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# Existence results for diffuse interface models describing phase separation and damage

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**Abstract**

In this paper we analytically investigate Cahn-Hilliard and Allen-Cahn systems which are coupled with elasticity and uni-directional damage processes. We are interested in the case where the free energy contains logarithmic terms of the chemical concentration variable and quadratic terms of the gradient of the damage variable. For elastic Cahn-Hilliard and Allen-Cahn systems coupled with uni-directional damage processes, an appropriate notion of weak solutions is presented as well as an existence result based on certain regularization methods and an higher integrability result for the strain.

# 1 Introduction

Phase separation and damage are common phenomena in many fields, including material sciences, hydrodynamics, biology and chemical reactions. Such microstructural processes take place to reduce the total free energy, which may include bulk chemical energy, interfacial energy and elastic strain energy.

The knowledge of the mechanisms inducing phase separation and damage processes is very important for technological applications as for instance in the area of micro-electronics due to the ongoing miniaturization. The materials used in this area are typically alloys consisting of mixtures of several components.

Phase separation and damage processes are usually described by two separate models in the mathematical literature. To describe phase separation processes for alloys, phase-field models of Cahn-Hilliard and Allen-Cahn type coupled with elasticity are well adapted. Damage processes for standard materials are often modeled as unilateral processes within a gradient-theory. A phase-field approach which describes *both* phase separation and damage processes in an unifying model has been recently introduced in [HK10].

Our aim here is to develop and investigate analytically a thermodynamically consistent diffuse phase field model where phase separation and damage processes may interact. We focus on a rate-dependent modeling for the damage field since damage processes in alloys are usually rate-independent. The phase separation and damage processes are described by a system of evolutionary equations of Cahn-Hilliard/Allen-Cahn type with elasticity and an evolutionary inclusion for the damage variable. In the following we will introduce the model formally.

The elastic material we want to consider in this work is an $N$ component alloy occupying a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^n$. To take phase separation, deformation and damage processes in *one* model into account, a state at a fixed time point is described by a triple $(u, c, z)$ where $u : \Omega \to \mathbb{R}^n$ denotes the deformation, $c : \Omega \to \mathbb{R}^N$ the vector of the chemical concentrations and $z : \Omega \to \mathbb{R}$ the damage variable. Each variable and its governing evolutionary process is explained in the following more explicitly.

### Elasticity.

Elastic behavior is modeled by a deformation variable $u$ so that each material point $x \in \Omega$ from the reference configuration is located at $x + u(t,x)$. We use the assumption that the strain $e$ is sufficiently small so that we can work with the linearized strain tensor given by $e(u) = \frac{1}{2}(\nabla u + (\nabla u)^t)$. The stress tensor $\sigma$ is defined by $W^{\mathrm{el}}_{,e}$, i.e. as the derivative of the elastic energy with respect to the strain. Hence for materials fulfilling Hook's law the stress tensor is determined by $\mathbb{C} : e(u)$. In this work we will neglect kinetic energies $\rho\ddot{u}$ and volume forces $b$ so that the momentum balance equation $\mathrm{div}(\sigma) + b = \rho\ddot{u}$ becomes a quasi-static force equation, i.e. $\mathrm{div}(\sigma) = 0$.

### Phase separation and phase transition.

The mixture of the alloy is described by a phase field vector $c = (c_1, \ldots, c_N)$, where the elements $c_k$, $k = 1, \ldots, N$, denotes the concentration of the component $k$. Therefore we will restrict the state space for $c$ to the physically meaningful conditions $\sum_{j=1}^{N} c_j = 1$ and $c_k \geq 0$ in $\Omega$ for every $k = 1, \ldots, N$.

If an alloy is cooled down below a critical temperature then usually spinodal decomposition and coarsening phenomena occur. Well established models for describing such effects are the Cahn-Hilliard and Allen-Cahn equations, which describe order parameter preserving and order parameter non-preserving phase separation in solids, cf. [Cah61, LC82, Gur96, BCD$^+$02, Gar00, CMP00, Gar05a, Gar05b, BP05] and [AC79, CNC94, BdS04, CP08, CGPGS10], respectively. The essential difference between these two equations is that that the Cahn-Hilliard equation is a fourth order parabolic evolutionary equation expressible as a $H^{-1}$ gradient flow of the free energy with respect to $c$ whereas the Allen-Cahn equation is a second order parabolic equation coming from an $L^2$ gradient flow. Analytical results for multi-component Cahn-Hilliard equations coupled with elastic deformations can be found in [Gar00]. These results have also been generalized to Allen-Cahn systems with elasticity [BW05]. It turns out that different elastic moduli of the phases in the mixture influence the rate of coarsening and the morphology of the phases decisively [DM00]. Numerical investigations are conducted in [GRW01]. Finite element error estimates of Cahn-Hilliard and Allen-Cahn equations with logarithmic free energies are recently derived in [BM10].

### Damage.

The damage process we want to consider here is uni-directional, i.e. it can only increase in time and the material is not able to heal itself. The phase field variable $0 \leq z \leq 1$ is interpreted as damage in a way that $z(x) = 1$ stands for a non-damaged and $z(x) = 0$ for a completely damaged material point $x \in \Omega$. These constraints lead to a differential inclusion formulation for the evolution of $z$ which relates the energy dissipation of the system due to increasing damage with the derivative of the free energy with respect to $z$. If the damage dissipation is given by a functional $\mathcal{R}$ which is homogenous of degree one, i.e. $\mathcal{R}(\alpha g) = \alpha\mathcal{R}(g)$ for all $\alpha > 0$, the system is called rate-independent and the well-developed framework of rate-independent systems [Mie05] may be applied to this case. See [MRZ10] for analytical results on rate-independent damage and numerical experiments. We also refer to [BSS05, EM06, FK09, MT10] for further analytical investigations of damage models.

The main objective of this work is to present existence results for systems where both Cahn-Hilliard (or Allen-Cahn) *and* rate-dependent damage differential inclusions for elastic materials are coupled. The work [HK10] introduces a unifying model for phase separation and damage processes for Cahn-Hilliard systems in the first instance. Here we generalize the results of [HK10] concerning the following aspects: Chemical free energies with logarithmic growth are studied. A gradient-of-damage theory is developed which does not depend on the space $n$, i.e. gradient terms $|\nabla z|^p$ in the free energy are not restricted to the assumption $p > n$. Multi-component systems are considered instead of binary alloys. In addition, elastic Allen-Cahn systems coupled with damage are investigated.

The systems we would like to consider in this work are governed by the following equations and inclusion, respectively:

$$\left.\begin{cases} \partial_t c = -\mathcal{S}w, & \text{on } \Omega_T \\ w = \mathbb{P}(-\mathrm{div}(\mathbf{\Gamma}\nabla c) + W^{\mathrm{ch,log}}_{,c}(c) + W^{\mathrm{el}}_{,c}(e(u),c,z)), & \text{on } \Omega_T \\ \mathrm{div}(\sigma) = 0, & \text{on } \Omega_T \\ \partial\rho(\partial_t z) - \Delta z + W^{\mathrm{el}}_{,z}(e(u),z) + \partial\iota_{[0,\infty)}(z) \ni 0, & \text{on } \Omega_T \end{cases}\right\} \tag{$\mathbf{S_0}$}$$

Here $W^{\mathrm{ch,log}}$ denotes the logarithmic chemical energy density, $W^{\mathrm{el}}$ the elastic energy density, $\sigma = W^{\mathrm{el}}_{,e}$ the stress tensor and $\mathbf{\Gamma}$ the gradient energy tensor which is a fourth order symmetric and positive definite tensor, mapping matrices from $\mathbb{R}^{N\times n}$ into itself. The matrix $\mathbb{P}$ on the other hand denotes the orthogonal projection of $\mathbb{R}^N$ onto the tangent space $T\Sigma = \{c \in \mathbb{R}^N \mid \sum_{k=1}^N c_k = 0\}$ (with $\Sigma$ given in (4)) The energy dissipation density due to damage progression is given by $\rho$ where we use the ansatz

$$\rho(\dot z) = -\alpha\dot z + \frac{\beta}{2}|\dot z|^2 + \iota_{(-\infty,0]}(\dot z) \tag{1}$$

with $\alpha, \beta > 0$. The operator $\mathcal{S}$ determines whether we have an *Allen-Cahn* or a *Cahn-Hilliard* evolution of the system. More precisely:

$$\mathcal{S} = \begin{cases} \mathbb{M} : L^2(\Omega;\mathbb{R}^N) \to L^2(\Omega;\mathbb{R}^N), & \text{for Allen-Cahn system,} \\ \qquad f \mapsto \mathbb{M}f, & \\ -\mathrm{div}(\mathbb{M}\nabla) : Y \to \mathcal{D}, & \text{for Cahn-Hilliard system,} \\ \qquad f \mapsto \langle \mathbb{M}\nabla f, \nabla\cdot\rangle_{L^2}\,, & \end{cases} \tag{2}$$

where $\mathbb{M}$ is a constant mobility matrix. The spaces $Y$ and $\mathcal{D}$ are defined as

$$\begin{aligned} Y &:= \left\{ c \in H^1(\Omega;\mathbb{R}^N) \mid \int_\Omega c = 0,\ \sum_{k=1}^N c_k = 0 \right\}, \\ \mathcal{D} &:= \Big\{ c^\star \in \left(H^1(\Omega;\mathbb{R}^N)\right)^\star \mid \langle c^\star, c\rangle_{(H^1)^\star\times H^1} = 0 \text{ for all } c = d(x)(1,\ldots,1), \\ &\qquad\qquad \text{where } d \in H^1(\Omega) \text{ and for all } c = e_k,\ k = 1,\ldots N \Big\}. \end{aligned}$$

We need to impose some restrictions to the mobility matrix $\mathbb{M}$. We assume that $\mathbb{M}$ is symmetric and positive definite on the tangent space $T\Sigma$. In addition, due to the constraint

$\sum_{k=1}^{N} c_k = 1$, $\mathbb{M}$ has to satisfy the property $\sum_{l=1}^{N} \mathbb{M}_{kl} = 0$ for all $k = 1, \ldots, N$. Note, that $\mathbb{M} = \mathbb{M}\,\mathbb{P}$.

The chemical free energy density is of the form

$$W^{\mathrm{ch,log}}(c) = \theta \sum_{k=1}^{N} c_k \log c_k + \frac{1}{2} c \cdot Ac \tag{3}$$

with $\theta > 0$ and symmetric $A \in \mathbb{R}^{N\times N}$. The inital-boundary conditions of our systems are summarized as follows:

$$\left.\begin{cases} c(0) = c^0, & \text{on } \Omega, \\ z(0) = z^0, & \text{on } \Omega, \\ u = b, & \text{on } \Gamma_T, \\ \sigma \cdot \overrightarrow{\nu} = 0, & \text{on } (\partial\Omega)_T \setminus \Gamma_T, \\ \mathbb{M}\nabla w \cdot \overrightarrow{\nu} = 0, & \text{on } (\partial\Omega)_T \text{ for C-H systems}, \\ \boldsymbol{\Gamma}\nabla c \cdot \overrightarrow{\nu} = 0, & \text{on } (\partial\Omega)_T, \\ \nabla z \cdot \overrightarrow{\nu} = 0, & \text{on } (\partial\Omega)_T, \end{cases}\right\} \tag{IBC}$$

where $\overrightarrow{\nu}$ is the unit normal on $\partial\Omega$ pointing outward, $\Gamma \subseteq \partial\Omega$ the Dirichlet part of the boundary with $\mathcal{H}^{n-1}(\Gamma) > 0$, $b$ the Dirichlet boundary value function, $0 \leq z_0 \leq 1$ a.e. on $\Omega$, $c^0 \in \Sigma \cap \mathbb{R}^N_{>0}$ with the affine plane

$$\Sigma := \Big\{ c \in \mathbb{R}^N \mid \sum_{l=1}^{N} c_k = 1 \Big\}, \tag{4}$$

$\Gamma_T := [0,T] \times \Gamma$ and $(\partial\Omega)_T := [0,T] \times \partial\Omega$.

The paper is organized as follows: In Chapter 2 we introduce an appropriate notion of weak solutions for the system ($\mathbf{S_0}$). To handle the differential inclusion rigorously, we develop the concept of energetic solutions originally introduced in the context of rate-independent systems (see for instance [Mie05]) to phase separation systems. This approach was firstly presented in [HK10]. In Chapter 3 we prove existence of weak solutions for a regularization of system ($\mathbf{S_0}$) expressed in classical formulation as

$$\left.\begin{cases} \partial_t c = -\mathcal{S}w, & \text{on } \Omega_T \\ w = \mathbb{P}(-\mathrm{div}(\boldsymbol{\Gamma}\nabla c) + W^{\mathrm{ch}}_{,c}(c) + W^{\mathrm{el}}_{,c}(e(u), c, z) + \varepsilon\partial_t c), & \text{on } \Omega_T \\ \mathrm{div}(\sigma) + \mathrm{div}(|\nabla u|^2 \nabla u) = 0, & \text{on } \Omega_T \\ \partial\rho(\partial_t z) - \Delta z - \varepsilon\mathrm{div}(|\nabla z|^{p-2}\nabla z) + W^{\mathrm{el}}_{,z}(e(u), z) + \partial\iota_{[0,\infty)}(z) \ni 0, & \text{on } \Omega_T \end{cases}\right\} \tag{$\mathbf{S}_\varepsilon$}$$

where $W^{\mathrm{ch}}$ and $W^{\mathrm{el}}$ satisfy certain polynomial growth conditions and $p > n$. The solutions of the regularized system have better regularity in $c \in H^1([0,T]; L^2(\Omega; \mathbb{R}^N))$ and better integrability in $\nabla u \in L^4(\Omega_T; \mathbb{R}^{n\times n})$ and in $\nabla z \in L^p(\Omega_T, \mathbb{R}^n)$ (with $p > n$ as above) and are constructed by adapting the approximation techniques developed in [HK10].

The limit problem $\varepsilon \searrow 0$ for ($\mathbf{S}_\varepsilon$) is solved in Chapter 4. The displacement field $u$ obtained in this process has $H^1(\Omega; \mathbb{R}^n)$-regularity in the first instance. To establish existence results for chemical free energies of logarithmic type, we prove a higher integrability result for $\nabla u$ in Chapter 5, which is based on some ideas similar to [Gar00, Gar05b]. Finally, Chapter 6 concerns about logarithmic free energies for the concentration $c$. Following the approach in [Gar00, Gar05b], we use a suitable regularization $W^{\mathrm{ch},\varepsilon}$ with polynomial growth of the logarithmic free energy density $W^{\mathrm{ch,log}}$ to obtain a solution for $(\mathbf{S_0})$. Using this regularization, the chemical components $c_k$ become strictly positive in the limit.

The notation we will use throughout this paper are collected in the following.

**Notation.**

Spaces and subsets

| | |
|---|---|
| $W^{1,r}_+(\Omega)$ | functions of $W^{1,r}(\Omega)$ which are non-negative almost everywhere |
| $W^{1,r}_-(\Omega)$ | functions of $W^{1,r}(\Omega)$ which are non-positive almost everywhere |
| $W^{1,r}_\Gamma(\Omega;\mathbb{R}^n)$ | functions of $W^{1,r}(\Omega;\mathbb{R}^n)$ which vanishes on $\Gamma$ in the sense of traces |
| $B(I,X)$ | space of everywhere bounded functions on the interval $I$ |
| $B_R(A)$ | open neighborhood of $A\subseteq\mathbb{R}^n$ with thickness $R$ |
| $Q_R(x_0)$ | cube $\{x\in\mathbb{R}^n \mid \|x-x_0\|_\infty<R\}$ |
| $\{f=\mu\}$ | level set of a function $f$ with value $\mu$ |
| $\Omega_T$ | the set $[0,T]\times\Omega$ |
| $N_{\mathrm{F}}(M;x)$ | the Fréchet normal cone at $x$ of the set $M$ |

Functions, operations and measures

| | |
|---|---|
| $[f]^+$ | maximum function $\max\{0,f\}$ |
| $\iota_M$ | indicator function of $M$ |
| $\chi_M$ | characteristic function of $M$ |
| $W_{,e}$ | derivative of a function $W$ with respect to the variable $e$ |
| $\langle A,f\rangle$ | dual pairing of $A\in(W^{1,r}(\Omega))^\star$ and $f\in W^{1,r}(\Omega)$ |
| $\partial^{\mathrm{Cl}}E$ | Clarke subdifferential of a functional $E$ |
| $\langle \mathrm{d}E,\zeta\rangle$ | Gâteaux differential of $E$ in direction $\zeta$ |
| $p^\star$ | Sobolev critical exponent $\frac{np}{n-p}$ for $n>p$ |
| $\mathrm{diam}(D)$ | diameter of $D$ |
| $\mathcal{H}^n$ | Hausdorff measure of dimension $n$ |
| $\mathcal{L}^n$ | Lebesgue measure of dimension $n$ |

# 2 Existence theorem

## 2.1 Weak formulation

The weak notion we will derive in this section for the doubly nonlinear differential inclusion occurring in $(\mathbf{S_0})$ is inspired by the concept of energetic solutions for rate-independent systems (see for instance [Mie05]). In the rate-independent setting the differential inclusion is formulated by a global stability condition and an energy inequality. In [HK10] we introduced an approach which uses an energy inequality and a variational inequality to handle the rate-dependence coming from the viscousity term $\beta|\partial_t z|^2$ in the damage dissipation density function $\rho$.

To derive a formulation, we introduce the following Gâteaux-differentiable free energy $\tilde{\mathcal{E}}$ and dissipation functional $\tilde{\mathcal{R}}$:

$$\begin{aligned}
&\tilde{\mathcal{E}} : H^1(\Omega;\mathbb{R}^n)\times H^1(\Omega;\mathbb{R}^N)\times(H^1(\Omega)\cap L^\infty(\Omega))\to\mathbb{R},\\
&\qquad(u,c,z)\mapsto\int_\Omega\frac{1}{2}\mathbf{\Gamma}\nabla c:\nabla c+\frac{1}{2}|\nabla z|^2+W^{\mathrm{ch,log}}(c)+W^{\mathrm{el}}(e,c,z)\,\mathrm{d}x,\\
&\tilde{\mathcal{R}}(\dot z) : L^2(\Omega)\to\mathbb{R},\\
&\qquad\dot z\mapsto\int_\Omega-\alpha\dot z+\frac{\beta}{2}|\dot z|^2\,\mathrm{d}x
\end{aligned}$$

with viscousity constants $\alpha,\beta>0$. To account for constraints, we extend the functionals above for analytical reasons by indicator functions:

$$\begin{aligned}
&\mathcal{E} : H^1(\Omega;\mathbb{R}^n)\times H^1(\Omega;\mathbb{R}^N)\times(H^1(\Omega)\cap L^\infty(\Omega))\to\mathbb{R},\\
&\qquad\mathcal{E}(u,c,z):=\begin{cases}\tilde{\mathcal{E}}(u,c,z), & \text{if } z\geq 0 \text{ a.e.},\\ \infty, & \text{else},\end{cases}\\
&\mathcal{R}(\dot z) : L^2(\Omega)\to\mathbb{R},\\
&\qquad\mathcal{R}(\dot z):=\begin{cases}\tilde{\mathcal{R}}(\dot z), & \text{if } \dot z\leq 0,\\ \infty, & \text{else}.\end{cases}
\end{aligned}$$

We first observe that the system ($\mathbf{S_0}$) can be rewritten using the functionals as

$$\partial_t c=-\mathcal{S}w, \tag{5a}$$

$$w=\mathbb{P}\mathrm{d}_c\mathcal{E}(u,c,z), \tag{5b}$$

$$\mathrm{d}_u\mathcal{E}(u,c,z)=0, \tag{5c}$$

$$\partial_z^{\mathrm{Cl}}\mathcal{E}(u,c,z)+\partial_{\dot z}\mathcal{R}(\dot z)\ni 0. \tag{5d}$$

The operator $\mathcal{S}$ is given in (2). If we equip the space $H^1(\Omega)\cap L^\infty(\Omega)$ with the norm $\|\cdot\|_{H^1\cap L^\infty}:=\|\cdot\|_{H^1}+\|\cdot\|_{L^\infty}$ the generalized subdifferential $\partial_z^{\mathrm{Cl}}\mathcal{E}$ at a point $(u,c,z)\in H^1(\Omega;\mathbb{R}^n)\times H^1(\Omega;\mathbb{R}^N)\times(H^1(\Omega)\cap L^\infty(\Omega))$ is

$$\partial_z^{\mathrm{Cl}}\mathcal{E}(u,c,z)=\left\{\mathrm{d}_z\tilde{\mathcal{E}}(u,c,z)+r\in(H^1(\Omega)\cap L^\infty(\Omega))^\star\,\middle|\,r\in N_{\mathrm{F}}(H^1_+(\Omega)\cap L^\infty(\Omega);z)\right\}.$$

The inclusion $L^1(\Omega)\subset(H^1(\Omega)\cap L^\infty(\Omega))^\star$ will be later used for the construction of a specific $r$.

The analytical basis for a formulation of a weak solution is the following proposition (a related version can be found in [HK10]):

**Proposition 2.1** *Let $(u,c,w,z)\in C^2(\Omega_T;\mathbb{R}^n\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R})$ be a smooth solution of* (5a)-(5c) *with initial-boundary conditions* (**IBC**)*. Then the following two conditions are equivalent:*

*(i)* $0\in\partial_z^{\mathrm{Cl}}\mathcal{E}(u(t),c(t),z(t))+\partial_{\dot z}\mathcal{R}(\dot z(t))$ *for all* $t\in[0,T]$,

*(ii) the energy inequality*

$$\begin{aligned}\mathcal{E}(u(t),c(t),z(t)) + \int_0^t \langle \mathrm{d}_{\dot z}\tilde{\mathcal{R}}(\partial_t z), \partial_t z\rangle \,\mathrm{d}s + \int_0^t \langle \mathcal{S}w(s), w(s)\rangle \mathrm{d}s \\ \leq \mathcal{E}(u(0),c(0),z(0)) + \int_{\Omega_t} W^{\mathrm{el}}_{,e}(e(u),c,z) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}s\end{aligned}$$

*for all* $0 \leq t \leq T$ *and the variational inequality*

$$0 \leq \left\langle \mathrm{d}_z\tilde{\mathcal{E}}(u(t),c(t),z(t)) + r(t) + \mathrm{d}_{\dot z}\tilde{\mathcal{R}}(\partial_t z(t)), \zeta \right\rangle$$

*for all* $\zeta \in H^1_-(\Omega) \cap L^\infty(\Omega)$ *and* $r(t) \in N_{\mathrm{F}}(H^1_+(\Omega) \cap L^\infty(\Omega); z(t))$ *and for all* $0 \leq t \leq T$.

*If one of the two conditions holds then the following energy balance equation is satisfied:*

$$\begin{aligned}\mathcal{E}(u(t),c(t),z(t)) + \int_0^t \langle \mathrm{d}_{\dot z}\tilde{\mathcal{R}}(\partial_t z), \partial_t z\rangle \,\mathrm{d}s + \int_0^t \langle \mathcal{S}w(s), w(s)\rangle \mathrm{d}s \\ = \mathcal{E}(u(0),c(0),z(0)) + \int_{\Omega_t} W^{\mathrm{el}}_{,e}(e(u),c,z) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}s\end{aligned}$$

We will see that in our approach the mathematical analysis of $(\mathbf{S_0})$ requires several regularization terms to establish the energy and variational inequality for the differential inclusion and to handle the logarithmic free energy. A transition to $\varepsilon \searrow 0$ will finally give us solution of the limit problem $(\mathbf{S_0})$.

Proposition 2.1 can also be formulated for the regularized system $(\mathbf{S}_\varepsilon)$ which may be written in a similar form:

$$\left\{\begin{array}{ll}\partial_t c = -\mathcal{S}w, & \text{on } \Omega_T \\ w = \mathbb{P}\mathrm{d}_c\mathcal{E}_\varepsilon(u,c,z) + \varepsilon\partial_t c, & \text{on } \Omega_T \\ \mathrm{d}_u\mathcal{E}_\varepsilon(u,c,z) = 0, & \text{on } \Omega_T \\ \partial_z^{\mathrm{Cl}}\mathcal{E}_\varepsilon(u,c,z) + \partial_{\dot z}\mathcal{R}(\dot z) \ni 0, & \text{on } \Omega_T\end{array}\right\}$$

with the regularized energy

$$\begin{aligned}\tilde{\mathcal{E}}_\varepsilon(u,c,z) &:= \int_\Omega \frac{1}{2}\mathbf{\Gamma}\nabla c : \nabla c + \frac{1}{2}|\nabla z|^2 + W^{\mathrm{ch}}(c) + W^{\mathrm{el}}(e,c,z) + \frac{\varepsilon}{4}|\nabla u|^4 + \frac{\varepsilon}{p}|\nabla z|^p \,\mathrm{d}x, \\ \mathcal{E}_\varepsilon(u,c,z) &:= \tilde{\mathcal{E}}_\varepsilon(u,c,z) + \int_\Omega \iota_{[0,\infty)}(z)\,\mathrm{d}x,\end{aligned}$$

and the same initial-boundary conditions (**IBC**). On should notice that $\mathbb{P}\partial_t c = \partial_t c$ because of $\partial_t c(t,x) \in \Sigma$. It is assumed that the energy densities $W^{\mathrm{ch}}$ and $W^{\mathrm{el}}$ in this system satisfy some polynomial growth conditions which will be specified later.

**Definition 2.2 (Weak solution for the regularized system $(\mathbf{S}_\varepsilon)$)** *We call a tuple* $q = (u,c,w,z)$ *a weak solution of the regularized system* $(\mathbf{S}_\varepsilon)$ *with initial-boundary conditions* (**IBC**) *if it satisfies the following properties:*

*(i) the components of $q$ are in the following spaces:*

$$u \in L^\infty([0,T]; W^{1,4}(\Omega;\mathbb{R}^n)),\ u|_{\Gamma_T} = b|_{\Gamma_T},$$
$$c \in L^\infty([0,T]; H^1(\Omega;\mathbb{R}^N)) \cap H^1([0,T]; L^2(\Omega;\mathbb{R}^N)),\ c(0) = c^0,$$
$$z \in L^\infty([0,T]; W^{1,p}_+(\Omega)) \cap H^1([0,T]; L^2(\Omega)),\ z(0) = z^0,\ \partial_t z \le 0,$$

*and*

$$w \in L^2([0,T]; H^1(\Omega;\mathbb{R}^N)), \qquad \textit{for C-H systems}$$
$$w \in L^2(\Omega_T;\mathbb{R}^N), \qquad \textit{for A-C systems}$$

*(ii) for all $\zeta \in H^1(\Omega;\mathbb{R}^N)$ and for a.e. $t \in [0,T]$:*

$$\int_\Omega \partial_t c(t)\cdot\zeta\,\mathrm{d}x = -\langle \mathcal{S}w(t),\zeta\rangle$$

*(iii) for all $\zeta \in H^1(\Omega;\mathbb{R}^N)$ and for a.e. $t \in [0,T]$:*

$$\begin{aligned}\int_\Omega w(t)\cdot\zeta\,\mathrm{d}x = &\int_\Omega \mathbb{P}\mathbf{\Gamma}\nabla c(t) : \nabla\zeta + \mathbb{P}W^{\mathrm{ch}}_{,c}(c(t))\cdot\zeta\,\mathrm{d}x\\ &+\int_\Omega \mathbb{P}W^{\mathrm{el}}_{,c}(e(u(t)),c(t),z(t))\cdot\zeta + \varepsilon\partial_t c(t)\cdot\zeta\,\mathrm{d}x\end{aligned}$$

*(iv) for all $\zeta \in W^{1,4}_\Gamma(\Omega;\mathbb{R}^n)$ and for a.e. $t \in [0,T]$:*

$$\int_\Omega W^{\mathrm{el}}_{,e}(e(u(t)),c(t),z(t)) : e(\zeta) + \varepsilon|\nabla u(t)|^2\nabla u(t) : \nabla\zeta\,\mathrm{d}x = 0$$

*(v) for all $\zeta \in W^{1,p}_-(\Omega)$ and for a.e. $t \in [0,T]$:*

$$\begin{aligned}\int_\Omega &(\varepsilon|\nabla z(t)|^{p-2}+1)\nabla z(t)\cdot\nabla\zeta + (W^{\mathrm{el}}_{,z}(e(u(t)),c(t),z(t)) - \alpha + \beta(\partial_t z(t)))\zeta\,\mathrm{d}x\\ &\ge -\langle r(t),\zeta\rangle,\end{aligned}$$

*where $r(t) \in (W^{1,p}(\Omega))^\star$ satisfies $\langle r(t), z(t)-\zeta\rangle \ge 0$ for all $\zeta \in W^{1,p}_+(\Omega)$*

*(vi) energy inequality for a.e. $t \in [0,T]$:*

$$\begin{aligned}&\mathcal{E}_\varepsilon(u(t),c(t),z(t)) - \mathcal{E}_\varepsilon(u(0),c(0),z(0)) + \int_\Omega \alpha(z^0 - z(t))\,\mathrm{d}x\\ &+\int_{\Omega_t}\beta|\partial_t z|^2 + \varepsilon|\partial_t c|^2\,\mathrm{d}x\mathrm{d}s + \int_0^t \langle \mathcal{S}w(s),w(s)\rangle\,\mathrm{d}s\\ &\qquad\le \int_{\Omega_t} W^{\mathrm{el}}_{,e}(e(u),c,z) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}s + \varepsilon\int_{\Omega_t}|\nabla u|^2\nabla u : \nabla\partial_t b\,\mathrm{d}x\mathrm{d}s\end{aligned}$$

**Definition 2.3 (Weak solution for the limit system ($\mathbf{S_0}$))** *A tuple $q = (u,c,w,z)$ is called a weak solution of the system* ($\mathbf{S_0}$) *with initial-boundary conditions* (**IBC**) *if it satisfies the following conditions:*

*(i) the components of $q$ are in the following spaces:*

$$\begin{aligned}
&u \in L^\infty([0,T]H^1(\Omega;\mathbb{R}^n)),\ u|_{\Gamma_T} = b|_{\Gamma_T},\\
&c \in L^\infty([0,T];H^1(\Omega;\mathbb{R}^N)),\ c_k > 0\\
&z \in L^\infty([0,T];H^1_+(\Omega)) \cap H^1([0,T];L^2(\Omega)),\ z(0) = z^0,\ \partial_t z \le 0
\end{aligned}$$

*and*

$$\begin{aligned}
&w \in L^2([0,T];H^1(\Omega;\mathbb{R}^N)) &&\textit{for C-H systems}\\
&w \in L^2(\Omega_T;\mathbb{R}^N) &&\textit{for A-C systems}
\end{aligned}$$

*(ii) for all $\zeta \in L^2([0,T];H^1(\Omega;\mathbb{R}^N))$ with $\partial_t\zeta \in L^2(\Omega_T;\mathbb{R}^N)$ and $\zeta(T) = 0$:*

$$\int_{\Omega_T} (c - c^0)\cdot\partial_t\zeta\,\mathrm{d}x = \int_0^T \langle \mathcal{S}w, \zeta\rangle\,\mathrm{d}t$$

*(iii) for all $\zeta \in H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)$ and for a.e. $t \in [0,T]$:*

$$\begin{aligned}
\int_\Omega w(t)\cdot\zeta\,\mathrm{d}x &= \int_\Omega \mathbb{P}\mathbf{\Gamma}\nabla c(t) : \nabla\zeta + \mathbb{P}W^{\mathrm{ch,log}}_{,c}(c(t))\cdot\zeta\,\mathrm{d}x\\
&\quad + \int_\Omega \mathbb{P}W^{\mathrm{el}}_{,c}(e(u(t)), c(t), z(t))\cdot\zeta\,\mathrm{d}x
\end{aligned}$$

*(iv) for all $\zeta \in H^1_\Gamma(\Omega;\mathbb{R}^n)$ and for a.e. $t \in [0,T]$:*

$$\int_\Omega W^{\mathrm{el}}_{,e}(e(u(t)), c(t), z(t)) : e(\zeta)\,\mathrm{d}x = 0$$

*(v) for all $\zeta \in H^1_-(\Omega) \cap L^\infty(\Omega)$ and for a.e. $t \in [0,T]$:*

$$\int_\Omega \nabla z(t)\cdot\nabla\zeta + (W^{\mathrm{el}}_{,z}(e(u(t)), c(t), z(t)) - \alpha + \beta(\partial_t z(t)))\zeta\,\mathrm{d}x \ge -\langle r(t), \zeta\rangle,$$

*where $r(t) \in (H^1(\Omega)\cap L^\infty(\Omega))^\star$ satisfies $\langle r(t), z(t) - \zeta\rangle \ge 0$ for all $\zeta \in H^1_+(\Omega)\cap L^\infty(\Omega)$*

*(vi) energy inequality for a.e. $t \in [0,T]$:*

$$\begin{aligned}
\mathcal{E}(u(t), c(t), z(t)) &+ \int_\Omega \alpha(z^0 - z(t))\,\mathrm{d}x + \int_{\Omega_t} \beta|\partial_t z|^2\,\mathrm{d}x\mathrm{d}s + \int_0^t \langle \mathcal{S}w(s), w(s)\rangle\,\mathrm{d}s\\
&\le \mathcal{E}(u(0), c(0), z(0)) + \int_{\Omega_t} W^{\mathrm{el}}_{,e}(e(u), c, z) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}s
\end{aligned}$$

**Remark 2.4** *(i) Both notions of weak solution imply chemical mass conservation, i.e.*

$$\int_\Omega c(t)\,\mathrm{d}x \equiv \mathit{const}, \qquad c \in \Sigma \textit{ a.e. in } \Omega_T.$$

*(ii) The dual pairing $\langle \mathcal{S}w(t), \zeta\rangle$ can be rewritten as follows:*

$$\begin{aligned}
&\langle \mathcal{S}w(t), \zeta\rangle = \int_\Omega \mathbb{M}\nabla w(t) : \nabla\zeta\,\mathrm{d}x, &&\textit{for C-H systems}\\
&\langle \mathcal{S}w(t), \zeta\rangle = \int_\Omega \mathbb{M}w(t)\cdot\zeta\,\mathrm{d}x, &&\textit{for A-C systems}
\end{aligned}$$

## 2.2 Assumptions and main result

Before we state the assumptions that are necessary for the proof of the existence theorem, all symbols and coefficient functions are collected in the following:

| | |
|---|---|
| Space dimension | $n = \{1, 2, 3\}$ |
| Components in the alloy | $N \in \mathbb{N}$ with $N \geq 2$ |
| Regularization exponent | $p > n$ |
| Viscousity factors | $\alpha, \beta > 0$ |
| Domain | $\Omega \subseteq \mathbb{R}^n$ bounded Lipschitz domain |
| Dirichlet boundary part | $\Gamma \subseteq \partial\Omega$ with $\mathcal{H}^{n-1}(\Gamma) > 0$ |
| Time interval | $[0, T]$ with $T > 0$ |
| Elastic energy density | $W^{\mathrm{el}} \in C^1(\mathbb{R}^{n\times n} \times \mathbb{R}^N \times \mathbb{R}; \mathbb{R}_+)$ |
| Chemical energy densities | $W^{\mathrm{ch}} \in C^1(\mathbb{R}^N; \mathbb{R})$ with $W^{\mathrm{ch}} \geq -C$,<br>$W^{\mathrm{ch,log}}(c) = \theta \sum_{k=1}^N c_k \log c_k + \frac{1}{2} c \cdot Ac$<br>with $\theta > 0$ and $A$ symmetric |
| Mobility matrix | $\mathbb{M} \in \mathbb{R}^{N\times N}$ symmetric and positive definite on $T\Sigma$<br>and $\sum_{l=1}^N \mathbb{M}_{kl} = 0$ for all $k = 1, \ldots, N$ |
| Gradient tensor | $\mathbf{\Gamma} \in \mathcal{L}(\mathbb{R}^{N\times n}; \mathbb{R}^{N\times n})$ symmetric and positive definite fourth order tensor |
| Projection matrix | $\mathbb{P}$ orthogonal projection from $\mathbb{R}^N$ to $T\Sigma$ |

The symmetry, convexity and growth assumptions we are using for $W^{\mathrm{ch}}$ and $W^{\mathrm{el}}$ throughout this paper are:

$$W^{\mathrm{el}}(e, c, z) = W^{\mathrm{el}}(e^t, c, z) \tag{A1}$$

$$W^{\mathrm{el}}(e, c, z) \leq C(|e|^2 + |c|^2 + 1) \tag{A2}$$

$$\eta |e_1 - e_2|^2 \leq (W^{\mathrm{el}}_{,e}(e_1, c, z) - W^{\mathrm{el}}_{,e}(e_2, c, z)) : (e_1 - e_2) \tag{A3}$$

$$|W^{\mathrm{el}}_{,e}(e_1 + e_2, c, z)| \leq C(W^{\mathrm{el}}(e_1, c, z) + |e_2| + 1) \tag{A4}$$

$$|W^{\mathrm{el}}_{,c}(e, c, z)| \leq C(|e|^2 + |c|^2 + 1) \tag{A5}$$

$$|W^{\mathrm{el}}_{,z}(e, c, z)| \leq C(|e|^2 + |c|^2 + 1) \tag{A6}$$

$$|W^{\mathrm{ch}}_{,c}(c)| \leq C(|c|^{2^\star/2} + 1) \tag{A7}$$

**Remark 2.5** *(i) We remark that in the work [HK10] we assumed the more restrictive growth condition $|W^{\mathrm{el}}_{,c}(e, c, z)| \leq C(|e| + |c|^2 + 1)$ instead of* (A5).

*(ii) If the space dimension $n$ is less than 3 the constant $2^\star$ can be replaced by a sufficiently large constant.*

The main result in this work is summarized in the following theorem:

**Theorem 2.6 (Existence theorem)** *Let the above assumptions be satisfied and, additionally, let $\mathbf{\Gamma} = \gamma\,\mathbf{Id}$ with a constant $\gamma > 0$. Then for every*

$$b \in W^{1,1}([0, T]; W^{1,\infty}(\Omega; \mathbb{R}^n)),$$

$$c^0 \in H^1(\Omega;\mathbb{R}^N) \text{ with } c^0 \in \Sigma \text{ and } c_k > 0 \text{ a.e. in } \Omega,$$
$$z^0 \in H^1(\Omega) \text{ with } 0 \le z^0 \le 1,$$

*there exists a weak solution $q$ of the system* ($\mathbf{S_0}$) *with initial-boundary condition* (**IBC**) *in the sense of Definition 2.3.*

# 3 Existence of weak solutions of ($\mathbf{S}_\varepsilon$)

The proof uses essential ideas from [HK10]. We concisely present the main ideas in this setting and focus on the modifications:

1. Step: constructing time-discrete solutions.
   Set $u_\varepsilon^0$ to be a minimizer of $u \mapsto \mathcal{E}_\varepsilon(u, c^0, z^0)$ defined on the space $W^{1,4}(\Omega)$ with the constraint $u|_\Gamma = b(0)|_\Gamma$ in the sense of traces.

   Since $\varepsilon$ is fixed in this section, we omit the $\varepsilon$-dependence in the notation, e.g. $\mathcal{E}$ always means $\mathcal{E}_\varepsilon$ and so on. Based on the initial triple $(u^0, c^0, z^0)$ we construct $(u_M^m, c_M^m, z_M^m)$ for $m = 1, \ldots, M$ recursively by minimizing the functional

$$\begin{aligned}&\mathbb{E}_M^m(u,c,z)\\ &:= \tilde{\mathcal{E}}(u,c,z) + \tau\tilde{\mathcal{R}}\left(\frac{z - z_M^{m-1}}{\tau}\right) + \frac{\tau}{2}\left\|\frac{c - c_M^{m-1}}{\tau}\right\|_X^2 + \frac{\varepsilon\tau}{2}\left\|\frac{c - c_M^{m-1}}{\tau}\right\|_{L^2(\Omega)}^2\end{aligned} \tag{6}$$

   on the closed subspace $\mathcal{Q}_M^m$ of $H^1(\Omega;\mathbb{R}^n) \times H^1(\Omega;\mathbb{R}^N) \times W^{1,p}(\Omega)$:

$$\mathcal{Q}_M^m := \begin{Bmatrix} u \in H^1(\Omega;\mathbb{R}^n), & u|_\Gamma = b(m\tau)|_\Gamma \\ c \in H^1(\Omega;\mathbb{R}^N), & \int_\Omega c - c^0 \,\mathrm{d}x = 0 \text{ for C-H systems} \\ z \in W^{1,p}(\Omega), & 0 \le z \le z_M^{m-1} \end{Bmatrix}$$

   where $X$ denotes the space $\mathcal{D}$ with the scalar-product

$$(c_1 \,|\, c_2)_X := (\mathbb{M}\nabla\mathcal{S}^{-1}c_1 \,|\, \nabla\mathcal{S}^{-1}c_2)_{L^2(\Omega;\mathbb{R}^N)}$$

   for Cahn-Hilliard systems and $X = L^2(\Omega;\mathbb{R}^N)$ with the scalar-product

$$(c_1 \,|\, c_2)_X := (\mathbb{M}c_1 \,|\, c_2)_{L^2(\Omega;\mathbb{R}^N)}$$

   for Allen-Cahn systems. Note, that the last regularization term in (6) is not necessary for Allen-Cahn equations due to the term with the $X$-norm. To use a uniform approach, we consider this term in both systems. By direct methods in the calculus of variations the functions

$$(u_M^m, c_M^m, z_M^m) := \underset{(u,c,z)\in\mathcal{Q}_M^m}{\arg\min}\ \mathbb{E}_M^m(u,c,z)$$

   are well-defined. Furthermore, we set

$$w_M^m := \begin{cases} -\mathcal{S}^{-1}\left(\frac{c_M^m - c_M^{m-1}}{\tau}\right) + \lambda_M^m, & \text{for C-H systems,} \\ -\mathcal{S}^{-1}\left(\frac{c_M^m - c_M^{m-1}}{\tau}\right), & \text{for A-C systems.}\end{cases}$$

with the Lagrange multiplier $\lambda_M^m$ (associated with the mass constraint) given by

$$\lambda_M^m := \fint_\Omega W_{,c}^{\mathrm{ch}}(c_M^m) + W_{,c}^{\mathrm{el}}(e(u_M^m), c_M^m, z_M^m)\,\mathrm{d}x.$$

We define the time incremental solutions as $q_M^m := (u_M^m, c_M^m, w_M^m, z_M^m)$ and introduce the constant interpolations $q_M$, $q_M^-, t_M, t_M^-$ and the linear interpolation $\hat{q}_M$ as

$$\begin{aligned}
&q_M(t) := q_M^m \text{ for } t \in \big((m-1)\tau, m\tau\big],\\
&q_M^-(t) := q_M^m \text{ for } t \in \big[m\tau, (m+1)\tau\big),\\
&t_M := \min\{m\tau \,|\, m \in \mathbb{N}_0 \text{ and } m\tau \geq t\},\\
&t_M^- := \min\{(m-1)\tau \,|\, m \in \mathbb{N}_0 \text{ and } m\tau \geq t\},\\
&\hat{q}_M(t) := \beta q_M^m + (1-\beta)q_M^{m-1} \text{ for } t \in \big[(m-1)\tau, m\tau\big) \text{ and } \beta = \frac{t}{\tau} - (m-1).
\end{aligned}$$

Due to the minimization properties of $q_M^m$ we establish the following variational formulas (cf. [HK10, Lemma 6.2]):

**Lemma 3.1 (Euler-Lagrange equation)** *The functions $q_M$, $q_M^-$ and $\hat{q}_M$ satisfy the following properties for all $t \in (0, T)$:*

*(i) for all $\zeta \in H^1(\Omega; \mathbb{R}^N)$:*

$$\int_\Omega (\partial_t \hat{c}_M(t)) \cdot \zeta \,\mathrm{d}x = -\langle \mathcal{S} w_M(t), \zeta\rangle \tag{7}$$

*(ii) for all $\zeta \in H^1(\Omega; \mathbb{R}^N)$:*

$$\begin{aligned}
\int_\Omega w_M(t) \cdot \zeta \,\mathrm{d}x = &\int_\Omega \mathbb{P}\boldsymbol{\Gamma}\nabla c_M(t) : \nabla\zeta + \mathbb{P}W_{,c}^{\mathrm{ch}}(c_M(t)) \cdot \zeta \,\mathrm{d}x\\
&+ \int_\Omega \mathbb{P}W_{,c}^{\mathrm{el}}(e(u_M(t)), c_M(t), z_M(t)) \cdot \zeta + \varepsilon\partial_t \hat{c}_M(t) \cdot \zeta \,\mathrm{d}x
\end{aligned} \tag{8}$$

*(iii) for all $\zeta \in W_\Gamma^{1,4}(\Omega; \mathbb{R}^n)$:*

$$\int_\Omega W_{,e}^{\mathrm{el}}(e(u_M(t)), c_M(t), z_M(t)) : e(\zeta) + \varepsilon|\nabla u_M(t)|^2 \nabla u_M(t) : \nabla\zeta \,\mathrm{d}x = 0 \tag{9}$$

*(iv) for all $\zeta \in W^{1,p}(\Omega)$ with $0 \leq \zeta + z_M(t) \leq z_M^-(t)$:*

$$\begin{aligned}
&\int_\Omega (\varepsilon|\nabla z_M(t)|^{p-2} + 1)\nabla z_M(t) \cdot \nabla\zeta + W_{,z}^{\mathrm{el}}(e(u_M(t)), c_M(t), z_M(t))\zeta \,\mathrm{d}x\\
&\qquad + \int_\Omega (-\alpha + \beta(\partial_t \hat{z}_M(t)))\zeta \,\mathrm{d}x \geq 0
\end{aligned} \tag{10}$$

2. Step: identifying convergent subsequences.

The recursive minimization scheme gives also rise to the energy estimate

$$\mathcal{E}(u_M(t), c_M(t), z_M(t)) + \int_0^{t_M} \int_\Omega -\alpha\partial_t \hat{z}_M + \frac{\beta}{2}|\partial_t \hat{z}_M|^2 + \frac{\varepsilon}{2}|\partial_t \hat{c}_M|^2 \,\mathrm{d}x\mathrm{d}s$$

$$
\begin{aligned}
&+\int_0^{t_M}\frac{1}{2}\langle \mathcal{S}w_M(s),w_M(s)\rangle\,\mathrm{d}s-\mathcal{E}(u^0,c^0,z^0)\\
&\qquad\le\int_0^{t_M}\int_\Omega W^{\mathrm{el}}_{,e}(e(u_M^-+b-b_M^-),c_M^-,z_M):e(\partial_t b)\,\mathrm{d}x\mathrm{d}s\\
&\qquad\quad+\varepsilon\int_0^{t_M}\int_\Omega|\nabla u_M^-+\nabla b-\nabla b_M^-|^2\nabla(u_M^-+b-b_M^-):\nabla\partial_t b\,\mathrm{d}x\mathrm{d}s \qquad (11)
\end{aligned}
$$

which leads to the boundedness (cf. [HK10, Lemma 6.3])

$$
\begin{aligned}
&\mathcal{E}(u_M(t),c_M(t),z_M(t))+\int_0^{t_M}\int_\Omega-\alpha\partial_t\hat{z}_M+\frac{\beta}{2}|\partial_t\hat{z}_M|^2+\frac{\varepsilon}{2}|\partial_t\hat{c}_M|^2\,\mathrm{d}x\mathrm{d}s\\
&+\int_0^{t_M}\frac{1}{2}\langle\mathcal{S}w_M(s),w_M(s)\rangle\,\mathrm{d}s\\
&\qquad\le C(\mathcal{E}(u^0,c^0,z^0)+1)
\end{aligned}
$$

for all $t\in[0,T]$ and all $M\in\mathbb{N}$ by growth condition (A4) and a Gronwall estimation argument. We obtain the a-priori estimates due to the energy estimate:

**Lemma 3.2 (A-priori estimates for the time incremental solutions)** *There exists a $C>0$ such that for all $M\in\mathbb{N}$ the following estimates holds:*

$$
\begin{aligned}
&\|u_M\|_{L^\infty([0,T];W^{1,4}(\Omega;\mathbb{R}^n))}\le C,\\
&\|c_M\|_{L^\infty([0,T];H^1(\Omega;\mathbb{R}^N))}\le C,\\
&\|\partial_t\hat{c}_M\|_{L^2(\Omega_T;\mathbb{R}^N)}\le C,\\
&\|w_M\|_{L^2([0,T];H^1(\Omega;\mathbb{R}^N))}\le C && \textit{for C-H systems},\\
&\|w_M\|_{L^2(\Omega_T;\mathbb{R}^N)}\le C && \textit{for A-C systems},\\
&\|z_M\|_{L^\infty([0,T];W^{1,p}(\Omega))}\le C,\\
&\|\partial_t\hat{z}_M\|_{L^2(\Omega_T)}\le C.
\end{aligned}
$$

By standard compactness arguments and a compactness theorem from Aubin and Lions [Sim86] we find a subsequence and an element $(u,c,w,z)=q$ satisfying (i) from Definition 2.2 such that

$$u_{M_k}\overset{\star}{\rightharpoonup}u\text{ in }L^\infty([0,T];W^{1,4}(\Omega))$$

and

$$
\begin{aligned}
&c_{M_k},c_{M_k}^-\overset{\star}{\rightharpoonup}c\text{ in }L^\infty([0,T];H^1(\Omega;\mathbb{R}^N)),\\
&c_{M_k}(t),c_{M_k}^-(t)\rightharpoonup c(t)\text{ in }H^1(\Omega;\mathbb{R}^N)\text{ for a.e. }t\in[0,T],\\
&c_{M_k},c_{M_k}^-\to c\text{ a.e. in }\Omega_T,\\
&\hat{c}_{M_k}\rightharpoonup c\text{ in }H^1([0,T];L^2(\Omega;\mathbb{R}^N))
\end{aligned}
$$

and

$$
\begin{aligned}
&w_{M_k}\rightharpoonup w\text{ in }L^2([0,T];H^1(\Omega;\mathbb{R}^N)) && \text{for C-H systems}\\
&w_{M_k}\rightharpoonup w\text{ in }L^2(\Omega_T;\mathbb{R}^N) && \text{for A-C systems}
\end{aligned}
$$

and

$$
\begin{aligned}
&z_{M_k}, z^-_{M_k} \overset{\star}{\rightharpoonup} z \text{ in } L^\infty([0,T];W^{1,p}(\Omega)),\\
&z_{M_k}(t), z^-_{M_k}(t) \rightharpoonup z(t) \text{ in } W^{1,p}(\Omega) \text{ for a.e. } t\in[0,T],\\
&z_{M_k}, z^-_{M_k} \to z \text{ a.e. in } \Omega_T,\\
&\hat{z}_{M_k} \rightharpoonup z \text{ in } H^1([0,T];L^2(\Omega))
\end{aligned}
$$

as $k\to\infty$ (cf. [HK10, Lemma 6.5]). For convenience we omit the index $k$ in the notation. Exploiting the Euler-Lagrange equations we can even prove stronger convergence properties. To proceed, we recall an approximation lemma from [HK10].

**Lemma 3.3 ([HK10, Lemma 4.2])** *Let $q\geq 1$, $p>n$ be real numbers and $f,\zeta\in L^q([0,T];W^{1,p}_+(\Omega))$ with $\{\zeta=0\}\supseteq\{f=0\}$. Furthermore, let $\{f_M\}_{M\in\mathbb{N}}\subseteq L^q([0,T];W^{1,p}_+(\Omega))$ be a sequence with $f_M(t)\rightharpoonup f(t)$ in $W^{1,p}(\Omega)$ as $M\to\infty$ for a.e. $t\in[0,T]$. Then there exists a sequence $\{\zeta_M\}_{M\in\mathbb{N}}\subseteq L^q([0,T];W^{1,p}_+(\Omega))$ and constants $\nu_{M,t}>0$ such that*

*(i)* $\zeta_M\to\zeta$ *in* $L^q([0,T];W^{1,p}(\Omega))$ *as* $M\to\infty$,

*(ii)* $\zeta_M\leq\zeta$ *a.e. in* $\Omega_T$ *for all* $M\in\mathbb{N}$,

*(iii)* $\nu_{M,t}\zeta_M(t)\leq f_M(t)$ *a.e. in* $\Omega$ *for a.e.* $t\in[0,T]$ *and for all* $M\in\mathbb{N}$.

*If, in addition, $\zeta\leq f$ a.e. in $\Omega_T$ then condition (iii) can be refined to*

*(iii)'* $\zeta_M\leq f_M$ *a.e. in* $\Omega_T$ *for all* $M\in\mathbb{N}$.

By using uniform convexity estimates we are now able to prove strong convergence results.

**Lemma 3.4 (Strong convergence of the time incremental solutions)** *There exists a subsequence $\{M_k\}$ such that*

*(i)* $u_{M_k}, u^-_{M_k}\to u$ *in* $L^4([0,T];W^{1,4}(\Omega;\mathbb{R}^n))$,
$u_{M_k}(t), u^-_{M_k}(t)\to u(t)$ *in* $W^{1,4}(\Omega;\mathbb{R}^n)$ *for a.e.* $t\in[0,T]$,
$u_{M_k}, u^-_{M_k}\to u$ *a.e. in* $\Omega_T$

*(ii)* $c_{M_k}, c^-_{M_k}\to c$ *in* $L^{2^\star}([0,T];H^1(\Omega;\mathbb{R}^N))$,
$c_{M_k}(t), c^-_{M_k}(t)\to c(t)$ *in* $H^1(\Omega;\mathbb{R}^N)$ *for a.e.* $t\in[0,T]$,
$c_{M_k}, c^-_{M_k}\to c$ *a.e. in* $\Omega_T$ *and*
$\hat{c}_{M_k}\rightharpoonup c$ *in* $H^1([0,T];L^2(\Omega;\mathbb{R}^N))$

*(iii)* $z_{M_k}, z^-_{M_k}\to z$ *in* $L^p([0,T];W^{1,p}(\Omega))$,
$z_{M_k}(t), z^-_{M_k}(t)\to z(t)$ *in* $W^{1,p}(\Omega)$ *for a.e.* $t\in[0,T]$,
$z_{M_k}, z^-_{M_k}\to z$ *a.e. in* $\Omega_T$ *and*
$\hat{z}_{M_k}\rightharpoonup z$ *in* $H^1([0,T];L^2(\Omega))$

*as* $k\to\infty$.

**Proof.** For (i) we refer to [HK10, Lemma 6.6]. We show (ii) and (iii), since the proofs in [HK10, Lemma 6.7, Lemma 6.8] need to be modified. We omit the index $k$ in the proof.

(ii) By Lebesgue's convergence theorem, Lemma 3.2 and the weak convergence properties we have $c_M \to c$ in $L^{2^\star/2+1}(\Omega_T;\mathbb{R}^N)$. Testing (8) with $\zeta = c_M(t)$ and with $\zeta = c(t)$ and integrate from $t=0$ to $t=T$ we have

$$\begin{aligned}\int_{\Omega_T} \mathbb{P}\mathbf{\Gamma}\nabla c_M : \nabla c_M \,\mathrm{d}x\mathrm{d}t &= \int_{\Omega_T} w_M \cdot c_M - \mathbb{P}W^{\mathrm{ch}}_{,c}(c_M)\cdot c_M \,\mathrm{d}x\mathrm{d}t\\ &\quad - \int_{\Omega_T} \mathbb{P}W^{\mathrm{el}}_{,c}(e(u_M),c_M,z_M)\cdot c_M + \varepsilon\partial_t\hat{c}_M\cdot c_M\,\mathrm{d}x\mathrm{d}t\\ \int_{\Omega_T} \mathbb{P}\mathbf{\Gamma}\nabla c_M : \nabla c \,\mathrm{d}x\mathrm{d}t &= \int_{\Omega_T} w_M \cdot c - \mathbb{P}W^{\mathrm{ch}}_{,c}(c_M)\cdot c \,\mathrm{d}x\mathrm{d}t\\ &\quad - \int_{\Omega_T} \mathbb{P}W^{\mathrm{el}}_{,c}(e(u_M),c_M,z_M)\cdot c + \varepsilon\partial_t\hat{c}_M\cdot c\,\mathrm{d}x\mathrm{d}t\end{aligned}$$

Passing to $M\to\infty$ and comparing the right sides of the equations shows

$$\int_{\Omega_T} \mathbb{P}\mathbf{\Gamma}\nabla c_M : \nabla c_M \,\mathrm{d}x\mathrm{d}t \to \int_{\Omega_T} \mathbb{P}\mathbf{\Gamma}\nabla c : \nabla c \,\mathrm{d}x\mathrm{d}t.$$

By using the properties $\mathbb{P}\nabla c_M = \nabla c_M$ and $\mathbb{P}\nabla c = \nabla c$ we eventually obtain $(\mathbf{\Gamma}\nabla c_M \,|\, \nabla c_M)_{L^2(\Omega_T;\mathbb{R}^N)} \to (\mathbf{\Gamma}\nabla c \,|\, \nabla c)_{L^2(\Omega_T;\mathbb{R}^N)}$. Since, by the assumptions on $\mathbf{\Gamma}$, $(\mathbf{\Gamma}\cdot \,|\, \cdot)_{L^2(\Omega_T;\mathbb{R}^N)}$ is a scalar product, it follows

$$(\mathbf{\Gamma}\nabla(c_M - c) \,|\, \nabla(c_M-c))_{L^2(\Omega_T;\mathbb{R}^N)} \to 0$$

and subsequently $\nabla c_M \to \nabla c$ in $L^2(\Omega_T;\mathbb{R}^N)$.

(iii) Applying Lemma 3.3 with $f = z$ and $f_M = z_M^-$ and $\zeta = z$ gives an approximation $\{\zeta_M\} \subseteq L^p([0,T];W^{1,p}_+(\Omega)) \cap L^\infty(\Omega_T)$ with the properties:

$$\zeta_M \to z \text{ in } L^p([0,T];W^{1,p}(\Omega)), \tag{12a}$$
$$0 \le \zeta_M \le z_M^- \text{ for all } M\in\mathbb{N} \tag{12b}$$

The estimate

$$\eta|\nabla z_M - \nabla z|^p \le (|\nabla z_M|^{p-2}\nabla z_M - |\nabla z|^{p-2}\nabla z)\cdot\nabla(z_M - z)$$

where $\eta > 0$ is a constant and equation (10) tested with $\zeta = \zeta_M(t) - z_M(t)$ (possible due to (12b)) yield:

$$\begin{aligned}&\eta\varepsilon\int_{\Omega_T} |\nabla z_M - \nabla z|^p\,\mathrm{d}x\mathrm{d}t + \int_{\Omega_T}|\nabla z_M - \nabla z|^2\,\mathrm{d}x\mathrm{d}t\\ &\quad\le \int_{\Omega_T} ((\varepsilon|\nabla z_M|^{p-2}+1)\nabla z_M - (\varepsilon|\nabla z|^{p-2}+1)\nabla z)\cdot\nabla(z_M - z)\,\mathrm{d}x\mathrm{d}t\\ &\quad\le \int_{\Omega_T} (\varepsilon|\nabla z_M|^{p-2}+1)\nabla z_M\cdot\nabla(z_M-\zeta_M)\,\mathrm{d}x\mathrm{d}t\\ &\qquad + \int_{\Omega_T} (\varepsilon|\nabla z_M|^{p-2}+1)\nabla z_M\cdot\nabla(\zeta_M - z)\,\mathrm{d}x\mathrm{d}t\\ &\qquad - \int_{\Omega_T} (\varepsilon|\nabla z|^{p-2}+1)\nabla z\cdot\nabla(z_M - z)\,\mathrm{d}x\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{\Omega_T} (W^{\mathrm{el}}_{,\mathrm{z}}(e(u_M), c_M, z_M) - \alpha + \beta\partial_t \hat{z}_M)(\zeta_M - z_M)\,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_{\Omega_T} (\varepsilon|\nabla z_M|^{p-2} + 1)\nabla z_M \cdot \nabla(\zeta_M - z)\,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_{\Omega_T} (\varepsilon|\nabla z|^{p-2} + 1)\nabla z \cdot \nabla(z_M - z)\,\mathrm{d}x\mathrm{d}t \\
&\leq \underbrace{\|W^{\mathrm{el}}_{,\mathrm{z}}(e(u_M), c_M, z_M) - \alpha + \beta\partial_t \hat{z}_M\|_{L^2(\Omega_T)}}_{\text{bounded}} \|\zeta_M - z_M\|_{L^2(\Omega_T)} \\
&\quad + \underbrace{(\varepsilon\|\nabla z_M\|^{p-1}_{L^p(\Omega_T)} + \|\nabla z_M\|_{L^{p/(p-1)}(\Omega_T)})}_{\text{bounded}} \|\nabla\zeta_M - \nabla z\|_{L^p(\Omega_T)} \\
&\quad - \int_{\Omega_T} (\varepsilon|\nabla z|^{p-2} + 1)\nabla z \cdot \nabla(z_M - z)\,\mathrm{d}x\mathrm{d}t
\end{aligned}
$$

Due to (12a) and $z_M \stackrel{\star}{\rightharpoonup} z$ in $L^\infty([0,T]; W^{1,p}(\Omega))$ each term on the right hand side converges to 0 as $M \to \infty$. ■

As already mentioned we omit the index $k$ in the following.

3. Step: establishing a precise energy inequality.

In this step we establish an asymptotic energy inequality, which is sharper than the energy inequality in (11). Note, that compared to (11) the factor 1/2 in front of $\langle \mathcal{S}w_M(s), w_M(s)\rangle$ is missing.

**Lemma 3.5** *For every $t \in [0,T]$:*

$$
\begin{aligned}
&\mathcal{E}(u_M(t), c_M(t), z_M(t)) + \int_0^{t_M}\int_\Omega -\alpha\partial_t\hat{z}_M + \beta|\partial_t\hat{z}_M|^2 + \varepsilon|\partial_t\hat{c}_M|^2\,\mathrm{d}x\mathrm{d}s \\
&+ \int_0^{t_M} \langle \mathcal{S}w_M(s), w_M(s)\rangle\,\mathrm{d}s - \mathcal{E}(u^0, c^0, z^0) \\
&\qquad \leq \int_0^{t_M}\int_\Omega W^{\mathrm{el}}_{,e}(e(u_M^- + b - b_M^-), c_M^-, z_M) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}s \\
&\qquad\quad + \varepsilon\int_0^{t_M}\int_\Omega |\nabla u_M^- + \nabla b - \nabla b_M^-|^2 \nabla(u_M^- + b - b_M^-) : \nabla\partial_t b\,\mathrm{d}x\mathrm{d}s + \kappa_M
\end{aligned}
$$

*with $\kappa_M \to 0$ as $M \to \infty$.*

**Proof.** Applying the estimate $\mathbb{E}^m_M(q^m_M) \leq \mathbb{E}^m_M(u^{m-1}_M + b^m_M - b^{m-1}_M, c^m_M, z^m_M)$ for $m = 1$ to $\frac{t_M}{\tau}$ yields (cf. [HK10, Lemma 6.10]):

$$
\begin{aligned}
&\mathcal{E}(u_M(t), c_M(t), z_M(t)) - \mathcal{E}(u^0, c^0, z^0) \\
&\qquad \leq \varepsilon\int_0^{t_M}\int_\Omega |\nabla(u_M^- + b(s) - b_M^-)|^2 \nabla(u_M^- + b(s) - b_M^-) : \nabla\partial_t b(s)\,\mathrm{d}x\mathrm{d}s \\
&\qquad\quad + \int_0^{t_M}\int_\Omega W^{\mathrm{el}}_{,e}(e(u_M^- + b - b_M^-), c_M^-, z_M^-) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&+\underbrace{\int_0^{t_M}\int_\Omega W^{\mathrm{el}}_{,c}(e(u_M^-+b_M-b_M^-),\hat c_M,z_M^-)\cdot\partial_t\hat c_M\,\mathrm{d}x\mathrm{d}s}_{(\star)_1}\\
&+\underbrace{\int_0^{t_M}\int_\Omega \mathbf{\Gamma}\nabla\hat c_M:\nabla\partial_t\hat c_M+W^{\mathrm{ch}}_{,c}(\hat c_M)\cdot\partial_t\hat c_M\,\mathrm{d}x\mathrm{d}s}_{(\star)_2}\\
&+\underbrace{\int_0^{t_M}\int_\Omega W^{\mathrm{el}}_{,z}(e(u_M^-+b_M-b_M^-),c_M,\hat z_M)\partial_t\hat z_M\,\mathrm{d}x\mathrm{d}s}_{(\star\star)_1}\\
&+\underbrace{\int_0^{t_M}\int_\Omega \varepsilon|\nabla\hat z_M|^{p-2}\nabla\hat z_M\cdot\nabla\partial_t\hat z_M+\nabla\hat z_M\cdot\nabla\partial_t\hat z_M\,\mathrm{d}x\mathrm{d}s}_{(\star\star)_2}
\end{aligned}
\tag{13}
$$

The elementary inequalities

$$
(|\nabla\hat z_M|^{p-2}\nabla\hat z_M-|\nabla z_M|^{p-2}\nabla z_M)\cdot\nabla\partial_t\hat z_M\le 0\quad\text{and}\quad(\nabla\hat z_M-\nabla z_M)\cdot\nabla\partial_t\hat z_M\le 0
$$

and (10) tested with $\zeta:=-\partial_t\hat z_M(t)\tau$ leads to the estimate:

$$
\begin{aligned}
&(\star\star)_1+(\star\star)_2\\
&\quad\le-\int_0^{t_M}\int_\Omega -\alpha\partial_t\hat z_M+\beta|\partial_t\hat z_M|^2\,\mathrm{d}x\mathrm{d}s\\
&\qquad+\underbrace{\int_0^{t_M}\int_\Omega(W^{\mathrm{el}}_{,z}(e(u_M^-+b_M-b_M^-),c_M,\hat z_M)-W^{\mathrm{el}}_{,z}(e(u_M),c_M,z_M))\partial_t\hat z_M\,\mathrm{d}x\mathrm{d}s}_{=:\kappa_M^3}
\end{aligned}
$$

Furthermore:

$$
\begin{aligned}
(\star)_1\le&\int_0^{t_M}\int_\Omega W^{\mathrm{el}}_{,c}(e(u_M),c_M,z_M)\cdot\partial_t\hat c_M\,\mathrm{d}x\mathrm{d}s\\
&+\underbrace{\int_0^{t_M}\int_\Omega(W^{\mathrm{el}}_{,c}(e(u_M^-+b_M-b_M^-),\hat c_M,z_M^-)-W^{\mathrm{el}}_{,c}(e(u_M),c_M,z_M))\cdot\partial_t\hat c_M\,\mathrm{d}x\mathrm{d}s}_{=:\kappa_M^1}
\end{aligned}
$$

Using the elementary estimate $\mathbf{\Gamma}(\nabla\hat c_M-\nabla c_M):\nabla\partial_t\hat c_M\le 0$:

$$
\begin{aligned}
(\star)_2\le&\int_0^{t_M}\int_\Omega\mathbf{\Gamma}\nabla c_M:\nabla\partial_t\hat c_M+W^{\mathrm{ch}}_{,c}(c_M)\cdot\partial_t\hat c_M\,\mathrm{d}x\mathrm{d}s\\
&+\underbrace{\int_0^{t_M}\int_\Omega(W^{\mathrm{ch}}_{,c}(\hat c_M)-W^{\mathrm{ch}}_{,c}(c_M))\cdot\partial_t\hat c_M\,\mathrm{d}x\mathrm{d}s}_{=:\kappa_M^2}
\end{aligned}
$$

Hence, applying equations (8) with $\zeta=\partial_t\hat c_M(t)$ and (7) with $\zeta=w_M(t)$ by noticing $\mathbb{P}\partial_t\hat c_M(t)=\partial_t\hat c_M(t)$ show:

$$
(\star)_1+(\star)_2\le-\int_0^{t_M}\langle\mathcal{S}w_M(s),w_M(s)\rangle\,\mathrm{d}s-\int_0^{t_M}\int_\Omega\varepsilon|\partial_t\hat c_M|^2\,\mathrm{d}x\mathrm{d}s+\kappa_M^1+\kappa_M^2.
$$

Lebesgue's generalized convergence theorem, growth conditions (A5)-(A7) and Lemma 3.4 shows $\kappa_M := \kappa_M^1 + \kappa_M^2 + \kappa_M^3 \to 0$ as $M \to \infty$. We would like to emphasize that we need the boundedness of $\nabla u_M$ in $L^4(\Omega_T;\mathbb{R}^{n\times n})$ and the boundedness of $\partial_t \hat{z}_M$ in $L^2(\Omega_T)$ with respect to $M$. ■

4. STEP: PASSING TO $M \to \infty$.

Using Lemma 3.4 and (7), (8) and (9) we establish (ii), (iii) and (iv) of Definition 2.2. Moreover, Lemma 3.5 implies:

$$
\begin{aligned}
&\mathcal{E}(u_M(t), c_M(t), z_M(t)) + \int_{\Omega_t} -\alpha\partial_t \hat{z}_M + \beta|\partial_t \hat{z}_M|^2 + \varepsilon|\partial_t \hat{c}_M|^2 \,\mathrm{d}x\mathrm{d}s \\
&+ \int_0^t \langle \mathcal{S}w_M(s), w_M(s)\rangle \,\mathrm{d}s - \mathcal{E}(u^0, c^0, z^0) \\
&\qquad\leq \int_0^{t_M}\int_\Omega W^{\mathrm{el}}_{,e}(e(u_M^- + b - b_M^-), c_M^-, z_M) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}s \\
&\qquad\quad + \varepsilon\int_0^{t_M}\int_\Omega |\nabla u_M^- + \nabla b - \nabla b_M^-|^2 \nabla(u_M^- + b - b_M^-) : \nabla\partial_t b\,\mathrm{d}x\mathrm{d}s + \kappa_M
\end{aligned}
$$

The energy estimate (vi) from Definition 2.2 follows from above by using Lemma 3.4 and weakly semi-continuity arguments.

It remains to show (v) of Definition 2.2. To proceed, take any test-function $\zeta \in L^p([0,T]; W^{1,p}_-(\Omega)) \cap L^\infty(\Omega_T)$ with $\{\zeta = 0\} \supseteq \{z = 0\}$. Lemma 3.3 gives a sequence $\{\zeta_M\} \subseteq L^p([0,T]; W^{1,p}_-(\Omega)) \cap L^\infty(\Omega_T)$ with $\zeta_M \to \zeta$ in $L^p([0,T]; W^{1,p}(\Omega))$ and $0 \geq \nu\zeta_M(t) \geq -z_M(t)$ where $\nu$ depends on $M$ and $t$. Therefore (10) holds for $\zeta = \zeta_M(t)$. Integration from 0 to $T$ and passing to $M \to \infty$ gives

$$
\int_{\Omega_T} (\varepsilon|\nabla z|^{p-2} + 1)\nabla z \cdot \nabla\zeta + (W^{\mathrm{el}}_{,z}(e(u), c, z) - \alpha + \beta(\partial_t \hat{z}))\zeta \,\mathrm{d}x\mathrm{d}t \geq 0.
$$

In other words

$$
\begin{aligned}
&\int_\Omega (\varepsilon|\nabla z(t)|^{p-2} + 1)\nabla z(t) \cdot \nabla\zeta + W^{\mathrm{el}}_{,z}(e(u(t)), c(t), z(t))\zeta \,\mathrm{d}x \\
&\qquad + \int_\Omega (-\alpha + \beta(\partial_t \hat{z}(t)))\zeta \,\mathrm{d}x \geq 0
\end{aligned}
$$

holds every $\zeta \in W^{1,p}_-(\Omega)$ with $\{\zeta = 0\} \supseteq \{z(t) = 0\}$ and a.e. $t \in [0,T]$. To finish the proof, we need a further lemma to extend the variational inequality to the whole space $W^{1,p}_-(\Omega)$.

**Lemma 3.6** *Let $f \in L^p(\Omega;\mathbb{R}^n)$, $g \in L^p(\Omega)$ and $z \in W^{1,p}(\Omega)$ with $f \cdot \nabla z \geq 0$ and $\{f = 0\} \supseteq \{z = 0\}$ a.e.. Furthermore, we assume that*

$$
\int_\Omega f \cdot \nabla\zeta + g\zeta \,\mathrm{d}x \geq 0 \quad \text{for all } \zeta \in W^{1,p}_-(\Omega) \text{ with } \{\zeta = 0\} \supseteq \{z = 0\}.
$$

*Then*

$$
\int_\Omega f \cdot \nabla\zeta + g\zeta \,\mathrm{d}x \geq \int_{\{z=0\}} [g]^+ \zeta \,\mathrm{d}x \quad \text{for all } \zeta \in W^{1,p}_-(\Omega).
$$

**Proof.** We assume $z \not\equiv 0$ on $\Omega$. Let $\zeta \in W^{1,p}_-(\Omega)$ be a test-function. For $\delta > 0$ small enough such that $\overline{\Omega} \setminus B_\delta(\{z = 0\}) \neq \emptyset$ we define

$$\zeta_\delta := \max\left\{\zeta, -z\|\zeta\|_{L^\infty} C_\delta^{-1}\right\}$$

with the constant

$$C_\delta := \inf\left\{z(x) \,|\, x \in \overline{\Omega} \setminus B_\delta(\{z = 0\})\right\} > 0.$$

We consider the following partition of $\overline{\Omega}$:

$$\overline{\Omega} = \Sigma_1 \cup \Sigma_2^{\leq} \cup \Sigma_2^{>}$$

with

$$\begin{aligned}
\Sigma_1 &:= \overline{\Omega} \setminus B_\delta(\{z = 0\}),\\
\Sigma_2^{\leq} &:= \overline{\Omega} \cap B_\delta(\{z = 0\}) \cap \{\zeta \leq -z\|\zeta\|_{L^\infty} C_\delta^{-1}\},\\
\Sigma_2^{>} &:= \overline{\Omega} \cap B_\delta(\{z = 0\}) \cap \{\zeta > -z\|\zeta\|_{L^\infty} C_\delta^{-1}\}.
\end{aligned}$$

By construction, the sequence $\{\zeta_\delta\}_{\delta \in (0,1]}$ satisfies

$$\zeta_\delta(x) = \begin{cases} \zeta(x), & \text{if } x \in \Sigma_1 \cup \Sigma_2^{>},\\ -z(x)\|\zeta\|_{L^\infty} C_\delta^{-1}, & \text{if } x \in \Sigma_2^{\leq}. \end{cases}$$

In particular $\zeta_\delta = 0$ on $\{z = 0\}$ for every $\delta \in (0, 1]$ and $\zeta_\delta \overset{\star}{\rightharpoonup} \zeta$ in $L^\infty(\{z > 0\})$ as $\delta \searrow 0$. Figure 1 visualizes some aspects of the idea of the construction of $\zeta_\delta = 0$.

We estimate

$$\begin{aligned}
&\int_\Omega f \cdot \nabla\zeta + g\zeta \,\mathrm{d}x - \int_{\{z=0\}} [g]^+ \zeta \,\mathrm{d}x\\
&\quad= \int_\Omega f \cdot \nabla(\zeta - \zeta_\delta) + g(\zeta - \zeta_\delta) \,\mathrm{d}x - \int_{\{z=0\}} [g]^+ \zeta \,\mathrm{d}x + \underbrace{\int_\Omega f \cdot \nabla\zeta_\delta + g\zeta_\delta \,\mathrm{d}x}_{\geq 0}\\
&\quad\geq \int_\Omega f \cdot \nabla(\zeta - \zeta_\delta) \,\mathrm{d}x + \int_{\{z>0\}} g(\zeta - \zeta_\delta) \,\mathrm{d}x\\
&\quad= \underbrace{\int_{\Sigma_1} f \cdot \nabla(\zeta - \zeta_\delta) \,\mathrm{d}x}_{=0} + \int_{\Sigma_2^{\leq}} f \cdot \nabla(\zeta - \zeta_\delta) \,\mathrm{d}x + \underbrace{\int_{\Sigma_2^{>}} f \cdot \nabla(\zeta - \zeta_\delta) \,\mathrm{d}x}_{=0}\\
&\qquad+ \int_{\{z>0\}} g(\zeta - \zeta_\delta) \,\mathrm{d}x\\
&\quad= \|\zeta\|_{L^\infty} C_\delta^{-1} \int_{\Sigma_2^{\leq}} f \cdot \nabla z \,\mathrm{d}x + \underbrace{\int_{\Sigma_2^{\leq}} f \cdot \nabla\zeta \,\mathrm{d}x}_{=\int_{\Sigma_2^{\leq} \setminus \{z=0\}} f \cdot \nabla\zeta \,\mathrm{d}x} + \int_{\{z>0\}} g(\zeta - \zeta_\delta) \,\mathrm{d}x\\
&\quad\geq \int_{\Sigma_2^{\leq} \setminus \{z=0\}} f \cdot \nabla\zeta \,\mathrm{d}x + \int_{\{z>0\}} g(\zeta - \zeta_\delta) \,\mathrm{d}x
\end{aligned}$$

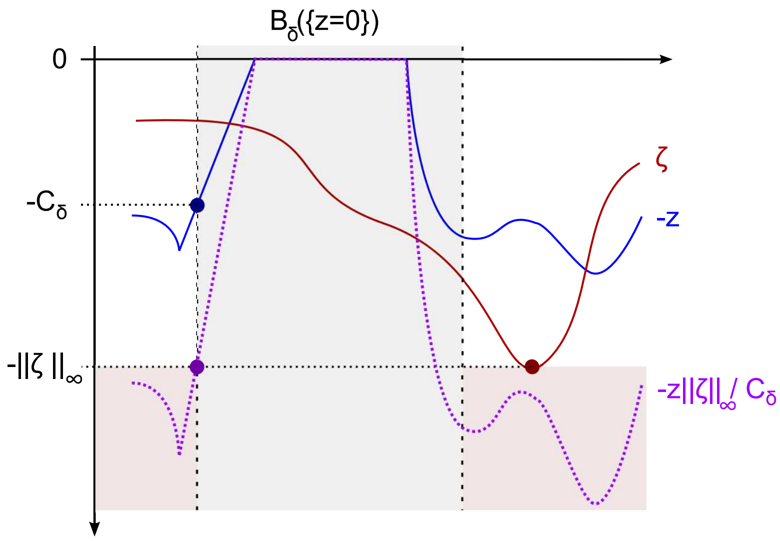


Figure 1: This example illustrates the construction of $\zeta_\delta$ for space dimension $n = 1$. The function $-z\|\zeta\|_{L^\infty}/C_\delta$ is less than $\zeta$ in $\bar{\Omega} \setminus B_\delta(\{z = 0\})$.

The terms on the right hand side converges to 0 as $\delta \searrow 0$. ■

Setting $f = (\varepsilon|\nabla z(t)|^{p-2} + 1)\nabla z(t)$ and $g = W^{\mathrm{el}}_{,z}(e(u(t)), c(t), z(t)) - \alpha + \beta(\partial_t \hat{z}(t))$, Lemma 3.6 shows

$$\begin{aligned}
\int_\Omega &(\varepsilon|\nabla z(t)|^{p-2} + 1)\nabla z(t) \cdot \nabla\zeta + (W^{\mathrm{el}}_{,z}(e(u(t)), c(t), z(t)) - \alpha + \beta(\partial_t \hat{z}(t)))\zeta \,\mathrm{d}x \\
&\geq \int_{\{z(t)=0\}} [W^{\mathrm{el}}_{,z}(e(u(t)), c(t), z(t)) - \alpha + \beta(\partial_t \hat{z}(t))]^+ \zeta \,\mathrm{d}x \\
&\geq \int_{\{z(t)=0\}} [W^{\mathrm{el}}_{,z}(e(u(t)), c(t), z(t))]^+ \zeta \,\mathrm{d}x.
\end{aligned}$$

Now, variational inequality (v) from Definition 2.2 follows by setting

$$r(t) := -\chi_{\{z(t)=0\}}[W^{\mathrm{el}}_{,z}(e(u(t)), c(t), z(t))]^+.$$

# 4 The limit case $\varepsilon \searrow 0$

In this chapter we show that an appropriate subsequence of the regularized solutions $\{q_\varepsilon\}_{\varepsilon\in(0,1]}$ of Definition 2.2 converges to a solution $q$ which satisfies the limit equations of Definition 2.2 for $\varepsilon = 0$. We will see that $r$ even lies in $L^1(\Omega)$ and therefore in $(L^\infty(\Omega))^\star$. In the next chapter we prove higher integrability of $\nabla u$, which even shows $r \in L^{1+\mu}(\Omega)$ for sufficiently small $\mu > 0$ since $r = -\chi[W^{\mathrm{el}}_{,z}(e(u(t)), c(t), z(t))]^+$ with $\chi \in L^\infty(\Omega)$.

For every $\varepsilon \in (0, 1]$ there exists a solution $q_\varepsilon$ of $(\mathbf{S}_\varepsilon)$ in the sense of Definition 2.2. In conclusion, the following integral equalities and inequalities hold for a.e. $t \in [0, T]$:

(i) for all $\zeta \in H^1(\Omega;\mathbb{R}^N)$:

$$\int_\Omega \partial_t c_\varepsilon(t)\cdot\zeta\,\mathrm{d}x = -\langle \mathcal{S}w_\varepsilon(t),\zeta\rangle, \tag{14}$$

(ii) for all $\zeta \in H^1(\Omega;\mathbb{R}^N)$:

$$\begin{aligned}\int_\Omega w_\varepsilon(t)\cdot\zeta\,\mathrm{d}x = &\int_\Omega \mathbb{P}\mathbf{\Gamma}\nabla c_\varepsilon(t):\nabla\zeta + \mathbb{P}W^{\mathrm{ch}}_{,c}(c_\varepsilon(t))\cdot\zeta\,\mathrm{d}x\\ &+\int_\Omega \mathbb{P}W^{\mathrm{el}}_{,c}(e(u_\varepsilon(t)),c_\varepsilon(t),z_\varepsilon(t))\cdot\zeta + \varepsilon\partial_t c_\varepsilon(t)\cdot\zeta\,\mathrm{d}x,\end{aligned} \tag{15}$$

(iii) for all $\zeta \in W^{1,4}_\Gamma(\Omega;\mathbb{R}^n)$:

$$\int_\Omega W^{\mathrm{el}}_{,e}(e(u_\varepsilon(t)),c_\varepsilon(t),z_\varepsilon(t)):e(\zeta)+\varepsilon|\nabla u_\varepsilon(t)|^2\nabla u_\varepsilon(t):\nabla\zeta\,\mathrm{d}x = 0, \tag{16}$$

(iv) for all $\zeta \in W^{1,p}_-(\Omega)$:

$$\begin{aligned}&\int_\Omega (\varepsilon|\nabla z_\varepsilon(t)|^{p-2}+1)\nabla z_\varepsilon(t)\cdot\nabla\zeta + (W^{\mathrm{el}}_{,z}(e(u_\varepsilon(t)),c_\varepsilon(t),z_\varepsilon(t)) - \alpha + \beta(\partial_t z_\varepsilon(t)))\zeta\,\mathrm{d}x\\ &\qquad\geq -\langle r_\varepsilon(t),\zeta\rangle,\end{aligned} \tag{17}$$

(v) where $r_\varepsilon(t)$ is given by

$$r_\varepsilon(t) = -\chi_{\{z_\varepsilon(t)=0\}}[W^{\mathrm{el}}_{,z}(e(u_\varepsilon(t)),c_\varepsilon(t),z_\varepsilon(t))]^+, \tag{18}$$

(vi) energy inequality:

$$\begin{aligned}&\mathcal{E}_\varepsilon(u_\varepsilon(t),c_\varepsilon(t),z_\varepsilon(t)) + \int_\Omega \alpha(z^0 - z_\varepsilon(t))\,\mathrm{d}x + \int_{\Omega_t}\beta|\partial_t z_\varepsilon|^2 + \varepsilon|\partial_t c_\varepsilon|^2\,\mathrm{d}x\mathrm{d}s\\ &+\int_0^t \langle \mathcal{S}w_\varepsilon(s),w_\varepsilon(s)\rangle\,\mathrm{d}s - \mathcal{E}_\varepsilon(u^0_\varepsilon,c^0,z^0)\\ &\qquad\leq \int_{\Omega_t} W^{\mathrm{el}}_{,e}(e(u_\varepsilon),c_\varepsilon,z_\varepsilon):e(\partial_t b)\,\mathrm{d}x\mathrm{d}s + \varepsilon\int_{\Omega_t}|\nabla u_\varepsilon|^2\nabla u_\varepsilon:\nabla\partial_t b\,\mathrm{d}x\mathrm{d}s\end{aligned} \tag{19}$$

Using the energy inequality and Gronwall's inequality we establish again the following energy estimate

$$\begin{aligned}&\mathcal{E}_\varepsilon(u_\varepsilon(t),c_\varepsilon(t),z_\varepsilon(t)) + \int_0^t\int_\Omega -\alpha\partial_t z_\varepsilon + \beta|\partial_t z_\varepsilon|^2 + \varepsilon|\partial_t c_\varepsilon|^2\,\mathrm{d}x\mathrm{d}s + \int_0^t\langle\mathcal{S}w_\varepsilon,w_\varepsilon\rangle\,\mathrm{d}s\\ &\qquad\leq C(\mathcal{E}_\varepsilon(u^0_\varepsilon,c^0,z^0)+1)\end{aligned}$$

for a.e. $t\in[0,T]$ and every $\varepsilon\in(0,1]$. Since $\mathcal{E}_\varepsilon(u^0_\varepsilon,c^0,z^0)\leq\mathcal{E}_\varepsilon(u^0_1,c^0,z^0)\leq\mathcal{E}_1(u^0_1,c^0,z^0)$, the left hand side is also uniformly bounded with respect to a.e. $t\in[0,T]$ and every $\varepsilon\in(0,1]$. This, on the other hand, results in the following a-priori estimates:

**Lemma 4.1 (A-priori estimates for $q_\varepsilon$)** *There exists a $C > 0$ such that for all $\varepsilon \in (0,1]$*

$$\begin{aligned}
&\varepsilon^{1/4}\|u_\varepsilon\|_{L^\infty([0,T];W^{1,4}(\Omega;\mathbb{R}^n))} \le C,\\
&\|u_\varepsilon\|_{L^\infty([0,T];H^1(\Omega;\mathbb{R}^n))} \le C,\\
&\|c_\varepsilon\|_{L^\infty([0,T];H^1(\Omega;\mathbb{R}^N))} \le C,\\
&\varepsilon^{1/2}\|\partial_t c_\varepsilon\|_{L^2(\Omega_T;\mathbb{R}^N)} \le C,\\
&\varepsilon^{1/p}\|z_\varepsilon\|_{L^\infty([0,T];W^{1,p}(\Omega))} \le C,\\
&\|z_\varepsilon\|_{L^\infty([0,T];H^1(\Omega))} \le C,\\
&\|\partial_t z_\varepsilon\|_{L^2(\Omega_T)} \le C,\\
&\left.\begin{aligned}&\|\partial_t c_\varepsilon\|_{L^2([0,T];(H^1(\Omega;\mathbb{R}^N))^\star)} \le C,\\ &\|w_\varepsilon\|_{L^2([0,T];H^1(\Omega;\mathbb{R}^N))} \le C,\end{aligned}\right\} \qquad \text{for C-H systems}\\
&\left.\begin{aligned}&\|\partial_t c_\varepsilon\|_{L^2(\Omega_T;\mathbb{R}^N)} \le C,\\ &\|w_\varepsilon\|_{L^2(\Omega_T;\mathbb{R}^N)} \le C.\end{aligned}\right\} \qquad \text{for A-C systems}
\end{aligned}$$

These estimates and the integral (in)equalities of $q_\varepsilon$ give rise to:

**Lemma 4.2 (Convergence properties of $q_\varepsilon$)** *There exists subsequence $\{\varepsilon_k\}$ with $\varepsilon_k \searrow 0$ as $k \to \infty$ and an element $(u,c,w,z) = q$ satisfying (i) of Definition 2.3 except $c_k > 0$ such that*

*(i)* $u_{\varepsilon_k} \to u$ *in* $L^2([0,T];H^1(\Omega;\mathbb{R}^n))$,
$u_{\varepsilon_k}(t) \to u(t)$ *in* $H^1(\Omega;\mathbb{R}^n)$ *for a.e.* $t \in [0,T]$ *and*
$u_{\varepsilon_k} \to u$ *a.e. in* $\Omega_T$

*(ii)* $c_{\varepsilon_k} \overset{\star}{\rightharpoonup} c$ *in* $L^\infty([0,T];H^1(\Omega;\mathbb{R}^N))$,
$c_{\varepsilon_k}(t) \rightharpoonup c(t)$ *in* $H^1(\Omega;\mathbb{R}^N)$ *for a.e.* $t \in [0,T]$,
$c_{\varepsilon_k} \to c$ *a.e. in* $\Omega_T$ *and*
$c_{\varepsilon_k} \rightharpoonup c$ *in* $H^1([0,T];L^2(\Omega;\mathbb{R}^N))$ *for A-C systems*

*(iii)* $w_{\varepsilon_k} \rightharpoonup w$ *in* $L^2([0,T];H^1(\Omega;\mathbb{R}^N))$ *for C-H systems*
$w_{\varepsilon_k} \rightharpoonup w$ *in* $L^2(\Omega_T;\mathbb{R}^N)$ *for A-C systems*

*(iv)* $z_{\varepsilon_k} \overset{\star}{\rightharpoonup} z$ *in* $L^\infty([0,T];H^1(\Omega))$,
$z_{\varepsilon_k}(t) \rightharpoonup z(t)$ *in* $H^1(\Omega)$ *for a.e.* $t \in [0,T]$,
$z_{\varepsilon_k} \to z$ *a.e. in* $\Omega_T$ *and*
$z_{\varepsilon_k} \rightharpoonup z$ *in* $H^1([0,T];L^2(\Omega))$

*as* $k \to \infty$.

**Proof.** Properties (ii), (iii) and (iv) follow from Lemma 4.1 and standard compactness theorems. For (i) we refer to [HK10, Lemma 6.14]. ■

As usual we omit the index $k$ in the subscripts.

**Remark 4.3** *We would like to mention that the arguments in [HK10, Lemma 6.14] cannot be adapted to prove strong convergence properties of $\nabla c_\varepsilon$ and $\nabla z_\varepsilon$ due to the usage of Lemma 3.3 and the lack of a compact embedding $H^1(\Omega) \not\subseteq C^{0,\alpha}(\overline{\Omega})$ for $n > 1$ as well as the more generous growth condition* (A5)*.*

We are now able to establish the limit equations of (14)-(19).

(i) Let $\zeta \in L^2([0,T]; H^1(\Omega;\mathbb{R}^N))$ with $\partial_t\zeta \in L^2(\Omega_T;\mathbb{R}^N)$ and $\zeta(T) = 0$. Integration from $t = 0$ to $t = T$ of (14) and integration by parts yields

$$\int_{\Omega_T} (c_\varepsilon - c^0) \cdot \partial_t\zeta \,\mathrm{d}x\mathrm{d}s = \int_0^T \langle \mathcal{S}w_\varepsilon, \zeta\rangle \,\mathrm{d}s.$$

Passing to $\varepsilon \searrow 0$ shows (ii) of Definition 2.3.

(ii) Let $\zeta \in L^2([0,T]; H^1(\Omega;\mathbb{R}^N)) \cap L^\infty(\Omega_T;\mathbb{R}^N)$. Integration from $t = 0$ to $t = T$ of (15) and passing to $\varepsilon \searrow 0$ yields

$$\int_{\Omega_T} w \cdot \zeta \,\mathrm{d}x\mathrm{d}s = \int_{\Omega_T} \mathbb{P}\mathbf{\Gamma}\nabla c : \nabla\zeta + (\mathbb{P}W^{\mathrm{ch}}_{,c}(c) + \mathbb{P}W^{\mathrm{el}}_{,c}(e(u), c, z)) \cdot \zeta \,\mathrm{d}x\mathrm{d}s.$$

Note that

$$\left|\int_{\Omega_T} \varepsilon\partial_t c_\varepsilon \cdot \zeta \,\mathrm{d}x\mathrm{d}s\right| \leq \varepsilon\|\partial_t c_\varepsilon\|_{L^2(\Omega_T;\mathbb{R}^N)}\|\zeta\|_{L^2(\Omega_T;\mathbb{R}^N)} \to 0$$

as $\varepsilon \searrow 0$. This shows (iii) of Definition 2.2 with $\varepsilon = 0$.

(iii) Let $\zeta \in W^{1,4}_\Gamma(\Omega;\mathbb{R}^n)$ be arbitrary. Passing to $\varepsilon \searrow 0$ in (16) yields for a.e. $t \in [0,T]$

$$\int_\Omega W^{\mathrm{el}}_{,e}(e(u(t)), c(t), z(t)) : e(\zeta) \,\mathrm{d}x = 0, \tag{20}$$

by noticing

$$\left|\int_\Omega \varepsilon|\nabla u_\varepsilon(t)|^2\nabla u_\varepsilon(t) : \nabla\zeta \,\mathrm{d}x\right| \leq \varepsilon\|\nabla u_\varepsilon(t)\|^3_{L^4(\Omega)}\|\zeta\|_{L^4(\Omega)} \to 0.$$

A density argument shows that (20) also holds for all $\zeta \in H^1_\Gamma(\Omega;\mathbb{R}^n)$. Therefore, (iv) of Definition 2.3 is shown.

(iv) The family $\{\chi_{\{z_\varepsilon=0\}}\}$ is bounded in $L^\infty(\Omega_T)$ with respect to $\varepsilon \in (0,1]$. We select a subsequence such that $\chi_{\{z_{\varepsilon_k}=0\}} \stackrel{\star}{\rightharpoonup} \chi$ in $L^\infty(\Omega_T)$ as $k \to \infty$. We omit the index $k$ in the notation. Integrating (17) from $t = 0$ to $t = T$ and passing to $\varepsilon \searrow 0$ shows

$$\int_{\Omega_T} \nabla z \cdot \nabla\zeta + (W^{\mathrm{el}}_{,z}(e(u), c, z) - \alpha + \beta(\partial_t z))\zeta \,\mathrm{d}x \geq -\int_{\Omega_T} \chi[W^{\mathrm{el}}_{,z}(e(u), c, z)]^+\zeta \,\mathrm{d}x\mathrm{d}s \tag{21}$$

for all $\zeta \in L^p([0,T]; W^{1,p}_-(\Omega)) \cap L^\infty(\Omega_T)$. We also use the fact that

$$\left|\int_{\Omega_T} \varepsilon|\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon \cdot \nabla\zeta \,\mathrm{d}x\mathrm{d}s\right| \leq \varepsilon\|\nabla z_\varepsilon\|^{p-1}_{L^p(\Omega_T)}\|\nabla\zeta\|_{L^p(\Omega_T)} \to 0.$$

It follows

$$\int_\Omega \nabla z(t)\cdot\nabla\zeta + (W^{\mathrm{el}}_{,z}(e(u(t)),c(t),z(t)) - \alpha + \beta(\partial_t z(t)))\zeta\,\mathrm{d}x$$
$$\geq -\int_\Omega \chi(t)[W^{\mathrm{el}}_{,z}(e(u(t)),c(t),z(t))]^+\zeta\,\mathrm{d}x$$

for all $\zeta\in H^1_-(\Omega)\cap L^\infty(\Omega)$ and a.e. $t\in[0,T]$. Set $r:=-\chi[W^{\mathrm{el}}_{,z}(e(u),c,z)]^+$. For every $\xi\in L^\infty([0,T])$ with $\xi\geq 0$ a.e. on $[0,T]$ and every $\zeta\in H^1_+(\Omega)\cap L^\infty(\Omega)$ we also have

$$\begin{aligned}0\geq \int_0^T\left(\int_\Omega r_\varepsilon(t)(\zeta - z_\varepsilon(t))\,\mathrm{d}x\right)\xi(t)\,\mathrm{d}t &= \int_{\Omega_T} r_\varepsilon(\zeta-z_\varepsilon)\xi\,\mathrm{d}x\mathrm{d}t\\ \to \int_{\Omega_T} r(\zeta-z)\xi\,\mathrm{d}x\mathrm{d}t &= \int_0^T\left(\int_\Omega r(t)(\zeta-z(t))\,\mathrm{d}x\right)\xi(t)\,\mathrm{d}t.\end{aligned}$$

This shows $\int_\Omega r(t)(\zeta-z(t))\,\mathrm{d}x\leq 0$ for a.e. $t\in[0,T]$. We obtain the inequalities (v) of Definition 2.3.

(v) Testing (16) with $\zeta = u_\varepsilon - b$ yields

$$\begin{aligned}\varepsilon\int_\Omega |\nabla u_\varepsilon(t)|^4\,\mathrm{d}x &= \varepsilon\int_\Omega |\nabla u_\varepsilon(t)|^2\nabla u_\varepsilon(t):\nabla b(t)\,\mathrm{d}x\\ &\quad - \int_\Omega W^{\mathrm{el}}_{,e}(e(u_\varepsilon(t)),c_\varepsilon(t),z_\varepsilon(t)) : e(u_\varepsilon(t)-b(t))\,\mathrm{d}x\\ &\to -\int_\Omega W^{\mathrm{el}}_{,e}(e(u(t)),c(t),z(t)) : e(u(t)-b(t))\,\mathrm{d}x = 0.\end{aligned}$$

Lemma 4.2 and weakly semi-continuity arguments leads to

$$\begin{aligned}&\liminf_{\varepsilon\searrow 0}\left(\mathcal{E}_\varepsilon(u_\varepsilon(t),c_\varepsilon(t),z_\varepsilon(t)) + \int_{\Omega_t}\beta|\partial_t z_\varepsilon|^2 + \underbrace{\varepsilon|\partial_t c_\varepsilon|^2}_{\geq 0}\,\mathrm{d}x\mathrm{d}s + \int_0^t\langle \mathcal{S}w_\varepsilon, w_\varepsilon\rangle\,\mathrm{d}s\right)\\ &\quad\geq \mathcal{E}(u(t),c(t),z(t)) + \int_{\Omega_t}\beta|\partial_t z|^2 + \int_0^t\langle\mathcal{S}w,w\rangle\,\mathrm{d}s.\end{aligned}$$

We obtain (vi) from Definition 2.3.

# 5 Higher integrability of the strain tensor

To prove existence results for chemical free energies of logarithmic type we establish a higher integrability result for the strain tensor, which is similar to [Gar00, Gar05b]. It It generalizes the higher integrability result for the strain tensor in [Gar00, Lemma 4.4] and [Gar00, Theorem 4.3] to the elliptic equation of the form

$$\left\{\begin{array}{ll}\mathrm{div}(W^{\mathrm{el}}_{,e}(e(u),c)), & \text{on } \Omega_T\\ W^{\mathrm{el}}_{,e}(e(u),c)\cdot\overrightarrow{\nu} = \sigma^\star\cdot\overrightarrow{\nu}, & \text{on } (\partial\Omega)_T\end{array}\right\}$$

to our setting with time-dependent boundary data $b$ and the additional damage variable $z$ in $(\mathbf{S_0})$. Before proving our integrability theorem we state two important types of inequalities.

**Theorem 5.1 (Sobolev-Poincaré type inequalities, cf. [Sob38], [Nir59], [BC07])** *Let $n \in \mathbb{N}$ with $n \geq 2$ and $p \in [1, n)$. There exists a constant $C > 0$ such that*

*(i) for all rectangles $D \subseteq \mathbb{R}^n$ and all $u \in W^{1,p}(D)$:*

$$\left( \fint_D |u - \fint_D u|^{p^\star} \right)^{\frac{1}{p^\star}} \leq C \left( \fint_D |\nabla u|^p \right)^{\frac{1}{p}} (\operatorname{diam} D)$$

*(ii) for all rectangles $D = \prod_{i=1}^n [a_i, b_i] \subseteq \mathbb{R}^n$ and all $u \in W^{1,p}(D)$ with $u(x) = 0$ for $x = (x_1, \ldots, x_{n-1}, a_n) \in D$ (in the sense of traces):*

$$\left( \fint_D |u|^{p^\star} \right)^{\frac{1}{p^\star}} \leq C \left( \fint_D |\nabla u|^p \right)^{\frac{1}{p}} (\operatorname{diam} D)$$

**Proof.**

(i) The case $1 < p < n$ was proven by Sobolev [Sob38] while Nirenberg [Nir59] gave a proof to $p = 1$.

(ii) The necessary modifications in this case can be found in a generalized form in [BC07]. ■

**Theorem 5.2 (Reverse Hölder inequality, see [Gia83])** *Let $Q \subseteq \mathbb{R}^n$ be a cube, $g \in L^q_{\mathrm{loc}}(Q)$ for a $q > 1$ and $g \geq 0$. Suppose that there exist a constant $b > 0$ and a function $f \in L^r_{\mathrm{loc}}(Q)$ with $r > q$ and $f \geq 0$ such that*

$$\fint_{Q_R(x_0)} g^q \,\mathrm{d}x \leq b \left( \fint_{Q_{2R}(x_0)} g \,\mathrm{d}x \right)^q + \fint_{Q_{2R}(x_0)} f^q \,\mathrm{d}x$$

*for each $x_0 \in Q$ and all $R > 0$ with $2R < \operatorname{dist}(x_0, \partial Q)$. Then $g \in L^s_{\mathrm{loc}}(Q)$ for $s \in [q, q + \varepsilon)$ for some $\varepsilon > 0$ and*

$$\left( \fint_{Q_R(x_0)} g^s \,\mathrm{d}x \right)^{\frac{1}{s}} \leq c \left( \left( \fint_{Q_{2R}(x_0)} g^q \,\mathrm{d}x \right)^{\frac{1}{q}} + \left( \fint_{Q_{2R}(x_0)} f^s \,\mathrm{d}x \right)^{\frac{1}{s}} \right)$$

*for all $x_0 \in Q$ and $R > 0$ such that $Q_{2R}(x_0) \subseteq Q$. The positive constants $c, \varepsilon > 0$ depend on $b$, $q$, $n$ and $r$.*

**Theorem 5.3 (Higher integrability)** *Consider the case $n \in \{2, 3\}$. Let $u \in H^1(\Omega; \mathbb{R}^n)$, $z \in L^\infty(\Omega)$ with $0 \leq z \leq 1$ a.e. in $\Omega$ and $c \in L^\mu(\Omega; \mathbb{R}^N)$ for a $\mu > 4$. We assume that for all $\zeta \in H^1_\Gamma(\Omega; \mathbb{R}^n)$ we have*

$$\int_\Omega W^{\mathrm{el}}_{,e}(e(u), c, z) : e(\zeta) \,\mathrm{d}x = 0 \tag{22}$$

*and* $u|_\Gamma = b|_\Gamma$ *for a* $b \in W^{1,\infty}(\Omega)$. *Then there exists a* $p \in (2, \mu/2]$ *such that* $u \in W^{1,p}(\Omega;\mathbb{R}^n)$ *and*

$$\|\nabla u\|_{L^p(\Omega;\mathbb{R}^{n\times n})} \leq C(\|\nabla u\|_{L^2(\Omega;\mathbb{R}^{n\times n})} + \|c\|^2_{L^{2p}(\Omega;\mathbb{R}^N)} + 1),$$

*where* $C$ *is independent of* $u$, $c$, $z$ *and* $p$ *is independent of* $c$.

**Proof.** The proof is based on [Gar00, Lemma 4.4 and Theorem 4.3] and uses a covering argument. Due to the non-constant boundary condition, we need to apply here a more general Sobolev-Poincaré inequality (see Theorem 5.1 (ii)).

(i) Higher integrability at the boundary.

Let $x_0 \in \partial\Omega$. Then there exists a bi-Lipschitz function $\tau : Q \to \mathbb{R}^n$ with $Q := Q_{R_0}(0)$ and $R_0 > 0$ such that

$$\tau(Q^+) \subseteq \Omega,$$
$$\tau(Q^-) \subseteq \mathbb{R}^n \setminus \overline{\Omega}.$$

where

$$Q^+ := \{x \in Q \,|\, x_n > 0\},$$
$$Q^- := \{x \in Q \,|\, x_n < 0\}.$$

Define the transformated functions $\tilde{u}, \tilde{b} \in H^1(Q^+;\mathbb{R}^n)$, $\tilde{c} \in H^1(Q^+)$ and $\tilde{z} \in L^\infty(Q^+)$ as

$$(\tilde{u}, \tilde{b}, \tilde{c}, \tilde{z})(x) := (u, b, c, z)(\tau(x))$$

To proceed, let $y_0 \in Q$ and $R < \frac{1}{2}\mathrm{dist}(y_0, \partial Q)$ and define for $R' > 0$

$$Q^+_{R'}(y_0) := \{x \in Q_{R'}(y_0) \,|\, x_n > 0\}.$$

We distinguish between three cases (cf. [Gar00, Theorem 4.3]):

**Case 1.** We first consider the case $Q^+_R(y_0) \neq \emptyset$ and $Q_{\frac{3}{2}R}(y_0) \cap Q^- \neq \emptyset$.

The bi-Lipschitz continuity of $\tau$ ensures

$$\mathrm{dist}(\tau(\partial Q^+_{2R}(y_0)) \cap \Omega, \tau(\partial Q^+_R(y_0)) \cap \Omega) > RC_1,$$

where $C_1 > 0$ is independent of $R$ and $y_0$. Let $\xi \in C_0^\infty(\Omega)$ be a cutoff function with the properties:

(i).1 $\xi = 0$ in $\Omega \setminus \tau(Q_{2R}(y_0))$,

(i).2 $0 \leq \xi \leq 1$ in $\Omega$,

(i).3 $\xi \equiv 1$ in $\tau(Q_R(y_0)) \cap \Omega$,

(i).4 $|\nabla\xi| \leq \frac{2}{C_1}R^{-1}$.

Now, choosing the test-function $\zeta = \xi^2(u-b)$ in (22) and using the computation

$$e(\zeta) = \xi^2 e(u) - \xi^2 e(b) + \xi((u-b)(\nabla\xi)^t + \nabla\xi(u-b)^t)$$

we obtain by (A1):

$$\begin{aligned}\int_\Omega \xi^2 W^{\mathrm{el}}_{,e}(e(u),c,z) : e(u)\,\mathrm{d}x \\ = \int_\Omega \xi^2 W^{\mathrm{el}}_{,e}(e(u),c,z) : e(b)\,\mathrm{d}x - 2\int_\Omega \xi W^{\mathrm{el}}_{,e}(e(u),c,z) : ((u-b)(\nabla\xi)^t)\,\mathrm{d}x\end{aligned} \tag{23}$$

By (A3), (A4) and (A2) we also have the estimates

$$\begin{aligned}&\eta|e(u)|^2 \le W^{\mathrm{el}}_{,e}(e(u),c,z) : e(u) + C(|c|^2+1)|e(u)|,\\ &|W^{\mathrm{el}}_{,e}(e(u),c,z) : ((u-b)(\nabla\xi)^t| \le \frac{C}{R}(|e(u)| + |c|^2 + 1)|u-b|,\\ &|W^{\mathrm{el}}_{,e}(e(u),c,z) : e(b)| \le (|e(u)| + |c|^2+1)|e(b)|.\end{aligned}$$

Therefore (23) can be estimated to

$$\begin{aligned}\eta\int_\Omega \xi^2|e(u)|^2\,\mathrm{d}x \le C\int_\Omega \xi^2(|c|^2+1)|e(u)|\,\mathrm{d}x + \frac{C}{R}\int_\Omega \xi(|e(u)|+|c|^2+1)|u-b|\,\mathrm{d}x\\ + C\int_\Omega \xi^2(|e(u)|+|c|^2+1)|e(b)|\,\mathrm{d}x.\end{aligned}$$

Young's inequality yields

$$c_1\int_\Omega \xi^2|e(u)|^2\,\mathrm{d}x \le C\int_\Omega \xi^2(|c|^4+1)\,\mathrm{d}x + \frac{C}{R^2}\int_{\tau(Q_{2R}(y_0))}|u-b|^2\,\mathrm{d}x. \tag{24}$$

We choose $\mu := ⨍_{Q^+_{2R}(y_0)} \tilde u\,\mathrm{d}x$. The calculation $e(\xi(u-\mu)) = \xi e(u) + \frac{1}{2}((u-\mu)(\nabla\xi)^t + \nabla\xi(u-\mu)^t)$ leads to

$$\int_\Omega |e(\xi(u-\mu))|^2\,\mathrm{d}x \le 2\left(\int_\Omega \xi^2|e(u)|^2\,\mathrm{d}x + \int_\Omega |u-\mu|^2|\nabla\xi|^2\,\mathrm{d}x\right). \tag{25}$$

Using (24) and (25) and applying Korn's inequality give

$$\begin{aligned}\int_\Omega |\nabla(\xi(u-\mu))|^2\,\mathrm{d}x \le C\int_\Omega \xi^2(|c|^4+1)\,\mathrm{d}x + \frac{C}{R^2}\int_{\tau(Q^+_{2R}(y_0))}|u-b|^2\,\mathrm{d}x\\ + \frac{C}{R^2}\int_{\tau(Q^+_{2R}(y_0))}|u-\mu|^2\,\mathrm{d}x.\end{aligned}$$

Because of $\nabla(\xi(u-\mu)) = \xi\nabla u + (u-\mu)(\nabla\xi)^t$ we derive by (a) and (c) the following type of Caccioppoli-inequality:

$$\begin{aligned}\int_{\tau(Q^+_R(y_0))}|\nabla u|^2\,\mathrm{d}x \le C\int_{\tau(Q^+_{2R}(y_0))}(|c|^4+1)\,\mathrm{d}x + \frac{C}{R^2}\int_{\tau(Q^+_{2R}(y_0))}|u-b|^2\,\mathrm{d}x\\ + \frac{C}{R^2}\int_{\tau(Q^+_{2R}(y_0))}|u-\mu|^2\,\mathrm{d}x.\end{aligned}$$

Integral transformation by $\tau$ implies

$$\int_{Q_R^+(y_0)} |\nabla\tilde{u}|^2 \,\mathrm{d}x \le C \int_{Q_{2R}^+(y_0)} (|\tilde{c}|^4 + 1) \,\mathrm{d}x + \frac{C}{R^2} \int_{Q_{2R}^+(y_0)} |\tilde{u} - \tilde{b}|^2 \,\mathrm{d}x$$
$$+ \frac{C}{R^2} \int_{Q_{2R}^+(y_0)} |\tilde{u} - \mu|^2 \,\mathrm{d}x.$$

The condition $Q_{\frac{3}{2}R}(y_0) \cap Q^- \neq \emptyset$ implies that $\tilde{u} - \tilde{b}$ vanishes on $\partial\big(Q_{2R}^+(y_0)\big) \cap \mathbb{R}^{n-1} \times \{0\}$. Therefore, we obtain by applying both variants of the Poincaré-Sobolev inequality (see Theorem 5.1)

$$\begin{aligned}\int_{Q_R^+(y_0)} |\nabla\tilde{u}|^2 \,\mathrm{d}x \le C \int_{Q_{2R}^+(y_0)} (|\tilde{c}|^4 + 1) \,\mathrm{d}x + \frac{C}{R^2} \mathcal{L}^n(Q_{2R}^+(y_0))^{-\frac{2}{n}} \mathrm{diam}(Q_{2R}^+(y_0))^2 \\ \cdot \left[ \left( \int_{Q_{2R}^+(y_0)} |\nabla\tilde{u} - \nabla\tilde{b}|^{\frac{2n}{n+2}} \,\mathrm{d}x \right)^{\frac{n+2}{n}} + \left( \int_{Q_{2R}^+(y_0)} |\nabla\tilde{u}|^{\frac{2n}{n+2}} \,\mathrm{d}x \right)^{\frac{n+2}{n}} \right].\end{aligned} \tag{26}$$

The estimates $\mathrm{diam}(Q_{2R}^+(y_0)) \le CR$ and $\mathcal{L}^n(Q_{2R}^+(y_0)) \ge R^n$ (for all $(y_0, R)$ with $Q_R^+(y_0) \neq \emptyset$) show

$$\mathcal{L}^n(Q_{2R}^+(y_0))^{-\frac{2}{n}} \mathrm{diam}(Q_{2R}^+(y_0))^2 \le C.$$

Now, dividing (26) by $\mathcal{L}^n(Q_R(y_0))$ and using

$$\frac{1}{R^2} \frac{1}{\mathcal{L}^n(Q_{2R}(y_0))} \le C \left( \frac{1}{\mathcal{L}^n(Q_{2R}(y_0))} \right)^{\frac{n+2}{n}}$$

gives

$$\begin{aligned}\frac{1}{\mathcal{L}^n(Q_R(y_0))} \int_{Q_R^+(y_0)} |\nabla\tilde{u}|^2 \,\mathrm{d}x &\le \frac{C}{\mathcal{L}^n(Q_{2R}(y_0))} \int_{Q_{2R}^+(y_0)} (|\tilde{c}|^4 + 1) \,\mathrm{d}x \\ &+ C \left( \frac{1}{\mathcal{L}^n(Q_{2R}(y_0))} \int_{Q_{2R}^+(y_0)} |\nabla\tilde{u}|^{\frac{2n}{n+2}} \,\mathrm{d}x \right)^{\frac{n+2}{n}} \\ &+ C \left( \frac{1}{\mathcal{L}^n(Q_{2R}(y_0))} \int_{Q_{2R}^+(y_0)} |\nabla\tilde{b}|^{\frac{2n}{n+2}} \,\mathrm{d}x \right)^{\frac{n+2}{n}}.\end{aligned}$$

Observe that

$$\left( \frac{1}{\mathcal{L}^n(Q_{2R}(y_0))} \int_{Q_{2R}^+(y_0)} |\nabla\tilde{b}|^{\frac{2n}{n+2}} \,\mathrm{d}x \right)^{\frac{n+2}{n}} \le \|\nabla b\|^2_{L^\infty(\Omega)}.$$

Defining

$$g(x) := \begin{cases} |\nabla\tilde{u}(x)|^{\frac{2n}{n+2}}, & \text{for } x \in Q^+, \\ 0, & \text{for } x \in Q \setminus Q^+, \end{cases}$$

and

$$f(x) := \begin{cases} C(|\tilde{c}|^4 + \|\nabla b\|^2_{L^\infty(\Omega)} + 1)^{\frac{n}{n+2}}, & \text{for } x \in Q^+, \\ 0, & \text{for } x \in Q \setminus Q^+. \end{cases}$$

We eventually get

$$\fint_{Q_R(y_0)} g^{\frac{n+2}{n}} \,\mathrm{d}x \le \fint_{Q_{2R}(y_0)} f^{\frac{n+2}{n}} \,\mathrm{d}x + C \left( \fint_{Q_{2R}(y_0)} g \,\mathrm{d}x \right)^{\frac{n+2}{n}}. \tag{27}$$

The reverse Hölder inequality (see Theorem 5.2) shows $g \in L^s_{\mathrm{loc}}(Q)$ for all $s \in \left[\frac{n+2}{n}, \frac{n+2}{n} + \varepsilon\right)$ and some $\varepsilon > 0$ depending on $R_0$ and $n$.

**Case 2.** Assume $Q_R^+(y_0) \neq \emptyset$ and $Q_{\frac{3}{2}R}(y_0) \cap Q^- = \emptyset$.

The case condition implies

$$\operatorname{dist}(\tau(\partial Q_{\frac{3}{2}R}(y_0)), \tau(\partial Q_R(y_0))) > RC_1,$$

where $C_1 > 0$ is independent of $R$ and $y_0$. We choose the cutoff function $\xi \in C_0^\infty(\Omega)$ to be

(i).1 $\xi = 0$ in $\Omega \setminus Q_{\frac{3}{2}R}(x_0)$,

(i).2 $0 \le \xi \le 1$ in $\Omega$,

(i).3 $\xi \equiv 1$ in $Q_R(x_0)$,

(i).4 $|\nabla \xi| \le \frac{2}{C_1} R^{-1}$.

Testing (22) with $\xi = \zeta^2(u - \mu)$ and $\mu := \fint_{Q_{2R}(x_0)} u \,\mathrm{d}x$ yields as in the previous case

$$\int_{\tau(Q_R(x_0))} |\nabla u|^2 \,\mathrm{d}x \le C \int_{\tau(Q_{2R}(x_0))} (|c|^4 + 1) \,\mathrm{d}x + \frac{C}{R^2} \int_{\tau(Q_{2R}(x_0))} |u - \mu|^2 \,\mathrm{d}x.$$

Consequently

$$\fint_{Q_R(x_0)} |\nabla \tilde{u}|^2 \,\mathrm{d}x \le C \fint_{Q_{2R}(x_0)} (|\tilde{c}|^4 + 1) \,\mathrm{d}x + C \left( \fint_{Q_{2R}(x_0)} |\nabla \tilde{u}|^{\frac{2n}{n+2}} \,\mathrm{d}x \right)^{\frac{n+2}{n}}.$$

Therefore, the inequality (27) is in this case also satisfied.

**Case 3.** Assume $Q_R^+(y_0) = \emptyset$.

In this case inequality (27) trivially holds.

(ii) Higher integrability in the interior.

This case follows with much less effort. Let $x_0 \in \Omega$ arbitrary and $R > 0$ such that $Q_{2R}(x_0) \subseteq \Omega$. We take a cutoff function $\xi \in C_0^\infty(\Omega)$ with

(ii).1 $\xi = 0$ in $\Omega \setminus Q_{2R}(x_0)$,

(ii).2 $0 \le \xi \le 1$ in $\Omega$,

(ii).3 $\xi \equiv 1$ in $Q_R(x_0)$,

(ii).4 $|\nabla\xi| \leq \frac{2}{R}$.

Testing (22) with $\xi = \zeta^2(u-\mu)$ and $\mu := \fint_{Q_{2R}(x_0)} u \,\mathrm{d}x$ yields as in the case (i) (also compare with [Gar00, Lemma 4.4]):

$$\int_{Q_R(x_0)} |\nabla u|^2 \,\mathrm{d}x \leq C \int_{Q_{2R}(x_0)} (|c|^4 + 1) \,\mathrm{d}x + \frac{C}{R^2} \int_{Q_{2R}(x_0)} |u-\mu|^2 \,\mathrm{d}x$$

Poincaré-Sobolev inequality implies

$$\fint_{Q_R(x_0)} |\nabla u|^2 \,\mathrm{d}x \leq C \fint_{Q_{2R}(x_0)} (|c|^4 + 1) \,\mathrm{d}x + C \left( \fint_{Q_{2R}(x_0)} |\nabla u|^{\frac{2n}{n+2}} \,\mathrm{d}x \right)^{\frac{n+2}{n}}.$$

Applying Theorem 5.2 with $g = |\nabla u|^{\frac{2n}{n+2}}$, $q = \frac{n+2}{n}$ and $f = C(|c|^4+1)^{\frac{n}{n+2}}$ finishes the proof. ■

# 6 Existence of weak solutions of $(\mathbf{S_0})$

The challenge here is to establish the integral equation (iii) in Definition 2.3 because the derivative of the logarithmic free chemical energy (3) becomes singular if one of the $c_k$'s approaches 0. We only sketch the proof in this section since all essential ideas can be found in [Gar00, Gar05b]. We use a regularization method suggested in [EL91] and also used in [Gar00, Gar05b].

The energy gradient tensor is assumed to be of the form $\mathbf{\Gamma} = \gamma\,\mathbf{Id}$ with a constant $\gamma > 0$. Define a $C^2(\mathbb{R}^N)$ regularization with the regularization parameter $\delta > 0$ as

$$W^{\mathrm{ch},\delta}(c) := \theta \sum_{k=1}^{N} \phi^\delta(c^k) + \frac{1}{2} c \cdot Ac,$$

with

$$\phi^\delta(x) := \begin{cases} x\log(x), & \text{for } d \geq \delta, \\ x\log(\delta) - \frac{\delta}{2} + \frac{x^2}{2\delta}, & \text{for } x < \delta. \end{cases}$$

Elliott and Luckhaus showed that the regularization $W^{\mathrm{ch},\delta}$ is uniformly bounded from below.

**Lemma 6.1 (cf. [EL91])** *There exists constants $\delta_0 > 0$ and $C > 0$ such that*

$$W^{\mathrm{ch},\delta}(c) \geq -C \qquad \textit{for all } c \in \Sigma,\ \delta \in (0,\delta_0).$$

Let $q_\delta$ denote a weak solution in the sense of Definition 2.3 except that we use the regularized free chemical energy $W^{\mathrm{ch},\delta}$ instead of $W^{\mathrm{ch},\log}$. Applying Lemma 6.1 and using Gronwall's inequality in the energy inequality (vi) of Definition 2.3 we can show a-priori estimates analogous as in Section 4 except the boundedness of $w_\delta$ in the space $L^2([0,T];H^1(\Omega))$ in the Cahn-Hilliard case. However, in the case of Cahn-Hilliard systems, we can use the following lemma, cf. [Gar00, Gar05b]:

**Lemma 6.2** *There exists a constant $C > 0$ such that for all $\delta \in (0, \delta_0)$*

$$\int_0^T \left( \fint_\Omega \mathbb{P}W^{\mathrm{ch},\delta}_{,c}(c_\delta(t))\,\mathrm{d}x \right)^2 \mathrm{d}t < C.$$

The proof of this lemma is similar to [Gar00, Lemma 4.3], since all arguments can be adapted to our case. Therefore we will omit the proof.

**Remark 6.3** *Within our notation we obtain*

$$W^{\mathrm{ch},\delta}_{,c}(c) = \theta \left( (\phi^\delta)'(c^k) \right)_{k=1}^N + \frac{1}{2} Ac$$

*for $c \in \mathbb{R}^N$.*

This lemma and the integral equation

$$\int_\Omega w_\delta(t)\,\mathrm{d}x = \int_\Omega \mathbb{P}W^{\mathrm{ch},\delta}_{,c}(c_\delta(t)) + \mathbb{P}W^{\mathrm{el}}_{,c}(e(u_\delta(t)), c_\delta(t), z_\delta(t))\,\mathrm{d}x$$

together with the already known boundedness properties shows

$$\int_0^T \left( \fint_\Omega w_\delta(t)\,\mathrm{d}x \right)^2 \mathrm{d}t < C$$

for a constant $C > 0$. Therefore $w_\delta$ is bounded in $L^2([0,T]; H^1(\Omega))$ by Poincaré's inequality. We can extract a subsequence $\{q_{\delta_k}\}$ such that we have the same convergence properties as in Lemma 4.2. We omit the subscript $k$. The remaining crucial step is to show that the limit $c$ satisfies $c_k > 0$ a.e. on $\Omega_T$ for all $k = 1, \dots, N$ and $W^{\mathrm{ch},\delta}_{,c}(c_\delta) \to W^{\mathrm{ch},\log}_{,c}(c)$ in $L^1(\Omega_T)$ as $\varepsilon \searrow 0$. Then the rest follows as in Section 4.

To proceed, we need an additional boundedness property.

**Lemma 6.4** *There exists constants $q > 1$ and $C > 0$ such that for all $\delta \in (0, \delta_0)$ and all $k = 1, \dots, N$*

$$\|(\phi^\delta)'(c^k_\delta)\|_{L^q(\Omega_T)} < C.$$

We omit the proof of this lemma. By utilizing Theorem 5.3 the arguments are analogous to [Gar00, Lemma 4.3].

It follows directly that

$$\lim_{\delta \searrow 0} (\phi^\delta)'(c^k_\delta) = \begin{cases} \log(c^k) + 1, & \text{if } \lim_{\delta \searrow 0} c^k_\delta = c^k > 0, \\ \infty, & \text{else.} \end{cases}$$

holds. Together with Lemma 6.4 we obtain

$$c_k > 0 \text{ a.e. on } \Omega_T$$

and

$$(\phi^\delta)'(c^k_\delta) \to \log(c^k) + 1 \text{ a.e. on } \Omega_T.$$

This and Lemma 6.4 further shows

$$(\phi^\delta)'(c^k_\delta) \to \log(c^k) + 1 \text{ in } L^1(\Omega_T)$$

by Vitali's convergence theorem. Finally, we can pass to $\delta \searrow 0$ in the equation

$$\int_{\Omega_T} w_\delta \cdot \zeta \,\mathrm{d}x\mathrm{d}t = \int_{\Omega_T} \gamma \nabla c_\delta : \nabla\zeta + \mathbb{P}W^{\mathrm{ch},\delta}_{,c}(c_\delta)\cdot\zeta + \mathbb{P}W^{\mathrm{el}}_{,c}(e(u_\delta), c_\delta, z_\delta)\cdot\zeta \,\mathrm{d}x\mathrm{d}t$$

and obtain (iii) from Definition 2.3. The remaining properties follow as in Section 4.

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