel Jabin (Pennsylvania State University) was well attended with around 50 participants, including more than 40 in-person. The participants involved a broad set of researchers, with 20 women and a large geographical representation.

We had 24 talks during the workshop. Those included an introductory presentation by E. Carlen on the first day and a general presentation for a broad audience on the last day by L. Desvillettes. Just as for the participants, the talks involved a mix of junior and more senior researchers, and covered a large variety of topics around mathematical kinetic theory, from classical to quantum systems.

The workshop showcased recent results on classical kinetic models, such as the Boltzmann equation, the Landau equation, the Vlasov equation for plasmas or other systems, Fokker-Planck equations or kinetic formulations of various macroscopic or hyperbolic systems. Those systems typically involve the free streaming of particles, binary collisions and/or long range interactions of mean-field type, and potentially interactions with a spatial boundary. A variety of mathematical models beyond physics was also discussed with applications in life sciences, social sciences, economy together with novel connections between kinetic theory and some aspects of data science.

Several important developments were introduced in the workshop:

- Those include strong progress on our understanding of the role of entropy and entropy dissipation in kinetic systems, and new approaches to the critical question of hypoellipticity, hypocoercivity.
- The multi-scale analysis of kinetic systems has also been one of the highlights of the workshop: from new results on the so-called mean-field limit to derive kinetic systems to the hydrodynamic limits from kinetic to macroscopic systems.
- New models have been derived to take the effects of complex particles agents better into account. Those can be non-spherical particles for which rotation is important or non-identical agents which appear naturally in biology or social sciences around the concept of graph limit.
- Innovative numerical schemes have also been proposed for a variety of complex and potentially ill-posed settings.

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Workshop: Classical and Quantum Mechanical Models of Many-Particle Systems

Table of Contents

Eric Carlen
On a Kac model with Exclusion 2183

Mikaela Iacobelli (joint with Megan Griffin-Pickering)
Stability and singular limits in plasma physics 2184

Amina Mecherbet (joint with Richard M. Höfer and Marta Leocata)
Homogenization of randomly rotating non spherical particles 2186

Clément Mouhot (joint with Fransesca Anceschi, Helge Dietert, Jessica Guerand, Amélie Loher, Annalaura Rebucci)
Trajectory approach to De Giorgi theory 2189

Peter Pickl (joint with Lea Boßmann, Sören Petrat and Avy Soffer)
Beyond Bogoliubov Dynamics 2190

Pierre Le Bris (joint with A. Guillin and P. Monmarché, as well as L. Colombani and C. Poquet)
Some recent results on propagation of chaos and mean-field limits 2191

Pierre Degond (joint with Amic Frouvelle)
Swarming rigid bodies in arbitrary dimensions 2193

Théophile Dolmaire (joint with Chiara Saffirio)
Derivation of the Boltzmann equation in the disk: quantitative controls of the recollisions 2195

Mitia Duerinckx (joint with Pierre-Emmanuel Jabin)
Corrections to mean field: collisional relaxation ... or not? 2198

Anna L. Mazzucato (joint with Christophe Lacave)
The Vanishing Viscosity Limit in Porous Media 2201

Maja Tasković (joint with Ioakeim Ampatzoglou, Irene M. Gamba, Nataša Pavlović)
On the binary-ternary Boltzmann equation 2203

Amit Einav
Can we have a little Order amidst all this Chaos? 2206

José A. Carrillo
Noise-driven bifurcations in a neural field system modelling networks of grid cells 2208

Ansgar Jüngel (joint with L. Chen and A. Holzinger)
Quantitative mean-field estimates for aggregation-diffusion equations and fluctuations 2208

Anne Nouri (joint with Leif Arkeryd)
On stationary solutions to the discrete velocity Boltzmann equation in the plane 2210

Miguel Escobedo
The Nordheim equation and the Wave Turbulence 2213

Nathalie Ayi
Graph limit for interacting particle systems on weighted deterministic and random graphs 2216

Yann Brenier
Monge-Ampère gravitation and Optimal transport 2219

Yan Guo (joint with Esposito, Marra and Wu)
Ghost Effect from Boltzmann Theory 2222

Qin Li
Two case studies of PDE-constrained optimization within kinetic framework 2227

Claudia Negulescu (joint with E. Lehman, F. Filbet)
Multiscale numerical treatment of the electron/ion kinetic model of fusion plasmas in the low eletron/ion mass-ratio regime 2229

Thierry Paul
From microscopic to macroscopic : the large number dynamics of agents and cells, possibly interacting with a chemical background and conversely 2231

Anton Arnold (joint with Jose A. Carrillo, Daniel Matthes)
All relative entropies for general nonlinear Fokker-Planck equations 2235

Laurent Desvillettes
Cercignani conjecture for the Landau equation 2238

Abstracts

On a Kac model with Exclusion

Eric Carlen

Recently, Colangeli, Pezzotti, and Pulvirenti [2] considered a Kac model [3] with an exclusion rule modeled on the Pauli exclusion principle. Their model is the original Kac model, but with one modification: the phase space is divided into a grid of cells of volume h^3, and the initial state of the process is such that no cell is doubly occupied. The process proceeds with pair collisions exactly as in the original Kac, but collision are disallowed if they would result in multiply occupies cells. Then using an BBGKY hierarchy analysis, it is shown in [2] that the resulting deterministic equation for the empirical distribution for this process is the Uehling-Uhlenbeck equation.

The work described here, which is joint work with Bernt Wennberg [1], concerns a related model, also with an exclusion rule, but this time with no partition of the phase space into cells. That is, we have exclusion without quantization. This type of process may be related to parking models or models in population biology.

Specifically, we consider a system of N particles evolving under pairwise energy conserving collisions. Let $x_j \in [0,\infty)$ denote the energy of the jth particle. The state of the system is given by the vector $(x_1,\dots,x_N)$, and for some fixed $\epsilon > 0$, we require that $|x_i - x_j| \geq \epsilon$ for all $i \neq j$. At random times to be specified, a pair $1 \leq i < j \leq N$ is selected, and energies x_i and x_j are updated to x_i^* and x_j^* in such a way that

$$x_i + x_j = x_i^* + x_j^* \,, \tag{1}$$

but the jump is suppressed if $|x_i^* - x_j^*| < \epsilon$. In this model, the likelihood for a jump to be successful depends very much on the distribution of gaps between occupied energies.

We study the system in the large N limit. For N particles, the minimum energy is $\epsilon N(N-1)/2$. We fix a parameter $0 < \alpha < 2$, and consider N-particle energies E_N such that with

$$\alpha_n := \frac{\epsilon n(n-1)}{E_n} \,, \quad \lim_{n\to\infty} \alpha_n = \alpha \,.$$

Note that $\alpha/2$ is the asymptotic fraction of the energy "used up" by the exclusion rule. The larger α is, the smaller the gaps will be, and the smaller will be the fraction of collisions that are permitted.

We rescale the variables x_j and ϵ with the average energy,

$$\tilde{x}_j = \frac{n}{E_n} x_j \quad \text{and} \quad \tilde{\epsilon}_n = \frac{\epsilon n}{E_n} = \frac{\alpha_n}{n-1} \,,$$

and define the empirical distribution

$$\mu_n(t) := \frac{1}{n} \sum_{j=1}^{n} \delta(x - \tilde{x}_j(t)) \,.$$

We are interested in when $\mu(t)$, which is a random probability measure, becomes deterministic in the large N limit so that

$$\lim_{N\to\infty} \mu_N(t) = f_t(x)\mathrm{d}x \ , \tag{2}$$

and then what equation f_t would satisfy. This question fits into the framework of "propagation of chaos", but there is a new feature: The sum defining the random probability measure $\mu_N(t)$ can only be expected to become deterministic through some form of the Law of Large numbers, and this requires some degree of statistical independence. The exclusion rule introduces correlations, and it is not immediately clear that one can even arrange at $t = 0$ for an interesting range of choices for f_0. We deal with this in the first part of the paper, and find a number of ways to construct initial data with this property.

However, because the dynamics depends very much on the distribution of gaps between occupied energies, one can get initially different evolutions of the density f_t for the same f_0. It is shown that the gap distribution becomes exponential, and that one can construct "chaotic" initial data with an exponential gap distribution. We then derive the resulting kinetic equation governing the evolution of f_t. This has the same general form as the Uehling-Uhlenbeck equation, except that the collision rate is even lower. Some open questions and possible extensions are discussed.

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Stability and singular limits in plasma physics

Mikaela Iacobelli

(joint work with Megan Griffin-Pickering)

A plasma is a gas that has undergone a process of ionization, whereby the particles making up the gas separate into electrons and positively charged ions. There are two commonly studied versions of the Vlasov-Poisson system: the classical Vlasov-Poisson system (VP), which describes the electrons in the plasma, and the Vlasov-Poisson system with massless electrons (VPME), which considers the point of view of the ions.

At large spatial and time scales, plasmas have the tendency to be quasineutral, i.e., the local charge disappears. On the other hand, at small spatial and time scales, the quasineutrality is no longer verified. The typical degeneracy scales are the oscillation frequency of the electrons and the Debye length. The Debye length is the distance at which electrons screen out electric fields, and it is often tiny

compared to the spatial observation length. Our works studied the quasineutral limit, i.e., the limit as the Debye length tends to zero, for the VP and VPME systems. Mathematically it corresponds to a certain hydrodynamic limit for the VP system, in which the formal limiting system is the so-called Kinetic Isothermal Euler system.

The mathematical study of the quasineutral limit for the VP equation began in the nineties with the pioneering works of Brenier and Grenier, leading to a full justification by Grenier of the quasineutral limit for initial data with uniform analytic regularity. Since then, results have been proven for other classes of initial data.

In particular, in a series of works started jointly with Daniel Han-Kwan and then pushed forward with Megan Griffin-Pickering, we tried to extend Grenier's result to the case of rough perturbations of uniformly analytic data, using Wasserstein stability estimates. These kinds of results are somehow surprising since for several classes of initial data, also very regular, counterexamples are available. In particular, it is known that polynomial smallness in the Debye length is not sufficient, so at least some exponential smallness is required.

After several contributions, in [2] we introduced a new class of Wasserstein-type distances specifically designed to tackle questions concerning stability and convergence to equilibria for kinetic equations. Thanks to these new distances, we obtained the validity of the quasineutral limit for VP under exponential smallness of the perturbation. We emphasize that exponential smallness is necessary, due to the presence of instabilities that render polynomial smallness inadequate.

Then, in a recent preprint with Megan Griffin-Pickering [1], we obtained the validity of the quasineutral limit also for the VPME system when dealing with rough initial data that are exponentially small perturbations of analytic data. The quantitative nature of this study presented us with distinct challenges, primarily stemming from the exponential Poisson coupling. Within the framework of this research paper, we introduced innovative tools and approaches tailored to tackle these challenges head-on. Our efforts extended the current theoretical foundation concerning the growth of characteristics in Vlasov systems characterized by nonlinear couplings. Additionally, we combined stability estimates using kinetic-Wasserstein distances with improved regularity bounds on the elliptic coupling. In the course of demonstrating our central result, we also enhanced the moment assumptions associated with the well-posedness of the ionic Vlasov-Poisson system.

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Homogenization of randomly rotating non spherical particles

Amina Mecherbet
(joint work with Richard M. Höfer and Marta Leocata)

The rheology of non spherical particles has been extensively studied in the last decades, it has been observed that such suspensions behave like a non Newtonian fluid due to the presence of a viscoelastic stress, we refer to [2, 8, 4] for more details on such complex fluids.

The Doi model is a Fokker-Planck equation coupled to a (Navier) Stokes equation describing suspension of Brownian (elongated) rod-like particles in a fluid which experiments a viscoelastic stress that depends on the particles density f. The equation writes

$$\begin{cases} \partial_t f + \mathrm{div}(uf) + \mathrm{div}_\xi(P_{\xi^\perp}\nabla_x u\xi f) = \dfrac{1}{\mathrm{De}}\Delta_\xi f + \dfrac{\lambda_1}{\mathrm{De}}\mathrm{div}_x((\mathrm{Id}+\xi\otimes\xi)\nabla_x f), \\ \mathrm{Re}(\partial_t u + (u\cdot\nabla)u) - \Delta u + \nabla p - \mathrm{div}\sigma = 0, \qquad \mathrm{div}u = 0 \\ \sigma = \sigma_v + \sigma_e = \lambda_2 \displaystyle\int_{\mathbb{S}^2} (Du : \xi\otimes\xi)\xi\otimes\xi f d\xi + \frac{\lambda_3}{\mathrm{De}}\int_{\mathbb{S}^2}(3\xi\otimes\xi - \mathrm{Id})f d\xi, \\ u(0,\cdot) = u_0, \quad f(0,\cdot) = f_0. \end{cases}$$

with (u,p) the fluid velocity and pressure, $f(t,x,\xi)$ the density of particles at time $t \geq 0$, position $x \in \mathbb{R}^3$ and orientation $\xi \in \mathbb{S}^2$, $P_{\xi^\perp}$ is the orthogonal projection on $\xi^\perp$, Du the symmetric gradient of u. De stands for the Deborah number, Re is the Reynolds number and λ_1, λ_2, λ_3 are dimensioneless parameters.

The two parts of the viscoelastic stress are sometimes referred to as the viscous part and the elastic part respectively. The viscous part σ_v is related to the celebrated Einstein effective viscosity problem and was extensively studied in the case of spherical particles. The elastic part σ_e arises, in this setting, from the Brownian rotational effect of the non spherical particles. Theoretical studies on such elastic stresses go back to the 1940s, see e.g. [12, 10, 9] and also the later works [11, 6, 1] and the references therein.

The present talk is based on [7] where we aimed to justify such a term by means of homogenization arguments starting from a microscopic description of non spherical Brownian particles suspended in a viscous fluid.

We emphasize that more recent results regarding the derivation of the elastic stress have been obtained in a different setting; namely the case of active particles where there is no Brownian effect, we refer to [5] for a homogenization result in this direction and to [3] for a full derivation of the Doi model in terms of a mean field limit result.

1. The rescaled microscopic model

In order to focus on the elastic stress term, in this work, we consider a simplified microscopic model in which we neglect the inertial effects of both the fluid and particles, the hydrodynamical interaction between the fluid and particles and moreover we assume that the particle positions are fixed and consider then only randomly rotating particles. We make precise the model below.

Let $N \in \mathbb{N}^*$ be the total number of particles and consider a reference particle $\mathcal{B}$ invariant under vertical rotation. Given N positions $x_i \in \mathbb{R}^3$ and N orientations $\xi_i \in \mathbb{S}^2$ we denote by $\mathcal{B}_i$ the ith particle given by

$$\mathcal{B}_i = x_i + rR(\xi_i)\mathcal{B}$$

where r is the radius of the particles and depends on N (see section 1.1), $R(\xi_i) \in SO(3)$ a rotation matrix satisfying $R(\xi_i)e_3 = \xi_i$. We consider then the following Stokes problem

$$\text{(1a)} \qquad \begin{cases} -\Delta u_n + \nabla p_n &= 0 \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{i=1}^n \mathcal{B}_i(t), \\ \operatorname{div} u_n &= 0 \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{i=1}^n \mathcal{B}_i(t), \\ u_n &= v_i + \omega_i \times (x - x_i) \quad \text{in } \mathcal{B}_i(t), \end{cases}$$

(1b)

$$\begin{cases} \displaystyle\int_{\partial \mathcal{B}_i(t)} \Sigma(u_n, p_n)\nu = 0 \quad 1 \le i \le n, \\ \displaystyle\int_{\partial \mathcal{B}_i(t)} [\Sigma(u_n, p_n)\nu] \times (x - x_i) = r^3 \sqrt{2\gamma_{rot}\mathcal{R}_2(\xi_i(t))} \circ \dot{B}_i(t) \quad 1 \le i \le n, \end{cases}$$

where $\Sigma(u_n, p_n) = 2D(u_n) - p_n \mathrm{Id}$, ν the unit outer normal on $\partial\mathcal{B}_i$, v_i and ω_i are the translational and angular velocity of the particle i and are unknown, $\circ$ stands for the Stratonovich integral, B_i three dimensional Brownian motion and $\dot{B}_i$ the corresponding white noise, γ_{rot} a constant related to $\mathcal{R}_2$ Stokes resistance matrix.

The particle positions x_i are fixed whereas we complete the equations with the motion of the orientations obtained after using Stratonovich to Ito conversion formula

$$\text{(2c)} \qquad \begin{cases} d\xi_i(t) &= \sigma_D(\xi_i) dB_i - 2\xi_i dt + P_{\xi_i^\perp} h(t, \xi_i(t), x_i) dt, \\ \xi_i(0) &= \xi_{i,0}. \end{cases}$$

where h a smooth external force, $\sigma_D(\xi_i)$ a skew symmetric matrix explicitly given in terms of ξ_i. The velocity u_n is defined in terms of infinite stochastic integral due to the presence of a white noise at the level of the torques in (1b):

Theorem 1 (Well posedness)**.** *Under the above assumptions on the separation between particles, for all $n \in \mathbb{N}$, there exists a unique solution $(\xi_1, \dots, \xi_n)$ to* (2c)*. Moreover, there exists $N_0 \in \mathbb{N}$ such that for all $n \ge N_0$, $s > 1/2$, $T > 0$*

$$u_n \in L^2(\Omega; H^{-s}(0, T; H^{-s}_{\mathfrak{s}}(\mathbb{R}^3))).$$

where $H^{-s}_{\mathfrak{s}}$ stands for the divergence free elements of H^{-s} and $(\Omega, \mathcal{F}, \mathbb{P})$ the filtered probability space.

1.1. **Assumptions.** Let $\xi_i(0)$ independent random variables, x_i satisfy

$$\min_{j \neq i} |x_i - x_j| \ge C n^{-1/3}, \quad \sup_n \max_{1 \le i \le n} |x_i| < +\infty$$

We assume that $\lim_{n\to\infty} \phi_n \log(n) = 0$ where $\phi_n = r^3 n$ the particles volume fraction. We assume that the empirical measure on $\mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{S}^2$ given by $S_n(t, x, \xi) = \frac{1}{n}\sum_i \delta_{x_i, \xi_i(t)}$ satisfies initially $\lim_{n\to\infty} \mathbb{E}\left[\mathcal{W}_1\left(f_0, S_n(0)\right)\right] = 0$ for some given $f_0 \in \mathcal{P}_1(\mathbb{R}^3 \times \mathbb{S}^2)$.

2. The limit model and main result

The mesoscopic model writes

$$
(2) \qquad \begin{cases} \partial_t f + \mathrm{div}_\xi(P_{\xi^\perp} h f - \nabla_\xi f) = 0, \ \text{on } \mathbb{R}^+ \times \mathbb{S}^2 \times \mathbb{R}^3 \\ \\ -\Delta u + \nabla p = \mathrm{div}\, \sigma, \ \mathrm{div} u = 0, \ \text{on } \mathbb{R}^+ \times \mathbb{R}^3 \\ \\ \sigma(t,x) = \gamma_{rot} \displaystyle\int_{\mathbb{S}^2} (3\xi \otimes \xi - \mathrm{Id})\, f(t,x,\xi) d\xi \end{cases}
$$

Theorem 2. *Under the previous assumptions, for all $t > 0$ and all $s > 1/2$ the following convergence in probability holds:*

$$
\forall \varepsilon > 0 \ \lim_{n\to\infty} \mathbb{P}\Big(\|\phi_n^{-1} u_n - u\|_{H^{-s}((0,t),H^{-s}(\mathbb{R}^3))} + \sup_{\tau\in[0,t]} W_1(S_n(\tau), f(\tau)) > \varepsilon \Big) = 0,
$$

where $f \in C([0,t], \mathcal{P}_1(\mathbb{R}^3 \times \mathbb{S}^2) \cap L^2(\mathbb{R}^3 \times \mathbb{S}^2))$ and $u \in L^2(0,t; \dot{H}^1_{\mathfrak{s}}(\mathbb{R}^3))$ satisfy (2).

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Trajectory approach to De Giorgi theory

Clément Mouhot
(joint work with Fransesca Anceschi, Helge Dietert, Jessica Guerand, Amélie Loher, Annalaura Rebucci)

The study of elliptic and parabolic equations with rough coefficients is now well-developed, following the seminal works of De Giorgi, Nash, Moser and Krüzkhov. By contrast the theory of the hypoelliptic equations that naturally arise from statistical mechanics with rough coefficients is only starting. These equations combine a degenerate diffusion in some directions (with rough coefficients, and possibly fractional) with a first-order hyperbolic operator, together with the Hörmander commutator conditions.

The paper [4] initiated an extension of the classical method of De Giorgi to such equations; however the argument in this paper was still non-constructive. It was later made constructive in [2] by the Moser-Krüzkhov approach, simplifying and clarifying [7], and in [3].

The focus of this talk is indeed on the trajectory approach to the classical De Giorgi theory introduced in [3], which provides a constructive version of [4]. This trajectory approach was extended to non-local operators in [5], and later improved and simplified in [6] and [1]. It yields new approaches even in the parabolic case.

The core ideas are:

(1) the construction of well-chosen trajectories along the Hörmander vector fields to prove weak forms of Poincaré inequalities and deduce Harnack inequalities, as in [3],
(2) the optimization of these trajectories by adjusting a "control function" so as to reduce the singularity caused by parametrizing these trajectories by one endpoint, as introduced in [6] and simplified in [1],
(3) an abstract linear algebra framework for constructing such control functions and trajectories for higher-order hypoelliptic operators, in [1].

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Beyond Bogoliubov Dynamics

Peter Pickl
(joint work with Lea Boßmann, Sören Petrat and Avy Soffer)

Interacting many body systems are the starting point for discussing many interesting effects in various sciences. Solving equations for many interacting particle-systems numerically or analytically is, however, often practically impossible. For quantum systems this is in particular difficult and with current techniques numerics is at its limits for systems of roughly ten particles in $\mathbb{R}^3$. To solve this problem, effective equations have been established in various situations that draw the main features of the system and are given by evolution equations on a much smaller space. Depending on the question one asks, in particular on the need of accuracy, finer descriptions are sought for which give a better approximation on the true dynamics with the price of increased – but still manageable – cost of solving the equation.

One example of such a finer description of an interacting system is the so called Bogoliubov equation [1]. While the leading order description one typically uses ignore correlations between particles, Bogoliubov found a closed equation for the first order correction which takes correlation between pairs of particles into account. This improves the description of an interacting quantum many-body system in the mean-field scaling regime: it gives a more accurate description compared to the Hartree equation, which effectively describes the respective system without reference to correlations. The Bogoliubov time evolution includes only terms that either keep the number of particles that are not in the state given by Hartree fixed or create respectively annihilate *pairs* of particles not in the Hartree state. One can show that the respective time evolution of these pairs can be given in a closed form.

In that sense the Bogoliubov time evolution fulfills the requirements we described above. It gives a more accurate description of the system. The price is that, next to solving the Hartree equation, one has to solve a differential equation describing the correlation of two particles, i.e. effectively a two particle system.

This rises the question if one can do better and find more accurate description which still keep the effective equations one has to solve on a space of a finite number of particles. The answer is "yes" as has been shown bei Paul and Pulvierenti [3] who give an iteration of effective equations which increase the complexity of the underlying space while improving the accuracy by a factor $N^{-1/2}$ in each step.

In the talk I will present a similar result [2] we found independently which has the advantage that the effective equations are all N-independent, whereas N still appears as parameter in [3]. It formulates the problem in second quantization and makes use of the theory of Bogoliubov transformations, which are a very convenient tool in this setting. More precisely: we iteratively construct corrections to the Bogoliubov dynamics that approximate the true N-body dynamics in norm to arbitrary precision. The N-independent corrections are given in terms of the solutions of the Bogoliubov and Hartree equations and satisfy a generalized form of

Wick's theorem. We determine the n-point correlation functions of the excitations around the condensate, as well as the reduced densities of the N-body system, to arbitrary accuracy, given only the knowledge of the two-point correlation functions of a quasifree state and the solution of the Hartree equation. In this way, the complex problem of computing all n-point correlation functions for an interacting N–body system is essentially reduced to the problem of solving the Hartree equation and the PDEs for the Bogoliubov two-point correlation functions.

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Some recent results on propagation of chaos and mean-field limits

Pierre Le Bris

(joint work with A. Guillin and P. Monmarché, as well as L. Colombani and C. Poquet)

Consider a system of N interacting particles described by the following system of Stochastic Differential Equations

$$dX_t^{i,N} = \frac{1}{N}\sum_{j=1}^{N} K(X_t^{i,N} - X_t^{j,N})dt + \sqrt{2\sigma_N}dB_t^i, \quad i \in \{1, ..., N\}. \tag{1}$$

Each particle is represented by a quantity $X \in \mathbb{R}^d$ (or on the torus $X \in \mathbb{T}^d$), most likely its position, where $X_t^{i,N}$ denotes the position at time t of the i-th particle. K is a function (which we will call an interaction kernel), σ_N is a diffusion coefficient that may or may not depend on the total number of particles, and $(B^i)_i$ are independent d-dimensional Brownian motions.

We are interested in the limit, as N goes to infinity, of this particle system and we wish to show that, as N grows, two given particles become "more and more" statistically independent. This phenomenon has been named *propagation of chaos*.

Denoting $\mu_t^N := \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^{i,N}}$ the empirical measure, we can rewrite the system of SDEs as follows

$$dX_t^{i,N} = K * \mu_t^N \left(X_t^{i,N}\right) dt + \sqrt{2\sigma_N}dB_t^i,$$

where $*$ denotes the convolution operation: $K * \mu(x) = \int K(x-y)\mu(dy)$. If the particles are indeed expected to become independent in the limit $N \to \infty$, as well as identically distributed, we can guess that μ_t^N will converge to a measure $\bar{\rho}_t$, the

law of a typical particle in the limit. Hence, very formally, a natural candidate for limit SDE

$$\begin{cases} d\bar{X}_t = K * \bar{\rho}_t(\bar{X}_t)dt + \sqrt{2\sigma}dB_t, \\ \bar{\rho}_t = \text{Law}(\bar{X}_t), \end{cases} \tag{2}$$

where $\sigma = \lim \sigma_N$. This non-linear SDE is said to be of *McKean-Vlasov* type because of the non-linearity induced by the interaction with its own law.

Let us denote $\rho_t^{k,N} = \text{Law}\left(X_t^{1,N}, ..., X_t^{k,N}\right)$ the joint law of the subset of the first k particles of the N particle system (with the convention $\rho_t^N = \rho_t^{N,N}$) and $\bar{\rho}_t^{\otimes k} = \bar{\rho}_t \otimes ....\otimes$ the nonlinear limit law $\bar{\rho}_t$ tensorized k times.

Our goal is to show a result of the form:

Propagation of chaos:

$$\forall k \in \mathbb{N}, \forall t \geq 0, \lim_{N\to\infty} \rho_t^{k,N} = \bar{\rho}_t^{\otimes k}, \text{ if it is true for } t = 0,$$

or equivalently

Mean field limit:

$$\forall t \geq 0, \lim_{N\to\infty} \mu_t^N = \bar{\rho}_t, \text{ if it is true for } t = 0.$$

We have two objectives in mind. First, we want to show results of quantitative *uniform in time* propagation of chaos. Second, we wish to be able to handle *singular Riesz-type* interactions.

During this talk we give some elements of comparison between a few methods used to prove propagation of chaos, seeing the advantages and limitations of each. The talk is thus organised as follows:

- we start with an introduction of the subject, the objects and the tools,
- we then talk about *coupling methods*, which are historically among the first methods used, about their benefits but also why they fail so far in dealing with singular interactions. Here we briefly mention the results of [3, 7, 1],
- Finally we show how we were able to obtain uniform in time propagation of chaos in some singular cases.

This last section, which makes up the biggest part of the talk, is structured around two main motivations.

We start by focusing on the *2D vortex model* where K is given in dimension 2 by $K(x) = \frac{1}{2\pi}\left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right)$. The approach, which is built upon the work of [5], consists in calculating the time evolution of the relative entropy of ρ_t^N with respect to $\bar{\rho}_t^{\otimes N}$, then using integration by parts to handle the singularity of K thanks to the regularity of the probability density $\bar{\rho}_t$. We improve this method, via the study of $\bar{\rho}_t$, to prove a uniform in time Sobolev inquality and vanishing bounds on the derivatives, and thus obtain a uniform in time result [2].

The second motivation is the *Dyson Brownian motion*, where $\sigma_N = \frac{1}{N}$ $K(x) = \frac{1}{x}$ in dimension 1. This example holds importance in random matrix theory. Here, we couple a N-particle system with a M-particle system, in order to obtain Cauchy

estimates on the sequence of empirical measures, and thus quantitative (possibly uniform in time) mean-field limit [4].

We then conclude with a few (to the best of our knowledge) open problems, notably obtaining probabilistic proofs (i.e via coupling methods) for propagation of chaos in singular cases, and understanding the optimal speed of convergence in N and how to obtain it as in [6].

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Swarming rigid bodies in arbitrary dimensions

Pierre Degond

(joint work with Amic Frouvelle)

This talk is based on [3] and is concerned with a collective dynamics model. Collective dynamics refers to systems of self-propelled particles, i.e. particles which produce their own movement by metabolizing some kind of energy (e.g. chemical energy). Collective dynamics produces large-scale self-organization which manifests itself in coordination, patterns or periodic oscillations.

Specifically, we consider a system of particles which consist of identical rigid bodies interacting through body attitude alignment. Each rigid-body is endowed with an orthonormal frame and moves at constant speed (supposed equal to 1 for simplicity) in the direction of the first vector of the frame. We define a reference frame and, for each particle, consider the unique rotation which maps this reference frame to its body frame. This rotation is subject to a stochastic differential equation which describes alignment of the body frame to the average body frame of the neighbouring particles on the one hand and noise on the other hand. Such systems may describe flocks of birds or schools of fish. They may also model complex-shaped objects interacting through volume exclusion such as sperm cells.

In dimensions larger than 3, this may also model a flow of data structured as n-dimensional rotations.

When the number of particles is large, the dynamics of the system can be described by a nonlinear Fokker-Planck equation for the probability distribution of the particles in the product space of positions and rotations. By assuming that the alignment frequency and noise intensity are large, we end-up with a perturbation problem in the Fokker-Planck equation with a small parameter denoted by ε. The problem tackled in this talk is to find (at least formally) what equations are satisfied in the limit $\varepsilon \to 0$. This limit is referred to as the hydrodynamic limit and the resulting system, the hydrodynamic equations.

We show that the hydrodynamic equations form a system for the mean particle density and mean body attitude, both being functions of space and time. The equation for the mean particle density has the form of a classical continuity equation. The equation for the mean body attitude (which is a space and time-dependent rotation) is less classical. It involves a material derivative which is balanced by terms reflecting the effect of the pressure force on the one hand and of an extra force which stems from spatial gradients of the mean body attitude on the other hand. This is a novel term compared with classical fluid dynamics. In [2], it is shown that this novel term generates milling solutions in which the particles spin in circles about an axis along which the mean body attitudes undergoes a twist. These solutions have non-trivial topology in that the twist of the mean body attitude defines an integral topological index. The non-trivial topology is preserved in the course of time by the hydrodynamic model. For the particle model, simulations show that the topology is preserved during a certain time but is eventually destroyed by an instability which may be attributed to the stochastic noise of the particle system.

The core of this work is the derivation of the hydrodynamic equations. Such derivation was done in dimension 3 in [4] using the parametrization of the rotation group by the Rodrigues formula, and in [5] using the quaternions. However, neither approach is generalizable to an arbitrary dimension. Later, in [1] the derivation of the hydrodynamic model was done in arbitrary dimension but for a simplified model where the Fokker-Planck operator is replaced by a BGK type operator. In this paper, the importance of Lie-group theoretic concepts, such as representation theory and the Weyl integration formula, was highlighted. The present talk and the paper [3] reports on the last step of this programme, namely the derivation of the hydrodynamic model in arbitrary dimension for the original Fokker-Planck model. Like in [1], Lie-group theoretic concepts play a crucial role, and particular the Weyl group which relates two elements of the maximal torus that are conjugate. They allow us to determine a key object needed in the hydrodynamic limit, namely the generalized collision invariants which extend the classical kinetic theory concept of collision invariants.

This work is one more step in a general program aiming at determining the structure of collective dynamics models where the inner variable of the particles (here the body attitude) belongs to a suitable manifold. In this work and the

previous one [1], it appears that Lie-group theoretic arguments seem to play a central role.

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Derivation of the Boltzmann equation in the disk: quantitative controls of the recollisions

Théophile Dolmaire

(joint work with Chiara Saffirio)

We presented the current state of our research on the derivation of the Boltzmann equation in the disk, from a deterministic system of hard spheres, interacting via specular reflections with the boundary of the domain.

The Boltzmann equation, that describes the time evolution of dilute gases, writes

$$\partial_t f + v \cdot \nabla_x f = Q(f,f), \tag{1}$$

with

$$Q(f,f) = \int_{v_* \in \mathbb{R}^d} \int_{\omega \in \mathbb{S}^{d-1}} [(v-v_*)\cdot\omega]_+ \left(f(v')f(v'_*) - f(v)f(v_*)\right) d\omega dv_*, \tag{2}$$

assuming that the microscopic particles composing the fluid interact according to the hard sphere model, with

$$\left\{ \begin{array}{rcl} v' & = & v - ((v-v_*)\cdot\omega)\,\omega, \\ v'_* & = & v_* + ((v-v_*)\cdot\omega)\,\omega, \end{array} \right. \tag{3}$$

and where $f = f(t,x,v)$ represents the number of particles of the gas lying at time t, at position $x \in \mathbb{R}^d$ and with velocity $v \in \mathbb{R}^d$. The equation (1) was first formally derived by Ludwig Boltzmann in 1872 ([2],[3]), considering at the microscopic scale a gas of hard spheres, subject to the Newton's laws, and in particular such a system is time reversible. On the other hand, it is well known that the solutions of the Boltzmann equation (1) present an irreversible behaviour, and have the trend to converge to some equilibrium, as stated by the H-theorem [20]. Therefore, there is an apparent paradox: how can irreversibility at a macroscopic scale can emerge from a reversible system at the microscopic scale? In particular, Loschmidt [17] and Zermelo [22] proposed very concrete versions of such a paradox. Therefore, justifying rigorously the validity of the Boltzmann equation from a more

fundamental model became a question of utmost importance, as it was suggested by Hilbert [11] when he stated his sixth problem at the International Congress of Mathematicians in 1900, not only in order to understand better the foundations of physics, but also in order to solve this fascinating conceptual paradox.

The first rigorous derivation of the Boltzmann equation is due to Lanford [16], providing in 1975 (that is, more than one century after Boltzmann wrote his equation) a mathematical description of the appearance of irreversibility from microscopic reversible systems. The proof of Lanford's theorem relies on the study of the marginals $f_N^{(s)}$ of the distribution function of a system of N hard spheres, since in particular, such marginals solve the so-called BBGKY hierarchy, named after Bogolyubov [1], Born and Green [4], Kirkwood [15] and Yvon [21]. Following Grad [10] and Cercignani [5], the formal limit of the BBGKY hierarchy in the Boltzmann-Grad scaling $N \to +\infty$, $N\varepsilon^{d-1} = 1$ (where d is the dimension of the domain, and ε is the diameter of the particles), provides the so-called Boltzmann hierarchy, that contains in particular the Boltzmann equation when the solution $\left(f^{(s)}\right)_{s\geq 1}$ of the latter hierarchy is tensorized. The idea of Lanford consists in solving the two hierarchies on the same (small) time interval, and to study the explicit formulae of the solutions. In particular, Lanford observed that the two solutions are both described as infinite sums of terms composed only with the initial data, and operators that have a natural geometrical interpretation. More explicitly, we can associate to the operators, for each of the two hierarchies, pseudo-trajectories, that are sort of generalized trajectories of subsets of the particle system. In particular, if one can prove the convergence of the pseudo-trajectories associated to the BBGKY hierarchy, towards the pseudo-trajectories associated to the Boltzmann hierarchy, one would obtain Lanford's result. The original proof of Lanford was supplemented over years by several authors, such as Cercignani, Illner and Pulvirenti ([13], [18], [14] and [7]), Uchiyama [19] or Gerasimenko and Petrina [6]. Nevertheless, the result obtained in these reference is restricted to the case of the whole Euclidean space on the one hand, and is not quantitative on the other hand. In 2013 was published the first quantitative version of Lanford's theorem, by Gallagher, Saint-Raymond and Texier [9], that applied, again, only for domains without boundary, namely, the whole Euclidean space, or the torus.

In our case, we restricted ourselves to the 2-dimensional case ($d = 2$). In order to obtain the first quantitative derivation of the Boltzmann equation in the disk, we proceeded to a careful control of the recollisions, which is the main obstruction to the convergence of the pseudo-trajectories. To do so, we adapted the strategy and the results of [9] and [8]. We presented in particular the quantitative estimates we obtained in the pre-collisional case.

To obtain such estimates, it was necessary to characterize completely the set of trajectories solving a shooting lemma, that is, the trajectories starting from a given point, and reaching a small given disk after bouncing n times, for any fixed number n of bouncings against the boundary of the domain (the unit disk). Such a characterization followed from considering a particular planar curve, the Holditch curve [12], which is defined as the envelope of the parts of trajectories,

all issued from the same starting point in the disk, and obtained after n specular reflections on the boundary of the unit disk. Since the Holditch curve is algebraic, we deduce the maximal number of trajectories solving the exact shooting problem (that is, when the targetted disk has a zero radius), uniformly in terms of the positions of the starting point and the target. On the other hand, the divergence of trajectories modified by small perturbations of the initial velocity is naturally controlled thanks to the Holditch curve, providing the characterization of all the trajectories solving the approximated shooting problem (that is when the target is a disk, with a non trivial, but small enough radius).

This central lemma is suffering some restriction, in particular it is necessary to consider targets that are on the one hand far enough from the singularities of the Holditch curve, and on the other hand far enough from the curve itself. All of these restrictions can be controlled thanks to delicate cut-offs, based in particular on an explicit control on the curvature of the Holditch curve.

This work in progress is a collaboration with Chiara Saffirio (Universität Basel).

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Corrections to mean field: collisional relaxation . . . or not?

Mitia Duerinckx
(joint work with Pierre-Emmanuel Jabin)

We consider a system of N point vortices in the 2-dimensional plane $\mathbb{R}^2$, interacting via a smooth force kernel $K = -\nabla^\perp W$, in some external force field $F = -\nabla^\perp V$, in the mean-field regime. More precisely, the particle trajectories are given by the following system of coupled ODEs,

$$\partial_t x_j = F(x_j) + \tfrac{1}{N} \textstyle\sum_{l:l\neq j}^N K(x_j - x_l), \qquad \text{for } 1 \le j \le N,$$

or, equivalently, the N-point density f_N satisfies the Liouville equation

$$\partial_t f_N + \textstyle\sum_{j=1}^N \Big(F(x_j) + \tfrac{1}{N} \sum_{l:l\neq j}^N K(x_j - x_l) \Big) \cdot \nabla_i f_N \,=\, 0.$$

In this setting, we now consider a tagged particle coupled to a set of background particles that are initially at equilibrium: this means that we consider an initial condition of the form

$$f_N|_{t=0} = f_N^\circ, \qquad f_N^\circ(x_1, \dots, x_N) \,=\, f_\circ(x_1)\, M_{N,\beta}(x_2, \dots, x_N),$$

where $M_{N,\beta}(x_2, \dots, x_N) \propto \exp\big[-\beta\big(\sum_{j>1}^N V(x_j) + \frac{1}{2N} \sum_{j,l>1}^N W(x_j - x_l)\big)\big]$ is the Gibbs thermal equilibrium for the background particles, and where $f_\circ$ is the initial density of the tagged particle. For $\beta \ll 1$ small enough, in the limit $N \uparrow \infty$, the mean-field theory ensures that the tagged particle density

$$f_N^1(x_1) = \textstyle\int_{(\mathbb{R}^d)^{N-1}} f_N(x_1, \dots, x_N)\, dx_2 \dots dx_N$$

converges to the solution f^1 of the linearized Vlasov equation

$$\partial_t f^1 + (F + K * M_\beta) \cdot \nabla f^1 \,=\, 0, \qquad f^1|_{t=0} = f_\circ,$$

where M_β is the mean-field equilibrium and is defined by the fixed-point equation $M_\beta \propto \exp[-\beta(V + W * M_\beta)]$. Focussing on the axisymmetric setting when V, W,

and $f_\circ$ are radial functions, the above mean-field equation becomes radially trivial in the sense that the radial density $\langle f^1\rangle(r) := \int_{\mathbb{S}^1} f^1(re)\,d\sigma(e)$ satisfies

$$\partial_t\langle f^1\rangle \,=\, 0, \qquad \langle f^1\rangle|_{t=0} = f_\circ,$$

hence $\langle f^1\rangle \equiv f_\circ$. We then aim to study the slow radial dynamics of the tagged particle that should occur as a correction to this trivial mean-field description. Due to its slow correlation with background particles, the tagged particle is generically expected to thermalize on the long timescale $t = O(N)$ — proportional to the number of background particles. More precisely, it has been conjectured that the time-rescaled radial density $\bar f^1_N(\tau) := \langle f^1_N(N\tau)\rangle$ converges to the solution of a radial Fokker–Planck equation

$$\partial_\tau \bar f^1 = \tfrac{1}{r}\partial_r\Big(ra_\beta(r)\big(\partial_r - (\log M_\beta)'(r)\big)\bar f^1\Big), \qquad \bar f^1|_{\tau=0} = f_\circ,$$

with some positive coefficient field a_β. This so-called "point-vortex diffusion" was first described in the physics literature by Chavanis [1, 2] and can be viewed as the equivalent for point-vortex systems of the celebrated Lenard–Balescu collisional relaxation for plasmas. In the spirit of our previous work with Saint-Raymond on the Lenard–Balescu theory [4], we justify this thermalization when the equilibrium measure M_β is not Gaussian and satisfies some non-degeneracy condition. As in [4], this derivation is limited to some intermediate timescale $1 \ll t \ll N^{1/18}$, thus failing to derive the relaxation on the relevant timescale $t = O(N)$, which is left as an open question in link with possible resonance issues.

Theorem (*Non-degenerate non-Gaussian case, see* [3])**.**
If the mean-field equilibrium measure M_β is not Gaussian and is non-degenerate in some suitable sense, then for all $0 < \sigma < \frac{1}{18}$ the subcritically-rescaled radial density $\bar f^1_N(\tau) = N^{1-\sigma}\langle f^1_N(N^\sigma\tau)\rangle$ satisfies

$$\partial_\tau \bar f^1_N \xrightarrow{N\uparrow\infty} \tfrac{1}{r}\partial_r\Big(ra_\beta(r)\big(\partial_r - (\log \mu_\beta)'(r)\big)f^\circ\Big).$$

However, we discovered that this thermalization of the tagged particle should only be expected in the non-Gaussian setting, while a completely different, dispersive behavior should occur otherwise. This is easily understood from a formal BBGKY analysis, as we now explain. On the one hand, the radial density of the tagged particle is checked to satisfy an equation of the form

$$\partial_t\langle f^1_N\rangle = \langle iS_2^- g^2_N\rangle, \tag{1}$$

where $g^2_N \in L^2_\beta((\mathbb{R}^2)^2)$ is the correlation of the tagged particle with one background particle, where L^2_β henceforth stands for the Lebesgue space with weight M_β, and where S_2^- is some "annihilation" operator $L^2_\beta((\mathbb{R}^2)^2) \to L^2_\beta(\mathbb{R}^2)$. On the other hand, higher-order correlation functions $\{g^m_N\}_{m\ge 2}$ for the tagged particle with the background satisfy a BBGKY-type hierarchy of equations of the form

$$(\partial_t + iL_m)g^m_N = iS^-_{m+1}g^{m+1}_N + \tfrac{1}{N}iS^+_{m-1}g^{m-1}_N, \qquad m \ge 2, \tag{2}$$

where iL_m is the m-particle linearized mean-field operator on $L^2_\beta((\mathbb{R}^2)^m)$, where $S^+_{m-1} : L^2_\beta((\mathbb{R}^2)^{m-1}) \to L^2_\beta((\mathbb{R}^2)^m)$ and $S^-_{m+1} : L^2_\beta((\mathbb{R}^2)^{m+1}) \to L^2_\beta((\mathbb{R}^2)^m)$ are

some "creation" and "annihilation" operators that are dual of one another, and where we have set $g_N^1 \equiv f_N^1$. As correlations are known to be a priori small, at least $g_N^2 = O(N^{-1/2})$, equation (1) gives a slow dynamics for $\langle f_N^1 \rangle$ and is coupled to the fast subdynamics (2) for correlations, which is driven by linearized mean-field operators $\{iL_m\}_m$. This subdynamics is thus expected to relax on the slow timescale of the tagged particle, and it is then clear that the limit should depend crucially on relaxation properties of linearized mean-field operators. Now it appears that the latter are radically different whether the equilibrium measure M_β is Gaussian or not, as the following shows.

Lemma (*see* [3]). *If M_β is not Gaussian and is non-degenerate in some suitable sense, then the linearized mean-field operators $\{iL_m\}_m$ have purely absolutely continuous spectrum on the orthogonal complement of their kernel. If instead M_β is Gaussian, then they reduce to compact operators.*

For a self-adjoint operator L, we recall the following elementary properties of long-time propagators: considering the solution h_ϵ of $(\epsilon\partial_t + iL)h_\epsilon = r$, $h_\epsilon|_{t=0} = 0$, in the limit $\epsilon \downarrow 0$, denoting by π the projection on $\ker(L)$ and $\pi^\perp := 1 - \pi$,

- if L has purely absolutely continuous spectrum close to 0 on $\ker(L)^\perp$, then we have $\pi^\perp h_\epsilon \to (0 + iL)^{-1}\pi^\perp r$;
- if L is compact, then the resolvent $(0+iL)^{-1}\pi^\perp r$ makes no sense as point spectrum accumulates at 0. Relaxation of long-time propagators fails and we find instead $\epsilon\partial_t h_\epsilon \to \pi r$.

Hence, when M_β is Gaussian, due to the compactness of linearized mean-field operators, the fast subdynamics (2) for correlations does not relax and the thermalization of the tagged particle fails for that precise reason. Instead, we formally find that the tagged particle should evolve on the timescale $t = O(N^{1/2})$ (rather than $t = O(N)$) and remain coupled to an infinite hierarchy of limiting correlations: more precisely, we expect that the time-rescaled radial density $\bar f_N^1(\tau) = \langle f_N^1(N^{1/2}\tau)\rangle$ and the time-rescaled correlations $\bar g_N^m(\tau) = N^{(m-1)/2} g_N^m(N^{1/2}\tau)$ converge to the solution of the following limiting hierarchy,

$$\begin{cases} \partial_\tau \bar f^1 = \langle iS_2^- \pi_2 \bar g^2 \rangle, \\ \partial_\tau \bar g^m = \pi_m (iS_{m+1}^-)\pi_{m+1}\bar g^{m+1} + \pi_m (iS_{m-1}^+)\pi_{m-1}\bar g^{m-1}, \quad m \geq 2, \end{cases}$$

where π_m stands for the projection on $\ker(L_m)$. Recalling that S_{m+1}^- and S_{m-1}^+ are dual, this reads as a unitary hierarchical evolution, which is reminiscent of quantum field theory: a Fock-space formalism can indeed be used for the set of limiting correlations describing background particles. This limiting evolution is related to so-called vector-resonant relaxation in the physics literature, e.g. [5].

Theorem (*Gaussian case*). *If M_β is Gaussian, then the above can be justified on the (almost optimal) intermediate timescale $1 \ll t \ll N^{1/2}$. Moreover, in the case $\beta = 0$, that is, for a uniform equilibrium, it can be justified on the optimal timescale $t = O(N^{1/2})$. A RAGE theorem can further be derived, showing that $\bar f^1$ splits into a sum of periodic evolutions and of a contribution that decays in time on every compact set, thus confirming the breakdown of any thermalization.*

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The Vanishing Viscosity Limit in Porous Media

Anna L. Mazzucato

(joint work with Christophe Lacave)

We consider the flow of a viscous, incompressible, Newtonian fluid in a perforated domain in the plane $\mathbb{R}^2$. The perforated domain Ω^ε is given by the complement of an array of obstacles, the pores, of size ε at distance $2d_\varepsilon$ from each other, each one the rescaled copy of a given obstacle, a compact, simply-connected domain $\mathcal{K}$ of class $C^{1,1}$. The obstacles are equally spaced on a given segment and then the segment is replicated a certain number of times, depending on the parameter $\mu > 0$, with $\mu = 0$ if there is only one segnment and $\mu = 1$ corresponding to a square lattice.

The viscous flow satisfies the incompressible Navier-Stokes equations (NSE) in Ω^ε with no-slip boundary conditions:

$$
(1) \qquad \begin{cases} u_t^{\nu,\varepsilon} - \nu\Delta u^{\nu,\varepsilon} + (u^{\nu,\varepsilon}\cdot\nabla)u^{\nu,\varepsilon} + \nabla p^{\nu,\varepsilon} = 0, & \text{on } (0,+\infty)\times\Omega^\varepsilon, \\ \operatorname{div} u^{\nu,\varepsilon} = 0, & \text{on } [0,+\infty)\times\Omega^\varepsilon, \\ u^{\nu,\varepsilon} = 0, & \text{on } (0,+\infty)\times\partial\Omega^\varepsilon, \end{cases}
$$

We study the simultaneous limit of vanishing pore size ε and inter-pore distance,d_ε and vanishing viscosity ν. Under suitable conditions on these parameters, we prove that $u^{\nu,\varepsilon}$ converges to a solution of the incompressible Euler equations, modeling perfect fluids, in the full plane. That is, the flow is not disturbed by the porous medium and becomes inviscid in the limit [5].

The Euler equations (EE) is the full plane are given by following system:

$$
(2) \qquad \begin{cases} u_t^E + (u^E\cdot\nabla)u^E + \nabla p^E = 0, & \text{on } (0,+\infty)\times\mathbb{R}^2, \\ \operatorname{div} u^E = 0, & \text{on } [0,+\infty)\times\mathbb{R}^2, \end{cases}
$$

The limit ε, $d_\varepsilon \to 0$ corresponds to the homogenization of the pore matrix. There are relatively few results for homogenization of the Euler and Navier-Stokes equations, since these are non-linear systems of equations and due to the role of the pressure. We refer to the recent article [2] for a discussion of the relevant literature. The vanishing viscosity limit, that is, whether solution of the Navier-Stokes

equation converge to solution of the Euler equations (usually strongly in the energy norm, that is, in L^2 uniformly in time) as $\nu \to 0$, is a singular limit even in the absence of boundaries. When boundaries are present, the problem is significantly more challenging, since a boundary layer, called the viscous boundary layer, forms where the solution can undergo rapid changes to fit the different boundary conditions for Euler and Navier-Stokes. Indeed, there are no general convergence results in bounded domains. When one combines the homogenization limit with the vanishing viscosity limit, the problem becomes even more challenging. The typical approach to singular limits in the presence of boundary layers is to introduce suitable boundary layer correctors, which accounts for the different behavior of the solution in the limit and can restore convergence. If there is only one shrinking obstacle kept at a fixed location, the simultaneous limit $\varepsilon, \nu \to 0$ was studied before (see [4] and references therein). In this work we identify a regime, where the viscous boundary layer is weak, because (informally) the obstacle shrinks faster than they can collide and faster than the rate at which viscosity vanishes. In this regime, it is only necessary to correct for the homogenization limit. Hence, the boundary correctors will only depend on ε.

We complement (1) and (2) with initial conditions. For NSE, we let $u^{\nu,\varepsilon}(0) = u_0^\varepsilon \in L^{2,\infty}(\Omega^\varepsilon)$, ensuring well-posedness of (1) in the exterior domain Ω^ε [3], while for EE we let

$$u_0(x) = K_{\mathbb{R}^2}[\omega_0](x) := \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \omega_0(y)\, dy, \tag{3}$$

where $K_{\mathbb{R}^2}$ denotes the Biot-Savart operator in the full plane and ω_0 is a smooth function, compactly supported away from the obstacle array, a technical condition. We can take the initial condition for NSE independent of ν and incompatible, that is, only the normal component is zero at the boundary of the obstacles, since the viscous layer is weak.

We compare $u^{\nu,\varepsilon}$ to u by extending $u^{\nu,\varepsilon}$ by zero to the interior of the obstacles, thanks to the no-slip boundary condition. Even though both $u^{\nu,\varepsilon}$ and $u \notin L^2$, their difference is and we prove convergence in the energy norm. Convergence will be obtained by constructing two correctors v^ε and h^ε. The first is needed to choose u_0^ε in such a way that u_0^ε, extended by zero, converges in $L^2(\mathbb{R}^2)$ to u_0, while the second is needed to control the difference between $u^{\nu,\varepsilon}$ and u. Both correctors are supported in a small neighborhood of the obstacles. The initial condition u_0^ε is formally obtained by truncating u_0 to be supported on the perforated domain. But this truncation is not divergence free and does not satisfy the no-penetration condition at the boundary. The role of the corrector v^ε is to restore both conditions. Similarly, the role of h^ε is to correct the truncated Euler solution on Ω^ε to satisfy the divergence-free condition and the no-slip boundary conditions. v^ε and h^ε must be constructed in such a way to ensure convergence in L^2. The construction is based on the Bogovskii operator and also adapts the approach in [1] for the homogenization of EE in a perforated domain. Once the correctors are obtained, convergence of the extended NSE solution to the EE solution follows readily via energy estimates, giving the following result.

Theorem 1 ([5]). *Given $\omega_0 \in C_c^\infty(\mathbb{R}^2)$, let u be the solution of (2) with initial condition u_0 as in (3). Let $u^{\nu,\varepsilon}$ be the solution of (1) with initial condition u_0^ε. Then, there exists a constant A depending only on $\mathcal{K}$ such that if*

$$\frac{\varepsilon}{d_\varepsilon^{(1+\mu)/2}} \leq \frac{A\nu}{\|\omega_0\|_{L^1\cap L^\infty(\mathbb{R}^2)}},$$

then for any time $T > 0$,

$$\sup_{0\leq t\leq T} \|u^{\nu,\varepsilon} - u\|_{L^2(\Omega^\varepsilon)} \leq B_T \frac{\sqrt{\nu}}{d_\varepsilon^{(1+\mu)/2}}, \tag{4}$$

where B_T depends only on T, $\|\omega_0\|_{L^1\cap W^{1,\infty}(\mathbb{R}^2)}$, and $\mathcal{K}$.

We do not know whether the convergence rate in (4) is optimal, but we observe that it is reduced, with respect to the optimal convergence rate for the vanishing viscosity limit of order $\nu^{1/4}$, by the factor $d_\varepsilon^{-(1+\mu)/2}$, which depends on the geometry of the obstacle array. Hence, (4) shows a "ghost" of the presence of the porous medium.

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On the binary-ternary Boltzmann equation

Maja Taskovi ć

(joint work with Ioakeim Ampatzoglou, Irene M. Gamba, Nataša Pavlović)

1. Introduction

In a series of works [7, 6, 3], Ampatzoglou and Pavlović rigorously derived a kinetic equation describing a toy model for systems of particles undergoing both binary and a certain type of ternary interactions. This new equation, referred to as the binary-ternary Boltzmann equation, generalizes the Boltzmann equation (cf. [12, 10, 14]) and has the following form (in the spatially homogeneous case):

$$\partial_t f = Q[f] := Q_2(f,f) + Q_3(f,f,f), \quad (t,v) \in (0,+\infty)\times\mathbb{R}^d. \tag{1}$$

The binary collision operator $Q_2(f,f)$ is given by

$$Q_2(f,f) = \int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} B_2(u,\omega)\,(f'f_1' - ff_1)\; d\omega\, dv_1, \tag{2}$$

where $u := v_1 - v$ is the relative velocity of the colliding particles, $f' := f(t,v')$, $f_1' := f(t,v_1')$, $f := f(t,v)$, $f_1 := f(t,v_1)$, and the post-collisional velocities v', v_1' are related to the pre-collisional velocities v, v_1 via the binary collisional law:

$$v' = v + (\omega\cdot u)\,\omega, \qquad v_1' = v_1 - (\omega\cdot u)\,\omega. \tag{3}$$

The ternary collisional operator $Q_3(f,f,f)$, introduced in [3, 7], is given by

$$\begin{aligned} Q_3(f,f,f) &= \int_{\mathbb{S}^{2d-1}\times\mathbb{R}^{2d}} B_3(\boldsymbol{u},\boldsymbol{\omega})\,(f^*f_1^*f_2^* - ff_1f_2)\; d\boldsymbol{\omega}\, dv_{1,2} \\ &\quad + 2\int_{\mathbb{S}^{2d-1}\times\mathbb{R}^{2d}} B_3(\boldsymbol{u_1},\boldsymbol{\omega})\left(f^{1*}f_1^{1*}f_2^{1*} - ff_1f_2\right)\; d\boldsymbol{\omega}\, dv_{1,2}, \end{aligned} \tag{4}$$

where $\boldsymbol{\omega} = \binom{\omega_1}{\omega_2} \in \mathbb{S}^{2d-1}$ is the impact directions vector and $\boldsymbol{u} := \binom{v_1-v}{v_2-v}$, $\boldsymbol{u_1} = \binom{v-v_1}{v_2-v_1}$ are the relative velocities of the colliding particles when the tracked particle is central or adjacent respectively for the ternary interaction happening. When the tracked particle is central, the collisional formulas are

$$v^* = v + \frac{\boldsymbol{u}\cdot\boldsymbol{\omega}}{1+\omega_1\cdot\omega_2}(\omega_1+\omega_2),\ v_1^* = v_1 - \frac{\boldsymbol{u}\cdot\boldsymbol{\omega}}{1+\omega_1\cdot\omega_2}\omega_1,\ v_2^* = v_2 - \frac{\boldsymbol{u}\cdot\boldsymbol{\omega}}{1+\omega_1\cdot\omega_2}\omega_2.$$

When the tracked particle is adjacent, similar collisional formulas hold.

The binary and ternary cross-sections are respectively given by:

$$B_2(u,\omega) = |u|^{\gamma_2} b_2(\hat{u}\cdot\omega), \quad u\neq 0, \quad \gamma_2\in[0,2],$$

$$B_3(\boldsymbol{u},\boldsymbol{\omega}) = |\widetilde{\boldsymbol{u}}|^{\gamma_3-\theta_3}|\boldsymbol{u}|^{\theta_3} b_3(\hat{\boldsymbol{u}}\cdot\boldsymbol{\omega},\omega_1\cdot\omega_2), \quad \boldsymbol{u}\neq 0, \quad \gamma_3\in[0,2], \quad \theta_3\geq 0,$$

where $\hat{u} = u/|u|$, $\hat{\boldsymbol{u}} = \boldsymbol{u}/|\boldsymbol{u}|$ and $|\widetilde{\boldsymbol{u}}| := (|v-v_1|^2 + |v-v_2|^2 + |v_1-v_2|^2)^{1/2}$. We assume that angular kernels b_2 and b_3 are integrable over the corresponding spheres and we consider hard potentials i.e.

$$\gamma := \max\{\gamma_2,\gamma_3\} > 0. \tag{5}$$

2. Global well posedness and moment estimates

In this talk we presented our recent work [4] on the global well-posedness and moment estimates for the spatially homogeneous binary-ternary Boltzmann equation (1). Let us mention that in our previous work [5] we established global well-posedness near vacuum for the spatially inhomogeneous binary-ternary equation.

The key tool is obtaining our results in [4] are the angular averaging estimates. We prove two types of such estimates, both for Q_2 and Q_3:

(1) One type provides an upper bound on the angular averaging part of the gain operator in terms of the total energy of the interaction. These estimates are used to prove the second type of angular averaging described below, and to establish the propagation and generation of moments. In the binary case, this type of estimates (also called Povzner-type estimates) was obtained, for example, in [8, 9].

(2) The second type provides estimates on a new decomposition of the angular averaging part of the collision operator. They are used in an inductive argument that establishes finiteness of moments. While these estimates are inspired by [15], they are novel even in the binary case. More precisely, results in [15] rely on the representation of post-collisional energies of particles as a sum of a convex combination of pre-collisional energies and a remainder. We, on the other hand, base our estimates on representing post-collisional energies as a fraction of the total energy of the interaction. This representation is especially suitable for higher order interactions, such as ternary, where pre-post collisional laws are more complex.

Moment estimates we prove show that the binary-ternary equation is "better behaved" than the Boltzmann equation $\partial_t f = Q_2(f, f)$ or the purely ternary equation $\partial_t f = Q_3(f, f, f)$ in the sense that the binary-ternary equation generates higher order exponential moment than the Boltzmann equation or purely ternary equation alone. As a consequence, the generation of exponential moments happens even if one of γ_2, γ_3 is zero (corresponding to the Maxwell molecules case) as long as the other one is strictly positive. This is in contrast with the binary Boltzmann equation for the Maxwell molecules, in which case generation of exponential moments is not known to happen.

Additionally, we show that exponential moments of solutions to (1) of order $s \in (0, 2]$, as well as polynomial moments of order $k > 2$, propagate in time, as was the case with the classical Boltzmann equation. Finally, polynomial moments of any order are generated in time as long as initial mass and energy are finite. The proof of propagation and generation of moments estimates is done in two phases:

(1) Phase 1: finiteness of moments. We show that any solution corresponding to initial data with finite mass and energy has finite and differentiable moments of any order $k > 2$. This is proven by an inductive argument that relies on the new decomposition of the collision operator and the novel angular averaging estimate.
(2) Phase 2: quantitative moment estimates. Here we use Povzner-type angular averaging estimates to obtain an ordinary differential inequality for polynomial moments, which results in quantitative estimates after a comparison to a Bernoulli-type ODE.

Finally, we prove that, as long as the initial data have $2 + \varepsilon$ finite moments, there exists a unique, global in time, solution to the equation (1). The proof of this result relies on techniques of the general theory for ODEs in Banach spaces, which was implemented in the context of the Boltzmann equation for the first time by Bressan in [11]. Subsequently, versions of this technique have been used in the context of the Boltzmann equation [1], systems of Boltzmann equations for gas mixtures [13] , as well as the quantum Boltzmann equation [2]. This technique boils down to verifying that the operator Q satisfies three conditions - Hölder continuity, one-sided Lipschitz condition and sub-tangent condition. We first verify this for initial data with $2 + 2\gamma$ moments, and then the generation of moments is used to relax the condition on the initial data to $2 + \varepsilon$ finite moments.

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Can we have a little Order amidst all this Chaos?

Amit Einav

Systems that involve many elements, be it a gas of particles or a herd of animals, are ubiquitous in our day to day lives. While such systems are of great interest to us, their investigation is hindered by their complexity the amount of (usually coupled) equations that are needed to be solved in order to understand them.

The mesoscopic approach, dating back to the golden age the investigation of kinetic gases, around the late 19th century, tried to simplify our dealing with such systems by finding an equation that describes the evolution of an average element said system. While widely used, the question of the validity of such equations remains an issue. One prime example to this is the proof of the validity of the Boltzmann equation - a problem so profound that it is included in Hilbert's 23

problem, proposed in the famous 1900 International Congress of Mathematicians (Hilbert's 6th problem).

In his 1956 work, [1], Kac has endeavoured to provide a probabilistic justification to Boltzmann's equation by considering an average model of dilute gas (i.e. an evolution equation for the probability density of the ensemble) and introducing the notion of chaos - an asymptotic correlation relation that refers to the fact that due to the rarity of the gas, any given group of particles become more and more independent as the number of particles in the system increases.

Besides fulfilling his original goal, and using his average model and the notion of chaoticity to give justification for the Boltzmann equation, Kac's work is the seed from which the mean field limit approach arose.

The mean field limit approach attempts to find the behaviour of a limiting average element in the system. Two ingredients are required to achieve this: an average model of the system and an asymptotic correlation relation that expresses the emerging phenomena we expect to get as the number of elements goes to infinity. Combining these ingredients gives us the ability to find a limit to the evolution equation of the first marginal of the total probability density, which is the desired equation of our limiting element.

While mean field limits of average models in various settings have been developed in recent centuries (see many examples in [4]), to date we use only chaoticity as our asymptotic correlation relation. This, however, doesn't seem reasonable in models that pertain to biological and societal phenomena. Such model, Choose the Leader model, was recently constructed and was shown to break the notion of chaoticity in the works of Carlen, Chatelin, Degond, and Wennberg [2, 3].

In our talk we outline the problem of having chaos as the sole asymptotic correlation relation and define the new asymptotic relation of *order*. We show that this is the right relation for the Choose the Leader model and highlight the importance of appropriate scaling in its investigation.

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Noise-driven bifurcations in a neural field system modelling networks of grid cells

José A. Carrillo

In this talk I reviewed several results in the modelling of grid cells. The activity generated by an ensemble of neurons is affected by various noise sources. It is a well-recognised challenge to understand the effects of noise on the stability of such networks. We demonstrate that the patterns of activity generated by networks of grid cells emerge from the instability of homogeneous activity for small levels of noise. This is carried out by upscaling a noisy grid cell model to a system of partial differential equations in order to analyse the robustness of network activity patterns with respect to noise. This is rigorously achieved by mean-field type arguments. Inhomogeneous network patterns are numerically understood as branches bifurcating from unstable homogeneous states for small noise levels. We prove that there is a phase transition occurring as the level of noise decreases. Our numerical study also indicates the presence of hysteresis phenomena close to the precise critical noise value. This talk is a summary of four papers/preprints in collaboration with A. Clini, H. Holden, P. Roux and S. Solem, see [1, 2, 3, 4].

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Quantitative mean-field estimates for aggregation-diffusion equations and fluctuations

Ansgar Jüngel

(joint work with L. Chen and A. Holzinger)

The aim of the talk is to prove a quantitative mean-field result in the $L^2(\mathbb{R}^d)$-norm associated to the following interacting stochastic particle system:

$$dX_i^{N,\eta}(t) = \frac{\kappa}{N}\sum_{j=1}^N \nabla V^\eta\big(X_i^{N,\eta}(t) - X_j^{N,\eta}(t)\big)dt + \sqrt{2\sigma}dB_i(t),$$
$$X_i^{N,\eta}(0) = \zeta_i \quad \text{in } \mathbb{R}^d,\ i = 1,\ldots,N,$$

where $X_i^{N,\eta}(t)$ is the position of the ith particle at time $t \geq 0$, N is the number of particles, V^η denotes the interaction potential with interaction radius $\eta > 0$, and $\sigma > 0$ is the diffusion coefficient. The parameter $\kappa = \pm 1$ models the type of interaction: The interactions are repulsive if $\kappa = -1$ and attractive if $\kappa = 1$. Furthermore, $(B_i(t))_{t\geq 0}$ are d-dimensional Brownian motions, and the initial data $\zeta_1, \ldots, \zeta_N$ are independent and identically distributed random variables with the common probability density function u_0. The interaction potential V^η approximates the Dirac distribution according to

$$V^\eta(x) = \eta^{-d} V\left(\frac{x}{\eta}\right), \quad x \in \mathbb{R}^d,$$

where $V : \mathbb{R}^d \to \mathbb{R}$ is a smooth, normalized, symmetric, and compactly supported function. One of our key assumptions is that $V = Z * Z$ is a convolution square. Then we can write $V^\eta = Z^\eta * Z^\eta$ for a suitable Z^η.

In the many-particle small-interaction limit $N \to \infty$, $\eta \to 0$, the stochastic processes $X_i^{N,\eta}$ converge in some sense to the stochastic process X_i, whose probability density $u(x,t)$ satisfies the aggregation-diffusion equation

$$\partial_t u = \sigma \Delta u - \kappa \operatorname{div}(u \nabla u) \quad \text{in } \mathbb{R}^d,\ t > 0, \quad u(0) = u_0.$$

Depending on the dependence of η on N, we distinguish between weak, moderate, and strong interactions. If $\eta = N^{-\beta}$ for some $\beta > 0$, we call the interactions moderate.

While the limit for the repulsive case $\kappa = 1$ was proved by Oelschläger (1985), the more delicate aggregation case $\kappa = -1$ was investigated by Stevens (2000) and later by Chen, Göttlich, and Knapp (2020). Stevens only proved the convergence in probability without rate, and Chen et al. used the (weaker) logarithmic scaling $\eta \geq C(\log N)^{-\beta}$ for some $C > 0$. We aim to generalize these results by allowing for the (more difficult) aggregation case, the (stronger) mean-square convergence in expectation, and the (stronger) algebraic rate $\eta = N^{-\beta}$ for some $\beta > 0$.

More precisely, we prove a convergence result for *smoothed* empirical measures, namely for

$$f^{N,\eta} = \mu^{N,\eta} * Z^\eta, \quad g^\eta = u^\eta * Z^\eta,$$

where $\mu^{N,\eta}(t) = N^{-1} \sum_{i=1}^N \delta_{X_i^{N,\eta}(t)}$ is the empirical measure associated to $X_i^{N,\eta}$ and u^η is the solution to the intermediate (nonlocal) diffusion system

$$\partial_t u^\eta = \sigma \Delta u^\eta - \kappa \operatorname{div}(u^\eta \nabla V^\eta * u^\eta) \quad \text{in } \mathbb{R}^d,\ t > 0, \quad u^\eta(0) = u_0.$$

We associate to this system an intermediate particle system for the processes X_i^η and the empirical measure $\mu^\eta = N^{-1} \sum_{i=1}^N \delta_{X_i^\eta}$.

Then, *assuming* the mean-field convergence in probability $\mu^\eta \to u^\eta$ with algebraic rate, for some $\beta > 0$, for any $T > 0$, there exist $\varepsilon > 0$ and $C > 0$ such that, for sufficiently small initial data (in a Sobolev space sense) and large $N > 0$,

$$\mathbb{E} \sup_{0<t<T} \|f^{N,\eta}(t) - g^\eta(t)\|_{L^2} \leq C N^{-1/2-\varepsilon}.$$

The proof is based on highly technical estimations, using the law of large numbers and Pickl's set decomposition[1].

Let us discuss two questions whose answers are work in progress.

When does the mean-field convergence in probability $\mu^\eta \to u^\eta$ hold true? This is the case if the potential is given by $V(x) = |x|^{-\alpha}$ for some suitable $\alpha > 0$. It is future work to verify this convergence for Keller–Segel-type potentials and potentials approximating the Dirac distribution.

Why smoothed empirical measures and why the rate $N^{-1/2-\varepsilon}$? We are interested in going beyond the mean-field limit by studying the fluctuations process $\xi^N(t) = \sqrt{N}(\mu^{\eta,N} - u)$. One may hope that $\mu^{\eta,N}$ behaves like $u + N^{-1/2}\xi$, where ξ is the limiting fluctuations process. If ξ is a Gaussian process, this would correspond to the central limit theorem. Unfortunately, we cannot expect that the convergence of $\mu^{N,\eta}$ to u is sufficiently "fast" for moderately interacting particles, due to the structural change of the limiting diffusion equation (local versus nonlocal). Oelschläger (1987) suggested to compare $\mu^{N,\eta}$ with the intermediate solution u^η instead of u. Thus, we are interested in the intermediate fluctuations process $\xi^{N,\eta} = \sqrt{N}(\mu^{N,\eta} - u^\eta)$. The particle dynamics is then given by the mean-field solution u, the intermediate fluctuations $\xi^{N,\eta}$, and the PDE approximation error $r^\eta = u^\eta - u$ (which is of order $\eta = N^{-\beta}$),

$$\mu^{N,\eta} = u + N^{-1/2}\xi^{N,\eta} + r^\eta.$$

Itô's lemma shows that the intermediate fluctuations process satisfies the stochastic differential equation

$$d\langle \xi N, \eta, \phi\rangle = \frac{\sqrt{2\sigma}}{\sqrt{N}} \sum_{i=1}^N \nabla\phi(X_i^{N,\eta}) dB_i + \sqrt{N}\langle |f^{N,\eta} - g^\eta|^2, \Delta\phi\rangle dt + R^N,$$

where ϕ is a test function and $R^N \to 0$ as $N \to \infty$ is an error term. We wish that the second term on the right-hand side also converges to zero as $N \to \infty$. This requires a mean-square convergence rate better than $N^{-1/2}$, which is provided by our result.

On stationary solutions to the discrete velocity Boltzmann equation in the plane

Anne Nouri

(joint work with Leif Arkeryd)

The Boltzmann equation is the fundamental mathematical model in the kinetic theory of gases. Replacing its continuum of velocities with a discrete set of velocities is a simplification, preserving the essential features of free flow and quadratic collision term. Besides this fundamental aspect, the discrete equations can approximate the Boltzmann equation with any given accuracy [2], and are thereby useful for approximations and numerics. In the quantum realm they can also be

[1]See the details in the PhD thesis of A. Holzinger; DOI 10.34726/hss.2019.63123.

more directly connected to microscopic quasi/particle models. A discrete velocity model of a kinetic gas is a system of partial differential equations having the form,

$$\frac{\partial f_i}{\partial t}(t,z) + v_i \cdot \nabla_z f_i(t,z) = Q_i(f,f)(t,z), \quad t > 0, \quad z \in \Omega, \quad 1 \le i \le p,$$

where $f_i(t,z)$, $1 \le i \le p$, are phase space densities at time t, position z and velocities v_i. The spatial domain is Ω. The given discrete velocities are v_i, $1 \le i \le p$. For $f = (f_i)_{1 \le i \le p}$, the collision operator $Q = (Q_i)_{1 \le i \le p}$ with gain part Q^+, loss part Q^-, and collision frequency ν, is given by

$$\begin{aligned} Q_i(f,f) &= \sum_{j,l,m=1}^{p} \Gamma_{ij}^{lm}(f_l f_m - f_i f_j) \\ &= Q_i^+(f,f) - Q_i^-(f,f), \\ Q_i^+(f,f) &= \sum_{j,l,m=1}^{p} \Gamma_{ij}^{lm} f_l f_m, \quad Q_i^-(f,f) = f_i \nu_i(f), \\ \nu_i(f) &= \sum_{j,l,m=1}^{p} \Gamma_{ij}^{lm} f_j, \quad i = 1, ..., p. \end{aligned}$$

The collision coefficients satisfy

$$\Gamma_{ij}^{lm} = \Gamma_{ji}^{lm} = \Gamma_{lm}^{ij} \ge 0. \tag{1}$$

If a collision coefficient Γ_{ij}^{lm} is non-zero, then the conservation laws for momentum and energy,

$$v_i + v_j = v_l + v_m, \quad |v_i|^2 + |v_j|^2 = |v_l|^2 + |v_m|^2, \tag{2}$$

are satisfied. We call interacting velocities any couple of velocities (v_i, v_j) such that for some $(l,m) \in \{1, \cdots, p\}^2$, $\Gamma_{ij}^{lm} > 0$. We consider

the generic case of normal coplanar velocity sets with
no pair of colinear interacting velocities (v_i, v_j). (3)

We consider stationary solutions to coplanar discrete velocity models satisfying (3), in a strictly convex bounded open subset $\Omega \subset \mathbb{R}^2$, with C^2 boundary $\partial\Omega$ and given boundary inflow. Denote by $n(Z)$ the inward normal to $Z \in \partial\Omega$. Denote the v_i-ingoing (resp. v_i-outgoing) part of the boundary by

$$\partial\Omega_i^+ = \{Z \in \partial\Omega\,;\, v_i \cdot n(Z) > 0\}, \qquad (\text{resp. } \partial\Omega_i^- = \{Z \in \partial\Omega\,;\, v_i \cdot n(Z) < 0\}).$$

Let

$$s_i^+(z) = \inf\{s > 0\,;\, z - s v_i \in \partial\Omega_i^+\}, \quad s_i^-(z) = \inf\{s > 0\,;\, z + s v_i \in \partial\Omega_i^-\}, \quad z \in \Omega.$$

Write

$$z_i^+(z) = z - s_i^+(z) v_i \quad (\text{resp. } z_i^-(z) = z + s_i^-(z) v_i) \tag{4}$$

for the ingoing (resp. outgoing) point on $\partial\Omega$ of the characteristics through z in direction v_i.
The stationary boundary value problem

$$v_i \cdot \nabla f_i(z) = Q_i(f,f)(z), \quad z \in \Omega, \tag{5}$$

$$f_i(z) = f_{bi}(z), \quad z \in \partial\Omega_i^+, \qquad 1 \le i \le p, \tag{6}$$

is considered in L^1 in one of the following equivalent forms ([3]);
the exponential multiplier form,

$$\begin{aligned} f_i(z) = {} & f_{bi}(z_i^+(z)) e^{-\int_0^{s_i^+(z)} \nu_i(f)(z_i^+(z)+sv_i)ds} \\ & + \int_0^{s_i^+(z)} Q_i^+(f,f)(z_i^+(z)+sv_i) e^{-\int_s^{s_i^+(z)} \nu_i(f)(z_i^+(z)+rv_i)dr} ds \\ & \text{for a.a. } z \in \Omega, \quad 1 \le i \le p, \end{aligned} \tag{7}$$

the mild form,

$$f_i(z) = f_{bi}(z_i^+(z)) + \int_0^{s_i^+(z)} Q_i(f,f)(z_i^+(z)+sv_i)ds, \text{ a.a. } z \in \Omega, \ 1 \le i \le p, \tag{8}$$

the renormalized form,

$$v_i \cdot \nabla \ln(1+f_i)(z) = \frac{Q_i(f,f)}{1+f_i}(z), \ z \in \Omega, \qquad f_i(z) = f_{bi}(z), \ z \in \partial\Omega_i^+, \quad 1 \le i \le p, \tag{9}$$

in the sense of distributions. Denote by $L^1_+(\Omega)$ the set of non-negative integrable functions on Ω. For a distribution function $f = (f_i)_{1\le i\le p}$, define its entropy (resp. entropy dissipation) by

$$\sum_{i=1}^p \int_\Omega f_i \ln f_i(z)dz, \quad \Big(\text{resp.} \quad \sum_{i,j,l,m=1}^p \Gamma_{ij}^{lm} \int_\Omega (f_l f_m - f_i f_j) \ln \frac{f_l f_m}{f_i f_j}(z)dz\Big).$$

The main result presented in the talk is

Theorem 1. *Consider a coplanar discrete velocity model and a non-negative ingoing boundary value f_b with mass and entropy inflows bounded,*

$$\int_{\partial\Omega_i^+} v_i \cdot n(z)\, f_{bi}(1+\ln f_{bi})(z)d\sigma(z) < +\infty, \quad 1 \le i \le p.$$

For the boundary value problem (5)-(6) *satisfying* (3)*, there exists a stationary mild solution in $\big(L^1_+(\Omega)\big)^p$ with finite mass and entropy-dissipation.*

The main difficulties are to prove that for a sequence of approximations, weak L^1 compactness holds and the limit of the collision operator equals the collision operator of the limit. Assumption (3) is crucial for proving L^1 compactness of the integrated collision frequencies, that is important for the convergence procedure. The limit procedure uses sub- and super-solutions as in the classical evolutionary frame for renormalized solutions to the Boltzmann equation [3].

For the continuous velocity evolutionary Boltzmann equation [3], the compactness properties of the collision frequency use in an essential way the averaging lemma, which is not available for the discrete velocity Boltzmann model. Here, the compactness properties are proven by the Kolmogorov-Riesz theorem [4]-[5].

A second result presented in the talk is a theorem of existence of stationary solutions to the Broadwell equation in the plane [1] . The Broadwell model, not included in Theorem 1, is a four-velocity model, with $v_1 + v_2 = v_3 + v_4 = 0$ and v_1, v_3 orthogonal. The proof uses in an essential way the constancy of the sums $f_1 + f_2$ and $f_3 + f_4$ along characteristics

We complete the talk by a theorem of existence of renormalized stationary solutions in the plane, with a proof that does not use any averaging lemma.

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The Nordheim equation and the Wave Turbulence

Miguel Escobedo

Under spatially homogeneous and isotropic assumptions, the Nordheim equation, reads

$$\begin{aligned}
\frac{\partial f}{\partial t}(t,\omega_1) = Q_3(f) &\equiv \int_{D(\omega_1)} M\left(\omega_1,\omega_3,\omega_4\right) q(f) d\omega_3 d\omega_4 \\
q\left(f\right) &= f_3 f_4 (1+f_1)(1+f_2) - f_1 f_2 (1+f_3)(1+f_4) \\
f_i &= f(t,\omega_i), \quad i = 1,2,3,4, \\
\omega_2 &= \omega_3 + \omega_4 - \omega_1 \\
D\left(\omega_1\right) &= \left\{(\omega_3,\omega_4) : \omega_3 > 0, \omega_4 > 0, \omega_3 + \omega_4 \geq \omega_1 > 0\right\} \\
M\left(\omega_1,\omega_3,\omega_4\right) &= \frac{\min\left(\sqrt{\omega_1},\sqrt{\omega_2},\sqrt{\omega_3},\sqrt{\omega_4}\right)}{\sqrt{\omega_1}},
\end{aligned}$$

deduced by L. W. Nordheim in [8]. It is known that for some regular initial data the solution f is such that $\sqrt{\omega} f(t,\omega)$ forms a Dirac delta at $\omega = 0$, after a finite time T_0. (cf. [6])

Question: Is it possible to go a little further?

$$f(t,\omega) = \frac{1}{\sqrt{\omega}} n(t)\delta_0 + h(t,\omega), \text{ where } h(t) \text{ is a function?} \tag{1}$$

If (1) is plugged in the equation and only homogeneous terms of order two and three are kept, (cf. [2] where such a question is already considered). Use of ideas and results from the Wave Turbulent theory allows attack rigorously questions like the existence of solutions and their regularity. ([9], [2] [7], [3], [4])

More about the "homogeneous" operators $\widetilde{Q}_3$ and $\widetilde{Q}_2$.
That type of operators arise and have been studied in a large variety of problems, in the context of the Wave Turbulence. (cf. [9], [2] and [7] and references therein). In particular, and quite remarkably, the existence of explicit stationary power law solutions is proved with important significance for the underlying system of waves, and their stability is studied.

Wave turbulence.
From the mathematical point of view, the starting point of all the problems is a set of nonlinear wave equations with weak nonlinearities (ε small measures the strength of the nonlinear interactions)

For $\varepsilon = 0$: a linear system of wave equations. If in the whole space $x \in \mathbb{R}^d$, for $t \in \mathbb{R}$ and the equations are invariant under space and time translations: $\longrightarrow$ solve the linearized problem using standard Fourier transform, but in principle the same ideas could be applied to non homogeneous systems.

The linearized problem admits solutions proportional to $e^{i(kx+\omega t)}$ with $\omega = \Omega(k)$, where Ω: the dispersion relation. In conservative problems, $\Omega(\cdot)$ is real.

The solution of the linear equation is then given as

$$u(t,x) = \int a(t,k)\, e^{i(kx-\omega t)} dk \quad \text{with} \quad u(x,0) = \int a(0,k)\, e^{ikx} dk$$

A crucial quantity is the density in the wavenumber space $F(t,k) = |a(t,k)|^2$. If Ω is real the function $F(t,k)$ is constant in the linearized problem. The dynamics of $F(t,k)$ becomes nontrivial for nonlinear equations due to the resonances between some specific wave numbers k. It is not possible to write a closed evolution equation for $F(t,k)$ because the dynamics of $a(t,\cdot)$ does not depend only on $|a(t,k)|$ but also in the phase of $a(t,\cdot)$. However, a key hypothesis in weak turbulence theory is: for some initial data u_0, it is possible to approximatethe evolution of f by means of a kinetic equation

$$\frac{\partial F(t,k)}{\partial t} = Q(F)(t,k) = \int\limits_{(R^d)^\ell} q(F)(k,k_1\cdots,k_\ell) W(k,k_1,\cdots,k_\ell) dk_1\cdots dk_\ell$$

Moreover, in the limit of weak nonlinear inteactions, it is possible to give an interpretation of the evolution of $F(t,k)$ by means of a particle model. The evolution of F is driven to the leading order in ε by resonances between linear modes with different values of k. This resonance condition can be given the interpretation of a collision between a number of particles, which results in another group of particles.

The resonance condition between modes can be understood as a condition for the conservation of the moment and energy of the particles in the collision process.

The precise conditions that allow to approximate the dynamics of wave equations by the kinetic models of weak turbulence is not known in general. However, for the cubic Schrödinger equation this has been obtained recently in [1].

One of the most relevant mathematical results for WTK equations was the discovery by V. E. Zakharov of stationary power law solutions with non zero fluxes for many models of weak turbulence and the study of their stability (cf. [9]).

Interesting aspects of the wave turbulence theory for the research in mathematics because:

- Application to a great variety of large systems of waves. Many different wave turbulence kinetic equations : water waves, plasmas, optic waves, acoustic waves, plates, gravity waves, ...
- Many results in physicist's literature: equilibria, KZ solutions, self similar solutions, stability,...
- Not as many results in the math literature, but more than some time ago, and much more to come...
- Many problems of mathematics around these equations: rigorous deduction, existence of solutions, stationary solutions, regularity, asymptotic behavior.

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Graph limit for interacting particle systems on weighted deterministic and random graphs

Nathalie Ayi

Opinion dynamics models focus on mathematically modeling the evolution of opinions and social interactions, with an emphasis on the mechanisms leading to consensus formation or polarization. In line with this perspective, the two papers presented here aims to contribute to the field by focusing on graph limits of interacting particles systems. Indeed, graph limits offer an innovative approach to studying the structure and properties of large complex networks, allowing for the analysis of the asymptotic behavior of increasingly large graph sequences.

In a first time, we are interested in a model which can be seen as a variant of the Hegelsman-Krause dynamics where we introduce some time-varying influence. The models writes

$$(\mathrm{D}_N) \qquad \begin{cases} \dfrac{d}{dt}x_i(t) = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} m_j(t)\phi(x_j(t) - x_i(t)) \\ \dfrac{d}{dt}m_i(t) = \psi_i(m(t), x(t)) \end{cases}$$

where $x_i \in \mathbb{R}^d$ is the state variable (opinion, position), $m_i \in \mathbb{R}^+$ is the agent's weight, $N = \sum_{i=1}^N m_i(0)$ is the (initial) total weight of the system, ϕ is the interaction function (often, $\phi(x_j - x_i) = a(\|x_i(t) - x_j(t)\|)(x_j(t) - x_i(t))$), ψ_i dictates the weight dynamics. We suppose $\sum_i \psi_i \equiv 0$. In the spirit of Medvedev in [4], we have established what is called the graph limit. Let us be more precise. Working under the following assumptions,

- **Hypothesis 1** The interaction function ϕ satisfies $\phi(0) = 0$ and $\phi \in \mathrm{Lip}(\mathbb{R}^d; \mathbb{R})$, with $\|\phi\|_{\mathrm{Lip}} = L_\phi$,
- **Hypothesis 2**

$$\begin{cases} \|\psi(\cdot, x_1, m_1) - \psi(\cdot, x_2, m_1)\|_{L^2(I)} \le L_\psi \|x_1 - x_2\|_{L^2(I)} \\ \|\psi(\cdot, x_1, m_1) - \psi(\cdot, x_1, m_2)\|_{L^2(I)} \le L_\psi \|m_1 - m_2\|_{L^2(I)}. \end{cases}$$

  and

$$|\psi(s, x, m)| \le C_\psi(1 + \|m\|_{L^\infty(I)}),$$

in [1], we obtain the following main result.

Theorem 1. *Suppose (H1) and (H2). We denote*

$$\begin{cases} x_N(s,t) = P_{\mathrm{c}}^N(x^N(t)) := \sum_{i=1}^N x_i^N(t)\mathbf{1}_{[\frac{i-1}{N}, \frac{i}{N})}(s) \\ m_N(s,t) = P_{\mathrm{c}}^N(m^N(t)) := \sum_{i=1}^N m_i^N(t)\mathbf{1}_{[\frac{i-1}{N}, \frac{i}{N})}(s). \end{cases}$$

where $(x_1,\dots,x_N)$ *and* $(m_1,\dots,m_N)$ *are solutions to the system* (D_N). *Then* (x_N,m_N) *is a solution to*

$$(\mathrm{C}_N)\qquad \begin{cases} \partial_t x_N(s,t) = \displaystyle\int_I m_N(s_*,t)\,\phi(x_N(s_*,t)-x_N(s,t))\,ds_*, \\ \partial_t m_N(s,t) = N\displaystyle\int_{\frac{1}{N}\lfloor sN\rfloor}^{\frac{1}{N}(\lfloor sN\rfloor+1)} \psi(s_*,x_n(\cdot,t),m_n(\cdot,t))\,ds_*, \end{cases}$$

with initial conditions $x_N(\cdot,0)=P_{\mathrm{c}}^N(P_{\mathrm{d}}^N(x^0))$ *and* $m_N(\cdot,0)=P_{\mathrm{c}}^N(P_{\mathrm{d}}^N(m^0))$ *and converges when* N *tends to infinity in the* $\mathcal{C}([0,T];L^2(I))$ *topology, i.e. there exists* $(x,m)\in\mathcal{C}([0,T];L^\infty(I,\mathbb{R}^d))\times\mathcal{C}([0,T];L^\infty(I,\mathbb{R}))$ *such that*

$$\|x-x_N\|_{\mathcal{C}([0,T];L^2(I,\mathbb{R}^d))}\xrightarrow[N\to+\infty]{}0 \quad \text{and} \quad \|m-m_N\|_{\mathcal{C}([0,T];L^2(I,\mathbb{R}))}\xrightarrow[N\to+\infty]{}0.$$

Moreover, the limit functions x *and* m *are solutions to*

$$(\mathrm{GL})\qquad \begin{cases} \partial_t x(s,t) = \displaystyle\int_I m(s_*,t)\phi(x(s,t)-x(s_*,t))ds_*; & x(\cdot,0)=x_0 \\ \partial_t m(s,t) = \psi(s,x(\cdot,t),m(\cdot,t)); & m(\cdot,0)=m_0. \end{cases}$$

In a second time, we are interested into weighted random graphs. We recall that a random graph is a graph which is generated by a random process. For instance, in the Erdos-Rényi graph, the edge between a pair of distinct nodes is inserted with probability p.

In [5], Medvedev took an interest in dynamical systems on what he called W-random graph.

Let $X=(X_1,X_2,X_3,\dots)$ and $X^N=(X_1,X_2,\dots,X_N)$ where $X_i, i\in\mathbb{N}$ are i.i.d. random variables with $\mathcal{L}(X_1)=\mathcal{U}(I)$. A W-random graph on N nodes generated by the random sequence X, denoted $G_N=\mathbb{G}(X_N,W)$ is such that the edges of G_N are selected at random and

$$\mathbb{P}((i,j)\in E(G_N))=W(X_i,X_j) \text{ for each } (i,j)\in\{1,\dots,N\}^2 \text{ for } i\neq j.$$

The decision wether to include a pair $(i,j)\in\{1,\dots,N\}^2$ is made independently as for the decisions of other pairs. He studied the following dynamical systems on W-random graph

$$\frac{d}{dt}u_i^N(t)=\frac{1}{N}\sum_{j=1}^N \xi_{ij}\phi(u_j^N(t)-u_i^N(t))$$

with $\mathcal{L}(\xi_{ij}|X)=\mathcal{B}(W(X_i,X_j))$ and obtains the convergence to the random graph limit equation

$$\partial_t u(s,t)=\int_I W(s,s_*)\phi(u(s_*,t)-u(s,t)ds_*.$$

Naturally, we were interested into a weighted variant of this model. In the literature, we have found some exemples as for instance in [3] where they study a weighted random graph model in which the probability of drawing an edge of discrete weight $w\in\mathbb{N}$ between vertices i and j is given by

$$\mathbb{P}(\xi_{ij}^N=w)=q_{ij}(w)=p^w(1-p).$$

In order to set a more general framework encompassing all the already existing models, we decided to define the following notions in [2]: a q-weighted random graph on N nodes generated by the random sequence X, denoted G_N, is such that the weight of an edge of G_N is randomly attributed. More precisely, the law for the weight of the edge (i,j) is $q(X_i, X_j, .)$ where

$$\begin{aligned} q: \quad I \times I \quad &\to \quad \mathcal{P}(\mathbb{R}_+) \\ (x,y) \quad &\mapsto \quad q(x,y;.). \end{aligned}$$

The decision of the attribution of the weight of a pair $(i,j) \in \{1,\dots,N\}^2$ is made independently from the decision for other pairs. Then, we also studied dynamical systems on q-weighted random graph i.e. in

$$(S_N^{\mathrm{r-r}}) \qquad \begin{cases} \dfrac{d}{dt}u_i^N(t) = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \xi_{ij} D(u_j^N(t) - u_i^N(t)), \\ u_i^N(0) = g(X_i^N), \ i \in \{1,\dots,N\} \end{cases}$$

with $\mathcal{L}(\xi_{ij}|X) = q(X_i, X_j; \cdot)$. We prove the convergence towards the continuum limit that we called the weighted random graph limit equation

$$(C) \qquad \begin{cases} \partial_t u(x,t) = \displaystyle\int_I \left(\int_{\mathbb{R}_+} w q(x,y;dw) \right) D(u(y,t) - u(x,t)) dy \\ u(x,0) = g(x), \ x \in I, \end{cases}$$

under the following assumptions:

- **Hypothesis 1** Let $D \in L^\infty(\mathbb{R})$ be bounded and Lipschitz continuous, with $\|D\|_{L^\infty(\mathbb{R})} := K$ and $\|D'\|_{L^\infty(\mathbb{R})} := L$.
- **Hypothesis 2** There exists $M > 0$ such that for all $(x,y) \in I^2$, for all $k \in \{1, \cdots, 4\}$,

$$\left(\int_{\mathbb{R}_+} w^k q(x,y;dw) \right)^{1/k} \leq M,$$

  i.e. the first four moments of the probability measure $q(x,y;\cdot)$ are bounded uniformly in x and y.

A more precise version of our result is the following:

Theorem 2. *Let D satisfy Hypothesis 1, let $g \in L^\infty(I)$ and let q be a weighted random graph law satisfying Hypothesis 2. Then, as N goes to infinity, solution u^N to the discrete system $(S_N^{\mathrm{r-r}})$ converges to the solution u of the continuous model (C). More precisely,*

$$\mathbb{P}\left[\sup_{t\in[0,T]} \|u^N(t) - \mathbf{P}_{\tilde{X}^N} u(\cdot,t)\|_{2,N} \geq \frac{C_1(T)}{\sqrt{N}} \right] \leq \frac{\tilde{C}_1}{N}$$

where the constants $C_1(T)$ and $\tilde{C}_1$ are respectively defined by $\tilde{C}_1 := 3M^4K^4 + 6$ and $C_1(T) := \sqrt{T}\sqrt{1 + M^2K^2} e^{(\frac{1}{2}+4ML)T}$, where we use the weighted Euclidean

inner product

$$(u,v)_N := \frac{1}{N}\sum_{i=1}^{N} u_i v_i, \quad u = (u_1, u_2, \ldots, u_N)^T, v = (v_1, v_2, \ldots, v_N)^T$$

and the corresponding norm $\|u\|_{2,N} := \sqrt{(u,u)_N}$.

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Monge-Ampère gravitation and Optimal transport

Yann Brenier

1. Early universe, optimal transport and Monge-Ampère gravitation

Uriel Frisch and coauthors [5] introduced a computational method based on optimal transport to reconstruct the history of the early universe from the observable distribution of clusters of galaxies, based on Peebles' semi-Newtonian model of gravitation and its Zeldovich approximation. (There is renewed interest for these topics after the launching of the James Webb Space Telescope, 25/12/21.) In Peebles' model, the trajectory $t \in \mathbb{R}_+ \to X(t,a) \in \mathbb{R}^3$ of each "particle" labelled by $a \in \mathbb{R}^3$ (mod $\mathbb{Z}^3$ for simplicity) is driven by

$$\frac{2t}{3}\frac{d^2X}{dt^2} + \frac{dX}{dt} + (\nabla\varphi)(t,X) = 0, \quad 1 + t\,\triangle\,\varphi = \rho = \int_{\mathbb{T}^3} \delta(x - X(t,a))da$$

where $\rho(t,x)$ and $\varphi(t,x)$ respectively denote the density field (supposed to be of unit average) and the gravitational potential.

General relativity is taken into account only through the term dX/dt and the coefficients depending explicitly on t which include Big Bang effects, everything else is Newtonian. (N.B. here, a "particle" roughly corresponds to a ...cluster of galaxies!) Because of the Big-Bang terms, the Peebles equations are degenerate at $t = 0$ where we get a continuum of particles of uniform density and velocity "slaved" by the gravitational potential:

$$\rho_0(x) = 1, \quad X_0(a) = a, \quad \frac{dX_0}{dt}(a) = -\nabla\varphi_0(a)$$

which is totally different from usual N-body Newton's gravitation. A very simple approximate solution (EXACT in 1D!) was proposed by Zeldovich in the '70s:

$$X(t,a) = a - t\nabla\varphi_0(a), \quad \triangle\varphi_0(x) = \lim_{t\to 0} \frac{\rho(t,x)-1}{t}$$

so that each particle just travels with a constant velocity due to the initial density fluctuation, until a collision ocurs (which is somewhat reminiscent of Lucretius' (99-55 BC) "De rerum natura"). In [3], the following correction to Peebles' model was suggested:

$$\rho(t,x) = \det(I + tD^2\varphi(t,x)) \text{ instead of } \rho(t,x) = 1 + t \triangle \varphi(t,x),$$

i) which is exact in 1d;
ii) asymptotically correct at early times and for weak fields;
iii) makes Zeldovich approximation exact (can be easily proven!);
iv) might be as good as the Poisson equation as an approximation to the Einstein equations (conjecture), based on the "vague" analogy

$$\frac{\text{Einstein equation}}{\text{Ricci curvature}} \sim \frac{\text{Monge}-\text{Ampere equation}}{\text{Gauss curvature}}$$

v) has a computational complexity similar to Poisson and allows large scale simulations of the Early Universe (performed at Institut d'Astrophysique de Paris and INRIA by P. Boldrini, B. Lévy and R. Mohayahee, Y.B., up to 512^3 particles, thanks to Lévy's 3D version of Q. Mérigot's algorithm for optimal transport, submitted work '23).

2. Discrete Monge-Ampère gravitation as a natural combinatorial approximation of Euler's incompressible fluid mechanics

According to V.I. Arnold '66, an incompressible fluid, confined in a compact Riemannian manifold D and moving according to the Euler equations, just follows a (constant speed) geodesic curve along the manifold of all possible incompressible (=volume preserving) maps of D. From a more concrete and computational viewpiont, it is worth considering the discrete version of an incompressible motion inside D: namely a succession of permutations of N sub-cells of equal volume of D. For $D = [0,1]^d$, we consider its dyadic decomposition by $N = 2^{nd}$ sub-cubes of barycenters $A(\alpha)$, $\alpha = 1, \cdots, N$.

A consistent penalized action for approximate geodesics (cf. Rubin-Ungar, CPAM '57, for smooth manifolds) reads

$$\int_{t_0}^{t_1} (\epsilon||\frac{dX_t}{dt}||^2 + \inf_{\sigma\in\mathfrak{S}_N} ||X_t - A_\sigma||^2)dt.$$

Here $X_t \in (\mathbb{R}^d)^N$ becomes the new, finite-dimensional, unknown, $||\cdot||$ denotes the euclidean norm in $(\mathbb{R}^d)^N$, $\mathfrak{S}_N$ is the set of all permutations of $\{1, \cdots, N\}$ and $A_\sigma(\alpha) = A(\sigma(\alpha))$, $\alpha = 1, ..., N$. Using the least action principle, we obtain for

$\alpha = 1, \cdots, N$

$$\epsilon \frac{d^2 X_t(\alpha)}{dt^2} = X_t(\alpha) - A(\sigma_{opt}(\alpha)), \quad \sigma_{opt} = \text{Arginf}_{\sigma \in \mathfrak{S}_N} \sum_{\alpha=1}^{N} |X_t(\alpha) - A(\sigma(\alpha))|^2$$

and finally, using Optimal Transport tools, the continuous version $N \to +\infty$

$$\epsilon \frac{d^2 X}{dt^2} + \nabla \varphi(t, X) = 0, \quad \det(I + D^2 \varphi) = \rho = \int \delta(x - X(t, a)) da$$

i.e. Monge-Ampère gravitation (up to the "big-bang" terms)!

3. Purely stochastic origin of Monge-Ampère gravitation [1]

We define a brownian cloud to be a finite set of N indistinguishable points in the euclidean space, i.e. as a point in the quotient space $(\mathbb{R}^d)^N / \mathfrak{S}_N$, initially located on the cubic lattice $\{A(\alpha) \in \mathbb{R}^d, \ \alpha = 1, \cdots, N\}$ and subject to N independent Brownian motions in $\mathbb{R}^d$, with uniform noise ν. In PDE terms, this corresponds to the heat equation in $\mathbb{R}^{Nd}$:

$$\frac{\partial \rho}{\partial t}(t, X) = \frac{\nu}{2} \triangle \rho(t, X), \quad \rho(t = 0, X) = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} \prod_{\alpha=1}^{N} \delta(X(\alpha) - A(\sigma(\alpha)))$$

where the initial data takes the relabeling symmetry into account so that $\rho(t, X)$ is the probability density of finding the brownian cloud at position X (up to a permutation of the labels) at time t

$$\rho(t, X) = \frac{1}{N!} (2\pi\nu t)^{-Nd/2} \sum_{\sigma \in \mathfrak{S}_N} \prod_{\alpha=1}^{N} \exp(-\frac{|X(\alpha) - A(\sigma(\alpha)|^2}{2\nu t}).$$

After solving the heat equation in the space of "clouds" $X \in \mathbb{R}^{Nd}$,

$$\frac{\partial \rho}{\partial t}(t, X) = \frac{\nu}{2} \triangle \rho(t, X), \quad \rho(t = 0, X) = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} \delta(X - A_\sigma),$$

we may solve the companion ODE in the same space $\mathbb{R}^{Nd}$

$$\frac{dX_t}{dt} = v(t, X_t), \quad v(t, X) = -\frac{\nu}{2} \nabla (\log \rho)(t, X).$$

(This is an adaptation of de Broglie's "onde pilote" idea. As a matter of fact, a similar calculation also works for the free Schrödinger equation: $(i\partial_t + \triangle)\psi = 0, \quad \psi(0, X) = \sum_\sigma \exp(-||X - A_\sigma||^2 / a^2), \quad v = \nabla \mathcal{I}m \log \psi.$) Setting $t = \exp(2\theta)$ and adding noise η:

$\frac{dX_\theta}{d\theta} = -\nabla_X \Phi_{\nu,\theta}(X_\theta) + \sqrt{\eta} \frac{dB_\theta}{d\theta}$ $\Phi_{\nu,\theta}(X) = \nu e^{2\theta} \log \sum_{\sigma \in \mathfrak{S}_N} \exp(\frac{-||X - A_\sigma||^2}{2\nu e^{2\theta}})$,

we first get a large deviation Vencel-Freidlin action for the limit $\eta \to 0$, while $\nu > 0$ is kept fixed. Then, we may pass to the limit $\nu \to 0$ and obtain as "$\Gamma-$limit"

$$\int_{\theta_0}^{\theta_1} ||\frac{dX_\theta}{d\theta} + \nabla \Phi(X_\theta)||^2 d\theta, \quad \Phi(X) = -\inf_{\sigma \in \mathfrak{S}_N} ||X - A_\sigma||^2 / 2$$

(cf. [1]). Using the least action principle, we finally recover the equations of discrete Monge-Ampère gravitation!

4. OT formulation of Einstein in vacuum

Finally, let us mention the following matrix-valued extension of the classical optimal transport optimal problem, with quadratic cost (in its Eulerian formulation, as in [2]. Find 4×4 matrix-valued fields $(C, V, M)(x, \xi)$ over the tangent bundle $(x, \xi) \in (\mathbb{R}^4)^2$ of $\mathbb{R}^4$, critical points of

$$\int \text{trace}(M(x,\xi)V(x,\xi) - C(x,\xi)V^2(x,\xi))dxd\xi$$

s.t. $\partial_{x^j} C^j_k + \partial_{\xi^j} M^j_k = 0$ and, for some vector-potential A, $C^j_k = \partial_{\xi^k} A^j - \partial_{\xi^\gamma} A^\gamma \ \delta^j_k$.

Theorem ([4])**.** *Let g be a smooth solution to the Einstein equations in vacuum, with Christoffel symbols $\Gamma =$ "$g^{-1}\partial g$". Set*

$$A^j(x,\xi) = \xi^j \ \text{det}g(x) \ \cos(\frac{g_{kq}(x)\xi^k\xi^q}{2}), \quad C^j_k(x,\xi) = \partial_{\xi^k} A^j(x,\xi) - \partial_{\xi^q} A^q(x,\xi) \ \delta^j_k,$$

$$V^j_k(x,\xi) = -\Gamma^j_{kq}(x)\xi^q, \quad M^j_k = C^j_\gamma V^\gamma_k + V^j_\gamma C^\gamma_k.$$

Then (C, M, V) is critical for the matrix-valued OT problem and

$$g^{ij}(x)\sqrt{-\text{det}g(x)} = cst \int (\xi^i A^j + \xi^j A^i)(x,\xi)d\xi.$$

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Ghost Effect from Boltzmann Theory

Yan Guo
(joint work with Esposito, Marra and Wu)

Consider the hydrodynamic limit $\varepsilon \to 0$ of a steady gas subject to wall temperature:

$$\begin{aligned} v \cdot \nabla_x F^\varepsilon &= \varepsilon^{-1} Q(F^\varepsilon, F^\varepsilon), \\ F^\varepsilon|_{n\cdot v<0} &= M_w \int_{n\cdot v>0} F^\varepsilon n \cdot v dv \end{aligned}$$

where the wall Maxwellian $M_w = \{2\pi\}^{-1}T_w^{-2}\exp\{-\frac{|v|^2}{2T_w}\}$ with a non-constant wall temperature:

$$T_w = 1 + O(|\nabla T_w|).$$

In the case of $|\nabla T_w| = O(\varepsilon)$: such a hydrodynamic limit has been established in leading to the derivation of the celebrated Fourier law (steady Navier-Stokes-Fourier theory) as

$$F^\varepsilon \backsim \mu + \varepsilon\mu\{\rho_1 + u_1\cdot v + T_1\frac{|v|^2-3T}{2}\},$$
$$\mu \equiv \{2\pi\}^{-3/2}\exp\{-\frac{|v|^2}{2}\}$$
$$\Delta T_1 = 0,$$
$$T_1|_{\partial\Omega} = T_w - 1. \text{ (Fourier Law with } u_1 \equiv 0)$$

In the case of $|\nabla T_w| = O(1)$: formal Hilbert expansion in ε [,4]:

$$F \backsim \mu + \varepsilon\mu\{\rho_1 + u_1\cdot v + T_1\frac{|v|^2-3T}{2}\} - \varepsilon\sqrt{\mu}A\cdot\frac{\nabla_x T}{2T^2} \text{ (new layer!)}$$
$$\mu \equiv: \{2\pi T\}^{-3/2}\rho\exp\{-\frac{|v|^2}{2T}\}, \qquad LA = \bar{A}.$$

Here μ is a local Maxwellian with non-constant $[\rho, T]$, $\bar{A} = v(|v|^2 - 5T)\sqrt{\mu}$, which satisfies the following hydrodynamic equations for ρ, T, and the *1st order velocity coefficient* u_1 :

(1)
$$P = \rho T,$$
$$\rho(u_1\cdot\nabla_x u_1) + \nabla p_1 = \nabla\cdot(\tau^{(1)} - \tau^{(2)}),$$
$$\nabla_x\cdot(\rho u_1) = 0,$$
$$\nabla_x\cdot(\kappa\frac{\nabla_x T}{2T^2}) = 5P(\nabla_x\cdot u_1).$$

with any given constant $P > 0$. Here the tensors $\tau^{(i)}$ are defined as

$$\tau^{(1)} = \lambda[\nabla_x u_1 + \{\nabla_x u_1\}^T + \frac{2}{3}(\nabla_x\cdot u)\mathbf{1}]$$
$$\tau^{(2)}(!) = \frac{\lambda^2}{P}[K_1(\nabla_x^2 T - \frac{1}{3}\Delta_x T\mathbf{1}) + \frac{K_2}{T}(\nabla_x T\otimes\nabla_x T - \frac{1}{3}|\nabla_x T|^2\mathbf{1})].$$

The velocity boundary condition is given by

$$u_1|_{\partial\Omega} = \beta_0\nabla_{||}T_w \neq 0.$$

and $\beta_0 \equiv \beta_0(T_w)$ is determined by the solution of a Milne problem [4]. Sone [4] refers such a mismatch of $[\rho, T]$ and u_1, or the influence of an infinitesimal velocity εu_1 on $[\rho, T]$ as the '*ghost effect*'. Moreover, the thermal stress $\tau^{(2)}$ is a purely kinetic effect from the Boltzmann theory, not predicted by any classical fluid theory.

We define

$$F^\varepsilon = \mu + \varepsilon\mu\{\rho_1 + u_1\cdot v + T_1\frac{|v|^2-3T}{2}\} - \varepsilon\sqrt{\mu}A\cdot\frac{\nabla_x T}{2T^2} + \varepsilon R$$
$$R \equiv: \{I-P\}R + (p + v\cdot b + c(|v|^2-5T))\sqrt{\mu}$$

Theorem 1. *(Guo-Esposito-Marra-Wu 2023 [3]) Assume general bounded domain. Assume $||\nabla T||_{W^{3,\infty}} \ll 1$ and a given constant $P > 0$. Then there there exists a remainder R such that $\int p dx = 0$, $||R||_X \ll 1$, $F \geq 0$ and*

$$||\langle F - \mu - \varepsilon\mu(u_1\cdot v), v\rangle_v||_{L^2_x} \lesssim \varepsilon^{3/2}.$$

Here

$$||R||_X \equiv: \varepsilon^{-1}\{||\{I-\bar{P}\}R||_\nu + ||e||_2 + ||p||_2 + ||\xi||\} + \varepsilon^{1/2}||wR||_\infty$$
$$+ \varepsilon^{-1/2}\{|\{I-P_\gamma\}R|_{2,\gamma} + ||b||_2 + ||\xi||_{H^2}\} + ||PR||_6$$

Such R is unique.

The equation for the remainder takes the form of

$$v\cdot\nabla_x R + (\mu^{-1/2}\frac{\bar{A}\cdot\nabla_x T}{4T^2})R + \varepsilon^{-1}LR = S,$$

with basic energy estimate:

$$\frac{1}{2\varepsilon}\int_\gamma \{I-P_\gamma\}R^2 + \frac{1}{\varepsilon^2}||\{I-P\}R||^2_\nu + \frac{1}{\varepsilon}(\mu^{-1/2}\frac{\bar{A}\cdot\nabla_x T}{4T^2}R, R) = \frac{1}{\varepsilon}(S, R),$$

where

$$\varepsilon^{-1}(\mu^{-1/2}\frac{\bar{A}\cdot\nabla_x T}{4T^2}R, R) \backsim \varepsilon^{-1}(\nabla_x T, bc).$$

There is a severe singularity of order ε^{-1} in light of the basic L^6 estimates in [1]

$$||Pf||_6 \backsim ||p||_6 + ||c||_6 + ||b||_6 \lesssim \varepsilon^{-1}||\{I-P\}R||_\nu.$$

Highlights of mathematical innovations

1. Hodge Decomposition, New Conservation Law

We first split $\{I-P\}R$ along $\bar{A}$ direction:

$$\{I-P\}R = d\cdot\bar{A} + \{I-\bar{P}\}R, \text{ and } (\bar{A}, \{I-\bar{P}\}R) = 0.$$

Furthermore, we define $A-$Hodge decomposition for d as

$$d = \nabla\xi + e, \qquad \nabla\cdot(\kappa e) = 0$$
$$\nabla\cdot(\kappa\nabla\xi) = \nabla_x\cdot(\kappa d), \quad \xi|_{\partial\Omega} = 0.$$

Recall local Conservation laws of mass, momentum and energy:

$$\nabla_x\cdot b \backsim 0$$
$$\nabla_x p + \nabla_x\langle v\otimes v, \{I-\bar{P}\}R\rangle \backsim 0$$
$$5P\nabla_x\cdot\{Tb\} + \nabla_x\{\kappa d\} \backsim 0.$$

• *New A− Conservation Law* (with A):

$$\kappa[\nabla_x c + \varepsilon^{-1} d] + \frac{\nabla T}{2T^2}(\kappa p + \sigma c) = ...$$
$$\kappa \nabla_x [c + \varepsilon^{-1}\xi] + \frac{\nabla T}{2T^2}(\kappa p + \sigma c) + \varepsilon^{-1}\kappa \mathbf{e} = ...$$

2. Reduced Energy Estimate

• Reduced Energy Estimate (missing ξ and PR!):

$$\varepsilon^{-1}\int_\gamma \{I - P_\gamma\}R^2 + \varepsilon^{-2}[|||\{I-\bar{P}\}R||_\nu^2 + ||e||_\nu^2] \lesssim o(1)|||R|||_X^2 + 1.$$

Sketch of Proof: A chain of identities to compute $\varepsilon^{-1}\langle \nabla_x T \cdot b, c\rangle$:

$$\begin{aligned}
&\backsim \varepsilon^{-1}\langle \nabla_x \cdot \{Tb\}, c\rangle \text{ (mass conservation } \nabla \cdot b \backsim 0)\\
&\backsim -\varepsilon^{-1}\langle \nabla_x \cdot \{\kappa d\}, c\rangle \text{ (energy conservation law)}\\
&\backsim -\varepsilon^{-1}\langle \nabla_x \{\kappa \nabla \xi\}, c\rangle \ \ (\nabla \cdot \{\kappa e\} = 0 \text{ Hodge decomposition})\\
&\backsim \varepsilon^{-1}\langle \kappa \nabla \xi, \nabla_x c\rangle \ (A \text{ conservation law})\\
&\backsim -\varepsilon^{-2}\langle \kappa \nabla_x \xi, \nabla_x \xi\rangle(\text{main}) - \varepsilon^{-1}\langle \kappa \nabla_x \xi, \frac{\nabla T}{2T^2}(\kappa p + \sigma c)\rangle
\end{aligned}$$

ignoring boundary contributions ($||\xi|| \backsim \varepsilon, ||\xi||_{H^2} \backsim \varepsilon^{1/2}$)

$$\varepsilon^{-1}\int_\gamma \kappa \nabla \xi \{I - P_\gamma\}R \lesssim o(1)\varepsilon^{-1}|\nabla \xi|_{L^2_\gamma}|\{I-P_\gamma\}R|_{L^2_\gamma} = o(1)||R||_X^2.$$

3. Improved Estimate for b

• $||p||_2 \lesssim \varepsilon$ from momentum conservation law.
• $||\varepsilon^{-1}\xi + c||_6 \lesssim 1$ from previous ellipticity estimate
• New Estimate for b **gain** $\varepsilon^{1/2}$:

$$\varepsilon^{-1/2}||b||_2 + ||b||_6 \lesssim o(1)|||R|||_X^2 + 1.$$

Sketch of Proof: Choose test function $\nabla_x \psi \otimes B$ with $\psi|_{\partial\Omega} = 0$ and $\nabla \cdot \psi = 0$

$$\begin{aligned}
&\varepsilon^{-1}\langle v \cdot \nabla_x R + \varepsilon^{-1} d \cdot A + \varepsilon^{-1} L\{I - \bar{P}\}R, \nabla \psi \otimes B\rangle\\
&\quad = \varepsilon^{-1}\langle v \cdot \nabla_x R, \nabla \psi \otimes B\rangle\\
&\quad \backsim \varepsilon^{-1}\langle v \cdot R, \nabla_x^2 \psi \otimes B\rangle\\
&\quad \backsim \varepsilon^{-1}\langle b, \Delta \psi\rangle + ...
\end{aligned}$$

Removal of $\varepsilon^{-1}\{I - \bar{P}\}R$ via momentum conservation:

$$\begin{aligned}
\varepsilon^{-1}\langle \psi, \nabla_x \{I - \bar{P}\}R \otimes B\rangle &\backsim \varepsilon^{-1}\langle \psi, \nabla p\rangle \text{ (momentum conservation)}\\
&\backsim 0 \text{ if } \nabla \cdot \psi = 0 \text{ and } \psi|_{\partial\Omega} = 0!
\end{aligned}$$

This motivates to seek a new test function $\nabla\psi : B+\varepsilon^{-1}\psi\cdot v\sqrt{\mu}$ with a dual *Stokes solution*

$$\begin{aligned} -\lambda\Delta\psi+\nabla q \text{ (up to lower order)} &\backsim b,\\ \nabla\cdot\psi &= 0,\\ \psi|_{\partial\Omega} &= 0. \end{aligned}$$

The artificial pressure q is harmless: solvability for constraint $\nabla\cdot\psi=0$!

4. Estimate for c: Dual Stokes-Poisson System

Choose test function $\nabla\phi\cdot A$ to obtain

$$\begin{aligned} &\langle v\cdot\nabla_x R+\varepsilon^{-1}d\cdot A,\nabla\phi\cdot A\rangle\\ &=\langle v\cdot\nabla_x R,\nabla\phi\cdot A\rangle\ (\backsim ||c||_{L^6}^6)+\langle\varepsilon^{-1}d\cdot A,\nabla\phi\cdot A\rangle\\ &\backsim -\langle\varepsilon^{-1}\nabla\cdot\{\kappa d\},\phi\rangle\\ &\backsim \langle\varepsilon^{-1}\nabla\cdot\{Tb\},\phi\rangle \text{ (energy conservation)}\\ &\backsim \langle\varepsilon^{-1}\{\nabla T\cdot b\},\phi\rangle\ (\nabla\cdot b\backsim 0,\ \text{singularity } \varepsilon^{-1}) \end{aligned}$$

Choose test function $\nabla_x\psi\otimes B$ with $\psi|_{\partial\Omega}=0$ and $\nabla\cdot\psi=0$

$$\begin{aligned} &\varepsilon^{-1}\langle v\cdot\nabla_x R+\varepsilon^{-1}d\cdot A+\varepsilon^{-1}L\{I-\bar{P}\}R,\nabla\psi\otimes B\rangle\\ &\quad=\varepsilon^{-1}\langle v\cdot\nabla_x R,\nabla\psi\otimes B\rangle\\ &\quad\backsim\varepsilon^{-1}\langle v\cdot R,\nabla_x^2\psi\otimes B\rangle\\ &\quad\backsim\varepsilon^{-1}\langle b,\Delta\psi\rangle+... \end{aligned}$$

We combine with ε^{-1} and choose ψ such that ε^{-1} terms involving b for test function $\psi\otimes B$ *cancel with* the singular $-\langle\varepsilon^{-1}\nabla T\cdot b\},\phi\rangle$! This requires $[\psi,\phi]$ solves a coupled Stokes-Poisson system:

$$\begin{aligned} -\Delta\psi+\nabla q &\backsim -5P\phi\nabla_x T\\ \nabla\cdot\psi &= 0,\\ -\nabla\cdot(\kappa\nabla\phi) &= c|c|^4. \end{aligned}$$

with new test function $\nabla\psi : B+\varepsilon^{-1}\psi\cdot v\sqrt{\mu}$.

- A loss of $\varepsilon^{-1/2}$ at boundary in this process:

$$-\varepsilon^{-1}\langle\nabla_x\psi : B,\{I-P_\gamma\}R\rangle|_{\gamma_+}.$$

5. Estimate for c (Boundary Layer)

To get rid of such a $\varepsilon^{-1/2}$ boundary loss, we construct a *ε-cutoff boundary layer*

$$g_B=g_B(\frac{x_\perp}{\varepsilon},x_{||},v)\chi_{|v_\perp|\geq\varepsilon}$$

such that

$$(g_B+g_I)_{\gamma_+}=-\nabla_x\psi : B\implies g_I=\sqrt{\mu}(v\cdot i),\ \nabla\cdot i=0$$

from $\partial_n\psi|_{\partial\Omega} = 0$ and parity of B! Fortunately, thanks to the special structure, there are exact cancellations for the interior solution g_I with *no singularity in* ε :

$$\varepsilon^{-1}\langle v\mu^{1/2}\cdot\nabla_x\{\mu^{-1/2}g_I\}, R\rangle = 0!$$

• Boundary layer interaction (Hardy's inequality): $x = nx_\perp + x_{||}$, for $h|_{\partial\Omega} = 0$ $(H^1 + \sqrt{\varepsilon}L^2!)$

$$(g_B, h) = (g_B n, \frac{1}{n}h) \lesssim ||g_B(\frac{n}{\varepsilon})n||_{L^2}||\frac{h}{n}||_{L^2} \lesssim \varepsilon\cdot\varepsilon^{1/2}||\nabla h||_2$$

• New BV Estimate to control $||\nabla g_B||_{L^1_{x,v}} \lesssim 1$ to avoid log loss (uniform in ε!):

$$|\partial_{v_\perp} g_B| \lesssim \frac{1}{|v_\perp|}.$$

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Two case studies of PDE-constrained optimization within kinetic framework

Qin Li

PDE-constrained optimization is a widely applied numerical strategy for PDE-based control and inverse problems. Recent years we have seen many such applications raising from kinetic theory / partial-differential-integral equation. In the talk, I gave two such examples and presented the obtained numerical results.

The first example emerges as a stability problem from plasma control. From the kinetic perspective, dynamics of plasma is governed by the Vlasov-Poisson (or more complicated Vlasov-Maxwell) equation. The Penrose condition determines the stability of the equilibrium state. For the equilibrium state that are unstable, we ask if it is possible to impose external electric field to stablize the dynamics. This comes down to solving the following problem:

$$\begin{aligned} &\min_H \quad J(f[H]) \\ &\text{s.t.} \quad \begin{cases} \partial_t f + v\partial_x f - (H + E[f])\partial_v f = 0 \\ E[f] = \partial_x G * (1 - \rho_f) \\ f(t=0,x,v) = f_0 = f^{\text{eq}} + \tilde{f}, \end{cases} \end{aligned} \tag{1}$$

where we set the initial condition to be a small perturbation from the equilibrium state, and E and H are self-generated and external electric fields respectively. The to-be-minimized objective function J is set to be stability criteria (can be $\|E(f)\|_2$

or $\|f - f^{\mathrm{eq}}\|_2$). The optimization problem returns the optimal external field H so that the instability is mostly suppressed. Since the feasible set is a whole function space and is computational challenging, numerically we use Fourier transform and confine ourselves to only a few Fourier modes. Numerical experiments suggest a highly oscillatory pattern of the objective function's dependence on the Fourier coefficients, see Figure 1.

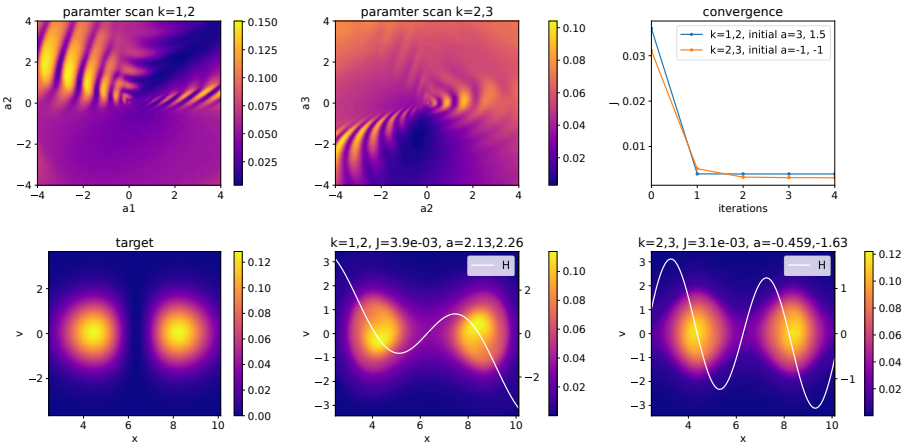

FIGURE 1. Parameter scan for the first three Fourier modes.

We suspect such oscillatory pattern originates from the wave-type instability of the plasma, resonating similar phenomenon from wave-inversion. More delicate analysis is necessary for a better understanding.

The second example is to rewrite the deep-neural-network into an integral form and translate DNN-training into the associated integral-equation-constrained optimization problem. In particular, the ResNet type DNN writes as:

$$z_{l+1} = z_l + \sum_m f(z_l, \theta_{l,m}), \quad z_0 = x,$$

with the output of the DNN to be $y(x) = z_L$. In the formulation f is the activation function and $\theta_{l,m}$ are coefficients for the m-the neurons (weights) on l-th layer. The DNN training is to find the proper tuning of parameters $\{\theta_{l,m}\}$ so that the DNN-generated data $y(x)$ agrees as much as possible to the given data.

This DNN training problem can be translated to a PDE-constrained optimization. To be specific, with a proper scaling, the DNN, in a continuous setting, corresponds to an equation that reads closely related to the homogeneous linearized Boltzmann equation:

$$\partial_t z = \int f(z, \theta)\rho(\theta, t)\mathrm{d}\theta,$$

and the training is then translated to minimizing the mismatch between the PDE-produced result and the given data. Once the to-be-optimized objective function

is spelled out, a gradient flow on ρ is then naturally formed. We gave a rigorous proof on the convergence of DNN to the limiting ordinary integral equation, and the training of DNN to the limiting gradient-flow equation, using the coupling method, borrowed from the mean-field analysis. This translation shines light into a more systematic manner of analyzing DNN training in its large limit.

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Multiscale numerical treatment of the electron/ion kinetic model of fusion plasmas in the low eletron/ion mass-ratio regime

Claudia Negulescu

(joint work with E. Lehman, F. Filbet)

The main concern of this talk (based on the two works [1, 2]) is the presentation of a multi-scale numerical scheme for an efficient resolution of the electron-ion dynamics in thermonuclear fusion plasmas. The starting mathematical model is based on a multi-species Fokker-Planck kinetic model, conserving mass, total momentum and energy, as well as satisfying Boltzmann's H-theorem, namely

$$(1)\qquad \begin{cases} \partial_t f_i + \mathbf{v}\cdot\nabla_{\mathbf{x}} f_i + \dfrac{e}{m_i}\,\mathbf{E}\cdot\nabla_{\mathbf{v}} f_i = \mathcal{Q}_{ii}(f_i,f_i) + \mathcal{Q}_{ie}(f_i,f_e)\,, \\ \partial_t f_e + \mathbf{v}\cdot\nabla_{\mathbf{x}} f_e - \dfrac{e}{m_e}\,\mathbf{E}\cdot\nabla_{\mathbf{v}} f_e = \mathcal{Q}_{ee}(f_e,f_e) + \mathcal{Q}_{ei}(f_e,f_i)\,, \end{cases}$$

associated to Poisson's equation for the description of the electrostatic potential

$$(2)\qquad -\Delta\phi = \frac{e}{\varepsilon_0}(n_i - n_e), \qquad \mathbf{E} = -\nabla_{\mathbf{x}}\phi\,,$$

with e the elementary charge, m_α the mass of species $\alpha \in \{e, i\}$ and ε_0 the vacuum permitivity. The magnetic field $\mathbf{B}$ is not considered here, as we shall focus in the following rather on the dynamics parallel to $\mathbf{B}$. The functions $f_\alpha(t,\mathbf{x},\mathbf{v})$ represent the particle distribution functions in the phase-space $(\mathbf{x},\mathbf{v}) \in \mathbb{T}^d \times \mathbb{R}^d$ ($\mathbb{T}^d$ being the d-dimensional torus) whereas the electron and ion macroscopic quantities are

given for $\alpha \in \{e,\, i\}$ by

$$\begin{cases} n_\alpha(t,\mathbf{x}) \,:=\, \displaystyle\int_{\mathbb{R}^d} f_\alpha(t,\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{v}\,, \\ n_\alpha \mathbf{u}_\alpha(t,\mathbf{x}) \,:=\, \displaystyle\int_{\mathbb{R}^d} \mathbf{v}\, f_\alpha(t,\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{v}\,, \\ d\,k_B\, n_\alpha\, T_\alpha(t,\mathbf{x}) \,:=\, m_\alpha \displaystyle\int_{\mathbb{R}^d} |\mathbf{v}-\mathbf{u}_\alpha|^2\, f_\alpha(t,\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{v}\,, \\ w_\alpha(t,\mathbf{x}) \,:=\, \dfrac{m_\alpha}{2} \displaystyle\int_{\mathbb{R}^d} |\mathbf{v}|^2\, f_\alpha(t,\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{v} = \dfrac{d}{2}\, k_B n_\alpha T_\alpha + \dfrac{m_\alpha}{2} n_\alpha |\mathbf{u}_\alpha|^2\,, \end{cases}$$

with k_B the Boltzmann constant. The collision operators describing the inter- and intra-species interactions are chosen of Fokker-Planck type, *i.e.* given for α, $\beta \in \{e, i\}$ by

$$\mathcal{Q}_{\alpha\beta}(f_\alpha, f_\beta) \,:=\, \nu_{\alpha\beta}\, \mathrm{div}_{\mathbf{v}} \left((\mathbf{v}-\mathbf{u}_{\alpha\beta}) f_\alpha + \frac{k_B T_{\alpha\beta}}{m_\alpha} \nabla_{\mathbf{v}} f_\alpha \right),$$

where $\nu_{\alpha\beta} > 0$ are the collisional frequencies corresponding to the couple (α, β) of particles. The choice of the inter-species mixed velocities $\mathbf{u}_{\alpha\beta}$ and temperatures $T_{\alpha\beta}$ is done such that to enforce the appropriate conservation laws and to ensure the H-theorem. For this we shall first of all require that

$$\mathbf{u}_{\alpha\beta} = \mathbf{u}_{\beta\alpha}\,, \qquad T_{\alpha\beta} = T_{\beta\alpha}\,, \qquad \nu_{ei} m_e n_e = \nu_{ie} m_i n_i\,. \tag{3}$$

These three requirements are fundamental and also physical. The justification of the last assumption comes from the Coulomb collisional frequency, given by

$$\nu_{\alpha\beta} = C\, e_\alpha^2\, e_\beta^2\, n_\beta\, \frac{m_\beta}{m_\alpha + m_\beta}\, \frac{1}{(v_{th,\alpha}^2 + v_{th,\beta}^2)^{3/2}}\,, \quad C > 0\,.$$

A unique choice of mixed velocities is then following, given by

$$\mathbf{u}_{ei} \,=\, \mathbf{u}_{ie} \,:=\, \frac{\mathbf{u}_e + \mathbf{u}_i}{2}\,, \tag{4}$$

as well as a unique choice of mixed temperatures, namely

$$T_{ei} \,=\, T_{ie} \,:=\, \frac{m_i\, T_e \,+\, m_e\, T_i}{m_e + m_i} \,+\, \frac{m_i\, m_e}{m_i \,+\, m_e} \frac{|\mathbf{u}_e - \mathbf{u}_i|^2}{2d\, k_B}\,. \tag{5}$$

Starting from this model, we firstly perform a physcial scaling characterizing the adiabatic regime of plasma dynamics and which is based mainly on the small electron/ion mass ratio $\epsilon^2 := \frac{m_e}{m_i} \ll 1$. A subsequent asymptotic limit is fulfilled, letting the electron/ion mass ratio go towards zero ($\epsilon \to 0$), in order to obtain a reduced model, consisting of a thermodynamic equilibrium state for the rapid electrons (adiabatic regime or Boltzmann relation), whereas the slow ions remain kinetic. The asymptotic analysis is performed formally [1] for the above mentioned model, and rigorously [2] in a simplified framework (linear Fokker-Planck collision operator). Our rigorous treatment uses the tools of *hypocoercivity theory*, more precisely, we decide to follow the so-called *"Auxiliary operator method"* build upon a weighted L^2-setting.

In a second step, we develop a first numerical scheme, based on a Hermite spectral method, and perform numerical simulations to investigate in more details this (kinetic towards macroscopic) asymptotic electron/ion limit. The specificity of this method is the fact that it permits considerable improvements in simulation time for small ϵ-values, as in such regimes very few Hermite modes have to be taken into account. For consistency reasons between the theoretical and numerical part, we carried out the theoretical analysis also in a Hermite spectral formalism, procedure which permitted to estimate the asymptotic decay of the Hermite-coefficients.

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From microscopic to macroscopic : the large number dynamics of agents and cells, possibly interacting with a chemical background and conversely

Thierry Paul

Understanding how a vision of he macroscopic world, namely the one in which we live, can be obtained out of a microscopic one, namely the one of "elementary" particles, is one of the oldest task in sciences, It relies in particular on how a continuous paradigm of the matter such as the one we actually perceive can be related to a discrete one like the one, say, of atoms or molecules. The scope of this equation is quite universal as it covers physics, biology, social sciences and more, all of these fields being hatted by a needed mathematical setting

In this talk we address the problematic 'Micro versus Macro¨, namely the question: how to recover the macroscopic scale, our scale, from a microscopic one, for example the cell's one?

Between Micro - living in N particles phase-space - and Macro - living in one particle configuration (physical) space - exists in fact an intermediate stage often called mesoscopic, involving a one particle dynamic on phase space. Vlasov equation lives at this stage, as the equation satisfied by the marginals of densities pushed forwarded by the N particles evolution, in the limit $N \to \infty$.

The novelties of the results presented here are twofolds. First, in the case of indistinguishable objects such as particles or cells, we consider not only self-propelled dynamics but also particles in interaction with a chemical environment, leading to dynamics non-local in time but rather depending of the full history of the dynamics. Second, in the case of agent systems, that is evolution of distinguishable objects, we consider not only agents subjects to a single, precise, deterministic opinion, but rather subjects to a stochastic opinion driven, for any single agent, by a density of probability. With respects of this extension of standard opinion systems, the Vlasov equation that we derive in, to our knowledge, new.

Passing from Meso to Macro, namely from one particle dynamics on phase space to the similar one on physical space, is established through a monokinetic hypothesis – that is fixing the velocity of particles/agents as function of their positions – on the initial data, that is in the case of agent, to a return to deterministic opinions. The novelty here is that – by this journey through hydrodynamic Euler paradigm – one gets the same limiting "graph limit" equation than the one derived with standard methods involving graphs.

The final novelty consists in somehow following the inverse way, that is associating to any PDE on physical space a system of agents whose graph limit is the given PDE. This is achieved by letting the generator of the agent dynamics becoming singular in the limit of large numbers of agents.

General dynamics

At the microscopic level, the Newton law gives rise to a trajectory $(x_1(t), \dots, x_N(t))$ (possibly NOT a flow) in $\mathbb{R}^{dN}$:

$$\Phi_N^t : \left\{ \begin{array}{rcl} \dot{x}_i(t) & = & \frac{v_i(t)}{m_i} \\ m_i \dot{v}_i(t) & = & F_i(t, x_1(t), \dots, x_N(t), v_1(t), \dots, v_N(t); \\ & & \quad \{x_i(s), v_i(s),\ i = 1 \dots, N,\ 0 \le s \le t\}). \end{array} \right.$$

Examples

- two body interaction $\dot{v}_i(t) = \frac{1}{N} \sum_{j=1}^N G(x_i(t), x_j(t))$ $\qquad i = 1, \dots, N$
- Cucker-Smale $\dot{v}_i(t) = \dfrac{\lambda}{N} \displaystyle\sum_{j=1}^N \left(1 + \frac{\|x_i(t) - x_j(t)\|^2}{R^2}\right)^{-\beta} (v_i(t) - v_j(t))$ $\qquad \beta, R > 0$
- topological interaction $\dot{v}_i(t) = \frac{1}{N} \sum_{j=1}^N K(M(x_i, |x_i - x_j|)(v_i(t) - v_j(t))$
- chemotaxis $\dot{v}_i(t) = \frac{1}{N} \displaystyle\sum_{j=1}^N F(x_i(t) - x_j(t), v_i(t) - v_j(t)) + \eta \nabla_x \varphi^t(x_i(t)),\ \partial_t \varphi^t(x) = \Delta \varphi^t(x) - \kappa \varphi^t + \frac{1}{N} \sum_{j=1}^N \chi(x - x_i(t))$,
- multi-agents (distinguishable particles) $\dot{\xi}_i(t) = \frac{1}{N} \sum_{j=1}^N G_{i.j}(\xi_i(t), \xi_j(t))$.

Methodology: one avoids the concept of empirical density, and rather consider a probabilistic distribution $\rho_N^{in}(x_1, \dots, x_N; v_1, \dots, v_N)$ equal to the

$$\boxed{\text{probability that each particle } i \text{ is at position } x_i \text{ with velocity } v_i},$$

propagated by Liouville equation to $\rho_\cdot^t = (\Phi_N^t)_* \rho_N^{t=0}$, an object intractable when N becomes large.

But averaging over all particles but one leads to the notion of *marginal*:

$$\rho_{N;1}^t(x, v) := \int \dots \int \rho_N^t(x, x_2, \dots, x_N; v, v_2, \dots, v_N) dx_2 \dots dx_N dv_2 \dots dv_N$$

satisfying

1. as $N \to \infty$, $\rho_{N;1}^t(x, v)$ has a limit $\rho^t(x, v)$
2. $\rho^t(x, v)$ satisfies the Vlasov equation

$$\partial_t \rho^t(x, v) + v \partial_x \rho^t(x, v) = \partial_v \left(\left(\int G(x, x'; v, v') \rho(x', v') dx' dv' + \eta \nabla_x \varphi^t(x) \right) \rho(x, v) \right),$$

$$\partial_s \psi^s(x) = \Delta \psi^s(x) + \psi^t s * \rho^s(x) \quad 0 \le s \le t.$$

Chemotaxis

Theorem ([Natalini, P. (2021)]). *Let W_2 the second order Wasserstein distance. Then, for ρ^t solution of Vlasov,*

$$W_2\left((\Phi_N^t \# (\rho^{in})^{\otimes N})_{N;1}, \rho^t\right)^2 \leq \tau(t) N^{-\frac{1}{d}} \quad (d > 2)$$

$$\int_{\mathbb{R}^{2dN}} \|\nabla \varphi^t_{Z^{in}} - \nabla \psi^t_{\rho^{in}}\|^2_\infty (\rho^{in})^{\otimes N}(dZ^{in}) \leq \tau_c(t) N^{-\frac{1}{d}} \quad (d > 2)$$

Theorem ([Natalini, P. (2021)]). *If μ^t, u^t solves the Euler system:*

$$\begin{cases} \partial_t \mu^t + \nabla(u^t \mu^t) = 0 \\ \partial_t(\mu^t u^t) + \nabla(\mu^t (u^t)^{\otimes 2}) = \mu^t \int f(\cdot - x'; u^t(\cdot) - u^t(x')\mu^t(x')dx' + \eta \nabla \psi^t \\ \partial_s \psi^s = \Delta \psi^s - \kappa \psi^s + \psi^s * \rho^s \end{cases}$$

then $\mu^t(x)\delta(v - u^t(x))$ solves the Vlasov system.

Numerical evidence morality[Menci, Natalini, P. (2023)]

- in many situations the most striking features of the particle level are preserved by passing to the one of Vlasov
- the fidelity of Euler versus Vlasov is increased by the presence of the chemical interaction
- and this even without monokineticity
- the nonlocal integral Euler system keeps memory of the interactions at the microscopic level
- adding an additional pressure term of size ε, still keeping the nonlocal integral term, helps
- existence of an optimal value ε realizing an improved correspondence between Vlasov moments and Euler solutions

Agent system:

$$\dot{\xi}_i(t) = \frac{1}{N}\sum_{j=1}^N G^N_{ij}(t, \xi_i(t), \xi_j(t)) = \frac{1}{N}\sum_{j=1}^N G(t, \tfrac{i}{N}, \tfrac{j}{N}, \xi_i(t), \xi_j(t)), \quad i = 1, \dots, N$$

- for Hamiltonian and Cucker Smale, G_{ij} does NOT depend on ij: (preserves *indistinguishability*)
- for opinion systems it does (do NOT preserve *indistinguishability*)
- implement main field methods by letting i, j be dynamical variables (although remaining *at rest*).

We can follow two points of view

- direct formal discrete-continuous limit $\xi_i(t) \sim y(t, i/N)$

$$\boxed{\textit{one opinion assigned to each agent i}}$$

$$\rightsquigarrow \ \partial_t y(t,x) = \int G(t, x, x', y(t,x), y(t,x'))\, d\nu(x')$$

- Liouville paradigm

$$\boxed{\textit{random opinion assigned to each agent i}}$$
$$+$$
$$\textit{average on all but one agent (marginal)}$$
$$+$$
$$\textit{hydrodynamic limit}$$
$$\rightsquigarrow \text{ same equation}$$

Theorem ([Trélat, P. (2022)]). *Let* $\rho^{in} = \overset{N}{\underset{i=1}{\otimes}} \delta_{\frac{i}{N}}(x_i) \overset{N}{\underset{i=1}{\otimes}} \mu^{in}(\frac{i}{N}, \xi_i)$.
Then, $W_1((\Phi_N^t \# \rho^{in})^s_{N,1}, \mu^t) \leq \frac{D(t)}{N}$, *where* μ^t *is solution to the Vlasov equation* $\partial_t \mu^t(x,\xi) = \partial_\xi \left(\int G(t,x,x',\xi,\xi') \mu^t(x',\xi') dx' d\xi' \mu^t(x,\xi) \right)$.

Moreover, if $\mu^{in}(x,\xi) := \delta(\xi - y^{in}(x))$ (monokinetic assumption), then $\mu^t(x,\xi) = \delta(\xi - y^t(x))$ and y^t is solution to the (nonlinear) Euler equation $\partial_t y^t(x) = \int_\Omega G(t,x,x',y^t(x),y^t(x'))\, dx'$, i.e. the continuous limit (graph limit) of the agent system $\dot{\xi}_i = \frac{1}{N} \sum_{j=1}^N G(\frac{i}{N}, \frac{j}{N}, \xi_i, \xi_j)$.

Particle approximations of PDEs: we have the following facts

$$G\text{-agent systems} \overset{N\to\infty}{\longrightarrow} \text{graph limit} = \text{(time derivative) integral equation}$$

$$? \qquad \overset{N\to\infty}{\longrightarrow} \text{general PDE}$$

$$G\text{-agent systems} \overset{N\to\infty}{\not\longrightarrow} \text{differential equation}$$

$$\text{but } G_\varepsilon\text{-agent systems} \overset{N\to\infty}{\underset{\varepsilon\to 0}{\longrightarrow}} \text{general PDEs} \qquad G_\varepsilon \rightsquigarrow x\text{-singular as } \varepsilon \to 0$$

And one can define "canonically" the following, for *any* quasilinear PDE,

$$G_\varepsilon^N(x,x',\xi,\xi') := \sum_{l=0}^L a_l(\xi,x)\xi' \left[(i\partial_{x'})^l \frac{e^{-\frac{(x-x')^2}{2\varepsilon}}}{(\pi\varepsilon)^{\frac{1}{2}}} \right] \overset{N\to\infty}{\underset{\varepsilon\to 0}{\rightsquigarrow}} \dot{y}(x) = \sum_{l=0}^L a_l(y(x),x) D_x^l y(x)$$

and we get the following (final) scheme

$$\boxed{\varepsilon\text{-particle system}} \qquad \overset{e^{\frac{\varepsilon}{\varepsilon}\frac{1}{\varepsilon}} N^{-1}}{\sim} \qquad \boxed{\varepsilon\text{-PDE}}$$

$$\boxed{\varepsilon\text{-PDE}} \qquad \overset{\sim}{\underset{?}{}} \qquad \boxed{\text{PDE}}$$

Theorem ([Trélat, P. (2022)]). *Let* $\dot{y}(x) = \sum_{l=0}^L a_l(x) D_x^l y(x)$ *develop a semigroup of solutions. Then there exists an explicit* N*-dependent particle system* ξ_j^t, $j = 1,\dots,N$, *(as before with* $\epsilon \sim (\log\log N)^{-\beta}, \beta > 0$*) such that*

$$\|y^t - y_N^t\|_{L^2} \leq C(\log\log N)^{-\alpha}, \alpha > 0,\ y_N^t(x) := \sum_{j=1}^N \xi_j^t 1_{[j/N,(j+1)/N)}(x).$$

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All relative entropies for general nonlinear Fokker-Planck equations

Anton Arnold

(joint work with Jose A. Carrillo, Daniel Matthes)

The topic of this talk is the analysis of the large-time behavior of nonlinear parabolic equations of the form

$$\begin{aligned} &\frac{\partial u}{\partial t} = \mathrm{div}_x\left(\nabla_x P(u) + u\nabla_x V(x)\right), \qquad x\in\mathbb{R}^d, t>0, \\ &u(x,t=0) = u_0(x) \geq 0, \end{aligned} \tag{1}$$

using the entropy method. This evolution equation is given in whole space $\mathbb{R}^d$ and is subject to a uniformly convex confinement potential V that satisfies:

$$\exists \lambda > 0: \quad \frac{\partial^2 V(x)}{\partial x^2} \geq \lambda \mathbf{I} \quad \forall x\in\mathbb{R}^d,$$

where $\mathbf{I}\in\mathbb{R}^{d\times d}$ denotes the identity matrix. The nonlinear function $P:\mathbb{R}_0^+\to\mathbb{R}_0^+$ is strictly increasing and satisfies $P(0)=0$, $P'(0+)\in[0,\infty]$. Hence, (1) can be either degenerate (for $P'(0)=0$) or not.

Exponential convergence, in relative entropy as $t\to\infty$, of the solution $u(x,t)$ of (1) towards the unique steady state u_∞ (having the same mass $\int u_\infty(x)\,dx$ as u_0) was established in [4] by a nonlinear generalization of the *entropy method* of Bakry and Émery [3]. The entropy method for *linear* Fokker-Planck equations can be applied to a large family of relative entropies of the form

$$H_\psi(u|u_\infty) := \int_{\mathbb{R}^d} \psi\Big(\frac{u}{u_\infty}\Big) u_\infty\, dx \geq 0, \tag{2}$$

with non-negative, strictly convex *entropy generators* $\psi:\mathbb{R}_0^+\to\mathbb{R}_0^+$ with $\psi(1)=0$ and ψ ranging between the logarithmic entropy $\psi_1(\sigma) := \sigma\ln\sigma - \sigma + 1$ and the quadratic entropy $\psi_2(\sigma) := (\sigma-1)^2$, see [3, 2]. By contrast, the analysis from [4] was only carried out for the analog of ψ_1 for nonlinear evolutions. This contrasting state-of-the-present-research leads to the **main goal of this talk**, and its underlying paper [1]:

Given a nonlinearity P, find are *all relative entropies* such that the generalized entropy method (à la Bakry-Émery) will yield exponential decay of such relative

entropy as $t \to \infty$. Typically, the corresponding convergence $u(.,t) \to u_\infty$ then has the rate λ obtained from the convexity of V.

The main challenge for this investigation is to find a practical definition (or parametrization) of relative entropies for nonlinear Fokker-Planck equations, as it turned out that (2) is not a convenient staring point. While degenerate and non-degenerate equations need slightly different definitions (see [1]), we shall confine ourselves here to the *degenerate case*.

As a preparation to discuss the equilibrium of (1), we define the increasing function $\phi : \mathbb{R}_0^+ \to \mathbb{R}$,

$$\phi(u) := \int_1^u \frac{P'(s)}{s}\, ds.$$

The equilibrium of (1), $u_\infty(x)$ has been characterized in [4]:

Lemma 1. *For any given mass $M > 0$, $\exists!$ equilibrium $u_\infty \geq 0$ with $\int u_\infty\, dx = M$, and it has compact support. Moreover, it can be characterized explicitly by:*

$$\exists!\, \bar{C} \in \mathbb{R} : \quad \phi\big(u_\infty(x)\big) + V(x) = \bar{C} \qquad \text{on } \operatorname{supp}(u_\infty).$$

For the definition of the entropies we also define the abbreviation

$$\xi := \phi(u) + V(x) - \bar{C} \geq \xi_{\min} = \phi(0+) - \bar{C} > -\infty,$$

where we used the normalization $V(0) = 0$ in the first estimate as well as the degeneracy of P.

Definition 2. *For given P and M, let the increasing scalar* generating function *$g : (\xi_{\min}, \infty) \to \mathbb{R}$ satisfy $g(0) = 0$, and define $G : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \times \mathbb{R}^d \to \mathbb{R}$ as*

$$G(a, b; x) := \int_a^b g(\phi(s) + V(x) - \bar{C})\, ds.$$

For non-negative functions u and u_∞, both having mass M, we define their relative g-entropy as

$$\mathcal{H}_g(u|u_\infty) := \int_{\mathbb{R}^d} G\big(u(x), u_\infty(x); x\big)\, dx \geq 0\,.$$

With this setup, the essence of the entropy method (just like in [3, 2, 4]) is to derive the following differential inequality for the entropy dissipation functional

$$-J_g := \frac{d}{dt}\mathcal{H}_g(u(t)|u_\infty) = -\int_{\mathbb{R}^d} u g'(\xi)|\nabla\xi|^2\, dx \leq 0.$$

For entropies $\mathcal{H}_g$ that are *admissible* for a given nonlinearity P, it satisfies

$$-\frac{d}{dt}J_g(u(t)) - 2\lambda J_g(u(t)) \geq 0, \tag{3}$$

and this yields exponential-in-time decay of J_g and subsequently of the relative entropy. The above mentioned admissibility of $\mathcal{H}_g$ is based on a careful analysis of the remainder term $\mathcal{R}$ appearing in the inequality (3). More precisely, we study (as in [3, 2]) the pointwise non-negativity of the integrand in this remainder term.

To formulate the final result we define the following scalar functions that only depend on P:

$$\gamma(u) := \frac{uP'(u)}{P(u)} - 1, \quad \kappa(u) := 1 + \frac{\gamma(u)}{8}\frac{d\gamma(u) - 8}{d\gamma(u) + 1}.$$

Theorem 3. *Let the nonlinearity P satisfy*

$$\gamma(u) \geq -\frac{1}{d} \quad \forall\, u > 0. \tag{4}$$

For such P given, a relative entropy $\mathcal{H}_g(u|u_\infty)$ is admissible (in the sense of yielding a non-negative remainder integrand, and hence making the entropy method "work") iff either

(1) $\gamma(u) = 0$ *for some* $u > 0$ *and* $g(\xi) \equiv \xi$,
(2) $\gamma(u) > 0\ \forall\, u > 0$ *and* $f := \ln g'$ *satisfies*

$$f''(\xi) - f'(\xi)\,\vartheta(\xi, f'(\xi)) \geq 0\,, \quad \xi > \xi_{\min},$$

with $\vartheta(\xi,\eta) := \sup_{0<u<\phi^{-1}(\xi+\bar{C})} \big\{\kappa(u)\eta - \frac{u}{P(u)}\big\}$, *which depends only on P.*

We remark that condition (4) already appeared in [5] for the displacement convexity of the standard entropy (i.e. free energy) for an interacting gas.

While the conditions in Theorem 3 may look involved, they can be simplified in may cases. For the prominent example of power law nonlinearities, $P(u) = u^m$, $m > 0$ it yields:

Theorem 4. (1) *There are no admissible relative entropies for* $0 < m < \frac{d-1}{d}$.
(2) *For* $\frac{d-1}{d} \leq m < 1$, *i.e. (non-degenerate) fast-diffusion equations (with* $\xi_{\min} = -\infty$*) only the standard entropy from* [4] *with* $g(\xi) \equiv \xi$ *is admissible.*
(3) *For* $m \geq 1$, *i.e. linear Fokker-Planck equations and (degenerate) porous-medium equations, there is a whole family of admissible relative entropies that can be written explicitly.*

For the fast-diffusion case (2), note that only $g(\xi) \equiv \xi$ yields the pointwise non-negativity of the integrand in the remainder $\mathcal{R}$, which we required for *admissibility*. So, possibly, there could still exist other entropies with exponential time decay that exploit cancellations in the integral form of the reminder $\mathcal{R}$.

We remark that these newly constructed entropies also yield new functional inequalities in the form $J_g(u|u_\infty) \geq \lambda \mathcal{H}_g(u|u_\infty)$. Moreover, they give new weighted x-moment estimates for the solution of degenerate diffusions equations.

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Cercignani conjecture for the Landau equation

Laurent Desvillettes

We write down the Landau operator appearing in kinetic theory (cf. [7]), defined by

$$Q_\gamma(f,f)(v) = \nabla_v \cdot \left\{ \int_{\mathbb{R}^3} |v-w|^{2+\gamma}\, \Pi(v-w) \left(f(w)\, \nabla f(v) - f(v)\, \nabla f(w) \right) dw \right\}, \tag{1}$$

where $\Pi_{ij}(z) = \delta_{ij} - \frac{z_i z_j}{|z|^2}$ is the i,j-component of the orthogonal projection Π onto $z^\perp := \{y \,/\, y \cdot z = 0\}$.

We focus on the case of so-called Maxwell molecules, hard spheres or hard potentials, that is

$$\gamma \in [0,1]. \tag{2}$$

The Landau operator satisfies (at the formal level) the conservation of mass, momentum and kinetic energy.

We also have the following property of nonnegativity of the so-called entropy dissipation associated to the Landau operator, namely (at the formal level), for any $f := f(v) > 0$,

$$\begin{aligned} D_\gamma(f) &:= -\int_{\mathbb{R}^3} Q_\gamma(f,f)(v) \ln f(v)\, dv \\ &= \frac{1}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} f(v)\, f(w)\, |v-w|^{2+\gamma}\, \Pi(v-w) \left(\frac{\nabla f}{f}(v) - \frac{\nabla f}{f}(w) \right) \\ &\qquad \cdot \left(\frac{\nabla f}{f}(v) - \frac{\nabla f}{f}(w) \right) dv dw \geq 0. \end{aligned} \tag{3}$$

In the last years, several works (cf. [5], [2], [3]) have been devoted to the proof of estimates from below of the quantity $D_\gamma(f)$ in terms of quantities like the Fisher information $F(f) := \int_{\mathbb{R}^3} \frac{|\nabla f(v)|^2}{f(v)}\, dv$, or the relative (with respect to a Maxwellian function $M := M(v)$) Fisher information $F_{rel}(f) := \int_{\mathbb{R}^3} \left| \frac{\nabla f}{f}(v) - \frac{\nabla M}{M}(v) \right|^2 f(v)\, dv$.

Those estimates (typically the ones related to $F(f)$) yield results of regularity for the solutions to the Landau equation (cf. [2]). When they are complemented with the logarithmic Sobolev inequality of Gross (cf. [6]), they also (typically the ones related to $F_{rel}(f)$) yield results for the large time behavior of the Landau equation (cf. [5], [1]), following the lines of the entropy-entropy dissipation method introduced in kinetic theory in the 90s (cf. [8] and [4] for historical details and an explanation of the general context).

We write down a typical estimate, taken from Corollary 2.5 of [3], which holds when γ satisfies (2), and $f > 0$ is normalized:

$$\int_{\mathbb{R}^3} \frac{|\nabla f(v)|^2}{f(v)}\, dv \leq 3072\, \Delta(f)^{-2} \left\{ 8448 + 48\, \sqrt{1+\pi}\, D_\gamma(f)\, ||f||_{L^2(\mathbb{R}^3)} \right\}, \tag{4}$$

where

$$\Delta(f) := \mathrm{Det} \left(\int_{\mathbb{R}^3} f(w)\, (1+|w|^2)^{-1/2} \begin{bmatrix} 1 & w_i & w_j \\ w_i & w_i^2 & w_i\, w_j \\ w_j & w_i\, w_j & w_j^2 \end{bmatrix} dw \right). \tag{5}$$

It is in fact possible to obtain estimates like (4) with the following improvements:

- No terms involving a determinant like (5) appears;
- The numerical constants are much smaller than the ones appearing in (4);
- The proof is short;
- These new estimates extend somewhat to the case of the Landau-Fermi-Dirac entropy dissipation.

The price to pay in order to get those improvements is the following restriction: the estimates that we will write down hold only when $D_\gamma(f)$ is not too large. It is possible to see however that this is not a problem when one wishes to prove the exponential decay towards equilibrium with explicit rate for the solutions to the Landau equation with hard potentials or hard spheres.

Indeed, one can prove the:

Theorem 1. *We consider γ satisfying (2). Then for all normalized $f := f(v) > 0$ such that*

$$||f||^2_{L^2_6(\mathbb{R}^3)}\, D_\gamma(f) \leq 0.062,$$

the following estimate holds:

$$\int_{\mathbb{R}^3} \left| \frac{\nabla f}{f}(v) + v \right|^2 f(v)\, dv \leq 200\, ||f||^2_{L^2_6(\mathbb{R}^3)}\, D(f).$$

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