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# C*-Algebras

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Abstract. Operator algebras form a very active area of mathematics which, since its inception in the 1940s, has always been driven by interactions with other fields of mathematics and physics. The scope of these interactions is very wide, ranging over dynamical systems, (non-commutative) geometry, functional analysis, (geometric) group theory, topology, random matrices, harmonic analysis and quantum information theory.

The goals of this workshop were to stimulate new collaborations across these fields of mathematics, to disseminate recent progress by giving participants a global view on the subject and to specially focus on two important developments: the solution of the Connes embedding problem by methods from quantum information theory and the progress on noncommutative dynamical systems, especially in the topological C*-algebra context.

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## Introduction by the Organizers

During the 2022 C*-algebras workshop at Oberwolfach, some of the most important recent results in Operator Algebras were presented and discussed. A distinct feature in most of the talks and discussion was the constant interaction between the two main research directions within the field, namely C*-algebras and von Neumann algebras, and with other parts of mathematics including quantum information theory and free probability theory.

The first main theme of the workshop was the classification theory of "sufficiently small" C*-algebras and their symmetry groups. The classification by $K$-theoretic invariants of all simple nuclear C*-algebras satisfying the appropriate

regularity assumption has been completed recently, after several decades of work by numerous researchers. James Gabe presented the key elements of a new and more direct proof of this classification theorem, which deduces isomorphism of the C*-algebras in an abstract way from an isomorphism between the invariants, without having to classify by hand numerous building block C*-algebras.

Since all amenable, minimal actions of discrete groups on compact Hausdorff spaces give rise to a nuclear C*-algebra by the crossed product construction, it is a key question to decide when these crossed products are classifiable. Two very general classifiability results were presented at the workshop, in a talk by David Kerr on actions of elementary amenable groups and in a talk by Shirly Geffen on amenable actions of a broad class of nonamenable groups.

The next step in the C*-algebra classification program is to fully classify actions of amenable groups on simple nuclear C*-algebras, again satisfying the appropriate regularity assumptions. Gábor Szabó presented a recent breakthrough in this direction, providing such a classification theorem in the purely infinite case, for actions of amenable groups on Kirchberg algebras.

Another important aspect of classification theory of C*-algebra consists of building good models for classifiable C*-algebras and, in particular, groupoid models. Progress in this direction was presented in several talks on groupoid C*-algebras by Astrid an Huef, Aidan Sims, Anna Duwenig and Becky Armstrong.

The second main focus of the workshop was the recent solution of the Connes Embedding Problem. While Connes' question is phrased in terms of von Neumann algebras, asking whether every $\mathrm{II}_1$ factor with separable predual embeds into an ultrapower of the hyperfinite $\mathrm{II}_1$ factor, the solution is entirely quantum information theoretic in nature. The operator algebra community still has a long way to go in order to fully understand this "MIP*=RE paper". The hopes are that such an operator algebraic understanding will lead to concrete examples of nonembeddable $\mathrm{II}_1$ factors and possibly even to discrete groups that are not hyperlinear, in particular not sofic. Talks by William Slofstra and Chris Schafhauser, who both also animated informal evening sessions, led to a significantly better understanding of the quantum complexity theory aspects of the proof and triggered numerous vibrant discussions.

A third theme of the workshop was rigidity theory. Rufus Willett presented the recent solution of the rigidity problem for uniform Roe algebras: for arbitrary uniformly locally finite metric spaces, the uniform Roe algebras are Morita equivalent if and only if the underlying metric spaces are coarsely equivalent. Connes' rigidity conjecture for group von Neumann algebras predicts that group $\mathrm{II}_1$ factors $L(\Gamma)$ and $L(\Lambda)$ of discrete groups with Kazhdan's property (T) are isomorphic if and only if the underlying groups are isomorphic. Until recently, no examples and no counterexamples to this conjecture were known. Adrian Ioana presented the first positive results confirming Connes' rigidity conjecture for wreath-like product groups with property (T).

Most of the rigidity theorems for $\mathrm{II}_1$ factors obtained in the past decades focus on the isomorphism problem for concrete families of von Neumann algebras. Sorin

Popa presented in his talk an overview of how his deformation/rigidity theory can be used to prove results about the embedding problem: when can a given $\mathrm{II}_1$ factor be embedded into another given $\mathrm{II}_1$ factor?

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# Abstracts

## $\mathcal{Z}$-stability and actions of elementary amenable groups

David Kerr
(joint work with Petr Naryshkin)

It has been a longstanding program, tracing back to the work of Putnam on Cantor minimal systems [12] and of Elliott and Evans on irrational rotations [5], to understand the structure of $C^*$-crossed products associated to groups actions on compact metrizable spaces and to determine criteria for the classifiability of such crossed products within the context of the Elliott program. While amenability of the action (which occurs for example if the group itself is amenable) is a precondition for classifiability, advances over the last several years, leading to and revolving around the final form of the classification theorem itself, have pinpointed finite nuclear dimension and $\mathcal{Z}$-stability as two equivalent expressions of the regularity hypothesis on which classifiability hinges in the setting of simple separable unital nuclear C*-algebras satisfying the UCT [7, 6, 13, 1].

These developments in C*-algebra theory have inspired an effort to identify dynamical analogues of finite nuclear dimension and $\mathcal{Z}$-stability with the goal of shifting the verification of classifiability as much as possible onto the dynamics itself. To this end the dynamical concept of almost finiteness was introduced as an analogue of the conjunction of $\mathcal{Z}$-stability and nuclearity [11, 8]. In the case of free minimal actions on compact metrizable spaces it implies $\mathcal{Z}$-stability and hence classifiability of the crossed product [2, 8].

Almost finiteness has been verified for free actions on finite-dimensional spaces of many amenable groups, including groups of subexponential growth [4], polycyclic groups [3], and the lamplighter group [3] (each time relying on the passage from zero-dimensional to finite-dimensional spaces from [10]). We have shown that this list can be expanded to include all elementary amenable groups [9]. The key technical step is to demonstrate that if $H \rtimes \mathbb{Z} \curvearrowright X$ is a free action on a compact metrizable space such that the restriction $H \curvearrowright X$ is almost finite then the action $H \rtimes \mathbb{Z} \curvearrowright X$ is itself almost finite. While there is no dimension assumption in this extension result, the fact that actions of the trivial group are almost finite only when the space is zero-dimensional means that we must restrict to such spaces when bootstrapping our way up to all elementary amenable groups. The usual appeal to [10] then yields the conclusion for finite-dimensional spaces.

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## Realizing quantum flag manifolds as graph C*-algebras

Karen Strung
(joint work with Tomasz Brzeziński, Ulrich Krähmer, Réamonn Ó Buachalla)

Graph C*-algebras are a particularly tractable class of C*-algebras, thanks to the presence of useful combinatorial tools coming from the underlying directed graph. If a C*-algebra $A$ has a graph C*-algebraic model, many of its structural properties are determined directly from the graph, such as whether $A$ is unital, its ideal structure, whether it is stably finite or purely infinite, and its $K$-theory.

Many well-known C*-algebras can be realized as graph C*-algebras, including the Cuntz algebras, $C(\mathbb{T})$, $M_n$, $\mathcal{K}$, and the Toeplitz algebra $\mathcal{T}$. In [2], many interesting examples of graph C*-algebras were constructed from quantum spaces, including quantum spheres and quantum projective spaces.

The meaning of *quantum space* refers to a $q$-deformation of the algebra of functions on a classical space which can be described by generators and relations. Given $q \in (0, 1]$, one replaces certain commutation relations with relations involving generators and functions in $q$. When $q = 1$, we recover the algebra of functions on the space. A canonical example is the $q$-deformation of $SU_2 \cong S^3$. The C*-algebra $C_q(SU_2) \cong C_q(S^3)$ is generated by two elements, $\alpha, \gamma$, subject to the relations

$$\begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \text{ is a unitary matrix .}$$

For compact connected simply connected Lie groups and their homogeneous spaces—for example the odd dimensional quantum spheres and quantum complex projective spaces—one can do this is in a precise way that allows one to keep much of the Lie theoretic structure. A particular class homogeneous spaces are *quantum*

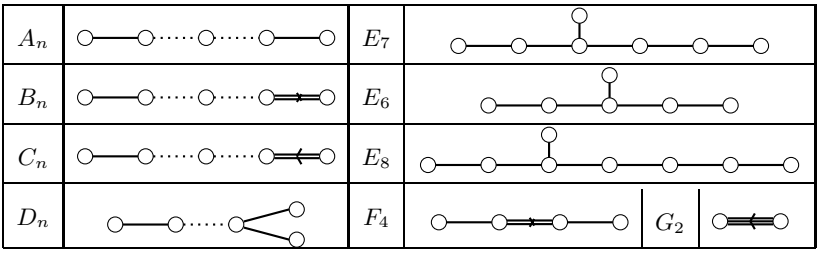
*flag manifolds*, which include quantum complex projective spaces. We show the following:

**Theorem 1.** *Let $q \in (0,1)$ and let $C_q(G/L_S)$ be the* C*-*algebra of a quantum flag manifold. Then there exists a directed graph $E$ such that $C^*(E) \cong C_q(G/L_S)$.*

In the classical setting, a flag manifold arises as a quotient of a simply connected compact semisimple Lie group. These Lie groups admit a particularly satisfying $q$-deformation: Given a complex semisimple Lie algebra $\mathfrak{g}$ and $q \in (0,1)$ the enveloping algebra $U(\mathfrak{g})$ has a $q$-deformation, $U_q(\mathfrak{g})$, which admits a Hopf algebra structure and has the same finite-dimensional representation theory as $U(\mathfrak{g})$. If $\mathfrak{g}$ is a Lie algebra of rank $r$, then $U_q(\mathfrak{g})$ is generated by elements $E_j, F_j, K_j$, $1 \le j \le r$ subject to certain relations, which can essentially be read off the relevant Dynkin diagram. The Dynkin diagram determines the Weyl group $W$ of $\mathfrak{g}$, which has one generator per node with relations

| | |
|---|---|
| $s_i^2 = 1$ | for every $1 \le i \le r$ |
| $s_i s_j = s_j s_i$ | if there is no edge between node $i$ and node $j$ |
| $(s_i s_j)^{3+\delta_{2k}+3\delta_{3k}} = 1$ | if node $i$ is connected to node $j$ by $k$ edges, $1 \le k \le 3$. |

TABLE 1. Dynkin diagrams




Dual to $U_q(\mathfrak{g})$ is the Hopf *-algebra $\mathcal{O}_q(G)$, which admits a C*-completion, $C_q(G)$. When $q = 1$, we get $C(G)$, the continuous functions on $G$ for $G$ the simply connected compact semisimple Lie group with Lie algebra $\mathfrak{g}$. While $C_q(G)$ is not a Hopf algebra, the coproduct extends to $\Delta : C_q(G) \to C_q(G) \otimes_{\min} C_q(G)$, giving a compact quantum group in the sense of Woronowicz. A quantum flag manifold of $G$ is a C*-subalgebra of $C_q(G)$ constructed in a canonical way from a subalgebra $\mathfrak{l}_S \subset \mathfrak{g}$ given by a subset $S$ of nodes on the relevant Dynkin diagram. Its C*-algebra is denoted $C_q(G/L_S)$. For example, quantum projective space $C_q(\mathbb{C}P^n)$ and the full quantum flag manifold $C_q(SU_3/\mathbb{T}^2)$ correspond to the diagrams in Figure 1.

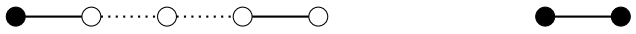


FIGURE 1. Dynkin diagrams of $C_q(\mathbb{C}P^n)$ (left) and $C_q(SU_3/\mathbb{T}^2)$ (right).

For every $1 \leq i \leq \operatorname{rank}(\mathfrak{g})$, there is a map from $U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{g})$ defined by $K \mapsto K_i$, $E \mapsto E_i$ and $F \mapsto F_i$. This induces a map $\sigma_i : C_q(G) \to C_q(SU_2)$. The elementary $^*$-representation $\pi_{s_i}$, for $s_i$ a generator of $W$, is given by

$$\pi_{s_i} := \rho \circ \sigma_i : C_q(G) \to \mathcal{B}(L^2(\mathbb{Z}_+)),$$

where $\rho : C_q(SU_2) \to \mathcal{B}(L^2(\mathbb{Z}_+)$ is the $^*$-representation constructed by Woronowicz, given on the generators $\alpha, \gamma$ of $C_q(SU_2)$ by

$$\rho(\alpha)(e_n) = (1 - q^{2n})^{1/2} e_{n-1}, \qquad \rho(\gamma)(e_n) = -q^n e_n.$$

For $C_q(G/L_S)$, let $W_S \subset W$ be the subgroup generated by $\{s_i \mid i \in S\}$, and let $W^S$ be the set of $W/W_S$ coset representatives of minimal length. For $w = s_{i_1} \cdots s_{i_k} \in W^S$, define $\pi_w := \pi_{s_{i_1}} \otimes \cdots \otimes \pi_{s_{i_k}} \circ \Delta^{k-1}$. In [3], it was shown that these representations are irreducible, do not depend on the choice of reduced word, and give a complete set of irreducible $^*$-representations. Thus $|\operatorname{Prim}(C_q(G/L_S))| < \infty$, allowing us to apply results in [1]:

**Theorem 2** (Eilers, Sørensen, Ruiz)**.** *Let $A$ be a $\mathrm{C}^*$-algebra with $\operatorname{Prim}(A)$ finite. Suppose that for each $x \in \operatorname{Prim}(A)$, the subquotient $A[x]$ is stably isomorphic to $\mathcal{K}$, and if $A[x]$ is unital, then $A[x] \cong \mathbb{C}$. Then there exists an amplified graph $E$ such that $A \cong C^*(E)$. Moreover, if $E$ is an amplified graph with finitely many vertices, then $A \cong \mathrm{C}^*(E)$ if and only if $\operatorname{Prim}^\tau(A) \cong \operatorname{Prim}^\tau(\mathrm{C}^*(E))$.*

Here, $A[x]$ is defined as follows. For an open subset $U \subset \operatorname{Prim}(A)$, set $A[U] := \bigcap_{p \in \operatorname{Prim}(A) \setminus U} p$. Let $U, V$ be open sets with $V \subset U$ and $\{x\} = U \setminus V$. Set $A[x] := A[U]/A[V]$.

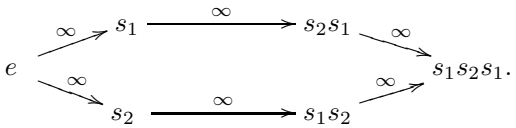
The *length* $\ell(w)$ of $w \in W$ is the number of generators appearing in any reduced form of $w$. If $u, v \in W$, $\ell(u) \geq 1$, satisfy $v = uw$ is in reduced form, write $w < v$.

**Proposition 3.** *Let $m$ be the length of the longest element in $W^S$. For $w \in W^S$ with length $\ell(w) < m$, for $U_w := \{\pi_v \in \operatorname{Prim}(C_q(G/L_S)) \mid v > w\}$ and $V_w := \{\pi_v \in \operatorname{Prim}(C_q(G/L_S)) \mid v < w\}$, we have $\{\pi_w\} = U_w \setminus V_w$, and $A[\pi_w] = A[U_w]/A[V_w] \cong \mathcal{K}$.*

This gives us Theorem 1. We can also construct a graph $E_S$ with the correct ideal structure so that $C_q(G/L_S) \cong \mathrm{C}^*(E_S)$. The vertices of $E_S$ are given by elements in $W^S$, and there are infinitely many arrows from $v \in E_S^0$ to $w \in E_S^0$, whenever $w = s_i v$ and $\ell(w) > \ell(v)$ for some generator $s_i$. Consider two examples based on the diagrams in Figure 1. For $C_q(\mathbb{C}P^n)$ we have $W^S = \{e, s_1, s_2 s_1, s_3 s_2 s_1, \ldots, s_n s_{n-1} \cdots s_1\}$, and the resulting graph is

$$e \xrightarrow{\infty} s_1 \xrightarrow{\infty} s_2 s_1 \xrightarrow{\infty} \cdots\cdots \xrightarrow{\infty} s_n \cdots s_2 s_1 \ .$$

For the full quantum flag manifold of $SU_3$, $C_q(SU_3/\mathbb{T}^2)$ we have $W^S = S_3 = \{e, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1\}$, giving




From the graph picture, we have $C_{q_1}(G/L_s) \cong C_{q_2}(G/L_S)$ for any $q_1, q_2 \in (0,1)$. We also find previously unknown isomorphisms. For example, label the nodes of $B_n$ and $C_n$ (left to right) as $1, \ldots, n$. Let $S \subset \{1, \ldots, n\}$. The Weyl groups of $B_n$ and $C_n$ are the same, so $C_q(SO_{2n+1}/L_S) \cong C_q(Sp_{2n}/L_S)$.

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## Groupoids as coordinate systems for algebras and operator algebras


Aidan Sims

(joint work with many authors — see below)


The idea of coordinatising operator algebras using groupoids goes back to Feldman-Moore in the 1970's [8, 9, 10]. A groupoid is like a group, except that the multiplication is only partially defined. For example, any equivalence relation $R \subseteq X \times X$, admits an associative partially-defined multiplication where the product $(w,x)(y,z)$ is defined if and only if $x = y$, in which case it is equal to $(w,z)$. Given such an equivalence relation endowed with a compatible Borel structure and a circle-valued Borel 2-cocycle, the space of measureable functions on $R$ is a von Neumann algebra under a twisted convolution formula, and contains the measurable functions on $X$ as a maximal abelian subalgebra. Feldman and Moore established a converse: they identified key properties of the abelian subalgebra that arises this way and proved that given any von Neumann algebra and any abelian subalgebra with these properties, there exist a Borel equivalence relation and Borel 2-cocycle so that the original von Neumann algebra and subalgebra are identical to those arising from the equivalence relation and cocycle. They called such subalgebras *Cartan subalgebras*. So, to paraphrase their theorem, Cartan subalgebras coordinatise von Neumann algebras.

In the $C^*$-algebraic setting some adjustments are needed, but the general idea still goes through. The Borel equivalence relation is replaced by an étale Hausdorff groupoid that is *effective* in the sense that it may have some nontrivial isotropy (elements that can be composed with themselves), but the interior of the isotropy must consist only of units. And the Borel 2-cocycle is replaced by a Fell line bundle over that groupoid. But once again, work of Renault [15, 16] and Kumjian [12] identified the correct notion of a Cartan subalgebra of a $C^*$-algebra so that

the convolution-algebra construction for Fell line bundles over appropriate étale Hausdorff groupoids becomes a bijection between isomorphism classes of Fell line bundles and isomorphism classes of Cartan inclusions of $C^*$-algebras.

A spectacular application of this theory came through the work of Matsumoto and Matui [13]. The Cuntz–Kriger algebras [7] are $C^*$-algebras that encode irreducible shifts of finite type, which are classified up to flow equivalence by their Bowen–Franks invariant [3]. In their seminal paper [7] Cuntz and Krieger showed that the $K$-theory of a Cuntz–Krieger algebra recovers the Bowen–Franks group of the shift space that it encodes, but this is not the whole invariant, and it was an open question whether isomorphism of Cuntz–Krieger algebras is equivalent to flow equivalence of their shifts of finite type until Rørdam answered the question in the negative by proving that $\mathcal{O}_2$ and $\mathcal{O}_{2,-}$ are isomorphic. Matsumoto and Matui's insight in [13] was that the Cartan subalgebra in a Cuntz–Krieger algebra is the key additional data required to recover the whole Bowen–Franks invariant. That is, two shift spaces are flow equivalent if and only if there is a *Cartan-preserving* isomorphism of the associated Cuntz–Krieger algebras.

This raises two very natural questions. Firstly the irreducible shifts of finite type appearing in Matui and Matsumoto's theorem are a very special class of dynamics, but the theory of Cartan subalgebras treats much more general dynamics, and results like the work of Giordano–Putnam–Skau on Cantor minimal systems [11], and the Boyle–Tomiyama theorem [4] both suggest that characterisations of continuous orbit equivalence of dynamical systems in terms of isomorphisms of $C^*$-algebras preserving suitable abelian subalgebras is feasible in significant generality. Secondly, the parallel theories of graph $C^*$-algebras [14] and Leavitt path algebras [1] first studied in the early 2000's demonstrated very interesting parallels between abstract algebras and $C^*$-algebras that seem to be best explained by the availability of a common groupoid model [5]. These parallels can be extended far beyond the classes of algebras in which they were first observed, and the natural question is to understand the limits of these parallels.

To answer each of these questions, one first needs to extend elements of Renault's theory to groupoids that are not effective, and also to develop a notion of Kumjian–Renault theory that applies to abstract algebras.

In this talk I will outline the construtions of Feldman–Moore [8] and of Kumjian–Renault [12, 15]. I will then outline recent results that partially extend Renault's theory to $C^*$-algebras of non-essential groupoids and that extend the full force of their theory to abstract algebras over very general totally disconnected groupoids:

**Theorem 1.** (Carlsen, Ruiz, S., Tomforde [6]) Let $G$ be a second-countable locally compact Hausdorff étale groupoid in which the interior of the isotropy consists of torsion-free abelian groups. Then $G$ can be reconstructed from $(C^*(G), C_0(G^{(0)}))$; in particular, two such groupoids are isomorphic if and only if there is an isomorphism of their $C^*$-algebras that preserves the canonical abelian subalgebras.

**Theorem 2.** (Armstrong, de Castro, Clark, Courtney, Lin, McCormick, Ramagge, S., Steinberg [2]) Let $R$ be a commutative ring (with identity) whose only

idempotent elements are $0, 1$. The twisted convolution algebra construction gives a bijection between the following:

- pairs $(B, G)$ where $G$ is a totally disconnected locally compact Hausdorff étale groupoid, and $B$ is a discrete $R$-twist over $G$ such that for a dense set of units $x$ of $G$, the twisted group ring of the reduction of $B$ to a fibre of the interior of the isotropy over $x$ has no nontrivial units; and
- quasi-Cartan pairs of $R$-algebras: $R$-algebras $A$ with commutative subalgebras $B$ generated by their idempotents such that $te = 0 \implies t = 0$ whenever $t \in R$ and $e \in B$ is a nonzero idempotent, such that $A$ is generated by normalisers of $B$ and there is a faithful conditional expectation from $A$ onto $B$ that is generated by projections.

I will also discuss applications, such as a generalisation of the Boyle–Tomiyama theorem to arbitrary actions: two integer actions on compact Hausdorff spaces decompose into conjugate and flip-conjugate components if and only if there is a diagonal-preserving isomorphism of the associated crossed-product $C^*$-algebras.

Some interesting open questions emerge:

- What $C^*$-algebraic properties characterise the pairs of $C^*$-algebras that appear in Theorem 1?
- Is is possible to obtain a version of Theorem 1 for twisted $C^*$-algebras over the same class of groupoids? Is there then a complete generalisation of Kumjian–Renault theory to this setting.
- Theorem 1 uses that for torsion-free abelian groups, the quotient of the unitary group of the group $C^*$-algebra by the connected component of the identity is isomorphic to the original group. But, more generally, as Stefaan Vaes pointed out during the workshop, no two non-isomorphic torsion-free groups are known to have isomorphic $C^*$-algebras. Is it possible to use this to weaken the hypothesis on the isotropy groups appearing in Theorem 1?

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## Wreath-like product groups and rigidity of their von Neumann algebras

Adrian Ioana
(joint work with Ionut Chifan, Denis Osin, Bin Sun)

The von Neumann algebra $\mathrm{L}(G)$ of a countable discrete group $G$ is defined as the weak operator closure of the complex group algebra $\mathbb{C}G$ acting on $\ell^2 G$ by left convolution. This algebra is a $\mathrm{II}_1$ factor exactly when $G$ is ICC, i.e., the conjugacy class of every nontrivial element of $G$ is infinite.

It is a central problem in operator algebras to classify group $\mathrm{II}_1$ factors $\mathrm{L}(G)$ in terms of the group $G$. The problem goes back to the seminal work of Murray and von Neumann [MvN43] who proved that there is a unique hyperfinite $\mathrm{II}_1$ factor, $R$, and deduced that $\mathrm{L}(G) \cong R$, for every ICC locally finite group $G$. On the other hand, they showed that $\mathrm{L}(\mathbb{F}_2) \ncong R$. In the 1970s, Connes remarkably proved that $\mathrm{L}(G) \cong R$, for every ICC amenable group $G$ [Con76]. The case of nonamenable groups, however, is significantly more challenging and was largely intractable for a long time. This has changed dramatically in the last 15 years following Popa's discovery of deformation/rigidity theory. Popa's theory has led to strong classification results for large families of nonamenable group $\mathrm{II}_1$ factors, see [Pop07a, Vae10, Ioa18]. For instance, [Pop06] showed that if $G, H$ are ICC groups with Kazhdan's property (T), then $\mathrm{L}(\mathbb{Z} \wr G) \cong \mathrm{L}(\mathbb{Z} \wr H)$ implies that $G \cong H$.

Despite this impressive progress, a far-reaching rigidity conjecture posed by Connes in 1980 [Con82] remained wide open. This conjecture predicts that for ICC groups $G, H$ with property (T), $\mathrm{L}(G) \cong \mathrm{L}(H)$ implies that $G \cong H$. Positive evidence is provided by a result of Cowling and Haagerup showing that if $m \neq n$, then any ICC lattices $G < \mathrm{Sp}(1, m), H < \mathrm{Sp}(1, n)$ (which have property (T)) give rise to nonisomorphic $\mathrm{II}_1$ factors [CH89]. By the main result of [CJ85], if $\mathrm{L}(G) \cong \mathrm{L}(H)$ and $G$ has property (T), then $H$ also has property (T). Connes' rigidity conjecture is therefore equivalent to asking whether every ICC property (T) group $G$ is W$^*$-superrigid, in the sense that any group $H$ with $\mathrm{L}(G) \cong \mathrm{L}(H)$ must be isomorphic to $G$. The first W$^*$-superrigid groups were found in 2010 by Ioana,

Popa and Vaes [IPV13]. This paper gave a large class of generalized wreath product groups of the form $G = \mathbb{Z}/2\mathbb{Z} \wr_I \Gamma$ which are W*-superrigid. Subsequently, several additional families of W*-superrigid groups were found, including in [BV14, CI18]. However, none of these groups have property (T), leaving open the problem of finding even a single example of a property (T) group which is W*-superrigid.

In recent joint work with Chifan, Osin and Sun [CIOS21], we solved this problem by providing a natural family of W*-superrigid groups with property (T):

**Theorem 1.** *Let $H$ be a torsion-free hyperbolic property (T) group and $h \in H \setminus \{e\}$.*

*Then for every $k \in \mathbb{N}$ sufficiently large, denoting $N = << h^k >>$, the quotient group $G := H/[N,N]$ is ICC, W*-superrigid and has property (T).* Here, $[N,N]$ denotes the commutator subgroup of the normal subgroup $N \lhd H$ generated by $h^k$.

The proof of Theorem 1 relies on the fact that $G$ has a wreath-like product structure in the following sense:

**Definition.** *We say that a group $G$ is a* wreath-like product *of groups $A$ and $B$ corresponding to an action of $B$ on a set $I$ if it is an extension of the form $\{e\} \longrightarrow \bigoplus_{i\in I} A_i \longrightarrow G \xrightarrow{\varepsilon} B \longrightarrow \{e\}$, where $A_i \cong A$ and the conjugation action of $G$ on $\bigoplus_{i\in I} A_i$ satisfies $gA_ig^{-1} = A_{\varepsilon(g)\cdot i}$ for all $g \in G$ and $i \in I$.*

The notion of a wreath-like product generalizes the ordinary (restricted) wreath product of groups. Conversely, if $G$ is a wreath-like product of $A$ and $B$, then $G \cong A \wr B$ whenever the extension provided by the above definition splits. The following theorem combines works of Dahmani, Guirardel and Osin [DGO17] and Sun [Sun20]: **Theorem 2.** *Let $H$ be a torsion-free hyperbolic group and $h \in H \setminus \{e\}$.*

*Then for any sufficiently large $k \in \mathbb{N}$, denoting $N = << h^k >>$, the group $H/N$ is hyperbolic* ICC*, and the group $G := H/[N,N]$ is a wreath-like product of $\mathbb{Z}$ with $H/N$ corresponding to a transitive action of $H/N$ with finite cyclic stabilizers.* Theorem 1 now follows from Theorem 2 and the following main result of [CIOS21]:

**Theorem 3.** *Let $A$ be a nontrivial abelian group, $B$ a nontrivial ICC subgroup of a hyperbolic group, and assume that $B$ acts on a set $I$ with amenable stabilizers. Let $G$ be a wreath-like product of $A$ and $B$ corresponding to the action of $B$ on $I$.*

*If $G$ has property (T), then it is W*-superrigid.* The proof of Theorem 3 is based on a deformation/rigidity strategy which plays property (T) against two properties ((i) and (ii) below) of wreath-like product groups that relate them to wreath product groups. Denote $\mathcal{M} = \mathrm{L}(G)$ and let $H$ be any other group such that $\mathcal{M} = \mathrm{L}(H)$. Let $(u_g)_{g\in G}$ and $(v_h)_{h\in H}$ be the canonical unitaries generating $\mathcal{M}$. For simplicity, assume that $G$ is a regular regular wreath-like product of $A$ and $B$. Denoting $\mathcal{P} = \mathrm{L}(A^{(B)})$, we have:

(i) The action $\sigma = (\mathrm{Ad}(u_g))_{g\in G}$ of $G$ on $\mathcal{P}$ is a generalized Bernoulli action.
(ii) $\mathcal{P} \subset \mathcal{M}$ is a Cartan subalgebra and $\mathcal{R}(\mathcal{P} \subset \mathcal{M})$ is the orbit equivalence relation of the Bernoulli action of $B$ on $\widehat{A}^B$, where $\widehat{A}$ is the dual of $A$.

Part (ii) implies the following "transfer principle". Let $\mathcal{N} = \mathrm{L}(A \wr B)$. If $\mathcal{P} \subset \mathcal{D} \subset \mathcal{M}$ is a subalgebra, then there is subalgebra $\mathcal{P} \subset \widetilde{\mathcal{D}} \subset \mathcal{N}$ such that the inclusions $\mathcal{P} \subset \mathcal{D}$ and $\mathcal{P} \subset \widetilde{\mathcal{D}}$ have isomorphic equivalence relations. In particular, $\widetilde{\mathcal{D}}$ is amenable iff $\mathcal{D}$ is amenable. This principle, which is a main novelty of our work, allows to go back and forth between (subalgebras of) $\mathcal{M}$ and $\mathcal{N}$.

Define the $*$-homomorphism $\Delta : \mathcal{M} \to \mathcal{M}\overline{\otimes}\mathcal{M}$ by letting $\Delta(v_h) = v_h \otimes v_h$, $h \in H$ [PV10]. The proof of Theorem 3 splits into three main parts.

In the first part of the proof, following [Ioa11, IPV13], we analyze $\Delta$ and show that $\mathcal{D} := \Delta(\mathcal{P})' \cap \mathcal{M}\overline{\otimes}\mathcal{M}$ is essentially unitarily conjugated to $\mathcal{P}\overline{\otimes}\mathcal{P}$. Since $\Delta(\mathcal{P})$ is amenable and has large normalizer, Popa and Vaes' structure theorem [PV14] allows us to essentially show that $\Delta(\mathcal{P}) \subset \mathcal{P}\overline{\otimes}\mathcal{P}$, after unitary conjugacy. Next, as in [BV14], we use solidity results for generalized Bernoulli crossed products. Thus, applying our transfer principle to $\mathcal{D}$ and extending the solidity theorem of [CI10], we derive that $\mathcal{D}$ is amenable. Another application of [PV14] implies that $\mathcal{D}$ is essentially unitarily conjugated to $\mathcal{P}\overline{\otimes}\mathcal{P}$.

The second part of the proof is a "discretization argument". By the first part, after unitary conjugacy, we may assume that $\Delta(\mathcal{P})' \cap \mathcal{M}\overline{\otimes}\mathcal{M} = \mathcal{P}\overline{\otimes}\mathcal{P}$. Hence, the group $\Delta(G) = (\Delta(u_g))_{g\in G}$ normalizes $\mathcal{P}\overline{\otimes}\mathcal{P}$. Moreover, the conjugation action of $\Delta(G)$ on $\mathcal{P}\overline{\otimes}\mathcal{P}$ descends to a free action of $\Delta(B)$. A second application of our transfer principle gives a free action of $\Delta(B) \cong B$ on $\widehat{A}^B \times \widehat{A}^B$ whose OE relation is contained in that of the product action of $B \times B$ on $\widehat{A}^B \times \widehat{A}^B$. As $B$ has property (T), generalizing a theorem from [Pop06] enables us to assume that $\Delta(B) \subset B \times B$, as groups of automorphisms of $\widehat{A}^B \times \widehat{A}^B$. This implies $\Delta(G)$ "discretizes" modulo $\mathcal{U}(\mathcal{P}\overline{\otimes}\mathcal{P})$: there are maps $\delta_1, \delta_2 : G \to G$ and $\omega : G \to \mathcal{U}(\mathcal{P}\overline{\otimes}\mathcal{P})$ such that

$$\Delta(u_g) = \omega_g(u_{\delta_1(g)} \otimes u_{\delta_2(g)}), \text{ for every } g \in G.$$

In the last part of the proof, we use the symmetry and associativity properties of $\Delta$ to show that we may take $\delta_1 = \delta_2 = \mathrm{Id}_G$. In other words,

$$\Delta(u_g) = \omega_g(u_g \otimes u_g), \text{ for every } g \in G.$$

Up to this point, we have only used that $B$, but not $G$, has property (T). Another main novelty of our work is the way we use property (T) for $G$. We start by noticing that as $G$ has property (T) and $\sigma$ is a generalized Bernoulli action (i), Popa's cocycle superrgidity theorem [Pop07b] implies that any 1-cocycle for $\sigma \otimes \sigma$ is cohomologous to a character of $G$. Since $(\omega_g)_{g\in G}$ is a 1-cocycle for $\sigma \otimes \sigma$, we can thus find a unitary $w \in \mathcal{P}\overline{\otimes}\mathcal{P}$ and a character $\rho : G \to \mathbb{T}$ such that $w\Delta(u_g)w^* = \rho(g)(u_g \otimes u_g)$, for every $g \in G$. But then a general result from [IPV13] implies that $G \cong H$, and the conclusion of Theorem 3 follows.

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## Tracial amenability and purely infinite $C^*$-algebras

Shirly Geffen
(joint work with Eusebio Gardella, Julian Kranz, Petr Naryshkin, Andrea Vaccaro)

One of the most remarkable achievements in $C^*$-algebra theory in the last decade is the completion of the *classification program* (see [8] for a survey). The main theorem identifies the optimal subclass of unital, simple, separable, nuclear $C^*$-algebras which can be classified, up to an isomorphism, by their $K$-theoretical and tracial data: those $C^*$-algebras which satisfy the Universal Coefficient Theorem (UCT) and are stable under tensoring with the Jiang-Su algebra $\mathcal{Z}$.

The question motivating this talk is the preservation of classifiability under formation of crossed products by nonamenable groups. More concretely, we ask the following question.

**Question.** Let $G$ be a nonamenable countable discrete group acting on a classifiable $C^*$-algebra $A$. When is $A \rtimes G$ again classifiable?

For unital commutative $C^*$-algebras of the form $C(X)$, for some compact Hausdorff space $X$, this question was recently considered in [2], where it is shown that for a large class of nonamenable groups, any minimal, amenable, topologically free action $G \curvearrowright X$ gives rise to a classifiable crossed product $C^*$-algebra $C(X) \rtimes G$. Using similar methods, we obtain an analogous result for classifiable $C^*$-algebras:

**Theorem.** Let $G$ be a nonamenable countable discrete group containing the free group $F_2$, let $A$ be a classifiable $C^*$-algebra, and let $G \curvearrowright A$ be an amenable outer action. Then, modulo the UCT, $A \rtimes G$ is classifiable.

The requirement that the action is *amenable* (see for example [1]) is necessary in order to obtain nuclearity of $A \rtimes G$. Moreover, *outerness* of the action is the canonical condition to put in order to guarantee simplicity of $A \rtimes G$ (see [3]). Therefore, what remained for us to prove was $\mathcal{Z}$-stability of $A \rtimes G$. In fact, under the above conditions on the system $(A, G)$, we prove that the crossed product $C^*$-algebra must be purely infinite (thus, $\mathcal{Z}$-stable), and therefore classifiable by its $K$-theory, using the Kirchberg-Phillips theorem from the 1990s [5].

We believe that the above theorem should hold in general for nonamenable groups, and not only for those containing a nonabelian free group. However, our methods use the paradoxical structure of $F_2$ (which is stronger than the paradoxical structure of general nonamenable groups obtained by Tarski).

Finally, we turn to examples of systems $(A, G)$ which satisfy the conditions mentioned in the theorem. Many examples were obtained by Ozawa-Suzuki in [4], when the underlying $C^*$-algebra $A$ is purely infinite (but, in this case, our theorem already followed using [3]). It turns out that when $A$ is stably finite, there are currently no examples. Let us state it as an open problem.

**Problem.** Are there any amenable actions of nonamenable groups on unital, simple, stably finite $C^*$-algebras?

This question has a positive answer when one relaxes the conditions on $A$. When asking the center of $A$ to be trivial, instead of simplicity, examples were provided in [6]. Moreover, non-unital examples were constructed in [7].

We define a broader notion of amenability, called *tracial amenability*, which allows us to obtain analogous results, and provide an abundance of examples. When the boundary of extreme traces $\partial_e T(A)$ is compact, tracial amenability of an actions $G \curvearrowright A$ is equivalent to amenability of the induced topological action $G \curvearrowright \partial_e T(A)$. Considering this concept, we obtain the following theorem.

**Theorem.** Let $G$ be a nonamenable countable discrete group containing the free group $F_2$, let $A$ be a classifiable $C^*$-algebra, and let $G \curvearrowright A$ be a tracially amenable outer action. Then, $A \rtimes G$ is a unital, simple, purely infinite $C^*$-algebra.

Note however that we get farther from classifiability (tracially amenable actions which are not amenable, give rise to non nuclear crossed products).

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## A substitute for Kazhdan's property (T) for universal non-lattices

Narutaka Ozawa

Kazhdan's property (T) for groups (which are discrete and finitely generated in this talk) is an important notion with a wide range of applications. A group $\Gamma$ has property (T) if in any unitary representation of $\Gamma$, any almost invariant vector is close to an invariant vector. This generalizes the rigidity aspect of being finite. The other important generalization of finiteness is amenability. These two notions generalize finiteness to perpendicular directions and only groups that have property (T) and amenability are finite groups. The most prominent examples of property (T) groups are the higher rank lattices, $\mathrm{SL}(n,\mathbb{Z})$, $n \geq 3$ [Kazhdan

1967]. This is generalized to the "universal lattice" $\mathrm{EL}(n,\mathcal{R})$, $n \geq 3$, for finitely generated ring $\mathcal{R}$ [Shalom + Vaserstein, Ershov–Jaikin-Zapirain 2006–08]. Here

$$\mathrm{EL}(n,\mathcal{R}) = \langle e_{ij}(r) : i \neq j,\, r \in \mathcal{R} \rangle$$

is the group generated by elementary matrices; $e_{ij}(r)$ is the matrix with 1's on the diagonal, $r$ on the $(i,j)$-th position, and zeros everywhere else. Another notable example of a property (T) group is $\mathrm{Aut}(F_n)$, $n \geq 4$ [Kaluba–Nowak–Ozawa ($n = 5$), Kaluba–Kielak–Nowak ($n \geq 6$), Nitsche ($n = 4$) 2017–20]. The proof heavily relied on computer calculations.

Let $\Gamma$ be a finitely generated group and $S$ be a finite generating subset. The corresponding Laplacian is $\Delta_S := \sum_{s\in S}(1-s)^*(1-s)$ in the full group C*-algebra $\mathrm{C}^*[\Gamma]$. The group $\Gamma$ has property (T) iff $\Delta_S$ has spectral gap, i.e., there is $\epsilon > 0$ such that $\mathrm{Spec}(\Delta_S) \subset \{0\} \cup [\epsilon, \infty)$. The latter is true iff $\Delta_S^2 - \epsilon\Delta_S \geq 0$ in $\mathrm{C}^*[\Gamma]$. Let's write the property (T) constant of $(\Gamma, S)$ by $\lambda_1(\Delta_S) := \inf(\mathrm{Spec}(\Delta_S) - \{0\})$. It is natural to ask: For which $\Gamma$, are the property (T) constants uniform over generating subset (of cardinality $d$), namely

$$\inf\{\lambda_1(\Delta_S) : S \text{ finite generating (with } |S| \leq d)\} > 0\,?$$

Usually this value is 0 [Gelander–Żuk, Osin 2002], but sometimes $> 0$ [Osin–Sonkin 2007]. It is not known for $\Gamma = \mathrm{SL}(n,\mathbb{Z})$ and, say, even for $S_{p,q} = \{e_{ij}(p), e_{ij}(q) : i \neq j\}$ where $(p,q)$ varies over coprime pairs. The "parent" of $(\mathrm{SL}(n,\mathbb{Z}), S_{p,q})_{p,q}$ is $\mathrm{EL}_n(\mathbb{Z}\langle x,y\rangle)$, where $\mathbb{Z}\langle x,y\rangle$ is the "*rng*" (a ring but without the identity) of polynomials with zero constant terms. Had $\mathrm{EL}_n(\mathbb{Z}\langle x,y\rangle)$ had property (T), it would follow that the property (T) constants for $(\mathrm{SL}(n,\mathbb{Z}), S_{p,q})_{p,q}$ are uniformly away from zero. However for the finitely generated rng $\mathcal{R} = \mathbb{Z}\langle x,y\rangle$ the group $\mathrm{EL}_n(\mathcal{R})$ fails property (T) as it has infinite nilpotent (and hence amenable) quotients $\mathrm{EL}_n(\mathcal{R}/\mathcal{R}^k)$. Thus we need to bridge between property (T) and nilpotency. Actually, both properties have many in common: Shalom's property $\mathrm{H_T}$ generalizes (T) and nilpotency; nilpotent groups tend to admit rather precise description of almost invariant vectors.

**Theorem** ([1])**.** *For every $d$, if $n$ is large enough, there is $\epsilon > 0$ such that $\mathrm{EL}_n(\mathbb{Z}\langle t_1,\dots,t_d\rangle)$ satisfies the following. For*

$$\Delta := \sum_r \sum_{i\neq j} (1 - e_{ij}(t_r))^*(1 - e_{ij}(t_r))$$

*and*

$$\Delta^{(2)} := \sum_{r,s} \sum_{i\neq j} (1 - e_{ij}(t_r t_s))^*(1 - e_{ij}(t_r t_s))$$

*one has*

$$\Delta^2 \geq \epsilon\Delta^{(2)}$$

*in* $\mathrm{C}^*[\mathrm{EL}_n(\mathbb{Z}\langle t_1,\dots,t_d\rangle)]$.

This means that for any finitely generated rng $\mathcal{R}$ and any unitary representation of $\mathrm{EL}_n(\mathcal{R})$, any $\delta$-almost invariant vector is close to a vector that is $C\delta^2$-almost invariant for $\mathrm{EL}_n(\mathcal{R}^2)$. By iterating this, one obtains the following corollary.

**Corollary.** *For $n$ large enough, the group $\mathrm{EL_n}(\mathbb{Z}\langle t_1,\dots,t_d\rangle)$ has property* (T) *w.r.t. the quotients of the form $\mathrm{EL_n}(\mathcal{S})$, where $\mathcal{S}$ is a finite ring (i.e., unital) quotient of $\mathbb{Z}\langle t_1,\dots,t_d\rangle$.*

Corollary implies that the property (T) constants for $(\mathrm{SL_n}(\mathbb{Z}/m\mathbb{Z}), S_{p,q})_{m,p,q}$ or $(\mathrm{SL_n}(\mathbb{Z}/q\mathbb{Z}), \{e_{ij}(p) : i \neq j\})_{p,q}$ are uniformly away from zero, that is to say, they form an expander family.

In the talk, I explained three things: (1) How the computer based proof [Kaluba–Kielak–Nowak] of property (T) for $\mathrm{Aut}(F_n)$ lead to a hypothetical inequality (which is so ad hoc and no human would have come up with without the computer's assistance) in the full group C*-algebra that proves Theorem. (2) The inequality necessarily involves the full group C*-algebra $\mathrm{C^*}[\Gamma]$, as opposed to the real (or complex) group algebra $\mathbb{R}[\Gamma]$ that is enough for the property (T), and hence Theorem cannot be proved by computer calculation (by the known methods). (3) The detailed analysis on the (irrational) rotation C*-algebras $\mathcal{A}_\theta$ [Boca–Zaharescu 2005] can be used to prove various inequalities in the full group C*-algebra $\mathrm{C^*}[\mathbf{H}]$ of the integral Heisenberg group $\mathbf{H}$ which can be assembled to a desired inequality in $\mathrm{C^*}[\mathrm{EL_n}(\mathbb{Z}\langle t_1,\dots,t_d\rangle)]$.

I suggested a stronger version of Theorem should hold true: (1) $n$ should not depend on $d$. (2) In fact, $n = 3$ should suffice. (3) Noncommutative rngs instead of the commutative rng $\mathrm{EL_n}(\mathbb{Z}\langle t_1,\dots,t_d\rangle)$.

See the paper [1] for details.

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## MIP*=RE: what it is and further directions for operator algebraists

William Slofstra

Connes' embedding problem (CEP) asks whether every separable finite von Neumann algebra embeds in $R^\Omega$. By well-known work of Kirchberg, it's also equivalent to the question of whether $C^*\mathbb{F}_n \otimes_{max} C^*\mathbb{F}_n = C^*\mathbb{F}_n \otimes_{min} C^*\mathbb{F}_n$, where $\mathbb{F}_n$ is the free group on $n$ generators, and $\otimes_{max}$ and $\otimes_{min}$ are the max and min $C^*$-tensor products respectively. In this equivalence, the group $\mathbb{F}_n$ can be replaced with other groups, such as $\mathbb{Z}_m^{*n}$, the $n$-fold free product of $\mathbb{Z}_m$. The relatively recent MIP$*$ = RE result of Ji, Natarajan, Vidick, Wright, and Yuen in quantum computational complexity theory resolves this problem (in the negative). In this abstract, we explain what the result is and the current status (including further directions) for operator algebraists.

To explain what the result is, it's helpful to know some terminology from computer science. An alphabet is a finite set $\Sigma$, and a language is a finite subset of $\Sigma^*$, the set of finite strings over $\Sigma$. Often we take $\Sigma = \{0, 1\}$, so languages are subsets of binary strings. A class is a set of languages. An example is the set of decidable languages, which are the languages $\mathcal{Z}$ for which there is a Turing machine $M$

such that for any $\alpha \in \Sigma^*$, if $\alpha \in \mathcal{Z}$, then $M(\alpha)$ (the Turing machine on input $\alpha$) accepts, and if $\alpha \notin \mathcal{Z}$ then $M(\alpha)$ rejects. The class P is defined exactly the same as the class of decidable languages, but now $M(\alpha)$ must halt in polynomial time in $|\alpha|$, the length of $\alpha$.

For proof systems, we change these definitions a little bit. RE is defined to be the set of languages $\mathcal{Z}$ for which there is a Turing machine $M$ such that (1) if $\alpha \in \mathcal{Z}$, then there is $\beta \in \Sigma^*$ such that $M(\alpha, \beta)$ accepts, and (2) if $\alpha \notin \mathcal{Z}$, then $M(\alpha, \beta)$ rejects for all $\beta$. The Turing machine $M$ is called a verifier for $\mathcal{Z}$. The idea is that a language is in RE if there is a way to convince the verifier to accept it. The language HALT of Turing machines which halt on empty input is a prototypical example of a language is RE which is not decidable. The class NP is defined similarly to RE, but $M(\alpha, \beta)$ must halt in polynomial time in $\alpha$. The class PCP is defined similarly to NP, but now $M$ can be a probabilistic Turing machine (in other words, a Turing machine which can flip a fair coin and branch on the outcome). Conditions (1) and (2) are also changed for this class: in condition (1), $M(\alpha, \beta)$ is required to accept with probability 1, and in condition (2), the probability that $M(\alpha, \beta)$ accepts should be bounded by $1/2$ (see also the closely related class MA of Merlin-Arthur proof systems). The PCP theorem of Arora, Lund, Motwani, Sudan, and Szedegy states that $\mathrm{NP} = \mathrm{PCP}(\log |\alpha|, O(1))$, meaning that every language in NP has a probabilistic verifier which tosses no more than $\log |\alpha|$ coins, and reads only a constant number of bits of the proof.

We are now closing in the definition of MIP$*$. An interactive proof system is defined exactly the same as a PCP, but instead of reading bits of a proof, the verifier has to ask a prover, who has knowledge of the earlier questions, for the $i$th bit. The class MIP is defined to be the set of languages with a (polynomial-time) multi-prover interactive proof system, meaning we allow the verifier to ask questions of multiple provers who are unable to communicate with each other. Allowing the provers to have adapt their answers to the verifier's questions interactively seems to change the power of these proof systems quite a bit: the class IP of interactive proofs is equal to PSPACE by a result of Shamir, and MIP = NEXP, the class of languages with an exponential size proof, by a result of Babai, Fortnow, and Lund.

In this last result, the verifier only needs to communicate with two provers, and only requires one round of communication. In such a protocol, the verifier sends questions $x$ and $y$ (drawn according to some distribution $\pi_\alpha$ on $X_\alpha \times X_\alpha$, where $X_\alpha$ is the finite set of possible questions), and receives answers $a$ and $b$ respectively drawn from some finite set of possible answers. The verifier then decides whether to accept or reject. The verifier's actions during this protocol are described by the tuple $G_\alpha = (X_\alpha, \pi_\alpha, A_\alpha, V_\alpha)$, where $V_\alpha$ is a function $V : A_\alpha \times A_\alpha \times X_\alpha \times X_\alpha \to \{0, 1\}$, such that $V(a, b, x, y) = 1$ if and only if the verifier accepts $(a, b, x, y)$. The tuple $G_\alpha$ is called a *nonlocal game*. For the verifier to be polynomial time, it must be possible to sample from $\pi_\alpha$ in time polynomial in $|\alpha|$, and $V_\alpha$ must be computable in polynomial time.

The provers' actions in such a protocol can be described by the probability $p(a,b|x,y)$ that the provers output $a$ and $b$ on inputs $x$ and $y$. The collection $p = \{p(a,b|x,y) : a,b,x,y \in A_\alpha \times A_\alpha \times X_\alpha \times X_\alpha\}$ is called a correlation. The probability that the verifier accepts on correlation $p$ is

$$\omega(G_\alpha, p) = \sum_{a,b,x,y} \pi_\alpha(x,y) V(a,b|x,y) p(a,b|x,y).$$

For a language $\mathcal{Z}$ to have a two-prover MIP protocol, there must be a mapping from strings $\alpha$ to games $G_\alpha$ (with a poly-time verifier) such that (1) if $\alpha \in \mathcal{Z}$, there is a correlation $p$ such that $\omega(G_\alpha, p) = 1$, and (2) if $\alpha \notin \mathcal{Z}$, then $\omega(G_\alpha, p) \leq 1/2$ for all correlations $p$.

The set of possible correlations is constrained by the fact that the provers cannot communicate. What exactly this means depends on our physical axioms. For the class MIP, we interpret this according to the rules of classical mechanics. The class MIP$^*$ is defined exactly the same as MIP, except that we use the axioms of quantum mechanics, in which the provers can share entanglement even if they can't communicate. Specifically, let $n = |X_\alpha|$ and $m = |A_\alpha|$, and consider the algebra $\mathbb{C}\mathbb{Z}_m^{*n} \times \mathbb{Z}_m^{*n}$. This algebra is generated by self-adjoint projections $p_a^x$ and $q_b^y$, $1 \leq x,y \leq n$, $1 \leq a,b \leq m$, such that $p_a^x q_b^y = q_b^y p_a^x$ for all $a,b,x,y$, and $\sum_a p_a^x = \sum_b q_b^y = 1$ for all $x,y$. A correlation $p$ is **quantum** (resp. **commuting operator**) if there is a state $f$ on $C^*\mathbb{Z}_m^{*n} \otimes_{min} C^*\mathbb{Z}_m^{*n}$ (resp. $C^*\mathbb{Z}_m^{*n} \otimes_{max} C^*\mathbb{Z}_m^{*n}$) such that $p(a,b,x,y) = f(p_a^x q_b^y)$ for all $x,y,a,b$. (This framework for correlations was worked out by Junge, Navascues, Palazuelos, Perez-Garcia, Scholz, and Werner, and Fritz.) Then MIP$^*$ is the class of languages with two-prover MIP protocols as defined above, where the provers have access to quantum correlations. There is another class, MIP$^{co}$, where the provers have access to commuting operator correlations.

We've explained the content of the result MIP$^*$ = RE. To see how this resolves the CEP, observe that if $C^*\mathbb{Z}_m^{*n} \otimes_{max} C^*\mathbb{Z}_m^{*n} = C^*\mathbb{Z}_m^{*n} \otimes_{min} C^*\mathbb{Z}_m^{*n}$, then MIP$^{co}$ = MIP$^*$. However, it is not hard to see that MIP$^{co} \subseteq$ coRE, the class of languages $\mathcal{Z}$ such that $\Sigma^* \setminus \mathcal{Z} \in$ RE. Since coRE does not contain RE, it is not possible for MIP$^{co}$ = MIP$^*$. It is worth noting that this can be made more concrete. The quantum value of a game $G$ is the supremum of $\omega(G,p)$ over quantum correlations $p$, and the commuting operator value of a game is defined similarly. It follows immediately from the above framework that the quantum value of a game is $\|\Phi_G\|_{min}$, where $\Phi_G = \sum_{a,b,x,y} \pi(x,y) V(a,b|x,y) p_a^x q_b^y$, and the commuting operator value is the max tensor norm $\|\Phi_G\|_{max}$. The proof of [**?**] explicitly gives a game $G$ such that $\|\Phi_G\|_{max} > \|\Phi_G\|_{min}$. This can be developed further to give an explicit $*$-algebra which has a tracial state, but no $*$-homomorphisms to $R^\Omega$.

The MIP$^*$ = RE result has raised a lot of questions. Stated broadly, the ones that seem to come up the most are:

(1) Can we make the current proof more algebraic?
(2) Can we make an explicit counterexample to the CEP?
(3) Are there similar types of results we should look for?
(4) Can we make a non-hyperlinear group?

Work is underway on (1), and it is worth looking at [3] and [7] in particular. As described above, the MIP$^*$ = RE result already gives a partial answer to (2). However, while it's possible to give an explicit $*$-algebra which is a counterexample to the Connes embedding problem, it is not yet possible to describe the tracial state on this $*$-algebra. For (3), the prospects seem quite good. In particular, a natural conjecture is that MIP$^{co}$ = coRE, and proving this seems to require adapting some of the ideas from [5] coming out of computer science to the setting of von Neumann algebras. Unfortunately, efforts on (4) do not seem to be making much progress at the moment; hopefully that will change.

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## Obstructions to matrix stability and $C^*$-stability of discrete groups

Marius Dadarlat

The realization that there are K-theory obstructions to perturbing approximate finite dimensional representations of $C^*$-algebras to genuine representations has emerged through work of Voiculescu [7], Connes, Gromov and Moscovici [1] and Connes and Higson [2]. In the realm of groups, it is natural to ask when an approximate representation is close to a genuine representation. We discuss primary topological obstructions for perturbing approximate representations of discrete groups to genuine representations with respect to the operator norm. Three forms of stability are considered for discrete countable groups $\Gamma$: (1) $\Gamma$ is *matricially stable* if for any sequence $\{\rho_n : \Gamma \to U(n)\}_n$ of unital maps, such that

$$\lim_{n\to\infty} \|\rho_n(st) - \rho_n(s)\rho_n(t)\| = 0, \quad \forall\, s, t \in \Gamma$$

there exist genuine representations $\{\pi_n : \Gamma \to U(n)\}_n$ satisfying

$$\lim_{n\to\infty} \|\rho_n(s) - \pi_n(s)\| = 0, \quad \forall\, s \in \Gamma.$$

(2) $\Gamma$ is *$C^*$-stable* if for any sequence $\{\rho_n : \Gamma \to U(B_n)\}_n$ of unital maps, where $B_n$ are unital $C^*$-algebras, such that

$$\lim_{n\to\infty} \|\rho_n(st) - \rho_n(s)\rho_n(t)\| = 0, \quad \forall\, s,t \in \Gamma$$

there exist homomorphisms $\{\pi_n : \Gamma \to U(B_n)\}_n$ satisfying

$$\lim_{n\to\infty} \|\rho_n(s) - \pi_n(s)\| = 0, \quad \forall\, s \in \Gamma.$$

If in the definition (2) we consider only $C^*$-algebras $B_n$ in a class $\mathcal{B}$, we say that the group $\Gamma$ is $C^*$-stable with respect to $\mathcal{B}$. (3) $\Gamma$ is *uniform-to-local stable* if for any sequence $\{\rho_n : \Gamma \to U(n)\}_n$ of unital maps, such that

$$\lim_{n\to\infty} \sup_{s,t\in\Gamma} \|\rho_n(st) - \rho_n(s)\rho_n(t)\| = 0,$$

there exist genuine representations $\{\pi_n : \Gamma \to U(n)\}_n$ s.t.

$$\lim_{n\to\infty} \|\rho_n(s) - \pi_n(s)\| = 0, \quad \forall\, s \in \Gamma.$$

Voiculescu has shown in [7] that $\mathbb{Z}^2$ is not matricially stable. A systematic study of matricial stability was undertaken by Eilers, Shulman and Sørensen in [5] where classes of matricially stable and matricially unstable groups were exhibited.

We showed in [4] that the nonvanishing of even dimensional rational cohomology in positive dimensions is a first obstruction to matricial stability for large classes of discrete groups.

A group $\Gamma$ is **MF** if it embeds in the unitary group $U\left(\frac{\prod_n M_n(\mathbb{C})}{\bigoplus_n M_n(\mathbb{C})}\right)$. The maximally almost periodic, the linear groups, the residually amenable groups, or more generally the groups that are locally embedable in amenable groups are MF by the theorem of Tikuisis, White and Winter. The class of groups which are uniformly embeddable in a Hilbert space includes the groups with the Haagerup property, the linear groups and the hyperbolic groups, among others.

**Theorem 1.** *Let $\Gamma$ be a countable discrete MF-group that admits a $\gamma$-element (e.g. $\Gamma$ is uniformly embeddable in a Hilbert space). If $H^{2k}(\Gamma, \mathbb{Q}) \neq 0$ for some $k \geq 1$, then $\Gamma$ is not matricially stable.*

The proof use ideas of Connes-Gromov-Moscovici, Kasparov, Yu, Tu and Kubota that emerged in work on the Novikov and the Baum-Connes conjectures and a technique based on the notion of quasidiagonal $K$-homology classes introduced in [3].

In response to a question of Dimitri Shlyakhtenko concerning the role of the odd dimensional cohomology groups, we obtained the following result.

**Theorem 2.** *Let $\Gamma$ be a countable discrete MF-group that admits a $\gamma$-element. If $H^k(\Gamma, \mathbb{Q}) \neq 0$ for some $k \geq 1$, then $\Gamma$ is not $C^*$-stable.*

In fact, under the assumptions of Theorem 2, the group $\Gamma$ is not $C^*$-stable with respect the class $\mathcal{B}$ consisting of $C^*$-algebras of the form $M_n(C(\mathbb{T}))$, $n \geq 1$.

Kazhdan [6] has shown that the surface groups of genus $\geq 2$ are not uniform-to-local stable. We exhibit other classes of groups which are not uniform-to-local stable.

**Theorem 3.** *Let $M$ be a closed connected Riemannian manifold with strictly negative sectional curvature such that at least one Betti number $\beta_{2i}(M) \neq 0$. If the fundamental group $\Gamma = \pi_1(M)$ is residually finite, then $\Gamma$ is not is not uniform-to-local stable.*

Let us note that if $M = M^{2n}$ is a closed connected oriented Riemannian manifold with sectional curvature $K(M) \leq -\kappa < 0$ and residually finite fundamental group, then $\pi_1(M)$ is not uniform-to-local stable since $\beta_{2n} = 1$ by orientability. In particular, if $\Gamma$ is a torsion free cocompact discrete subgroup of the Lorentz group $SO_0(2n, 1)$, then $\Gamma$ is not uniform-to-local stable.

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## Twisted groupoid C*-algebras and nuclear dimension


Astrid an Huef

(joint work with Kristin Courtney, Anna Duwenig, Magda Georgescu, Maria Grazia Viola)


Nuclear dimension was introduced by Winter and Zacharias in [13]: it is a noncommutative analogue of topological covering dimension in the sense that if $X$ is second-countable, locally compact and Hausdorff space, then the nuclear dimension of $C_0(X)$ is the topological covering dimension $\dim(X)$ of $X$. Finite nuclear dimension is a key concept in the dividing line between C*-algebras that can be classified by K-theoretic data and those that cannot be [12, 11]. The classification programme for separable, unital, simple and infinite-dimensional C*-algebras with finite nuclear dimension which satisfy the UCT was recently completed in [11]. The UCT problem, which asks if every separable and nuclear C*-algebras satisfies the UCT, remains open. By [1] separable and nuclear C*-algebras with a Cartan subalgebra belong to the UCT class [1]; since these are twisted groupoid C*-algebras by [10] there is renewed interest in twists and their C*-algebras.

Here we consider a twist over an étale and principal groupoid $G$. Our main theorem, Theorem 3 below, says that the nuclear dimension of the twisted groupoid C*-algebra is bounded by a number depending on the dynamic asymptotic dimension of the groupoid and the topological covering dimension of its unit space. This generalizes a similar theorem by Guentner, Willett and Yu for the C*-algebra of $G$ [4, Theorem 8.6]. Here we describe the three main ideas of the proof of Theorem 3 and discuss an application to groupoids with potentially large abelian stability subgroups.

Throughout $G$ is a second-countable, locally compact, Hausdorff groupoid with unit space $G^{(0)}$. A groupoid is *etale* if the range map from $G$ to $G^{(0)}$ is a local homeomorphism. We start by recalling the notion of twist.

**Definition 1.** Regard $G^{(0)} \times \mathbf{T}$ as a trivial group bundle with fibres $\mathbf{T}$. A *twist* over $G$ is a triple $(E, \iota, \pi)$ or a sequence

$$G^{(0)} \times \mathbf{T} \xrightarrow{\iota} E \xrightarrow{\pi} G$$

where $E$ is a locally compact and Hausdorff groupoid, and $\iota\colon G^{(0)} \times \mathbf{T} \to E$ and $\pi\colon E \to G$ are continuous groupoid homomorphism such that

(1) $\iota$ is injective and $\pi$ is surjective, and both restrict to homeomorphisms of unit spaces (thus we identify $E^{(0)}$ and $G^{(0)}$);
(2) every $\alpha \in G$ has a neighbourhood $U$ which is a bisection and there exists a continuous $S\colon U \to E$ such that $\pi \circ S = \mathrm{id}_U$ and the map $(\beta, z) \to \iota(r(\beta), z)S(\beta)$ is a homeomorphism of $U \times \mathbf{T}$ onto $\pi^{-1}(U)$;
(3) $\iota(G^{(0)} \times \mathbf{T})$ is central in the sense that $\iota(r(e), z)e = e\iota(s(e), z)$ for all $e \in E$ and $z \in \mathbf{T}$;
(4) $\pi^{-1}(G^{(0)}) = \iota(G^{(0)} \times \mathbf{T})$.

As a consequence of the definition of a twist, there is a continuous action of $\mathbf{T}$ on $E$ defined by $z \cdot e = \iota(r(e), z)e$ and $e \cdot z = e\iota(s(e), z)$ for $z \in \mathbf{T}$ and $e \in E$. Also, there is an induced Haar system $\sigma$ on $E$. The *reduced twisted groupoid* C*-*algebra* $\mathrm{C}^*_{\mathrm{r}}(E; G)$ is the closure of the ideal

$$C_c(E; G) := \{f \in C_c(E) : f(z \cdot e) = zf(e) \text{ for } e \in E, z \in \mathbf{T}\}$$

in $\mathrm{C}^*_{\mathrm{r}}(E, \sigma)$; it is in fact a direct summand of $\mathrm{C}^*(E, \sigma)$. Similarly, the *full twisted groupoid* C*-*algebra* $\mathrm{C}^*(E; G)$ is the closure of $C_c(E; G)$ in $\mathrm{C}^*(E, \sigma)$.

The following is [4, Definition 5.1].

**Definition 2.** Let $G$ be an étale groupoid. Then $G$ has *dynamic asymptotic dimension* $d \in \mathbf{N}$ if $d$ is the smallest natural number with the property that for every open and precompact subset $V \subset G$, there are open subsets $U_0, U_1, \ldots, U_d$ of $G^{(0)}$ that cover $s(V) \cup r(V)$ such that for each $i$, the set $\{\gamma \in V : r(\gamma), s(\gamma) \in U_i\}$ generates a precompact subgroupoid of $G$. In this case, we write $\mathrm{DAD}(G) = d$.

**Theorem 3.** Let $G$ be a second-countable, locally compact, Hausdorff, principal and étale groupoid, and let $(E, \iota, \pi)$ be a twist over $G$. Suppose that $G^{(0)}$ has topological covering dimension $N$ and that $G$ has dynamic asymptotic dimension $d$. Then the nuclear dimension of $\mathrm{C}^*_{\mathrm{r}}(E; G) = \mathrm{C}^*(E; G)$ is at most $(N+1)(d+1)-1$.

By taking $E$ to be the cartesian product $\mathbf{T} \times G$ we recover [4, Theorem 8.6] which gives a bound for the nuclear dimension of $\mathrm{C}^*_{\mathrm{r}}(G)$. The first main idea of our proof of Theorem 3 is a reduction to the case where $1 \in \mathrm{C}^*_{\mathrm{r}}(E; G)$, that is, to the case where the unit space of $G$ is compact.

**Proposition 4.** Let $(E, \iota, \pi)$ be a twist over an étale groupoid $G$ and suppose that the unit space of $G$ is not compact. Then there exists an étale groupoid $\widetilde{G}$ with compact unit space such that $\mathrm{DAD}(G) = \mathrm{DAD}(\widetilde{G})$ and $\dim(G^{(0)}) = \dim(\widetilde{G}^{(0)})$. Further, there is a twist $(\widetilde{E}, \iota, \pi)$ over $\widetilde{G}$ such that the minimal unitization of $\mathrm{C}^*_{\mathrm{r}}(E; G)$ is isomorphic to $\mathrm{C}^*_{\mathrm{r}}(\widetilde{E}; \widetilde{G})$. Similarly, the minimal unitization of $\mathrm{C}^*(E; G)$ is isomorphic to $\mathrm{C}^*(\widetilde{E}; \widetilde{G})$.

The groupoid $\widetilde{G}$ has already appeared in the literature, for example, in [5].

The nuclear dimension of a non-unital C*-algebra is equal to the nuclear dimension of its minimal unitization by [13]. Thus Proposition 4 and Theorem 3 for groupoids $G$ with compact unit space implies the theorem for groupoids $G$ with non-compact unit space.

The second main ingredient is to study the C*-algebras of the subgroupoids arising from applying the definition of finite asymptotic dimension: that they are precompact groupoids translates to them having a finite open cover of bisections.

**Proposition 5.** Let $H$ be a second-countable, locally compact, Hausdorff, principal and étale groupoid, and let $(F, \iota, \pi)$ be a twist over $H$. Let $M \in \mathbf{N}$ and suppose that $H$ has a finite open cover of $M$ bisections.

(1) Then $H$ is amenable.
(2) The primitive ideal space of $\mathrm{C}^*(F; H)$ is homeomorphic to the orbit space $H^{(0)}/H$ and each irreducible representation of $\mathrm{C}^*(F; H)$ has dimension at most $M$.
(3) For $m \in \mathbf{N}$, let $\mathrm{Prim}_m(\mathrm{C}^*(F; H))$ denote the set of primitive ideals of irreducible representations of dimension $m$ and let
$$H^{(0)}_m = \{x \in H^{(0)} \colon |r(s^{-1}(x))| = m\}.$$
Then $H^{(0)}_m/H$ is locally compact and Hausdorff, and
$$\dim(H^{(0)}_m) = \dim(H^{(0)}_m/H).$$
(4) We have $\dim(H^{(0)}) = \max_{1 \le m \le M} \dim(H^{(0)}_m)$.
(5) The decomposition rank of $\mathrm{C}^*(F; H)$ is at most $\dim(H^{(0)})$.

For example, the first part of item (2) follows because $\mathrm{C}^*(F; H)$ is liminal with spectrum $H^{(0)}/H$ by [2]. The second part of item (2) follows from work on irreducible representations in [8]. The proof of item (4) is much more complicated than the analogous proof in [4] because our $\mathrm{C}^*(H)$ may not have continuous trace (see the proof of [4, Theorem 8.13]).

In the language of [9, Theorem 2.16], Proposition 5 implies that $\mathrm{C}^*(F; H)$ has a recursive subhomogeneous decomposition with maximum matrix size at most $M$ and topological dimension at most $\dim(H^{(0)})$.

The third main ingredient in our proof of Theorem 3 is an abstraction of proof ideas used in, for example, [4, Theorem 8.6] and [6, Theorem 6.2]: both pick up on a colored version of local subhomogeneity [3, Definition 1.5]. The following proposition makes this explicit.

**Proposition 6.** Let $A$ be a unital C*-algebra, let $1_A \in A_0 \subset A$ be a dense *-subalgebra and let $d, N \in \mathbf{N}$. Let $\mathcal{F} \subset A_0$ be a finite subset and let $\varepsilon > 0$. Suppose that for $0 \leq i \leq d$ there exist C*-subalgebras $B_i \subset A$ with nuclear dimension at most $N$ and there exist $b_i \in A$ with $\|b_i\| \leq 1$ such that $b_i \mathcal{F} b_i^* \subset B_i$, and for all $a \in \mathcal{F}$ we have

$$\|a - \sum_{i=0}^{d} b_i a b_i^*\| < \varepsilon \|a\|.$$

Then the nuclear dimension of $A$ is at most $(d+1)(N+1) - 1$.

In the proof of Theorem 3 we apply Proposition 6 to the C*-algebras $B_i$ ($0 \leq i \leq d$) of the groupoids $H_i$ arising from the definition of $\mathrm{DAD}(G) = d$ together with functions $b_i$ adapted from [4].

Our application is to non-principal groupoids $G$, and is based on an isomorphism of $\mathrm{C}^*(G)$ with a twisted groupoid C*-algebra [7, 2].

**Corollary 7.** Let $G$ be a second-countable, locally compact, Hausdorff and étale groupoid. Assume that the orbits of $G$ are closed in $G^{(0)}$ and that the stability subgroups of $G$ are abelian. Let $\mathcal{A}$ be the isotropy groupoid and let $\widehat{\mathcal{A}}$ be the spectrum of the commutative C*-algebra $\mathrm{C}^*(\mathcal{A})$. Suppose that the quotient groupoid $\mathcal{R}$ of $G$ by the isotropy groupoid $\mathcal{A}$ is étale, that the topological dimension of $\widehat{\mathcal{A}}$ is at most $N$ and that $\mathcal{R}$ has finite dynamic asymptotic dimension at most $d$. Then the nuclear dimension of $\mathrm{C}^*_{\mathrm{r}}(G) = \mathrm{C}^*(G)$ is at most $(N+1)(d+1) - 1$.

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## A conjugate system for $q$-Gaussian operators for all $q$

Roland Speicher
(joint work with Akihiro Miyagawa)

Fix a real number $-1 \leq q \leq 1$. The *$q$-Gaussian distribution* is the noncommutative distribution of a collection $\{X(f) \mid f \in H_{\mathrm{real}}\}$ of selfadjoint operators $X(f)$, which are indexed by a real Hilbert space $H_{\mathrm{real}}$ and whose moments, with respect to a trace $\tau$, are given by the following $q$-deformed version of the Wick (or Isserlis) formula:

$$\tau[X(f_1)\cdots X(f_n)] = \sum_{\pi\in P_2(n)} \Big( \prod_{(l,r)\in\pi} \langle f_l, f_r\rangle \Big) \cdot q^{\mathrm{cr}(\pi)},$$

where $P_2(n)$ denotes the set of pairings (or matchings) of the set $\{1,\ldots,n\}$ and where $\mathrm{cr}(\pi)$ is the number of crossings of the pairs of $\pi$.

For $q = 1$ this is the classical Wick/Isserlis formula for a Gaussian family of classical random variables, whereas for $q = 0$ the factor $q^{\mathrm{cr}(\pi)}$ suppresses effectively all crossings and one gets the formula over non-crossing pairings for a semicircular family.

By the GNS construction with respect to $\tau$ one can realize those $q$-Gaussian operators in the form $X(f) = a(f) + a^*(f)$, where $a(f)$ and $a^*(f)$ are annihilation and creation operators, respectively, given on a $q$-deformed Fock space as follows. Let $H$ be the complexification of $H_{\mathrm{real}}$ and

$$\mathcal{F}_q(H) = \overline{\bigoplus_{n\geq 0} H^{\otimes n}}^{\langle\cdot,\cdot\rangle_q}$$

the completion of the algebraic Fock space with respect to the inner product

$$\langle f_1\otimes\cdots\otimes f_n, g_1\otimes\cdots\otimes g_m\rangle_q := \delta_{mn} \sum_{\sigma\in S_n} \prod_{i=1}^{n} \langle f_i, g_{\sigma(i)}\rangle q^{\mathrm{inv}(\sigma)},$$

where $\mathrm{inv}(\sigma)$ denotes the number of inversions of the permutation $\sigma$.

The *creation operator* $a^*(f)$ is defined by

$$a^*(f) f_1\otimes\cdots\otimes f_n = f\otimes f_1\otimes\cdots\otimes f_n$$

and the *annihilation operator* $a(f)$, which is the adjoint of $a^*(f)$ with respect to the $q$-inner product, is given by

$$a(f)\Omega = 0$$

$$a(f)f_1 \otimes \cdots \otimes f_n = \sum_{r=1}^{n} q^{r-1} \cdot \langle f, f_r \rangle \cdot f_1 \otimes \cdots \otimes f_{r-1} \otimes f_{r+1} \otimes \cdots \otimes f_n$$

$H^{\otimes 0} = \mathbb{C}\Omega$ is here a one-dimensional Hilbert space with a distinguished vector $\Omega$ of norm 1, called *vacuum*. The trace $\tau$ is then the vector state corresponding to $\Omega$, i.e., $\tau(b) = \langle \Omega, b\Omega \rangle$.

Consider now $d := \dim H < \infty$ and let $e_1, \ldots, e_d$ be an orthonormal basis of $H$. We put then $X_i := X(e_i) = a_i + a_i^*$. The operators $a_i$ satisfy the $q$-commutation relations $a_i a_j^* - q a_j^* a_i = \delta_{ij} 1$. Thus our construction interpolates between the CAR (for $q = -1$) and the CCR (for $q = +1$).

We are mainly interested in the $X_i$ and their generated von Neumann algebra $\Gamma_q(\mathbb{R}^d) := W^*(X_1, \ldots, X_d)$. The big questions are about regularity properties of the $q$-distributions, i.e., the noncommutative distribution of $(X_1, \ldots, X_d)$, and how the von Neumann algebra $\Gamma_q(\mathbb{R}^d)$ depends on $q$. The central case $q = 0$ is generated by free semicircular elements and free probability tools give then easily that $\Gamma_0(\mathbb{R}^d)$ is isomorphic to the free group factor $L(\mathbb{F}_d)$. So the main question is whether the $q$-Gaussian algebras $\Gamma_q(\mathbb{R}^d)$ are, for $-1 < q < 1$, isomorphic to the free group factor.

Over the years it has been shown that these algebras share many properties with the free group factors. For instance, for all $-1 < q < 1$ the $q$-Gaussian algebras are $\mathrm{II}_1$-factors, non-injective, prime, and have strong solidity. Here is an incomplete list of papers proving these properties [4], [14], [13], [11], [12], [1].

A partial answer to the isomorphism problem was achieved by Guionnet and Shlyakhtenko [8], who proved that the $q$-Gaussian algebras are isomorphic to the free group factors for small $|q|$ (where the size of the interval depends on $d$ and goes to zero for $d \to \infty$). However, it is still open whether this is true for all $-1 < q < 1$.

In our paper [10] we are looking at noncommutative derivatives with respect to the $X_i$ and corresponding conjugate systems.

**Definition.** $\xi_1, \ldots, \xi_d \in L^2(X_1, \ldots, X_d) = \mathcal{F}_q(\mathbb{R}^d)$ is a *conjugate system* if

$$\tau[\xi_i Q(X_1, \ldots, X_d)] = \tau \otimes \tau[(\partial_i Q)(X_1, \ldots, X_d)],$$

where $\partial_i$ is the noncommutative derivative, given by

$$\partial_i X_{i_1} \cdots X_{i_m} = \sum_{r=1}^{m} \delta_{i,i_r} X_{i_1} \cdots X_{i_{r-1}} \otimes X_{i_{r+1}} \cdots X_{i_m}$$

In terms of inner products this means

$$\langle Q(X_1, \ldots, X_d)\Omega, \xi \rangle = \langle (\partial_i Q)(X_1, \ldots, X_d)\Omega, \Omega \otimes \Omega \rangle, \qquad \text{or} \qquad \xi_i = \partial_i^* \Omega \otimes \Omega.$$

In [10] we can now prove the following statements.

**Proposition.** For $\eta_1, \eta_2 \in H^{\otimes m}$ we have

$$\langle \partial_i \eta_1 \otimes e_j \otimes \eta_2, \Omega \otimes \Omega \rangle_q = \delta_{ij} (-1)^m q^{m(m+2)/2} \langle \eta_1, \eta_2 \rangle_q.$$

This allows us to derive the following formula for the conjugate variables in the $q$-case.

**Theorem.** The conjugate variables for the $q$-Gaussians are given by

$$\xi_i = \sum_{m=0}^{\infty} \sum_{i_1,\ldots,i_m=1}^{d} (-1)^m q^{m(m+2)/2} \cdot r^*_{ii_1\ldots i_m} e_{i_1} \otimes \cdots \otimes e_{i_m},$$

where $r^*$ is the adjoint of the undeformed right annihilation operator. The sum above converges, for all $-1 < q < 1$, in operator norm. Furthermore $(\xi_1, \ldots, \xi_d)$ is Lipschitz conjugate, i.e., all $\partial_j \xi_i$ exist and are actually bounded operators

One should note that we do not have a concrete combinatorial description for the operator $r^*$, still results of Bozejko allow to estimate its norm. Hence the above formula for $\xi_i$ is a mixture of concrete combinatorial factors and more abstract ones. A crucial point is that the combinatorial factor $q^{m(m+2)/2}$ is responsible for the uniform estimates on the sum in the whole interval $(-1, 1)$ for $q$.

Having the existence of conjugate systems for all $q$ with $-1 < q < 1$ has then, by general results, many consequences for all such $q$; like, for any $-1 < q < 1$, non-$\Gamma$ of $q$-Gaussian algebras, by [6], or that any non-constant selfadjoint rational function over $q$-Gaussians has no atom in its distribution, by [9]. In Lemma 37 of [5], algebraic freeness of noncommutative power series over $q$-Gaussians is proved.

There are also quite some applications of the fact that our conjugate system is Lipschitz conjugate. By [5], the existence of a Lipschitz conjugate system and Connes embeddability (which is given for our $q$-Gaussians, for all $q$) imply the maximality of the micro-states free entropy dimension. As a consequence of this or a direct application of Theorem 1.3 in [7], we can recover the fact that $\Gamma_q(\mathbb{R}^d)$ has no Cartan subalgebra for any $-1 < q < 1$, which has been already shown by Avsec [1] by other methods. Furthermore, the paper by Banna and Mai [2] gives us Hölder continuity of cumulative distribution functions of noncommutative polynomials in the $q$-Gaussians.

In the following corollary we collect the most important consequences of our result.

**Theorem.** For all $-1 < q < 1$ we have the following properties for the $q$-Gaussian operators $(X_1, \ldots, X_d)$.

- The division closure of the $q$-Gaussians in the unbounded operators affiliated to $W^*(X_1, \ldots, X_d)$ is isomorphic to the free field. This implies that any noncommutative rational function $r$ in $d$ noncommuting variables can be applied to the $q$-Gaussians, yielding a (possibly unbounded) operator $r(X_1, \ldots, X_d)$. If $r$ is not the zero rational function, then this operator has trivial kernel; i.e., for any selfadjoint $r$ which is different from a constant the corresponding distribution has no atoms.

- There is no non-zero noncommutative power series $\sum_{w\in[d]^*}\alpha_w A^w$ of radius of convergence $R > \|X_i\|$ such that $\sum_{w\in[d]^*}\alpha_w X^w = 0$.
- For any selfadjoint noncommutative polynomial $Y = p(X_1,\ldots,X_d)$, its cumulative distribution function $\mathcal{F}_Y$ is Hölder continuous with exponent $1/(2^{\deg Y}-1)$ where $\deg Y$ is the degree of $p$.
- The $q$-Gaussian operators have finite non-microstates free Fisher information and maximal microstates free entropy dimension,

$$\Phi^*(X_1,\ldots,X_d) < \infty, \quad \text{and} \qquad \delta_0(X_1,\ldots,X_d) = d.$$

- $\Gamma_q(\mathbb{R}^d)$ does not have property $\Gamma$ and it does not have a Cartan subalgebra.
- A free Gibbs potential (in the sense of [8]) exists for the $q$-Gaussians.

Unfortunately, we are not able to use our result for adding anything to the isomorphism problem. However, the fact that the free entropy dimension is maximal for all $q$ in the whole interval is another indication that they might all be isomorphic to the free group factor.

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## Nuclear C*-algebras as inductive limits of finite dimensional C*-algebras

Kristin Courtney
(joint work with Wilhelm Winter)

Inductive limits are a natural vehicle for "bootstrapping" nice properties from well-understood classes of objects to slightly less tangible ones. For operator algebraists, inductive limits of finite-dimensional algebras (so-called AFD von Neumann algebras and AF C*-algebras) are particularly tractable, and give rise to famous classification results, such as the uniqueness of the AFD $\mathrm{II}_1$ factor [3] or the classification of AF C*-algebras [4]. Dual to these classification results are structural results, which aim to describe more operator algebras as inductive limits of well-understood classes of algebras. The most influential of these is due to Alain Connes [1] who showed that a separable von Neumann algebra is AFD iff it is semi-discrete, meaning its identity map approximately factorizes though finite-dimensional von Neumann algebras. Just as AF is the C*-analogue to AFD, semi-discreteness has a C*-analogue called nuclearity, which has been the subject of intensive research over the past several decades. Since many C*-algebras, such as the Cuntz algebras are not AF (e.g., Cuntz algebras), there is no direct C*-analogue to Connes' hyperfiniteness theorem. This is neither surprising nor deterring: one must often adapt von Neumann properties, results, and techniques to find viable C*-analogues. Likewise, we shall adjust our notion of inductive limits.

In [2], the authors consider inductive systems of C*-algebras where the connecting *-homomorphisms are replaced by certain positivity preserving maps called cpc maps. They show that any separable C*-algebra is nuclear and quasidiagonal iff it arises as the limit of a system of finite-dimensional C*-algebras with asymptotically multiplicative cpc connecting maps. However, there exist non-quasidiagonal nuclear C*-algebras, including Cuntz algebras. By generalizing asymptotic multiplicativity to asymptotic orthogonality preservation, we are able describe all nuclear C*-algebras as inductive limits of finite dimensional C*-algebras.

**Theorem 1** (C.-Winter)**.** *A separable* C*-*algebra is nuclear if and only if it arises as the inductive limit of a system of finite dimensional* C*-*algebras with asymptotically orthogonality preserving cpc maps.*

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## Groupoids, topological full groups, algebraic K-theory spectra, and infinite loop spaces

Xin Li

Topological full groups have recently attracted attention because they led to solutions of several outstanding open problems in group theory (see [7], [8] and [15]).

Topological groupoids and their topological full groups arise in a variety of settings, for instance from Cantor minimal systems, from shifts of finite type, or more generally, from graphs (see for instance [11]), from self-similar groups or actions and from higher rank graphs (see for instance [12, 14]). In this context, there is an interesting connection to C*-algebra theory because topological groupoids serve as models for C*-algebras (see [17]) such as Cuntz algebras, Cuntz-Krieger algebras, graph C*-algebras or higher rank graph C*-algebras, many of which play important roles in the classification programme for C*-algebras. There is also an interesting link to group theory because Thompson's group $V$ (see [4]) and many of its generalizations and variations [6, 20, 1] can be described as topological full groups of corresponding topological groupoids. In the case of $V$ this observation goes back to [13].

While general structural properties [10, 11, 12, 16, 9] and rigidity results have been developed [18, 11], and several deep results have been established for particular examples of topological full groups [7, 8, 15, 19, 21], it would be desirable to create a dictionary between dynamical properties and invariants of topological groupoids on the one hand and group-theoretic properties and invariants of topological full groups on the other hand. This would allow us to study topological full groups – which are very interesting but in many aspects still remain mysterious – through the underlying topological groupoids which are often much more accessible. In my talk, I described recent work which develops this programme in the context of homological invariants by establishing a link between groupoid homology and group homology of topological full groups.

For the particular example class of Thompson's group $V$ and its generalizations, the study of homological invariants and properties has a long history [3, 2]. It was shown in [2] that $V$ is rationally acyclic. Only recently it was established in [21] that $V$ is even integrally acyclic. The new approach in [21] also allows for many more homology computations for Higman-Thompson groups. However, for other classes of topological full groups, very little is known about homological invariants.

Inspired by [21], we have developed an approach to homological invariants of topological full groups. Let us now formulate our main results. Let $G$ be a topological groupoid, i.e., a topological space which is at the same time a small category with invertible morphisms, such that all operations (range, source, multiplication and inversion maps) are continuous. We always assume the unit space $G^{(0)}$ consisting of the objects of $G$ to be locally compact and Hausdorff. In addition, suppose that $G$ is ample, in the sense that it has a basis for its topology given by compact open bisections. If $G^{(0)}$ is compact, then the topological full group $\boldsymbol{F}(G)$ is defined as the group of global compact open bisections. In the general case, $\boldsymbol{F}(G)$ is the inductive limit of topological full groups of restrictions of $G$ to compact open

subspaces of $G^{(0)}$. Given an ample groupoid $G$ as above, we construct a small permutative category $\mathfrak{B}_G$ of compact open bisections of $G$ . Let $\mathbb{K}(\mathfrak{B}_G)$ be the algebraic K-theory spectrum of $\mathfrak{B}_G$ and $\Omega^\infty\mathbb{K}(\mathfrak{B}_G)$ the associated infinite loop space.

Our first main result identifies reduced homology of $\mathbb{K}(\mathfrak{B}_G)$ with groupoid homology of $G$ as introduced in [5] and studied in [10].

**Theorem 1.** *Let $G$ be an ample groupoid with locally compact Hausdorff unit space $G^{(0)}$. Then we have*

$$\tilde{H}_*(\mathbb{K}(\mathfrak{B}_G)) \cong H_*(G).$$

For the second main result, we need the assumption that $G$ is minimal, i.e., every $G$-orbit is dense in $G^{(0)}$. We also require $G$ to have comparison, which roughly means that $G$-invariant measures on $G^{(0)}$ control when one compact open subspace of $G^{(0)}$ can be transported into another by compact open bisections of $G$. In this setting, we can identify group homology of the topological full group $\boldsymbol{F}(G)$ with the homology of $\Omega^\infty_0\mathbb{K}(\mathfrak{B}_G)$, the connected component of the base point in $\Omega^\infty\mathbb{K}(\mathfrak{B}_G)$.

**Theorem 2.** *Let $G$ be an ample groupoid, with locally compact Hausdorff unit space $G^{(0)}$ without isolated points. Assume that $G$ is minimal and has comparison. Then we have*

$$H_*(\boldsymbol{F}(G)) \cong H_*(\Omega^\infty_0\mathbb{K}(\mathfrak{B}_G)).$$

Among other things, our results lead to

- a complete description of rational group homology for large classes of topological full groups,
- general vanishing and acyclicity results, explaining and generalizing the result that $V$ is acyclic in [21].

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## From quantum information theory to type III von Neumann algebras

Marius Junge
(joint work with Haojian Li, Li Gao, Nick LaRacuente, Yidong Chen)

We consider an array of particles, with an interaction of energy states of a particle of some probability $p$. The energy released by these transitions can be measured as photon transmission. Taking the mathematical model leads to a semigroup of completely positive maps and a generator satisfying the 0-detailed balance condition. For infinitely many particles, we find a type III factor where the modular group is used to obtain the correct Laplacian. We prove uniform spectral gap and entropy decay for the tracial case. This however fails in the type $\mathrm{III}_\lambda$ situation. A slight change in the form of the Laplacian is stable and admits uniform decay to equilibrium.

## Rigidity of Roe algebras

Rufus Willett

(joint work with Florent Baudier, Bruno de Mendonça Braga, Ilijas Farah, Ana Khukhro, Alessandro Vignati)

Uniform Roe algebras are $C^*$-algebras associated to metric spaces; they see only the large-scale (or 'coarse') geometry of the underlying space. It is natural to ask how 'rigid' the Roe algebra construction is, or in other words, how much of the underlying coarse geometry it remembers. Somewhat surprisingly, the answer turns out to be essentially 'all of it'; it was the goal of my talk to explain this.

We now give formal definitions. The metric spaces we are interested in are those with *bounded geometry* (also called *uniformly locally finite* metric spaces): for every $r > 0$, the number of points in an $r$-ball is uniformly bounded. Important examples include finitely generated discrete groups equipped with word metrics, and discretizations of Riemannian manifolds. A map $f : X \to Y$ between two (bounded geometry) metric spaces it *coarse* if for any $r > 0$ there is $s > 0$ such that $d_X(x_1, x_2) \leq r$ implies $d_Y(f(x_1), f(x_2)) \leq s$. Metric spaces $X$ and $Y$ are *coarsely equivalent* if there are coarse maps $f : X \to Y$ and $g : Y \to X$ such that $\sup_{x\in X} d_X(g(f(x)), x)$ and $\sup_{y\in Y} d_Y(f(g(y)), y)$ are both finite. One thinks of coarsely equivalent spaces as having 'the same large scale geometry'.

We now associate a $C^*$-algebra to a bounded geometry space $X$. Let $\mathbb{C}_u[X]$ be the $*$-algebra of $X$-by-$X$ indexed matrices $a = (a_{xy})_{x,y\in X}$ with uniformly bounded entries, and so that the *propagation* defined by

$$\mathrm{prop}(a) := \sup\{d(x, y) \mid a_{xy} \neq 0\}$$

is finite. The bounded geometry condition implies that there is a well-defined action of $\mathbb{C}_u[X]$ on $\ell^2(X)$ by bounded operators. The *uniform Roe algebra* of $X$, denoted $C^*_u(X)$, is the completion of $\mathbb{C}_u[X]$ in this representation. If $X = \Gamma$ is a finitely generated discrete group, then $C^*_u(X)$ is canonically isomorphic to $\ell^\infty(\Gamma) \rtimes_r \Gamma$.

Going back to a suggestion of Gromov [10, pages 262-3], if $X$ and $Y$ are coarsely equivalent, then $C^*_u(X)$ and $C^*_u(Y)$ are Morita equivalent: this was established by Brodzki, Niblo, and Wright in [9, Theorem 4]. It then becomes natural to ask how "rigid" the uniform Roe algebra construction is: precisely, is the converse to the result of Brodzki-Niblo-Wright true?

This question has been worked on by several authors over the last ten years, starting with [12] by Špakula and the author, who established a rigidity result under the geometric assumption that the spaces involved satisfy Yu's *property A*. A conceptual step forward was taken by Braga and Farah in [4], where those authors isolated some of the key conditions needed to prove rigidity, and weakened the geometric conditions needed on the spaces involved. Further weakenings of the geometric conditions needed to imply rigidity were achieved subsequently by Braga, Chyuan, and Li [3], and by Li, Špakula, and Zhang [11]. However, the methods used in all these papers face fundamental obstructions coming from the

existence of expander graph type phenomena in exotic metric spaces; getting over these obstructions needed a new idea. (I should also mention that at the same time there was other work developing these techniques, and employing new ones, to attack variants of the rigidity problem and otherwise elucidate the structure of Roe algebras: see for example [5] on coronas, [13] on Cartan subalgebras, [6] on embeddings, [8] on automorphisms, [2] on Banach algebras, and [7] on the non-metrizable case).

In my talk, I explained the recent *unconditional* solution to the rigidity problem [1]:

**Theorem 1** (Baudier, Braga, Farah, Khukhro, Vignati, Willett)**.** *If $C^*_u(X)$ and $C^*_u(Y)$ are Morita equivalent, then $X$ and $Y$ are coarsely equivalent.*

A purely group-theoretic variant of this shows that if $\Gamma$ and $\Lambda$ are finitely generated groups, then $\ell^\infty(\Gamma) \rtimes_r \Gamma$ is isomorphic to $\ell^\infty(\Lambda) \rtimes_r \Lambda$ if and only if $\Gamma$ and $\Lambda$ are bi-Lipschitz equivalent. Several other variants cover the case of coronas, and of Roe-type algebras of operators on more general Banach spaces.

The key new idea needed for the proof of the theorem above is a quantitative version of Lyapunov's theorem on the convexity of the range of a finite-dimensional vector measure, which in turn is based on the Shapley-Folkman theorem from economics. This can be combined with earlier uniform approximability results and some basic observations about projections to deduce uniform estimates on the size of certain matrix entries; these estimates are exactly what one needs to apply the machinery established in [12] and [4] and deduce the result.

I should finish by commenting that there are some interesting variants of the rigidity problem that are still open. One of these is the question of whether isomorphism of $C^*_u(X)$ and $C^*_u(Y)$ is equivalent to *bijective* coarse equivalence of the underlying spaces; this is known for groups [1] and in the property A case [13], but not in general. Another asks whether rigidity holds for the *Roe algebras* $C^*(X)$: these are $C^*$-algebras built analogously to $C^*_u(X)$ but with the matrix entries being compact operators rather than complex numbers; the state of the art is due to Li, Špakula, and Zhang [11]. Finally, the rigidity problem is also open in the non-metrizable case; here the best known results are due to Braga, Farah, and Vignati [7].

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## Cartan subalgebras for non-effective twisted groupoid C*-algebras

Anna Duwenig

(joint work with E. Gillaspy, R. Norton, S. Reznikoff, D. P. Williams, S. Wright, J. Zimmerman)

A maximal abelian subalgebra $B$ of a C*-algebra $A$ is *Cartan* if there exists a faithful conditional expectation $A \to B$ and if $B$ is *regular*, i.e., $A$ is generated by $n \in A$ for which $nBn^*, n^*Bn \subseteq B$ ([7], [11], [10]). The existence of a Cartan has important implications ([1], [2], [3], [8], [9]). An easy example is the C*-subalgebra of either of the universal unitaries that generate the irrational rotation algebra $A_\theta$.

Of particular interest is the example due to J. Renault: Given a second countable, locally compact Hausdorff étale groupoid $G$ with twist $\Sigma$, $C_0(G^{(0)})$ embeds as a regular commutative subalgebra into $\mathrm{C}_r^*(G;\Sigma)$ with conditional expectation. This subalgebra is maximal abelian (and hence Cartan) if the groupoid is *effective*, i.e., $\mathrm{Iso}(G)^\circ = G^{(0)}$, because an element $a$ of $\mathrm{C}_r^*(G;\Sigma)$ commutes with $C_0(G^{(0)})$ iff its open support is contained in $\mathrm{Iso}(G)^\circ$, while $a$ is in $C_0(G^{(0)})$ iff its open support is contained in $G^{(0)}$. Miraculously, any separable C*-algebra with Cartan subalgebra arises in this fashion [11].

Now think of $A_\theta$ as $\mathrm{C}_r^*(\mathbb{Z}, \mathbf{c}_\theta)$ with 2-cocycle $\mathbf{c}_\theta((m,n),(k,l)) = \mathrm{e}^{2\pi i\theta nk}$. Since non-trivial groups are never effective, the above argument does not apply. It is not hard to see, however, that for $\theta \notin \mathbb{Q}$ the two Cartan subalgebras mentioned above correspond to the subgroups $\mathbb{Z} \times \{0\}$ and $\{0\} \times \mathbb{Z}$. It is thus a natural question to ask, under which conditions the C*-algebra of a subgroupoid $S$ of a twisted (not necessarily effective) groupoid $G$ is a Cartan subalgebra. Renault's proof that $C_0(G^{(0)})$ is a regular subalgebra with a conditional expectation goes through for $\mathrm{C}_r^*(S;\Sigma|_S)$, as long as $S$ is clopen and normal. And if the twist restricted to $S$ is abelian, then the algebra is abelian. Therefore, the crux in answering the question lies in finding conditions on $S$ that force maximality.

Clearly, one needs $S$ to be maximal among certain subgroupoids. In particular, one must have $G^{(0)} \subset S \subset \mathrm{Iso}(G)$, and so in the case of an effective groupoid $G$,

the only open such $S$ would be $G^{(0)} = \mathrm{Iso}(G)^{(0)}$, as in Renault's result. But in general, this maximality assumption alone is not enough: It is easy to construct a group $G$ with a maximal subgroup $S$ and elements $\nu, \mu \notin S$ for which $\{\nu, \mu\} = \{\alpha\nu\alpha^{-1} : \alpha \in S\}$. Consequently, the sum $\delta_\mu + \delta_\nu \notin \mathrm{C}_r^*(S, \mathbf{c})$ of point masses is a commutator of $\mathrm{C}_r^*(S, \mathbf{c})$ that lies outside of the subalgebra, showing that the subalgebra is not maximal abelian. To avoid this issue, we require $S$ to satisfy an additional condition, which arises in many situations, e.g., when $\mathrm{Iso}(G)^\circ$ is abelian or when each isotropy group $S_u^u$ has the unique roots property.

**Definition 1.** A subgroupoid $S$ is *immediately centralizing* if, for any $\nu \in \mathrm{Iso}(G)^\circ$,

$$|\{\alpha\nu\alpha^{-1} : \alpha \in S\}| > 1 \implies |\{\alpha\nu\alpha^{-1} : \alpha \in S\}| = \infty.$$

These sets have already been considered in the study of Cartans (cf. [4, p. 360]).

**Theorem 1.** *Let $G$ be a second countable, locally compact Hausdorff, étale groupoid with twist $\Sigma$. Suppose that $S \trianglelefteq G$ is clopen and maximal among the subgroupoids of $\mathrm{Iso}(G)^\circ$ whose restricted twist is abelian. If $S$ is immediately centralizing, then $\mathrm{C}_r^*(S; \Sigma|_S)$ is a Cartan subalgebra of $\mathrm{C}_r^*(G; \Sigma)$.*

For 2-cocycle twists, the above is one of two main results in [6]; for general twists, it is part of on-going work with D. P. Williams and J. Zimmerman. Note that, if $G$ is untwisted, then $S$ can only be maximal and immediately centralizing if every isotropy group $G_u^u$ is *icc relative to the subgroup* $S_u^u$ in the sense of Rørdam.

**Example 1.** For $(\mathbb{Z}^2, \mathbf{c}_\theta)$ with $\theta \notin \mathbb{Q}$, one can choose $S$ as $\mathbb{Z} \times \{0\}$ or $\{0\} \times \mathbb{Z}$, and these give rise to the Cartan subalgebras in $A_\theta$ described above; no other subgroups fall into the scope of Theorem 1. If $\theta = p/q \in \mathbb{Q}$ with $p, q$ relatively prime, then $S$ can be chosen to be any $m\mathbb{Z} \times n\mathbb{Z}$ with $mn = q$.

In other examples, Theorem 1 picks up on multiple non-isomorphic Cartan subalgebras within the same groupoid C*-algebra.

If $S \trianglelefteq G$ and $\Sigma$ are such that $\mathrm{C}_r^*(S; \Sigma|_S)$ is Cartan in $\mathrm{C}_r^*(G; \Sigma)$ (for example, as in Theorem 1), then the other direction of Renault's result yields the so-called Weyl groupoid with twist: an *effective* groupoid $W$ whose twisted C*-algebra is isomorphic to $\mathrm{C}_r^*(G; \Sigma)$ such that the canonical Cartan $C_0(W^{(0)})$ is mapped to $\mathrm{C}_r^*(S; \Sigma|_S)$. For $\Sigma$ coming from a 2-cocycle $\mathbf{c}$, the main results in [5] then describe the relationship between $S, G, \mathbf{c}$ and $W$.

**Theorem 2.** *There exists an action of the quotient groupoid $G/S$ on the spectrum of $\mathrm{C}_r^*(S, \mathbf{c})$, and the corresponding transformation groupoid is isomorphic to $W$. If the quotient map $G \to G/S$ has a continuous section, then the twist on $W$ is given by a 2-cocycle.*

For $\Sigma$ more general than a 2-cocycle, such a description is work in progress.

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## Wasserstein distance and ultraproducts


David Jekel

(joint work with Wilfrid Gangbo, Kyeongsik Nam, Dimitri Shlyakhtenko)


Motivated by works on transport of measure in free probability [5, 3, 6, 7], the authors studied the properties of optimal couplings in the setting of tracial non-commutative probability [4]. While the analogy with classical Monge-Kantorovich duality can be pushed quite far, the space of laws with the non-commutative Wasserstein distance is at the same time very different than the classical case.

Non-commutative laws describe the distribution of $d$-tuples of self-adjoint operators in tracial W*-algebras. $\Sigma_{d,R}$ will denote the space of traces on the universal free product $C[-R,R]^{*d}$. Given $\mu \in \Sigma_{d,R}$, a self-adjoint $d$-tuple from $\mathcal{M} = (M, \tau_M)$, we say that $X \sim \mu$ if $\mu(p) = \tau(p(X))$ for all non-commutative polynomials $p$. Biane and Voiculescu [2] defined the non-commutative analog of the Wasserstein distance between $\mu, \nu \in \Sigma_{d,R}$, namely $d_W(\mu, \nu)$ is the infimum of $\|X - Y\|_{L^2(\mathcal{M})}$ over all tracial W*-algebras $\mathcal{M}$ and all self-adjoint $d$-tuples $X, Y$ in $\mathcal{M}$ with $X \sim \mu$ and $Y \sim \nu$.

Although in classical probability theory, the Wasserstein distance metrizes the weak-$*$ topology on the space of laws, this is far from being true in the non-commutative case. First, one can show using similar techniques to Ozawa [8] that $\Sigma_{d,R}$ is not separable with respect to $d_W$. Hence the topology generated by $d_W$ is very different than the weak-$*$ topology, since $\Sigma_{d,R}$ is weak-$*$ compact. We further

investigated *why* and *how* these two topologies are different by expressing them in operator-algebraic terms.

While weak-$*$ convergence of the law of $X_n$ in $\mathcal{M}_n$ to the law of $X$ fits naturally with the study of embeddings of $\mathcal{M} = \mathrm{W}^*(X)$ into the ultraproduct $\prod_{n\to\mathcal{U}} \mathcal{M}_n$, we found that Wasserstein convergence relates to embeddings into ultraproducts which lift to complete positive maps $\Phi_n : \mathcal{M} \to \mathcal{M}_n$ such that $\Phi_n$ is factorizable in the sense of Anantharaman-Delaroche [1], meaning that it factors as a trace-preserving inclusion into $\mathcal{M} \to \mathcal{N}_n$ followed by a trace-preserving conditional expectation $\mathcal{N}_n \to \mathcal{M}_n$. As a consequence, we have the following result:

**Proposition 1** ([4])**.** *Let $\mathcal{M}$ be a tracial $\mathrm{W}^*$-algebra generated by $X = (X_1, ..., X_d)$, and let $\mu$ be the non-commutative law of $X$. The following are equivalent:*

*(1) The weak-$*$ and Wasserstein topologies agree at the point $\mu$ (meaning every neighborhood in the one topology contains a neighborhood in the other).*

*(2) Every embedding of $\mathcal{M}$ into an ultraproduct $\prod_{n\to\mathcal{U}} \mathcal{M}_n$ of tracial $\mathrm{W}^*$-algebras lifts to a sequence of factorizable completely positive maps $\Phi_n : \mathcal{M} \to \mathcal{M}_n$.*

*Assuming that $\mathcal{M}$ is Connes embeddable, then the following two conditions are also equivalent to (1) and (2) above:*

*(3) $\mu$ is in the Wasserstein closure of the set of laws $\Sigma_{d,R,fin}$ that are realized in finite-dimensional tracial $\mathrm{W}^*$-algebras.*

*(4) $\mathcal{M}$ is semi-discrete / amenable.*

Of course, (4) relies on Connes' characterizations of amenability. It is an open question whether there exist non-Connes-embeddable tracial $\mathrm{W}^*$-algebras that satisfy (1) and (2).

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### W*-rigidity paradigms for embeddings of $\mathrm{II}_1$ factors

Sorin Popa
(joint work with Stefaan Vaes)

In this talk I present some recent work with Stefaan Vaes, in which we undertake a systematic study of W*-rigidity paradigms for the embeddability relation between separable $\mathrm{II}_1$ factors and its stable, weaker version, involving amplifications of factors. We notably obtain a series of concrete families of non-stably isomorphic $\mathrm{II}_1$ factors that are mutually embeddable (many to one paradigm) and large families of $\mathrm{II}_1$ factors that are mutually non-stably embeddable (disjointness paradigm). I will comment on the proofs, which employ some "hard" deformation-rigidity arguments.

### Miscellaneous about commutants mod

Dan-Virgil Voiculescu

If $\tau$ is a $n$-tuple of bounded operators on an $\infty$-dimensional separable Hilbert space and $(\mathcal{J}, |\cdot|_{\mathcal{J}})$ is a normed ideal of compact operators , the commutant of $\tau$ modulo $\mathcal{J}$ is a Banach algebra $\mathcal{E}(\tau;\mathcal{J})$. The quasicentral modulus $k_{\mathcal{J}}(\tau)$ is the liminf of $|[\tau, A]|_{\mathcal{J}}$ when the $A$'s are the finite rank positive contractions converging to $I$. The number $k_{\mathcal{J}}(\tau)$ plays a key role in questions about perturbations of operators, like the invariance of absolutely continuous parts and also in the structure of the $\mathcal{E}(\tau;\mathcal{J})$ . Such commutants $mod\mathcal{J}$ can be associated with a smooth compact manifold using $n$-tuples of multiplication operators arising from nice embeddings in $\mathbb{R}^n$ . We discussed certain results about the $K$-theory and structure of these algebras. Returning to $k_{\mathcal{J}}(\tau)$, there is a noncommutative analogy with condenser capacity in nonlinear potential theory and a noncommutative variational problem associated with the condenser. For background material on commutants mod and quasicentral modulus see [1]. About commutants mod associated with compact smooth manifolds see [2]. The heuristics of the analogy with condenser capacity in nonlinear potential theory are in [3], while the technical development of this idea and generalizations of some of the results to a semifinite factors context (i.e beyond the type$I$ case of $\mathcal{B}(\mathcal{H})$ are in [4] . It is natural to wonder how far the noncommutative analogy with capacity in nonlinear potential theory may go.

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## Elements of classifying $C^*$-algebras

James Gabe
(joint work with José Carrión, Chris Schafhauser, Aaron Tikuisis, and Stuart White)

The Elliott programme of classifying nuclear $C^*$-algebras has been a hugely successful endeavour over the last 30 years. After the work of many hands (a very incomplete list includes Kirchberg [10], Phillips [11], Elliott, Gong, Lin, Niu [7], [9], Tikuisis, White, and Winter [14]) the classification of simple nuclear $C^*$-algebras is complete (modulo settling the UCT problem) with the following capstone result.

**Theorem 1** (Many hands)**.** *Let $A$ and $B$ be separable nuclear simple unital $\mathcal{Z}$-stable $C^*$-algebras satisfying the UCT. Then $A \cong B$ if and only if they have the same $K$-theory and traces.*

$\mathcal{Z}$-stability in the theorem is a regularity property resembling the McDuff property for $\mathrm{II}_1$-factors, meaning that $A \cong A \otimes \mathcal{Z}$ where $\mathcal{Z}$ is the Jiang–Su algebra. As predicted by the Toms–Winter conjecture, $\mathcal{Z}$-stability has several striking equivalent formulations (for separable nuclear simple $C^*$-algebras) such as having finite nuclear dimension [4], or having strict comparison and Uniform Property $\Gamma$ [3].

In collaboration with Carrión, Schafhauser, Tikuisis and White [1], we have come up with a completely new approach to proving this theorem. The purpose of this talk is to give an idea of some elements of this new proof (while sweeping some of the crucial technicalities under the rug).

We will cheat a bit and restrict to the case where the $C^*$-algebras have a *unique trace*. This eliminates some of the difficulties which were overcome in [4] and [2]. Recall that a trace $\tau_A$ on a $C^*$-algebra $A$ induces a homomorphism $(\tau_A)_*\colon K_0(A) \to \mathbb{R}$.

We assume from now on that $A$ and $B$ are as in Theorem 1, have a unique trace, say $\tau_A$ and $\tau_B$, and that there is a given isomorphism $\alpha_*\colon K_*(A) \xrightarrow{\cong} K_*(B)$ such that $\alpha_0([1_A]) = [1_B]$ and $(\tau_B)_* \circ \alpha_0 = (\tau_A)_*$. The goal is to show (or at least give a rough idea) that $A \cong B$. Let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ and define

$$B_\omega = \frac{\{(b_k)_k \in \prod_{\mathbb{N}} B : \sup_k \|b_k\| < \infty\}}{\{(b_k)_k : \lim_{k\to\omega} \|b_k\| = 0\}}, \quad B^\omega = \frac{\{(b_k)_k \in \prod_{\mathbb{N}} B : \sup_k \|b_k\| < \infty\}}{\{(b_k)_k : \lim_{k\to\omega} \tau_B(b_k^* b_k) = 0\}}.$$

As both the induced tracial von Neumann algebras $\pi_{\tau_A}(A)''$ and $\pi_{\tau_B}(B)''$ are isomorphic to the hyperfinite $\mathrm{II}_1$-factor $\mathcal{R}$ by Connes' classification theorem [5], it follows that $B^\omega \cong \mathcal{R}^\omega$ and that there is a unique unital $*$-homomorphism $\theta\colon A \to B^\omega$ up to unitary equivalence.

Let $q_B\colon B_\omega \to B^\omega$ denote the obvious surjection. The next step in the proof is to lift $\theta$ to a unital $*$-homomorphism $\phi\colon A \to B_\omega$, i.e. so that $\theta = q_B \circ \phi$. For this we will use the following result which was essentially proved by Schafhauser in [12] and [13] (using a Weyl–von Neumann type absorption theorem by Elliott–Kucerovsky [8]). It shows that it suffices to lift $\theta$ in $KK$-theory.

**Theorem 2** (Schafhauser)**.** *Suppose there is an element $\kappa \in KK(A, B_\omega)$ such that $(q_B)_*(\kappa) = [\theta]_{KK} \in KK(A, B^\omega)$ and $\kappa_0([1_A]) = [1_{B_\omega}]$. Then there exists a unital $*$-homomorphism $\phi\colon A \to B_\omega$ such that $q_B \circ \phi = \theta$ and $[\phi]_{KK} = \kappa$.*

An element $\kappa \in KK(A, B_\omega)$ with the given property will be produced by using the UCT (twice!). Recall that $A$ satisfying the UCT means (by definition) that for any $C^*$-algebra $D$ there is a short exact sequence

$$0 \to \mathrm{Ext}(K_*(A), K_{1-*}(D)) \to KK(A, D) \to \mathrm{Hom}(K_*(A), K_*(D)) \to 0.$$

It remains a major open problem (the UCT problem) whether every separable nuclear $C^*$-algebra satisfies the UCT. Let $\iota\colon B \to B_\omega$ be the canonical embedding. We first apply the UCT with $D = B_\omega$ to lift $\iota_* \circ \alpha_* \in \mathrm{Hom}(K_*(A), K_*(B_\omega))$ to an element $\kappa \in KK(A, B_\omega)$. As $\alpha_0([1_A]) = [1_B]$ it follows that $\kappa_0([1_A]) = [1_{B_\omega}]$.

Secondly, we use the UCT with $D = B^\omega \cong \mathcal{R}^\omega$. As this is a $\mathrm{II}_1$-factor we have $K_0(B^\omega) = \mathbb{R}$ and $K_1(B^\omega) = 0$. As $\mathbb{R}$ is divisible this implies that the Ext-group vanishes in the UCT short exact sequence, and thus $KK(A, B^\omega) \cong \mathrm{Hom}(K_0(A), \mathbb{R})$. The element $[\theta]_{KK}$ is canonically identified with $(\tau_A)_*$ via this isomorphism, and similarly $(q_B \circ \iota_B)_0 = (\tau_B)_*\colon K_0(B) \to \mathbb{R}$. Hence (with obvious abuse of notation)

$$[\theta]_{KK} \cong (\tau_A)_* = (\tau_B)_* \circ \alpha_0 = (q_B)_0 \circ (\iota_0 \circ \alpha_0) \cong (q_B)_*(\kappa).$$

By Schafhauser's theorem from above, there is a $*$-homomorphism $\phi\colon A \to B_\omega$ lifting $\theta$ such that $[\phi]_{KK} = \kappa$. In particular, $\phi_* = \iota_* \circ \alpha_*$.

Suppose now that $\psi\colon A \to B_\omega$ is a unital $*$-homomorphism such that $[\psi]_{KK} = \kappa = [\phi]_{KK}$ and $q_B \circ \psi = \theta = q_B \circ \phi$. By the Cuntz pair picture of $KK$-theory, we obtain an element $[\phi, \psi]_{KK} \in KK(A, J_B)$ where $J_B = \ker q_B$. The proof of the following uses $\mathcal{Z}$-stability and a theorem of Dadarlat and Eilers [6].

**Theorem 3** ($\mathcal{Z}$-stable $KK$-uniqueness)**.** *If $[\phi, \psi]_{KK} = 0$ then $\phi$ and $\psi$ are unitarily equivalent. More generally:*

*Let $C, D$ and $E$ be separable $C^*$-algebras such that $E$ contains $D$ as a two-sided closed ideal, and let $\Phi, \Psi\colon C \to E$ be $*$-homomorphisms agreeing modulo $D$ (so they induce $[\Phi, \Psi]_{KK} \in KK(C, D)$). Assume that $\Phi$ and $\Psi$ are absorbing relative to $D$ and that $E$ is $\mathcal{Z}$-stable. Then $[\Phi, \Psi]_{KK} = 0$ if and only if $\Phi$ and $\Psi$ are asymptotically unitarily equivalent via a path of unitaries in $\tilde{D}$.*

**Question 1.** Is $\mathcal{Z}$-stability necessary in the above theorem?

This theorem can (after making some adjustments to $\phi$ and applying the UCT again) be used to show that $\phi$ is unitarily equivalent to all possible reindexations of itself (again up to some cheating). This implies that there is a $*$-homomorphism $\rho\colon A \to B$ such that $\phi$ and $\iota_B \circ \rho$ are unitarily equivalent, and hence $\rho_* = \alpha_*$. After sweeping some more details under the rug, we use an intertwining argument to show that $\rho$ is approximately unitarily equivalent to an isomorphism $A \cong B$, as desired.

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## The dynamical Kirchberg–Phillips theorem

Gábor Szabó
(joint work with James Gabe)

In this talk I reported on the main outcomes of the recent works [6, 7] about a dynamical version of the celebrated Kirchberg–Phillips theorem. The basic motivation for this line of research is to try to come up with a satisfactory dynamical counterpart to the Elliott program. The latter, which aims to explore the fine structure of simple nuclear C*-algebras, is now nearly completed due to the (frequently labeled) "many hands theorem" classifying all simple nuclear Jiang–Su stable C*-algebras by $K$-theory and traces. One of the big early milestones in the Elliott program was the traceless subcase of this classification program, namely the classification of separable simple nuclear purely infinite C*-algebras (known as Kirchberg algebras) due to Kirchberg and Phillips [12, 15]. We recall:

**Theorem** (Kirchberg–Phillips)**.** *Let $A$ and $B$ be two stable Kirchberg* C*-*algebras. Then:*

*(1) Every invertible element $x \in KK(A,B)^{-1}$ lifts to an isomorphism $A \cong B$.*
*(2) If $A$ and $B$ both satisfy the UCT, then $A \cong B$ if and only if $K_*(A) \cong K_*(B)$ as pairs of countable abelian groups.*

Stated this way, one can split the work towards the full conclusion of this theorem into two independent parts. The first part about the isomorphism theorem via $KK$-theory can be labeled the *analytic* part of the work, which involves all the heavy lifting related to analytic techniques in C*-algebra theory, culminating in the abstract construction of the desired isomorphisms. The second part, which here is the prior work of Rosenberg–Schochet in disguise [17], can be labeled the *algebraic* part of the work, which involves techniques from algebraic topology to understand $KK$-theory via the (much easier to compute) ordinary $K$-theory groups.

Our work aims to give a satisfactory generalization of the analytic aspect above for group actions on Kirchberg algebras. For the rest of the abstract we fix a second-countable, locally compact group $G$. We shall introduce the two important dynamical properties that we impose on the actions we classify.

**Definition.** Given an action $\alpha : G \curvearrowright A$ on a C*-algebra, we form the action $\bar{\alpha} : G \curvearrowright \mathcal{C}_c(G,A)$ via $\bar{\alpha}_g(f)(h) = \alpha_g(f(g^{-1}h))$. We equip $\mathcal{C}_c(G,A)$ with the obvious $A$-bimodule structure and the standard $A$-valued inner product $\langle \cdot \mid \cdot \rangle$ inducing the norm $\| \cdot \|_2$. We say that $\alpha$ is amenable, if there exists a net of unit vectors $\zeta_i \in \mathcal{C}_c(G,A)$ satisfying

$$\langle \zeta_i \mid \zeta_i \rangle a \to a, \quad \|a\zeta_i - \zeta_i a\|_2 \to 0, \quad \max_{g \in K} \|\zeta_i - \bar{\alpha}_g(\zeta_i)\|_2 \to 0$$

for all $a \in A$ and every compact set $K \subseteq G$.

In this generality, the above notion of amenability for actions on C*-algebras has only recently been studied and shown to be equivalent to various other notions of amenability [3, 2, 13], which goes back to earlier such notions for actions on von Neumann algebras [1]. Clearly every action of an amenable group is amenable on any C*-algebra, but for general $G$, this is a genuine property to be studied for its own sake.

**Definition.** Let $\alpha : G \curvearrowright A$ be an action on a separable C*-algebra. We equip the Hilbert space $L^2(G)$ with the unitary $G$-action given by the left-regular representation of $G$. We say $\alpha$ is isometrically shift-absorbing, if there exists a linear equivariant map

$$s : L^2(G) \to F_\infty(A) = (A_\infty \cap A')/(A_\infty \cap A^\perp)$$

satisfying the equation $s(\xi)^* s(\eta) = \langle \xi \mid \eta \rangle \cdot \mathbf{1}$ for all $\xi, \eta \in L^2(G)$. (Note that $F_\infty(A)$ carries the algebraic $G$-action coming from applying $\alpha$ componentwise to representing sequences in $A$.)

This property acts as a dynamical non-triviality property that also encompasses pure infiniteness in a certain way. For instance, it is easy to see under the assumption of the above property for $G \neq \{1\}$ that $A$ must tensorially absorb the Cuntz

algebra $\mathcal{O}_\infty$, since the images of two orthogonal unit vectors under $s$ yields two isometries in $F_\infty(A)$ with orthogonal range projections. Although isometric shift-absorption is (at least a priori) a rather technical property in full generality, there are some important subcases where it restricts to something familiar:

**Proposition** ([9]). *Suppose $G$ is discrete and $\alpha : G \curvearrowright A$ is an action on a Kirchberg algebra. Then $\alpha$ is isometrically shift-absorbing if and only if $\alpha$ is (pointwise) outer.*

As a culmination of various sources in the literature [14, 16, 13, 7], it is possible to come to the conclusion that there are many $G$-actions on Kirchberg algebras that are both amenable and isometrically shift-absorbing:

**Theorem.** *Let $\alpha : G \curvearrowright A$ be an amenable action on a separable nuclear* C*-*algebra. Then there exists a stable Kirchberg algebra $B$ and an amenable, isometrically shift-absorbing action $\beta : G \curvearrowright B$ such that $(A, \alpha) \sim_{KK^G} (B, \beta)$. If $G$ satisfies the Haagerup property, then the conclusion is true even without the assumption that $\alpha$ is amenable.*

With this in mind, our main result is the classification of such actions via equivariant Kasparov theory:

**Theorem.** *Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be actions on stable Kirchberg algebras. Suppose that $\alpha$ and $\beta$ are amenable and isometrically shift-absorbing. Then every invertible element $x \in KK^G(\alpha, \beta)^{-1}$ lifts to a cocycle conjugacy between $(A, \alpha)$ and $(B, \beta)$.*

In my talk I supplied a few more conceptual comments on aspects of the methodology, which involves the idea of existence and uniqueness theorems for so-called proper cocycle embeddings between such actions. This relies on the categorical framework for C*-dynamics developed in [18]. In order to access equivariant Kasparov theory for this purpose, we rely crucially on Thomsen's dynamical version of the Cuntz picture from [19]. A key ingredient towards our main result is a stable uniqueness theorem [6] in the spirit of Lin and Dadarlat–Eilers [4, 5].

At the end of my talk, I explained a few consequences of our main result. For example, the recent work of Meyer [16] shows that it recovers and generalizes the recent work of Izumi–Matui [10, 11] for outer actions of poly-$\mathbb{Z}$ groups, and in fact proves Izumi's conjecture formulated in [8] for all torsion-free amenable groups.

Regarding future work, there remain a few challenges and open problems. The most obvious one is to complete the algebraic part of the classification problem, i.e., to find reasonably computable $K$-theory invariants that uniquely determine the $KK^G$-class of an action under the assumption that an action belongs to the equivariant bootstrap class; see [16]. Another open problem would be to determine whether isometric shift-absorption is actually a more familiar property in disguise. I am tempted to ask the following naive question: Suppose $\alpha : G \curvearrowright A$ is an amenable action on a Kirchberg algebra such that for every closed subgroup $H \subseteq G$, the crossed product $A \rtimes H$ is a Kirchberg algebra. Does it follow that $\alpha$ is isometrically shift-absorbing?

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## Quantum graphs and colorings

Priyanga Ganesan
(joint work with Michael Brannan and Samuel Harris)

Quantum information theory has attracted a lot of attention from mathematicians in recent years due to its intriguing connections to operator algebras. In particular, techniques from quantum information theory have been used to show a refutation of the long standing Connes embedding problem in operator algebra [1] recently. Non-local games (a cooperative game played by two spatially separated non-communicating players) and their quantum and classical strategies are closely

related to the negative solution of the Connes embedding problem. An operator algebra friendly proof of the Connes embedding problem may be obtained by finding examples of nonlocal games that exhibit a gap between $qa-$ and $qc-$ correlation sets. With this motivation, it is useful to study the non-local graph coloring game and investigate its possible generalizations with the hope of finding graphs that exhibit a separation between $qc-$chromatic number and $qa-$ chromatic number.

The non-local graph coloring game is played by two separated players who cooperatively try to convince an interrogator with certainty that they have a coloring for the given graph. During each round of the game, the interrogator randomly selects two (possibly same) vertices from the graph and sends one to each of the players. The players then individually respond with a color assignment for their vertex. They win the round if their responses jointly satisfy certain conditions (implying a coloring for the given graph). The players cannot communicate with each other directly during the round, but can share an entangled state and use different strategies $(loc, q, qa, qs, qc)$ to correlate their answers. The least number of colors that the players use to win the game is called the $t$-chromatic number $(\chi_t)$ of that graph $(t \in \{loc, q, qa, qs, qc\})$. There are known examples of graphs [2, 3] whose quantum chromatic number is strictly smaller than its classical chromatic number, thus exhibiting the power of quantum entanglement. In [4], we extended this idea of quantum coloring to the setting of *quantum graphs*.

Quantum graphs are an operator space generalization of classical graphs that have emerged in different disguises in operator algebras, non-commutative topology and quantum information theory. Mathematically, quantum graphs can be described as operator spaces satisfying a certain bimodule property [5]. Alternately, they may be viewed as a finite dimensional C*-algebra equipped with additional structure induced by a *quantum adjacency operator* [6, 7] that mimics the structure of a classical adjacency matrix. Quantum graphs also serve as a quantum analogue of the confusability graph of classical channels [9], and hence play an important role in zero-error quantum communication. Motivated by non-local games and coloring problems, we introduce a non-local coloring game with quantum inputs and classical outputs that generalizes the classical graph coloring game [2] and develop a double quantization of the chromatic number, namely the quantum chromatic number of a quantum graph. We prove that the quantum chromatic number of a quantum graph is always finite, while its classical chromatic number may not be. We also obtain a combinatorial characterization of quantum coloring using the winning strategies of the nonlocal quantum-to-classical graph coloring game.

A natural question to ask is how can we estimate the quantum chromatic numbers? In general, computing the chromatic number of a graph is an NP-hard problem. So, inequalities involving the eigenvalues of the adjacency matrix are often used in estimating the chromatic number. In [7], Elphick & Wocjan prove many spectral lower bounds on the quantum chromatic number of a classical graph. We extend their results to quantum graphs in [10] using the eigenvalues of a "quantum adjacency operator" and obtain five lower bounds for the quantum chromatic

number of quantum graphs. In particular, we prove a quantum analogue of the Hoffman's bound and demonstrate the tightness of the bound in the case of complete quantum graphs.

A larger goal is to find a counterexample to the Connes-Kirchberg conjecture by showing a separation between $\chi_{qc}$ and $\chi_{qa}$ chromatic numbers. We hope to find examples of quantum graphs with the analogous property. Alternately, finding spectral bounds that separate $\chi_{qc}$ and $\chi_{qa}$ would also be very interesting. It might be easier to do this with quantum graphs than classical graphs since the former provides a larger class of examples and allows the use of operator algebraic techniques.

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## Connes's Embedding Problem and Tsirelson's Conjecture

Chris Schafhauser

The recent result of Ji, Natarajan, Vidick, Wright, and Yuen that MIP$^*$=RE implies a negative solution to Tsirelson's conjecture, and hence to the Connes Embedding Problem. In fact, the proof of MIP$^*$=RE provides a strong failure of Tsirelson's conjecture giving a synchronous correlation realized in the quantum commuting model but not in the quantum model. This allows for a simplified proof that Connes Embedding Problem is false which avoids Kirchberg's tensor product conjecture. The proof is based on ideas of Dykema-Paulsen and Kim-Paulsen-S.

## The Williams problem through the lens of Cuntz-Krieger algebras

Søren Eilers
(joint work with Toke M. Carlsen and Adam Dor-On)

By steady progress spanning more than four decades, it is now known that Cuntz-Krieger algebras $\mathcal{O}_A$ [7] capture essential notions of sameness for the associated two-sided shifts of finite types $\mathsf{X}_A$, encompassing Williams' notions of *(strong) shift equivalence* [18] as well as *flow equivalence* (formally defined in [2] but traceable to [14]). For this purpose, as realized in early work by Bratteli and Kishimoto [3] on the one hand, and by Rørdam [17] on the other, it is necessary to consider these $C^*$-algebras along with further structure, involving the canonical gauge action $\gamma : \mathbb{T} \to \mathrm{Aut}(\mathcal{O}_A)$ and the canonical diagonal subalgebra $\mathcal{D}_A \subseteq \mathcal{O}_A$. We introduce the following convenient notation:

**Definition 1.** For $\mathsf{y}, \mathsf{z} \in \{0, 1\}$ we say that two essential matrices $A$ and $B$ are equivalent up to $\mathsf{yz}$-equivalence, and write $(A, B) \in \overline{\mathsf{yz}}$, when there is a $*$-isomorphism

$$\varphi : \mathcal{O}_A \otimes \mathbb{K} \to \mathcal{O}_B \otimes \mathbb{K}$$

which additionally satisfies

$$\varphi \circ (\gamma_z \otimes \mathrm{id}) = (\gamma_z \otimes \mathrm{id}) \otimes \varphi$$

for all $z \in \mathbb{T}$ when $\mathsf{y} = 1$, and

$$\varphi(\mathcal{D}_A \otimes \ c_0) = \mathcal{O}_B \otimes c_0$$

when $\mathsf{z} = 1$.

We will consider our relations as subsets of $\mathcal{M} \times \mathcal{M}$ with $\mathcal{M}$ denoting all essential[1] square matrices with nonnegative integer entries, sometimes specializing to $\mathcal{P} \times \mathcal{P}$ with $\mathcal{P}$ denoting those matrices that are *primitive* in the sense that some power of it has only positive entries.

Using $\overline{\mathsf{SSE}}$,$\overline{\mathsf{SE}}$,$\overline{\mathsf{FE}}$ to denote the relations induced by the above-mentioned notions in symbolic dynamics of the shift spaces given by such matrices, the state of the art is as indicated in Figure 1, where solid arrows indicate containments known for all SFTs (i.e., in $\mathcal{M} \times \mathcal{M}$) and dashed ones indicate relations only known to hold in the case where the shift spaces and Cuntz-Krieger algebras are given by primitive[2] matrices, rendering the dynamics mixing and the $C^*$-algebras (gauge) simple. We know of no counterexamples for the dashed arrows to hold in general, but it is known, notably due to [12], that no further arrows may be reversed.

The results summarized above show that *Williams' problem* concerning the very elusive difference between $\overline{\mathsf{SSE}}$ and $\overline{\mathsf{SE}}$ can be recast as a $C^*$-algebraic problem in the primitive case, and it is a pressing question to decide if that is possible in general. The talk discussed recent attempts, thus far unsuccessful, to extend the remaining relations to general shift spaces, and to relate the two open problems. It should be noted that primitivity enters into the proof of the two inclusions

[1] No zero rows or columns.

[2] In fact, one may show that irreducibility suffices.

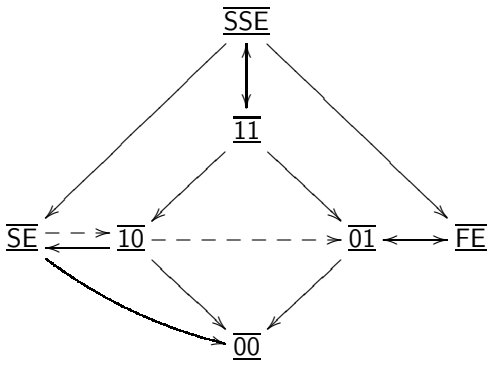


FIGURE 1. Rigidity results of [3, 5, 6, 13]

not known in general in very different ways. For $\overline{\mathsf{SE}} \cap (\mathcal{P} \times \mathcal{P}) \subseteq \overline{\mathsf{01}}$, Bratteli and Kishimoto argue via the Rokhlin property on a certain $AF$-algebra which is simple and has a unique trace up to scaling only in this case, and for $\overline{\mathsf{10}} \cap (\mathcal{P} \times \mathcal{P}) \subseteq \overline{\mathsf{01}}$ one needs to involve the *Bowen-Franks invariant* which is only complete (by a result of Franks [9]) there.

Whereas we have little to say about how the relation $\overline{\mathsf{10}} \subseteq \overline{\mathsf{01}}$ might be seen to hold in general, or even how to show it in the primitive case without a detour into symbolic dynamics, the recent paper [4] addresses $\overline{\mathsf{SE}} \subseteq \overline{\mathsf{01}}$. A very promising approach, initiated in [15] and further developed in [10], involves the Cuntz-Pimnser construction [16] which allows $\mathcal{O}_A$ to be defined as the $C^*$-algebra associated to a certain $C^*$-correspondence $X(A)$. To show that $\overline{\mathsf{SE}} \subseteq \overline{\mathsf{00}}$ in general, Kakariadis and Katsoulis consider

$$X(A) \overset{SE}{\sim} X(B) \overset{(*)}{\Longrightarrow} X(A)_\infty \overset{SE}{\sim} X(B)_\infty \Longrightarrow X(A)_\infty \overset{\text{Morita}}{\sim} X(B)_\infty$$

where "$\overset{SE}{\sim}$" indicates a natural extension of Williams' notion to $C^*$-correspondences, and $X(-)_\infty$ is the *Pimsner dilation*, a Hilbert $C^*$-bimodule which defines the same $C^*$-algebra as $X(-)$. In joint work with Carlsen and Dor-On we were able to show that in fact $(*)$ implies even the desired containment $\overline{\mathsf{SE}} \subseteq \overline{\mathsf{10}}$, but in the process of proving our results, we uncovered a very subtle error in [10]. The claim has now been retracted, cf. [11].

In [4], we show that $(*)$ would follow if one assumes that the given shift equivalence is represented on a Hilbert space in a precise sense we denote *representable* shift equivalence ($\overline{\mathsf{RSE}}$). It is easy to see that this follows from another, purely combinatorial and seemingly very modest, strengthening of shift equivalence that we call *compatible* shift equivalence ($\overline{\mathsf{CSE}}$), but to our surprise we could show the third inclusion in

$$\overline{\mathsf{SSE}} \subseteq \overline{\mathsf{CSE}} \subseteq \overline{\mathsf{RSE}} \subseteq \overline{\mathsf{11}} \subseteq \overline{\mathsf{SSE}},$$

which entails that all these notions coincide, and offer alternative descriptions of Williams' most restrictive notion of strong shift equivalence.

The realization that $\overline{\mathsf{CSE}} = \overline{\mathsf{SSE}}$ appears to be new, and we are currently studying it as a way to attack well known concrete open problems in symbolic dynamics, but the result implies that such an approach to establishing $(*)$ is not possible. Until this important open problem has been resolved, one can at least see that $\overline{\mathsf{SE}} \subseteq \overline{\underline{00}}$ by combining [8] with [1].

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## Consequences of the recent resolution of the Peterson-Thom conjecture

Ben Hayes

(joint work with David Jekel and Srivatsav Kunnawalkam Elayavalli)

Given a finite von Neumann algebra $M$, we say that $M$ has *unique maximal amenable extensions* if for every diffuse, amenable subalgebra $Q \leq M$ there is a *unique* maximal amenable $P \leq M$ with $Q \subseteq P$. In [14], Peterson-Thom conjectured that any diffuse, amenable subalgebra of a free group factor has unique maximal amenable extensions, a conjecture that came to be known the *Peterson-Thom conjecture*. This conjecture was motivated by Peterson-Thom work results on $L^2$-Betti numbers as well as prior work of previous work of Ozawa-Popa, Peterson, and Jung [11, 13, 10].

This formulation in terms of unique maximal amenable extensions does not lend itself well to testing examples, but another formulation in terms of absorption of amenable subalgebras does. We say that a diffuse $P \leq M$ has the *absorbing amenability property* if whenever $Q \leq M$ is amenable, and $P \cap Q$ is diffuse, then $Q \leq P$. The Peterson-Thom conjecture is then equivalent to the statement that every maximal amenable subalgebra of a free group factor has the absorbing amenability property. This reformulation inspired many papers establishing the absorbing amenability property (and other absorption properties such as Gamma stability) in many examples (see [7, 22, 2, 12, 6]).

In [5], the first-named author formulated a conjecture on random matrices which he showed to imply the Peterson-Thom conjecture. Several works in random matrices made progress towards this random matrix conjecture. A recent breakthrough of Belinschi and Capitaine [1] proves this random matrix conjecture, thus resolving the Peterson-Thom conjecture in the positive. The aim of this talk was to talk about applications of this resolution to general structure of free group factors.

The reduction of the Peterson-Thom conjecture to a random matrix problem uses Voiculescu's microstates free entropy dimension theory, namely the 1-bounded entropy implicitly defined by Jung [10] and explicitly by the author in [4]. Voiculescu's microstates free entropy dimension was a crucial tool in Voiculescu's breakthrough result on absence of Cartan subalgebras [21]. This was itself generalized for algebras much more general than free group factors by Ozawa-Popa [11] using Popa's deformation/rigidity theory. For our first application, we use several weakenings of the normalizer. We recall some of these weakenings here. The first is the 1-sided quasi-normalizer defined in [9, 15, 17] (building off of ideas in [16]),

$$q^1\mathcal{N}_M(N) = \left\{ x \in M : \text{ there exists } x_1, \cdots, x_n \in M \text{ so that } xN \subseteq \sum_{j=1}^{n} Nx_j \right\},$$

we also consider the $wq$-normalizer, defined in [18, 19, 8, 3],

$$\mathcal{N}_M^{wq}(N) = \{u \in \mathcal{U}(M) : uNu^* \cap N \text{ is diffuse}\}$$

and its cousin the very weak quasi-normalizer

$$\mathcal{N}_M^{vwq}(N) = \{u \in \mathcal{U}(M) : \text{there exists } v \in \mathcal{U}(M) \text{ so that } uNv \cap N \text{ is diffuse.}\}.$$

Since $uNv \cap N$ is not an algebra, the phrase "$uNv \cap N$ is diffuse" should be interpreted as saying that there is a sequence of unitaries $v_n \in uNv \cap N$ which tend to zero weakly. We also consider the *weak interwining space* $wI_M(Q, Q)$ due to Popa [20]. As shown in [4, Proposition 3.2], all of these are contained in the anti-coarse subspace,

$$\mathcal{H}_{anti-c}(N \leq M) = \bigcap_{T \in \mathrm{Hom}_{N-N}(L^2(M), L^2(N) \otimes L^2(N))} \ker(T).$$

Here $\mathrm{Hom}_{N-N}(L^2(M), L^2(N) \otimes L^2(N)$ is the space of bounded, linear, $N$-$N$ bimodular maps $T\colon L^2(M) \to L^2(N) \otimes L^2(N)$.

**Theorem 1.** *Let $t > 1$ and let $Q \leq L(\mathbb{F}_t)$ be a diffuse, amenable subalgebra. Then $W^*(\mathcal{H}_{anti-c}(Q \leq L(\mathbb{F}_t)))$ remains amenable. In particular, for any*

$$X \subseteq q^1\mathcal{N}_{L(\mathbb{F}_t)}(Q) \cup \mathcal{N}^{wq}_{L(\mathbb{F}_t)}(Q) \cup wI_{L(\mathbb{F}_t)}(Q, Q) \cup \mathcal{N}_M^{vwq}(Q)$$

*we have that $W^*(X)$ is amenable.*

We can list a generalization of this in the ultraproduct setting.

**Theorem 2.** *Let $t \in (1, +\infty)$ and let $\omega$ be a free ultrafilter on $\mathbb{N}$. Suppose that $Q \leq L(\mathbb{F}_t)^\omega$ is a diffuse, amenable subalgebra. Suppose we are given Neumann subalgebras $Q_\alpha$ defined for ordinals $\alpha$ satisfying the following:*

- $Q_0 = Q$,
- *if $\alpha$ is a successor ordinal then $Q_\alpha = W^*(X_\alpha)$ where*

$$X_\alpha \subseteq q^1\mathcal{N}_{L(\mathbb{F}_t)^\omega}(Q_{\alpha-1}) \cup \mathcal{N}^{wq}_{L(\mathbb{F}_t)^\omega}(Q_{\alpha-1}) \cup wI_{L(\mathbb{F}_t)^\omega}(Q_{\alpha-1}, Q_{\alpha-1}) \cup \mathcal{N}_M^{vwq}(Q_{\alpha-1})$$

- *if $\alpha$ is a limit ordinal, then $Q_\alpha = \overline{\bigcup_{\beta<\alpha} Q_\beta}^{SOT}$.*

*Then for any ordinal $\alpha$ we have $Q_\alpha \cap L(\mathbb{F}_t)$ is amenable. In particular, $L(\mathbb{F}_t)$ has the following Gamma stability property: if $Q \leq L(\mathbb{F}_t)^\omega$ is diffuse, and if $Q' \cap L(\mathbb{F}_t)^\omega$ is diffuse, then $Q \cap L(\mathbb{F}_t)$ is amenable.*

This recovers the previous Gamma stability results from [7].

The case of the weak intertwining space itself leads to a dichotomy in terms of Popa's deformation/ridigity theory for maximal amenable subalgebras of free group factors. Recall that if $M$ is finite von Neumann algebra, and $P, Q \leq M$ we say that *a corner of $Q$ embeds into $P$ inside of $M$* and write $Q \preceq P$ if there are nonzero projections $f \in Q$, $e \in P$, a unital $*$-homomorphism $\Theta\colon fQf \to ePe$ and a nonzero partial isometry $v \in M$ so that:

- $xv = v\Theta(x)$ for all $x \in fQf$,
- $vv^* \in (fQf)' \cap fMf$,
- $v^*v \in \Theta(fqf)' \cap eMe$.

This should be thought as $Q, P$ being "unitarily conjugate up to cutting by a corner". This definition is due to Popa [18].

**Theorem 3.** *Fix $t > 1$, and let $Q, P$ be maximal amenable subalgebras of $L(\mathbb{F}_t)$. Then exactly one of the following occurs:*

*(1) either there are nonzero projections $e \in Q, f \in P$ and a unitary $u \in L(\mathbb{F}_t)$ so that $u^*(ePe)u = fQf$, or*
*(2) for any diffuse $Q_0 \leq Q$ we have that no corner of $Q_0$ embeds into $Q$ inside of $L(\mathbb{F}_t)$ (in the sense of Popa).*

Our next application is a positive resolution of the *coarseness conjecture* independently due to the first-named author and Popa [4, Conjecture 1.12] and [20, Conjecture 5.2]. If $M$ is a von Neumann algebra, an $M$-$M$ bimodule $\mathcal{H}$ is a Hilbert space with normal left and right actions of $M$ which commute. We use ${}_M\mathcal{H}_M$ to mean that $\mathcal{H}$ is an $M$-$M$ bimodule. If $\mathcal{H}, \mathcal{K}$ are $M$-$M$ bimodules, we use ${}_M\mathcal{H}_M \leq_M \mathcal{K}_M$ to mean that there is $M$-bimodular unitary map $\mathcal{H} \to \mathcal{K}$. If $\mathcal{H}_1 \leq \mathcal{H}_2$ are Hilbert spaces, we use $\mathcal{H}_2 \ominus \mathcal{H}_1$ for $\mathcal{H}_1^{\perp} \cap \mathcal{H}_2$.

**Theorem 4.** *Let $t > 1$. For any maximal amenable subalgebra $P \leq L(\mathbb{F}_t)$ we have*

$$ {}_P[L^2(L(\mathbb{F}_t)) \ominus L^2(P)]_P \leq (L^2(P) \otimes L^2(P))^{\oplus \infty}. $$

In [20], this property is referred to as *coarseness* of the inclusion $P \leq L(\mathbb{F}_t)$. As explained in the introduction to [20], we may think of coarseness as the "most random" position a subalgebra can be in. It is of interest to specialize theorem 4 to the case where $P$ is abelian.

Suppose $(M, \tau)$ is a tracial von Neumann algebra, and $A \leq M$ is a maximal abelian $*$-subalgebra. Write $A = L^\infty(X, \mu)$ for some compact Hausdorff space $X$ and some Borel probability measure $\mu$ on $X$. The representation

$$ \pi \colon C(X) \otimes C(X) \to B(L^2(M) \ominus L^2(A)) $$

given by

$$ \pi(f \otimes g)\xi = f\xi g, $$

gives rise to a spectral measure $E$ on $X \times X$ whose marginals are Radon-Nikodym equivalent to $\mu$. We say that $\nu \in \mathrm{Prob}(X \times X)$ is a *left/right measure* of $A \leq M$ if it is Radon-Nikodym equivalent to $E$. One often abuses terminology and refers to *the* left/right measure to refer to any element of this equivalence class of measures.

**Theorem 5.** *Let $M = L(\mathbb{F}_t)$ for $t > 1$. Suppose that $A \leq M$ is abelian and a maximal amenable subalgebra of $M$. Write $A = L^\infty(X, \mu)$ for some compact metrizable space $X$ and some Borel probability measure on $X$. Then the left/right measure of $A \leq M$ is absolutely continuous with respect to $\mu \otimes \mu$.*

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## Conjugacy of local homeomorphisms via groupoids and C*-algebras

Becky Armstrong

(joint work with Kevin Aguyar Brix, Toke Meier Carlsen, and Søren Eilers)

A *(rank-one) Deaconu–Renault system* consists of a locally compact Hausdorff space $\mathsf{X}$ and a partially defined local homeomorphism $\sigma_{\mathsf{X}}\colon \mathrm{dom}(\sigma_{\mathsf{X}}) \to \mathrm{ran}(\sigma_{\mathsf{X}})$ between open subsets of $\mathsf{X}$. Each such system gives rise to an associated amenable locally compact Hausdorff étale groupoid $\mathcal{G}_{\mathsf{X}}$ (called a *Deaconu–Renault groupoid*) that is graded by the integers, along with an associated groupoid C*-algebra $\mathrm{C}^*(\mathcal{G}_{\mathsf{X}})$. Examples of such systems include homeomorphisms and covering maps on locally compact Hausdorff spaces, one-sided shifts of finite type, and one-sided shifts on the boundary-path spaces of directed graphs and topological graphs. The

class of C*-algebras associated to Deaconu–Renault systems is quite large, and includes C*-crossed products by actions of the integers, graph C*-algebras, and all Kirchberg algebras satisfying the universal coefficient theorem.

In 2008 Renault showed that every separable C*-algebra containing a Cartan subalgebra can be realised as a reduced twisted groupoid C*-algebra $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G};\mathcal{E})$ with Cartan subalgebra $\mathrm{C}_0(\mathcal{G})$, and conversely, that given a twist $\mathcal{E}$ over a second-countable locally compact Hausdorff étale groupoid $\mathcal{G}$, $\mathrm{C}_0(\mathcal{G})$ is a Cartan subalgebra of $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G};\mathcal{E})$ if and only if $\mathcal{G}$ is topologically principal (see [11, Theorems 4.2(ii), 5.2, and 5.9]). Although this result is extremely useful for recovering groupoid data from a large class of C*-algebras, it is not necessarily applicable in our setting, because a Deaconu–Renault groupoid $\mathcal{G}_{\mathsf{X}}$ is topologically principal if and only if the set of non-periodic points (with respect to $\sigma_{\mathsf{X}}$) is dense in $\mathsf{X}$, which is not a condition that we insist on in our setting. A natural question to ask instead in the setting of Deaconu–Renault groupoids is whether the Deaconu–Renault system can be recovered from its associated groupoid or C*-algebra, at least up to some type of equivalence.

With this goal in mind, in my talk I introduced the following notion of (topological) conjugacy of Deaconu–Renault systems. I then showed how the conjugacy class of a Deaconu–Renault system may be recovered from its associated groupoid or groupoid C*-algebra. (The details can be found in [1].)

**Definition** ([1, Definition 2.1])**.** Let $(\mathsf{X},\sigma_{\mathsf{X}})$ and $(\mathsf{Y},\sigma_{\mathsf{Y}})$ be Deaconu–Renault systems. We call a homeomorphism $h\colon \mathsf{X}\to\mathsf{Y}$ a *conjugacy* if $h(\sigma_{\mathsf{X}}(x)) = \sigma_{\mathsf{Y}}(h(x))$ for all $x\in\mathrm{dom}(\sigma_{\mathsf{X}})$, and $h^{-1}(\sigma_{\mathsf{Y}}(y)) = \sigma_{\mathsf{X}}(h^{-1}(y))$ for all $y\in\mathrm{dom}(\sigma_{\mathsf{Y}})$. We say that the systems $(\mathsf{X},\sigma_{\mathsf{X}})$ and $(\mathsf{Y},\sigma_{\mathsf{Y}})$ are *conjugate* if there exists a conjugacy $h\colon \mathsf{X}\to\mathsf{Y}$.

Suppose that $(\mathsf{X},\sigma_{\mathsf{X}})$ is a Deaconu–Renault system and that $\Gamma$ is a locally compact abelian group. Fix $f\in \mathrm{C}(\mathsf{X},\Gamma)$. Let $c_f\colon \mathcal{G}_{\mathsf{X}}\to\Gamma$ be the continuous 1-cocycle given by

$$c_f(x,m-n,y) \coloneqq \sum_{i=0}^{m-1} f\big(\sigma_{\mathsf{X}}^i(x)\big) - \sum_{j=0}^{n-1} f\big(\sigma_{\mathsf{X}}^j(y)\big),$$

for $(x,m-n,y)\in\mathcal{G}_{\mathsf{X}}$ satisfying $\sigma_{\mathsf{X}}^m(x) = \sigma_{\mathsf{X}}^n(y)$. Let $\gamma^{\mathsf{X},f}\colon \widehat{\Gamma}\curvearrowright \mathrm{C}^*(\mathcal{G}_{\mathsf{X}})$ be the *weighted action* satisfying

$$\gamma^{\mathsf{X},f}_{\chi}(\xi)(x,m-n,y) \coloneqq \chi(c_f(x,m-n,y))\,\xi(x,m-n,y),$$

for $\chi\in\widehat{\Gamma}$, $\xi\in\mathrm{C_c}(\mathcal{G}_{\mathsf{X}})$, and $(x,m-n,y)\in\mathcal{G}_{\mathsf{X}}$ satisfying $\sigma_{\mathsf{X}}^m(x)=\sigma_{\mathsf{X}}^n(y)$.

In my talk I presented the following simplified version of the main theorem that appears in our paper. (See [1, Theorem 3.1] for the complete statement of the theorem.)

**Theorem.** Let $(\mathsf{X}, \sigma_{\mathsf{X}})$ and $(\mathsf{Y}, \sigma_{\mathsf{Y}})$ be second-countable Deaconu–Renault systems. Let $\Gamma$ be any locally compact abelian group that is *separating* for $\mathsf{X}$ and $\mathsf{Y}$, in the sense of [1, Definition 3.5][1]. The following statements are equivalent.

(1) The systems $(\mathsf{X}, \sigma_{\mathsf{X}})$ and $(\mathsf{Y}, \sigma_{\mathsf{Y}})$ are conjugate.
(2) There exists a groupoid isomorphism $\psi\colon \mathcal{G}_{\mathsf{X}} \to \mathcal{G}_{\mathsf{Y}}$ satisfying the following three equivalent conditions:
   (i) there is a conjugacy $h\colon \mathsf{X} \to \mathsf{Y}$ such that $\psi(x,p,y) = (h(x),p,h(y))$ for all $(x,p,y) \in \mathcal{G}_{\mathsf{X}}$;
   (ii) $c_g \circ \psi = c_{g\circ\psi|_{\mathsf{X}}}$ for all $g \in \mathrm{C}(\mathsf{Y},\Gamma)$;
   (iii) there is a homeomorphism $h\colon \mathsf{X} \to \mathsf{Y}$ that satisfies $c_{g\circ h} = c_g \circ \psi$ for all $g \in \mathrm{C}(\mathsf{Y},\Gamma)$.
(3) There is a $*$-isomorphism $\varphi\colon \mathrm{C}^*(\mathcal{G}_{\mathsf{X}}) \to \mathrm{C}^*(\mathcal{G}_{\mathsf{Y}})$ satisfying the following two equivalent conditions:
   (i) $\varphi(\mathrm{C}_0(\mathsf{X})) = \mathrm{C}_0(\mathsf{Y})$, and there is a conjugacy $h\colon \mathsf{X} \to \mathsf{Y}$ such that $\varphi(f) = f \circ h^{-1}$ for all $f \in \mathrm{C}_0(\mathsf{X})$ and $\varphi \circ \gamma^{\mathsf{X},g\circ h}_{\chi} = \gamma^{\mathsf{Y},g}_{\chi} \circ \varphi$ for all $\chi \in \widehat{\Gamma}$ and $g \in \mathrm{C}(\mathsf{Y},\Gamma)$;
   (ii) there is a homeomorphism $h\colon \mathsf{X} \to \mathsf{Y}$ (which is not necessarily a conjugacy) such that $\varphi \circ \gamma^{\mathsf{X},g\circ h}_{\chi} = \gamma^{\mathsf{Y},g}_{\chi} \circ \varphi$ for all $\chi \in \widehat{\Gamma}$ and $g \in \mathrm{C}(\mathsf{Y},\Gamma)$.

The above theorem generalises much of the previous work on the subject, which has mainly focused on specific examples of Deaconu–Renault systems arising from directed graphs and irreducible $\{0,1\}$-matrices (see, for instance, [5, 3, 7, 8, 9, 10]). Applying our theorem in the setting of shifts of finite type provides a strengthening of a recent theorem of Matsumoto; c.f. [10, Theorem 1]. The proof of our theorem was inspired by Matsumoto's methods and also complements and applies results of Carlsen, Ruiz, Sims, and Tomforde [4].

In future work [2], we shall approach conjugacy of directed graphs from an algorithmic and combinatorial point of view related to [6].

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## Characterizing traces on crossed products of noncommutative C*-algebras

Dan Ursu

Having a nice description of the traces on a C*-algebra is often of interest, for example due to the fact that traces form part of the Elliott invariant in classification theory. Given a unital C*-algebra $A$ and a discrete group $G$ acting on $A$ by *-automorphisms, we give complete descriptions of the tracial states on both the universal and reduced crossed products $A \rtimes G$ and $A \rtimes_\lambda G$, respectively, in terms of the $G$-invariant traces on $A$.

**Theorem 1.** *Let $\tau \in T_G(A)$ be a $G$-invariant trace on $A$, let $\pi_\tau : A \to B(H_\tau)$ be the GNS representation, and let $M = \pi_\tau(A)''$ be the enveloping von Neumann algebra of $A$ under this representation, with faithful normal trace $\tau_M$. The following are in canonical bijection:*

*(1) Traces $\sigma \in T(A \rtimes G)$ satisfying $\sigma|_A = \tau$.*
*(2) Maps from $G$ to $M$ sending $t \in G$ to $x_t \in M$ such that:*
  *(a) $x_e = 1$*
  *(b) $x_t y = (t \cdot y) x_t$ for all $y \in M$*
  *(c) $s \cdot x_t = x_{sts^{-1}}$*
  *(d) The matrix $[x_{st^{-1}}]_{s,t \in G}$ is positive, in the sense that it is positive on finite submatrices.*

*Given coefficients $\{x_t\}_{t \in G}$, the corresponding trace $\sigma \in T(A \rtimes G)$ is given by $\sigma(au_t) = \tau_M(\pi_\tau(a)x_t)$.*

For the reduced crossed product $A \rtimes_\lambda G$, it is a result of Bryder and Kennedy in [3, Theorem 5.2] that traces on $A \rtimes_\lambda G$ concentrate on $A \rtimes_\lambda R_a(G) = A \rtimes R_a(G)$, where $R_a(G)$ is the amenable radical of $G$. Hence, any result on the universal crossed product automatically gives a similar corresponding result on the reduced crossed product, and we will not repeat them here.

It is has been well-known for some time that if the action of $G$ on the enveloping von Neumann algebra $\pi_\tau(A)''$ is properly outer, then the trace $\tau \in T_G(A)$ has unique extension to the crossed product (the converse is true if $A$ is commutative, but not if it is noncommutative). See for example [1, Section 2]. It is easy to read this off from condition 2b in Theorem 1, as proper outerness is equivalent to the coefficients $x_t$ being zero for $t \neq e$. One can also interpret Theorem 1 to get an *if and only if* condition on when the trace has unique extension to the crossed product. In addition, by taking polar decomposition $x_t = u_t|x_t|$,

we obtain corresponding conditions on unitaries on corners of the von Neumann algebra $\pi_\tau(A)''$. Condition 2d, however, does not result in anything nice. One might therefore ask if there are special cases where the conditions simplify, and the case of FC groups is one of them.

**Theorem 2.** *Assume $G$ is an FC group, i.e. each conjugacy class of $G$ is finite. Let $\tau \in T_G(A)$ be a $G$-invariant trace on $A$, let $\pi_\tau : A \to B(H_\tau)$ be the GNS representation, and let $M = \pi_\tau(A)''$. Then $\tau$ has unique tracial extension to the crossed product $A \rtimes G$ if and only if the following scenario is impossible:*

*There is some $t \neq e$ in $G$, some nonzero $t$-invariant central projection $p \in M$, and some unitary $u \in U(Mp)$ such that $t$ acts by $\operatorname{Ad} u$ on $Mp$ and $s \cdot u = u$ for all $s \in C_G(t)$.*

Observe that if the final condition of "$s \cdot u = u$ for all $s \in C_G(t)$" were omitted, then the above statement would simply be that $G$ acts on $\pi_\tau(A)''$ properly outer. In the case of abelian groups $G$, old results of Bédos [2, Proposition 11] and Thomsen [4, Theorem 4.3] more or less claim results along the lines of this being the case, i.e. unique extension of the trace is equivalent to proper outerness of the action. Unfortunately, this is seen to be false with a finite-dimensional counterexample.

**Example 1.** Let $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle u \rangle \times \langle v \rangle$ act on $M_2$ by

$$u = \operatorname{Ad} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad v = \operatorname{Ad} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Clearly, this action is not properly outer. However, the crossed product $M_2 \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ is isomorphic to $M_4$ (a simple computation will show that the crossed product has trivial center), and therefore has a unique trace.

Theorem 2 provides a fix to the aforementioned results of Bédos and Thomsen. However, given that the specific case of $G = \mathbb{Z}$ is often the case of interest for many people, it would be quite nice if their results still indeed hold in that case. As it turns out, the answer is yes:

**Theorem 3.** *Assume $G = \mathbb{Z}$, and $\tau \in T_{\mathbb{Z}}(A)$ is an invariant trace on $A$. Let $\pi_\tau : A \to B(H_\tau)$ be the GNS representation, and let $M = \pi_\tau(A)''$. Then $\tau$ has unique extension to the crossed product $A \rtimes \mathbb{Z}$ if and only if the action of $\mathbb{Z}$ on $M$ is properly outer.*

It would also be interesting to know if there is some cohomological reason why this works for $G = \mathbb{Z}$, and if there are any other groups for which this works.

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## Inclusions of $C^*$-algebras

Mikael Rørdam

In analogy with irreducible inclusions of von Neumann algebra factors, an inclusion $A \subseteq B$ of unital simple $C^*$-algebras is said to be $C^*$-*irreducible* if all its intermediate $C^*$-algebras are simple. This can be shown to be equivalent to each non-zero positive element $b$ in $B$ being full relatively to $A$, i.e., $\sum_{j=1}^n x_j^* b x_j \geq 1$, for some $x_j \in B$. An inclusion of type $\mathrm{II}_1$ factors is $C^*$-irreducible if and only if it is irreducible and has finite index (this largely follows from results of Popa). An inclusion of unital simple $C^*$-algebras of finite index wrt some conditional expectation is $C^*$-irreducible if and only if it is irreducible (by a theorem of Izumi, [3]).

Besides determining when an inclusion of simple $C^*$-algebras is $C^*$-irreducible, it is an important and challenging problem to classify all intermediate $C^*$-algebras of such inclusions. There are surprisingly many strong results in this direction, in some cases providing Galois type correspondances between intermediate $C^*$-algebras and other related substructures.

Many interesting examples of inclusions of $C^*$-algebras arise from groups and dynamical systems. For example, if $\Gamma$ is a discrete group acting on a unital $C^*$-algebra $A$ one obtains inclusions $A \subseteq A \rtimes_r \Gamma$ and $C^*_\lambda(\Gamma) \subseteq A \rtimes_r \Gamma$, and more generally $A \rtimes_r \Lambda \subseteq A \rtimes_r \Gamma$, whenever $\Lambda$ is a subgroup of $\Gamma$. We have full understanding of when the former two types of inclusions are $C^*$-irreducible. It is particularly interesting to decide when inclusions of the form $C^*_\lambda(\Lambda) \subseteq C^*_\lambda(\Gamma)$ are $C^*$-irreducible. One can think of this problem as a relative version of $C^*$-simplicty of groups, and indeed, existence of a (topological) free actions of $\Gamma$ on a compact Hausdorff space, which is boundary relatively to the subgroup (in a suitable way) can be shown to imply $C^*$-irreducibility of the inclusion. By results of Ursu and Bedos–Omland, the reverse implication also holds in the case when the subgroup is normal, see [1].

With Echterhoff, we proved in [2] when inclusions of the form $A^H \subseteq A \rtimes_r G$, the $C^*$-analog of inclusions of $\mathrm{II}_1$-factors considered by Bisch and Haagerup, are $C^*$-irreducible, where $A$ is a simple $C^*$-algebra and $G$ and $H$ are groups acting (outerly) on $A$, with $H$ finite.

There are several techniques available to study inclusions of $C^*$-algebras, such as the relative Dixmier property, introduced by Popa in [4]. In [5], Popa introduced an averaging property for automorphisms on $C^*$-algebras which provides a powerful tool to study inclusions arising from crossed products. More specifically, an automorphism $\alpha$ on a unital $C^*$-algebra $A$ has Popa's averaging property if 0 belongs to the closed convex hull of $\{ub\alpha(u)^* : u \in U(A)\}$, for all $b \in A$. (It is also interesting to consider the weaker property that 0 belongs to the closed convex

hull of $\{u\alpha(u)^* : u \in U(A)\}$.) We described which automorphisms possess these properties, and mentioned several applications thereof.

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