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Abstract

In amateur or youth sports leagues, the teams play all matches during their leisure time. Thus, a schedule with a smaller number of game days is preferred and the teams are willing to partly renounce on the fairness for this by playing minitournaments instead of single matches. In this format, multiple teams meet at one of them and play against each other, reducing the number of necessary game days and required referees at the cost of unevenly distributed home field advantages. The travel times of all teams now depend on their assignment to the respective minitourmanets and the choice of the home team. We present a binary linear optimization model to schedule a sports league as a double Round Robin tournament with minitournaments and most evenly distributed home field advantages, yielding a feasible league schedule with minimal total traveling distances for all teams. After adjusting orbital shrinking to break the occurring symmetries in the possible assignments, we discuss the computational efficiency and evaluate an existing schedule for the „Basketball Senioren Landesliga Brandenburg“ amateur basketball league in Germany.

1 Introduction

Scheduling a sports league is a complex task, especially for professional leagues. Several stakeholders, ranging from the teams over sponsors and broadcasting companies to their fans, with sometimes competing interests are involved and must be satisfied. At the same time, there are many possible objectives to choose from. An extensive introduction to this field with basic terms, problems, and related solution approaches is given in [11] and [12]. One could think that the scheduling problem for amateur sport leagues gets easier since there are smaller groups and fewer restrictions compared to professional leagues, but [11] and [16] disagree with this assumption. In amateur sports leagues, the restrictions and objectives of the participating groups can be completely different compared to professional business, as we see in this work for the efficiency. Since the teams play all matches during their leisure time, they are willing to sacrifice some fairness to reduce the overall number of game days and avoid traveling to each match separately. This leads to the game format of minitournaments, where several teams meet at the venue of one of them and play against each other.

We consider in this work the „Basketball Senioren Landesliga Brandenburg“, an amateur sports league with three teams per minitournament. The season is a double Round Robin tournament, consisting of two legs, but these legs do not need to be separated. The second match against any team can take place before all first encounters are complete. Since there are more away teams than home teams per game day in this type of schedule, the home field advantages cannot be distributed evenly. Nevertheless, they should be divided as evenly as possible during the season to maintain some fairness. For the same reason, the same three teams must not meet at the same home team twice. Applying this scenario, the travel distances of the teams are now no longer equal and we are looking for a feasible league schedule for the complete season with minimal total traveling distances.

Using the categories introduced in [12], we have a 2-Round-Robin tournament (2RR) with a time-constrained (C) or time-relaxed (R) schedule, depending on the number of teams, and no symmetry ($\emptyset$). Due to the most even distribution of the home field advantages, there are constraints of type CA1. Furthermore, the number of possible minitournaments per game day and the unique use of each minitournament yield constraints of type GA1. All these constraints are hard, therefore they are annotated by the superscript „H“. The objective of minimizing the total traveled distances is included in the category „TR“. Thus, we can describe our setting in a compact way by the three-field notation [12] (2RR,C,$\emptyset$|CA1^H,GA1^H|TR) or (2RR,R,$\emptyset$|CA1^H,GA1^H|TR), depending on the number of teams. However, this classification is not able to display the format of minitournaments.

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In the literature, a recent review about tournament scheduling is given in [2]. Thus, we restrict here to literature about our case of basketball league scheduling, although our resulting model is not specific for one type of sport. There are several works for different leagues. Nemhauser and Trick [9] derive a three-stage solution procedure of generating home/away/bye patterns, assigning games to these patterns, and then assigning the teams to calculate a league schedule for the Atlantic Coast Conference (ACC) in college basketball. Within the single stages of their method, integer programming and enumeration techniques are used, incorporating restrictions to the patterns, separation of the two games between the same teams, broadcaast contracts, and opponent orderings. Later, this problem was again considered by Henz [6] and solved by constraint programming.
Wright [15] tackles the league scheduling problem for the National Basketball League in New Zealand (NBL) with 18 teams and only 16 time intervals to play. The focus is therefore on scheduling the intervals and teams for the respective double game intervals and the exact game days in the intervals. Furthermore, uncertainty at the availability of the arenas is incorporated by a rating. The resulting problem is solved with a simulated annealing heuristic for several competing objectives.
Westphal [14] presents and compare different modeling approaches for the German Basketball League „Basketball Bundesliga“ (BBL), incorporating location availability, successive home and away games, broadcast slots, and traveling distances. Therein, it turned out that mirrored schedules violate more of the given restrictions than schedules with nonmirrored legs. The results were implemented in scheduling software and applied to the real-world scheduling.
Lu and Chen [8] give an integer programming approach to schedule the regular season of the National Basketball League in the United States (NBA). Incorporating the official regulations as constraints, they aim for a fair schedule by minimizing the difference between all overall winning percentages of the participating teams.
Durán *et al.* [3] formulate the scheduling problem for the National League „Liga Nacional de Básquetbol“ (LNB) of Argentina as a variant of the Traveling Tournament Problem. They consider arena availability and successive home and away games, but the teams can make suggestions for their trips and week days with home games. The objective is to maximize the number of suggested trips that are taken into the league schedule. For the solution of the problem, a two-stage procedure is developed, where, in a first stage, the trips of all teams are chosen with approximate game days and, in a second stage, the games are assigned to exact dates. The results are compared to existing, manually planned league schedules and reveal significant benefits in terms of travel distances.
Regarding the incorporation of minitournaments, we are aware of only one work where this topic occurs. In [10], Nurmi, Goossens and Kyngäs present the scheduling problem of the Finnish national youth ice hockey league as a three-round robin tournament, where some game days are planned as minitournaments of five teams. Therein, also the question after a fair distribution of the home advantages arises in the presence of shared venues for multiple teams, an upper bound on the difference of played games between all teams, and a lower bound on the number of game days before two teams can meet again. However, these minitournaments have no single-round robin format, i.e., they contain only three games per team, and the travel distance is only partly taken into account by explicit constraints to design the minitournaments. To generate a good schedule in this complex setting, an extended local search heuristic is used and the results outperform the manually created schedules from the past.
This paper is structured as follows. In Section 2, we present an integer program for the considered problem, which suffers from symmetries. To reduce their effect in the solution process, orbital shrinking is applied and adapted at Section 3. The comparison of the computational performance of the different solution approaches and a qualitative evaluation of an existing league schedule with this regime can be found in Section 4. We conclude our work and give directions for future research in Section 5.

2 Binary Optimization Model

We formulate the league scheduling problem with minitournaments of three teams as a binary linear program (BLP). A list of all used sets, parameters, and variables is given in Table 1 and Table 2. Consider a league with $n \in \mathbb{N}, n \geq 3$, teams and minitournaments by three of them at each game day. Then, there are $k = \left\lfloor \frac{n}{3} \right\rfloor$ minitournaments per game day. Furthermore, we incorporate the possibility of an additional single match per game day for the two teams not assigned to a minitournament in the case of $n = 3h+2$, $h \in \mathbb{N}$, to reduce the number of required game days. This single match can be seen as a minitournament with an artificial team $n + 1$. Thus, we denote in the following with $\mathcal{M} = \{1, \ldots, n\}$ the set of all real teams and define the

Table 1: Overview of the sets used in the model.

Symbol	Index	Definition
$\mathcal{M}$	i	set of all teams
$\mathcal{T}$	t	set of all game days
$\mathcal{Q}$	$q = (q_1, q_2, q_3)$	set of possible minitournaments

Table 2: Overview of the parameters and variables used in the model.

Symbol	Domain	Definition
$d_{i,j}$	$\mathbb{R}_+$	distance between the home locations of teams i and j
$\underline{h}$	$\mathbb{R}_+$	minimum number of game days with home field advantage
$\overline{h}$	$\mathbb{R}_+$	maximum number of game days with home field advantage
k	$\mathbb{N}$	number of minitournaments per game day
n	$\mathbb{N}$	number of teams in the league
$x_{q,t}$	$\{0, 1\}$	binary variable, indicating whether the minitournament q is assigned to game day t

set of all possible minitournaments by

$$\mathcal{Q} = \{(i_1, i_2, i_3) \in \mathcal{M} \times \mathcal{M} \times (\mathcal{M} \cup \{n+1\}) \mid i_1 \neq i_2 \neq i_3, i_2 < i_3\}, \tag{1}$$

where we assume the home team to be on the first position and arrange the away teams in ascending order to avoid duplicates. Since the artificial team $n + 1$ has no home field advantages, it can only occur in the third position.
The number of necessary game days depends on the number of participating teams and can be calculated beforehand.

Proposition 2.1. *For a league with $n \in \mathbb{N}$, $n \geq 3$, teams, minitournaments with three of them, and two legs, the minimal number of game days is given by*

$$T(n) = n - 1 + (n \bmod 3). \tag{2}$$

Proof. For each leg, $\frac{n(n-1)}{2}$ matches are necessary, yielding $n(n-1)$ matches for the whole season. Since $3k$ matches are played during a game day, we can distinguish three cases, depending on the number of participating teams:

$n = 3k$: $T(3k) = \frac{3k(3k-1)}{3k} = 3k - 1 = n - 1$

$n = 3k + 1$: $T(3k + 1) = \frac{(3k+1)(3k)}{3k} = 3k + 1 = n$

$n = 3k + 2$: $T(3k + 2) = \frac{(3k+2)(3k+1)}{3k} = 3k + 3 + \frac{2}{3k} = n + 1 + \frac{2}{3k}$

Now, we exploit for $n = 3k + 2$ the possibility of additional single matches. Since it holds $n + 1 > 2$ for all $n \in \mathbb{N}$, $n \geq 3$, we enforce these two remaining matches as additional single matches within the $n + 1$ game days with minitournaments. Merging these three cases, the minimal number of game days is given by (2). □

Following Proposition 2.1, we use $\mathcal{T} = \{1, \dots, T(n)\}$ to denote the set of all game days of a season. Thus, we get a time-constraint league schedule for $n \in \{3k, 3k + 1\}$ teams, $k \in \mathbb{N}$, and a time-relaxed schedule for $n = 3k + 2$ teams, due to the additional single matches.
In a feasible league schedule, it is relevant which minitournaments are carried out at every game day. Thus, we introduce binary variables $x_{q,t} \in \{0, 1\}$, indicating whether minitournament $q \in \mathcal{Q}$ is assigned to game

day $t \in \mathcal{T}$. Every team can be assigned to at most one minitournament per game day. Thus, it must hold

$$\sum_{\substack{q\in\mathcal{Q}\\ i\in q}} x_{q,t} \leq 1 \qquad \forall i \in \mathcal{M}\ \forall t \in \mathcal{T}. \tag{3}$$

To increase the fairness, three teams must not meet with the same home field advantage twice. Thus, each minitournament can occur at most once and we have

$$\sum_{t\in\mathcal{T}} x_{q,t} \leq 1 \qquad \forall q \in \mathcal{Q} : q_3 \leq n, \tag{4}$$

$$\sum_{t\in\mathcal{T}} x_{q,t} \leq \chi_{\{i\in\mathbb{N}\ \mid\ i \bmod 3=2\}}(n) \qquad \forall q \in \mathcal{Q} : q_3 = n+1, \tag{5}$$

where $\chi_{\{i\in\mathbb{N}\ \mid\ i \bmod 3=2\}} : \mathbb{N} \to \{0,1\}$ is the characteristic function of the set of all positive integers leaving a remainder of two at division by three. Note, that we could formulate the desired restriction also by applying (4) for all $q \in \mathcal{Q}$, but the separation into (4) and (5) allows us to remove all variables with the artificial team by fixing them to zero if there are no additional single matches.
Depending on $n \in \mathbb{N}$, the number of minitournaments per game day can vary by one, due to the additional single matches for $n = 3k+2$, yielding

$$\sum_{q\in\mathcal{Q}} x_{q,t} \geq k \qquad \forall t \in \mathcal{T}, \tag{6}$$

$$\sum_{q\in\mathcal{Q}} x_{q,t} \leq k + \chi_{\{i\in\mathbb{N}\ \mid\ i \bmod 3=2\}}(n) \qquad \forall t \in \mathcal{T}, \tag{7}$$

Since two arbitrary teams must play against each other exactly twice, it holds

$$\sum_{\substack{q\in\mathcal{Q}\\ i_1,i_2\in q}} \sum_{t\in\mathcal{T}} x_{q,t} = 2 \qquad \forall (i_1,i_2) \in \mathcal{M}^2 :\ i_1 < i_2. \tag{8}$$

It remains to incorporate the fairest distribution of the home field advantages. The possible values can also be calculated beforehand.

Proposition 2.2. *For a league with $n \in \mathbb{N}$ teams, each team has either $\lfloor \frac{n-1}{3} \rfloor$ or $\lceil \frac{n-1}{3} \rceil$ home games in a most fair distribution of the home field advantages.*

Proof. Consider a league with $n \in \mathbb{N}$ teams. At each game day, there are $k = \lfloor \frac{n}{3} \rfloor$ minitournaments chosen. Thus, we can calculate the average number of home games per team $\frac{kT(n)}{n}$, depending on n:

$n = 3k$: It holds

$$\frac{kT(3k)}{3k} = \frac{k(3k-1)}{3k} = k - \frac{1}{3} = \frac{n-1}{3}.$$

Since this number is not integer, each team has either $\lfloor \frac{n-1}{3} \rfloor$ or $\lceil \frac{n-1}{3} \rceil$ home games.

$n = 3k+1$: It holds

$$\frac{kT(n)}{n} = \frac{k(3k+1)}{3k+1} = k = \frac{n-1}{3}.$$

Thus, each team has exactly $\frac{n-1}{3}$ home field advantages.

$n = 3k+2$: We have two additional single matches in this case, yielding

$$\frac{kT(n)+2}{n} = \frac{k(3k+3)+2}{3k+2} = k + \frac{k+2}{3k+2} = \frac{n-1}{3} + \frac{4}{3n}.$$

Since $\frac{4}{3n} < 1$, $n \geq 3$, each team has either $\lfloor \frac{n-1}{3} \rfloor$ or $\lceil \frac{n-1}{3} \rceil$ home games.

Combining all three cases yields the desired result, where we use for $n = 3k+1$ the identical lower and upper bound $\lfloor \frac{n-1}{3} \rfloor = \lceil \frac{n-1}{3} \rceil = \frac{n-1}{3}$. $\square$

Since we later variate these values, we introduce parameters $\underline{h} = \left\lfloor \frac{n-1}{3} \right\rfloor$ and $\overline{h} = \left\lceil \frac{n-1}{3} \right\rceil$ as lower and upper bound for the home field advantages of each team, respectively. According to Proposition 2.2, the most fair distribution is guaranteed by

$$\sum_{\substack{q \in \mathcal{Q} \\ q_1 = i}} \sum_{t \in \mathcal{T}} x_{q,t} \geq \underline{h} \qquad \forall i \in \mathcal{M}, \tag{9}$$

$$\sum_{\substack{q \in \mathcal{Q} \\ q_1 = i}} \sum_{t \in \mathcal{T}} x_{q,t} \leq \overline{h} \qquad \forall i \in \mathcal{M}. \tag{10}$$

Due to the incorporation of minitournaments, the total travel distance in one season for each team is no longer equal. This allows the minimization of the total traveled distance for all teams over the season to achieve a more efficient schedule by reducing travel times and emissions. Using $d_{i,j} \in \mathbb{R}_+$ to denote the distance between the teams $i \in \mathcal{M}$ and $j \in \mathcal{M}$, the objective is given by

$$\min \qquad \sum_{t \in \mathcal{T}} \left(\sum_{\substack{q \in \mathcal{Q} \\ q_3 \leq n}} (d_{q_1,q_2} + d_{q_1,q_3}) x_{q,t} + \sum_{\substack{q \in \mathcal{Q} \\ q_3 = n+1}} d_{q_1,q_2} x_{q,t} \right). \tag{11}$$

Remark 2.3. *The BLP* (3) – (11) *simplifies for $n = 3k$ and $n = 3k + 1$ teams. In both cases, it holds $\chi_{\{i \in \mathbb{N} \mid i \bmod 3 = 2\}}(n) = 0$. Thus, constraint* (5) *gets obsolete by using*

$$\mathcal{Q}' = \left\{ (i_1, i_2, i_3) \in \mathcal{M}^3 \mid i_1 \neq i_2 \neq i_3, i_2 < i_3 \right\}$$

as the set of possible minitournaments and (6) *and* (7) *can be combined into an equality constraint. For $n = 3k$, also all teams are assigned to a minitournament at every game day, yielding additional equality in* (3)*.*

Interestingly, there are two numbers of participating teams that cannot be modeled with the previously given restrictions.

Theorem 2.4. *For $n \in \{5, 6\}$, the BLP* (3) – (11) *is infeasible.*

Proof. For $n = 5$ teams, it holds $T(5) = 6$. We have one minitournament per game day and two additional single matches during the season. Thus, there are two game days where all teams are playing, while at the remaining four game days two teams have a bye. In total, each team must play eight matches. Between two consecutive game days, either one or two teams in the minitournament must be exchanged. Thus, all possible assignments of the five teams into a triple and a pair, representing the minitournament and the additional single match or the resting teams, can be achieved from any other assignment. We assume, without loss of generality, that the two additional single matches are scheduled at the first two game days. We distinct now two cases, based on the number of exchanged teams in the minitournament after the first game day.
First, if only one team is replaced between the first two game days, the two teams remaining in the minitournament played after the second game day already twice against each other. We denote them in the following by i_1 and i_2. At the same time, the team that remained in the additional single matches, denoted by i_5, has played two matches and did not met i_1 and i_2, respectively. In the remaining four game days, there is one minitournament and two teams have a bye. Thus, i_5 must be part of the minitournament for exactly three game days to complete its six remaining matches and must play in this time against i_1 and i_2 twice. This is not possible without assigning i_1 and i_2 to the minitournament at the same game day, violating (8). Second, if two teams are changed in the minitournament between the first two game days, there are two pairs of teams that played already twice against each other, while the remaining team played against all others exactly once. Let i_1 be this aforementioned team and (i_2, i_3) and (i_4, i_5) the respective other team pairs. In this situation, i_1 must be assigned to the minitournament at two of the remaining four game days to play its last four matches. Thus, one must set up two minitournaments only with $i_2, \ldots, i_5$, which is not possible without a third match between i_2 and i_3 or i_4 and i_5, respectively. This again violates (8). Thus, the problem must be infeasible.
For $n = 6$ teams, it holds $T(6) = 5$. There are two minitournaments and each team has two matches per game day. Between two consecutive game days, the new minitournaments can be build by exchanging either one or two teams between them. As displayed in Table 3, both possibilities yield the same arrangement, given the same initial configuration. Thus, we restrict in the following on exchanging always one team.

Table 3: Possibilities to generate new minitournaments between consecutive game days for a league with six teams. Exchanged teams are marked red.

Game day	Exchange one team		Exchange two teams	
t	(i_1, i_2, i_3)	(i_4, i_5, i_6)	(i_1, i_2, i_3)	(i_4, i_5, i_6)
$t+1$	(i_1, i_5, i_3)	(i_4, i_2, i_6)	(i_4, i_2, i_6)	(i_1, i_5, i_3)

Exchanging one team per game day, any team played after the third game day against two other teams twice and these three teams met once in the same minitournament. Otherwise, it would not be possible to play against two teams twice within three game days. Assume in the following, that, without loss of generality, i_1 played already twice against i_2 and i_3. Then, i_2 and i_3 must be in the same minitournament at the fourth game day, separated from i_1 and playing their second match against each other. Thus, at the fifth game day, i_1, i_2, and i_3 cannot be distributed into two minitournaments without meeting each other, violating (8). Therefore, the problem must be infeasible. □

Although we already list each possible minitournament in $\mathcal{Q}$ only once by applying a special assignment of the teams to the components in (1), the solutions of the model are still highly symmetric since each permutation of the game days remains feasible. An intuitive approach to reduce symmetries is adding valid inequalities to enforce an ordering of the variables and forbid permutations of a solution [7]. Therefore, we exploit that each pair of teams must play against each other twice and restrict the games between the first team and the others to their earliest possible game day. Since there are always three teams in a minitournament, we only can give a range for the possible game days, yielding

$$\sum_{\substack{t\in\mathcal{T}\\ t>2i}} \sum_{\substack{q\in\mathcal{Q}\\ 1\in q\wedge i\in q}} x_{q,t} = 0 \qquad \forall i \in \left\{2,\dots,\left\lfloor \frac{T(n)}{2} \right\rfloor\right\}. \tag{12}$$

3 Orbital Shrinking

A more complex approach, called orbital shrinking, is given by Fischetti *et al.* in [4]. The main idea is to merge variables with the same orbit into a single new one and set up a decomposition method. Therein, the formulation with the merged variables serves as a master problem, while in a subproblem the feasibility of an optimal solution of the master problem for the original problem is checked by splitting the merged variables up again. If it is feasible, the obtained solution is also optimal for the original problem. Otherwise, it must be cut by additional constraints and the next iteration starts.

The approach in [3] is similar to the idea of orbital shrinking by separating the matches and the game days to avoid symmetric solutions and solving the problem in two stages. We apply and adjust orbital shrinking to our league scheduling problem. Since an arbitrary permutation of the game days in any feasible solution remain feasible, all variables $x_{q,t} \in \{0,1\}$ for $t \in \mathcal{T}$ and a fixed minitournament $q \in \mathcal{Q}$ have the same orbit with size $T(n)$. Thus, we introduce new variables $z_q \in \mathbb{Z}_+$ and replace $x_{q,t} = \frac{z_q}{T(n)}$ for all $q \in \mathcal{Q}$ and $t \in \mathcal{T}$. This yields the master problem

$$\min \quad \sum_{\substack{q\in\mathcal{Q}\\ q_3\le n}} (d_{q_1,q_2}+d_{q_1,q_3})z_q + \sum_{\substack{q\in\mathcal{Q}\\ q_3>n}} d_{q_1,q_2} z_q, \tag{13}$$

$$\sum_{\substack{q\in\mathcal{Q}\\ i\in q}} z_q \le T(n) \qquad \forall i \in \mathcal{M}, \tag{14}$$

$$\sum_{q\in\mathcal{Q}} z_q \ge T(n)k, \tag{15}$$

$$\sum_{q\in\mathcal{Q}} z_q \le T(n)(k+\chi_{\{i\in\mathbb{N}\ \mid\ i \bmod 3=2\}}(n)), \tag{16}$$

$$\sum_{\substack{q\in\mathcal{Q}\\ i_1,i_2\in q}} z_q = 2 \qquad \forall (i_1,i_2)\in\mathcal{M}^2 :\ i_1<i_2, \tag{17}$$

$$\sum_{\substack{q\in\mathcal{Q}\\ q_1=i}} z_q \geq \underline{h} \qquad \forall i \in \mathcal{M}, \tag{18}$$

$$\sum_{\substack{q\in\mathcal{Q}\\ q_1=i}} z_q \leq \overline{h} \qquad \forall i \in \mathcal{M}, \tag{19}$$

$$z_q \leq \chi_{\{i\in\mathbb{N}\ \mid\ i \bmod 3=2\}}(n) \qquad \forall q \in \mathcal{Q}:\ q_3 > n, \tag{20}$$

$$z_q \in \{0,1\} \qquad \forall q \in \mathcal{Q}. \tag{21}$$

Remark 3.1. *After the substitution,* (4) *and* (5) *ensure z_q to be binary and therefore are partly incorporated in* (21)*. Furthermore, the created master problem yields in the considered case an exact reformulation of the original problem (see Corollary 2.6 in [4]), since the original problem is a BLP and therefore convex.*

For every solution $z^* \in \{0,1\}^{|\mathcal{Q}|}$, we set up a subproblem, consisting of (3) – (10) with the additional constraint

$$\sum_{t\in\mathcal{T}} x_{q,t} = z^*_q \qquad \forall q \in \mathcal{Q} \tag{22}$$

to couple the binary variables $x_{q,t}$ with the respective solution values in z^*. If the subproblem has a feasible solution, it is the optimal solution for the original BLP. Otherwise, we remove the solution z^* from the master problem with a cut of the form

$$\sum_{\substack{q\in\mathcal{Q}\\ z^*_q=0}} z_q \geq 1 \tag{23}$$

and solve the master problem again to obtain a new solution candidate for the BLP.
At this point, the procedure suffers from the structure of the subproblem (3) – (10) and (22) since it contains again all the symmetries of the original problem. Thus, we adjust the orbital shrinking method by simplifying the subproblem, exploiting some properties of the solution $z^* \in \{0,1\}^{|\mathcal{Q}|}$. We start with a useful observation about the number of additional single matches.

Proposition 3.2. *For every optimal solution $x^* \in \{0,1\}^{|\mathcal{Q}|\times|\mathcal{T}|}$ of* (3) – (11) *and $n \in \mathbb{N}$, $n \geq 3$, it holds*

$$\sum_{t\in\mathcal{T}} \sum_{\substack{q\in\mathcal{Q}\\ q_3>n}} x^*_{q,t} = 2\chi_{\{i\in\mathbb{N}\ \mid\ i \bmod 3=2\}}(n). \tag{24}$$

Proof. Let $x^* \in \{0,1\}^{|\mathcal{Q}|\times|\mathcal{T}|}$ denote an optimal solution of (3) – (11) for $n \in \mathbb{N}$, $n \geq 3$. Given $n \neq 3k+2$, $k \in \mathbb{N}$, it holds $\chi_{\{i\in\mathbb{N}\ \mid\ i \bmod 3=2\}}(n) = 0$. Thus, equation (24) is correct since (5) enforces $x^*_{q,t} = 0$ for all $q \in \mathcal{Q}$, $q_3 > n$, and $t \in \mathcal{T}$.
It remains to consider $n = 3k+2$ for $k \in \mathbb{N}$. We know from Proposition 2.1, that $n(n-1) = 9k^2 + 9k + 2$ matches must be played in a season. Taking only the minitournaments into account, there can be at most $3k(n+1) = 9k^2 + 9k$ matches scheduled. Thus, there must be at least two additional single matches, yielding

$$\sum_{t\in\mathcal{T}} \sum_{\substack{q\in\mathcal{Q}\\ q_3>n}} x^*_{q,t} \geq 2. \tag{25}$$

The lower bound (25) is attained since the replacement of a minitournament $(q_1, q_2, q_3) \in \mathcal{Q}$ by additional single matches would cause a traveled distance of $d_{q_1,q_2} + d_{q_1,q_3} + d_{q_2,q_3} > d_{q_1,q_2} + d_{q_1,q_3}$ in the objective function, compared to the traveled distance of the replaced minitournament. This contradicts the optimality of x^*. $\square$

With this in mind, the solution $z^* \in \{0,1\}^{|\mathcal{Q}|}$ of the master problem (13) – (21) can be characterized.

Theorem 3.3. *For every optimal solution $z^* \in \{0,1\}^{|\mathcal{Q}|}$ of* (13) – (21)*, $n \in \mathbb{N}$, $n \geq 3$, and $k = \lfloor \frac{n}{3} \rfloor$ it holds*

$$\sum_{q\in\mathcal{Q}} z^*_q = T(n)k + 2\chi_{\{i\in\mathbb{N}\ \mid\ i \bmod 3=2\}}(n). \tag{26}$$

Proof. Let $z^* \in \{0,1\}^{|\mathcal{Q}|}$ denote an optimal solution of (13) – (21) for $n \in \mathbb{N}$, $n \geq 3$, and $k = \lfloor \frac{n}{3} \rfloor$. If $n \neq 3k+2$, the combination of (15) and (16) yield (26) directly.

Consider $n = 3k + 2$. Since every solution of the master problem (13) – (21) is a candidate for the optimal solution of the original BLP (3) – (11), constraint (24) must also hold in the master problem. Transforming it accordingly yields

$$\sum_{\substack{q \in \mathcal{Q} \\ q_3 > n}} z_q^* = 2\chi_{\{i \in \mathbb{N} \mid i \bmod 3 = 2\}}(n). \tag{27}$$

Furthermore, summing (17) for all $(i_1, i_2) \in \mathcal{M}^2$ with $i_1 < i_2$ and applying (27) gives

$$\begin{aligned} & \sum_{\substack{(i_1,i_2) \in \mathcal{M}^2 \\ i_1 < i_2}} \sum_{\substack{q \in \mathcal{Q} \\ i_1, i_2 \in q}} z_q^* = \sum_{\substack{(i_1,i_2) \in \mathcal{M}^2 \\ i_1 < i_2}} 2 \\ \Leftrightarrow \quad & 3 \sum_{\substack{q \in \mathcal{Q} \\ q_3 \leq n}} z_q^* + \sum_{\substack{q \in \mathcal{Q} \\ q_3 = n+1}} z_q^* = 3k \underbrace{(3k+3)}_{=T(n)} + 2 \\ \Leftrightarrow \quad & \sum_{\substack{q \in \mathcal{Q} \\ q_3 \leq n}} z_q^* = T(n)k. \end{aligned} \tag{28}$$

The sum of (27) and (28) yields the desired result. □

Theorem 3.4. *For every optimal solution $z^* \in \{0,1\}^{|\mathcal{Q}|}$ of (13) – (21), $n \in \mathbb{N}$, $n \geq 3$, and $k = \lfloor \frac{n}{3} \rfloor$, let $\mathcal{Q}^* = \left\{ q \in \mathcal{Q} \mid z_q^* = 1 \right\}$ denote the set of all chosen minitournaments and $\mathcal{R} \subset \mathcal{Q}^{*k} \cup \mathcal{Q}^{*k+1}$ the set of all possible game days, built from $\mathcal{Q}^*$. Then, it holds*

$$z^* \text{ feasible for BLP} \Leftrightarrow \exists \mathcal{R}' \subseteq \mathcal{R} : \; |\mathcal{R}'| = T(n) \; \wedge \; (\forall q \in \mathcal{Q}^* \; \exists r \in \mathcal{R}' : \; q \in r) \,. \tag{29}$$

Proof. „⇐“: Let $z^* \in \{0,1\}^{|\mathcal{Q}|}$ be an optimal solution of (13) – (21), such that for $n \in \mathbb{N}$, $n \geq 3$, and $k = \lfloor \frac{n}{3} \rfloor$

$$\exists \mathcal{R}' \subseteq \mathcal{R} : \; |\mathcal{R}'| = T(n) \; \wedge \; (\forall q \in \mathcal{Q}^* \; \exists r \in \mathcal{R}' : \; q \in r) \,. \tag{30}$$

for every $q \in \mathcal{Q}^*$, we find a game day that contains it. By $|\mathcal{R}'| = T(n)$ we must use all game days from $\mathcal{R}'$ to create a feasible schedule. Together with $|\mathcal{Q}^*| = T(n)k + 2\chi_{\{i \in \mathbb{N} \mid i \bmod 3 = 2\}}(n)$ by Theorem 3.3, this guarantees that every $q \in \mathcal{Q}^*$ appear in any possible schedule of the BLP exactly once. Let, without loss of generality, denote $r_t \in \mathcal{R}'$ the elements of $\mathcal{R}'$ for $t \in \mathcal{T}$. We define such a league schedule by $x_{q,t} = 1$ if, and only if, $q \in r_t$ for all $q \in \mathcal{Q}$ and $t \in \mathcal{T}$. Following the previous arguments, this schedule fulfills (4) or (5). Furthermore, also (9) – (10) and (8) hold since these conditions are incorporated by (17) – (19) in the master problem and we use its complete solution. By the definition of $\mathcal{R}'$, there are either k or $k+1$ minitournaments in every game day and each team can occur at most once in every game day, fulfilling (3) and (6) – (7). Thus, this league schedule is feasible for the BLP.

„⇒“: We prove this part by contraposition. Thus, assume

$$\forall \mathcal{R}' \subseteq \mathcal{R} : \; |\mathcal{R}'| \neq T(n) \; \vee \; (\exists q \in \mathcal{Q}^* \; \forall r \in \mathcal{R}' : \; q \notin r) \,. \tag{31}$$

The set of all subsets $\mathcal{R}' \subseteq \mathcal{R}$ can be partitioned into two cases. First, for all $\mathcal{R}'$ with $|\mathcal{R}'| < T(n)$, any feasible schedule must repeat at least one game day, contradicting (4) or (5) by scheduling some minitournaments more than once. Second, for all $\mathcal{R}'$ with $|\mathcal{R}'| \geq T(n)$, we use Theorem 3.3, yielding $|\mathcal{Q}^*| = T(n)k + 2\chi_{\{i \in \mathbb{N} \mid i \bmod 3 = 2\}}(n)$. Since this is also the minimal number of required minitournaments and additional single games for the whole season, each $q \in \mathcal{Q}^*$ must appear in some game day. Otherwise, we must repeat at least one minitournament, contradicting again (4) or (5). By assumption, there is a $q \in \mathcal{Q}^*$, not appearing in any game day. Thus, we have a contradiction to (8) since some teams meet only once. In conclusion, any resulting league schedule cannot be feasible, which is the desired result.

□

Following Theorem 3.4, we can perform the feasibility check of an optimal solution $z^* \in \{0,1\}^{|\mathcal{Q}|}$ of (13) – (21) in the subproblem by constructing from $\mathcal{Q}^*$ the set $\mathcal{R}$ of feasible game days and solving the equality

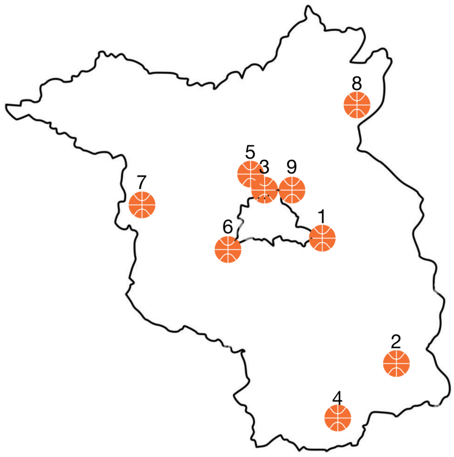

Figure 1: Participants of the „Basketball Senioren Landesliga Brandenburg“ (season 2023/2024).

Table 4: Optimal league schedule for the „Basketball Senioren Landesliga Brandenburg“ (season 2023/2024). Home teams are marked with red.

Game day	Tourn. 1	Tourn. 2	Tourn. 3
1	9 - 1 - 4	7 - 2 - 3	5 - 6 - 8
2	9 - 2 - 5	1 - 7 - 8	6 - 3 - 4
3	1 - 5 - 9	2 - 4 - 7	3 - 6 - 8
4	7 - 4 - 5	8 - 3 - 9	2 - 1 - 6
5	5 - 1 - 3	6 - 7 - 9	4 - 2 - 8
6	8 - 1 - 4	6 - 2 - 5	3 - 7 - 9
7	5 - 7 - 8	1 - 2 - 3	4 - 6 - 9
8	9 - 2 - 8	7 - 1 - 6	3 - 4 - 5

system

$$\sum_{r \in \mathcal{R}} y_r = T(n), \tag{32}$$

$$\sum_{\substack{r \in \mathcal{R} \\ q \in r}} y_r = 1 \qquad \forall q \in \mathcal{Q}^*, \tag{33}$$

$$y_r \in \{0, 1\} \qquad \forall q \in \mathcal{Q}^*. \tag{34}$$

4 Computational Results

All presented models and solution procedures were implemented in Python 3.10 [13] and solved on a Mac Pro (2019) with an Intel Xeon W running 32 threads parallel at 3.2 GHz clock speed and 768 GB RAM using Gurobi 12.0.1 [5]. The relative and absolute gap were always set to 0 and a time limit of 10,800 seconds for each instance was used.

4.1 Comparison to Existing Schedule

In this experiment, we compare our results with the existing, manually planned schedule of the „Basketball Senioren Landesliga Brandenburg“ for the season 2023/2024 [1]. The distribution of the nine participating teams across the federal state of Brandenburg is displayed in Figure 1. Given these teams, 324 different minitournaments are possible, while only 24 are chosen. Fixing in the BLP the variables representing the executed league schedule yields a total travel distance of 5,443.8km over all teams after the season. The solution of the instance without fixed variables results in the schedule displayed in Table 4 with a 7.1% smaller total travel distance of 5,056.2km. One can see, that minimizing the total traveled distance disadvantages

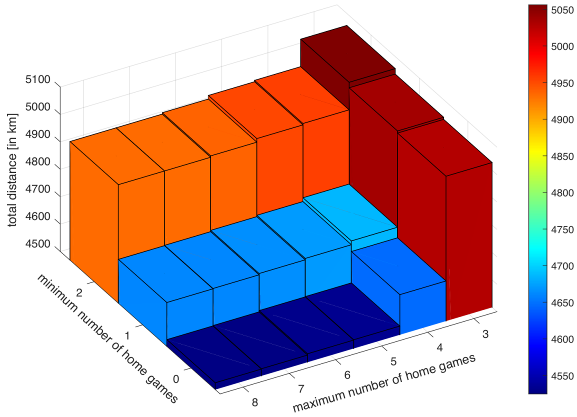

Figure 2: Total traveled distance (in km) for varying values of the minimum and the maximum number of home field advantages.

the teams 2,4, and 8, which are located more on the outside and have one home field advantage less, since it is better that they travel more often instead of two teams traveling to them more than necessary.

4.2 Effect of Uneven Home Games

Since the sacrifice of even distributed home field advantages allows potential savings, we examine in this experiment its magnitude. Therefore, we again took the data of the „Basketball Senioren Landesliga Brandenburg“ and solved the instance for all possible combinations of the parameters $\underline{h} \in \{0, 1, 2\}$ and $\overline{h} \in \{3, \ldots, 8\}$. The results are displayed in Figure 2. As one would expect, the minimum number of home field advantages is a driving factor for the total traveled distance at one season. If some teams would renounce on their home field advantages, it could be reduced by 16.9% to 4,9249km, compared to the existing league schedule in Section 4.1. At the same time, an increase of the maximum number of home field advantages yield only minor savings.

4.3 Computational Analysis

In this experiment, we compare the computational performance of the presented solution approaches. Therefore, we created for each number from seven to 17 teams 15 instances with randomly generated distances and solved them with the BLP (3) – (11), the BLP (3) – (11) with additional variable fixing (12), orbital shrinking with the subproblem (3) – (10) and (22), and orbital shrinking with the smaller subproblem (32) – (34). For the last mentioned approach, the set of feasible game days $\mathcal{R}$ is constructed by enumerating all possibilities with a depth-first search. To compare the performance of the different methods with computation time $c \in [0, 10800]$ and remaining gap $g \in [0, 100]$, we introduce a score function

$$\varphi\colon\ [0, 10800] \times [0, 100] \to [0, 2]\,, \quad \varphi(c, g) = \frac{c}{10,800} + \frac{g}{100}. \tag{35}$$

Thus, an instance was solved to proven global optimality if $\varphi(c, g) \leq 1$ and a score value of $\varphi(c, g) = 2$ means that no feasible solution was found within the given time limit. The median score value of the different methods is displayed in Figure 3, grouped by instances with the same number of teams with a bye. A detailed overview of the number of solved instances and the average computation times is given in Table 5.

Table 5: Detailed results of the comparison between the different solution methods for a varying number of teams.

number of teams	BLP		BLP variable fixing		Orbital Shrinking complete subproblem		Orbital Shrinking small subproblem	
	#solved (fastest)	average time (sec)	#solved (fastest)	average time (sec)	#solved (fastest)	average time (sec)	#solved (fastest)	average time (sec)
7	15 (0)	0.03	15 (0)	0.02	15 (0)	0.02	15 (**15**)	0.01
8	15 (**7**)	1,660.42	15 (3)	763.43	15 (0)	751.40	15 (5)	748.19
9	15 (3)	2.02	15 (**7**)	2.32	15 (0)	33.54	15 (5)	29.70
10	15 (0)	14.26	15 (0)	11.31	15 (1)	3.0081	15 (**14**)	0.46
11	0 (0)	10,800.00	0 (0)	10,800.00	15 (**10**)	143.97	15 (5)	143.88
12	11 (2)	5,961.30	13 (4)	5,602.66	11 (0)	5,420.40	11 (**9**)	5,403.18
13	15 (0)	2,577.83	15 (0)	869.73	15 (2)	1.24	15 (**13**)	0.73
14	0 (0)	10,800.00	0 (0)	10,800.00	6 (**4**)	5,438.71	6 (2)	5,440.06
15	0 (0)	10,800.00	0 (0)	10,800.00	0 (0)	10,800	15 (**15**)	409.00
16	0 (0)	10,800.00	1 (0)	7,965.50	15 (**14**)	28.09	15 (1)	35.37
17	0 (0)	10,800.00	0 (0)	10,800.00	0 (0)	10,800	0 (0)	10,800.00

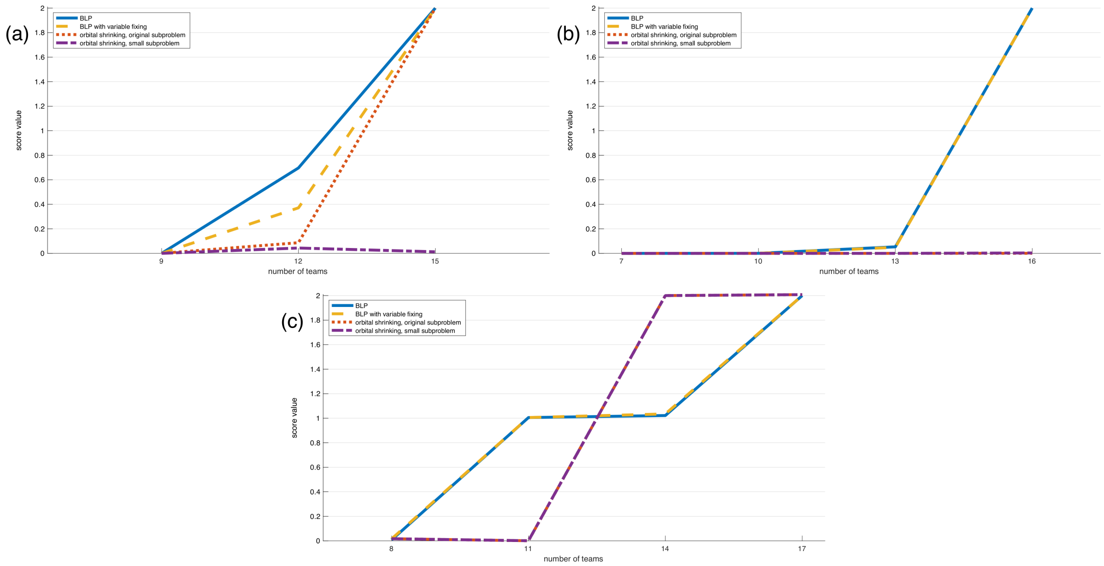

Figure 3: Median score values of the BLP, the BLP with variable fixing, orbital shrinking with original subproblem, and orbital shrinking with small subproblem for instances with $n = 3k$ teams *(a)*, $n = 3k + 1$ teams *(b)*, and $n = 3k + 2$ teams *(c)*, $k \in \{2, \ldots, 5\}$.

To solve all 165 instances with the BLP, Gurobi required 939,838 seconds. One can see, that, next to number of participating teams, the number of teams with a bye has a significant effect on the computation times. Instances with $n = 3k + 2$, $k \in \{2, \ldots, 5\}$, teams seem to be hard since also for eleven teams, no instance was solved to proven global optimality within the time limit. Furthermore, the first instances that were not solved within seconds occurred for eight teams, while all instances with nine teams were solved almost immediately again. Adding the variable fixing to the BLP accelerates the overall solution process for instances with $n = 3k$ teams, $k \in \{2, \ldots, 5\}$, yielding a total computation time of 892,187 seconds for all instances. Nevertheless, there is the same behavior for instances with $n = 3k + 1$ or $n = 3k + 2$ teams, $k \in \{2, \ldots, 5\}$. Interestingly, with additional variable fixings, there is only for one instance with 16 teams a feasible solution. Without variable fixings, in three of them a feasible solution was found.
Changing the solution procedure to an iterative method by orbital shrinking reduces the total computation time for all instances significantly. With orbital shrinking and the complete subproblem, it took 500,668 seconds, while the orbital shrinking method with the reduced subproblem required only 336,396 seconds. Note, that both methods either find the proven global optimal solution within the time limit or terminate with no feasible solution. Thus, either there is $\varphi(c, g) \leq 1$ in Figure 3 or $\varphi(c, g) = 2$ for any instance. For both orbital shrinking variants, the first instance without a feasible solution occurred for twelve teams, where the first stage hit the time limit without calling the subproblem. This behavior was also the reason that no instance with 17 teams was solved to optimality. The main difference between both approaches is

visible at the instances with twelve and 15 teams. Due to the faster solution of the subproblem, the orbital shrinking variant with the smaller subproblem required less computation time to solve these instances or even allows multiple iterations, while orbital shrinking with the complete subproblem was not able to decide the feasibility of the first solution candidate within the time limit. Although, there are three instances with twelve teams, that were solved by the BLP with variable fixing and not by the orbital shrinking approaches.

5 Conclusion and Future Work

In this paper, we introduced a new variant of a sports league scheduling problem with minitournaments of three teams instead of single matches per game day and derived a binary linear program to generate feasible league schedules with a minimal total traveling distance. Due to the high symmetry of the problem, variable fixing and orbital shrinking as methods to reduce its effect in the solution process were applied. Furthermore, we adjusted the orbital shrinking method to the structure of our scheduling problem to reduce the size of the subproblem. The different solution approaches were evaluated in a computational study. It revealed, that the orbital shrinking approach is superior to solving the BLP in nearly all considered instances. Furthermore, the reduction of the subproblem allows the solution of several instances that cannot be handeled by orbital shrinking with the complete subproblem. Nevertheless, orbital shrinking has still its limitations at the computation of the master problem for larger team numbers. A comparison of our optimization approach to an existing league schedule of the „Basketball Senioren Landesliga Brandenburg“ for the season 2023/2024 revealed, that significant savings are possible.
There are several directions for future work on this problem. The presented model is a pure economic view. One could also take a look from a fairness-based perspective, where at first the meaning of fairness in this context must be clarified. Furthermore, it would be interesting if there are other numbers of teams, which yield an infeasible problem. Regarding the solution procedure, a closer look on the problem structure of instances with additional single matches could give insights how they can be solved more efficiently. The same holds for the master problem of the orbital shrinking method, which is computationally expensive for larger team numbers. As a natural generalization one could also think of larger minitournaments with more participating teams, but then another model formulation seems necessary since enumerating all possible minitournaments is no longer an reasonable option.

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